2.10: A, p and q for General Reflectance Rules

Again, consider a sphere of radius a centred in an O*xyz* frame with corresponding directional spherical coordinates (Θ, Φ), and let the sphere be irradiated with flux density **F** from the *z*-direction. For the geometrical albedo the phase angle α is zero and the incident and reflected radiation are given by $\mu_0 = \mu = \cos\Theta$, so that

$$p=\int_0^{2\pi}\int_0^{\pi/2} f_r(\cos\theta,\cos\Theta,0;\ldots)\cos^2\Theta\sin\theta d\Theta d\Phi, \tag{2.10.1}$$

resulting in

$$p=2\pi\int_0^1 f_r(\mu,\mu,0;\ldots)\mu^2 d\mu. \tag{2.10.2}$$

Using the same geometry for the Bond albedo, for each point on the irradiated hemisphere we have $\mu_0 = \cos\Theta$, so that the directional hemispherical reflectance is

$$\rho\left(\mu_0\right)=\int_0^{2\pi}\int_0^1 f_r\left(\mu_0,\mu,\alpha;\ldots\right)\mu d\mu d\phi_r, \tag{2.10.3}$$

where the phase angle is that between the incident and reflected radiation at each stage of the integral,

$$\cos\alpha=\mu_0\mu+\sqrt{\left(1-\mu_0^2\right)\left(1-\mu^2\right)}\cos\phi_r, \tag{2.10.4}$$

where φ_r is the azimuth of the reflected radiation. The Bond albedo is then given by

$$A=\frac{1}{\pi}\int_0^{2\pi}\int_0^{\pi/2}\rho(\cos\Theta)\cos\Theta\sin\Theta d\Theta d\Phi, \tag{2.10.5}$$

which reduces to

$$A=2\int_0^t \rho(\mu)\mu d\mu. \tag{2.10.6}$$

The phase integral, equation (20), may be computed from equations (14), (15) and (16), the factor a^2 **F** disappearing in the process, so that we may write, for the purposes of computation, equation (16) as

$$I(\alpha)=\int_{\alpha-\pi/2}^{\pi/2}\int_0^{\pi} f_r\left(\mu_0,\mu,\alpha;\ldots\right)\mu_0\mu\sin\Theta d\Theta d\Phi, \tag{2.10.7}$$

where it can be seen that for Φ the range of integration is from α - $\pi/2$, the limb, to $\pi/2$, the terminator.

In these equations it can be seen that the geometrical albedo is just a single integral and thus may be quickly and accurately integrated numerically with just about any method. The Bond albedo and the phase integral are, however, triple integrals, so that a method which combines the advantages of speed and accuracy is required; for this reason *Gaussian Quadrature* is the chosen method. In the following section we present this method in algorithmic form and discuss its application to the integrals at hand.

For the theory and examples of Gaussian Quadrature, its performance compared to other methods of integration as well as tabulations of the *roots* and *coefficients* needed, the reader is referred to astrowww.phys.uvic.ca/~tatum/ *Celestial Mechanics*, Chap. 1.