4.4: The Heat Conduction Equation

The situation described in Section 4.2 and in figure IV.1 was a *steady-state* situation, in which the temperature was a function of x but not of time. We are now going to consider a more general situation in which the temperature may vary in time as well as in space.

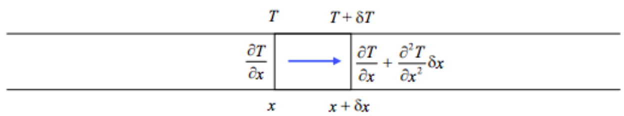

FIGURE IV.3

In this case the temperature gradient is written as a partial derivative, \(\frac{\partial T}{\partial x} and is not uniform down the length of the rod. We'll suppose it is \frac{ \deta T}{\partial x} at *x* and $\frac{\partial T}{\partial x}+\frac{\partial^2 T}{\partial x^2}\delta x$ at x + δx.

Consider the heat flow into and out of the portion between x and x + δx. The rate of flow into this portion at *x* is $-KA\frac{\partial T}{\partial x}$, and the rate of flow out at x + δx is $-KA\left(\frac{\partial T}{\partial x}+\frac{\partial^2 T}{\partial x^2}\delta x\right)$, so that the net flow of heat into that portion is $KA\frac{\partial^2 T}{\partial x^2}\delta x$. This must be equal to $C\rho A\delta x\frac{\partial T}{\partial t}$, where ρ is the density (and hence ρ*A*δ*x* is the mass of the portion), and *C* is the specific heat capacity.

Therefore

$$C\rho\frac{\partial T}{\partial t}=K\frac{\partial^2 T}{\partial x^2}. \tag{4.4.1}$$

This can be written

$$\frac{\partial T}{\partial t}=D\frac{\partial^2 T}{\partial x^2}, \tag{4.4.2}$$

where

$$D=\frac{K}{C\rho} \tag{4.4.3}$$

is the *thermal diffusivity* ($m^2\ s^{-1}$).

Equation 4.3.2 is the *heat conduction equation*. In three dimensions it is easy to show that it becomes

$$T=D\nabla^2 T. \tag{4.4.4}$$

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