

13.1: Coefficient of Expansion

Notation: In an ideal world, I'd use α , β , γ respectively for the coefficients of linear, area and volume expansion. Unfortunately we need γ for the ratio of heat capacities. Many people use β for volume expansion, so I'll follow that. What, then, to use for area expansion? I'll use b , so we now have α , b , β , which is very clumsy. However, we shall rarely need b , so maybe we can survive.

Coefficient of linear expansion: α

Coefficient of area expansion: b

Coefficient of volume expansion: β

For small ranges of temperature, the increases in length, area and volume with temperature can be represented by

$$l_2 = l_1 [1 + \hat{\alpha} (T_2 - T_1)] \quad (13.1.1)$$

$$A_2 = A_1 [1 + \hat{b} (T_2 - T_1)] \quad (13.1.2)$$

and

$$V_2 = V_1 [1 + \hat{\beta} (T_2 - T_1)] \quad (13.1.3)$$

Here $\hat{\alpha}$, \hat{b} and $\hat{\beta}$ are the approximate coefficients of linear, area and volume expansion respectively over the temperature range T_1 to T_2 . For all three, the units are degree⁻¹ – that is C^{o-1} or K⁻¹.

For anisotropic crystals, the coefficient may be different in different directions, but for isotropic materials we can write

$$A_2 = l_2^2 = l_1^2 [1 + \hat{\alpha} (T_2 - T_1)]^2 = A_1 [1 + 2\hat{\alpha} (T_2 - T_1) + \dots] \quad (13.1.4)$$

$$V_2 = l_2^3 = l_1^3 [1 + \hat{\alpha} (T_2 - T_1)]^3 = V_1 [1 + 3\hat{\alpha} (T_2 - T_1) + \dots] \quad (13.1.5)$$

Thus for small expansions, $\hat{b} \approx 2\hat{\alpha}$ and $\hat{\beta} \approx 3\hat{\alpha}$.

Equations 13.1.1, 2 and 3 define the approximate coefficients over a finite temperature range. The coefficients at a particular temperature are defined in terms of the derivatives, i.e.

$$\alpha = \frac{1}{l} \left(\frac{\partial l}{\partial T} \right)_P, \quad (13.1.6)$$

$$b = \frac{1}{A} \left(\frac{\partial A}{\partial T} \right)_P \quad (13.1.7)$$

$$\beta = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P. \quad (13.1.8)$$

The relations $b = 2\alpha$ and $\beta = 3\alpha$ are exact.

We specify “at constant pressure” because obviously we don't want, in our definition, to prevent the material from expanding by increasing the pressure on it when we heat it.

For solids, the coefficient of linear expansion is usually the appropriate parameter; for liquids and gases the volume coefficient is usually appropriate. For most familiar common metals the coefficient of linear expansion is of order 10⁻⁵ K⁻¹. Alloys such as the nickel-steel alloy, “invar”, used in clock construction, may have much smaller coefficients. Ordinary glass has a coefficient only a little less than that of metals; pyrex and fused quartz have a much smaller expansion – hence their use in telescope mirrors. For liquids and gases it is usually the volume coefficient that is quoted. The volume coefficient of mercury is about 0.00018 K⁻¹. Water actually contracts between 0 and 4 °C, and expands above that temperature. The volume coefficient of air at 0 °C is 0.0037 K⁻¹.

At room temperatures and above, the coefficient of linear expansion of metals doesn't vary a huge amount with temperature, but at low temperatures the coefficient of expansion varies much more rapidly with temperature – and so does the specific heat capacity (see Section 8.10). Indeed, for a given metal, the variation of expansion coefficient and the specific heat capacity vary with temperature in a rather similar manner, so that, for a given metal, the ratio α/C_P is constant over a large temperature range.

Exercise: A square metal plate has a circular hole of area 300 cm^2 in the middle of it. If the coefficient of linear expansion is $2 \times 10^{-5} \text{ C}^{-1}$, calculate the area of the hole when the temperature of the plate is raised through 100 degrees.

Exercise: Show that the coefficient of volume expansion of an ideal gas is $1/T$. Compare this with the numerical value for air given above.

Although classical thermodynamics does not deal with detailed microscopic processes, it is of interest to ask *why* a solid material expands upon heating. Let us imagine a crystalline solid to be made up of atoms connected to each other by little springs, and each spring is governed by Hooke's Law, and consequently each atom is vibrating in a parabolic potential well and is moving in simple harmonic motion. If we increase the temperature, we increase the *amplitude* of the vibrations, but we *do not change the mean positions of the atoms*. Consequently, in such a model, we would not expect any expansion upon heating. However, the real potential is not parabolic, but is shaped, at least qualitatively, something like the Lennard-Jones or Morse potentials mentioned in Chapter 6, Section 6.8. If the material is heated, the amplitude of the vibrations increases, and, because of the higher-order terms in the potential, which give the potential its asymmetric anharmonic shape, the mean separation of the atoms does indeed increase, and so we have expansion. Thus the expansion upon heating of a solid material is a consequence of the anharmonicity of the atomic vibrations and the asymmetry of the potential in which they are moving.

In the next two exercises, I shall be thinking of the expansion of a metal rod as the temperature is increased, and the pressure will be assumed to be constant at all times. Thus I am going to assume that pressure is not a variable in the discussion, and I shall define the coefficient of linear expansion as $\alpha = \frac{1}{l} \frac{dl}{dT}$ rather than the more general $\frac{1}{l} \left(\frac{\partial l}{\partial T} \right)_P$. A small point that I make at this stage is this: Suppose that the length of a metal rod increases linearly with temperature, so that $\frac{dl}{dT}$ is constant. This does *not* mean that the coefficient of expansion is independent of temperature. And if α is independent of temperature, l does *not* increase linearly with temperature. The next two exercises will illustrate that, and will also illustrate how the exact coefficient $\alpha = \frac{1}{l} \frac{dl}{dT}$ is related to what I have called (for want of a better term) the "approximate" coefficient $\hat{\alpha} = \frac{1}{l_1} \frac{l_2 - l_1}{T_2 - T_1}$.

Exercise. Suppose that the length of a metal rod increases with temperature according to $l = l_0 (1 + \alpha_0 T)$ where l_0 is the length at 0 K, and α_0 is the coefficient at 0 K. This means that $\frac{dl}{dT}$ and $l\alpha$ are independent of temperature, and each is equal to $l_0\alpha_0$. Show that the coefficient at temperature T is given by

$$\alpha = \frac{\alpha_0}{1 + \alpha_0 T}. \quad (13.1.9)$$

Show that $\hat{\alpha}$, the approximate coefficient over the temperature range T_1 to T_2 , is equal to the exact coefficient α evaluated at $T = T_1$.

Exercise. Suppose that the coefficient α is independent of temperature. Show that the length of the rod increases with temperature according to $l = l_0 e^{\alpha T}$, where l_0 is the length at 0 K. Show also that $\hat{\alpha} = \frac{e^{\alpha(T_2 - T_1)} - 1}{T_2 - T_1}$.

By this time, it may have occurred to the reader that what we have called α , for all its usefulness in the equation $l_2 = l_1 [1 + \hat{\alpha} (T_2 - T_1)]$, is not "the" coefficient of expansion at temperature T_1 , nor is it the mean coefficient in the temperature range T_1 to T_2 . The *mean* coefficient in this range must be defined by $\bar{\alpha} (T_2 - T_1) = \int_{T_1}^{T_2} \alpha dT$. So now, one more exercise:

Exercise. Suppose that the length of a metal rod increases with temperature according to $l = l_0 (1 + \alpha_0 T)$, where l_0 is the length at 0 K, and α_0 is the coefficient at 0 K. Show that

$$\bar{\alpha} = \frac{1}{(T_2 - T_1)} \ln \left(\frac{1 + \alpha_0 T_2}{1 + \alpha_0 T_1} \right). \quad (13.1.10)$$

Summary

If $\frac{dl}{dT}$ is constant	If α is constant
$l = l_0 (1 + \alpha_0 T)$	$l = l_0 e^{\alpha T}$
$\alpha = \frac{\alpha_0}{1 + \alpha_0 T}$	$\alpha = \alpha_0$
$\hat{\alpha} = \alpha_1$	$\hat{\alpha} = \frac{e^{\alpha(T_2 - T_1)} - 1}{T_2 - T_1}$
$\bar{\alpha} = \frac{1}{(T_2 - T_1)} \ln \left(\frac{1 + \alpha_0 T_2}{1 + \alpha_0 T_1} \right)$	$\bar{\alpha} = \alpha$

(13.1.11)

Of course, you may feel that this distinction between α , α_0 , $\hat{\alpha}$ and $\bar{\alpha}$ is splitting hairs. Let us discover for ourselves how much they differ, by putting in some numbers. Let us suppose that $\alpha_0 = 1.7 \times 10^{-5} \text{ K}^{-1}$ and that $l_0 = 1 \text{ m}$. Then, assuming that $T_1 = 280 \text{ K}$ (6.85 °C) and $T_2 = 380 \text{ K}$ (106.85 °C), we obtain

If $\frac{dl}{dT}$ is constant	If α is constant
$l_1 = 1.004760 \text{ m}$	1.004771 m
$\alpha(280 \text{ K}) = 1.691946 \times 10^{-5} \text{ K}^{-1}$	$\alpha(280 \text{ K}) = 1.700000 \times 10^{-5} \text{ K}^{-1}$
$\hat{\alpha} = 1.691946 \times 10^{-5} \text{ K}^{-1}$	$\hat{\alpha} = 1.701446 \times 10^{-5} \text{ K}^{-1}$
$\bar{\alpha} = 1.690516 \times 10^{-5} \text{ K}^{-1}$	$\bar{\alpha} = 1.700000 \times 10^{-5} \text{ K}^{-1}$

In general, if the length at T_1 is l_1 , the length l_2 at T_2 will be given by

$$l_2 = l_1 \exp\left(\int_{T_1}^{T_2} \alpha dT\right). \quad (13.1.12)$$

In the case where dl/dT is constant, so that $\alpha = \frac{\alpha_0}{1+\alpha_0 T}$, this becomes

$$l_2 = l_1 \left(\frac{1 + \alpha_0 T_2}{1 + \alpha_0 T_1} \right) = l_1 (1 + \alpha_0 (T_2 - T_1) - \alpha_0^2 T_1 (T_2 - T_1) + \dots). \quad (13.1.13)$$

In the case where α is constant, so it becomes

$$l_2 = l_1 \exp(\alpha (T_2 - T_1)) = l_1 \left(1 + \alpha (T_2 - T_1) + \frac{1}{2} \alpha^2 (T_2 - T_1)^2 + \dots \right) \quad (13.1.14)$$

Thus to the first order of small quantities, all varieties of α are equal.

Coefficient of Expansion as a Tensor Quantity. In Chapter 4, I briefly mentioned that, in the case of an anisotropic crystal, the coefficient of thermal conduction is a tensor quantity. The same is true, for an anisotropic crystal, of the coefficient of expansion. Thus, if, during an physics examination, you were asked to give examples of tensor quantities, you could give these as examples – though a small risk might be involved if your teacher had not thought of these as tensors! The coefficient of expansion of an anisotropic crystal may vary in different directions. (In Iceland Spar – calcium carbonate – in one direction the coefficient is actually negative.) If you cut an anisotropic crystal in the form of a cube, whose edges are not parallel to the crystallographic axis, the sample, upon heating, will not only expand in volume, but it will change in shape to become a non-rectangular parallelepiped. However, it is possible to cut the crystal in the form of a cube such that, upon heating, the sample expands to a rectangular parallelepiped. The edges of the cube (and the resulting parallelepiped) are then parallel to the *principal axes of expansion*, and the coefficients in these directions are the *principal coefficients of expansion*. These directions will be parallel to the crystallographic axes if the crystal has one of more axes of symmetry (but obviously not otherwise)

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