University of Victoria

Map: Planetary Photometry (Tatum & Fairbairn)

Jeremy Tatum

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This text was compiled on 01/09/2025

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About this Book

Max Fairbairn, who was my first ever graduate student, obtained an M.Sc. in Astronomy at the University of Victoria, Canada, in 1971, after which he returned to his native Australia. It was some years later when I heard from him again. Apart from wanting to renew a friendship, he had found himself interested in a paper that I and two other Victoria students, Tim Lester and Marshall McCall, had written on the subject of planetary photometry. This led to some fruitful correspondence on the subject, during which it became evident to me that Max understood the subject far better than I had ever done. This is, of course, the greatest compliment that a former student can ever pay to his erstwhile professor. His email letters to me on the subject were so clear that I suggested that it would be useful to put his material on the Web so that others could benefit from the thought he had put into it.

Max had just started to put his material together when he died, after a painful illness, in January 2005, at the young age of 58. At that time, he had put one chapter, "Principles of Planetary Photometry", on the Web, though there were some problems with the diagrams, which had not reproduced well on the Web, and he had written, but not posted, a second chapter, "Albedo". He had planned further chapters, leading up to the determination of asteroid shapes from their lightcurves, but was not able to complete these before his death.

Nevertheless, although this work is far from complete, I felt that the first two chapters, which deal with fundamental principles and definitions, described in a clear and unambiguous manner, would be of interest and use not only to students who are just starting out on the subject, but quite likely also to more experienced astronomers who are actively working in the field. I have therefore repaired the computer-mangled defective drawings that appeared on the Web posting of Chapter 1 and have done some minor editing, and have copy-edited the unpublished Chapter 2, and I have called upon the computer expertise of Jason Stumpf to post these two chapters. We also found a document entitled *A Brief History of the Lommel-Seeliger Law*, which I have treated as a third chapter. Although there is much more to the subject, I am hopeful that some readers will find this introductory material of interest and use, and that Max's efforts will then not have been in vain.

Comments and notification of errors can be directed to the undersigned at jtatum@uvic.ca

CHAPTER OVERVIEW

1: Principles of Planetary Photometry

Thumbnail: Surface of Mercury (Public Domain; NASA).

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1.1: Introduction

The subject of planetary photometry is, in substantial part, a subset of that branch of mathematical physics known as *radiative transfer*, for which the classical and definitive work is that of Chandrasekhar (1960).

Here we present this aspect of the subject in a modern context and although we have adhered as much as possible to the symbols, nomenclature and notation of Chandrasekhar, the following changes and additions have been made.

1. The quantity called by Chandrasekhar *intensity* I is here called *radiance* L. I make no apology for this since it conforms with modern international radiometric standards.
2. A plane parallel beam of radiation is specified by its *radiant flux density* $\mathbf{F}$ rather than its *net flux* πF, the latter being a more generally defined quantity.
3. Shorthands for incident, reflected and transmitted radiation, with subscripts i, r and t have been introduced.
4. Reflectance functions in addition to Chandrasekhar's formulations are presented.

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1.2: Radiance and the Equation of Transfer

Radiance may be regarded as the fundamental quantity of radiative transfer.

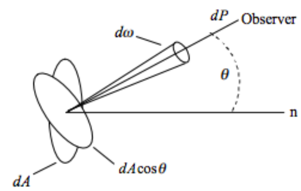

Fig.1.

Consider, as shown in figure 1, an emitting (or reflecting) surface of area dA which emits dP watts of power into solid angle $d\omega$ about the direction of an observer at angle θ to the surface normal vector n of dA, the latter presenting a *projected area* $dA\cos\theta$ to the observer. The radiance detected by the observer is then

$$L = \frac{dP}{d\omega\, dA\cos\theta} \tag{1.2.1}$$

Radiance then has the following properties:

- It is defined at a point and in a specified direction.
- It is independent of the distance from which it is observed
- Its SI units are watts per square metre per steradian (W m^{-2} sr^{-1}). This can be interpreted in either of two ways. Either, it is the power projected into unit solid angle from unit projected area of an extended surface (i.e. projected on a plane at right angles to the line of sight from the observer); or it is the power arriving per unit area at the observer from unit solid angle (subtended at the observer) of the extended source. That these are equivalent is shown in reference (2).

If radiance is the fundamental quantity of radiative transfer, then the fundamental law is the *equation of transfer*

$$-\frac{dL}{\kappa\rho ds} = L - \mathfrak{J}. \tag{1.2.2}$$

Here $\frac{dL}{ds}$ is the rate of change of radiance with, and in the direction of, position s in a given medium, r is the density of the medium (kg m^{-3}) and κ is the *mass attenuation coefficient* (m^2 kg^{-1}). Here attenuation refers to any process which reduces the brightness of a beam of radiation, and so includes absorption and scattering. Some authors use the word extinction for attenuation, and some (particularly in the field of stellar atmospheres) use the word opacity to refer to the mass attenuation coefficient.

Equation 1.2.2 could be read as follows: as a beam of radiance L traverses the distance ds it will be diminished in radiance by the amount $krLds$ and enhanced by the amount $\kappa\rho\mathfrak{J}\delta s$. The quantity $\mathfrak{J}$ is called the *source function* and, as we shall see, a typical problem of planetary photometry is to find a solution for this quantity before solving the equation of transfer as a whole.

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1.3: Diffuse Reflection and Transmission

The fundamental problem of planetary photometry is the diffuse reflection and transmission of a *plane parallel* beam of radiation by a *scattering medium*, which we would understand as a planetary atmosphere and/or surface or a planetary layer such as the rings of Saturn. Such media may be idealised as locally plane parallel strata in which physical properties are uniform throughout a given layer. In such cases we may use a hybrid Cartesian and spherical frame of reference in which the Oxy plane is the surface and z-axis points in the direction of the surface normal. Directions are then specified by the polar and azimuthal angles ϑ and φ ("curly theta" and "curly phi") respectively. Further, with problems of this kind, rather than working in actual physical distances it is preferable to work in terms of normal optical thickness t , measured downwards from $z = 0$, such that $d\tau = -\kappa\rho dz$. Radiation that has traversed a path of optical thickness t is attenuated by a factor of exp(- t)

Using the direction cosine m = cos ϑ the *standard form* of the equation of transfer for *plane parallel media* is

$$\mu \frac{dL(\tau,\mu,\varphi)}{d\tau} = L(\tau,\mu,\varphi) - \mathfrak{J}(\tau,\mu,\varphi) \tag{1.3.1}$$

For a scattering medium, the only contribution to the source function is the scattering of that radiation which has been incident on the medium from *external* sources, so that, by totalling the contributions impinging on level τ from all directions, the source function is

$$\mathfrak{J}(\tau,\mu,\varphi) = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} p\left(\mu,\varphi;\mu',\varphi'\right) L\left(\tau,\mu',\varphi'\right) d\varphi' d\mu' \tag{1.3.2}$$

where p is the *normalised phase function* which determines the angular distribution of the scattering. A convenient way to think of p is that $\frac{p}{4\pi}d\omega$ is the probability that a photon travelling in the direction (μ', φ') would be scattered into an elemental solid angle dw in the direction (μ, φ).

Radiation traversing a normal optical thickness dt in the direction (μ, φ) will be attenuated by the amount $\delta L = L\delta\tau/\mu$. Of this amount, a fraction can be attributed to that caused by scattering alone – this fraction is called the *single scattering albedo* ϖ_0. It then follows that the phase function p must be normalised according to

$$\int_{4\pi} \frac{p}{4\pi} d\omega = 0 \leq \varpi_0 \leq 1 \tag{1.3.3}$$

and if p is a constant, then $p = \varpi_0$ and the scattering is isotropic.

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1.4: Directions and Notation

The strength of a plane parallel beam of radiation is specified by the radiant flux density **F** watts per square metre such that **F** = dP/dA, where A is the area perpendicular to the direction of propagation. Thus **F** is equal to the net flux πF used by Chandrasekhar, with the important exception that **F** is used only for a plane parallel beam.

Since the equation of transfer deals only in radiances, we will now address the rather intriguing question, "What is the radiance of a plane parallel beam?"

Figure 2 shows a ray of a plane parallel beam of flux density **F** incident on the surface of a scattering medium. We shall let the resulting incident radiance be L_i, which, using Chandrasekhar's notation would be specified in position and direction as

$$L_i = L\left(0, -\mu_0, \varphi_0\right), \quad \mu_0 = \left|\cos\vartheta_0\right| \tag{1.4.1}$$

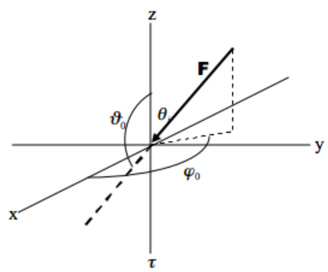

Fig. 2.

Many authors specify the polar direction of this radiation in terms of an *angle of incidence*, say i or θ_i, as the angle between the surface normal and the incident ray, such that $i = \pi - \vartheta_0$ and define μ_0 as $\mu_0 = \cos i$.

The angular distribution of **F** (or, rather, its lack of it) may be specified analytically by making use of the Dirac delta function, which has the property that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

and, perhaps more importantly, for any $\varepsilon > 0$

$$\int_{a-\varepsilon}^{a+\varepsilon} f(x)\delta(x-a)dx = f(a).$$

The incident radiance on the surface in the direction $(-\mu_0, \varphi_0)$ is then[1]

$$L_i = \mathbf{F}\delta\left(\mu-\mu_0\right)\delta\left(\varphi-\varphi_0\right) \tag{1.4.2}$$

Figure 3 shows a reflected beam of radiance L_r where

$$L_r = L(0, +\mu, \varphi), \quad \mu = \cos\vartheta \tag{1.4.3}$$

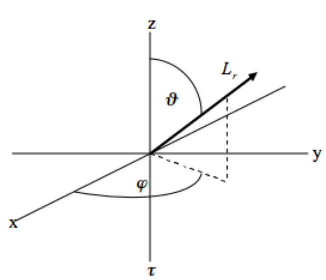

Fig. 3.

Again, many authors define the polar direction of reflection in terms of an *angle of reflection*, the angle between the surface normal and the reflected ray, say θ_r (which is the same as ϑ), and define μ as $\cos\theta_r$.

Figure 4 shows a transmitted ray of radiance L_t where

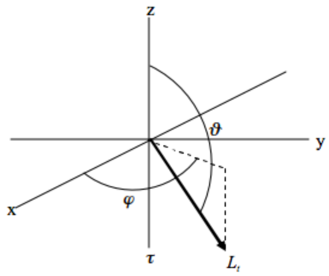

Fig. 4.

The question then arises: are the values of μ_0 and μ, introduced by Chandrasekhar and taken up by others, albeit with different definitions, always positive? The answer to this is, mostly, yes, but care needs to be taken depending on the context in which they occur, as, for example, in the following cases.

Consider the problem of determining the phase angle $0 \leq a \leq p$ between incident and reflected or transmitted directions μ_0 and μ, where

$$\cos\alpha = \mu_0\mu + \sqrt{\left(1-\mu_0^2\right)\left(1-\mu^2\right)}\cos(\varphi - \varphi_0). \tag{1.4.4}$$

For reflection both μ and μ_0 are positive, but for transmitted rays μ must be negative. The phase function *p* is often expressed in terms of a or cos a . Or, consider, as in figure 5, the optical path lengths of reflected and transmitted radiation from a depth within a medium of optical thickness *t*. The total optical path for the reflected ray is $\tau/\mu_0 + \tau/\mu_r$ and for the transmitted ray $\tau/\mu_0 + (t - \tau)/\mu_t$ in which μ_0, μ_r and μ_t are all positive.

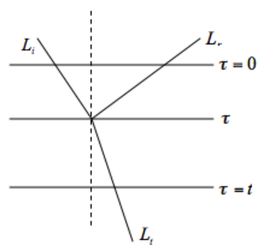

Fig. 5.

[1]If equation 1.4.2 bothers you in that its right hand side does not appear to have units of radiance, then do not worry; in chapter 2 we will demonstrate that indeed it does have such units.

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1.5: Reflectance Functions

In the most general case of diffuse reflection, the reflectance of a surface will depend on both the direction of the incident radiation and that of the reflected radiation. The *bidirectional reflectance distribution function*, f_r, links the *irradiance* E to the reflected radiance, such that

$$L_r = f_r\left(\mu, \varphi; \mu_0, \varphi_0\right) E\left(\mu_0, \varphi_0\right). \tag{1.5.1}$$

For a surface irradiated with flux density **F**, the irradiance is simply the component of the flux density perpendicular to the surface

$$E = \mu_0 \mathbf{F}, \tag{1.5.2}$$

so that, abbreviated, we can write

$$L_r = f_r \mu_0 \mathbf{F} \tag{1.5.3}$$

One of the simplest examples of a reflectance rule is that of a Lambertian reflecting surface for which the radiance is isotropic, so that

$$L_r = \frac{\lambda_0}{\pi} \mu_0 \mathbf{F}, \tag{1.5.4}$$

where λ_0 is sometimes referred to as the Lambertian albedo . Although it is not strictly physically correct, it is convenient (Chandrasekhar, p147) to identify λ_0 with the single scattering albedo ϖ_0, so for Lambert's law the BRDF is

$$f_r = \frac{\varpi_0}{\pi}. \tag{1.5.5}$$

For the most part, we shall refer all reflectance rules used to a BRDF; alternative reflectance functions will be discussed in Section 1.7.

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1.6: Diffuse Reflection - the Lommel-Seeliger Law

The Lommel-Seeliger reflectance rule is a time-honoured law which is still very much in use today. It is based on a model which is possibly the simplest from which a solution may be readily obtained for the source function and the equation of transfer. As such it is a *single scattering* model in which the scattering is *isotropic*, *i.e.* $p = \varpi_0$.

Consider a surface irradiated by flux density as shown in figure 3, so that the incident radiance is given by equation (7). Of this incident radiation, only a fraction will penetrate to optical depth τ without being scattered or absorbed

$$L(\tau,\mu,\varphi) = \mathbf{F} e^{-\tau/\mu_0} \delta(\mu-\mu_0)\delta(\varphi-\varphi_0) \tag{1.6.1}$$

and the source function is

$$\begin{aligned}\mathfrak{J}(\tau,\mu,\varphi) &= \frac{1}{4\pi}\int_{-1}^{1}\int_{-1}^{2\pi} \varpi_0 \mathbf{F} e^{-\tau/\mu_0}\delta(\mu'-\mu_0)\delta(\varphi'-\varphi_0)\,d\varphi' d\mu' \\ &= \frac{\varpi_0}{4\pi}\mathbf{F} e^{-\tau/\mu_0}\end{aligned}$$

Thus the contribution to the radiance from isotropic scattering in the direction (+μ, φ) from a layer of thickness *d*τ at a depth τ is

$$dL = \frac{\varpi_0 \mathbf{F} e^{-\tau/\mu_0}}{4\pi\mu} d\tau \tag{1.6.2}$$

so that the radiance emerging from the surface, without incurring any further absorption or scattering, is

$$dL(0,\mu,\varphi) = \frac{\varpi_0 \mathbf{F} e^{-\tau/\mu_0}}{4\pi\mu} e^{-\tau/\mu} d\tau \tag{1.6.3}$$

Note that *dL* is the contribution to the total radiance from the layer resulting from *single scattering*. The Lommel-Seeliger model considers only the scattering of the collimated incident light. It does not take into account scattering of diffuse light which has made its way indirectly to the same position by being scattered one or more times, i.e. it does not consider multiple scattering.

For a planetary surface, the layer is “semi-infinite” (t = ∞) and the total radiance in the direction μ is

$$\begin{gathered} L_r = \tfrac{\varpi_0 \mathbf{F}}{4\pi u} \times \\ \int_0^\infty \exp\left[-\tau\left(\tfrac{1}{\mu_0}+\tfrac{1}{\mu}\right)\right] d\tau \end{gathered} \tag{1.6.4}$$

resulting in

$$L_r = \frac{\varpi_0 \mathbf{F}}{4\pi\mu}\frac{\mu_0\mu}{\mu+\mu_0} \tag{1.6.5}$$

and since the irradiance is E = $\mathbf{F}\mu_0$ and $L_r = f_r E$ it follows that the bidirectional reflectance distribution function (BRDF) which defines the Lommel-Seeliger reflectance rule is

$$f_r = \frac{\varpi_0}{4\pi}\frac{1}{\mu+\mu_0}. \tag{1.6.6}$$

It is interesting to note that Chandrasekhar never *quite* derives the Lommel-Seeliger law formally and explicitly; indeed the name is not even mentioned. However, he does come very close on at least two occasions - see Chandrasekhar p146 and p217.

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1.7: Other Reflectance Functions

It is important to distinguish between a reflectance function and the reflectance law it represents. So far we have only considered one such function, the BRDF, so that the Lommel-Seeliger law expressed in terms of the BRDF is given by Equation 1.7.1 and the specific equation for the radiance is given by

$$L_r = \frac{\varpi_0}{4\pi} \frac{1}{\mu_0 + \mu} \mu_0 \mathbf{F} \tag{1.7.1}$$

Chandrasekhar takes a quite different approach, linking the radiance to the incident flux density through a factor $1/4\mu$, providing a consistent set of **scattering functions** S and **transmission functions** T, so that in the case of reflection from a semi-infinite surface we have

$$L_r = \frac{F}{4\mu} S(\mu, \varphi; \mu_0, \varphi_0) = \frac{\mathbf{F}}{4\pi\mu} S(\mu, \varphi; \mu_0, \varphi_0), \tag{1.7.2}$$

where Chandrasekhar *always* uses πF for incident radiant flux density **F**. Although, at least at first sight, this formulation may seem strange, even counterintuitive, there is a reason for it; the μ in the denominator is used to satisfy the *Helmholtz principle of reciprocity* (Chandrasekhar, p171), so that

$$S(\mu, \varphi; \mu_0, \varphi_0) = S(\mu_0, \varphi_0; \mu, \varphi). \tag{1.7.3}$$

Comparing Equations 1.7.1 and 1.7.2, it follows that for the Lommel-Seeliger law the Chandrasekhar scattering function is

$$S(\mu, \mu_0) = \frac{\varpi_0 \mu_0 \mu}{\mu_0 + \mu}, \tag{1.7.4}$$

where it can be seen that the reciprocity principle does indeed hold.

Another function to be found in the literature is the **bidirectional reflectance** r, which links the radiance to the incident flux density, so that the Lommel-Seeliger law is then

$$r(\mu_0, \mu) = \frac{\varpi_0}{4\pi} \frac{\mu_0}{\mu_0 + \mu}, \quad L_r = r\mathbf{F} \tag{1.7.5}$$

So, which, if any, of the above functions is the "best" for planetary applications? There does not appear to be any "standard" in use in the literature, indeed the situation would seem to be quite the opposite, many authors making up their own *ad hoc* "reflectance" or "scattering" functions to suit the problem at hand. (This can make for very frustrating reading, especially when words such as "flux", "intensity" and "brightness" are used loosely, as, sadly, is often the case).

The author can see no compelling reason to prefer one function over another. What is important is for authors to state clearly and without ambiguity the properties of the reflectance function and rule(s) which they are using.

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1.8: Diffuse Reflection and Transmission

A scattering layer of finite optical thickness t may be used to model e.g. a planetary ring. If we use the Lommel-Seeliger model, then the reflected radiance of such a layer may be determined by changing the upper limit of the integral in equation (20) so that

$$L_r = \frac{\varpi_0 \mathbf{F}}{4\pi\mu} \times \int_0^t \exp\left[-\tau\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right)\right] d\tau \tag{1.8.1}$$

resulting in

$$L_r = \frac{w_0}{4\pi} \frac{1}{\mu + \mu_0} \times \left[1 - \exp\left\{-t\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right)\right\}\right] \mu_0 \mathbf{F} \tag{1.8.2}$$

For the transmitted radiance, it is readily shown that

$$dL_t = \frac{\varpi_0 \mathbf{F} e^{-\tau/\mu_0}}{4\pi\mu} e^{-(t-\tau)/\mu} d\tau \tag{1.8.3}$$

and in the special case $\mu = \mu_0$, integration results in

$$L_t = \frac{\varpi_0 \mathbf{F} t}{4\pi\mu_0} e^{-t/\mu_0} \tag{1.8.4}$$

and otherwise

$$L_t = \frac{\varpi_0 \mathbf{F}}{4\pi} \frac{\mu_0}{\mu - \mu_0} \left[e^{-t/\mu} - e^{-t/\mu_0}\right] \tag{1.8.5}$$

In all cases the values of μ and μ_0 are positive; some authors even explicitly put in absolute value symbols to emphasise this point!

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1.9: Radiances of Planetary Spheres

We conclude this chapter by applying some of the work covered so far to a planetary situation. We will consider two hypothetical planets idealised as smooth spheres. One planet will have a surface reflecting according to Lambert's law, the other the Lommel-Seeliger law. The observer is able to resolve both planets equally well, so that we may compare and contrast the distribution of radiance across the projected discs at various phase angles.

Consider a unit sphere centred at the origin of an O*xyz* coordinate system irradiated with flux density **F** from the *x*-direction. A distant observer in the *xy* plane detects the radiance at phase angle α (the angle Sun-planet-Earth). Using the spherical coordinates (1, Θ, Φ) for the surface of the sphere, it can be shown for the *angles of incidence and reflection*

$$\begin{aligned} \mu_0 &= \sin\Theta\cos\Phi \\ \mu &= \sin\Theta\cos(\Phi-\alpha) \end{aligned} \tag{1.9.1}$$

The projected sphere, the disc seen by the observer, will have projected coordinates

$$\begin{aligned} y' &= \sin\Theta\sin(\Phi-\alpha) \\ z &= \cos\Theta \end{aligned} \tag{1.9.2}$$

such that $y'^2 + z^2 \leq 1$. Except at zero phase, not all the illuminated surface will be visible, since for each point on the disc both the condition $\mu_0 > 0$ and $\mu > 0$ must be satisfied in order that the point be both irradiated and not obscured from the observer.

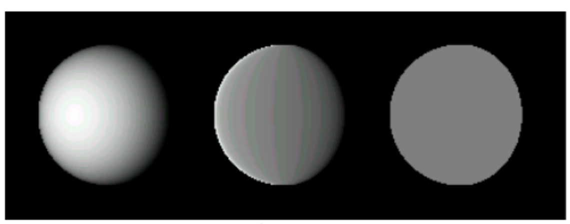
Defining a quantity *relative radiance*, $\pi L/\varpi_0 \mathbf{F}$, we can directly compare the radiances of the two spheres, as shown in the table.

Relative Radiances of Spheres

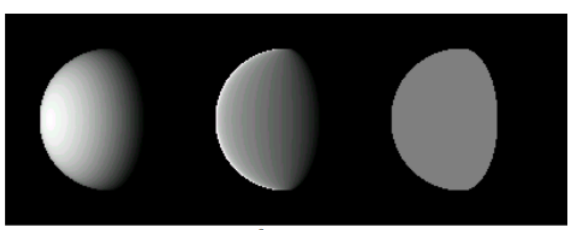
Lambertian	μ_0
Lommel-Seeliger	$\frac{1}{4}\frac{\mu_0}{\mu_0+\mu}$

The resulting images are readily programmed. In those that follow (le ftmost Lambertian, middle Lommel-Seeliger), each planet has been adjusted so that the maximum relative radiance is *white*. The rightmost image shows the outline of the lune visible to the observer.

$\alpha = 0$



$\alpha = 30^o$



$\alpha = 60^o$

$\alpha = 90^o$

$\alpha = 105^o$

$\alpha = 120^o$

At opposition the Lambertian sphere is limb-darkened, whereas the Lommel-Seeliger sphere is uniformly bright. As the phase angle increases from zero the LommelSeeliger sphere becomes darkened towards the terminator and brightened at the limb. For phases greater than ninety degrees, the cusps of the Lommel-Seeliger sphere are more persistant than the Lambertian. Without commenting any further, the images *do* make interesting comparisons with the phases of the Moon.

Reference Notes.

Sections 2, 4, 5, 7, 8 and 9 are based on the author's interpretation of Chandrasekhar's book, chapters I, III, VI and IX.

1. Chandrasekhar, S., 1960, *Radiative Transfer*, Dover, New York.

The ideas of a quantity **F** defined only for a plane parallel beam and the use of the BRDF (bi-directional reflectance distribution function) for astronomical applications are taken from

2. Lester. P. L., McCall, M. L. & Tatum, J. B., 1979, *J. Roy. Astron. Soc. Can.*, **73**, 233

who use F for flux density, which clashes with the F of the πF used by Chandrasekhar – this is the reason for using **F**. (See also Nicodemus, F.E., *Applied Optics*, 4, 767 (1965) and **9**, 1474 (1970) – JBT)

Section 10 is a revised and corrected adaptation of an article by the author

3. Fairbairn, M. B., 2002, J. Roy. Astron. Soc. Can., **96**, 18.

Depending on the hardware used, the images shown may display some spurious contouring

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CHAPTER OVERVIEW

2: Albedo

Albedo is a measurement, expressed as a fraction, of the amount of radiation scattered from a surface or an object. In this chapter we describe those albedos most commonly used and describe methods to calculate them in cases where analytical solutions are difficult, if not impossible, to obtain. In order to do this we introduce two more photometric quantities, namely *exitance* M and *intensity* I. A comprehensive summary of the photometric quantities is also presented.

Thumbnail: Surface of Mercury (Public Domain; NASA).

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2.2: Scattering and Absorption

The reduction in radiance when a beam passes through a given medium by any process in which the radiation is converted to heat or excitation energy is called *absorption*. A process by which the radiance is reduced by redirection of part of the radiation (by reflection, refraction or diffraction, or by being absorbed and immediately re-radiated in all directions) is called *scattering*, for which *reflectance* is often used as a synonym. The total effect of absorption and scattering is called *extinction*, although the author prefers the less often used alternative, *attenuation*.

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2.3: Absorption, Scattering and Attenuation Coefficients

The decrease in radiance $-dL$ as a beam of radiance L passes through a medium of thickness ds as a result of absorption is

$$-dL = \alpha L ds \tag{2.3.1}$$

where α is the **linear absorption coefficient**. With similar equations we can define the *linear scattering coefficient* σ and the *linear attenuation (extinction) coefficient* ε. The SI units of α, σ and ε are m^{-1} and $\varepsilon = \sigma + \alpha$.

The *mass absorption coefficient, mass scattering coefficient* and *mass extinction coefficient* each with units m^2 kg^{-1} are defined respectively as α/ρ, σ/ρ and ε/ρ, where ρ is the density (kg m^{-3}) of the medium. Chandrasekhar uses κ for the mass extinction coefficient, which, in the theory of stellar atmospheres, is also known as the **opacity**.

The *atomic* (or *molecular*) *absorption*, *scattering* and *extinction coefficients* are respectively α/N, σ/N and ε/N, where N is the number density (atoms or molecules per unit volume), with units of m^2/atom (or molecule). Because of these units the coefficients are often referred to as **cross-sections**.

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2.4: Surfaces - Single-scattering Albedo

We have already encountered a bare-boned, but nonetheless adequate, definition of single scattering albedo in Chapter 1. The loss of radiance from a beam of radiance L traversing a thickness ds of a medium is

$$dL = -\varepsilon L ds = -(\alpha + \sigma) L ds \tag{2.4.1}$$

and the single scattering albedo *is that fraction of the loss which can be attributed to scattering alone. i.e.*

$$\varpi_0 = \frac{\sigma}{\alpha + \sigma} = \frac{\sigma}{\varepsilon} \tag{2.4.2}$$

and the single scattering albedo is thus the ratio of the scattering coefficient to the extinction coefficient.

Single scattering albedo is the property of a *surface* or a *layer*, and may be regarded as the fundamental albedo, since all albedos that will be derived here from a given definition or reflectance rule will contain at least one instance of ϖ_0.

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2.5: Surfaces - Normal Albedo

If a lossless (conservative) Lambertian reflector (ϖ_0 = 1) is irradiated normally with flux density F, then its radiance in any direction will be F/π. The **normal albedo** p_n of a point on a surface is the ratio of the normally observed radiance to that of the Lambertian surface, so that

$$p_n = \pi f_r \left(\mu = \mu_0 = 1\right). \tag{2.5.1}$$

The author has found two definitions of normal albedo in the literature. In one, the surface must be radiated normally and observed normally ($\mu = \mu_0 = 1$) and the other in which it can be irradiated from any direction, in which case p_n is a function of μ_0.

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2.6: Net Flux and Exitance

Formerly known as *emittance*, the **exitance** M refers to a point on a reflecting or emitting surface and is defined as the total power emitted in all directions per unit *physical* area, so that

$$M=\int_0^{2\pi}\int_0^{\pi/2}L(\vartheta,\varphi)\sin\vartheta\cos\vartheta d\vartheta d\varphi \tag{2.6.1}$$

where it may be seen from the limits of integration that "in all directions" means over a hemisphere. The factor $\sin\vartheta d\vartheta d\varphi$ is an element of solid angle, $d\omega$, and the factor $\cos\vartheta$ is needed to convert the projected area of radiance back into physical area. Using the notation of Chapter 1., i.e. let $\mu=\cos\vartheta,\ d\mu=-\sin\vartheta d\vartheta$, we have

$$M=\int_0^{2\pi}\int_0^{1}L(\mu,\ \varphi)\mu d\mu d\varphi. \tag{2.6.2}$$

If we compare M to Chandrasekhar's quantity the *net flux* πF, which, in particular, he uses for a plane parallel beam of radiation

$$\begin{aligned}\pi F\quad &=\textstyle\int_0^{2\pi}\int_0^{2\pi}L(\vartheta,\ \varphi)\sin\vartheta\cos\vartheta d\vartheta d\varphi\\ &=\textstyle\int_0^{2\pi}\int_{-1}^{1}L((\mu,\ \varphi)\mu d\mu d\varphi\end{aligned} \tag{2.6.3}$$

we see that the net flux is indeed the result of integration over all directions, *i.e.* over a *sphere*. It follows that net flux and exitance are *not the same thing* (although there may be situations in which they amount to the same), and nor does πF always mean the strength of a plane parallel beam of radiant flux density $\mathbf{F}$. Indeed, we can calculate the net flux of a plane parallel beam incident on a surface in the direction $(\mu_0,\ \varphi_0)$, using the radiance of a plane parallel beam given by Chapter1, equation (7), as

$$\pi F=\int_0^{2\pi}\int_{-1}^{1}\mathbf{F}\delta\left(\mu-\mu_0\right)\delta\left(\varphi-\varphi_0\right)\mu d\mu d\varphi, \tag{2.6.4}$$

which results in

$$\pi F=\mathbf{F}\boldsymbol{\mu_0}, \tag{2.6.5}$$

this result being the irradiance E of the surface, as we knew it should be!

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2.7: Surfaces - Hemispherical Albedo

Also known as the *directional hemispherical reflectance*, the hemispherical albedo ρ refers to a point on a reflecting surface, and is defined as the ratio of the exitance *M* to the irradiance *E*, so that

$$\rho\left(\mu_0, \varphi_0\right)=\frac{M}{E\left(\mu_0, \varphi_0\right)}, \tag{2.7.1}$$

and in terms of the BRDF, we have

$$\rho\left(\mu_0, \varphi_0\right)=\int_0^{2 \pi} \int_0^1 f_r\left(\mu, \varphi ; \mu_0, \varphi_0\right) \mu d \mu d \varphi . \tag{2.7.2}$$

Unlike the single scattering albedo, ρ and the other albedos that we will encounter do not necessarily have in principle a maximum possible value of unity. (See *A Brief History of the Lommel-Seeliger Law*). The scattering properties of the surfaces that we have studied so far are summarised in Table I, from which, for the Lommel-Seeliger law, it can be seen that the maximum possible value for ρ is ½ and 0.125 for the normal albedo.

Table I. Properties of Surfaces

	Lambertian	Lommel-Seeliger
f_r	ϖ_0/π	$\frac{\varpi_0}{4\pi}\frac{1}{\mu_0+\mu}$
ρ	ϖ_0	$\frac{\varpi_0}{2}[1-\mu_0\ln(1+1/\mu_0)]$
p_n	ϖ_0	$\frac{\varpi_0}{8}$

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2.8: Intensity

The *intensity* of a source in a given direction is the power radiated per unit solid angle about the specified direction, *i.e.*

$$I = dP/d\omega. \tag{2.8.1}$$

The SI units are watts per steradian (W sr^{-1}). The intensity of an element of area is the product of its radiance and its *projected* area., and the intensity of a surface in a given direction is the integral of the radiance over the projected area of the surface. As an example, the shape of an irregularly shaped asteroid can be approximated as a set of connected planar triangular facets; two such facets are shown in figure 1.

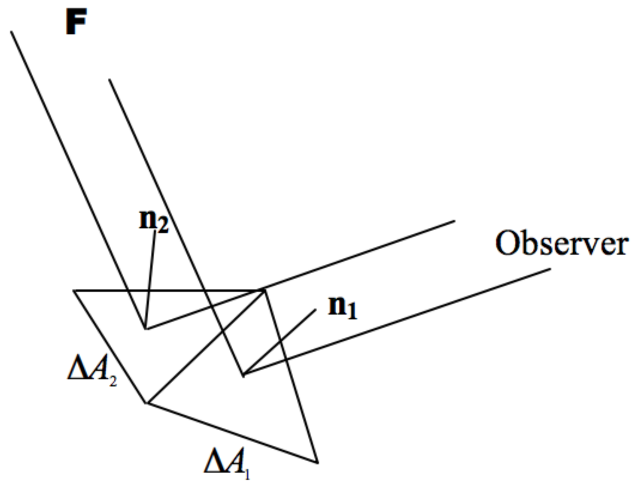

Fig.1.

For each facet of area ΔA_k the contribution to the intensity in the direction of the observer is

$$\Delta I_k = L_{obs,k} \Delta A_k \cos\theta_k \tag{2.8.2}$$

where θ_k is the angle between the surface normal vector $\mathbf{n}_k$ and the (fixed) direction to the observer. The total intensity (in the direction toward the observer) of the asteroid is then

$$I = \sum_{k=1}^{N} \Delta A_k \tag{2.8.3}$$

where N is the total number of facets both irradiated and visible to the observer.

Of particular interest is the intensity of a *sphere* as a function of solar phase angle α. If we consider a sphere of radius α centred in an O*xyz* frame with directional spherical coordinates (Θ, Φ) irradiated from the *x*-direction with flux density **F**, an element of surface area is $\alpha^2 \sin \Theta d\Theta d\Phi$ and its projected area in the direction μ is $\mu\alpha^2 \sin \Theta d\Theta d\Phi$.

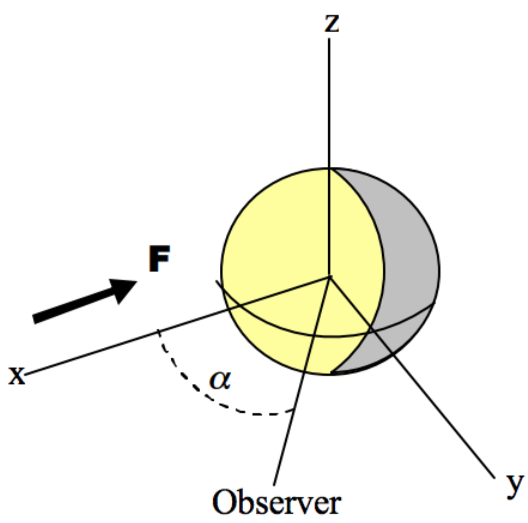

Fig. 2.

The irradiance of a point (α, Θ, Φ) of a point on the surface is $E = \mathbf{F}\mu_0$, where it may be shown that

$$\mu_0 = \sin\Theta\cos\Phi, \tag{2.8.4}$$

and for an observer at phase angle α in the *xy*-plane

$$\mu = \sin\Theta\cos(\alpha-\Phi), \tag{2.8.5}$$

in which case the intensity as a function of phase angle is given by

$$I(\alpha) = \alpha^2\mathbf{F}\int_{\alpha-\pi/2}^{\pi/2}\int_0^{\pi} f_r\mu_0\mu\sin\Theta d\Theta d\Phi. \tag{2.8.6}$$

We will return to this equation, with more detail, in **§9**.

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2.9: Spheres - Bond Albedo, Phase Integral and Geometrical Albedo

Originally defined for a sphere, the Bond albedo is defined as the ratio of the total power P_r scattered by the sphere to the total power P_i intercepted by it.

If we let the intensity of the sphere as a function of solar phase angle α be I(α) watts per steradian, then the total scattered flux may be obtained by multiplying by 2π sin α dα and integrating over α from 0 to π

$$P_r = 2\pi \int_0^\pi I(\alpha) \sin\alpha d\alpha, \tag{2.9.1}$$

which can be expressed in terms of the *normalised phase law* ψ(α) = I(α)/I(0)

$$P_r = 2\pi I(0) \int_0^\pi \phi(\alpha) \sin\alpha d\alpha. \tag{2.9.2}$$

For a sphere of radius α, the intercepted flux is simply P_i = π α^2 **F**, so that the Bond albedo may be expressed as

$$A = \frac{I(0)}{\alpha^2 \mathbf{F}} \times 2 \int_0^\pi \phi(\alpha) \sin\alpha d\alpha = p \times q \tag{2.9.3}$$

in which it may be seen as the product of two factors, the second of which,

$$q = 2 \int_0^\pi \phi(\alpha) \sin\alpha d\alpha, \tag{2.9.4}$$

is called the *phase integral*, which depends only on the directional reflecting properties of the planet. The first factor

$$p = \frac{I(0)}{\alpha^2 \mathbf{F}} \tag{2.9.5}$$

depends only on the geometrical and photometric properties of the planet when observed at full phase. The quantity p is itself a (kind of) albedo since α^2**F** can be seen as the intensity, scattered back towards the source, of a normally irradiated *lossless* (ϖ_0=1) Lambertian disc of the same radius as the planet. The factor p is called the *geometrical albedo*. [When albedo is used without qualification in the context of the photometry of asteroids it (usually) means geometrical albedo, in particular that observed in the Johnson V-band, p_V, the *visual geometrical albedo*].

For the reflectance rules we have considered so far, *i.e.* Lambert's law and the Lommel-Seeliger law, analytical expressions for A, p and q are readily found, as summarised in Table II.

Table II. Properties of Spheres

	Lambertian	Lommel-Seeliger
q	$\frac{3}{2}$	$\frac{16}{3}(1-\ln 2)$
p	$2\varpi_0/3$	$\varpi_{0/8}$
A	ϖ_0	$\frac{3}{2}\varpi_0(1-\ln 2)$

More complicated reflectance laws, in particular those which address the problem of the *opposition effect* for atmosphereless bodies do not readily lend themselves to analytical solutions. In general, such laws exhibit a BRDF which depends on phase angle α and a possible set of *reflectance parameters*, symbolised by the ellipsis, so that the BRDF would be generally expressed in the form

$$f_r = f_r(\mu_0, \mu, \alpha; \ldots), \tag{2.9.6}$$

where the dependence on φ and φ_0 has been replaced by α, the angle between the incident and scattered radiation, *i.e.* α does not always mean *solar* phase angle.

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2.10: A, p and q for General Reflectance Rules

Again, consider a sphere of radius a centred in an O*xyz* frame with corresponding directional spherical coordinates (Θ, Φ), and let the sphere be irradiated with flux density **F** from the *z*-direction. For the geometrical albedo the phase angle α is zero and the incident and reflected radiation are given by $\mu_0 = \mu = \cos\Theta$, so that

$$p=\int_0^{2\pi}\int_0^{\pi/2} f_r(\cos\theta,\cos\Theta,0;\ldots)\cos^2\Theta\sin\theta d\Theta d\Phi, \tag{2.10.1}$$

resulting in

$$p=2\pi\int_0^1 f_r(\mu,\mu,0;\ldots)\mu^2 d\mu. \tag{2.10.2}$$

Using the same geometry for the Bond albedo, for each point on the irradiated hemisphere we have $\mu_0 = \cos\Theta$, so that the directional hemispherical reflectance is

$$\rho\left(\mu_0\right)=\int_0^{2\pi}\int_0^1 f_r\left(\mu_0,\mu,\alpha;\ldots\right)\mu d\mu d\phi_r, \tag{2.10.3}$$

where the phase angle is that between the incident and reflected radiation at each stage of the integral,

$$\cos\alpha=\mu_0\mu+\sqrt{\left(1-\mu_0^2\right)\left(1-\mu^2\right)}\cos\phi_r, \tag{2.10.4}$$

where φ_r is the azimuth of the reflected radiation. The Bond albedo is then given by

$$A=\frac{1}{\pi}\int_0^{2\pi}\int_0^{\pi/2}\rho(\cos\Theta)\cos\Theta\sin\Theta d\Theta d\Phi, \tag{2.10.5}$$

which reduces to

$$A=2\int_0^t\rho(\mu)\mu d\mu. \tag{2.10.6}$$

The phase integral, equation (20), may be computed from equations (14), (15) and (16), the factor a^2 **F** disappearing in the process, so that we may write, for the purposes of computation, equation (16) as

$$I(\alpha)=\int_{\alpha-\pi/2}^{\pi/2}\int_0^{\pi} f_r\left(\mu_0,\mu,\alpha;\ldots\right)\mu_0\mu\sin\Theta d\Theta d\Phi, \tag{2.10.7}$$

where it can be seen that for Φ the range of integration is from α - π/2, the limb, to π/2, the terminator.

In these equations it can be seen that the geometrical albedo is just a single integral and thus may be quickly and accurately integrated numerically with just about any method. The Bond albedo and the phase integral are, however, triple integrals, so that a method which combines the advantages of speed and accuracy is required; for this reason *Gaussian Quadrature* is the chosen method. In the following section we present this method in algorithmic form and discuss its application to the integrals at hand.

For the theory and examples of Gaussian Quadrature, its performance compared to other methods of integration as well as tabulations of the *roots* and *coefficients* needed, the reader is referred to astrowww.phys.uvic.ca/~tatum/ *Celestial Mechanics*, Chap. 1.

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2.11: Gaussian Triple Integral Algorithm

To approximate the integral

$$I=\int_a^b\int_c^d\int_e^f F(x,y,z)dzdydx \tag{2.11.1}$$

where it is assumed that the roots R and coefficients C are stored in two-dimensional arrays.

```
BEGIN
    h1 = (b – a)/2
    h2 = (b + a)/2
    I = 0
    FOR i = 1, 2,..., m DO
        Ix = 0
        x = h1*R[m][i] + h2
        k1 = (d – c)/2
        k2 = (d + c)/2
        FOR j = 1, 2,..., n DO
            Iy = 0
            y = k1*R[n][i] + k2
            l1 = (f – e)/2
            l2 = (f + e)/2
            FOR k = 1, 2,..., p DO
                z = l1*R[p][k] + l2
                Iy = Iy + C[p][k]*F(x, y, z)
            END FOR { k-loop }
            Ix = Ix + C[n][j]*l1*Iy
        END FOR { j-loop }
        I = I + C[m][i]*k1*Ix
    END FOR { i-loop }
    I = h1*I
    PRINT I
END
```

This algorithm may be generalised further by allowing limits e and f to be functions e(x,y) and f(x,y) and c and d to be functions c(x) and d(x). For our purposes the limits of integration are fixed values.

Applying this algorithm to equation (28) for the Bond albedo and identifying μ with x, we see that

$$\frac{A}{2}=\int_0^1\int_0^{2\pi}\int_0^1 \times f_r(x,\mu,\alpha;\ldots)\mu d\mu d\phi dx \tag{2.11.2}$$

and by further identifying *z* with μ and *y* with φ

$$\mathrm{F(x,y,z)=2xz}f_r(\mathrm{x,z},\alpha;\ldots) \tag{2.11.3}$$

where a is itself a function of x,y and z [cf. equation (26)]

$$\alpha=\cos^{-1}\left[\mathrm{xz}+\sqrt{\left(1-\mathrm{x}^{2}\right)\left(1-\mathrm{z}^{2}\right)}\cos \mathrm{y}\right]. \tag{2.11.4}$$

For the phase integral, there is no need to invoke the likes of equation (32) since the intensity I(α) is explicitly expressed in terms of α and one stage of the integration is with respect to α. The parameters, … , are, of course, not variables since they retain their values for the duration of the integration.

When applying these integrals it is strongly suggested that *A*, *p* and *q* each be calculated independently in order to verify that the relationship *A* = *p q* holds. Taking shortcuts may bury insidious bugs, some possibly as simple as a typo., inside a program and result in at least two undetected erroneous results.

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2.12: Summary of Photometric Quantities

With this chapter we have completed the description of the basic photometric quantities used in planetary photometry (although we have yet to embrace *magnitude*). These are summarised in Table III, in which those names in the first column correspond to those in *standard usage*, the exception being flux density **F**. The third and fourth columns correspond to standard symbols and units. In the second column may be found some names commonly, and not so commonly, used in astronomical literature.

Table III. Photometric Quantities

Name	Synonyms	Symbol	SI Units
Radiant Flux	Radiant Power	P, Φ	W
Radiant Flux Density	Collimated Intensity	$\mathbf{F}$	$W \cdot m^{-2}$
Irradiance	Insolation	E	$W \cdot m^{-2}$
Exitance	Emittance	M	$W \cdot m^{-2}$
Radiance	Surface Brightness Specific Intensity Intensity	L	$W \cdot m^{-2}\ sr^{-1}$
Intensity	Integrated Brightness	I	$W \cdot sr^{-1}$

The author has seen the term "collimated intensity" used by only one author (Hapke) when referring to a plane parallel beam, and he finds it a more meaningful term than "flux density", so much so that in standard usage the term "collimated radiance" would make a splendid alternative.

The symbols have been used in their most general sense, without any subscripting or other embellishments so that *e.g.* L could mean L_λ, the radiance in the wavelength interval $[\lambda, \lambda + d\lambda]$, or L_V, the "visual radiance" in the Johnson V-band or indeed it could mean the radiance integrated over all wavelengths, the "bolometric radiance".

Reference Notes.

Much of the content of this chapter is an adaptation from, and an extension to, the *Theory of Planetary Photometry* by

1. Lester. P. L., McCall, M. L. & Tatum, J. B., 1979, *J. Roy. Astron. Soc. Can.*, **73**, 233.

Further definitions, and interesting insights into the photometric quantities and standard usage may be found in the above reference, as well as in

2. astrowww.phys.uvic.ca/~tatum/ *Stellar Atmospheres, Chap. 1*.

Sections 9 and 10 are based on an article by the author

3. Fairbairn, M. B., 2004, *J. Roy. Astron. Soc. Can.*, **98**, 149

in which a numerical example may be found in the appendix.

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CHAPTER OVERVIEW

3: A Brief History of the Lommel-Seeliger Law

Thumbnail: Surface of Mercury (Public Domain; NASA).

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3.1: A Brief History of the Lommel-Seeliger Law

Introduction. The use of the Lommel-Seeliger law is an enduring aspect of planetary photometry. It has the advantage of analytical simplicity as well as, in many cases, being an excellent first approximation to diffuse reflection. In spite of its shortcomings, in particular its inability to display an opposition effect, it is still very much in use today in applications as diverse as lightcurve inversion (the determination of asteroid poles and shapes from their lightcurves (Kaasalainen, 2003)), to the prediction of photometric signatures of unresolved ringed extrasolar planets (Arnold & Schneider, 2004). Indeed, it is the topic of exoplanets which has recently generated an interest in planetary photometry by astronomers who would otherwise not be concerned with the subject.

Here we present the Lommel-Seeliger law in some detail and as a result point out the existence and consequences of an insidious error, which has percolated down through the literature.

Description. The Lommel-Seeliger law is based on a simple physical model of diffuse reflection. As such it is a *single scattering* model in which the scattering is *isotropic*.

The model assumes that light penetrates the surface, being attenuated exponentially as it does so. Here *attenuation* refers to any process which reduces the brightness of a beam of light, and thus includes scattering and absorption. Each element of volume encountered by the attenuated beam scatters part of it isotropically, *i.e.* equally in all directions into the 4π steradians (the imaginary sphere, if you like) surrounding it. Thus, of this diffuse scattered radiation, only *half* is directed back towards the surface, and this fraction will be further attenuated before emerging as diffuse reflected radiation.

Derivation. The following derivation is intended to be more illustrative than entirely rigorous and contains a few shortcuts. It is nonetheless correct; for a more robust and general proof see Chapter 1.

Consider, as shown in Figure $3.1.1$, a diffuse reflecting (and transmitting) layer of *normal optical thickness* t, in which the optical thickness includes attenuation by both scattering and absorption. In problems of this nature, it is more convenient to work in terms of optical thickness than actual physical thickness. Light traversing a path of optical thickness τ is attenuated by a factor $e^{-\tau}$.

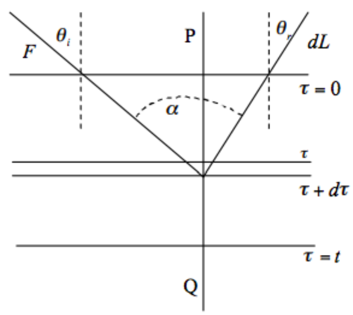

Figure $3.1.1$: In this diagram, the incident beam, the line PQ and the reflected beam need not be in the same plane; imagine one half rotated about PQ with respect to the other half. The angle between the incident and reflected beams is the (solar) phase angle, α. The value of α is relevant only for anisotropic scattering.

The surface (τ = 0) is irradiated by a plane parallel beam of radiant flux density F at an angle of incidence θ_i, so that the irradiance is $E = F\cos\theta)i$. We are concerned with the resulting radiance in the direction of an angle of reflection $\theta_r < 90^o$. Now let $\mu_0 = \cos\theta_i$ and $\mu = \cos\theta_r$ and consider the layer between τ and $\tau + d\tau$. The incident flux density which has penetrated to this level is $Fe^{-\tau/\mu_0}$.

The contribution to the diffuse radiance in the direction μ by isotropic scattering is thus $\dfrac{\varpi_0}{4\pi}Fe^{-\tau/\mu_0}\dfrac{d\tau}{\mu}$ where ϖ_0 is the single scattering albedo. This radiation will be further attenuated by the factor $e^{-\tau/\mu}$ before emerging from the surface, so that the contribution to the radiance in the direction μ is

$$dL = \frac{\varpi_0}{4\pi}Fe^{-\tau/\mu_0}\frac{d\tau}{\mu}e^{-\tau/\mu}. \tag{3.1.1}$$

Note that dL is the contribution to the total radiance from the layer resulting from *single scattering*. The Lommel-Seeliger model considers only the scattering of the collimated incident light. It does not take into account scattering of diffuse light which has made its way indirectly to the same position by being scattered one or more times, *i.e.* it does not consider *multiple scattering*.

For a planetary surface, the layer is "semi-infinite" ($t=\infty$) and the total radiance in the direction μ is

$$L=\frac{\varpi_0 F}{4\pi\mu}\times\int_0^\infty \exp\left[-\tau\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)\right]d\tau. \tag{3.1.2}$$

resulting in

$$L=\frac{\varpi_0 F}{4\pi\mu}\frac{\mu_0\mu}{\mu+\mu_0}; \tag{3.1.3}$$

and, since the irradiance is $E=F\mu_0$ and $L=f_r E$, it follows that the **bidirectional reflectance distribution function** (BRDF) which defines the Lommel -Seeliger reflectance rule is

$$f_r=\frac{\varpi_0}{4\pi}\frac{1}{\mu+\mu_0}. \tag{3.1.4}$$

The use of the Lommel-Seeliger model is not restricted to planetary surfaces. For a layer of finite optical thickness *t*, *e.g*. an (exo)planetary ring, the reflected radiance is

$$L_R=\frac{\varpi_0 F}{4\pi\mu}\times\int_0^t \exp\left[-\tau\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)\right]d\tau, \tag{3.1.5}$$

resulting in

$$L_R=\frac{\varpi_0}{4\pi}\frac{1}{\mu+\mu_0}\times\left[1-\exp\left\{-t\left(\frac{1}{\mu_0}+\frac{1}{\mu}\right)\right\}\right]\mu_0 F; \tag{3.1.6}$$

and by similar reasoning the radiance L_T transmitted through the layer may be determined (Arnold & Schneider, 2004).

Errors in the Literature. The oldest error which the author has been able to detect dates back to 1916 in a paper on planetary albedos by (no less than) Henry Norris Russell (Russell, 1916). In that paper the BRDF is implicitly expressed as

$$f_r=\frac{\gamma}{\mu_0+\mu} \tag{3.1.7}$$

in which γ is a constant. From this Russell derives the directional hemispherical reflectance (hemispherical albedo)

$$\rho(\mu_0)=2\pi\gamma\quad\times\quad[1-\mu_0\ln(1+1/\mu_0)], \tag{3.1.8}$$

where it may be seen that the expression in brackets varies monotonically from 0.308 (μ_0 = 1) to unity (μ_0 = 0); he then argues " Since (ρ) can never exceed unity it follows that πγ cannot be greater 0.5 nor (the Bond albedo) *A* than 0.409. Hence a planet for which (the geometrical albedo) *p* exceeds 0.25 cannot reflect light in strict accordance with the Lommel-Seeliger law".

Although this argument sounds entirely plausible, it is wrong. While it is true that ρ, like any albedo, cannot exceed unity, in the case of the Lommel-Seeliger law it cannot exceed $\frac{1}{2}$ and therefore the maximum value of γ is $\frac{1}{4\pi}$ not $\frac{1}{2\pi}$; the value of γ is $\frac{\varpi_0}{4\pi}$. Such an error can affect albedo calculations by a fact or of two. Unfortunately, this error has filtered down into subsequent publications, e.g. Lester, McCall & Tatum (1979), Fairbairn (2002, 2004). The correct properties of the LommelSeeliger law are summarised in Table I.

Table I. Properties of the Lommel-Seeliger law for surfaces and spheres. Maximum possible values are shown in parentheses in the first and third columns. pn is the normal albedo and q the phase integral.

Properties of Surfaces	the Lommel-Seeliger Reflectance Law	Spheres	
f_r	$\frac{\varpi_0}{4\pi}\frac{1}{\mu_0+\mu}$	q	$\frac{16}{3}(1-\ln 2)$
ρ (0.5)	$\frac{\varpi_0}{2}[1-\mu_0\ln(1+1/\mu_0)]$	p (0.125)	$\frac{\varpi_0}{8}$
p_n (0.125)	$\frac{\varpi_0}{8}$	A (0.2046)	$\frac{2}{3}\varpi_0(1-\ln 2)$

For some applications, the error of a factor of two is of no consequence. In cases where only relative magnitudes matter, so that the offset is arbitrary, the factor disappears into the offset. Asteroid lightcurve profiles are such examples.

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