4.5: A Solution of the Heat Conduction Equation

Methods of solving the heat conduction equation are commonly given in courses on partial differential equations. Here we shall look at a simple one-dimensional example.

A long copper bar is initially at a uniform temperature of 0 $^{\circ}$C. At time t = 0, the left hand end of it is heated to 100 $^{\circ}$C. Draw graphs of temperature versus distance x from the hot end of the bar (up to x = 100 cm) at t = 4, 16, 64, 256 and 1024 seconds. Draw also a graph of temperature versus time at x = 5 cm, up to 1024 seconds. Assume no heat is lost from the sides of the bar.

Data for copper:

K = 400 W m^{-1} K^{-1}

C = 395 J kg^{-1} K^{-1}

ρ = 8900 kg m^{-3}

whence

D = 1.137 × 10^{-4} m^2 s^{-1}

The equation to be solved is

$$D\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{4.5.1}$$

From the form of this equation, it is obvious (once it has been pointed out!) that a solution could be found in which $T(x, t)$ is solely a function of x^2/t, or, for that matter, $x/t^{1/2}$. Thus, let

$$u = x/t^{1/2}, \tag{4.5.2}$$

and you will see that equation 4.4.1 reduces to the second order total differential equation

$$D\frac{d^2 T}{du^2} = -\frac{u}{2}\frac{dT}{du}. \tag{4.5.3}$$

Let $T' = dT/du$, and it becomes even easier − a first order equation:

$$D\frac{dT'}{du} = -\frac{1}{2}uT'. \tag{4.5.4}$$

Upon integration, we obtain

$$\ln T' = -\frac{u^2}{4D} + \ln A, \tag{4.5.5}$$

where ln A is an integration constant, to be determined by the initial and boundary conditions. (What are the dimensions of A?)

That is,

$$T' = A\exp\left[-u^2/(4D)\right]. \tag{4.5.6}$$

We have to integrate again, with respect to u:

$$T = A\int \exp\left[-u^2/(4D)\right] du. \tag{4.5.7}$$

Now, T = 100 $^{\circ}$C at x = 0 for any t > 0. That is, T = 100 for u = 0.

And T = 0 $^{\circ}$C at t = 0 for any x > 0. That is, T = 0 for u = ∞.

Therefore

$$100 - 0 = A\int_\infty^0 \exp\left[-u^2/(4D)\right] du. \tag{4.5.8}$$

The integral is slightly difficult though well known. I'll just state the answer here; it is $-\sqrt{\pi D}$. From this, we find that the integration constant is

$$A=-5284 \mathrm{K} \mathrm{m}^{-1} \mathrm{s}^{1/2}. \tag{4.5.9}$$

We now have

$$100-T(x,\ t)=A\int_{xt^{-1/2}}^{0}\exp\left[-u^2/(4D)\right]du. \tag{4.5.10}$$

The error function erf(r) is defined by

$$\operatorname{erf}(r)=\frac{2}{\sqrt{\pi}}\int_0^r\exp(-s^2)ds, \tag{4.5.11}$$

so that equation 4.4.10 can be written

$$T(x,\ t)=100+A\sqrt{\pi D}\operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right)=100\left[1-\operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right)\right]. \tag{4.5.12}$$

This function is easy to plot provided that your computer will give you the erf function. The solutions are shown in figures IV.4 and 5.