2.7.2: Solving Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve inequalities using the Subtraction and Addition Properties of inequality
- Solve inequalities using the Division and Multiplication Properties of inequality
- Solve inequalities that require simplification
- Translate to an inequality and solve

Solve Inequalities using the Subtraction and Addition Properties of Inequality

The Subtraction and Addition Properties of Equality state that if two quantities are equal, when we add or subtract the same amount from both quantities, the results will be equal.

PROPERTIES OF EQUALITY

$$\begin{array}{ll} \textbf{Subtraction Property of Equality} & \textbf{Addition Property of Equality} \\ \text{For any numbers } a, b, \text{ and } c, & \text{For any numbers } a, b, \text{ and } c \\ \text{if} \qquad a=b, & \text{if} \qquad a=b \\ \text{ then } a-c=b-c. & \text{ then } a+c=b+c \end{array} \tag{2.7.2.1}$$

Similar properties hold true for inequalities.

Table 2.7.2.1

For example, we know that −4 is less than 2.	$-4<2$
If we subtract 5 from both quantities, is the left side still less than the right side?	$-4-5\ ?\ 2-5$
We get −9 on the left and −3 on the right.	$-9\ ?\ -3$
And we know −9 is less than −3.	$-9<-3$
	The inequality sign stayed the same.

Similarly we could show that the inequality also stays the same for addition.

This leads us to the Subtraction and Addition Properties of Inequality.

PROPERTIES OF INEQUALITY

$$\begin{array}{ll} \textbf{Subtraction Property of Inequality} & \textbf{Addition Property of Inequality} \\ \text{For any numbers } a, b, \text{ and } c, & \text{For any numbers } a, b, \text{ and } c \\ \text{if} \qquad a<b & \text{if} \qquad a<b \\ \text{ then } a-c<b-c. & \text{ then } a+c<b+c \\ & \\ \text{if} \qquad a>b & \text{if} \qquad a>b \\ \text{ then } a-c>b-c. & \text{ then } a+c>b+c \end{array} \tag{2.7.2.2}$$

We use these properties to solve inequalities, taking the same steps we used to solve equations. Solving the inequality $x+5>9$, the steps would look like this:

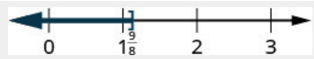
$$\begin{array}{rrcl} & x+5 & > & 9 \\ \text{Subtract 5 from both sides to isolate } x. & x+5-5 & > & 9-5 \\ & x & > & 4 \end{array} \tag{2.7.2.3}$$

Any number greater than 4 is a solution to this inequality.

? Exercise 2.7.2.7

Solve the inequality $n-\frac{1}{2} \leq \frac{5}{8}$, graph the solution on the number line, and write the solution in interval notation.

Answer

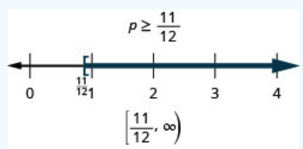
	$n-\frac{1}{2} \leq \frac{5}{8}$
Add $\frac{1}{2}$ to both sides of the inequality.	$n-\frac{1}{2}+\frac{1}{2} \leq \frac{5}{8}+\frac{1}{2}$
Simplify.	$n \leq \frac{9}{8}$
Graph the solution on the number line.	
Write the solution in interval notation.	

? Exercise 2.7.2.8

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$p-\frac{3}{4} \geq \frac{1}{6}$

Answer

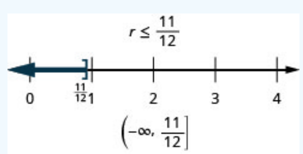


? Exercise 2.7.2.9

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$r-\frac{1}{3} \leq \frac{7}{12}$

Answer



Solve Inequalities using the Division and Multiplication Properties of Inequality

The Division and Multiplication Properties of Equality state that if two quantities are equal, when we divide or multiply both quantities by the same amount, the results will also be equal (provided we don’t divide by 0).

> PROPERTIES OF EQUALITY

$$\begin{array}{llll}\textbf{Division Property of Equality} & & \textbf{MUltiplication Property of Equality} & \\ \text{For any numbers a, b, c, and c} \neq 0 & & \text{For any numbers a, b, c} & \\ \text{if} & a=b & \text{if} & a=b \\ \text{then} & \frac{a}{c}=\frac{b}{c} & \text{then} & ac=bc\end{array} \tag{2.7.2.4}$$

Are there similar properties for inequalities? What happens to an inequality when we divide or multiply both sides by a constant?

Consider some numerical examples.

Table 2.7.2.2

	$10 < 15$		$10 < 15$
Divide both sides by 5.	$\frac{10}{5} \; ? \; \frac{15}{5}$	Multiply both sides by 5.	$10(5) \; ? \; 15(5)$
Simplify.	$2 \; ? \; 3$		$50 \; ? \; 75$
Fill in the inequality signs.	$2 < 3$		$50 < 75$

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

Table 2.7.2.3

	$10 < 15$		$10 < 15$
Divide both sides by -5.	$\frac{10}{-5} \; ? \; \frac{15}{-5}$	Multiply both sides by -5.	$10(-5) \; ? \; 15(-5)$
Simplify.	$-2 \; ? \; -3$		$-50 \; ? \; -75$
Fill in the inequality signs.	$-2 > -3$		$-50 > -75$

The inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

Here are the Division and Multiplication Properties of Inequality for easy reference.

> DIVISION AND MULTIPLICATION PROPERTIES OF INEQUALITY

For any real numbers a,b,c

$$\begin{array}{ll}\text{if } a<b \text{ and } c>0, \text{ then} & \frac{a}{c}<\frac{b}{c} \text{ and } ac<bc \\ \text{if } a>b \text{ and } c>0, \text{ then} & \frac{a}{c}>\frac{b}{c} \text{ and } ac>bc \\ \text{if } a<b \text{ and } c<0, \text{ then} & \frac{a}{c}>\frac{b}{c} \text{ and } ac>bc \\ \text{if } a>b \text{ and } c<0, \text{ then} & \frac{a}{c}<\frac{b}{c} \text{ and } ac<bc\end{array} \tag{2.7.2.5}$$

When we **divide or multiply** an inequality by a:

- **positive** number, the inequality stays the **same**.
- **negative** number, the inequality **reverses**.

Exercise 2.7.2.10

Solve the inequality $7y < 42$, graph the solution on the number line, and write the solution in interval notation.

Answer

	$7y < 42$
Divide both sides of the inequality by 7. Since $7 > 0$, the inequality stays the same.	$\frac{7y}{7} < \frac{42}{7}$
Simplify.	$y < 6$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

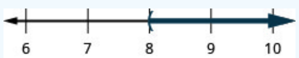
Exercise 2.7.2.11

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$9c > 72$

Answer

$c > 8$



$(8, \infty)$

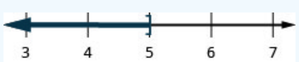
Exercise 2.7.2.12

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$12d \leq 60$

Answer

$d \leq 5$



$(-\infty, 5]$

Exercise 2.7.2.13

Solve the inequality $-10a \geq 50$, graph the solution on the number line, and write the solution in interval notation.

Answer

	$-10a \geq 50$
Divide both sides of the inequality by -10. Since $-10 < 0$, the inequality reverses.	$\frac{-10a}{-10} \leq \frac{50}{-10}$
Simplify.	$a \leq -5$
Graph the solution on the number line.	

Write the solution in interval notation.	$(-\infty, -5]$

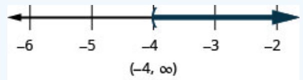
Exercise 2.7.2.14

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-8q < 32$

Answer

$q > -4$

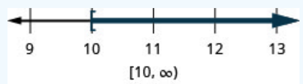


Exercise 2.7.2.15

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-7r \leq -70$

Answer



SOLVING INEQUALITIES

Sometimes when solving an inequality, the variable ends up on the right. We can rewrite the inequality in reverse to get the variable to the left.

$$x > a \text{ has the same meaning as } a < x \tag{2.7.2.6}$$

Think about it as "If Xavier is taller than Alex, then Alex is shorter than Xavier."

Exercise 2.7.2.16

Solve the inequality $-20 < \frac{4}{5}u$, graph the solution on the number line, and write the solution in interval notation.

Answer

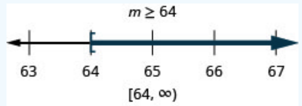
	$-20 < \frac{4}{5}u$
Multiply both sides of the inequality by $\frac{5}{4}$. Since $\frac{5}{4} > 0$, the inequality stays the same.	$\frac{5}{4}(-20) < \frac{5}{4}\left(\frac{4}{5}u\right)$
Simplify.	$-25 < u$
Rewrite the variable on the left.	$u > -25$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-25, \infty)$

Exercise 2.7.2.17

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$24 \leq \frac{3}{8}m$

Answer

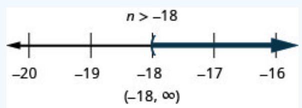


Exercise 2.7.2.18

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$-24 < \frac{4}{3}n$

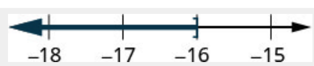
Answer



Exercise 2.7.2.19

Solve the inequality $\frac{t}{-2} \geq 8$, graph the solution on the number line, and write the solution in interval notation.

Answer

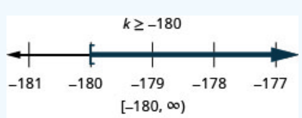
	$\frac{t}{-2} \geq 8$
Multiply both sides of the inequality by -2. Since $-2 < 0$, the inequality reverses.	$-2\left(\frac{t}{-2}\right) \leq -2(8)$
Simplify.	$t \leq -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

Exercise 2.7.2.20

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{k}{-12} \leq 15$

Answer

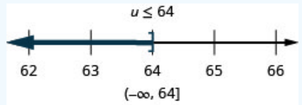


Exercise 2.7.2.21

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{u}{-4} \geq -16$

Answer



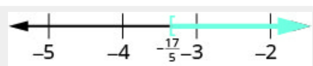
Solve Inequalities That Require Simplification

Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but be sure to pay close attention during multiplication or division.

Exercise 2.7.2.22

Solve the inequality $4m \leq 9m + 17$, graph the solution on the number line, and write the solution in interval notation.

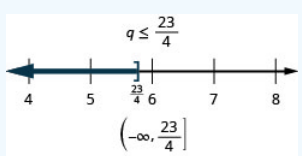
Answer

	$4m \leq 9m + 17$
Subtract 9m from both sides to collect the variables on the left.	$4m - 9m \leq 9m - 9m + 17$
Simplify.	$-5m \leq 17$
Divide both sides of the inequality by −5, and reverse the inequality.	$\frac{-5m}{-5} \geq \frac{17}{-5}$
Simplify.	$m \geq -\frac{17}{5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

Exercise 2.7.2.23

Solve the inequality $3q \geq 7q - 23$, graph the solution on the number line, and write the solution in interval notation.

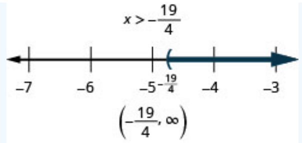
Answer



Exercise 2.7.2.24

Solve the inequality $6x < 10x + 19$, graph the solution on the number line, and write the solution in interval notation.

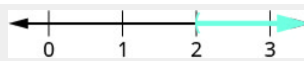
Answer



? Exercise 2.7.2.25

Solve the inequality $8p+3(p-12)>7p-28$ graph the solution on the number line, and write the solution in interval notation.

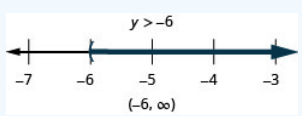
Answer

Simplify each side as much as possible.	8p+3(p−12)>7p−28
Distribute.	8p+3p−36>7p−28
Combine like terms.	11p−36>7p−28
Subtract 7p from both sides to collect the variables on the left.	11p−36−7p>7p−28−7p
Simplify.	4p−36>−28
Add 36 to both sides to collect the constants on the right.	4p−36+36>−28+36
Simplify.	4p>8
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4}>84$
Simplify.	$p>2$
Graph the solution on the number line.	
Write the solution in interval notation.	$(2,\infty)$

? Exercise 2.7.2.26

Solve the inequality $9y+2(y+6)>5y-24$, graph the solution on the number line, and write the solution in interval notation.

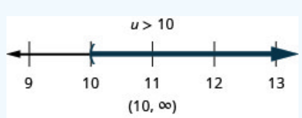
Answer



? Exercise 2.7.2.27

Solve the inequality $6u+8(u-1)>10u+32$, graph the solution on the number line, and write the solution in interval notation.

Answer



Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

Exercise 2.7.2.28

Solve the inequality $8x-2(5-x)<4(x+9)+6x$, graph the solution on the number line, and write the solution in interval notation.

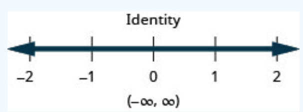
Answer

Simplify each side as much as possible.	8x−2(5−x)<4(x+9)+6x
Distribute.	8x−10+2x<4x+36+6x
Combine like terms.	10x−10<10x+36
Subtract 10x from both sides to collect the variables on the left.	10x−10−10x<10x+36−10x
Simplify.	−10<36
The xx’s are gone, and we have a true statement.	The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty,\infty)$

Exercise 2.7.2.29

Solve the inequality $4b-3(3-b)>5(b-6)+2b$, graph the solution on the number line, and write the solution in interval notation.

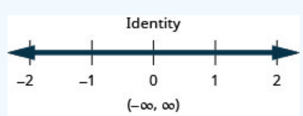
Answer



Exercise 2.7.2.30

Solve the inequality $9h-7(2-h)<8(h+11)+8h$, graph the solution on the number line, and write the solution in interval notation.

Answer



Exercise 2.7.2.31

Solve the inequality $\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

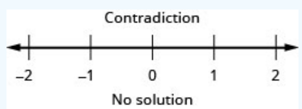
Answer

	$\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a - \frac{1}{8}a\right) > 24\left(\frac{5}{24}a + \frac{3}{4}\right)$
Simplify.	$8a - 3a > 5a + 18$
Combine like terms.	$5a > 5a + 18$
Subtract 5a from both sides to collect the variables on the left.	$5a - 5a > 5a - 5a + 18$
Simplify.	$0 > 18$
The statement is false!	The inequality is a contradiction.
	There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

? Exercise 2.7.2.32

Solve the inequality $\frac{1}{4}x - \frac{1}{12}x > \frac{1}{6}x + \frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

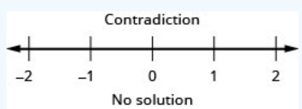
Answer



? Exercise 2.7.2.33

Solve the inequality $\frac{2}{5}z - \frac{1}{3}z < \frac{1}{15}z - \frac{3}{5}$, graph the solution on the number line, and write the solution in interval notation.

Answer



Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like ‘more than’ and ‘less than’. But others are not as obvious.

Think about the phrase ‘at least’ – what does it mean to be ‘at least 21 years old’? It means 21 or more. The phrase ‘at least’ is the same as ‘greater than or equal to’.

Table 2.7.2.4shows some common phrases that indicate inequalities.

Table 2.7.2.4

$>$	$\geq$	$<$	$\leq$
" data-valign="middle" class="lt-math-15134">is greater than	is greater than or equal to	is less than	is less than or equal to

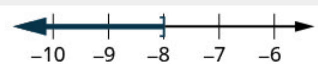
$>$	$\geq$	$<$	$\leq$
" data-valign="middle" class="lt-math-15134">is more than	is at least	is smaller than	is at most
" data-valign="middle" class="lt-math-15134">is larger than	is no less than	has fewer than	is no more than
" data-valign="middle" class="lt-math-15134">exceeds	is the minimum	is lower than	is the maximum

Exercise 2.7.2.34

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twelve times *c* is no more than 96.

Answer

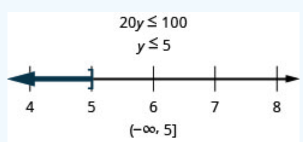
Translate.	Twelve times *c* is no more than 96 $12c \leq 96$
Solve—divide both sides by 12.	$\frac{12c}{12} \leq \frac{96}{12}$
Simplify.	$c \leq 8$
Write in interval notation.	$(-\infty, -8]$
Graph on the number line.	

Exercise 2.7.2.35

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twenty times *y* is at most 100

Answer

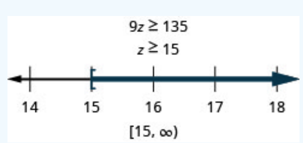


Exercise 2.7.2.36

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nine times *z* is no less than 135

Answer



Exercise 2.7.2.37

Translate and solve. Then write the solution in interval notation and graph on the number line.

Thirty less than x is at least 45.

Answer

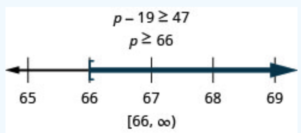
Translate.	Thirty less than x is at least 45. $x-30 \geq 45$
Solve—add 30 to both sides.	$x-30+30 \geq 45+30$
Simplify.	$x \geq 75$
Write in interval notation.	$[75, \infty)$
Graph on the number line.	

Exercise 2.7.2.38

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nineteen less than p is no less than 47

Answer

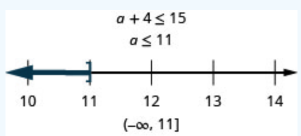


Exercise 2.7.2.39

Translate and solve. Then write the solution in interval notation and graph on the number line.

Four more than a is at most 15.

Answer



Key Concepts

- **Subtraction Property of Inequality**
  For any numbers a, b, and c,
  if $a<b$ then $a-c<b-c$ and
  if $a>b$ then $a-c>b-c$.
- **Addition Property of Inequality**
  For any numbers a, b, and c,
  if $a<b$ then $a+c<b+c$ and
  if $a>b$ then $a+c>b+c$.
- **Division and Multiplication Properties of Inequality**
  For any numbers a, b, and c,
  if $a<b$ and $c>0$, then $ac<bc$ and $ac>bc$.

if a>b and c>0, then ac>bc and ac>bc.
if a<b and c<0, then ac>bc and ac>bc.
if a>b and c<0, then ac<bc and ac<bc.

- When we **divide or multiply** an inequality by a:
  - **positive** number, the inequality stays the **same**.
  - **negative** number, the inequality **reverses**.

Practice Makes Perfect

Everyday Math

Exercise 2.7.2.75

Safety A child’s height, *h*, must be at least 57 inches for the child to safely ride in the front seat of a car. Write this as an inequality.

Exercise 2.7.2.76

Fighter pilots The maximum height, *h*, of a fighter pilot is 77 inches. Write this as an inequality.

Answer

$h \leq 77$

Exercise 2.7.2.77

Elevators The total weight, *w*, of an elevator’s passengers can be no more than 1,200 pounds. Write this as an inequality.

Exercise 2.7.2.78

Shopping The number of items, *n*, a shopper can have in the express check-out lane is at most 8. Write this as an inequality.

Answer

$n \leq 8$

Writing Exercises

Exercise 2.7.2.79

Give an example from your life using the phrase ‘at least’.

Exercise 2.7.2.80

Give an example from your life using the phrase ‘at most’.

Answer

Answers will vary.

Exercise 2.7.2.81

Explain why it is necessary to reverse the inequality when solving $-5x > 10$

Exercise 2.7.2.82

Explain why it is necessary to reverse the inequality when solving $\frac{n}{-3} < 12$

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph inequalities on the number line.			
solve inequalities using the Subtraction and Addition Properties of Inequality.			
solve inequalities using the Division and Multiplication Properties of Inequality.			
solve inequalities that require simplification.			
translate to an inequality and solve.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?