

## 4.5: A Solution of the Heat Conduction Equation

Methods of solving the heat conduction equation are commonly given in courses on partial differential equations. Here we shall look at a simple one-dimensional example.

A long copper bar is initially at a uniform temperature of 0 °C. At time  $t = 0$ , the left hand end of it is heated to 100 °C. Draw graphs of temperature versus distance  $x$  from the hot end of the bar (up to  $x = 100$  cm) at  $t = 4, 16, 64, 256$  and 1024 seconds. Draw also a graph of temperature versus time at  $x = 5$  cm, up to 1024 seconds. Assume no heat is lost from the sides of the bar.

Data for copper:

$$K = 400 \text{ W m}^{-1} \text{ K}^{-1}$$

$$C = 395 \text{ J kg}^{-1} \text{ K}^{-1}$$

$$\rho = 8900 \text{ kg m}^{-3}$$

whence

$$D = 1.137 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$$

The equation to be solved is

$$D \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad (4.5.1)$$

From the form of this equation, it is obvious (once it has been pointed out!) that a solution could be found in which  $T(x, t)$  is solely a function of  $x^2/t$ , or, for that matter,  $x/t^{1/2}$ . Thus, let

$$u = x/t^{1/2}, \quad (4.5.2)$$

and you will see that equation 4.4.1 reduces to the second order total differential equation

$$D \frac{d^2 T}{du^2} = -\frac{u}{2} \frac{dT}{du}. \quad (4.5.3)$$

Let  $T' = dT/du$ , and it becomes even easier – a first order equation:

$$D \frac{dT'}{du} = -\frac{1}{2} u T'. \quad (4.5.4)$$

Upon integration, we obtain

$$\ln T' = -\frac{u^2}{4D} + \ln A, \quad (4.5.5)$$

where  $\ln A$  is an integration constant, to be determined by the initial and boundary conditions. (What are the dimensions of  $A$ ?)

That is,

$$T' = A \exp \left[ -u^2 / (4D) \right]. \quad (4.5.6)$$

We have to integrate again, with respect to  $u$ :

$$T = A \int \exp \left[ -u^2 / (4D) \right] du. \quad (4.5.7)$$

Now,  $T = 100$  °C at  $x = 0$  for any  $t > 0$ . That is,  $T = 100$  for  $u = 0$ .

And  $T = 0$  °C at  $t = 0$  for any  $x > 0$ . That is,  $T = 0$  for  $u = \infty$ .

Therefore

$$100 - 0 = A \int_{\infty}^0 \exp \left[ -u^2 / (4D) \right] du. \quad (4.5.8)$$

The integral is slightly difficult though well known. I'll just state the answer here; it is  $-\sqrt{\pi D}$ . From this, we find that the integration constant is

$$A = -5284 \text{K m}^{-1} \text{s}^{1/2}. \quad (4.5.9)$$

We now have

$$100 - T(x, t) = A \int_{xt^{-1/2}}^0 \exp[-u^2/(4D)] du. \quad (4.5.10)$$

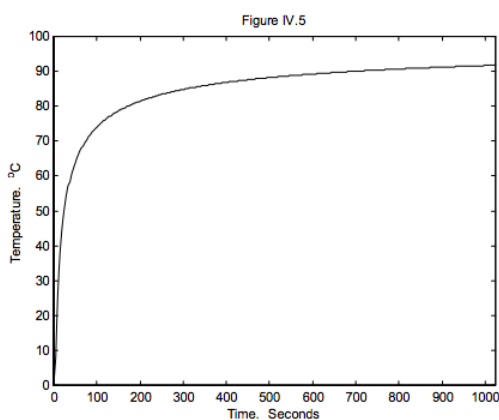
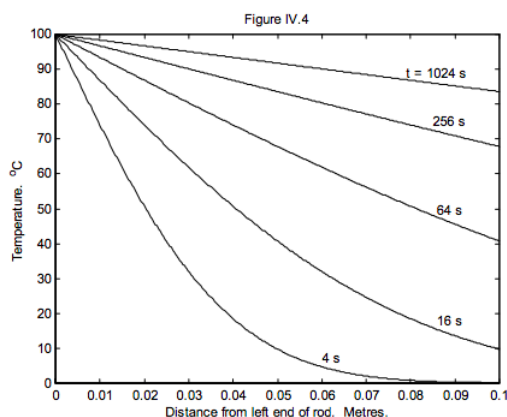
The error function  $\text{erf}(r)$  is defined by

$$\text{erf}(r) = \frac{2}{\sqrt{\pi}} \int_0^r \exp(-s^2) ds, \quad (4.5.11)$$

so that equation 4.4.10 can be written

$$T(x, t) = 100 + A\sqrt{\pi D} \text{erf}\left(\frac{x}{2\sqrt{Dt}}\right) = 100 \left[1 - \text{erf}\left(\frac{x}{2\sqrt{Dt}}\right)\right]. \quad (4.5.12)$$

This function is easy to plot provided that your computer will give you the erf function. The solutions are shown in figures IV.4 and 5.



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