2.5: Impedance

We need to remind ourselves of one other thing from electromagnetic theory before we can proceed, namely the meaning of *impedance* in the context of electromagnetic wave propagation. The impedance Z is merely the ratio E/H of the electric to the magnetic field. The SI units of E and H are V/m and A/m respectively, so the SI units of Z are V/A, or ohms, Ω. We are now going to see if we can express the impedance in terms of the permittivity and permeability of the medium in which an electromagnetic wave is travelling.

Maxwell's equations are

$$\nabla \cdot \mathbf{D} = \rho \tag{2.5.1}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{2.5.2}$$

$$\nabla \times \mathbf{H} = \dot{\mathbf{D}} + \mathbf{J}. \tag{2.5.3}$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}. \tag{2.5.4}$$

In an isotropic, homogeneous, nonconducting, uncharged medium (such as glass, for example), the equations become:

$$\nabla \cdot \mathbf{E} = 0 \tag{2.5.5}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{2.5.6}$$

$$\nabla \times \mathbf{H} = \epsilon \dot{\mathbf{E}}. \tag{2.5.7}$$

$$\nabla \times \mathbf{H} = -\mu \dot{\mathbf{H}}. \tag{2.5.8}$$

If you eliminate $\mathbf{H}$ from these equations, you get

$$\nabla^2 \mathbf{E} = \epsilon\mu \ddot{\mathbf{E}}, \tag{2.5.9}$$

which describes an electric wave of speed

$$\frac{1}{\sqrt{\epsilon\mu}}. \tag{2.5.10}$$

In free space, this becomes

$$\frac{1}{\sqrt{\epsilon_0\mu_0}} \tag{2.5.11}$$

which is $2.998 \times 10^8\ m\ s^{-1}$.

The ratio of the speeds in two media is

$$\frac{v_1}{v_2} = \frac{n_2}{n_1} = \sqrt{\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}}, \tag{2.5.12}$$

and if, as is often the case, the two permeabilities are equal (to μ_0), then

$$\frac{v_1}{v_2} = \frac{n_2}{n_1} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}. \tag{2.5.13}$$

In particular, if you compare one medium with a vacuum, you get: $n = \sqrt{\frac{\epsilon}{\epsilon_0}}$.

Light is a high-frequency electromagnetic wave. When a dielectric medium is subject to a high frequency field, the polarization (and hence D) cannot keep up with the electric field E. D lags behind E. This can be described mathematically by ascribing a complex value to the permittivity. The amount of lag depends, unsurprisingly, on the frequency - i.e. on the color - and so the permittivity and hence the refractive index depends on the wavelength of the light. This is *dispersion*.

If instead you eliminate $\mathbf{E}$ from Maxwell's equations, you get

$$\nabla^2 \mathbf{H} = \epsilon\mu \ddot{\mathbf{H}}. \tag{2.5.14}$$

This is a magnetic wave of the same speed.

If you eliminate the time between 2.5.9 and 2.5.14, you find that $\frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}}$, which, in free space, has the value $\sqrt{\frac{\mu_0}{\epsilon_0}} = 377\Omega$, which is the impedance of free space. In an appropriate context I may use the symbol Z_0 to denote the impedance of free space, and the symbol Z to denote the impedance of some other medium.

The ratio of the impedances in two media is

$$\frac{Z_1}{Z_2} = \sqrt{\frac{\epsilon_2 \mu_1}{\epsilon_1 \mu_2}}, \tag{2.5.15}$$

and if, as is often the case, the two permeabilities are equal (to μ_0), then

$$\frac{Z_1}{Z_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{n_2}{n_1} = \frac{v_1}{v_2}. \tag{2.5.16}$$

We shall be using this result in what follows.