2.8.4: Properties_of_Logarithms

Learning Objectives

By the end of this section, you will be able to:

- Use the properties of logarithms
- Use the Change of Base Formula

Before you get started, take this readiness quiz.

1. Evaluate: a. a^0 b. a^1.
   If you missed this problem, review Example 5.14.
2. Write with a rational exponent: $\sqrt[3]{x^2 y}$.
   If you missed this problem, review Example 8.27.
3. Round to three decimal places: 2.5646415
   If you missed this problem, review Example 1.34.

Use the Properties of Logarithms

Now that we have learned about exponential and logarithmic functions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $a^0 = 1$, we can convert this to logarithmic form and get $\log_a 1 = 0$. Also, since $a^1 = a$, we get $\log_a a = 1$.

Definition 2.8.4.1

Properties of Logarithms

$$\log_a 1 = 0 \quad \log_a a = 1$$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

Example 2.8.4.1

Evaluate using the properties of logarithms:

a. $\log_8 1$
b. $\log_6 6$

Solution:

a.

$$\log_8 1$$

Use the property, $\log_a 1 = 0$.

$$0 \quad \log_8 1 = 0$$

b.

$$\log_6 6$$

Use the property, $\log_a a = 1$.

$$1 \quad \log_6 6 = 1$$

Exercise 2.8.4.1

Evaluate using the properties of logarithms:

a. $\log_{13} 1$
b. $\log_9 9$

Answer

a. 0
b. 1

Exercise 2.8.4.2

Evaluate using the properties of logarithms:

a. $\log_5 1$
b. $\log_7 7$

Answer

a. 0
b. 1

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation $a^{\log_a x} = x$ converts to the logarithmic equation $\log_a x = \log_a x$, which is a true statement for positive values for x only.

The logarithmic equation $\log_a a^x = x$ converts to the exponential equation $a^x = a^x$, which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power "undoes" the log and taking the log "undoes" raising to a power. These two properties show the **composition** of functions. Both ended up with the identity function which shows again that the exponential and logarithmic functions are inverse functions.

Definition 2.8.4.2

Inverse Properties of Logarithms

For $a > 0, x > 0$ and $a \neq 1$,

$$a^{\log_a x} = x \quad \log_a a^x = x$$

In the next example, apply the inverse properties of logarithms.

Example 2.8.4.2

Evaluate using the properties of logarithms:

a. $4^{\log_4 9}$
b. $\log_3 3^5$

Solution:

a.

$$4^{\log_4 9}$$

Use the property, $a^{\log_a x} = x$.

$$9 \quad 4^{\log_4 9} = 9$$

b.

$$\log_3 3^5$$

Use the property, $a^{\log_a x} = x$.

$$5 \quad \log_3 3^5 = 5$$

Exercise 2.8.4.3

Evaluate using the properties of logarithms:

a. $5^{\log_5 15}$
b. $\log_7 7^4$

Answer

a. 15
b. 4

Exercise 2.8.4.4

Evaluate using the properties of logarithms:

a. $2^{\log_2 8}$
b. $\log_2 2^{15}$

Answer

a. 8
b. 15

There are three more properties of logarithms that will be useful in our work. We know exponential functions and logarithmic function are very interrelated. Our definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents, $a^m \cdot a^n = a^{m+n}$, we see that to multiply the same base, we add the exponents. The **Product Property of Logarithms**, $\log_a M \cdot N = \log_a M + \log_a N$ tells us to take the log of a product, we add the log of the factors.

Definition 2.8.4.3

Product Property of Logarithms

If $M > 0, N > 0, \text{a} > 0$ and $\text{a} \neq 1$, then

$$\log_a (M \cdot N) = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

Example 2.8.4.3

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3 7x$
b. $\log_4 64xy$

Solution:

a.

$$\log_3 7x$$

Use the Product Property, $\log_a (M \cdot N) = \log_a M + \log_a N$.

$$\log_3 7 + \log_3 x$$

$$\log_3 7x = \log_3 7 + \log_3 x$$

b.

$$\log_4 64xy$$

Use the Product Property, $\log_a (M \cdot N) = \log_a M + \log_a N$.

$$\log_4 64 + \log_4 x + \log_4 y$$

Simplify be evaluating, $\log_4 64$.

$$3 + \log_4 x + \log_4 y$$

$$\log_4 64xy = 3 + \log_4 x + \log_4 y$$

Exercise 2.8.4.5

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3 3x$
b. $\log_2 8xy$

Answer

a. $1 + \log_3 x$
b. $3 + \log_2 x + \log_2 y$

Exercise 2.8.4.6

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_9 9x$
b. $\log_3 27xy$

Answer

a. $1 + \log_9 x$
b. $3 + \log_3 x + \log_3 y$

Similarly, in the Quotient Property of Exponents, $\frac{a^m}{a^n} = a^{m-n}$, we see that to divide the same base, we subtract the exponents. The **Quotient Property of Logarithms**, $\log_a \frac{M}{N} = \log_a M - \log_a N$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

Definition 2.8.4.4

Quotient Property of Logarithms

If $M > 0, N > 0, \mathrm{a} > 0$ and $\mathrm{a} \neq 1$, then

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

Note that $\log_a M = \log_a N \neq \log_a (M - N)$.

We use this property to write the log of a quotient as a difference of the logs of each factor.

Example 2.8.4.4

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_5 \frac{5}{7}$
b. $\log \frac{x}{100}$

Solution:

a.

$$\log_5 \frac{5}{7}$$

Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.

$$\log_5 5 - \log_5 7$$

Simplify.

$$1 - \log_5 7$$

$$\log_5 \frac{5}{7} = 1 - \log_5 7$$

b.

$$\log \frac{x}{100}$$

Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.

$$\log x - \log 100$$

Simplify.

$$\log x - 2$$

$$\log \frac{x}{100} = \log x - 2$$

Exercise 2.8.4.7

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_4 \frac{3}{4}$
b. $\log \frac{x}{1000}$

Answer

a. $\log_4 3 - 1$
b. $\log x - 3$

Exercise 2.8.4.8

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_2 \frac{5}{4}$
b. $\log \frac{10}{y}$

Answer

a. $\log_2 5 - 2$
b. $1 - \log y$

The third property of logarithms is related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$, we see that to raise a power to a power, we multiply the exponents. The **Power Property of Logarithms**, $\log_a M^p = p \log_a M$ tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

Definition 2.8.4.5

Power Property of Logarithms

If $M>0, \mathrm{a}>0, \mathrm{a} \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power as the product product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

Example 2.8.4.5

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_5 4^3$
b. $\log x^{10}$

Solution:

a.

$$\log_5 4^3$$

Use the Power Property, $\log_a M^p = p \log_a M$.

$$3 \log_5 4$$

$$\log_5 4^3 = 3 \log_5 4$$

b.

$$\log x^{10}$$

Use the Power Property, $\log_a M^p = p \log_a M$.

$$10 \log x$$

$$\log x^{10} = 10 \log x$$

Exercise 2.8.4.9

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_7 5^4$
b. $\log x^{100}$

Answer

a. $4 \log_7 5$
b. $100 \cdot \log x$

Exercise 2.8.4.10

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_2 3^7$
b. $\log x^{20}$

Answer

a. $7 \log_2 3$
b. $20 \cdot \log x$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

Properties of Logarithms

If $M>0, \mathrm{a}>0, \mathrm{a} \neq 1$ and p is any real number then,

Table 10.4.1

Property	Base a	Base e
	$\log_a 1=0$	$\ln 1=0$
	$\log_a a=1$	$\ln e=1$
Inverse Properties	$a^{\log_a x}=x$ $\log_a a^x=x$	$e^{\ln x}=x$ $\ln e^x=x$
Product Property of Logarithms	$\log_a(M \cdot N)=\log_a M+\log_a N$	$\ln(M \cdot N)=\ln M+\ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N}=\log_a M-\log_a N$	$\ln \frac{M}{N}=\ln M-\ln N$
Power Property of Logarithms	$\log_a M^p=p \log_a M$	$\ln M^p=p \ln M$

Now that we have the properties we can use them to "expand" a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

Example 2.8.4.6

Use the Properties of Logarithms to expand the logarithm $\log_4\left(2x^3y^2\right)$. Simplify, if possible.

Solution:

Use the Product Property, $\log_a M \cdot N=\log_a M+\log_a N$.

Use the Power Property, $\log_a M^p=p \log_a M$, on the last two terms. Simplify.

Exercise 2.8.4.11

Use the Properties of Logarithms to expand the logarithm $\log_2\left(5x^4y^2\right)$. Simplify, if possible.

Answer

$\log_2 5+4\log_2 x+2\log_2 y$

Exercise 2.8.4.12

Use the Properties of Logarithms to expand the logarithm $\log_3\left(7x^5y^3\right)$. Simplify, if possible.

Answer

$\log_3 7+5\log_3 x+3\log_3 y$

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

Example 2.8.4.7

Use the Properties of Logarithms to expand the logarithm $\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$. Simplify, if possible.

Solution

$$\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$$

Rewrite the radical with a rational exponent.

$$\log_2 \left(\frac{x^3}{3y^2z}\right)^{\frac{1}{4}}$$

Use the Power Property, $\log_a M^p = p\log_a M$.

$$\frac{1}{4}\log_2\left(\frac{x^3}{3y^2z}\right)$$

Use the Quotient Property, $\log_a M \cdot N = \log_a M - \log_a N$.

$$\frac{1}{4}\left(\log_2\left(x^3\right) - \log_2\left(3y^2z\right)\right)$$

Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$, in the second term.

$$\frac{1}{4}\left(\log_2\left(x^3\right) - \left(\log_2 3 + \log_2 y^2 + \log_2 z\right)\right)$$

Use the Power Property, $\log_a M^p = p\log_a M$, inside the parentheses.

$$\frac{1}{4}(3\log_2 x - (\log_2 3 + 2\log_2 y + \log_2 z))$$

Simplify by distributing.

$$\frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z)$$

$$\log_2 \sqrt[4]{\frac{x^3}{3y^2z}} = \frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z)$$

? Exercise 2.8.4.13

Use the Properties of Logarithms to expand the logarithm $\log_4 \sqrt[5]{\frac{x^4}{2y^3z^2}}$. Simplify, if possible.

Answer

$\frac{1}{5}\left(4\log_4 x - \frac{1}{2} - 3\log_4 y - 2\log_4 z\right)$

? Exercise 2.8.4.14

Use the Properties of Logarithms to expand the logarithm $\log_3 \sqrt[3]{\frac{x^2}{5yz}}$. Simplify, if possible.

Answer

$\frac{1}{3}(2\log_3 x - \log_3 5 - \log_3 y - \log_3 z)$

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

✓ Example 2.8.4.8

Use the Properties of Logarithms to condense the logarithm $\log_4 3 + \log_4 x - \log_4 y$. Simplify, if possible.

Solution:

The log expressions all have the same base, 4.

The first two terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M : N$.

Since the logs are subtracted, we use the Quotient Property, $\log_a M - \log_a N = \log_a \frac{M}{N}$.

Exercise 2.8.4.15

Use the Properties of Logarithms to condense the logarithm $\log_2 5 + \log_2 x - \log_2 y$. Simplify, if possible.

Answer

$\log_2 \frac{5x}{y}$

Exercise 2.8.4.16

Use the Properties of Logarithms to condense the logarithm $\log_3 6 - \log_3 x - \log_3 y$. Simplify, if possible.

Answer

$\log_3 \frac{6}{xy}$

Example 2.8.4.9

Use the Properties of Logarithms to condense the logarithm $2\log_3 x + 4\log_3(x+1)$. Simplify, if possible.

Solution:

The log expressions have the same base, 3.

$$2\log_3 x + 4\log_3(x+1)$$

Use the Power Property, $\log_a M + \log_a N = \log_a M \cdot N$.

$$\log_3 x^2 + \log_3(x+1)^4$$

The terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.

$$\log_3 x^2(x+1)^4$$

$$2\log_3 x + 4\log_3(x+1) = \log_3 x^2(x+1)^4$$

Exercise 2.8.4.17

Use the Properties of Logarithms to condense the logarithm $3\log_2 x + 2\log_2(x-1)$. Simplify, if possible.

Answer

$\log_2 x^3(x-1)^2$

Exercise 2.8.4.18

Use the Properties of Logarithms to condense the logarithm $2\log x + 2\log(x+1)$. Simplify, if possible.

Answer

$\log x^2(x+1)^2$

Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the **Change-of-Base Formula**. We will show how this is derived.

Suppose we want to evaluate $\log_a M$	$\log_a M$
Let $y = \log_a M$.	$y = \log_a M$
Rewrite the epression in exponential form.	$a^y = M$
Take the $\log_b$ of each side.	$\log_b a^y = \log_b M$
Use the Power Property.	$y \log_b a = \log_b M$
Solve for y.	$y = \frac{\log_b M}{\log_b a}$
Substiture $y = \log_a M$.	$\log_a M = \frac{\log_b M}{\log_b a}$

The Change-of-Base Formula introduces a new base b. This can be any base b we want where $b > 0, b \neq 1$. Because our calculators have keys for logarithms base 10 and base e, we will rewrite the Change-of-Base Formula with the new base as 10 or e.

Definition 2.8.4.6

Change-of-Base Formula

For any logarithmic bases a, b and $M > 0$,

$\log_a M = \frac{\log_b M}{\log_b a}$	$\log_a M = \frac{\log M}{\log a}$	$\log_a M = \frac{\ln M}{\ln a}$
new base b	new base 10	new base e

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol $(\approx)$.

Example 2.8.4.10

Rounding to three decimal places, approximate $\log_4 35$.

Solution:

Table 10.4.2

	$\log_4 35$
Use the Change-of-Base Formula.	$\log_a M = \frac{\log_b M}{\log_b a}$
Identify a and M. Choose 10 for b.	$\log_4 35 = \frac{\log 35}{\log 4}$
Enter the expression $\frac{\log 35}{\log 4}$ in the calculator using the log button for base 10. Round to three decimal places.	$\log_4 35 \approx 2.565$

Exercise 2.8.4.19

Rounding to three decimal places, approximate $\log_3 42$.

Answer

3.402

Exercise 2.8.4.20

Rounding to three decimal places, approximate $\log_5 46$.

Answer

2.379

In the previous section, we derived two important properties of logarithms, which allowed us to solve some basic exponential and logarithmic equations.

properties of logs

Inverse Properties:

$$\log_b(b^x) = x \tag{2.8.4.1}$$

$$b^{\log_b x} = x \tag{2.8.4.2}$$

Exponential Property:

$$\log_b(A^r) = r\log_b(A) \tag{2.8.4.3}$$

Change of Base:

$$\log_b(A) = \frac{\log_c(A)}{\log_c(b)} \tag{2.8.4.4}$$

While these properties allow us to solve a large number of problems, they are not sufficient to solve all problems involving exponential and logarithmic equations.

properties of logs

Sum of Logs Property:

$$\log_b(A) + \log_b(C) = \log_b(AC) \tag{2.8.4.5}$$

Difference of Logs Property:

$$\log_b(A) - \log_b(C) = \log_b\left(\frac{A}{C}\right) \tag{2.8.4.6}$$

It's just as important to know what properties logarithms do not satisfy as to memorize the valid properties listed above. In particular, the logarithm is not a linear function, which means that it does not distribute:

$$\log A + B \neq \log A + \log B. \tag{2.8.4.7}$$

To help in this process we offer a proof of Equation 2.8.4.7 to help solidify our new rules and show how they follow from properties you've already seen.

Proof

Let $a = \log_b(A)$ and $c = \log_b(C)$.

By definition of the logarithm, $b^a = A$ and $b^c = C$.

Using these expressions,

$$AC = b^a b^c$$

Using exponent rules on the right,

$$AC = b^{a+c}$$

Taking the log of both sides, and utilizing the inverse property of logs,

$$\log_b(AC) = \log_b(b^{a+c}) = a + c$$

Replacing a and c with their definition establishes the result

$$\log_b(AC) = \log_b A + \log_b C$$

The proof for the difference property is very similar.

With these properties, we can rewrite expressions involving multiple logs as a single log, or break an expression involving a single log into expressions involving multiple logs.

Example 2.8.4.1

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Solution

Using the sum of logs property on the first two terms,

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40)$$

This reduces our original expression to

$$\log_3(40)-\log_3(2)$$

Then using the difference of logs property,

$$\log_3(40)-\log_3(2)=\log_3\left(\frac{40}{2}\right)=\log_3(20)$$

Example 2.8.4.2

Evaluate $2\log(5)+\log(4)$ without a calculator by first rewriting as a single logarithm.

Solution

On the first term, we can use the exponent property of logs to write

$$2\log(5)=\log\left(5^2\right)=\log(25)$$

With the expression reduced to a sum of two logs, $\log(25)+\log(4)$, we can utilize the sum of logs property

$$\log(25)+\log(4)=\log(4\cdot 25)=\log(100)$$

Since $100=10^2$, we can evaluate this log without a calculator:

$$\log(100)=\log\left(10^2\right)=2$$

Exercise 2.8.4.1

Without a calculator evaluate by first rewriting as a single logarithm:

$$\log_2(8)+\log_2(4)$$

Answer

$$\log_2(8\cdot 4)=\log_2(32)=\log_2\left(2^5\right)=5$$

Example 2.8.4.3

Rewrite $\ln\left(\frac{x^4y}{7}\right)$ as a sum or difference of logs

Solution

First, noticing we have a quotient of two expressions, we can utilize the difference property of logs to write

$$\ln\left(\frac{x^4y}{7}\right)=\ln\left(x^4y\right)-\ln(7)$$

Then seeing the product in the first term, we use the sum property

$$\ln\left(x^4 y\right)-\ln(7)=\ln\left(x^4\right)+\ln(y)-\ln(7)$$

Finally, we could use the exponent property on the first term

$$\ln\left(x^4\right)+\ln(y)-\ln(7)=4\ln(x)+\ln(y)-\ln(7)$$

Interestingly, solving exponential equations was not the reason logarithms were originally developed. Historically, up until the advent of calculators and computers, the power of logarithms was that these log properties reduced multiplication, division, roots, or powers to be evaluated using addition, subtraction, division and multiplication, respectively, which are much easier to compute without a calculator. Large books were published listing the logarithms of numbers, such as in the table to the right. To find the product of two numbers, the sum of log property was used. Suppose for example we didn't know the value of 2 times 3. Using the sum property of logs:

value	log(value)
1	0.0000000
2	0.3010300
3	0.4771213
4	0.6020600
5	0.6989700
6	0.7781513
7	0.8450980
8	0.9030900
9	0.9542425
10	1.0000000

$$\log(2\cdot 3)=\log(2)+\log(3)$$

Using the log table,

$$\log(2\cdot 3)=\log(2)+\log(3)=0.3010300+0.4771213=0.7781513$$

We can then use the table again in reverse, looking for 0.7781513 as an output of the logarithm. From that we can determine:

$$\log(2\cdot 3)=0.7781513=\log(6).$$

By using addition and the table of logs, we were able to determine$2\cdot 3=6$.

Likewise, to compute a cube root like $\sqrt[3]{8}$

$$\log(\sqrt[3]{8})=\log\left(8^{1/3}\right)=\frac{1}{3}\log(8)=\frac{1}{3}(0.9030900)=0.3010300=\log(2)$$

So $\sqrt[3]{8}=2$.

Although these calculations are simple and insignificant, they illustrate the same idea that was used for hundreds of years as an efficient way to calculate the product, quotient, roots, and powers of large and complicated numbers, either using tables of logarithms or mechanical tools called slide rules.

These properties still have other practical applications for interpreting changes in exponential and logarithmic relationships.

Example 2.8.4.4

Recall that in chemistry, the pH scale is used for quantifying acidic

$$pH=-\log\left(\left[H^+\right]\right).$$

If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid, so $P=-\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH=-\log(2C)$$

Using the sum property of logs,

$$pH=-\log(2C)=-(\log(2)+\log(C))=-\log(2)-\log(C)$$

Since $P=-\log(C)$, the new pH is

$$pH=P-\log(2)=P-0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by 0.301.

Log properties in Solving Equations

The logarithm properties often arise when solving problems involving logarithms. First, we'll look at a simpler log equation.

Example 2.8.4.5

Solve $\log(2x-6)=3$.

Solution

To solve for x, we need to get it out from inside the log function. There are two ways we can approach this.

Method 1: Rewrite as an exponential.

Recall that since the common log is base 10, $\log(A)=B$ can be rewritten as the exponential $10^B=A$. Likewise, $\log(2x-6)=3$ can be rewritten in exponential form as

$$10^3=2x-6$$

Method 2: Exponentiate both sides.

If $A=B$, then $10^A=10^B$. Using this idea, since $\log(2x-6)=3$, then $10^{\log(2x-6)}=10^3$. Use the inverse property of logs to rewrite the left side and get $2x-6=10^3$.

Using either method, we now need to solve $2x-6=10^3$. Evaluate 10^3 to get

$$2x-6=1000$$

Add 6 to both sides

$$2x=1006$$

Divide both sides by 2

$$x=503$$

Occasionally the solving process will result in extraneous solutions – answers that are outside the domain of the original equation. In this case, our answer looks fine.

Example 2.8.4.6

Solve $\log(50x+25)-\log(x)=2$.

Solution

In order to rewrite in exponential form, we need a single logarithmic expression on the left side of the equation. Using the difference property of logs, we can rewrite the left side:

$$\log\left(\frac{50x+25}{x}\right)=2$$

Rewriting in exponential form reduces this to an algebraic equation:

$$\frac{50x+25}{x}=10^2=100$$

Multiply both sides by x

$$50x+25=100x$$

Combine like terms

$$25=50x$$

Divide by 50

$$x=\frac{25}{50}=\frac{1}{2}$$

Checking this answer in the original equation, we can verify there are no domain issues, and this answer is correct.

? Exercise 2.8.4.2

Solve $\log(x^2-4)=1+\log(x+2)$.

Answer

$$\log(x^2-4)=1+\log(x+2)$$

Move both logs to one side

$$\log\left(x^2-4\right)-\log(x+2)=1$$

Use the difference property of logs

$$\log\left(\frac{x^2-4}{x+2}\right)=1$$

Factor

$$\log\left(\frac{(x+2)(x-2)}{x+2}\right)=1$$

Simplify

$$\log(x-2)=1$$

Rewrite as an exponential

$$10^1=x-2$$

Add 2 to both sides

$$x=12$$

✓ Example 2.8.4.7

Solve $\ln(x+2)+\ln(x+1)=\ln(4x+14)$.

Solution

$$\ln(x+2)+\ln(x+1)=\ln(4x+14)$$

Use the sum of logs property on the right

$$\ln((x+2)(x+1))=\ln(4x+14)$$

Expand

$$\ln\left(x^2+3x+2\right)=\ln(4x+14)$$

We have a log on both side of the equation this time. Rewriting in exponential form would be tricky, so instead we can exponentiate both sides.

$$e^{\ln\left(x^2+3x+2\right)}=e^{\ln(4x+13)}$$

Use the inverse property of logs

$$x^2+3x+2=4x+14$$

Move terms to one side

$$x^2 - x - 12 = 0$$

Factor

$$(x+4)(x-3) = 0$$

$$x = -4 \text{ or } x = 3$$

Checking our answers, notice that evaluating the original equation at $x = -4$ would result in us evaluating $\ln(-2)$, which is undefined. That answer is outside the domain of the original equation, so it is an extraneous solution and we discard it. There is one solution: $x = 3$.

More complex exponential equations can often be solved in more than one way. In the following example, we will solve the same problem in two ways – one using logarithm properties, and the other using exponential properties.

Example $2.8.4.8a$

In 2008, the population of Kenya was approximately 38.8 million, and was growing by 2.64% each year, while the population of Sudan was approximately 41.3 million and growing by 2.24% each year(World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 24, 2010). If these trends continue, when will the population of Kenya match that of Sudan?

Solution

We start by writing an equation for each population in terms of t, the number of years after 2008.

$$Kenya(t) = 38.8(1+0.0264)^t$$
$$Sudan(t) = 41.3(1+0.0224)^t$$

To find when the populations will be equal, we can set the equations equal

$$38.8(1.0264)^t = 41.3(1.0224)^t$$

For our first approach, we take the log of both sides of the equation.

$$\log\left(38.8(1.0264)^t\right) = \log\left(41.3(1.0224)^t\right)$$

Utilizing the sum property of logs, we can rewrite each side,

$$\log(38.8) + \log\left(1.0264^t\right) = \log(41.3) + \log\left(1.0224^t\right)$$

Then utilizing the exponent property, we can pull the variables out of the exponent

$$\log(38.8) + t\log(1.0264) = \log(41.3) + t\log(1.0224)$$

Moving all the terms involving t to one side of the equation and the rest of the terms to the other side,

$$t\log(1.0264) - t\log(1.0224) = \log(41.3) - \log(38.8)$$

Factoring out the t on the left,

$$t\left(\log(1.0264) - \log(1.0224)\right) = \log(41.3) - \log(38.8)$$

Dividing to solve for t

$$t = \frac{\log(41.3) - \log(38.8)}{\log(1.0264) - \log(1.0224)} \approx 15.991$$

It will be 15.991 years until the populations will be equal.

Example 2.8.4.8b

Solve the problem above by rewriting before taking the log.

Solution

Starting at the equation

$$38.8(1.0264)^t = 41.3(1.0224)^t$$

Divide to move the exponential terms to one side of the equation and the constants to the other side

$$\frac{1.0264^t}{1.0224^t} = \frac{41.3}{38.8}$$

Using exponent rules to group on the left,

$$\left(\frac{1.0264}{1.0224}\right)^t = \frac{41.3}{38.8}$$

Taking the log of both sides

$$\log\left(\left(\frac{1.0264}{1.0224}\right)^t\right) = \log\left(\frac{41.3}{38.8}\right)$$

Utilizing the exponent property on the left,

$$t\log\left(\frac{1.0264}{1.0224}\right) = \log\left(\frac{41.3}{38.8}\right)$$

Dividing gives

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} \approx 15.991 \text{ years}$$

While the answer does not immediately appear identical to that produced using the previous method, note that by using the difference property of logs, the answer could be rewritten:

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} = \frac{\log(41.3) - \log(38.8)}{\log(1.0264) - \log(1.0224)}$$

While both methods work equally well, it often requires fewer steps to utilize algebra before taking logs, rather than relying solely on log properties.

Exercise 2.8.4.3

Tank A contains 10 liters of water, and 35% of the water evaporates each week. Tank B contains 30 liters of water, and 50% of the water evaporates each week. In how many weeks will the tanks contain the same amount of water?

Answer

Tank A: $A(t) = 10(1-0.35)^t$. Tank B: $B(t) = 30(1-0.50)^t$

Solving A(t) = B(t),

$$10(0.65)^t = 30(0.5)^t$$

Using the method from Example 8b

$$\frac{(0.65)^t}{(0.5)^t} = \frac{30}{10}$$

Regroup

$$\left(\frac{0.65}{0.5}\right)^t = 3$$

Simplify

$$(1.3)^t = 3$$

Take the log of both sides

$$\log\left((1.3)^t\right) = \log(3)$$

Use the exponent property of logs

$$t\log(1.3) = \log(3)$$

Divide and evaluate

$$t = \frac{\log(3)}{\log(1.3)} \approx 4.1874 \text{ weeks}$$

Important Topics of this Section

- Inverse
- Exponential
- Change of base
- Sum of logs property
- Difference of logs property

Access these online resources for additional instruction and practice with using the properties of logarithms.

- Using Properties of Logarithms to Expand Logs
- Using Properties of Logarithms to Condense Logs
- Change of Base

Key Concepts

- $\log_a 1 = 0 \quad \log_a a = 1$
- **Inverse Properties of Logarithms**
    - For $a > 0, x > 0$ and $a \neq 1$

$$a^{\log_a x} = x \quad \log_a a^x = x$$

- **Product Property of Logarithms**
    - If $M > 0, N > 0, a > 0$ and $a \neq 1$, then,

$$\log_a M \cdot N = \log_a M + \log_a N$$

  The logarithm of a product is the sum of the logarithms.

- **Quotient Property of Logarithms**
    - If $M > 0, N > 0, \mathrm{a} > 0$ and $a \neq 1$, then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

  The logarithm of a quotient is the difference of the logarithms.

- **Power Property of Logarithms**
    - If $M > 0, a > 0, a \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power is the product of the power times the log of the number.

- **Properties of Logarithms Summary**
  If $M > 0, a > 0, a \neq 1$ and p is any real number then,

Table 10.4.1

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

- **Change-of-Base Formula**
  For any logarithmic bases a and b, and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \log_a M = \frac{\log M}{\log a} \qquad \log_a M = \frac{\ln M}{\ln a}$$

$$\text{new base } b \qquad \text{new base } 10 \qquad \text{new base } e$$