GEOMETRIC OPTICS

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CHAPTER OVERVIEW

1: Reflection and Refraction

Thumbnail: The larger the angle to the normal, the smaller is the fraction of light transmitted rather than reflected, until the angle at which total internal reflection occurs. The color of the rays is to help distinguish the rays, and is not meant to indicate any color dependence. (CC BY-SA 3.0; Clément 421138).

1.1: Introduction

This "book" is not intended to be a vast, definitive treatment of everything that is known about geometric optics. It covers, rather, the geometric optics of first-year students, whom it will either help or confuse yet further, though I hope the former. The part of geometric optics that often causes the most difficulty, particularly in getting the right answer for homework or examination problems, is the vexing matter of sign conventions in lens and mirror calculations. It seems that no matter how hard we try, we always get the sign wrong! This aspect will be dealt with in Chapter 2. The present chapter deals with simpler matters, namely reflection and refraction at a plane surface, except for a brief foray into the geometry of the rainbow. The rainbow, of course, involves refraction by a spherical drop. For the calculation of the radius of the bow, only Snell's law is needed, but some knowledge of physical optics will be needed for a fuller understanding of some of the material in Section 1.7, which is a little more demanding than the rest of the chapter.

1.2: Reflection at a Plane Surface

The **law of reflection** of light is merely that the angle of reflection r is equal to the angle of incidence i. There is really very little that can be said about this, but I'll try and say what little need be said.

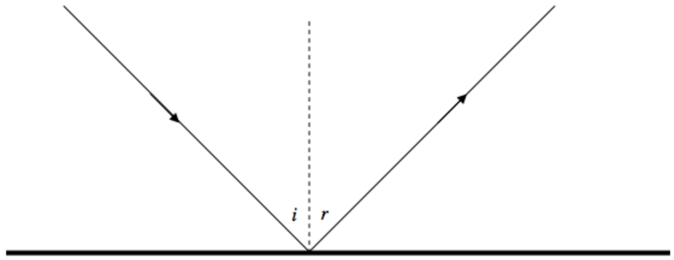


FIGURE I.1

i. It is customary to measure the angles of incidence and reflection from the normal to the reflecting surface rather than from the surface itself.
ii. Some curmudgeonly professors may ask for the law**S** of reflection, and will give you only half marks if you neglect to add that the incident ray, the reflected ray and the normal are coplanar.
iii. A plane mirror forms a *virtual image of a real object*:

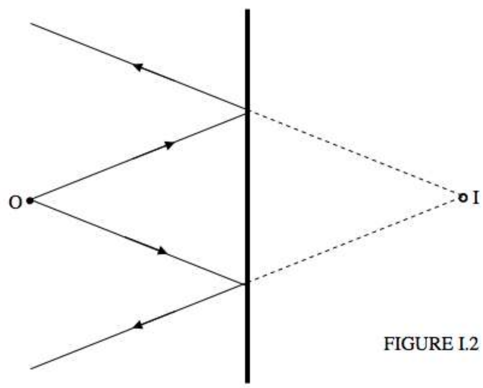


FIGURE I.2

or a *real image of a virtual object*:

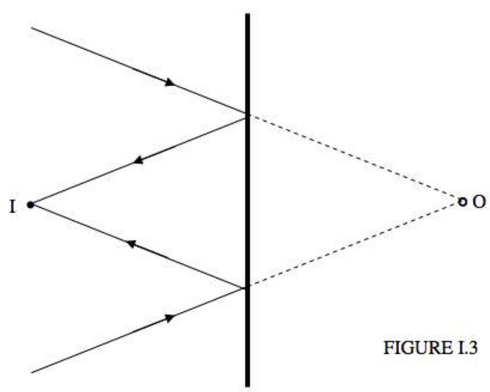
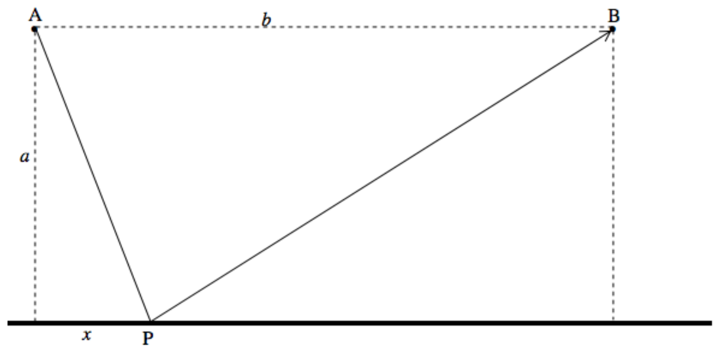


FIGURE I.3

iv. It is usually said that the image is as far behind the mirror as the object is in front of it. In the case of a virtual object (i.e. light converging on the mirror, presumably from some large lens somewhere to the left) you'd have to say that the image is as far *in front of* the mirror as the object is *behind it!*

v. If the mirror were to move at speed v away from a real object, the virtual image would move at speed $2v$. I'll leave you to think about what happens in the case of a virtual object.
vi. If the mirror were to rotate through an angle θ (or were to rotate at an angular speed ω), the reflected ray would rotate through an angle 2θ (or at an angular speed 2ω).
vii. Only smooth, shiny surfaces reflect light as described above. Most surfaces, such as paper, have minute irregularities on them, which results in light being scattered in many directions. Various equations have been proposed to describe this sort of scattering. If the reflecting surface looks equally bright when viewed from all directions, the surface is said to be a perfectly diffusing Lambert's law surface. Reflection according to the $r = i$ law of reflection, with the incident ray, the reflected ray and the normal being coplanar, is called **specular reflection** (Latin: speculum, a mirror). Most surfaces are intermediate between specular reflectors and perfectly diffusing surfaces. This chapter deals exclusively with specular reflection.
viii. The image in a mirror is reversed from left to right, and from back to front, but is not reversed up and down. Discuss.
ix. If you haven't read *Through the Looking-glass and What Alice Found There*, you are missing something.
x.

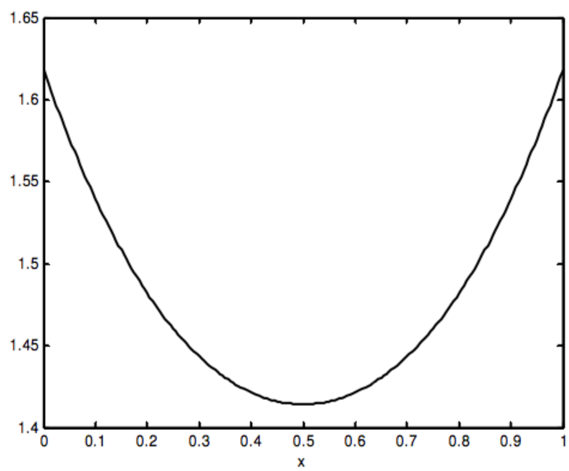


Light goes from A to B via reflection from a point P in a mirror.

The distance s traveled is given by

$$s = \sqrt{a^2 + x^2} + \sqrt{a^2 + (b - x)^2} \tag{1.2.1}$$

Here is the distance traveled as a function of the position of the point P:



The path that the light actually takes is the path such that the distance traveled is a minimum, which is such that P is horizontally halfway between A and B. You can see this from the graph, or by differentiating the above expression for s. This means that the angle of reflection is equal to the angle of incidence. You may regard this observation as a slightly interesting trivium, or as a fundamental principle of the deepest significance. Whichever you choose, you will come across lots of other examples of nature operating with Least Action. And you won't have to wait long. There's another one in the next section.

1.3: Refraction at a Plane Surface

When a ray of light enters a denser medium it is refracted towards the normal in such a manner than the ratio of the sine of the angle of incidence to the sine of the angle of refraction is constant, this constant being called the **refractive index** n.

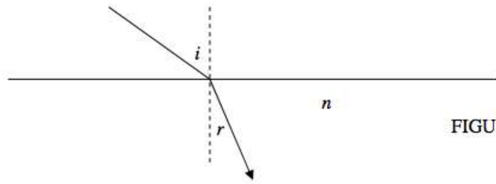


FIGURE I.4

This is all right as far as it goes, but we may be able to do better.

i. Remember the curmudgeonly professor who will give you only half marks unless you also say that the incident ray, the refracted ray and the normal are coplanar.
ii. The equation

$$\frac{\sin i}{\sin r} = n, \tag{1.3.1}$$

where n is the refractive index of the medium, is all right as long as the light enters the medium from a vacuum. The refractive index of air is very little different from unity. Details on the refractive index of air may be found in Section 7.1 of Stellar Atmospheres and Section 11.3.3 of Celestial Mechanics. If light is moving from one medium to another, the law of refraction takes the form

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{1.3.2}$$

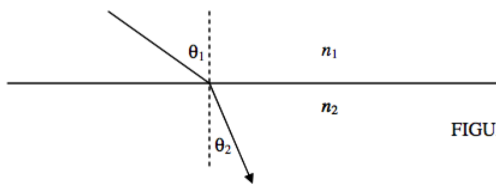


FIGURE I.5

iii. The statement of Snell's law as given above implies, if taken literally, that there is a one-to-one relation between refractive index and density. There must be a formula relating refractive index and density. If I tell you the density, you should be able to tell me the refractive index. And if I tell you the refractive index, you should be able to tell me the density. If you arrange substances in order of increasing density, this will also be their order of increasing refractive index.

Not Quite True

This is not quite true, and, if you spend a little while looking up densities and refractive indices of substances in, for example, the *CRC Handbook of Physics and Chemistry*, you will find many examples of less dense substances having a higher refractive index than more dense substances. It is true in a general sense usually that denser substances have higher indices, but there is no one-to-one correspondence.

In fact light is bent towards the normal in a "denser" medium as a result of its slower speed in that medium, and indeed the speed v of light in a medium of refractive index n is given by

$$n = c/v, \tag{1.3.3}$$

where c is the speed of light in *vacuo*. Now the speed of light in a medium is a function of the electrical permittivity ϵ and the magnetic permeability μ:

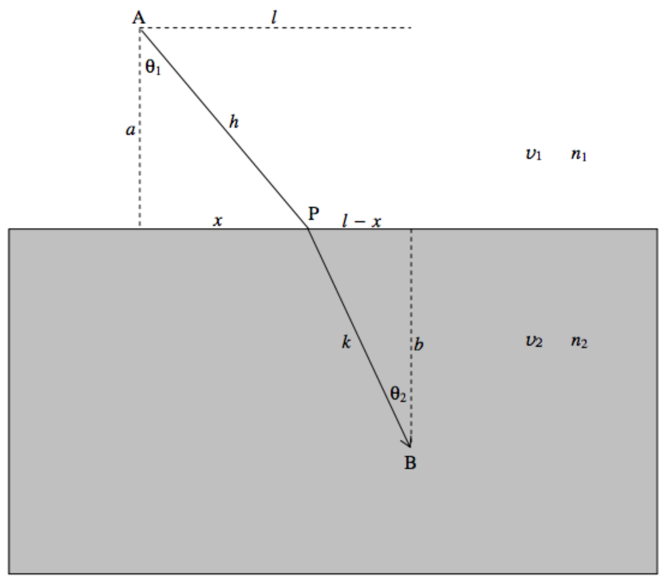
$$v = 1/\sqrt{\epsilon\mu}. \tag{1.3.4}$$

The permeability of most nonferromagnetic media is very little different from that of a vacuum, so the refractive index of a medium is given approximately by

$$n \approx \sqrt{\frac{\epsilon}{\epsilon_0}} \tag{1.3.5}$$

Thus there is a much closer correlation between refractive index and relative permittivity (dielectric constant) than between refractive index and density. Note, however, that this is only an approximate relation. In the detailed theory there is a small dependence of the speed of light and hence refractive index on the frequency (hence wavelength) of the light. Thus the refractive index is greater for violet light than for red light (*violet* light is refracted more *violently*). The splitting up of white light into its constituent colours by refraction is called *dispersion.*

Here is a ray of light travelling from one medium to another:



It moves faster in the upper medium than in the lower medium.

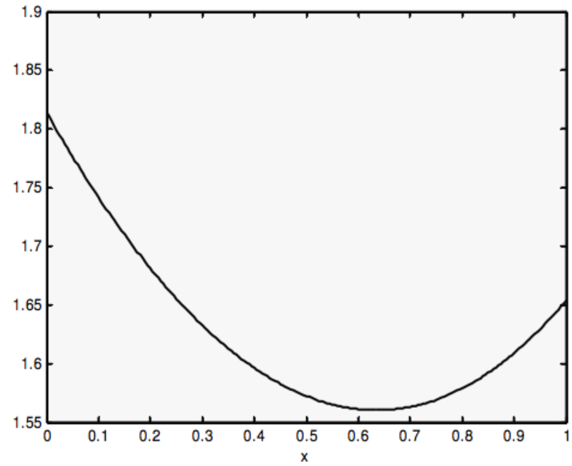
Time taken to get from A to B:

$$t = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (1 - x)^2}}{v^2}. \tag{1.3.6}$$

That is:

$$ct = n_1 \sqrt{a^2 + x^2} + n_2 \sqrt{b^2 + (l - x)^2}\,. \tag{1.3.7}$$

Here is the time taken as a function of the position of P, calculated for $n_2/n_1 = 1.5$.



As you see, it goes through a minimum. You can find where it is by differentiating Equation 1.3.7:

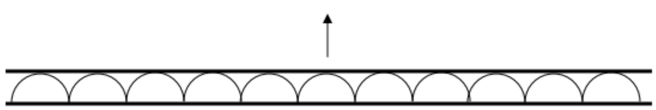
$$c\frac{dt}{dx} = \frac{n_1 x}{\sqrt{a^2+x^2}} - \frac{n_2(l-x)}{\sqrt{b^2+(l-x)^2}} = n_1\sin\theta_1 - n_2\sin\theta_2, \tag{1.3.8}$$

This is zero when $n_1\sin\theta_1 - n_2\sin\theta_2 = 0$. Thus Snell's law is such that the path actually taken is the path that takes the shortest time. Trivial, or profound?

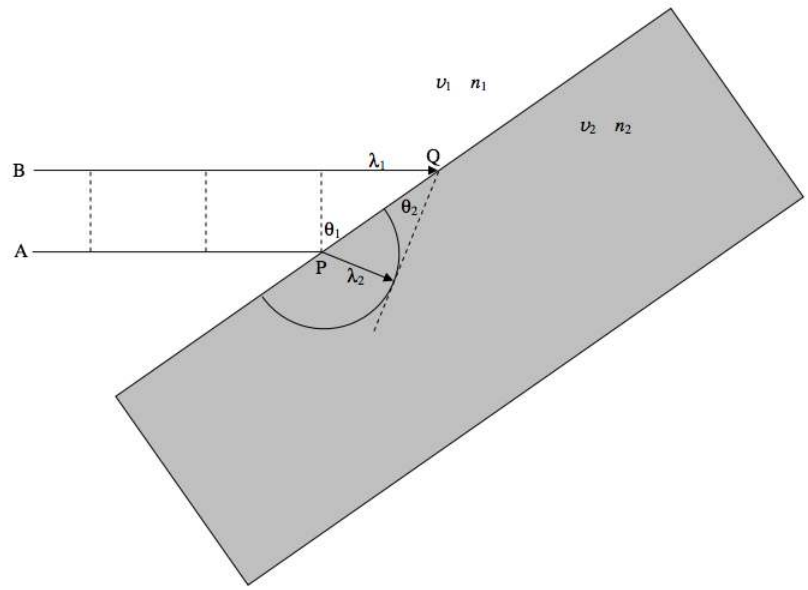
Huygens' Construction

Here is a wavefront moving upwards. "Light "rays" are normals to the wavefront.

Huygens' construction is a way of prediction what will happen next. It says that you can imagine every point on the wavefront to be a source that generates a little wavelet. Then, after a little time the wavelets look like this - and the new wavefront is the common tangent to all the wavelets.



This may sound trivial at first, although much has been written about it - i.e. whether it represents reality, or is merely a convenient construction. And, if real, what happens to the wavelets in the backwards direction? We'll not pursue that here, but we can use the Huygens construction as an interesting way to think about Snell's law.



A beam of light of wavelength λ_1 is approaching a glass block from the left at speed v_1. The dashed lines represent the wavefronts. Ray A reaches the block first, at P. A wavelet is generated at P, moving with speed v_2. The drawing is made for the instant when ray B reaches the point Q. The new wavefront is the tangent from Q to the little wavelet that started at P. The geometry will show that $\frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$ and therefore $\sin\theta_1 = n_2 \sin\theta_2$.

Example 1.3.1: Refraction through a glass block

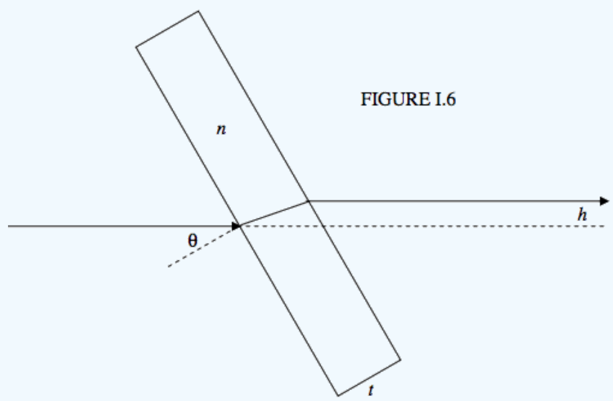


Figure I.6 shows a ray of light passing through a rectangular glass block of thickness t and refractive index n (taken to be 1.5 in the drawing). The normal to the surface of the block makes an angle θ with the incoming ray. It is a matter of simple geometry (do it!) to show that the *lateral displacement* h of the ray is given by

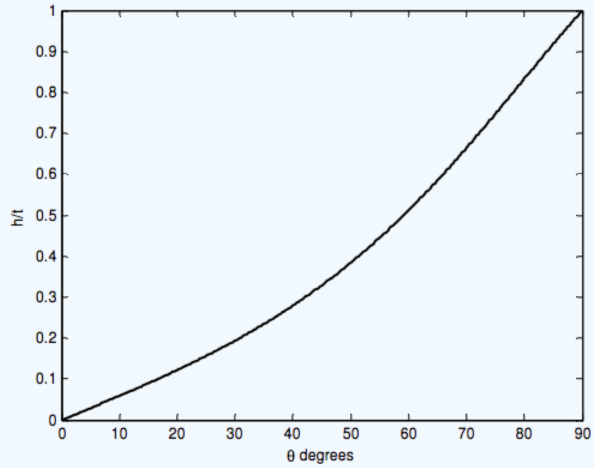
$$h = t(\sin\theta - \cos\theta\tan\phi), \tag{1.3.9}$$

where ϕ is the angle of refraction, given by $\sin\theta = n\sin\phi$. In terms of θ, n and t, this is

$$h = t\sin\theta\left(1 - \frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}}\right). \tag{1.3.10}$$

Figure I.7 is drawn for $n = 1.5$.

FIGURE 1.7



One might imagine making use of this to measure the distance between two points close together. For example suppose that you have a photograph of some stars on an old photographic plate, and it includes a close pair of stars, and you want to measure the distance between the two star images. (Today the photograph would be on a CCD detector, and the distance between the two images would be recorded electronically, which is why I specify an old photographic plate.) You look at the photograph through a microscope and see one of the stars bisected by a crosshair in the microscope eyepiece. But you have a glass plate in front of the photograph, and you tilt the plate in order to displace the images so that the images move and the second star is now bisected by the crosshair. From the large angle through which you tilt the plate you can work out the tiny distance between the two images. You'll want to use monochromatic light, and you'd need to know the refractive index at that wavelength.

How will you measure the angle through which the plate has turned? Well, you could shine a laser beam off it; the reflected light will move at twice the speed of the plate, and you could let it illuminate a sheet of graph paper several feet away. Thus the tiny distance between the images will correspond to a large distance on the graph paper. There may be one or two other practical details that you'd want to think about. For example, how thick would you want the glass plate to be? A thin microscope slide, maybe, or something much thicker than that? Would it be better to work at angles θ less than about 40º where the slope of Figure I.7 is small, or at angles greater than 50º where the slope is larger?

You might also wish to move a laser beam sidewise through a small and controlled amount. You could put a glass block on a turntable which could be rotated through a tiny measurable angle and thus move the laser beam laterally and accurately through a very tiny amount.

Let's continue with the glass block.

Example 1.3.2: Refraction through a moving glass block

Show that, if the glass block were to be rotated counterclockwise with angular speed ω, the laser beam would move upwards at a speed

$$\dot{h} = t\left[\cos\theta - \frac{n^2 - 2n^2\sin^2\theta + \sin^4\theta}{(n^2 - \sin^2\theta)^{3/2}}\right]\omega \tag{1.3.11}$$

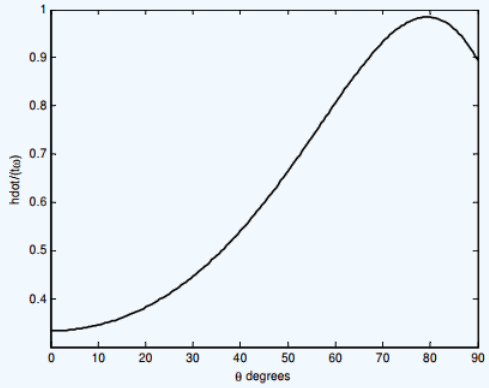
If we write $\sin^2\theta$ as $1-\cos^2\theta$, we'll be able to express this entirely in terms of $\cos\theta$. And if, for illustrative purposes, I take $n = 1.5$, the equation becomes

$$\frac{\dot{h}}{t\omega} = c + \frac{1.25 - 2.5c^2 - c^4}{(1.25 + c^2)^{3/2}}, \tag{1.3.12}$$

where $c = \cos\theta$. For ease in computation, I'll set $C = c^2$. Equation 1.3.12 then becomes

$$\frac{\dot{h}}{t\omega} = c + \frac{1.25 - C(2.5 + C)}{(1.25 + C)^{3/2}}. \tag{1.3.13}$$

This is easy to compute, and the result is shown in the graph below.



We see, perhaps to our surprise, that $\dot{h}$ & goes through a maximum at about θ = 79º. To obtain Equation 1.3.12, we had to differentiate Equation 1.3.7. Now, to find out where $\dot{h}$ goes through a maximum, we are going to have to differentiate again, although mercifully we can differentiate Equation 1.3.13 with respect to C rather than to θ. If we do this, and then set the derivative to zero, we find, after some simplification,

$$\frac{1}{2C^{1/2}} = \frac{5+1.25C+0.5C^2}{(1.25+C)^{5/2}} \tag{1.3.14}$$

If we square this and collect powers of C, we arrive at a quartic equation in C:

$$625-17980C-6240C^2-2176C^3+256C^4=0.$$

where $C=c^2=\cos^2\theta$.

The solution to this equation is $C=0.0343465490$, corresponding to

<u>θ = 79º.319 731 1</u>

1.4: Real and Apparent Depth

When we look down into a pool of water from above, the pool looks less deep than it really is. Figure I.6 shows the formation of a virtual image of a point on the bottom of the pool by refraction at the surface.

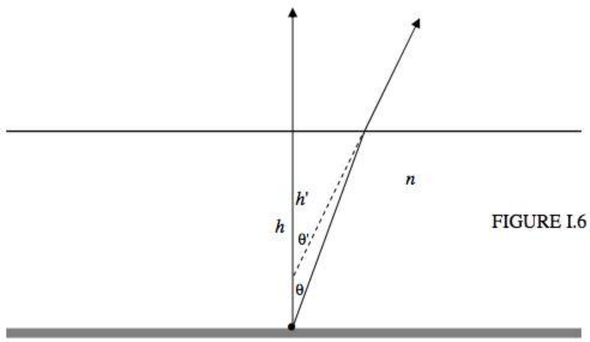


FIGURE I.6

The diameter of the pupil of the human eye is in the range 4 to 7 mm, so, when we are looking down into a pool (or indeed looking at anything that is not very close to our eyes), the angles involved are small. Thus in Figure I.6 you are asked to imagine that all the angles are small; actually to draw them small would make for a very cramped drawing. Since angles are small, I can approximate Snell's law:

$$n = \frac{\sin\theta'}{\sin\theta} \tag{1.4.1}$$

$$\approx \frac{\tan\theta'}{\tan\theta} \tag{1.4.2}$$

and hence

$$\frac{\text{real depth}}{\text{apparent depth}} = \frac{h}{h'} = \frac{\tan\theta'}{\tan\theta} = n. \tag{1.4.3}$$

For water, n is about $\frac{4}{3}$, so that the apparent depth is about $\frac{3}{4}$ of the real depth.

Exercise 1.4.1

An astronomer places a photographic film, or a CCD, at the primary focus of a telescope. He then decides to insert a glass filter, of refractive index n and thickness t, in front of the film (or CCD). In which direction should he move the film or CCD, and by how much, so that the image remains in focus?

Now if Snell's law really were given by Equation 1.4.2, all refracted rays from the object would, when produced backwards, appear to diverge from a single point, namely the virtual image. But Snell's law is really Equation 1.4.1, so what happens if we do not make the small angle approximation?

We have

$$\frac{h}{h'} = \frac{\tan\theta'}{\tan\theta} \tag{1.4.4}$$

and, if we apply the trigonometric identity

$$\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \tag{1.4.5}$$

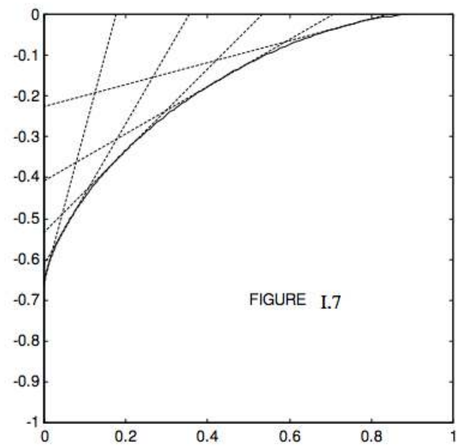
and apply Snell's law (Equation 1.4.1), we find that

$$\frac{h}{h'}=\frac{n\cos\theta}{\sqrt{1-n^2\sin^2\theta}} \tag{1.4.6}$$

Exercise 1.4.2

Show that, to first order in θ that Equation 1.4.6 becomes $h/h'=n$.

Equation 1.4.6 shows h' as a function of θ − and that the refracted rays, when projected backwards, do not all appear to come from a single point. In other words, a point object does not result in a point image. Figure I.7 shows (for $n=1.5$ – i.e. glass rather than water) the backward projections of the refracted rays for θ' = 15, 30, 45, 60 and 75 degrees, together with their envelope or "caustic curve". The "object" is at the bottom left corner of the frame, and the surface is the upper side of the frame.



Exercise 1.4.3

(for the mathematically comfortable). Show that the parametric equations for the caustic curve are

$$x-y\tan\theta'-h\tan\theta=0$$

and

$$ny\sec^3\theta'+h\sec^3\theta=0.$$

Here, $y=0$ is taken to be the refracting surface, and θ and θ' are related by Snell's law.

Thus refraction at a plane interface produces an *aberration* in the sense that light from a point object does not diverge from a point image. This type of aberration is somewhat similar to the type of aberration produced by reflection from a spherical mirror, and to that extent the aberration could be referred to as "spherical aberration". If a point at the bottom of a pond is viewed at an angle to the surface, rather than perpendicular to it, a further aberration called "astigmatism" is produced. This will be discussed in Chapter 4.

1.5: Reflection and Refraction

We have described reflection and refraction, but of course when a ray of light encounters an interface between two transparent media, a portion of it is reflected and a portion is refracted, and it is natural to ask, even during an early introduction to the subject, just what fraction is reflected and what fraction is refracted. The answer to this is quite complicated, for it depends not only on the angle of incidence and on the two refractive indices, but also on the initial state of polarization of the incident light; it takes us quite far into electromagnetic theory and is beyond the scope of this chapter, which is intended to deal largely with just the *geometry* of reflection and refraction. However, since it is a natural question to ask, I can give explicit formulas for the fractions that are reflected and refracted in the case where the incident light is unpolarized.

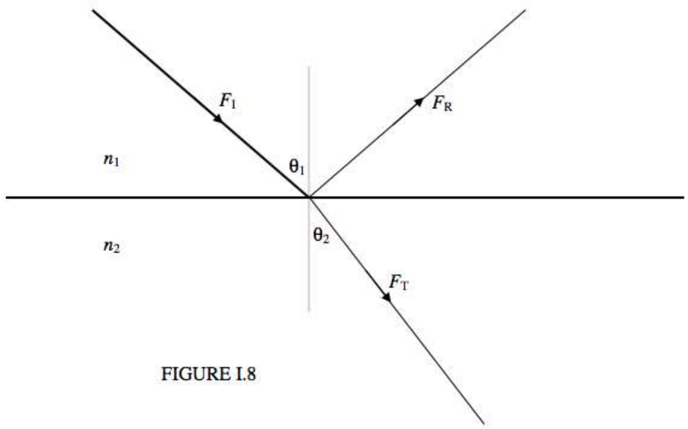


FIGURE I.8

Figure I.8 shows an incident ray of energy flux density (W m^{-2} normal to the direction of propagation) F_I arriving at an interface between media of indices n_1 and n_2. It is subsequently divided into a reflected ray of flux density F_R and a transmitted ray of flux density F_T. The fractions transmitted and reflected (t and r) are

$$t=\frac{F_T}{F_I}=2n_1n_2\cos\theta_1\cos\theta_2\left(\frac{1}{(n_1\cos\theta_1+n_2\cos\theta_2)^2}+\frac{1}{(n_1\cos\theta_1+n_2\cos\theta_1)^2}\right) \tag{1.5.1}$$

and

$$r=\frac{F_T}{F_I}=\frac{1}{2}\left[\left(\frac{n_1\cos\theta_1-n_2\cos\theta_2}{(n_1\cos\theta_1+n_2\cos\theta_2)^2}\right)^2+\left(\frac{n_1\cos\theta_1-n_2\cos\theta_1}{(n_1\cos\theta_1+n_2\cos\theta_1)^2}\right)^2\right] \tag{1.5.2}$$

Here the angles and indices are related through Snell's law, Equation 1.3.2. If you have the energy, show that the sum of these is 1. Both the transmitted and the reflected rays are partially plane polarized. If the angle of incidence and the refractive index are such that the transmitted and reflected rays are perpendicular to each other, the reflected ray is completely plane polarized – but such details need not trouble us in this chapter.

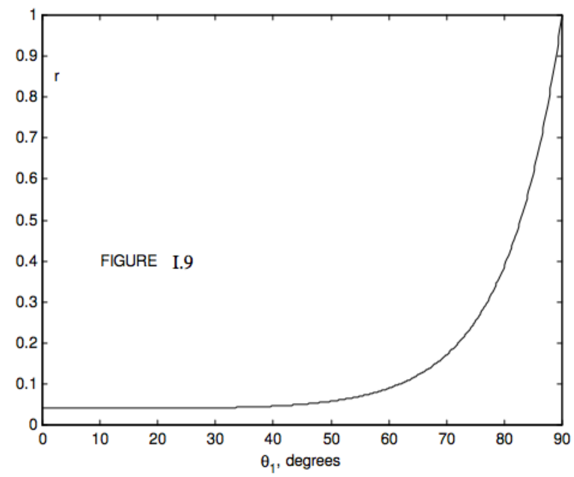
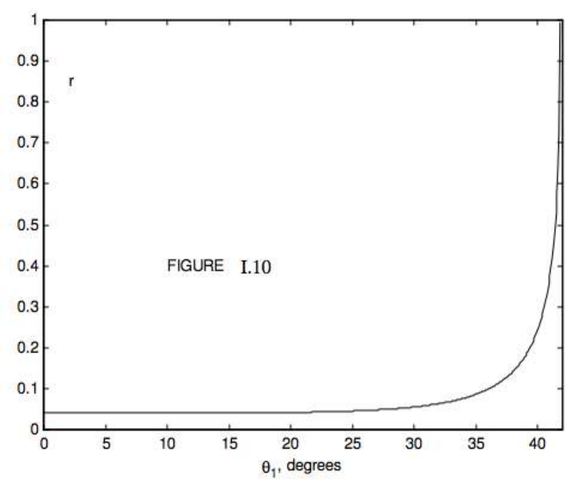


Figure I.9 shows the reflection coefficient as a function of angle of incidence for unpolarized incident light with $n_1 = 1.0$ and $n_2 = 1.5$ (e.g. glass). Since $n_2 > n_1$, we have *external reflection*. We see that for angles of incidence less than about 45 degrees, very little of the light is reflected, but after this the reflection coefficient increases rapidly with angle of incidence, approaching unity as $\theta_1 \rightarrow 90°$ (grazing incidence). If $n_1 = 1.5$ and $n_2 = 1.0$, we have internal reflection, and the reflection coefficient for this case is shown in Figure I.10. For internal angles of incidence less than about 35°, little light is reflected, the rest being transmitted. After this, the reflection coefficient increases rapidly, until the internal angle of incidence θ_1 approaches a critical angle C, given by

$$\sin C = \frac{n_2}{n_1}, \tag{1.5.3}$$

This corresponds to an angle of emergence of 90°. For angles of incidence greater than this, the light is *totally internally reflected*. For glass of refractive index 1.5, the critical angle is 41°.2, so that light is totally internally reflected inside a 45° prism such as is used in binoculars.



1.6: Refraction by a Prism

Figure I.11 shows an isosceles prism of angle α and a ray of light passing through it.

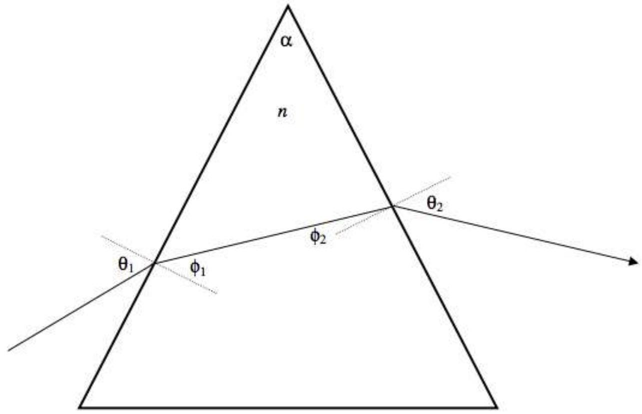


FIGURE I.11

I have drawn just one ray of a single color. For white light, the colors will be *dispersed*, the violet light being deviated by the prism more than the red light. We'll choose a wavelength such that the refractive index of the prism is n. The deviation D of the light from its original direction is $\theta_1-\phi_1+\theta_2-\phi_2$. I want to imagine, now, if we keep the incident ray fixed and rotate the prism, how does the deviation vary with angle of incidence θ_1? By geometry, $\phi_2=\alpha-\phi_1$, so that the deviation is

$$D=\theta_1+\theta_2-\alpha. \tag{1.6.1}$$

Apply Snell's law at each of the two refracting surfaces:

$$\frac{\sin\theta_1}{\sin\phi_1}=n \quad \text{and} \quad \frac{\sin\theta_2}{\sin(\alpha-\phi_1)}=n, \tag{1.6.2}$$

and eliminate ϕ_1:

$$\sin\theta_2=\sin\alpha\sqrt{n^2-\sin^2\theta_1}-\cos\alpha\sin\theta_1. \tag{1.6.3}$$

Equations 1.6.1 and 1.6.3 enable us to calculate the deviation as a function of the angle of incidence θ_1. The deviation is least when the light traverses the prism symmetrically, with $\theta_1=\theta_2$, the light inside the prism then being parallel to the base. Putting $\theta_1=\theta_2$ in equation shows that minimum deviation occurs for an angle of incidence given by

$$\sin\theta_1=\frac{n\sin\alpha}{\sqrt{2(1+\cos\alpha)}}=n\sin\frac{1}{2}\alpha. \tag{1.6.4}$$

The angle of minimum deviation $D_{\min}$ is $2\theta_1-\alpha$, where θ_1 is given by Equation 1.6.4, and this leads to the following relation between the refractive index and the angle of minimum deviation:

$$n=\frac{\sin\frac{1}{2}(D_{\min}+\alpha)}{\sin\frac{1}{2}\alpha}. \tag{1.6.5}$$

Of particular interest are prisms with α = 60° and α = 90°. I have drawn, in Figure I.12 the deviation versus angle of incidence for 60- and 90-degree prisms, using (for reasons I shall explain) $n=1.31$, which is approximately the refractive index of ice. For the 60° ice prism, the angle of minimum deviation is 21°.8, and for the 90° ice prism it is 45°.7.

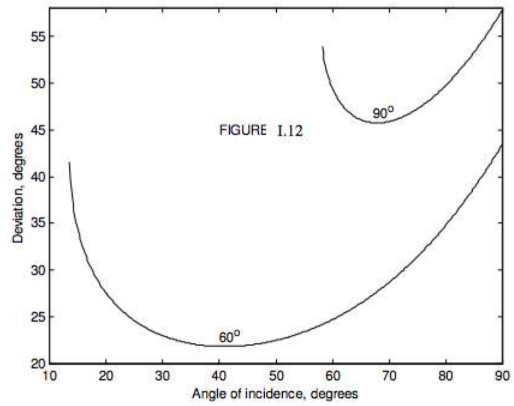


Figure I.12:

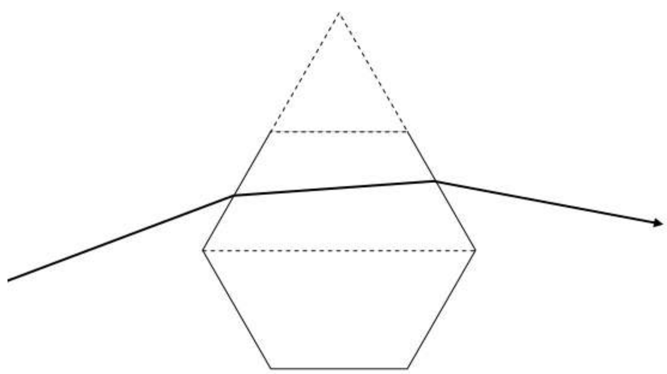


FIGURE I.13

Solar Halo

When hexagonal ice crystals are present in the atmosphere, sunlight is scattered in all directions, according to the angles of incidence on the various ice crystals (which may or may not be oriented randomly). However, the rate of change of the deviation with angle of incidence is least near minimum deviation; consequently much more light is deviated by 21°.8 than through other angles. Consequently we see a halo of radius about 22° around the Sun.

Seen sideways on, a hexagonal crystal is rectangular, and consequently refraction is as if through a 90° prism (Figure I.14):

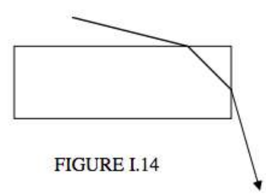


FIGURE I.14

Again, the rate of change of deviation with angle of incidence is least near minimum deviation, and consequently we may see another halo, of radius about 46°. For both haloes, the violet is deviated more than the red, and therefore both haloes are tinged violet on the outside and red on the inside.

Figure: Sun halos with a larger and fainter 46° halo and a 22° halo with an upper tangent arc. (CC BY 2.0; NOAA Photo Library).

1.7: The Rainbow

I do not know the exact shape of a raindrop, but I doubt very much if it is drop-shaped! Most raindrops will be more or less spherical, especially small drops, because of surface tension. If large, falling drops are distorted from an exact spherical shape, I imagine that they are more likely to be flattened to a sort of horizontal pancake shape rather than drop shaped. Regardless, in the analysis in this section, I shall assume drops are spherical, as I am sure small drops will be.

We wish to follow a light ray as it enters a spherical drop, is internally reflected, and finally emerges. See Figure I.15. We shall refer to the distance b as the *impact parameter.*

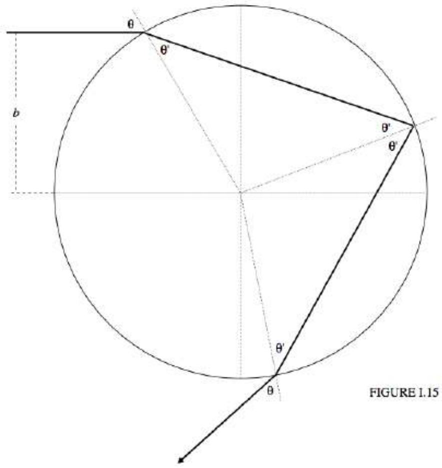


FIGURE I.15

We see a ray of light headed for the drop, which I take to have unit radius, at impact parameter b. The deviation of the direction of the emergent ray from the direction of the incident ray is

$$D=\theta-\theta'+\pi-2\theta'+\theta-\theta'=\pi+2\theta-4\theta'. \tag{1.7.1}$$

However, we shall be more interested in the angle $r=\pi-D$. A ray of light that has been deviated by D will approach the observer from a direction that makes an angle r from the centre of the bow, which is at the anti-solar point (Figure I.16)

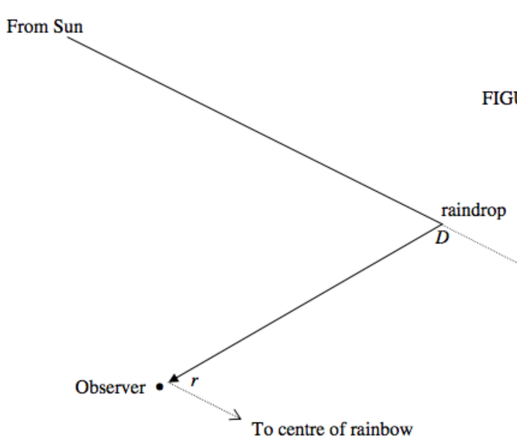


FIGURE I.16

We would like to find the deviation D as a function of impact parameter. The angles of incidence and refraction are related to the impact parameter as follows:

$$\sin\theta=b, \tag{1.7.2}$$

$$\cos\theta=\sqrt{1-b^2}, \tag{1.7.3}$$

$$\sin\theta'=b/n, \tag{1.7.4}$$

and

$$\cos\theta = \sqrt{1 - b^2/n^2}. \tag{1.7.5}$$

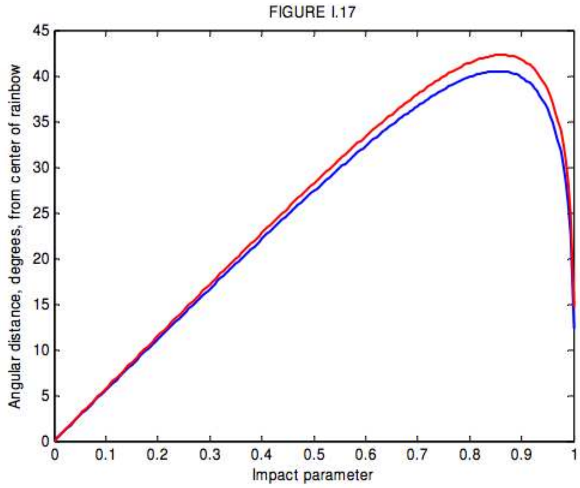
These, together with Equation 1.7.1, give us the deviation as a function of impact parameter. The deviation goes through a minimum – and r goes through a maximum. The deviation for a light ray of impact parameter b is

$$D = \pi + 2\sin^{-1} b - 4\sin^{-1}(b/n) \tag{1.7.6}$$

The angular distance r from the centre of the bow is $r = \pi - D$, so that

$$r = 4\sin^{-1}(b/n) - 2\sin^{-1} b. \tag{1.7.7}$$

This is shown in Figure I.17 for n = 1.3439 (blue - γ = 400 nm) and n = 1.3316 (red - γ = 650 nm).



Differentiation gives the maximum value, R, of r - i.e. the radius of the bow – or the minimum deviation D_{min}. We obtain for the radius of the bow

$$R = 4\sin^{-1}\sqrt{\frac{4-n^2}{3n^2}} - 2\sin^{-1}\sqrt{\frac{4-n^2}{3}}. \tag{1.7.8}$$

For n = 1.3439 (blue) this is 40° 31' and for n = 1.3316 (red) this is 42° 17'. Thus the blue is on the inside of the bow, and red on the outside.

For grazing incidence (impact parameter = 1), the deviation is $2\pi - 4\sin^{-1}(1/n)$, or 167° 40' for blue or 165° 18' for red. This corresponds to a distance from the centre of the bow $r = 4\sin^{-1}(1/n) - \pi$, which is ' 12° 20' for blue and 14° 42' for red. It will be seen from figure I.17 that for radii less than R (i.e. inside the rainbow) but greater than 12° 20' for blue and 14° 42' for red there are two impact parameters that result in the same deviation, i.e. in the same position inside the bow. The paths of two rays with the same deviation are shown in Figure I.18. One ray is drawn as a full line, the other as a dashed line. They start with different impact parameters, and take different paths through the drop, but finish in the same direction. The drawing is done for a deviation of 145°, or 35° from the bow centre. The two impact parameters are 0.969 and 0.636. When these two rays are recombined by being brought to a focus on the retina of the eye, they have satisfied all the conditions for interference, and the result will be brightness or darkness according as to whether the path difference is an even or an odd number of half wavelengths.

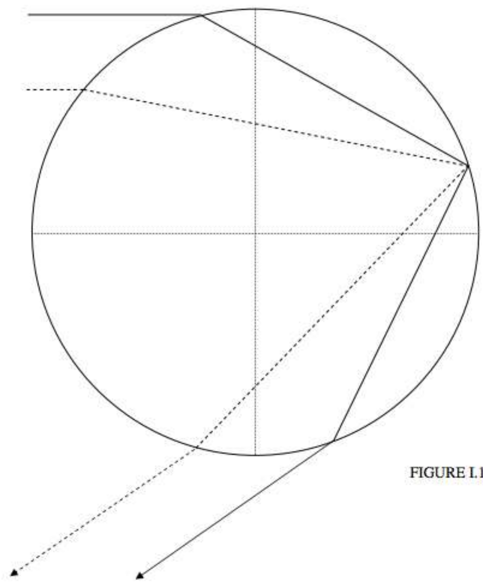


FIGURE I.18

If you look just inside the inner (blue) margin of the bow, you can often clearly see the interference fringes produced by two rays with the same deviation. I haven't tried, but if you were to look through a filter that transmits just one colour, these fringes (if they are bright enough to see) should be well defined. The optical path difference for a given deviation, or given r, depends on the radius of the drop (and on its refractive index). For a drop of radius a it is easy to see that the optical path difference is

$$2a(\cos\theta_2 - \cos\theta_1) - 4n(\cos\theta'_2 - \cos\theta'_1),$$

where θ_1 is the larger of the two angles of incidence. Presumably, if you were to measure the fringe spacing, you could determine the size of the drops. Or, if you were to conduct a Fourier analysis of the visibility of the fringes, you could determine, at least in principle, the size distribution of the drops.

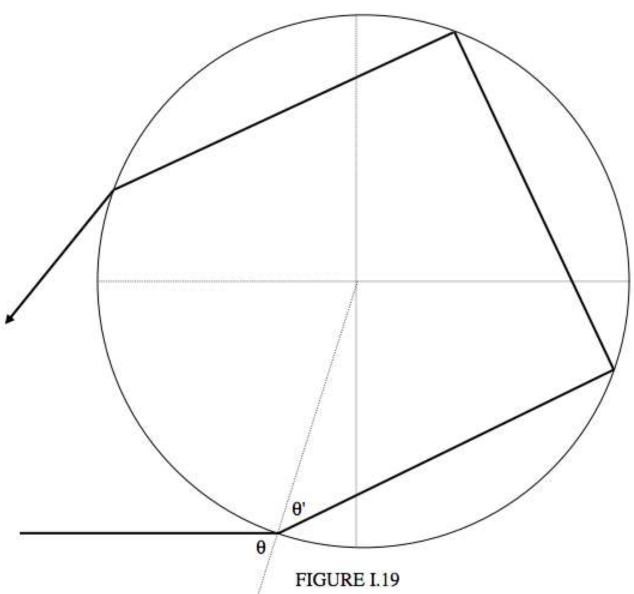
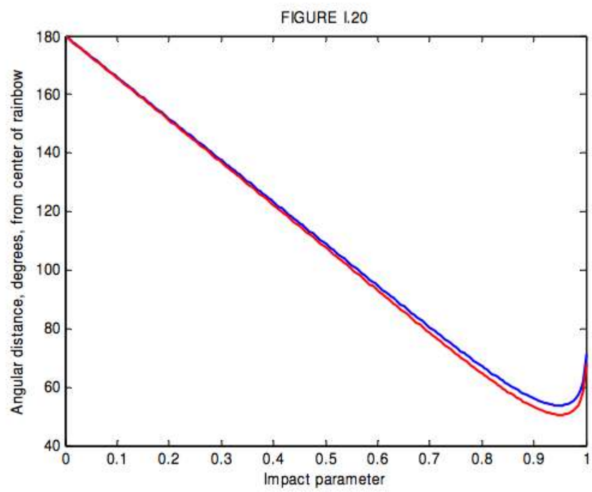


FIGURE I.19

Some distance outside the primary rainbow, there is a secondary rainbow, with colours reversed – i.e. red on the inside, blue on the outside. This is formed by two internal reflections inside the drop (Figure I.19). The deviation of the final emergent ray from the direction of the incident ray is $(\theta - \theta') + (\pi - 2\theta') + (\pi - 2\theta') + (\theta - \theta')$, or $2\pi + 2\theta - 6\theta'$ counterclockwise, which amounts to $D = 6\theta' - 2\theta$ clockwise. That is,

$$D = 6\sin^{-1}(b/n) - 2\sin^{-1} b. \tag{1.7.9}$$

clockwise, and, as before, this corresponds to an angular distance from the centre of the bow $r = \pi - D$. I show in Figure I.20 the angular distance from the centre of the bow versus the impact parameter b. Notice that D goes through a *maximum* and hence r has a *minimum* value. There is no light scattered outside the primary bow, and no light scattered inside the secondary bow. When the full glory of a primary bow and a secondary bow is observed, it will be seen that the space between the two bows is relatively dark, whereas it is brighter inside the primary bow and outside the secondary bow.



Differentiation shows that the least value of r, (greatest deviation) corresponding to the radius of the secondary bow is

$$R = 6\sin^{-1}\sqrt{\frac{3-n^2}{2n^2}} - 2\sin^{-1}\sqrt{\frac{3-n^2}{2}} \tag{1.7.10}$$

For $n = 1.3439$ (blue) this is 53° 42' and for $n = 1.3316$ (red) this is 50° 31'. Thus the red is on the inside of the bow, and blue on the outside.

In principle a tertiary bow is possible, involving three internal reflections. I don't know if anyone has observed a tertiary bow, but I am told that the primary bow is blue on the inside, the secondary bow is red on the inside, and "therefore" the tertiary bow would be blue on the inside. On the contrary, I assert that the tertiary bow would be red on the inside. Why is this?

Let us return to the primary bow. The deviation is (Equation 1.7.1) $D = \pi + 2\theta - 4\theta'$. Let's take $n = 4/3$, which it will be for somewhere in the middle of the spectrum. According to Equation 1.7.8, the radius of the bow $(R = \pi - D_{\min})$ is then about 42°. That is, $2\theta' - \theta$ = 21°. If we combine this with Snell's law, $3\sin\theta = 4\sin\theta'$, we find that, at minimum deviation (i.e. where the primary bow is), θ = 60°.5 and θ' = 40°.8. Now, at the point of internal reflection, not all of the light is reflected (because θ' is less than the critical angle of 36°.9), and it will be seen that the angle between the reflected and refracted rays is (180 − 60.6 − 40.8) degrees = 78°.6. Those readers who are familiar with Brewster's law will understand that when the reflected and transmitted rays are at right angles to each other, the reflected ray is completely plane polarized. The angle, as we have seen, is not 90°, but is 78°.6, but this is sufficiently close to the Brewster condition that the reflected light, while not completely plane polarized, is strongly polarized. Thus, as can be verified with a polarizing filter, the rainbow is strongly plane polarized.

I now want to address the question as to how the brightness of the bow varies from centre to circumference. It is brightest where the slope of the deviation versus impact parameter curve is least – i.e. at minimum deviation (for the primary bow) or maximum deviation (for the secondary bow). Indeed the radiance (surface brightness) at a given distance from the centre of the bow is (among other things) inversely proportional to the slope of that curve. The situation is complicated a little in that, for deviations between $D_{\min}$ and $2\pi - 4\sin^{-1}(1/n)$, (this latter being the deviation for grazing incidence), there are two impact parameters giving rise to the same deviation, but for deviations greater than that (i.e. closer to the centre of the bow) only one impact parameter corresponds to a given deviation.

Let us ask ourselves, for example, how bright is the bow at 35° from the centre (deviation 145°)? The deviation is related to impact parameter by Equation 1.7.6. For $n = 4/3$, we find that the impact parameters for deviations of 144, 145 and 146 degrees are as follows:

D°	b
144	0.6583 and 0.9623
145	0.6366 and 0.9693
146	0.6157 and 0.9736

Figure I.21 shows a raindrop seen from the direction of the approaching photons.

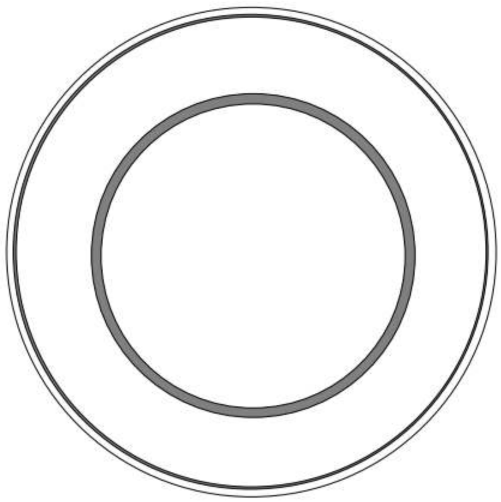
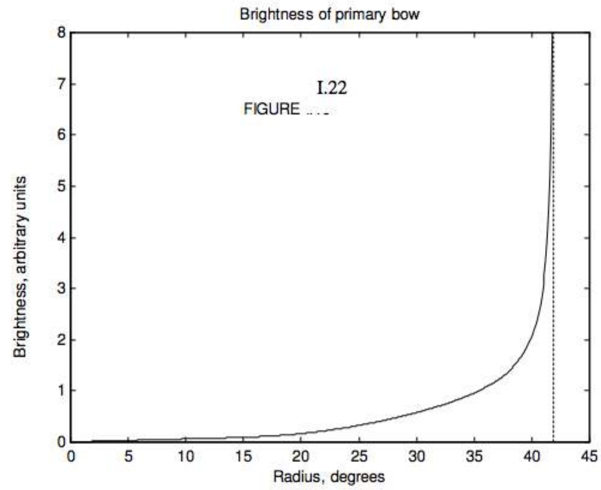


FIGURE I.21

Any photons with impact parameters within the two dark annuli will be deviated between 144° and 146°, and will ultimately approach the observer at angular distances between 36° and 34° from the centre. The radiance at a distance of 35° from the centre will be proportional, among other things, to the sum of the areas of these two annuli.

I have said "among other things". Let us now think about other things. I have drawn Figure I.15 as if all of the light is transmitted as it enters the drop, and then all of it is internally reflected within the drop, and finally all of it emerges when it leaves the drop. This is not so, of course. At entrance, at internal reflection and at emergence, some of the light is reflected and some is transmitted. The fractions that are reflected or transmitted depend on the angle of incidence, but, for minimum deviation, about 94% is transmitted on entry to and again at exit from the drop, but only about 6% is internally reflected. Also, after entry, internal reflection and exit, the percentage of polarization of the ray increases. The formulas for the reflection and transmission coefficients (Fresnel's equations) are somewhat complicated (Equations 1.5.1 and 1.5.2) are for unpolarized incident light), but I have followed them through as a function of impact parameter, and have also taken account of the sizes of the one or two annuli involved for each impact parameter, and I have consequently calculated the variation of surface brightness for one color $(n = 4/3)$ from the centre to the circumference of the bow. I omit the details of the calculations, since this chapter was originally planned as an elementary account of reflection and transmission, and we seem to have gone a little beyond that, but I show the results of the calculation in Figure I.22. I have not, however, taken account of the interference phenomena, which can often be clearly seen just within the primary bow.


Brightness of primary bow
I.22
FIGURE
Brightness, arbitrary units
Radius, degrees
0
1
2
3
4
5
6
7
8
0
5
10
15
20
25
30
35
40
45

1.8: Differential Form of Snell's Law

Snell's law in the form $n \sin \theta$ = constant is useful in calculating how a light ray is bent in travelling from one medium to another where there is a discrete change of refractive index. If there is a medium in which the refractive index is changing continuously, a differential form of Snell's law may be useful. This is obtained simply by differentiation of $n \sin \theta$ = constant, to obtain the differential form of Snell's law

$$\cot \theta d\theta = -\frac{dn}{n}. \tag{1.8.1}$$

If we express this in terms of the complementary angle Ψ (see Figure I.24), this equation takes the form

$$\tan \Psi d\Psi = \frac{dn}{n}. \tag{1.8.2}$$

Let us see how this might be used. Let us suppose, for example, that we have some medium in which the refractive index diminishes linearly with the y coordinate according to

$$n = n_0 - \frac{(n_0 - 1)y}{a}. \tag{1.8.3}$$

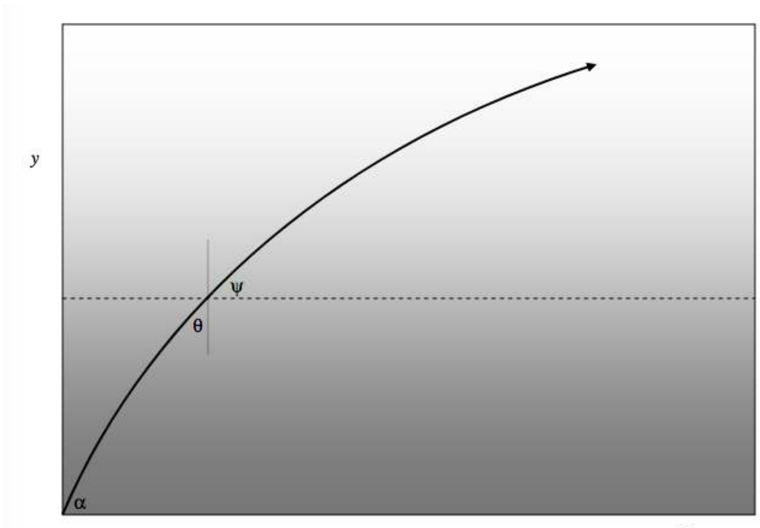


FIGURE I.24

This form has $n = n_0$ at $y = 0$, decreasing linearly to $n = 1$ at $y = a$. We suppose that $n = 1$ everywhere beyond $y = a$. The equation is simplified if we write μ for $n - 1$ and for μ_0 for $n_0 - 1$, so that

$$\mu = \mu_0 \left(1 - \frac{y}{a}\right), \tag{1.8.4}$$

which has μ decreasing linearly from μ_0 to zero. Let us suppose that we direct a light ray upwards from the origin in a direction making an angle α with the horizontal, and we wish to trace the ray through the medium as the refractive index continuously changes. See Figure I.24. With the refractive index changing as in Equation 1.8.4, Snell's law takes the form

$$tan\Psi d\Psi = -\frac{dy}{k - y}, \quad \text{where} \quad k = \frac{n_0 a}{\mu_0}. \tag{1.8.5}$$

On integration, this becomes

$$y = k\left(1 - \frac{\cos \alpha}{\cos \Psi}\right) \quad \text{or} \quad \cos \Psi = \frac{k \cos \alpha}{k - y}. \tag{1.8.6}$$

This is the (y, Ψ) equation to the path taken by the light. One can see from this equation that the path becomes horizontal when $y = k(1 - \cos \alpha$. To find the (x, y) equation, we use the identities $\cos \Psi = (1 + \tan^2 \Psi)^{-1/2}$ and $\tan \Psi = \frac{dy}{dx}$. Substitution of these into Equation 1.8.6 gives the differential equation to the path in (x, y) coordinates:

$$\left(\frac{dy}{dx}\right)^2 = \frac{(k-y)^2 - l^2}{l^2}, \quad \text{where} \quad l = k\cos\alpha. \tag{1.8.7}$$

This can be readily integrated to give the (x, y) equation to the path:

$$x = l\ln\left[\frac{k(1+\sin\alpha)}{k-y\pm\sqrt{(k-y)^2-l^2}}\right]. \tag{1.8.8}$$

In Figure I.25 I have taken $a = 1$ (in other words, I am expressing all distances in units of a), and I am taking $n_0 = 1.5$ (hence $k = 3$), and I calculate the paths for α = 15°, 30°, 45°, 60°, 75°.

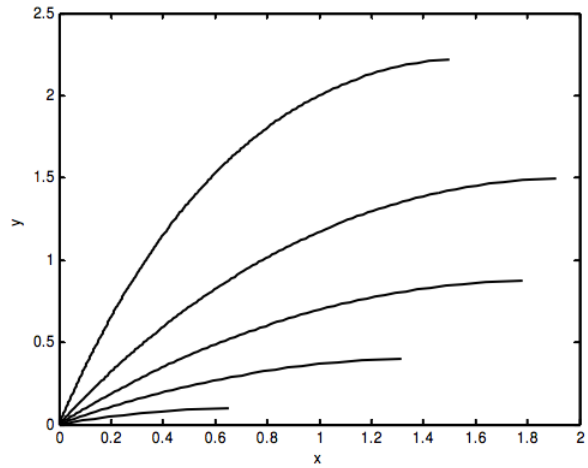


FIGURE I.25

Figure I.25 and Equation 1.8.8 show the paths of the light if the refractive index is varying from n_0 at $y = 0$, to 1 at $y = a$ as described by Equation 1.8.3. But now suppose that the refractive index is varying with height according to

$$n = \frac{n_0 a}{(n_0 - 1)y + a}. \tag{1.8.9}$$

In this model, too, the refractive index goes from n_0 at $y = 0$, to 1 at $y = a$, but the variation is not linear. In fact you might like to convince yourself that it is the speed of light that is increasing linearly from $y = 0$ to $y = a$. See if you can trace the paths of the light rays in this situation. I think they are arcs of circles, and you might be able to calculate the radii and the coordinates of the centres of the circles. Here's another one:

$$n = \sqrt{\frac{n_0^2 a}{(n_0^2 - 1)y + a}}. \tag{1.8.10}$$

Here, too, the refractive index goes from n_0 at $y = 0$, to 1 at $y = a$. You might like to try tracing the rays in this model. I think they may be arcs of cycloids. Of course these examples may seem to be very unlikely. Can you imagine a glass block of width a, made of glass whose refractive index varies continuously from 1.5 at one edge to zero at the other? Not very likely, yet there is a situation that comes to mind in which there is a continuous variation of refractive index from some basal value n_0 to zero. I am thinking of Earth's atmosphere (or indeed the atmosphere of any planet). As light from a star travels through Earth's atmosphere, it moves not in a straight line, but in a slight curve, so that it is deviated through a few arc minutes before it reaches the astronomer's telescope. For a star low down near the horizon, the refraction amounts to almost half a degree. This has to be taken into account when astronomers are making precise positional measurements. And sound waves, passing through the atmosphere, are also subject to refraction via the differential form of Snell's law. The speed of sound (and hence the refractive index) varies with the temperature of the atmosphere, and hence with height in the atmosphere. Among the many applications of this sort of theory is the path of sound waves from meteorites hurtling through the atmosphere. This is discussed in a paper published in *Meteoritics and Planetary Science* 34, 572-585 (1999).

CHAPTER OVERVIEW

2: Lens and Mirror Calculations

Thumbnail: Parallel rays coming into a parabolic mirror are focused at a point F. The vertex is V, and the axis of symmetry passes through V and F. For off-axis reflectors (with just the part of the paraboloid between the points P1 and P3), the receiver is still placed at the focus of the paraboloid, but it does not cast a shadow onto the reflector. (Public Domain).

2.1: Introduction to Lens and Mirror Calculations

The equation that relates object distance p, image distance q and focal length f is

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}. \tag{2.1.1}$$

Or is it? Should that not be a minus sign on the left hand side? Or should it be a plus sign for mirrors and minus for lenses? ("**M**ore for a **M**irror; **L**ess for a **L**ens."?)

As all who have ever tried lens and mirror calculations know, the biggest difficulty we have to face is that of the dreaded *sign conventions*. It seems that no two professors or teachers or books use the same convention. No sooner have we mastered one than we encounter a professor who insists upon another. Each professor thinks that his is far superior to anyone else's, or is even seemingly unaware that there could be any convention other than his own. We rapidly become discouraged. Indeed, when we try to use Equation 2.1.1 there is just one chance in eight that we choose the correct sign for all three symbols. In fact the situation is even worse than that. You might choose the signs correctly for all the symbols and get the right answer, −15 cm, according to your own convention, yet your professor, who uses a different convention, marks it wrong. You might be perfectly clear in your own mind that, since the image is a virtual image, the answer must be minus fifteen. But your professor may interpret the minus sign as meaning that the image is to the left of the lens, or on the opposite side of the lens from the object, or is inverted, and he consequently marks it wrong.

Truly, of course, an answer "−15 cm" means nothing unless we are all certain as to exactly what it meant by the minus sign. I therefore suggest that you do not leave the "answer" is this ambiguous form. If you are asked where the image is and what is its magnification, be very explicit and make it clear, in words (not just by plus or minus signs) whether the image

1. is on the same side of the lens as the object is, or on the opposite side;
2. is real or virtual;
3. is erect or inverted;
4. is magnified or diminished.

In this chapter I shall, needless to say, introduce my own sign convention, and needless to say my own convention is vastly superior to anyone else's and quite different from any that you may already be used to or that your own teacher uses, or that you have ever seen in any book. Worse, I am not going to make use of Equation 2.1.1 at all. Instead, I am going to use a technique referred to as the *convergence method*. At first, you are not going to like it at all, and you may give up impatiently after just a few minutes. I hope, however, that you will persist. Let us look, for example, at the following problem:

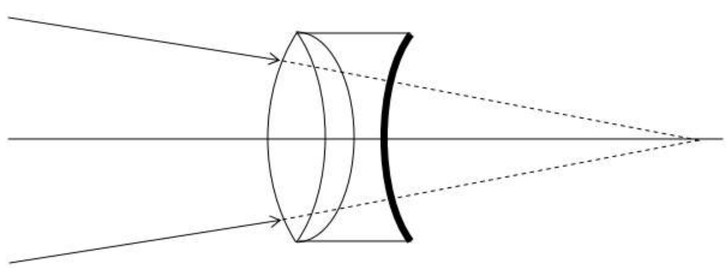


FIGURE II.1

The three lenses all have different refractive indices, all radii of curvature are different, the whole thing is immersed in water, the last surface is a mirror, and the object is a virtual object. Perhaps you are asked to find the image. Or you may be told where the image is and asked to find one of the radii of curvature, or one of the refractive indices. At present, this looks like a hopelessly difficult problem to be avoided before all others in an examination. There is scarcely any chance of getting the right answer.

However, I now assert that, if you take a few minutes to understand the convergence method, you will be limited in your ability to solve problems like these, correctly, *solely by the speed at which you can write*. As soon as you see the problem you will immediately and confidently know how to do it. You just have to make sure that you know where to find the $1/x$ button on your calculator.

2.2: Limitations

Before describing the convergence method, I would say a few words about image formation – words that are equally valid whether you choose to use the convergence method or to stick to conventional equations such as Equation 2.1.1.

We are assuming that a lens or mirror will form a point image of a point object, and that a parallel beam entering a lens will come to a point focus. You are probably aware – even if unfamiliar with all the fine details – that this is not exactly so, and you will be aware that if the diameter of a lens or mirror is comparable to its focal length or to the object or image distance, light will not come to a focus at a point, but the image will suffer from *spherical aberration*. Also, if the angular size of the object or image is large, so that light enters or leaves a lens or mirror at a large angle, additional aberrations such as *coma* and *astigmatism* appear. (It is not always realized that both spherical aberration and astigmatism also occur with refraction at a *plane* interface; neither are phenomena associated solely with curved refractive interfaces or curved reflective surfaces.) Although angles in this chapter are assumed to be small, I shall rarely draw them as small, because to do so would make for very cramped drawings.

In this chapter, I am going to ignore lens and mirror aberrations. I may possibly prepare a separate chapter on lens and mirror aberrations sometime, but that is not the topic of the present chapter. Thus, in this chapter, I am going to assume that all angles are *small*. How small depends on how large an aberration we are prepared to tolerate. Generally it means that I shall be satisfied with the approximation $\sin x = \tan x = x$. This approximation is known as the *paraxial* approximation. It means that none of the light rays make very large angles with the axis of the optical system.

You will also be aware that the refractive index varies with wavelength, and as a result lenses are affected by *chromatic aberration*. This, too, I shall ignore, except for a brief foray in Section 2.9, and if I say that the refractive index of a lens (or rather of the glass of which it is made) is 1.5, I am referring to a particular wavelength or color.

Initially, I shall also make the approximation that lenses are thin. That is to say, I assume that I can neglect the thickness of the lens compared with its focal length or with the object or image distance. However, in Section 2.11, I shall relax this restriction, and I shall deal with "thick" lenses.

2.3: Real and Virtual

Most people have no trouble understanding what a real object is or the distinction between a real image and a virtual image. A virtual *object*, however, may take one by surprise – so let's look at all of them here

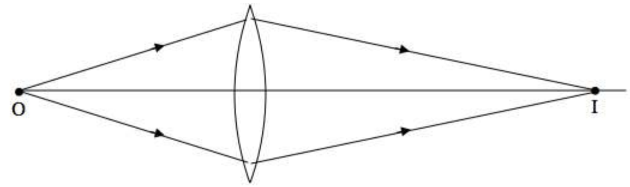


FIGURE II.2

Figure II.2 shows a lens forming a real image of a real object, and I think it requires little explanation. Light diverges from a real object and it converges to a real image. Real photons of light depart from a real object, and real photons of light arrive at a real image.

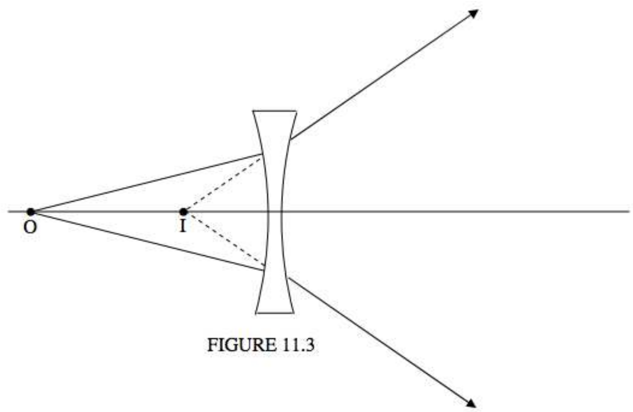


FIGURE 11.3

Figure II.3 shows a lens forming a virtual image of a real object. As before, light diverges from the real object, but no light converges to a real image. After refraction through the lens, the light is diverging from a point where the photons have never visited. The light is diverging from a virtual image.

Whereas you can project a real image on to a piece of card or a photographic film, you cannot do this with a virtual image. The reason that you can *see* a real image with your eye is that the additional optics of your eye bends the diverging light from the virtual image and makes it converge on to a real image on your retina.

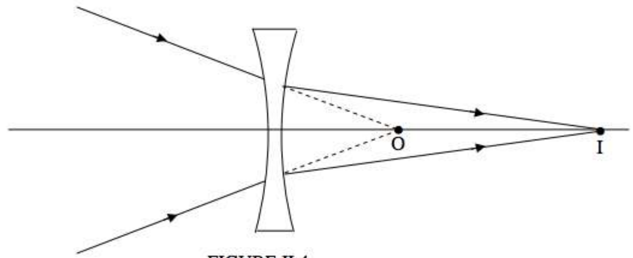


FIGURE II.4

Figure II.4 illustrates what is meant by a virtual object. Light is coming from the left – perhaps from a big lens beyond the left hand edge of the paper (or your computer screen). It is converging to the point O, and, if the concave lens had not got in the way, it would have formed a real image at O. However, as far as the concave lens of Figure II.3 is concerned, the point O to which the light was converging before it reached the lens is a *virtual object.* No photons reach that point. The lens bends the light, which eventually comes to a focus at a real image, I.

You will see that light *converges to a real image or to a virtual object*, and light *diverges from a real object or from a virtual image.*

This is not a sign "convention"; it is just a statement of fact, or an explanation of what are meant by "real object", virtual object", "real image" or "virtual image".

2.4: Convergence

Figure II.5 shows a lens made of glass of refractive index 1.50. To the left of the lens is air (refractive index 1.00). To the right of the lens is water (refractive index 1.33). A converging beam of light is incident upon the lens directed toward a virtual object O that is 60 cm from the lens. After refraction through the lens, the light converges to a real image I that is 20 cm from the lens.

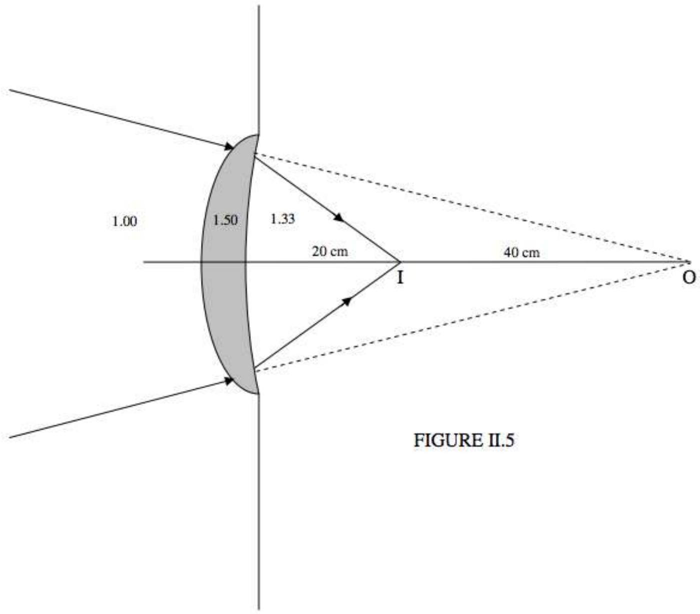


FIGURE II.5

I am not at this stage going to ask you to calculate the radii of curvature of the lens. (You can't – you need one more item of information.) I just want to use this diagram to define what I mean by *convergence.*

The convergence of the light at the moment when it is incident upon the lens is called the *initial convergence* C_1, and it is defined as follows:

$$initial\ convergence = \frac{Refractive\ index}{Object\ distance}. \tag{2.4.1}$$

The convergence of the light at the moment when it leaves the lens is called the *final convergence* C_2, and it is defined as follows:

$$final\ convergence = \frac{Refractive\ index}{Image\ distance}. \tag{2.4.2}$$

Sign convention

- Converging light has positive convergence;
- Diverging light has negative convergence.

Example 2.4.1

Initial convergence = $+\frac{1.00}{60} = +0.01667\ \text{cm}^{-1}$.

Final convergence = $+\frac{1.33}{20} = +0.06650\ \text{cm}^{-1}$.

Notice that, before the light enters the lens, it is in a medium of refractive index 1.00. Thus the relevant refractive index is 1.00, even though the virtual object is in the water.

2.5: Power

It will be evident that the function of a lens is to *change the convergence* of a beam of light. Indeed the difference between the initial and final convergence is called the *power P* of the lens, or of a refracting interface, or of a reflecting mirror. Thus, here is the only equation you need to know in geometric optics. (Well, maybe not quite true.)

Final convergence = initial convergence plus power,

or

$$C_2 = C_1 + P. \tag{2.5.1}$$

In order to solve a question in geometric optics, then, it is necessary to know the power of the optical system.

There are three basic optical elements for which we need to know the power, namely a lens, a refracting interface, and a reflecting surface.

I am now going to *tell you*, without proof, what the powers of these elements are. I shall supply proofs later. For the moment, I want us to become used to using the formulas, accurately and at speed.

1. The power of a lens of focal length f is

$$P = \frac{1}{f}. \tag{2.5.2}$$

Note that by the focal length of a lens I mean the focal length of the lens when it is in a vacuum, or, what amounts to almost the same thing, when it is in air.

Sign convention:

The focal length of a <u>converging</u> lens is <u>positive</u>;

The focal length of a <u>diverging</u> lens is <u>negative</u>.

2. The power of a refracting interface, of radius of curvature r, separating media of refractive indices n_1 and n_2, is

$$P = \frac{n_2 - n_1}{r}. \tag{2.5.3}$$

Sign convention:

The radius of curvature of a <u>convex</u> surface or interface is <u>positive</u>;

The radius of curvature of a <u>concave</u> surface or interface is <u>negative.</u>

3. The power of a reflecting spherical surface of radius of curvature r immersed in a medium of refractive index n is

$$P = -\frac{2n}{r}. \tag{2.5.4}$$

Power can be expressed in cm^{-1} or in m^{-1}. In this connection a m^{-1} is sometimes called a <u>diopter</u>. Thus a lens of focal length 5 cm has a power of 20 diopters.

2.6: Magnification

Magnification is, of course, defined as

$$\text{Magnification} = \frac{\text{Image space height}}{\text{Object space height}}. \tag{2.6.1}$$

Strictly speaking, this is the *linear transverse (or lateral) magnification*. There are other "sorts" of magnification, such as angular magnification and longitudinal magnification, but we shan't deal with these just yet, and the term "magnification" will be assumed to mean the lateral linear magnification.

I now assert *without proof*, (but I shall prove later) that the magnification can be calculated from

$$\text{Magnification} = \frac{\text{Inital space convergence}}{\text{Final space convergence}} = \frac{C_1}{C_2}. \tag{2.6.2}$$

Sign convention

- If the magnification is *positive*, the image is *erect*
- If the magnification is *negative*, the image is *inverted*

2.7: Examples

Example 2.7.1

A real object is 15 cm from a converging lens of focal length 25 cm. Where is the image? Describe it.

Light diverges from a real object, so the initial convergence is negative. $C_1 = -1/15$ cm $^{-1}$. The power of the converging lens is P = + 1/25 cm^{-1}. The final convergence is

$$C_2 = -\frac{1}{15} + \frac{1}{25} = -\frac{2}{75} \text{ cm}^{-1}.$$

The image is 37.5 cm from the lens. Light is diverging after it leaves the lens. The image is on the same side of the lens as the object is. It is a virtual image. The magnification is $C_1/C_2 = +2.5$. The image is erect and magnified in size.

Example 2.7.2

The faces of a biconvex lens have radii of curvature 20 cm and 30 cm, and the refractive index of the glass is 1.5. What is the focal length of the lens?

Refer to Figure II.6.

The initial convergence is zero. The final convergence will be $1/f$. The power of the first surface is $\frac{1.5-1.0}{+20}$ cm^{-1}.The power of the second surface is $\frac{1.0-1.5}{-30}$ cm $^{-1}$. *Note that the radius of curvature of the second surface, when encountered by the light, is negative.*

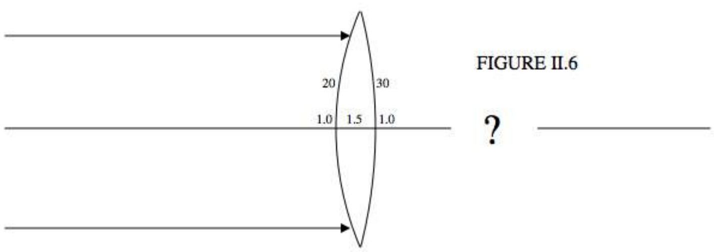


FIGURE II.6

Therefore

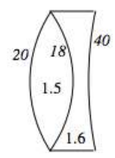
$$\frac{1}{f} = \frac{1.5-1.0}{+20} + \frac{1.0-1.5}{-30}.$$

$$f = 24 \text{ cm.}$$

The lens is a converging lens.

Example 2.7.3

What is the focal length of this lens, in which I have marked the radii of curvature in cm and the refractive indices?



The power, which is the reciprocal of the focal length, is the sum of the powers of the three interfaces:

$$\frac{1}{f} = \frac{1.5-1.0}{20} + \frac{1.6-1.5}{-18} + \frac{1.0-1.6}{40} = +0.004 \text{ cm}^{-1}. \qquad \therefore\ f = +225.0 \text{ cm.}$$

Example 2.7.4

Let's now go straight to the impossibly difficult problem of Section 2.1

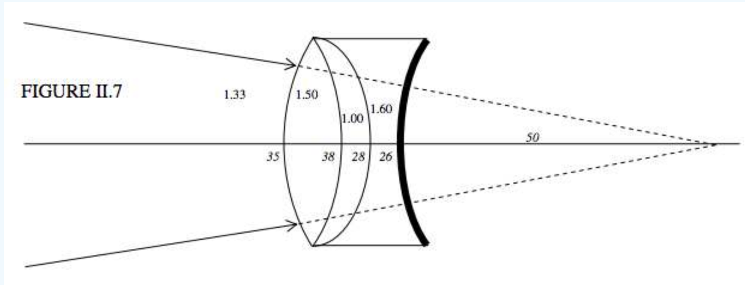


FIGURE II.7

I have marked in the several refractive indices, and, in italics, the radii of curvature and the distance of the virtual object, in cm. Remember that, notwithstanding the drawing, we are assuming that all lenses are *thin* – that is to say that their thicknesses are negligible compared with other distances.

The system is immersed in water, so the initial convergence is +1.33/50. We are going to find the final convergence. To the initial convergence we are going to add, successively, the powers of the first three refracting interfaces, then the reflecting surface, and then the three refracting surfaces again on the way out. Watch for the refractive indices and the signs of the radii of curvature in each term. The calculation goes like this – as fast as you can write: Final convergence =

$$+\frac{1.33}{50}+\frac{1.50-1.33}{35}+\frac{1.00-1.50}{-38}+\frac{1.60-1.00}{-28}+\frac{-2\times1.60}{+26}+\frac{1.00-1.60}{+28}+\frac{1.50-1.00}{+38}+\frac{1.33-1.50}{-35}\quad \mathrm{cm}^{-1}.$$

You can almost double the speed when you realize that the power of a refracting interface is the same whichever way you go (from left to right or from right to left).

We obtain:

Final convergence = $-0.103304\ \mathrm{cm}^{-1}$.

Final convergence is refractive index divided by image distance, so the distance of the image from the lens (remember that it's a thin lens, so don't ask which part of the lens) is 1.33 ÷ 0.103304, or <u>12.9 cm</u>.

The light is diverging after it leaves the lens. It is on the <u>same</u> side of the lens as the virtual object is. It is a *virtual* image. The magnification is initial convergence ÷ final convergence and is therefore −0.257. The image is <u>inverted</u> and <u>diminished</u> in size.

This example perhaps shows the greatest power (pun not intended) of the convergence method – i.e. in dealing with many optical elements one after the other. The is no need for convoluted arguments such as "the real image formed by the first element acts as a virtual object for the second element, and then...".

Example 2.7.5

Three observations are performed on a lens in order to determine the radii of curvature of its two surfaces and the refractive index of its glass.

i.) A real object is placed 40 cm to the left of the lens, and a real image is formed 300 cm to the right.

The question doesn't tell us what sort of a lens it is. Let's suppose that it is biconvex; we'll soon find out if it isn't.

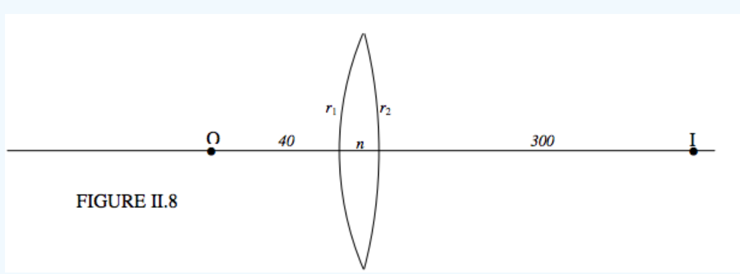


FIGURE II.8

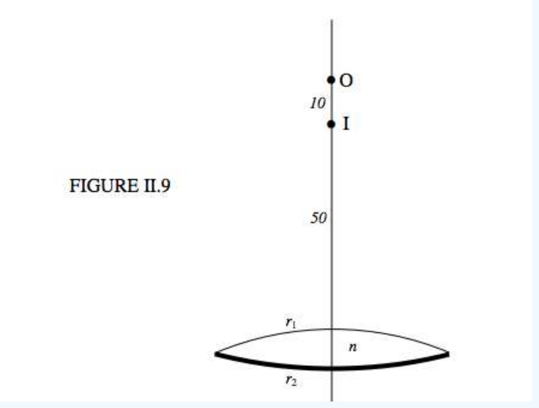
The first experiment tells us:

$$\frac{1}{+300} = -\frac{1}{40} + \frac{n-1}{r_1} + \frac{1-n}{-r_2}.$$

That is

$$(n-1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = 0.0283\text{cm}^{-1} \tag{2.7.1}$$

ii.) The lens is floated on the surface of mercury, r_1-side up. A real object is placed 60 cm above it, and a real image is formed 50 cm above it.



The second experiment tells us:

$$+\frac{1}{50} = -\frac{1}{60} + \frac{n-1}{r_1} + \frac{-2n}{-r_2} + \frac{1-n}{-r_1} \quad .$$

That is:

$$\frac{n-1}{r_1} + \frac{n}{r_2} = 0.0183. \tag{2.7.2}$$

iii.) The lens is floated on the surface of mercury, r_2-side up. A real object is placed 60 cm above it, and a real image is formed 6 cm above it. (Figure II.10.)

It is necessary to remind ourselves that, the drawing notwithstanding, the lens is thin and all angles are small. The third experiment tells us:

$$+\frac{1}{6} = -\frac{1}{6-} + \frac{n-1}{r_2} + \frac{-2n}{-r_1} + \frac{1-n}{-r_2}.$$

That is

$$\frac{n-1}{r_2} + \frac{n}{r_1} = 0.0916. \tag{2.7.3}$$

Thus we have three nonlinear equations to solve for the three unknowns. Three nonlinear equations have been known to make grown men tremble in their shoes, but fortunately, these three are trivial to solve. It might help to let $s = 1/r_1$ and $t = r_2$, when the equations become

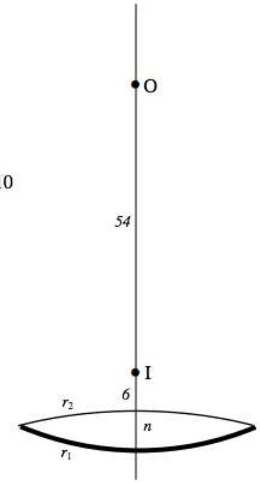


FIGURE II.10

$$(n-1)(s+t)=0.0283. \tag{2.7.4}$$

$$(n-1)s+nt=0.0183. \tag{2.7.5}$$

$$(n-1)t+ns=0.0916. \tag{2.7.6}$$

One soon arrives at: $\underline{n=1.53,\ r_1=15.8 \text{ cm},\ r_2=-100.0 \text{ cm}}$.

Our assumption that the lens is biconvex was wrong. The second surface is the other way round, and the lens is a meniscus converging lens.

Exercise 2.7.1

A converging lens forms a real image of a real object. Show that the least distance between real object and real image is $4f$, and that the magnification is then -1. Remember this when you are trying to show slides in your living room, and you can't seem to focus the projector on the screen.

Exercise 2.7.2

A screen is at a fixed distance from a real object. A converging lens is placed between object and screen so as to throw a magnified inverted real image on the screen. The lens is then moved towards the screen, and, after it has moved a distance d, it is seen to throw another real, inverted image on the screen, but this time diminished. Show that, if the distance between object and screen is w, the focal length of the lens is

$$f=\frac{w^2-d^2}{4w}$$

Exercise 2.7.3

A beetle on the axis of a converging lens and at a distance greater than $2f$ from it runs towards the lens at a speed v. Show that its real image moves at a speed m^2v, where m is the transverse magnification. In which direction does the image move – towards or away from the lens?

Exercise 2.7.4

Two media of refractive indices n_1 and n_2 are separated by a spherical refractive interface or by a lens – it doesn't matter which. An object of length Δp lies along the axis in the n_1 side. As a result, the length of the image is Δq. The ratio $\Delta q/\Delta p$ is called the l*ongitudinal magnification.* Show that

$$m_{\text{long}}=\frac{n_2}{n_1}m_{\text{lat}}^2.$$

Exercise 2.7.5

When a converging lens is placed in water (refractive index = $\frac{3}{4}$) its focal length is twice what it is when it is in air. What is the refractive index of the glass of which the lens is made?

2.8: Derivation of the Powers

Up to this point I have defined what is meant by convergence, and I have defined power as the difference between the final and initial convergences. I *asserted without proof* formulas for the powers of a lens, a refracting interface, and a mirror. It is now time to derive them. Remember that in this chapter I am dealing with small angles only (indeed if angles are not small, a point object will not result in a point image) and consequently I am going to assume that any angle is equal to its tangent or to its sine, and I am going to write Snell's law in the form

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \;\; or \; n_1 \tan\theta_1 = n_2 \tan\theta_2 \;\; or \; n_1\theta_1 = n_2\theta_2$$

as the spirit moves me and at my convenience.

Topic hierarchy

2.8A: Power of a Lens

I have drawn two rays emanating from the tip of the object. One is parallel to the axis; after refraction it passes through the focus. The other goes through the centre ("pole") of the lens; since the lens is "thin", this ray is neither deviated not displaced. The two rays cross at the tip of the image.

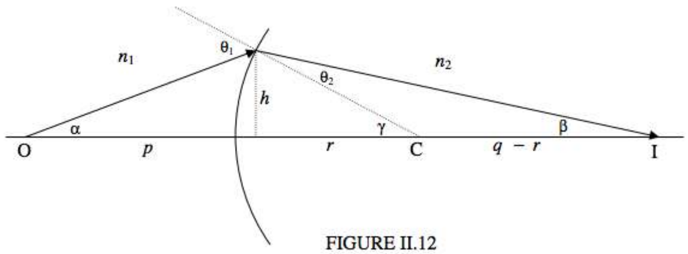


FIGURE II.12

From two obvious pair of similar triangles, we see that

$$\frac{h'}{h} = \frac{q}{p} = \frac{q-f}{f}. \tag{2.8A.1}$$

From this we immediately obtain

$$\frac{1}{q} = -\frac{1}{p} + \frac{1}{f}. \tag{2.8A.2}$$

Since the initial and final convergences are $-1/p$ and $1/q$, it follows that the power is $1/f$. You might want to draw the cases where the real object is at a distance less than $2f$ from the lens (and hence forms a virtual image) or for a virtual object, or the corresponding situations for a diverging lens. You will reach the same conclusion in each case.

2.8B: Power of a Refracting Interface

Figure II.12 shows a refracting interface of radius of curvature r separating media of indices n_1 and n_2.

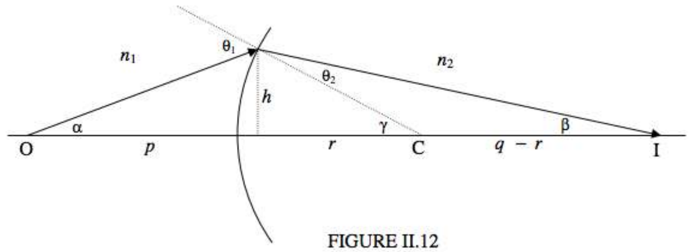


FIGURE II.12

I show a real object at O, a real image at I and the centre of curvature at C. Remember that angles are small and the "lens" is thin. We see that $h = \alpha p = \beta q = \gamma r$. By Euclid, $\theta_1 = \alpha + \gamma$ and $\theta_2 = \gamma - \beta$, and by Snell, $n_1\theta_1 = n_2\theta_2$. From these we obtain

$$\frac{n2}{q} = -\frac{n_1}{p} + \frac{n_2 - n_1}{r}. \tag{2.8B.1}$$

Thus the power is $\frac{n_2 - n_1}{r}$. The reader should try this for other situations (virtual object, virtual image, concave interface, and so on) to see that you always get the same result.

2.8C: Power of a Mirror

In Figure II.13 shows a reflecting surface of radius of curvature r submerged in a medium of index n. I show a real object at O, a virtual image at I and the centre of curvature at C. We see that $h = \alpha p = \beta q = \gamma r$. By Euclid, $\theta = \alpha + \gamma$ and $2\theta = \alpha + \beta$. Remember again that all angles are supposed to be small (even β!), in spite of the drawing. From these we obtain

$$\frac{1}{q} = \frac{1}{p} + \frac{2}{r}. \tag{2.8C.1}$$

On multiplying this by $-n$, we find that the power is $-2n/r$. Again the reader should try this for other situations, such a concave mirror, or a real image, and so on. The same result will always be obtained.

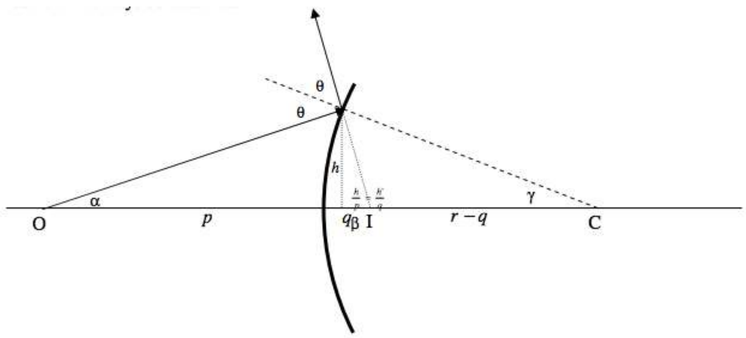


FIGURE 11.13

2.9: Derivation of Magnification

Figure II.14 shows an optical element separating media of indices n_1 and n_2. I have drawn the element as an interface, though it could equally well be a lens (or, if I were to fold the drawing, a mirror). An image of height h' is formed at a distance q of an object of height h at a distance p.

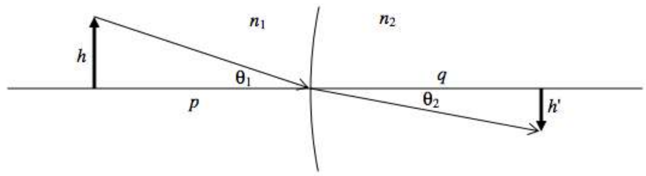


FIGURE II.14

Assuming, as ever, that angles are small, we have

$$\text{magnification} = \frac{\theta_2 q}{\theta_1 p}. \tag{2.9.1}$$

But Snell's law, for small angles, is $n_1\theta_1 = n_2\theta_2$, and therefore

$$\text{magnification} = \frac{n_1 q}{n_2 p} = \frac{C_1}{C_2}. \tag{2.9.2}$$

2.10: Designing an Achromatic Doublet

It is not the intention of this chapter to study lens aberrations. However, the design of an achromatic doublet lens lends itself to the sort of calculation we are doing in this chapter.

A combination of two lenses in contact, a converging lens made of crown glass and a weaker diverging lens made of flint glass, can be designed so that the combination is a converging lens that is almost achromatic. Flint glass is a little denser than crown glass, and has a higher refractive index and a greater dispersive power.

The *dispersive power* ω of glass is usually defined as

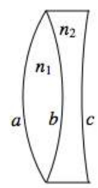
$$\omega=\frac{n^{(F)}-n^{(C)}}{n^{(D)}-1}. \tag{2.10.1}$$

Here C, D and F refer to the wavelengths of the C, D and F Fraunhofer lines in the solar spectrum, which are respectively, Hα (656.3 nm), Na I (589.3 nm), Hβ (486.1 nm), and which may be loosely referred to as “red”, “yellow” and “blue”. A typical value for a crown glass would be about 0.016, and a typical value for a flint glass would be about 0.028.

An achromatic doublet is typically made of a positive crown glass lens whose power is positive but which decreases with increasing wavelength (i.e. toward the red), cemented to a weaker flint glass lens whose power is negative and also decreases (in magnitude) with increasing wavelength. The sum of the two powers is positive, and varies little with wavelength, going through a shallow minimum. Typically, in designing an achromatic doublet, there will be two requirements to be satisfied: 1. The power or focal length in yellow will be specified, and 2. You would like the power in red to be the same as the power in blue, and to vary little in between.

Consider the doublet illustrated in Figure II.15, constructed of a biconvex crown lens and a biconcave flint lens.

FIGURE II.15



I have indicated the indices and the radii of curvature. The power (reciprocal of the focal length) of the first lens by itself is

$$P_1=(n_1-1)\left(\frac{1}{a}+\frac{1}{b}\right), \tag{2.10.2}$$

and the power of the second lens is

$$P_2=-(n_2-1)\left(\frac{1}{b}+\frac{1}{c}\right). \tag{2.10.3}$$

I shall write these for short, in obvious notation, as

$$P_1=k_1(n_1-1), \qquad P_2=-k_2(n_2-1). \tag{2.10.4}$$

But we need equations like these for each of the three wavelengths, thus:

$$P_1^{(C)}=k_1(n_1^{(C)}-1), \qquad P_2^{(C)}=-k_2(n_2^{(C)}-1), \tag{2.10.5}$$

$$P_1^{(D)}=k_1(n_1^{(D)}-1), \qquad P_2^{(D)}=-k_2(n_2^{(D)}-1), \tag{2.10.6}$$

$$P_1^{(F)}=k_1(n_1^{(F)}-1), \qquad P_2^{(F)}=-k_2(n_2^{(F)}-1). \tag{2.10.7}$$

Now we want to satisfy two conditions. One is that the total power be specified:

$$P_1^{(D)}+P_2^{(D)}=P^{(D)}. \tag{2.10.8}$$

The other is that the total power in the red is to equal the total power in the blue, and I now make use of equations 2.10.5 and 2.10.7:

$$k_1(n_1^{(C)}-1)-k_2(n_2^{(C)}-1)=k_1(n_1^{(F)}-1)-k_2(n_2^{(F)}-1). \tag{2.10.9}$$

On rearrangement, this becomes

$$k_1(n_1^{(F)}-n_1^{(C)})=k_2(n_2^{(F)}-n_2^{(C)}). \tag{2.10.10}$$

Now, making use of equations 2.10.1 and 2.10.6, we obtain the condition that the powers will be the same in red and blue:

$$\omega_1 P_1+\omega_2 P_2=0. \tag{2.10.11}$$

For example, suppose that we want the focal length in yellow to be 16 cm ($P^{(D)}=0.0625$ cm^{-1}) and that the dispersive powers are 0.016 and 0.028. Equations 2.10.8 and 2.10.11 then tell us that we must have $P_1^{(D)}=0.14583$ cm^{-1} \) and $P_2^{(D)}=-0.083$ cm^{-1}. ($f_1=6.86$ cm and $f_2=-12.0$ cm).

If we want to make the first lens equibiconvex, so that $a=b$, and if $n_1=1.5$, Equation 2.10.2 tells us that a = 6.86 cm. If $n_2=1.6$, Equation 2.10.3 then tells us that $c=-144$ cm. That c is negative tells us that our assumption that the flint lens was concave to the right was wrong; it is convex to the right.

Exercise 2.10.1

Suppose that, instead of making the crown lens equibiconvex, you elect to make the last surface flat – i.e. c = ∞. What, then, must a and b be?

Answers. a = 6.55 cm, b = 7.20 cm.

2.11: Thick Lenses

Figure II.16 shows a thick lens of index n_2, an object O and an image I. For good measure I have put a medium of index n_1 to the left of the lens and a medium of index n_3 to the right of the lens.

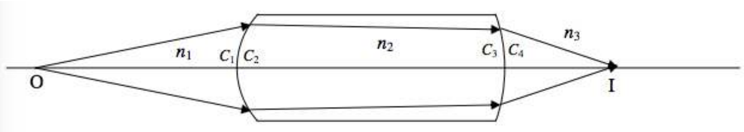


FIGURE II.16

If you are given the position of O, can you calculate the position of the image?

Well, it is easy to calculate the convergence C_1 when the light arrives at the first surface. Then we can easily calculate the convergence C_2 merely by adding the power of the first interface. And, if we know C_3 (Aye, there's the rub) we can easily calculate C_4. We see that the key to solving thick lens problems is to know how convergence changes with distance, so we shall make that our next aim.

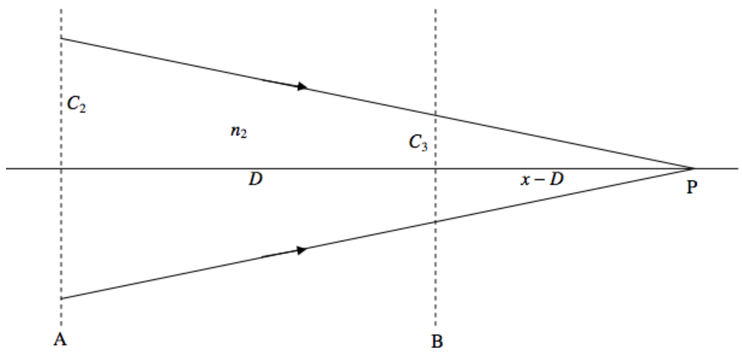


FIGURE II.17

Figure II.17 shows a beam of light, in a medium of index n_2, converging to the point P, which is at a distance x from the plane A. The convergence of the beam as it leaves the plane A is $C_2 = n_2/x$. When it arrives at the plane B, which is at a distance D from the plane A, its convergence is $C_3 = n_2/(x-D)$. When we eliminate x, we obtain

$$C_3 = \frac{n_2 C_2}{n_2 - DC_2} \tag{2.11.1}$$

for the formula that tells us how convergence changes with distance.

Let us now return to the problem of Figure II.16.

Example 2.11.1

Let's suppose that the radii of curvature of the first and second faces are 15 and 25 cm respectively, and the distance between the faces is 50 cm. The refractive index of the glass is n_2 = 1.60. We'll suppose that there is water (n_1 = 1.33) to the left of the lens, and, to the right of the lens there is some liquid with refractive index 1.42. The object is 30 cm to the left of the first face. Where is the image?

Solution

The calculation goes as follows (Equation 2.11.1):

$$C_1 = -\frac{1.33}{30} = -0.044333\,\mathrm{cm}^{-1}$$

$$C_2 = C_1 + \frac{1.60-1.33}{+15} = -0.026333\,\mathrm{cm}^{-1}$$

$$C_3 = \frac{1.60 \times C_2}{1.60 - 50 \times C_2} = -0.014446\,\mathrm{cm}^{-1}.$$

Notice that the light is diverging by the time that it reaches the second face.

$$C_4 = C_3 + \frac{1.42-1.60}{-25} = -0.007246\,\mathrm{cm}^{-1}.$$

The light is still diverging, so the image is virtual. The distance of the image from the second face is 1.42 ÷ 0.007 246 = 196 cm, and it is to the left of the second face.

The magnification of a thick lens is easily found. The magnification produced by the first face is, as usual, C_1/C_2, and then there is a further magnification of C_3/C_4 produced by the second face. Thus the overall magnification is $\frac{C_1C_3}{C_2C_4}$, which is this case is +3.356. The image is magnified in size and it is erect.

This method for thick lenses can also be used for separated lenses and mirrors. Here's one:

Example 2.11.2

Figure II.18 shows a thin lens separated from a mirror, and an object 14 cm from the lens. Where is the image?

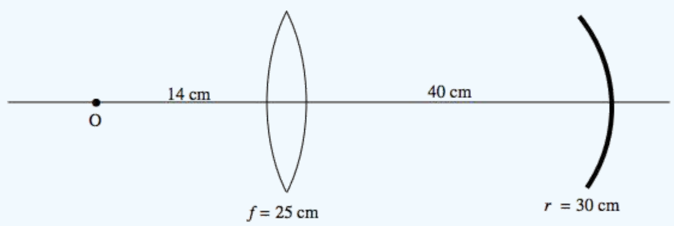


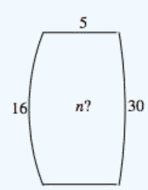
FIGURE II.18

Solution

$$C_1 = -1/14 = -0.071420\,\mathrm{cm}^{-1}.$$
$$C_2 = C_1 + 1/25 = -0.031429\ \mathrm{cm}^{-1}.$$
$$C_3 = \frac{C_2}{1-40\times C_2} = -0.013924\ \mathrm{cm}^{-1}.$$
$$C_4 = C_3 + \frac{-2}{-30} = +0.053743\ \ \mathrm{cm}^{-1}.$$

The image is real. It is 18.96 cm to the left of the mirror. The magnification is −0.60. The image is inverted and diminished.

Of course those who set examinations can think of all sorts of unpleasant questions. For example, we might have a thick lens and an object, but, instead of being asked to find the image, we may be told the image distance and asked to find the refractive index, or the thickness, or one of the radii. Or, even worse, we might not be told the image distance, but we might be told its magnification and whether it is real or virtual, or erect or inverted, and asked to find something else. There are endless possibilities! Here's one.



The lens shown has radii of curvature 16 and 30 cm, and is 5 cm thick. An object is 36 cm to the left of the 16 cm face. Its image is 50 cm to the right of the 30 cm surface. Show that the refractive index is the positive solution of

$$1845n^2 - 2417n - 520 = 0.$$

Here's another one.

Example 2.11.2

The lens shown is 4 cm thick and the refractive index is 1.6. The radius of curvature of the first face is 15 cm. An object is 32 cm to the left of the 15 cm face. Its image is real, inverted and magnified by 22. Determine the radius of curvature of the second face.

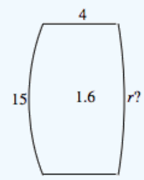


FIGURE II.19

Solution

Hints. The image is real. Which side of the lens is it? You can easily calculate C_1, C_2 and C_3, so you should be able to get C_4 from the magnification. The answer, by the way, is 80.1 cm – but is it convex to the right, as shown, or is it concave to the right?

One more.

Example 2.11.4

The two lenses are made of a very light solid whose refractive index is only 1.3. (I'm not sure if there *is* such a stuff!) and they are immersed in a liquid of index 1.4. That means that the convex lens is diverging. The second surface of the second lens is a reflecting mirror. I have indicated the radii of curvature, and the lenses are 40 cm apart. Parallel light comes from the left. Where does it come to a focus?

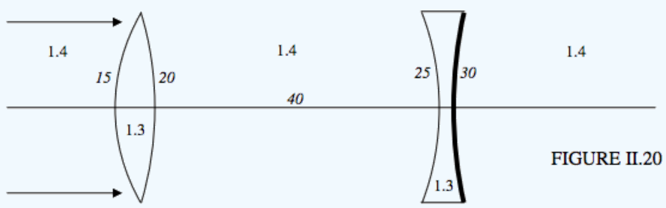


FIGURE II.20

Solution

The initial convergence $C_1 = 0$. I'll calculate the convergence after the light arrives at or leaves each surface or interface. I hope the notation will be clear. All convergences are in cm^{-1}.

$$C_2 = 0 + \frac{1.3 - 1.4}{+15} = -0.006.$$

$$C_3 = -0.006 + \frac{1.4 - 1.3}{-20} = -0.0116.$$

$$C_4 = \frac{1.4C_3}{1.4 - 40C_3} = -0.00875.$$

$$C_5 = -0.00875 + \frac{1.3 - 1.4}{-25} = -0.00475.$$

$$C_6 = -0.00475 + \frac{-2 \times 1.3}{30} = -0.091416.$$

$$C_7 = -0.091416 + \frac{1.4 - 1.3}{+25} = -0.087416.$$

$$C_8 = \frac{1.4C_7}{1.4 - 40C_7} = -0.02499319265.$$

$$C_9 = C_8 + \frac{1.3 - 1.4}{-20} = -0.02999319265.$$

$$C_{10} = C_9 + \frac{1.4 - 1.3}{-15} = -0.03665985931.$$

Finally, $C_{10} = \frac{1.4}{x}$, so $x = -38.18890815 \left(= -\frac{22035}{577}\right)$ cm.

That is, the focus is 38.2 cm to the right of the convex lens, or 1.8 cm to the left of the concave lens.

Exercise 2.11.1

Compose a problem in which a student is given the focal length of two lenses, and the positions of object and image, and the student is asked to calculate the distance between the lenses.

2.12: Principal Planes

Consider a thick lens, or a system of two separated lenses. In Figure II.21, F_1 is the first focal point and H_1 is the first principal plane. In Digure II.22, F_2 is the second focal point and H_2 is the second principal plane

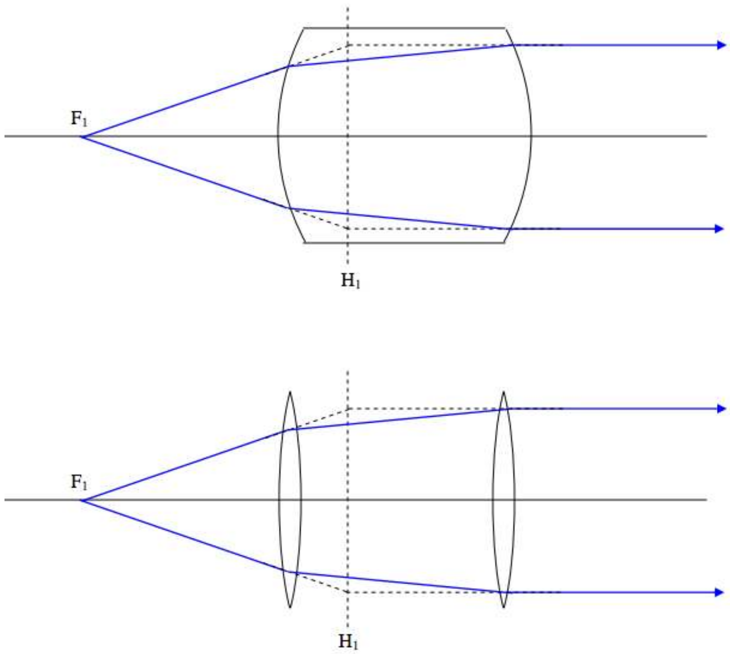


FIGURE II.21

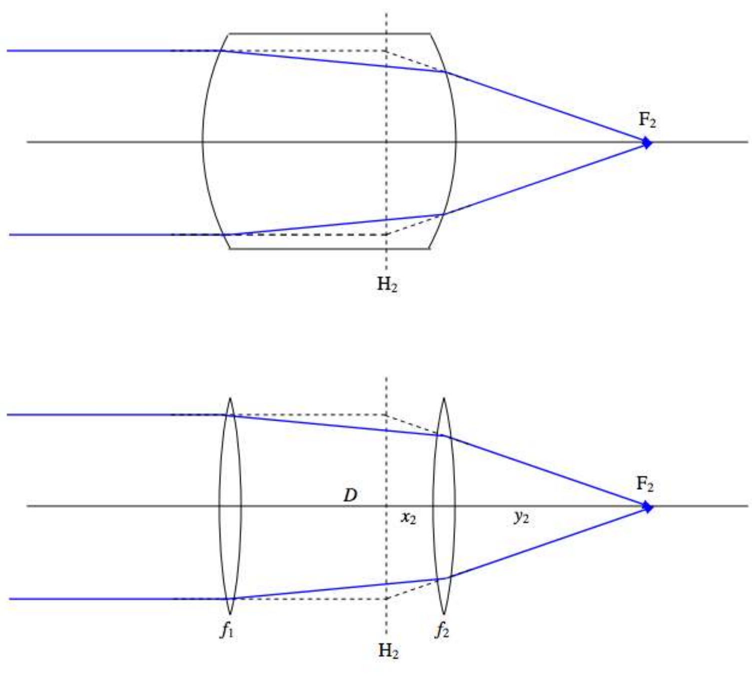


FIGURE II.22

I refer now to the second part of FFigure II.22, and I suppose that the focal lengths of the two lenses are f_1 and f_2, and the distance between them is D. I now invite the reader to calculate the distances x_2 and y_2. The distance x_2 can be calculated by consideration of some similar triangles (which the reader will have to add to the drawing), and the distance y_2 can be calculated by calculating the convergences C_1, C_2, C_3, C_4 in the manner which is by now familiar. You should get

$$x_2 = \frac{Df_2}{f_1 + f_2 - D}. \tag{2.12.1}$$

and

$$y_2 = \frac{f_2(f_1 - D)}{f_1 + f_2 - D}. \tag{2.12.2}$$

I further invite the reader to imagine that the two lenses are to be replaced by a single lens situated in the plane H_2 so as to bring the light to the same focus F_2 as was obtained by the two original lenses. The question is: what must be the focal length f of this single lens? The answer is obviously $x_2 + y_2$, which comes to

$$f = \frac{f_1 f_2}{f_1 + f_2 - D}. \tag{2.12.3}$$

The eyepiece of an optical instrument such as a telescope or a microscope is generally a combination of two (or more) lenses, called the *field lens* and the *eye lens*. They are generally arranged so that the distance between the two is equal to half the sum of the focal lengths of the two lenses. We shall now see that this arrangement, with two lenses made of the same glass, is relatively free from chromatic aberration.

Let us remind ourselves that the power of a lens in air is given by

$$P = \frac{1}{f} = (n-1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \tag{2.12.4}$$

Here r_1 and r_2 are the radii of curvature of the two surfaces, and n is the refractive index of the glass. For short, I am going to write Equation 2.12.4 as

$$P = \frac{1}{f} = \mu S, \tag{2.12.5}$$

where $\mu = n - 1$ and $S = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)$. That being so, Equation 2.12.3 can be written

$$P = \mu(S_1 + S_2) - \mu^2 SD_1 S_2 \tag{2.12.6}$$

This equation shows how the position of the focus F_2 varies with colour. In particular,

$$\frac{dP}{d\mu} = S_1 + S_2 - 2\mu DS_1 S_2, \tag{2.12.7}$$

which shows that the position of F_2 doesn't vary with colour provided that the distance between the lenses is

$$D = \frac{S_1 + S_2}{2\mu S_1 S_2}. \tag{2.12.8}$$

On going back to Equation 2.12.5, we see that this translates to

$$\underline{\underline{D = \frac{1}{2}(f_1 + f_2).}} \tag{2.12.9}$$

2.13: The Lazy Way

The convergence and power method has great advantages when you have a complex systems of many lenses, mirrors and interfaces in succession. You just add the powers one after the other. But I expect there are some readers who don't want to be bothered with all of that, and just want to do simple single-lens calculations with a simple formula that they are accustomed to, in particular the well-known $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$, , which is appropriate for the "real is positive" sign convention – and they want to get the calculation over with as soon as possible and with as little effort as possible. This section is for them! I have drawn a simple diagram in Figure II.23. It is not extremely accurate – it is the best I can do with this infernal machine that I am sitting in front of. All you need in order to draw a really good version of it is a sheet of graph paper. There are three axes, labelled p, q and f. For any particular problem, to solve the above equation, all you do is to lay the edge of a ruler across the figure. For example: p = 40 cm, f = 26 cm. What is q? The dashed line gives the answer: $q=75$cm. Another example: $p=33$ cm, $q=-60$ cm. What is f? The dotted line gives the answer: $f=73$ cm.

This diagram can also be used for resistors in parallel, capacitors in series, synodic and sidereal periods of planets...

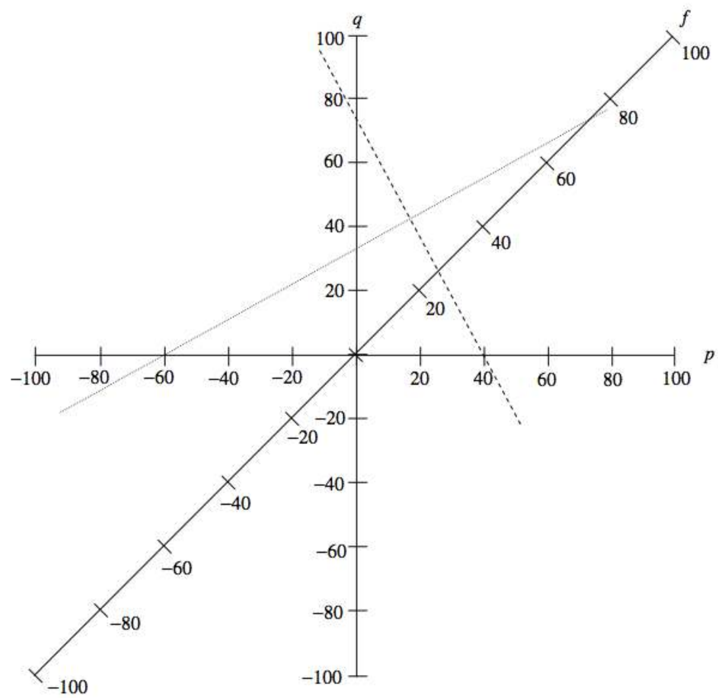


FIGURE II.23

2.14: Exercise

Exercise 2.14.1

An object is placed 90 cm to the left of a thin lens in air. The image is real and is 99 cm to the right of the lens.

However, if the medium to the right of the lens is water (refractive index 1.33), the image is virtual and is 76 cm to the left of the lens.

And if the medium to the left of the lens is water (and to the right is air) the image is real and 47 cm to the right of the lens.

Calculate the two radii of curvature and the refractive index of the glass.

Exercise 2.14.2

An object is placed 100 cm to the left of the first surface (A) of a thick lens (thickness = 10 cm) in air. The image is real and is 25 cm to the right of the second surface (B).

However, if the medium to the right of the second surface (B) is water (refractive index 1.33), the image is real and 41 cm to the right of surface B.

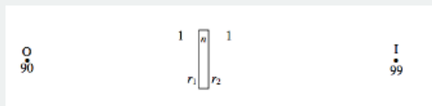
And if the medium to the left of surface A is water (and to the right of B is air) the image is real and 92 cm to the right of the lens.

Calculate the two radii of curvature and the refractive index of the glass.

Neither of these two problems is likely to turn up in a practical situation – but they are very good practice for difficult lens problems! Solutions on the next page – but no peeking until you have tried them!

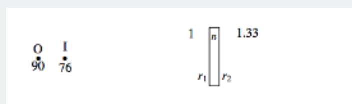
Solutions

1. First:



$$\frac{1}{99} = -190 + \frac{n-1}{r_1} + \frac{1-n}{r_2}. \tag{2.14.1}$$

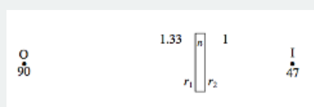
Second:



$$-\frac{-1.33}{76} = -\frac{1}{90} + \frac{n-1}{r_1} + \frac{1.33-n}{r_2}. \tag{2.14.2}$$

(You might be tempted to think that the left hand side of this equation should be $\frac{1}{76}$. Make sure that you understand why this is wrong.)

Third:



$$\frac{1}{47} = -\frac{1.33}{90} + \frac{n-1.33}{r_1} + \frac{1-n}{r_2}. \tag{2.14.3}$$

The physics is now finished. All that has to be done is to solve the three equations for the three unknowns. I would start by letting $x = r_1, y = r_2, z = n$. The equations then become:

$$z(x-y) - x + y = 0.021212121, \tag{2.14.4}$$

$$z(x-y) - x + 1.33y = -0.006388889 \tag{2.14.5}$$

and

$$z(x-y) - 1.33x + y = 0.036054374. \tag{2.14.6}$$

These are easy to solve:

$$x = -0.044977, \quad y = -0.083639, \quad z = +1.54864.$$

Thus

$$r_1 = 22.23\text{cm}, \qquad r_2 = 11.96\text{cm}, \qquad n = 1.5549\text{cm}$$

The lens is a positive meniscus lens (i.e. thicker in the middle), both surfaces being convex to the right. It looks like this:

2.

First:

$$C_1 = -\frac{1}{100} \tag{2.14.7}$$

$$C_2 = -\frac{1}{100} + \frac{n-1}{r_1} = \frac{100n - 100 - r_1}{100r_1} \tag{2.14.8}$$

$$C_3 = \frac{nC_2}{n - 10C_2} = \frac{100n^2 - 100n - r_1 n}{100r_1 n + 10r_1 - 1000n + 1000} \tag{2.14.9}$$

$$C_4 = \frac{100n^2 - 100n - r_1 n}{100r_1 n + 10r_1 - 1000n + 1000} + \frac{1-n}{r_2} = \frac{1}{25} \tag{2.14.10}$$

Second:

$$C_1 = -\frac{1}{100} \tag{2.14.11}$$

$$C_2 = -\frac{1}{100} + \frac{n-1}{r_1} = \frac{100n - 100 - r_1}{100r_1} \tag{2.14.12}$$

$$C_3 = \frac{nC_2}{n - 10C_2} = \frac{100n^2 - 100n - r_1 n}{100r_1 n + 10r_1 - 1000n + 1000} \tag{2.14.13}$$

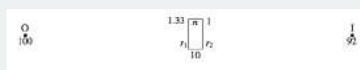
$$C_4 = \frac{100n^2 - 100n - r_1 n}{100r_1 n + 10r_1 - 1000n + 1000} + \frac{1.33-n}{r_2} = \frac{1.33}{41} \tag{2.14.14}$$

So far, we have obtained two complicated-looking Equations (2.14.10 and \(
\ref{eq:2.14.14\)) in the three unknowns \(r_1, r_2\) and \(n\), and we are just about to embark on obtaining a third equation from the third experiment, after which we shall have to face the unpleasant task of solving the th
\) and 2.14.14 I make it

$$r_2 = -43.64516129\text{cm} \tag{2.14.15}$$

so that the second surface is concave to the left – i.e. it bulges towards the right. This was an unexpected piece of good fortune! We can now move on to the third experiment.

Third:



$$C_1 = -\frac{1.33}{100} \tag{2.14.16}$$

$$C_2 = -\frac{1.33}{100} + \frac{n-1.33}{r_1} = \frac{100n - 133 - 1.33r_1}{100r_1} \tag{2.14.17}$$

$$C_3 = \frac{nC_2}{n - 10C_2} = \frac{100n^2 - 133n - 1.33r_1 n}{100r_1 n + 13.3r_1 - 1000n + 1330} \tag{2.14.18}$$

$$C_4 = \frac{100n^2 - 133n - 1.33r_1 n}{100r_1 n + 13.3r_1 - 1000n + 1300} + \frac{1-n}{r_2} = \frac{1}{92} \tag{2.14.19}$$

We can now solve Equations 2.14.10 and 2.14.19, or 2.14.14 and 2.14.19 for r_1 and n. The very conscientious will want to solve them using 2.14.10 and 2.14.19 and then repeat the solution using 2.14.14 and 2.14.19, and verify that they give the same answer, and will then further verify that the correct solutions have been obtained by substitution in each of the three equations in turn. Being slightly less conscientious, I am going to use Equations 2.14.10 and 2.14.19, and I shall then verify that the solutions obtained satisfy Equation 2.14.14.

I find it easier to solve equations in x and y rather than in r_1 and n, so I am going to let $x = r_1$ and $y = n$. Then, bearing in mind that we have already found that $r_2 = -43.64516129$, Equations 2.14.10 and 2.14.19 become, respectively, after a little algebra and arithmetic,

$$\frac{100y^2 - 100y - xy}{100xy + 10x - 1000y + 1000} = by + c \tag{2.14.20}$$

and

$$\frac{100y^2 - 100ay - axy}{100xy + 10ax - 1000y + 1000a} = by + d, \tag{2.14.21}$$

where

a = +1.33

b = -0.02291204730

c = +0.06291204630

d =+0.03379161252

After a little more slightly tedious but routine algebra and arithmetic, these become

$$Axy^2 + By^2 + C_1xy + D_1x + E_1y + F_1 = 0 \tag{2.14.22}$$

and

$$Axy^2 + By^2 + C_2xy + D_2x + E^2y + F_2 = 0, \tag{2.14.23}$$

where

A = 2.291204730

B = 77.08795270

C_1 = -7.062084 257

D_1 = -0.6291204730

E_1 = -14.17590540

F_1 = -62.91204730

C_2 = -4.403431023

D_2 = -0.449295447

E_2 = -68.74536457

F_2 = -44.92954465.

Then we have to solve these two equations! These can be solved, for example, by the method described in Section 1.9 of Chapter 1 of the notes on Celestial Mechanics. Since I already have a computer program that does that, I used it and got $x = 15.386908$and $y = .1518865$. Thus the solution for the lens is

$$r_1 = +15.39\mathrm{cm} \qquad r_2 = -43.65\mathrm{cm} \qquad n = 1.519$$

The first surface is convex to the left, and the second surface is convex to the right. I.e. the lens is “fat”, bulging in the middle.

As a check that our arithmetic is all right, we can verify that this solution also satisfies Equation 2.14.14. (It does!)

As a further check, the reader might now like to start with these numbers, and an object distance of 100 cm, and see if it results in the three image distances given in the original problem. (It does!)

Another way to solve equations 2.14.22and 2.14.23is to subtract the former from the latter to obtain

$$axy + bx + cy + d = 0, \tag{2.14.24}$$

when

a = 2.658653234

b = 0.179825026

c = -54.56945917

d = 17.98260265.

You can then express x and a function of y and substitute into Equation 2.14.22(or into 2.14.23, or both as a check). You then have a single cubic equation in y, rather than two simultaneous equations in x and y, as follows:

$$(Ba - Ac_y^3 + (E_1 a + Bb - C_1 c - Ad)y^2 + (F_1 a + E_1 b - D_1 c - C_1 d)y + F_1 b - D_1 c = 0. \tag{2.14.25}$$

Numerically, this is

$$329.9799377y^3 - 450.4024164y^2 - 77.14660950y - 6.294 \times 10^{-5} = 0. \tag{2.14.26}$$

The only positive real root of this is $y(=n) = 1.518864$, which is the same as we obtained before. The value of $x(=r_1)$ is then readily found, from Equation 2.14.24, to be 15.3869 cm, as before.

Exercise 2.14.3

A converging lens has a focal length of 40 cm in air. What is its focal length when it is immersed in water, of refractive index 1.333?

After a moment’s thought you will demand that you be told the refractive index of the glass. After further thought, you will conclude that not only do you need to know the refractive index of the glass, but you also need to know the shape (radii of curvature of the surfaces) of the lens.

So, here’s the question properly set.

A biconvex lens is made of glass of refractive index 1.5. The radii of curvature of its surfaces are 25 cm and 100 cm. What is its focal length in air? What would be its focal length if immersed in water of refractive index 4/3? What would be its focal length if immersed in carbon bisulphide of refractive index 5/3?

Exercise 2.14.4

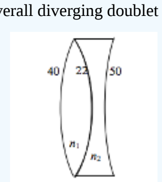
A lens has a focal length in air of 30 cm.

A block of glass with the same refractive index as the above lens has an air bubble inside it of exactly the same size and shape of the above lens. What is the focal length of this lens-shaped bubble?

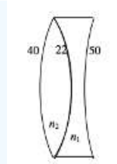
You may be asking yourself if you need to know the shape of the lens, or the refractive index of the glass. I’ll let you ponder.

Exercise 2.14.5

A thin cemented doublet is made of two thin lenses cemented together, as shown in the drawing below. The radii of curvature in cm are indicated in the drawing. The refractive index of the left hand lens is n_1, and that of the right hand lens is n_2. The combination results in an overall diverging doublet lens of focal length 127 cm.



As a result of a manufacturing error, the two types of glass are inadvertently exchanged, and a doublet lens as shown below is made:



This combination results in an overall *converging* doublet lens of focal lens 72 cm. Calculate the refractive indices of the two types of glass.

Here are my solutions to problems **3** to **5**

Solutions

3. The focal length in air is given by

$\cdot = 0 + \frac{1.5-1.0}{25} + \frac{1.0-1.5}{-100}, \qquad \text{whence} \underline{\underline{f = 40 \text{cm}}}$

The focal length in water is given by

$\frac{\frac{4}{3}}{f} = 0 + \frac{\frac{3}{2}-\frac{4}{3}}{25} + \frac{\frac{4}{3}-\frac{3}{2}}{-100}, \qquad \text{whence} \underline{\underline{f = 160 \text{cm}}}$

The focal length in CS_2 is given by

$\frac{\frac{5}{3}}{f} = 0 + \frac{\frac{3}{2}-\frac{5}{3}}{25} + \frac{\frac{5}{3}-\frac{3}{2}}{-100}, \qquad \text{whence} \underline{\underline{f = -200 \text{cm}}}$

4. Regardless of the shape of the lens, the focal length of the glass lens in air is given by

$$\frac{1}{f_{\text{lens}}} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

whereas the focal length of the bubble in glass is given by

$$\frac{1}{f_{\text{bubble}}} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Thus

$$\underline{\underline{f_{\text{bubble}} = -n \times f_{\text{lens}} = -30n \quad \text{cm}}}$$

5. The focal lengths of the two doublets are related to the refractive indices by

$$-\frac{1}{127} = \frac{n_1-1}{40} + \frac{n_2-n_2}{-22} + \frac{1-n_2}{50}$$

and

$$\frac{1}{72} = \frac{n_2-1}{40} + \frac{n_1-n_2}{-22} + \frac{1-n_1}{50}$$

These equations can be rewritten

$$1968n_1 - 18288n_2 + 803 = 0$$

and

$$1296n_1 - 1395n_2 + 374 = 0,$$

with solutions

$$\underline{\underline{n_1 = 1.521 \qquad n_2 = 1.682}}$$

CHAPTER OVERVIEW

3: Optical Instruments

The title of this chapter is to some extent false advertising, because the instruments described are the instruments of first-year optics courses, not optical instruments of the real world of optical technology. Thus a telescope consists of a long focal length lens called the object glass and a short focal length lens called the eyepiece, and the magnification is equal to the ratio of the focal lengths. Someone whose experience with telescopes is limited to this concept of a telescope would scarcely recognize a real telescope. A real telescope would consist of an overwhelming mass of structural engineering intertwined with a bewildering array of electronics, wires and flashing lights. There would be no long focal length lens. Instead there would be a huge mirror probably with a hole in the middle of it. There would be no eyepiece, nor anyone to look through it. The observer would be sitting in front of a computer terminal, quite possibly in another continent thousands of miles away.

Thus the intent of the chapter is mainly to give a little bit of help to beginning students who are struggling to answer examination question of the type “A microscope consists of two lenses of such-and-such focal lengths. What is the magnification?” None of this means, however, that the simple and fundamental principles described in this chapter do not apply to real instruments. They most certainly do apply. This is just a beginning.

Thumbnail: The photographer can see the subject before taking an image by the mirror. When taking an image the mirror will swing up and light will go to the sensor instead. Camera lens Reflex mirror Focal-plane shutter Image sensor Matte focusing screen Condenser lens Pentaprism/pentamirror Viewfinder eyepiece. (CC BY-SA 3.0; Cburnett).

3.1: The Driving Mirror

The mirror inside a car above the driver's head and the outside mirror on the driver's side are usually plane mirrors. The mirror I have in mind for this section, however, is the outside mirror on the passenger's side. This is usually a convex mirror with some words inscribed on it that say something like "OBJECTS IN MIRROR ARE CLOSER THAN THEY APPEAR".

The image formed by the convex mirror is actually an erect, diminished, virtual image, and it "appears" just a few inches behind the surface of the mirror. The object is much further away that it "appears" to be!

That, however, is not the main purpose of discussing this important scientific instrument. The reason that the outside mirror on the passenger side is convex is to give the driver a large *field of view*, so that this gives us an opportunity to think about the *field of view* of an optical system.

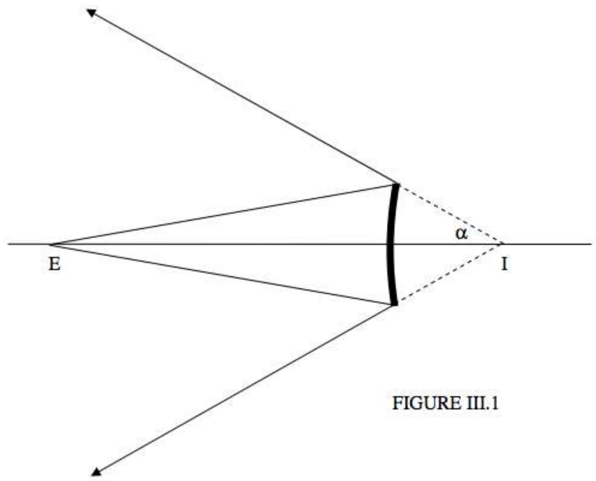


FIGURE III.1

In Figure III.1, we see a convex mirror, and the observer's eye is at E. (As with previous chapters, angles are supposed to be small, my artistic efforts notwithstanding.) The angle α is evidently the *radius of the field of view*. How do we calculate it? Well, I hope it is clear from the drawing that the point I is actually the *virtual image of the eye* formed by the mirror. That being so, we can say:

> *The angular size of the field of view is equal to the angle subtended by the mirror at the image of the eye.*

This is true of a concave mirror as well as of a convex or indeed a plane mirror, and is equally true when we look through a lens. (Draw the corresponding diagrams to convince yourself of this.)

Example 3.1.1

Your eye is 50 cm in front of a convex mirror whose diameter is 4 cm and whose radius of curvature is 150 cm. What is the angular diameter of the field of view?

Solution

First we need to find the position of the image of the eye. Suppose it is at a distance q behind the mirror.

Final convergence = Initial convergence + power

The light is diverging before and after reflection, so both convergences are negative. The power of a mirror is $-2n/r$, and here $n = 1$ and $r = +150$ cm, because the surface is convex. Thus

so the image is 30 cm behind the mirror. The diameter of the mirror is 4 cm, so that angular diameter of the mirror from I (i.e. the field of view) is 4/30 = 0.1333 rad = 7°38'.

3.2: The Magnifying Glass

Two points about a magnifying glass to begin with. First, apparently rather few people understand how to use this complicated scientific instrument. The correct way to use it is to *hold it as close to your eye as possible*. The second point it that it doesn't magnify at all. The angular size of the image is exactly the same as the angular size of the object.

Before examining the magnifying glass, it is probably useful to understand just a little about the workings of the human eye. I am not a biologist, I am very squeamish about any discussion of eyes, so I'll keep this as basic as possible. When light enters the front surface or *cornea* of the eye, it is refracted in order to come to a focus on the back surface of the retina. The image on the *retina* is a real, inverted image, but the brain somehow corrects for that, so that objects look the right way up. While most of the refraction takes place at the cornea, some adjustment in the effective focal length is made possible by a flexible lens, whose power can be adjusted by means of *ciliary muscles*. The adjustment of this lens enables us to *accommodate* or bring to a focus objects that are at varying distances from us.

For an eye in good condition in a young person, the eye and the ciliary muscles are most relaxed when the eye is set to bring to a focus light from an infinitely-distant object – that is, when the eye is set to receive and bring to a focus light that is parallel before it enters the eye. In order to focus on a nearby object, the ciliary muscles have to make a bit of an effort to increase the power of the lens. They can increase the power of the lens only so far, however, and most people cannot focus on an object that is closer than a certain distance known as the *near point*. For young people the near point is usually taken to be 10 inches or 25 cm in calculations. The actual real point may differ from person to person; the figure of 25 cm is a "standard" near point. With older people, the near point recedes, so that 25 cm is too close for comfort, and the lens becomes less flexible.

When we use a magnifying glass properly (by holding it very close to the eye) we automatically place it so that the object we are looking at is at the focal point of the lens, and consequently parallel light emerges from the lens before it enters our eye. We don't think about this. It is just that the ciliary muscles of the eye are most relaxed when they are set to bring to bring parallel light to a focus. It is merely the most comfortable thing to do. Figure III.2 shows a magnifying glass at work. As usual, angles are small and the lens is thin.

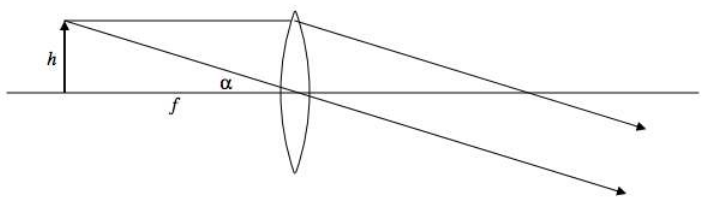


FIGURE III.2

The object is in the focal plane of the lens. I draw two rays from the tip of the object. One is parallel to the axis, and, after passing through the lens, it passes through the focus on the other side of the lens. The other goes through the center of the lens. (Since the lens is thin, this ray is not laterally displaced.) Parallel rays emerge from the lens. The eye is immediately to the right of the lens, and it easily brings the parallel rays to a focus on the retina.

Although the lens does not actually produce an image, it is sometimes said that the lens produces "a virtual image at infinity". The angular size of this virtual image is α, which is also the angular size of the object, namely $\alpha = h/f$. Thus the angular size of the image is the same as the angular size of the object, and the lens hasn't magnified at all!

However, if you put the object at a distance f (perhaps a few cm) from the eye without using the lens, you simply couldn't focus your eye on it. Without the lens, the closest that you can put the object to your eye would be D, the distance to the near point - 25 cm for a young eye. The angular size of the object would then be only h/D.

The angular magnification of a magnifying glass is therefore *defined* as

$$\frac{\text{angular size of the image (which is } h/f)}{\text{angular size of the object when the object is at the near point (which is } h/D)} \tag{3.2.1}$$

Hence the magnification is equal to D/f. The near point is taken to be 25 cm, so that a lens of focal length 2.5 cm has an angular magnification of 10.

If you bring the object just a little inside the focal plane, the light emerging on the other side will diverge, as it were from a virtual image that is no longer at infinity. (Figure III.3).

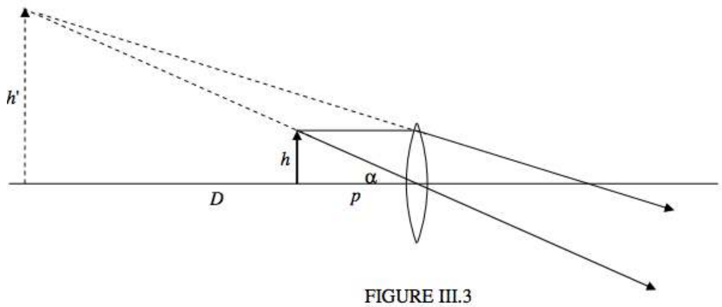


FIGURE III.3

There is no point, however, in bringing the image closer than the near point. If you bring it to the near point, what must the object distance p be? A simple lens calculation shows that $p = \frac{fD}{f+D}$. The angular size of the image is therefore $\frac{h(f+D)}{fD}$. Since the angular size of the object when the object is at the near point is h/D, the angular magnification is now $\frac{D}{f}$ + 1 <u>when the image is at the near point</u>. This, for our f = 2.5 cm lens, the angular magnification is then 11.

3.3: Spectacle Lenses

The less time I spend thinking about eyes the better. However, for a number of different reasons it may happen that, when parallel light enters the relaxed eye, it may be brought to a focus *before* the retina. In effect the lens, or the cornea, of the eye is too strong, or perhaps the eyeball is too deep. It is easy to see objects that are close up, but light from more distant objects is brought to a focus too soon. The eye is said to be myopic or *shortsighted* or *near-sighted*. All that is needed is a weak diverging lens in front of the eye.

Perhaps when parallel light enters the eye, it is brought to a focus *behind* the retina. Maybe the lens or the cornea is too weak, or the eyeball isn't deep enough. By contracting the ciliary muscles you can bring parallel light to a focus, and may even be able to focus on distant objects. But you just cannot focus on nearby objects. Your near point is much more distant than the standard 25 cm. In that case the eye is *hypermetropic*, or *long-sighted* or *far-sighted.* It is easily corrected with a weak converging lens in front of the eye. It is normal for the near point to recede with age, and weak convex glasses are required. Such glasses to not "magnify"; they merely enable you to focus on objects that are closer than your near point – just as a so-called "magnifying glass" does. If you are hypermetropic, looking at large print won't help! Large print won't come to a focus any more than small print will.

Other eye defects, such as astigmatism, aren't so easily corrected with a simple lens, and require specially shaped (and expensive!) lenses.

3.4: The Camera

The camera is a box with a lens in one side of it and a photographic film or a CCD on the opposite side. The distance between camera lens and film can be changed so as to focus on objects at various distances. The aperture can also be changed. In dim light you need to open the aperture up to let a lot of light in; but this makes the image less sharp, and you have a smaller depth of field.

The *aperture* of a lens is merely its diameter, and it is usually expressed as a fraction of the focal length. Thus an aperture of f/22 is a small aperture. You can use this only in strong light, but you will then have nice sharp images and a large depth of field. An aperture of f/6.3 is wide open; the cone of light inside the camera is quite steep, and focussing is then quite critical. You use such a wide aperture only if you are forced to by dim light. The apertures typically available on a camera are often in steps with a ratio of approximately $\sqrt{2}$ from one to the next. As you increase the aperture by a factor of $\sqrt{2}$, you get twice as much light on the film (because this depends on the area of the exposed lens), so presumably you can cut the exposure time by one half. This is probably true for a CCD camera; the degree of blackening of a photographic film is not quite proportional to the product of the illuminance and the time, but at least it serves as a rough guide.

How is the *depth of focus* related to the aperture? Let us suppose that we have a lens that is free of aberrations such as spherical aberration, and that a point object produces a point image in the focal plane. If your film or CCD is not exactly in the plane, it will be illuminated not by a point image but by a small circle of finite diameter. If this circle is smaller than the grain or pixel size, you may wish to regard it as not seriously out of focus. So the question is: How far can you move the film away from the focal plane in either direction without the image being seriously out of focus? This range is the *depth of focus.*

In Figure III.4 we see a cone of light converging from a lens of radius R to a focal point at distance f. Let us suppose that we place a film or CCD at the plane indicated by the dotted line at a distance x from the focal point, and that we are prepared to tolerate an out-of-focus "image" of radius r. From similar triangles we see that $x/r = f/R$. Or, if D is the diameter of the lens, and d is the diameter of the tolerable out-of-focus circle, $x/d = f/D$. Thus we can place the film at a distance fd/D on either side of the true focal plane without appreciable degradation of the image. For example, if the aperture is $D = f/6.3$, and you are prepared to tolerate an out-of-focus diameter d = 0.1 mm, the depth of focus will be ±6.3d or ±0.63 mm. On the other hand if you "stop down" to $D = f/22$, your depth of focus will be 2.2 mm. Note that we have not been considering here the effect of spherical aberration, but of course this, too, increases with aperture, as well as merely the "out-of-focus" effect.

Notice that the tangent of the semi angle of the converging cone is R/f, or $D/2f$. For apertures of $f/6.3$ and $f/22$, the semi angles are 4°.5 and 2°.6 respectively. This may give some comfort to those readers who have been uncomfortable with our assumption that angles are small. I have not been able to draw the lens and mirror drawings in these chapters with realistically small angles, because the drawings would be too cramped. I hope you will understand this shortcoming; you are welcome to try yourself!

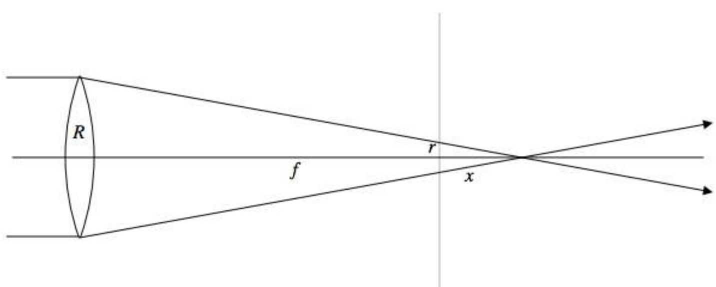


FIGURE III.4

Depth of focus is not the same thing as *depth of field*. Suppose we want to photograph an object at a distance p from the camera lens, and that we are prepared to tolerate an out-offocus "image" of diameter up to d, or radius r. Any object at a distance within the range $p \pm \Delta p$ may satisfy this, and we now want to find Δp. Figure III.5 shows, with full lines, light from an object at distance p coming to a focus at a distance q, and with dashed lines, light from an object at a distance Δp closer to the lens coming to a focus at a distance Δq further from the lens. The position of the film is indicated by the dotted line, and the radius of the out-of-focus dashed "image" is r.

We have

$$\frac{1}{q}=-\frac{1}{q}=-\frac{1}{p}+\frac{1}{f}, \tag{3.4.1}$$

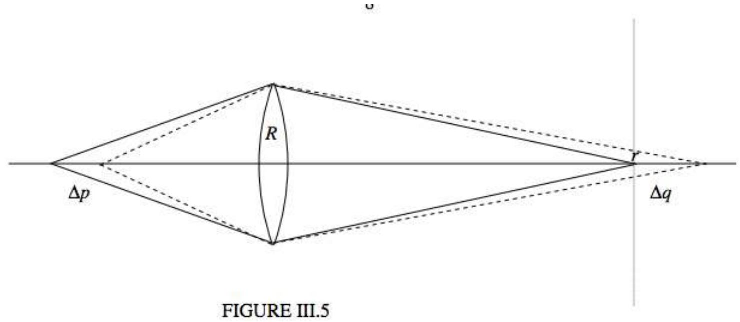


FIGURE III.5

So that

$$q=\frac{pf}{p-f} \tag{3.4.2}$$

and, without regard to sign

$$\Delta q=-\left(\frac{f}{p-f}\right)^2\Delta p. \tag{3.4.3}$$

From similar triangles we see that

$$\frac{R}{q+\Delta q}=\frac{r}{\Delta q}. \tag{3.4.4}$$

Elimination of q and Δq results in

$$\Delta p=\frac{pr(p-f)}{f(R-r)}, \tag{3.4.5}$$

or, in terms of diameters rather than radii,

$$\Delta p=\frac{pd(p-f)}{f(D-d)}. \tag{3.4.6}$$

For example, suppose the focal length is $f=25$ cm and you want to photograph an object at a distance of p = 400 cm. You are prepared to regard an out-of-focus "image" tolerable if its diameter is no larger than d = 0.1 mm. If the aperture is $D=f/6.3$, you can photograph objects in the range (400 ± 15) cm, whereas if you "stop down" to $D=f/22$, you can photograph objects in the range (400 ± 53) cm.

To the approximation that $d<<D$ and $f<<p$, Equation 3.4.6 becomes

$$\Delta p\approx\frac{p^2d}{fD}. \tag{3.4.7}$$

3.5: The Telescope

Our purpose here is not to describe at length of the details of modern telescope design, but just to give the basic principles of a simple telescope at a level needed to answer first-year examination questions and not necessarily to describe a telescope that one might actually be able to see anything through! An advanced astronomy student wanting details of real telescopes will have to search elsewhere. That said, the basic principles of a simple telescope still apply to real telescopes. Figure III.6, then, illustrates a telescope in its simplest form. Because of the difficulty of drawing diagrams with small angles, the telescope looks very stubby compared with a real one. To make a more realistic drawing, most of the angles should be less than about one degree.

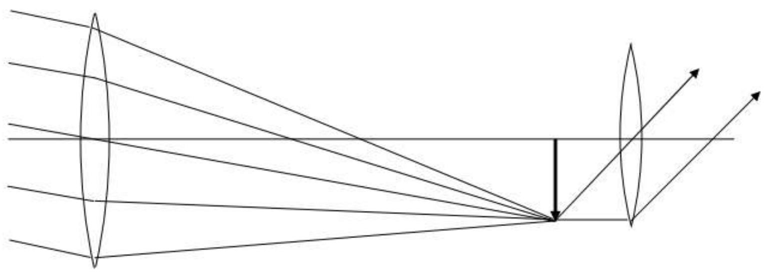


FIGURE III.6

We see at the left hand side of the figure a parallel beam of light coming in from a distant object off-axis. The first lens that it encounters is the *object glass*. Its function is to produce a real image in its focal plane, and the distance between the object glass and this primary image is f_1, the focal length of the object glass. In a real bird-watching telescope, the object glass in reality is a crown-flint achromatic doublet that brings all colours to almost the same focus. In a large astronomical telescope, instead of a lens, the primary image is formed by a large concave mirror that is often paraboloidal rather than spherical in shape.

If the telescope is an astronomical telescope intended for photography, that is all there is to it. There is no second lens. The primary image falls directly on to a photographic plate or film or CCD. Let is suppose that we are looking at the Moon, whose angular radius is about a quarter of a degree and whose actual linear radius is about 1740 km. The distance of the Moon is about 384,000 km. We'll suppose that the telescope is pointed straight at the centre of the Moon, and that the beam of light coming in from the left of figure III.6 is coming from the upper limb of the Moon. The image of the upper limb of the Moon is the tip of the thick arrow. The radius of the image of the Moon (i.e. the length of the thick arrow) is $f_1 \tan \frac{1}{4}^{\alpha}$. We'll suppose that we are using a fairly large telescope, with a focal length of ten metres. The radius of the primary image is then 4.4 cm, whereas the radius of the object (the Moon) is 1740 km. Thus the function of the object glass is to produce an image that is very, very, very much smaller that the object. The linear magnification is 4.4/174,000,000 or 2.5×10^{-8}. This is also equal to image distance divided by object distance, which is 10/384,000,000. And you thought that a telescope magnifies!

However, rather than using the telescope for photography, we want to "look through" the telescope. We don't want a photographic plate at the position of the real image. Instead, all we have to do is to look at the real image with a magnifying glass, and that is what the second lens in Figure III.6 is. This second lens, which is just a magnifying glass (which, we have seen in Section 3.3, doesn't magnify either!) is called the *eyepiece*. As is usual with a magnifying glass, the thing we are looking at (which is the primary image produced by the object glass, but which serves as an object for the eyepiece) is placed in the focal plane of the eyepiece, so that parallel light emerges from the eyepiece. As explained in Section 3.3 you don't have to think about this – your ciliary muscles are most relaxed when the eye is ready to receive parallel light. The eyepiece of a telescope can usually be moved in and out until the image appears sharp to your relaxed eye. Thus the primary image is in the focal plane of the object glass and also of the eyepiece, and the distance between object glass and eyepiece is $f_1 + f_2$, where f_1 and f_2 are the focal lengths of object glass and eyepiece respectively. I have drawn the usual two rays from the primary image (which is the object for the eyepiece), namely one that goes straight through the centre of the lens, and one parallel to the axis, which subsequently passes through the focal point of the eyepiece.

Figure III.7 is Figure III.6 redrawn with all but two rays removed, namely the ray that passes through the centre of the object glass and the ray that passes through the centre of the eyepiece.

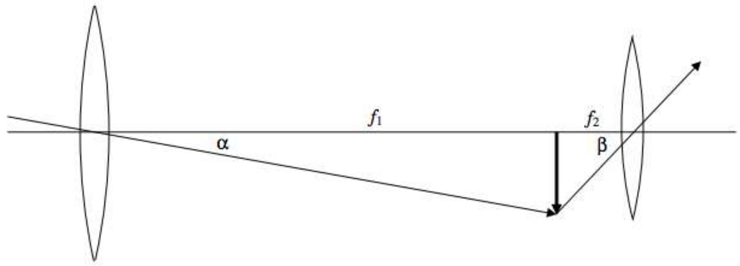


FIGURE III.7

Although, as we have seen, the linear size of the primary image is very much smaller than (i.e. centimetres rather than thousands of kilometres!) the object, what counts when we are looking through a telescope is the *angular magnification*, which is the ratio of the angular size of the image to the angular size of the object – that is the ratio β/α. And since, as usual, we are dealing with small angles (the angular diameter of the Moon is only about half a degree) – even though it is difficult to draw a realistic diagram with such small angles − this ratio is just equal to f_1/f_2. Note that the *definition* of the angular magnification is the ratio of the angular size of the image to the angular size of the object (and this time we don't add "when the object is at the near point"!), while f_1/f_2 is how we can calculate it. Thus, if you are asked what is meant by the angular magnification of a telescope, and you say "f_1/f_2" you will get nought out of ten – and deservedly so.

In any case, for large magnification, you need an object glass of long focal length and an eyepiece of short focal length. Generally you have a choice of several eyepieces to choose from.

It should be pointed out that magnification is not the most important attribute of a large astronomical telescope. Large astronomical telescopes have large primary mirrors mainly to collect as much light as possible.

Exercise $3.5.1$

A telescope is used with an eyepiece that magnifies 8 times. The angular magnification of the telescope when used with this eyepiece is 200. What is the distance between object glass and eyepiece?

Answer

628.125 cm.

One thing is odd about the "telescope" described so far – the image is upside down! In fact for astronomical purposes this doesn't matter at all, and there is nothing "wrong". For a telescopes designed for terrestrial use, however, such as for bird-watching, we want the image to be the right way up. In older telescopes this was done with two additional lenses; in modern telescopes the image is reversed with additional prisms.

The astute reader may notice that there is something else wrong with Figures III.6 and 7. The object glass produces a real primary image, and then we examine that real primary image with a magnifying glass. But look at the ray that goes from the tip of the primary image through the centre of the eyepiece. Where did it come from? It doesn't seem ever to have passed through the object glass! Part of the answer to this is that angles in the drawings are grossly exaggerated (it is too difficult to draw diagrams with realistically small angles), and that if the angles were correctly drawn, this rogue ray would indeed be seen to have passed through the object glass. But this is only part of the answer, and a telescope with just the two lenses shown would have a very small field of view.

In practice an *eyepiece* consists of two lenses separated by a short distance. These two lenses are called the *field lens* and the *eye lens*. In one arrangement the field lens coincides with the primary image – i.e. the primary image formed by the object glass falls exactly on the field lens. The field lens does not affect the magnification at all; it merely serves to bend some of the light from the object glass into the eye lens. The rogue ray to which we have called attention has been bent towards the eye lens by the field lens. This arrangement would work, although one problem that would arise is that bits of dust on the surface of the field lens would be in sharp focus when viewed with the eye lens. Thus the field lens is often arranged so as not to coincide exactly with the primary image. It can also be shown (Section 2.12) that if the separation of the field and eye lenses is equal to half the sum of their focal lengths, the eyepiece is free of chromatic aberration. Eyepiece design could easily occupy an entire chapter, and it is not uncommon for a good eyepiece to have six or more components; we just mention this particular problem to illustrate some of the points to be considered in optical design.

Let us return to our simple telescope of just two lenses. Let us look at things from the point of view of the eyepiece (which, in our simple telescope, consists of just the eye lens). If we now regard the object glass as an *object*, we can understand that the eyepiece will produce a real image of this "object". See Figure III.8.

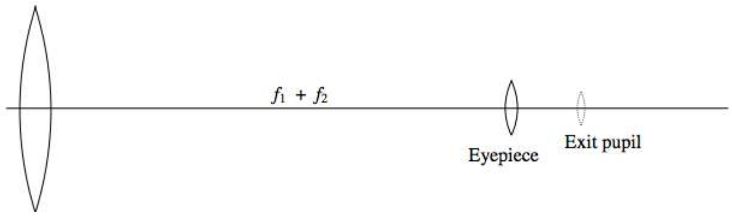


FIGURE III.8

The real image of the object glass produced by the eyepiece is called the *exit pupil* of the telescope, and the object glass is the *entrance pupil* of the telescope. All light that passes through the entrance pupil also passes through the exit pupil. You can easily see the exit pupil a few millimetres from the eyepiece if you hold a pair of binoculars in front of you at arm's length. The notation such as "10 % 50", which you see on a pair of binoculars means that the angular magnification is 10 and the diameter of the object glass is 50 mm. If you look at the exit pupils of a pair of binoculars that you are considering buying, make sure that they are *circular* and not square. If they are square, some of the light that passed through the entrance pupil is being obstructed, probably by inadequate prisms inside the binoculars, and you are not getting your full 50 millimetres' worth. The size of the exit pupil should be approximately equal to the size of the entrance pupil of your eye. This is about 4mm in sunlight and about 7 mm at night – so you have to consider whether you are going to be using the binoculars mainly for birdwatching or mainly for stargazing.

Just where is the exit pupil, and how big is it? "Where?" is just as important a question as "how big?" – the distance between the eyepiece and the exit pupil is the *eye relief*. You want this distance to be small if you do not wear glasses. If you are merely myopic or hypermetropic, there is no need for you to wear your glasses when using binoculars or a telescope – you can merely adjust the focus of the telescope. If you wear glasses to correct for astigmatism, however, you will still need your glasses when using the binoculars or telescope, so you will need a larger eye relief.

To find the eye relief, or distance of the exit pupil from the eye lens, recall that the distance between object glass and eyepiece is $f_1 + f_2$, and the focal length of the eyepiece is f_2. The eye relief is therefore given by

$$\frac{1}{q} = -\frac{1}{f_1 + f_2} + \frac{1}{f_2} \tag{3.5.1}$$

or

$$q = \frac{f_2(f_1 + f_2)}{f_1}. \tag{3.5.2}$$

The ratio of the size of the entrance pupil to the size of the exit pupil is equal to the ratio of their distances from the eyepiece. This is just f_1/f_2, which is the angular magnification of the telescope. Thus the diameter of the exit pupil of a pair of 10×50 binoculars is 5 mm – just divide the diameter of the object glass by the magnification.

3.6: The Microscope

The front lens of a microscope is generally called the **"objective" lens**, rather than the "object glass". In contrast to the telescope, the objective is a small lens with a short focal length. The object is placed *just outside* the focal point of the objective, and a magnified real inverted primary image is formed quite some distance away. This is examined with an eyepiece in the same way that the primary image formed in a telescope is examined with an eyepiece. As discussed for the telescope eyepiece, the eyepiece in reality has a second lens (the "field lens"), which I have not drawn, which almost (but not quite) coincides with the primary in order to bend that vexing ray towards the centre of the eye lens. The primary image is in the focal plane of the eyepiece, but (unlike for the telescope) it is not in the focal plane of the objective,

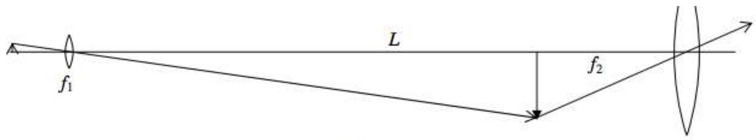


FIGURE III.9

Everyone knows how to calculate the angular magnification produced by a magnifying glass (D/f) and by a telescope (f_1/f_2). A microscope isn't quite so easy, which is why, in an exam, you will be asked for the magnification of a microscope rather than of a magnifying glass or a telescope. When you are focussing a telescope, you pull the eyepiece in and out until the image appears in focus for your relaxed eye. When you are focussing a microscope, however, rather than moving just the eyepiece, you move the whole microscope tube up and down, in such a manner that the distance L between the two lenses is constant. What we need, then, is to find the magnification in terms of the two focal lengths and the distance L between the lenses.

Recall the way a microscope works. First, the objective produces a magnified real image of the object. Then you look at this primary image with an eyepiece. The overall magnification, then, is the *product* of the *linear magnification produced by the objective* and the angular magnification produced by the eyepiece. We shall address ourselves to these two in turn.

To find the linear magnification produced by the objective, we need to know the object and image distances. The image distance is just $L - f_2$, and, since the focal length of the objective is f_1, it doesn't take us a moment to find that the object distance is

$$\frac{f_1(L-f_2)}{L-f_1-f_2}. \tag{3.6.1}$$

Therefore the linear magnification produced by the objective is

$$\frac{L-f_1-f_2}{f_1}. \tag{3.6.2}$$

And the angular magnification produced by the eyepiece is just D/f_2, where D is the distance to the near point (25 cm). Thus the overall angular magnification is

$$\frac{L-f_1-f_2}{f_1} \times \frac{D}{f_2}. \tag{3.6.3}$$

Voilà! It's easy!

CHAPTER OVERVIEW

4: Optical Aberrations

Thumbnail: Coma. (CC BY-SA 3.0; HHahn).

4.1: Introduction to Optical Aberrations

We have hitherto made the assumption that a lens or a curved mirror is able to form a point image of a point object. This may be approximately true if the depth of the mirror or the thickness of the lens is small compared with other distances, and if the angle that all rays make with axis of the mirror or lens is small, and if we are using monochromatic light. Usually none of these conditions is satisfied exactly, and consequently the image formed by a lens or curved mirror suffers from several aberrations.

There are five *geometrical* aberrations, given the names

- Spherical aberration
- Astigmatism
- Coma
- Curvature of field
- Distortion (pincushion or barrel distortion).

In addition, unless we are using monochromatic light, lenses (but not mirrors) exhibit *chromatic aberration* (longitudinal and transverse).

It may be possible to minimize some of these aberrations by careful choice of the radii of curvature of a lens system ("bending the lens"), although the condition for minimizing one aberration may be different from minimizing another. Consequently some sort of compromise must be reached, which may depend on which aberrations are important, and which are not so important, for a particular application.

4.2: Spherical Aberration

We'll begin by looking at the spherical aberration resulting from reflection from a spherical mirror. We have hitherto assumed that a parallel beam of light, after reflection from a spherical mirror, comes to a focus at a point, and that the distance of the focal point from the surface of the mirror is half the radius of curvature of the mirror, as in Figure IV.1:

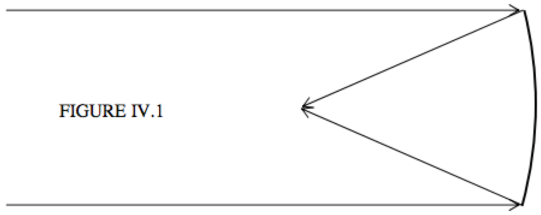
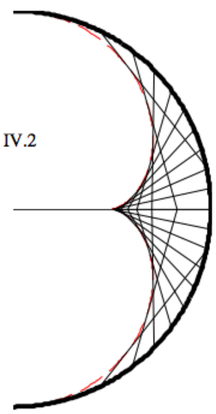


FIGURE IV.1

This is approximately true for a small aperture mirror ("aperture" meaning the ratio of the diameter to the focal length). This is not the case, however, for a large aperture mirror. In Figure IV.2 I have drawn a hemispherical mirror. I assume that there is an incident beam of light (not drawn) coming in horizontally from the left, and I have drawn the rays after reflection from the mirror. (Some of the rays will be reflected a second time from the surface before eventually escaping, but I have not drawn the rays after a second reflection because they would only clutter up the diagram and are not pertinent in describing what I want to describe. You can see that the reflected rays are bounded by an envelope known as a caustic curve, shown as a dashed red curve in Figure IV.2.

FIGURE IV.2



Can we find the equation to this caustic curve?

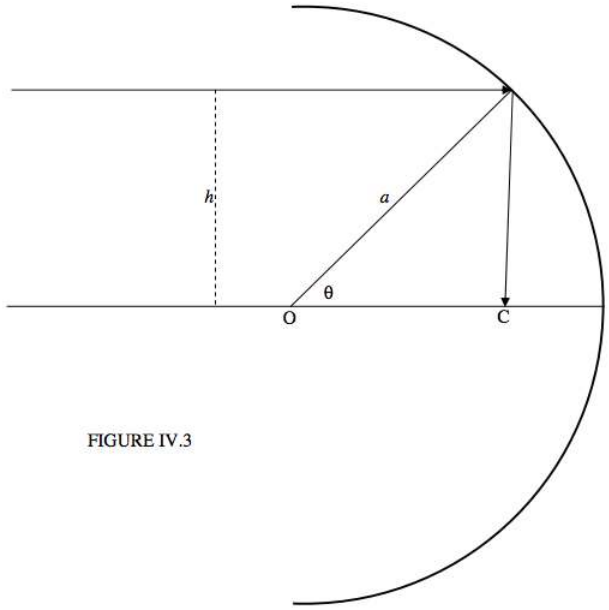


FIGURE IV.3

We'll take the centre of curvature of the mirror as origin O of coordinates, and suppose that the radius of curvature of the mirror is a. Let us consider the adventures of a ray of light coming in parallel to the horizontal (x) axis and at a height h from it. The equation to the incoming light ray is just $y=h$, and the equation to the mirror surface is $x^2+y^2=a^2$. A little bit of coordinate geometry will enable us to determine that the equation to the reflected ray is

$$y=\frac{h}{a^2-2h^2}\left[\left(2\sqrt{a^2-h^2}\,x-a^2\right)\right], \tag{4.2.1}$$

and that it crosses the x-axis at a point C such that

$$OC=\frac{a^2}{2\sqrt{a^2-h^2}}. \tag{4.2.2}$$

It is also convenient to write these formulas in terms of the angle θ, which is given by $h=a\sin\theta$. After a little algebra and application of some trigonometric identities, we obtain

$$y=x\tan 2\theta-\frac{a\sin\theta}{\cos 2\theta} \tag{4.2.3}$$

for the equation to the reflected ray, and

$$OC=\frac{1}{2}a\sec\theta. \tag{4.2.4}$$

We can write Equation 4.2.3 as

$$f(x,y;\theta)=x\tan 2\theta-\frac{a\sin\theta}{\cos 2\theta}-y=0. \tag{4.2.5}$$

From our long-forgotten, yellowed and mildewy mathematics notes, we recall that to find the equation to the envelope of a family of curves of the form $f(x,y;\theta)=0$, we have to eliminate the parameter θ from that equation and the equation $\frac{\partial f}{\partial\theta}=0$. After some more algebra and more application of trigonometric identities, we find that the latter equation comes to

$$x=a\cos\theta.\left(\frac{3}{2}-\cos^2\theta\right). \tag{4.2.6}$$

So, all we have to do is to eliminate the parameter θ from Equations 4.2.3 and 4.2.6, and this would give us the x,y equation to the caustic curve. These two equations are, in fact, the parametric equations to the caustic curve. Now I don't know how easy it would be to eliminate θ. Since Equation 4.2.6 is a cubic equation in $\cos\theta$, I suspect that it might not be particularly easy. But (as is often the case with two parametric equations to a curve) we can happily plot the curve numerically, without having to eliminate the parameter algebraically. Thus, in order to plot the red curve in Figure IV.2, I varied θ from −90° to +90°, and calculated x from Equation 4.2.6, and I then calculated y from Equation 4.2.3.

To avoid spherical aberration, telescope mirrors can be made in a paraboloidal shape. It can be shown that an incident beam of light, coming in parallel to the axis of a paraboloidal mirror, after reflection will come to single focal point, namely at the focus of the parabola. A proof of this is given in Section 2.4 of Chapter 2 of my Celestial Mechanics notes and is not repeated there. In that Chapter, it is also shown that, if a bucket of liquid is rotated about a vertical axis, the surface of the liquid will take up a paraboloidal shape, and mention is made there of two applications to the manufacture of paraboloidal mirrors. In one, a vat of molten glass is rotated, and is gradually cooled down until the glass solidifies into a paraboloidal shape. In the other, a container of mercury is rotated, the surface of the mercury taking up a paraboloidal shape, and this liquid paraboloid is then used as the main mirror of a reflecting telescope. While it can observe only close to the zenith, some excellent results have been obtained. I shan't repeat it here, but you might want to refer to the above-mentioned notes, since it is pertinent here.

This property (of light being reflected from the surface of a parabola to a single focal point) applies only to light coming in parallel to the axis of the paraboloid. Consequently paraboloidal telescope mirrors have only a rather narrow field of view. A *Schmidt* telescope uses a spherical mirror (hence a large field of view) and, to avoid spherical aberration, a corrector plate is mounted in front of the mirror. Typically the spherical mirror is at the "bottom end" of the telescope tube, and the corrector plate is at the "top end". The corrector plate causes light that is coming in parallel to the telescope tube, but some distance from the axis of the tube, to diverge slightly from the axis before reaching the spherical mirror. In this manner all of the incoming light, after reflection from the mirror, comes to a focus at a single point.

A lens also suffers from spherical aberration, of course, but it does not lend itself to such simple analysis as for a spherical mirror. One needs to perform detailed numerical ray-tracing to find the exact shape of the caustic curve for a lens. We showed, however, in Section 1.4 of Chapter 1, that refraction even at a plane surface produces spherical aberration.

One might wonder, given that a paraboloidal mirror when used on axis is free of spherical aberration, whether a lens made with paraboloidal surfaces, is also free of spherical aberration. Alas, that is not so.

One can, however, design a lens with spherical surfaces that minimize the spherical aberration, by suitable choice of the radii or curvature of the lens surfaces. This is called "bending the lens".

For example, Figure IV.4 shows five lenses, in which I have written, beside each surface, its radius of curvature in cm. In what follows I assume that the lens is "thin" in the sense that its thickness is very small compared with any other distances under discussion. If the refractive index is 1.6, each of these lenses has a focal length of 20 cm.

You can characterize the shape of a lens by means of its *shape factor.*

$$q = \frac{r_1 + r_2}{r_1 - r_2} \tag{4.2.7}$$

In Figure IV.4 I have written the shape factor above each lens.

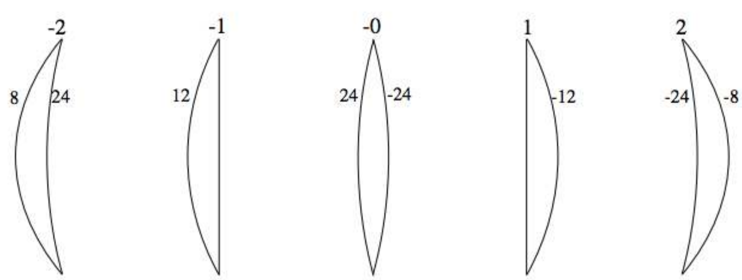


FIGURE IV.4

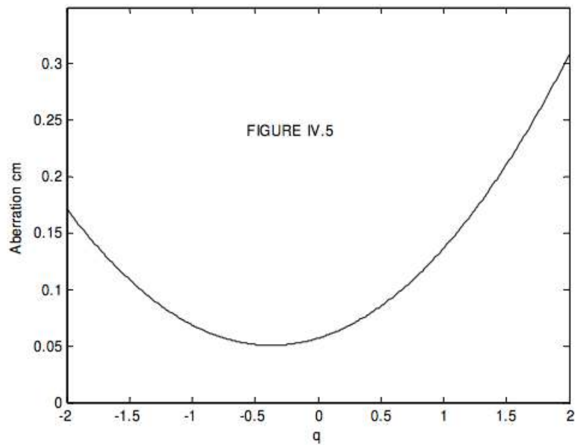
For light coming in horizontally near the axis, the focal length of each of these lenses is 20 cm. However, light coming in horizontally at some distance from the axis, after passage through the lens, falls a little short of 20 cm. We may characterize the spherical aberration by the amount it falls short. Assuming that the lenses are thin (compared with any other distances under consideration) I calculated the shortfall for a ray of light coming in from the left at a height of 1 cm from the axis. This is shown in

Figure IV.5, in which I have drawn the shortfall (labelled "Aberration" in the figure) versus shape factor q. It is seen that the aberration is least for a shape factor of about $q = -0.38$. The radii of curvatures of the lens must satisfy equation 4.2.7 as well as $q = -0.38$

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \tag{4.2.8}$$

so that, for f = 20 cm and q = −0.38, the radii of curvature for least spherical aberration should be r_1 = 17.4 cm and r_2 = −38.7 cm.

Of course, you have to use the lens the right way round! If you turn it round, or if light is coming in from the right, the shape factor is +0.38, and the spherical aberration is not at a minimum. Mind you, the minimum is fairly shallow, so you can vary the shape factor a fair amount without grossly increasing the spherical aberration.



4.3: Astigmatism

In Greek, *stigma* means a mark - in particular the mark made by the prick of a pointed instrument. An ideal optical instrument produces an image of a point source, which is also a point. If the image is not a point, then it is astigmatic. However, the use of the word astigmatic to describe an image of a point source that is not also a point is restricted to the kind of optical aberration described in this section.

The easiest way to understand the phenomenon of astigmatism is to imagine a lens (or mirror) whose surfaces are not exactly spherical but for which the radius of curvature (and hence focal length) in one plane is different from the radius of curvature in a plane at right angles to the first.

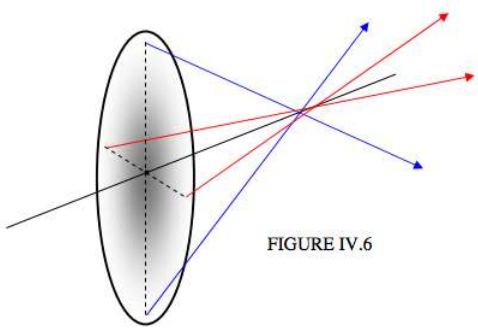


FIGURE IV.6

This is going to stretch my poor artistic abilities to the limit. The ellipse is intended to represent a lens seen somewhat from behind, at an angle. The black line is its optical axis. The lens is supposed to be illuminated from the left with a beam of light parallel to the optical axis. I have drawn two transmitted rays in the vertical plane by means of two blue arrows converging on to the optical axis at some distance from the lens. I have drawn two transmitted rays in the horizontal plane by means of two red arrows converging on to the optical axis at a slightly greater distance from the lens. (The colours of the arrows are not intended to means different colours of light. We'll suppose that the light is all monochromatic.) Evidently the rays from different points around the circumference of the lens come to a confused mess on the optical axis; the image is decidedly astigmatic. In fact at one point on the optical axis, a line image is formed; a little further along the axis, another line image, at right angles to the first, is formed. In Figure IV.7, I repeat Figure IV.6, but I add these two line images.

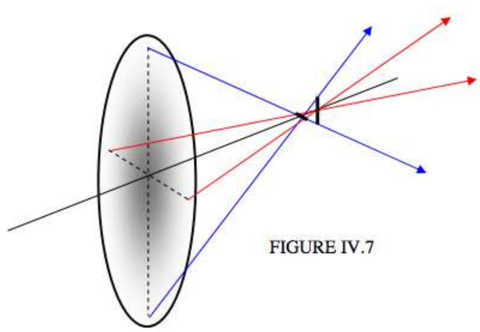


FIGURE IV.7

As you move along the optical axis, the image changes from a horizontal line to a vertical line in a sequence that looks something like this:

Somewhere about half way between the two linear images is the "circle of least confusion".

I have explained the aberration of astigmatism by supposing that the lens has a different focal length in one plane than in the other. This may be the easiest way for an introductory explanation of the aberration. However, in practice it is unlikely that a lens has different focal lengths in two orthogonal planes; indeed it would be quite difficult to make such a lens.

In most cases astigmatism is caused, as we shall see, by using a perfectly good lens or mirror *off-axis.*

If you look at a star through a telescope, and if you move the eyepiece in and out as you look through it, you may see the star image going through a series of astigmatic images such as illustrated above. This is not usually caused by a bad lens, but is caused if the object glass (in a refracting telescope) or lens (in a reflecting telescope) is crooked in its cell, so that you are using it off-axis. Indeed, doing this little test is a good way of telling whether the object glass or the mirror is crooked in its cell.

Although different radii of curvature in different planes is not the usual cause of astigmatism, there is an exception - namely, the human eye. If the radii of curvature of the cornea, or of the lens, is different in different planes, then the image on the retina will be astigmatic even on-axis.

We saw in Chapter 1 that refraction at a plane surface produces *spherical aberration*. It is not always appreciated that refraction at a plane surface produces *astigmatism* when the surface is viewed at an angle. If you visit an aquarium and look into glass side of a tank at an angle, you will see that the fish look a little blurred because of this astigmatism.

In Figure IV.8 I have drawn two rays from a point O at the bottom of a glass block, making angles of 20° and 30° with the normal to the upper surface. With a refractive index of 1.6 the angles that the emerging rays make with the normal are 33° and 53°. I refer to the plane of the paper (or your computer screen) as the *tangential plane.* A vertical plane perpendicular to the plane of the paper is the *sagittal plane*. You will see that the two rays in the tangential plane diverge, after refraction, from a point T in the tangential plane. If we take the height of the glass block to be 1, we can calculate that the (x, y) coordinates of the point T in the tangential plane are (0.145, 0.666).

To anticipate, the image at T is not a point; rather, it is a short horizontal line in the sagittal plane, perpendicular to the plane of the paper.

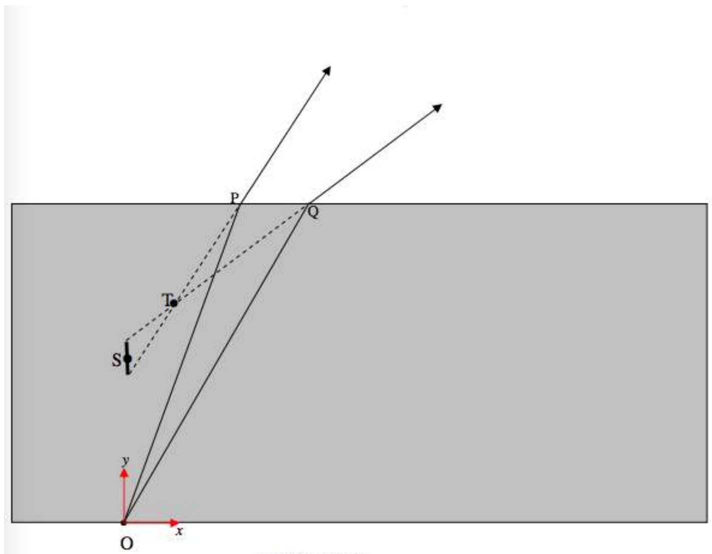


FIGURE IV.8

Now let’s look at the glass block from above:

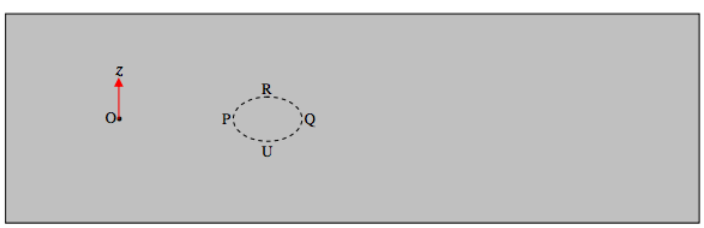


FIGURE IV.9

I have drawn (accurately, I hope, after some calculation) the ellipse where the cone of light coming from O intersects the upper surface of the block. The point P and Q are in the tangential plane, and light emerging from P and Q appears to diverge from T. The points R and U are in the sagittal plane. Tracing the rays OR and OU after emergence from the block doesn’t look very easy, but it will probably be agreed that they do not diverge from T, as the rays in the tangential plane did. Indeed, after further thought, you’ll probably see that the rays OR and OU after emergence will be diverging from a point S on the y-axis; that is to say, directly above O.

For reference, the coordinates of the several points on the drawing, if my calculations are correct, are:

O: (0.000 000 000 , 0.000 000 000 , 0.000 000 000)

P: (0.363 970 234 , 1.000 000 000 , 0.000 000 000)

Q: (0.577 350 269 , 1.000 000 000 , 0.000 000 000)

R: (0.470 660 252 , 1.000 000 000 , 0.069 344 256)

U: (0.470 660 252 , 1.000 000 000 , −0.069 344 256)

T: (0.145 831 216 , 0.666 360 298 , 0.000 000 000)

S: (0.000 000 000 , 0.497 301 940 , 0.000 000 000)

The angle of incidence of the ray at R is 25.442 358 40 degrees to the normal, and the angle of refraction is 43.421 850 83 degrees.

The net result of this is that there is a short linear "image" at T perpendicular to the tangential plane, and a short linear "image" at S perpendicular to the sagittal plane, and, somewhere in between, there is a circle of least confusion. One way of looking at the situation is to recognize that the wavefront of the emergent cone is nonspherical - its radii of curvature are different in the tangential and sagittal planes.

Thus refraction at a plane surface results in both spherical aberration and astigmatism. Refraction through a glass prism, as in a prism spectrograph, also produces astigmatism, and it can be shown that the astigmatism is least when the light passes through the prism symmetrically in the position of minimum deviation. This is one reason why prism spectrographs are normally used in the position of minimum deviation.

We have seen that a lens does not produce a point image of a point object on the axis of the lens, but the image is subject to spherical aberration. The spherical aberration is small if the aperture of the lens is small compared with its focal length and object and image distances, so that the angles that the various rays make with the optic axis are small enough that one can make the approximation $\sin\theta \approx \tan\theta \approx \theta$, and is small also if the shape of the lens is suitably designed as in the example in Section 4.2. For a point object on the axis, the image is free of astigmatism (presuming that the radii of curvature of the lens in the tangential and sagittal planes are equal). However, for a point object *off*-axis, in which the light passes through the lens at an oblique angle, the refracted cone gives rise to an astigmatic image in just the same way as for oblique refraction at a plane surface. This there will be a line "image" normal to the tangential plane, and, at a different distance, there will be another line normal to the sagittal plane , and a circle of least confusion between them. The further off-axis the object, the greater will be the distance between the tangential and sagittal lines. (The distance will be zero for a point object on axis.) Unlike the case for spherical aberration, the amount of astigmatism (the distance between T and S) is not greatly improved by changing the shape of the lens, and a third lens component is often used to correct for the astigmatism.

We mentioned, however, that astigmatism in the eye is generally caused by different tangential and sagittal curvatures of the cornea, and it is evident on axis as well as off axis. It may be corrected by a single lens, which is designed to have different tangential and sagittal curvatures. Such lenses are not easy to make, and they are generally fairly expensive

4.4: Coma

Coma, like astigmatism, is another aberration that appears off axis, near the edge of an image field. If you look at a wide-field photograph of some stars taken with a photographic telescope, the stars near the centre of the field should be points, but, at the very edge of the photograph, if the telescope is less than perfect, the stars may appear like little comets, with a sharp nucleus, but each with a fuzzy tail directed away from the centre of the photograph. This aberration is called “coma”. The word “coma”, as well as the word “comet”, comes from the Latin *coma*, meaning “hair”, from a fanciful resemblance of a comet, or of a comatic image of a star, to the head of a girl with her long hair streaming out behind her.

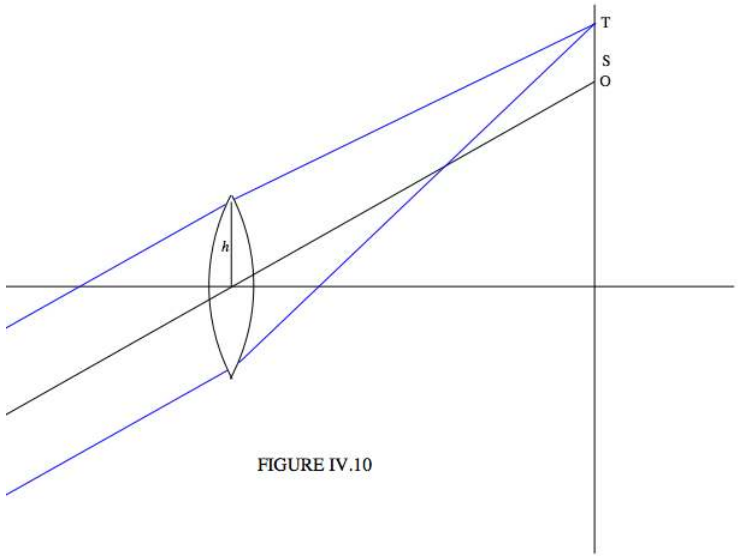


FIGURE IV.10

In Figure IV.10, we see a parallel bunch of rays entering the lens obliquely from the left. The central ray, in black, goes straight through to a point O. Two rays in the tangential plane (i.e. the plane of the computer screen, or the paper, if you have printed it out) converge not to the point O, however, but to a point T as shown. If I could draw two rays equally far from the centre of the lens but in the sagittal plane (i.e. a vertical plane perpendicular to the plane of the paper), they would converge to a point S, about a third of the way between O and T.

If I could draw the rays entering the lens all around the zone of radius h on the lens, each pair of opposite rays would converge to a point on the *comatic circle*. See Figure IV.11.

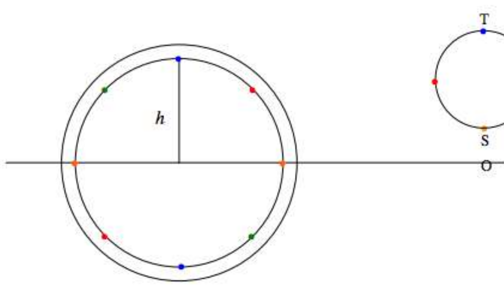
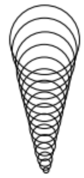


FIGURE IV.11

The radius and height of the comatic circle is different for each zone on the lens that produces it, with the result that the “image” appears as a superposition of all the comatic circles produced by all the zones on the lens, something like the drawing below. That, at least is a qualitative description of the phenomenon.



To go further is a bit of a specialist skill, so I'll leave it here. Suffice to say that the degree of coma and the degree of spherical aberration depend on the shape factor of the lens, and fortunately the shape that gives least spherical aberration is not very different from the shape that gives least coma.

The aberrations discussed so far are aberrations that result when the lens or mirror does not produce a point image of a point object. If, somehow, we manage to get rid of spherical aberration, astigmatism and coma, then a point object will result in a point image. But will that image be in the right place? There are two further aberrations that are concerned with where the image is formed. These aberrations are *curvature of field* and *distortion*.

4.5: Curvature of Field

Suppose we have a lens that we have managed to correct for (or at least to minimize) spherical aberration, astigmatism and coma, say by a combination of choosing the right shape of the lens and not going too far off-axis. (I.e. we might close the lens to an aperture of $f/11$ rather than opening it up to $f/5.6$.) Nothing that we know about refraction and lenses and mirrors tells us that that light coming in at different angles to the axis forms point images conveniently situated in a plane, as illustrated hopefully in Figure IV.13.

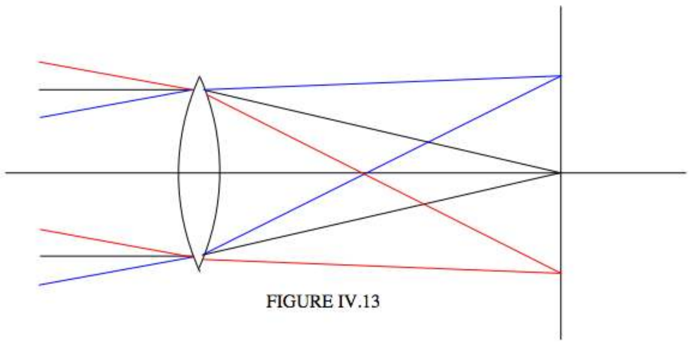


FIGURE IV.13

Alas, life is not as simple as that, and light doesn't generally come to a focus in a focal plane, but rather in a curved focal surface (sometimes called the *Petzval surface*), as in Figure IV.14.

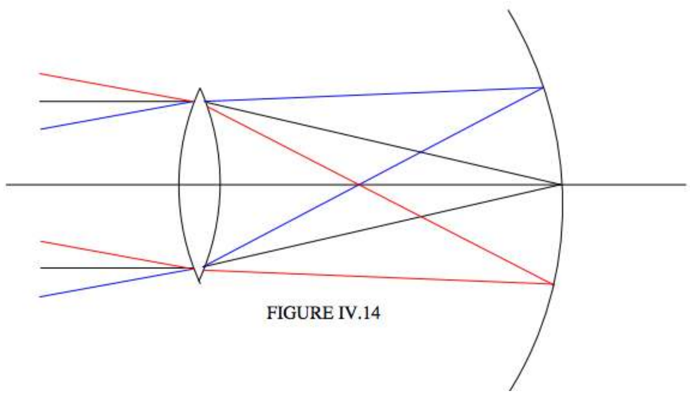


FIGURE IV.14

This doesn't matter a great deal in a telescope designed merely for looking through, since the eye can rapidly accommodate for slightly different image distance, but it obviously matters in a photographic telescope. One effective way of dealing with this problem, particularly if your detector is a flexible film, is to shape the filmholder so that the film fits along the Petzval surface. This is often done, for example, with Schmidt astronomical telescopes.

In designing a lens or lens system, the problems of astigmatism and curvature of field are often closely related. For example a meniscus lens tends to suffer from astigmatism, and there is a focal surface for the tangential image, and a focal surface for the sagittal image, and the tangential and sagittal surfaces curve in opposite senses. With luck, or more likely with some careful design, the surface (C) for the loci of the circles of least confusion is between the tangential (T) and sagittal (S) surfaces and is approximately planar (Figure IV.15).

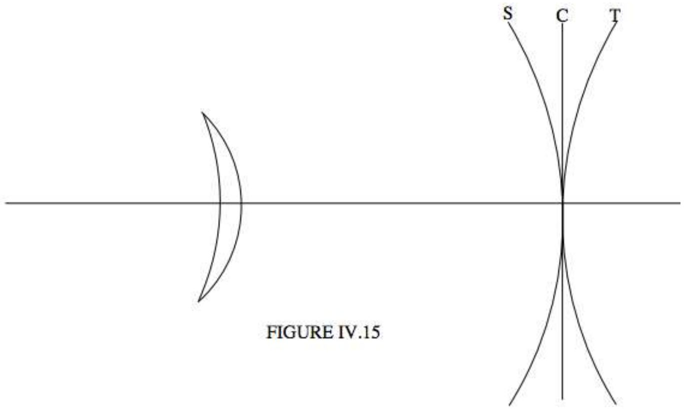


FIGURE IV.15

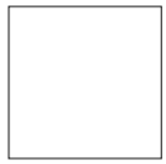
It has been shown that, if you have a doublet lens, made of two lenses, one a converging lens of focal length f_1 and refractive index n_1, and the other a diverging lens of focal length f_2 and refractive index n_2, curvature of field will be least if $\frac{1}{n_1 f_2} + \frac{1}{n_2 f_2} = 0$. For example if you have two glasses, of refractive indices n_1 = 1.51 and the other of refractive index n_2 = 1.67, and you want to make a doublet lens of focal length 100 cm, what should be the focal lengths of the two components of the doublet if you want to minimize curvature of field?

Answer: The lenses need to satisfy $\frac{1}{n_1 f_1} + \frac{1}{n_2 f_2} = 0$ and $\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f}$. It's probably easier to work in terms of powers rather than focal lengths, so we have to solve $1.67P_1 + 1.51P_2 = 0$ and $P_1 + P_2 = 0.01$. This gives $P_1 = -0.094375$ cm^{-1} and $P_2 = +0.104375\,\text{cm}^{-1}$, or f_1 = 10.60 cm and f_2 = 9.58 cm. You will then have to design the lenses so that the faces of the two lenses that are in contact have the same radius of curvature, and we leave that to the reader.

For a similar problem concerning a doublet with minimum chromatic aberration, see Chapter 2, Section 2.10.

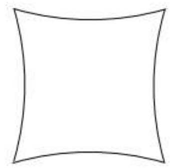
4.6: Distortion

Let us suppose that, by dint of great labour and overcoming many obstacles, we have finally designed a lens system that is free from chromatic aberration, spherical aberration, astigmatism, coma and curvature of field, or at least have minimized these aberrations or have come to a tolerable compromise for a particular purpose, can we at last relax? Unfortunately, no, we cannot. The magnification of an image is image distance divided by object distance, and image distance is different off-axis than on-axis, so the image magnification varies with distance from the axis. This means that the image of an object like this:



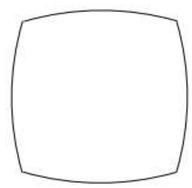
may look like this:

(pincushion distortion)

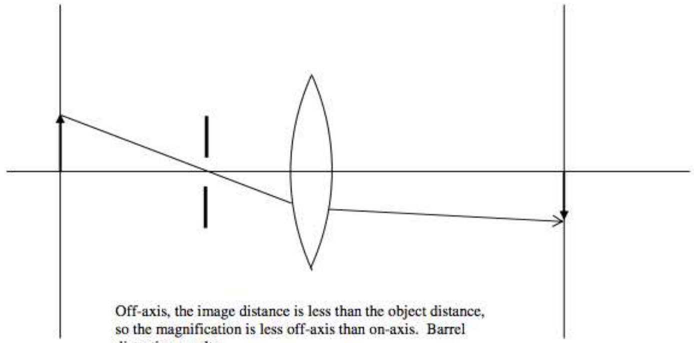


or like this:

(barrel distortion)



If the distortion is quite small, it may not be noticed in ordinary pictorial photography, but if one is using a photograph for precise positional measurements (for example, in astrometry) it is necessary to correct for the distortion. Often barrel distortion is introduced into a lens system if a stop is placed in front of a lens, while pincushion distortion results if a stop is placed behind a lens. The drawing below, in which I have exaggerated the situation by drawing a very small stop, may explain the reason why. I have placed the object at twice the focal distance from the lens, so that, on axis, the image and object distances are equal, and the magnification is unity. A symmetric air-spaced doublet with a stop half way between the two components minimises distortion.


Off-axis, the image distance is less than the object distance, so the magnification is less off-axis than on-axis. Barrel distortion results.

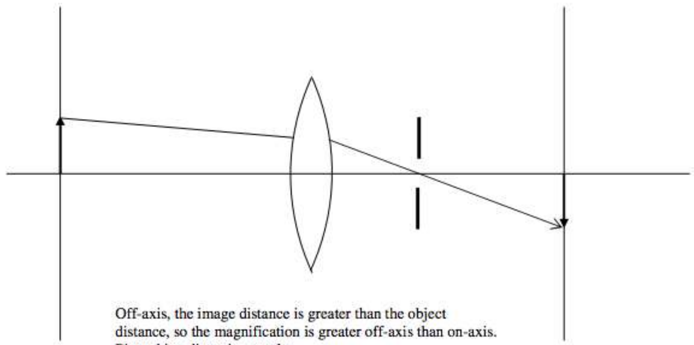

Off-axis, the image distance is greater than the object distance, so the magnification is greater off-axis than on-axis. Pincushion distortion results.

FIGURE IV.16

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