2.6: Euler's Theorem for Homogeneous Functions

There is a theorem, usually credited to Euler, concerning homogenous functions that we might be making use of.

A homogenous function of degree n of the variables x, y, z is a function in which all terms are of degree n. For example, the function $f(x,\ y,\ z) = Ax^3 + By^3 + Cz^3 + Dxy^2 + Exz^2 + Gyx^2 + Hzx^2 + Izy^2 + Jxyz$ is a homogenous function of x, y, z, in which all terms are of degree three.

The reader will find it easy to evaluate the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x}$ and equally easy (if slightly tedious) to evaluate the expression $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$. Tedious or not, I do urge the reader to do it. You should find that the answer is $3Ax^3 + 3By^3 + 3Cz^3 + 3Dxy^2 + 3Exz^2 + 3Fyz^2 + 3Gyx^2 + 3Hzx^2 + 3Izy^2 + 3Jxyz$.

In other words, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 3f$. If you do the same thing with a homogenous function of degree 2, you will find that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 2f$. And if you do it with a homogenous function of degree 1, such as $Ax + By + Cz$, you will find that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = f$. In general, for a homogenous function of x, y, z... of degree n, it is always the case that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} + \ldots = nf. \tag{2.6.1}$$

This is Euler's theorem for homogenous functions.