# 2.7: Undetermined Multipliers

Let $\psi(x, y, z)$ be some function of $x$, $y$ and $z$. Then if $x$, $y$ and $z$ are independent variables, one would ordinarily understand that, where $\psi$ is a maximum, the derivatives are zero:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial z} = 0. \tag{2.7.1}$$

However, if x, y and z are not completely independent, but are related by some constraining equation such as $f(x, y, z) = 0$, the situation is slightly less simple. (In a thermodynamical context, the three variables may be, for example, three "intensive state variables", $P$, $V$ and $T$, and $\psi$ might be the entropy, which is a function of state. However the intensive state variables may not be completely independent, since they are related by an "equation of state", such as $PV = RT$.)

If we move by infinitesimal displacements $dx$, $dy$, $dz$ from a point where $\psi$ is a maximum, the corresponding changes in $\psi$ and $f$ will both be zero, and therefore both of the following equations must be satisfied.

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial z}dz = 0, \tag{2.7.2}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0. \tag{2.7.3}$$

Consequently any linear combination of $\psi$ and $f$, such as $\Phi = \psi + \lambda f$, where $\lambda$ is an arbitrary constant, also satisfies a similar equation. The constant $\lambda$ is sometimes called an "undetermined multiplier" or a "Lagrangian multiplier", although often some additional information in an actual problem enables the constant to be identified.

In summary, the conditions that $\psi$ is a maximum (or minimum or saddle point), if $x$, $y$ and $z$ are related by a functional constraint $f(x, y, z) = 0$, are

$$\tfrac{\partial\Phi}{\partial x} = 0 \quad \tfrac{\partial\Phi}{\partial y} = 0, \quad \tfrac{\partial\Phi}{\partial z} = 0, \tag{2.7.4}$$

where

$$\Phi = \psi + \lambda f. \tag{2.7.5}$$

Of course, if $\psi$ is a function of many variables $x_1$ , $x_2$ , $x_3$..., and the variables are subjected to several constraints, such as $f = 0$, $g = 0$, $h = 0$, etc., where $f$, $g$, $h$, etc., are functions connecting all or some of the variables, the conditions for $\psi$ to be a maximum (etc.) are

$$\frac{\partial\psi}{\partial x_i} + \lambda\frac{\partial\psi}{\partial x_i} + \mu\frac{\partial\psi}{\partial x_i} + \nu\frac{\partial\psi}{\partial x_i} + \ldots = 0, \; i = 1, \; 2, \; 3 \tag{2.7.6}$$