2.2: Partial Derivatives

The equation

$$z = z(x,\ y) \tag{2.2.1}$$

represents a two-dimensional surface in three-dimensional space. The surface intersects the plane y = constant in a plane curve in which *z* is a function of *x*. One can then easily imagine calculating the slope or gradient of this curve in the plane *y* = constant. This slope is $\left(\frac{\partial z}{\partial x}\right)_y$ - the partial derivative of z with respect to x, with y being held constant. For example, if

$$z = y \ln x, \tag{2.2.2}$$

then

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{y}{x}, \tag{2.2.3}$$

y being treated as though it were a constant, which, in the plane *y* = constant, it is. In a similar manner the partial derivative of *z* with respect to *y*, with *x* being held constant, is

$$\left(\frac{\partial z}{\partial y}\right)_x = \ln x \tag{2.2.4}$$

When you have only three variables – as in this example – it is usually obvious which of them is being held constant. Thus *∂z/∂y* can hardly mean anything other than at constant *x*. For that reason, the subscript is often omitted. In thermodynamics, there are often more than three variables, and it is usually (I would say always) essential to indicate by a subscript which quantities are being held constant.

In the matter of pronunciation, various attempts are sometimes made to give a special pronunciation to the symbol ∂. (I have heard "day", and "dye".) My own preference is just to say "partial dz by dy".

Let us suppose that we have evaluated z at *(x , y)*. Now if you increase *x* by δx, what will the resulting increase in *z* be? Obviously, to first order, it is $\frac{\partial x}{\partial x}\delta x$. And if *y* increases by δ*y*, the increase in *z* will be $\frac{\partial z}{\partial y}\delta y$. And if both *x* and *y* increase, the corresponding increase in *z*, to first order, will be

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y \tag{2.2.5}$$

No great and difficult mathematical proof is needed to "derive" this; it is just a plain English statement of an obvious truism. The increase in *z* is equal to the rate of increase of *z* with respect to *x* times the increase in *x* plus the rate of increase of *z* with respect to *y* times the increase in *y*.

Likewise if *x* and *y* are increasing with time at rates $\frac{dx}{dt}$ and $\frac{dy}{dt}$, the rate of increase of *z* with respect to time is

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{2.2.6}$$