2.4: Product of Three Partial Derivatives

Suppose x, y and z are related by some equation and that, by suitable algebraic manipulation, we can write any one of the variables explicitly in terms of the other two. That is, we can write

$$x = f(y,\ z), \tag{2.4.1}$$

or

$$y = y(z,\ x), \tag{2.4.2}$$

or

$$z = z(x,\ y). \tag{2.4.3}$$

Then

$$\delta x = \frac{\partial x}{\partial y}\delta y + \frac{\partial x}{\partial z}\delta z, \tag{2.4.4}$$

$$\delta y = \frac{\partial y}{\partial z}\delta z + \frac{\partial y}{\partial x}\delta x \tag{2.4.5}$$

and

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y. \tag{2.4.6}$$

Eliminate δy from Equations 2.4.4 and 2.4.5:

$$\delta x\left(1 - \frac{\partial x}{\partial y}\frac{\partial y}{\partial x}\right) = \delta z\left(\frac{\partial x}{\partial z} + \frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\right), \tag{2.4.7}$$

and δz from Equations 2.4.4 and 2.4.6:

$$\delta x\left(1 - \frac{\partial x}{\partial z}\frac{\partial z}{\partial x}\right) = \delta y\left(\frac{\partial x}{\partial y} + \frac{\partial x}{\partial z}\frac{\partial z}{\partial y}\right). \tag{2.4.8}$$

Since z and x can be varied independently, and x and y can be varied independently, the only way in which Equations 2.4.7 and 2.4.8 can always be true is for all of the expressions in parentheses to be zero. Equating the left-hand parentheses to zero shows that

$$\frac{\partial x}{\partial y} = 1/\frac{\partial y}{\partial x} \tag{2.4.9}$$

and

$$\frac{\partial x}{\partial z} = 1/\frac{\partial z}{\partial x}. \tag{2.4.10}$$

These results may seem to be trivial and "obvious" – and so they are, *provided that the same quantity is being kept constant in the derivatives of both sides of each equation*. In thermodynamics we are often dealing with more variables than just x, y and z, and we must be careful to specify which quantities are being held constant. If, for example, we are dealing with several variables, such as u, v, w, x, y, z, it is not in general true that $\frac{\partial u}{\partial y} = 1/\frac{\partial y}{\partial u}$, unless the same variables are being held constant on both sides of the equation.

Return now to Equation 2.4.7. The left hand parenthesis is zero, and this, together with Equation 2.4.10, results in the important relation:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1. \tag{2.4.11}$$

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