2.9.2: Differentiation Rules

Learning Objectives

- State the constant, constant multiple, and power rules.
- Apply the sum and difference rules to combine derivatives.
- Use the product rule for finding the derivative of a product of functions.
- Use the quotient rule for finding the derivative of a quotient of functions.
- Extend the power rule to functions with negative exponents.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.
- State the chain rule for the composition of two functions.
- Apply the chain rule together with the power rule.
- Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
- Recognize the chain rule for a composition of three or more functions.
- Describe the proof of the chain rule.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit.

The process that we could use to evaluate $\frac{d}{dx}(\sqrt[3]{x})$ using the definition, while similar, is more complicated.

In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

2.9.2.1 The Basic Rules

The functions $f(x)=c$ and $g(x)=x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

2.9.2.2 The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x)=c$. For this function, both $f(x)=c$ and $f(x+h)=c$, so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{c-c}{h} \\ &= \lim_{h\to 0}\frac{0}{h} \\ &= \lim_{h\to 0} 0 = 0. \end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

The Constant Rule

Let c be a constant. If $f(x)=c$, then $f'(x)=0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example 2.9.2.1: Applying the Constant Rule

Find the derivative of $f(x)=8$.

Solution

This is just a one-step application of the rule: $f'(8)=0$.

Exercise 2.9.2.1

Find the derivative of $g(x)=-3$.

Hint

Use the preceding example as a guide

Answer

0

2.9.2.3 The Power Rule

We have shown that

$$\frac{d}{dx}\left(x^2\right)=2x \quad \text{and} \quad \frac{d}{dx}\left(x^{1/2}\right)=\frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}(x^n)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x)=x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$. As we go through this derivation, pay special attention to the portion of the expression in boldface, as the technique used in this case is essentially the same as the technique used to prove the general case.

Example 2.9.2.2: Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

Solution:

$\frac{d}{dx}\left(x^3\right)=\lim\limits_{h\to 0}\frac{(x+h)^3-x^3}{h}$	
$=\lim\limits_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$	Notice that the first term in the expansion of $(x+h)^3$ is x^3 and the second term is $3x^2h$. All other terms contain powers of h that are two or greater
$=\lim\limits_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$	In this step the x^3 terms have been cancelled, leaving only terms containing h.
$=\lim\limits_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}$	Factor out the common factor of h.
$=\lim\limits_{h\to 0}(3x^2+3xh+h^2)$	After cancelling the common factor of h,the only term not containing h is $3x^2$.
$=3x^2$	Let h go to 0.

Exercise 2.9.2.2

Find $\frac{d}{dx}\left(x^4\right)$.

Hint

Use $(x+h)^4=x^4+4x^3h+6x^2h^2+4xh^3+h^4$ and follow the procedure outlined in the preceding example.

Answer

$\frac{d}{dx}\left(x^4\right)=4x^3$

As we shall see, the procedure for finding the derivative of the general form $f(x)=x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x)=x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x)=3^x$.

The Power Rule

Let n be a positive integer. If $f(x)=x^n$,then

$$f'(x)=nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}(x^n)=nx^{n-1.}$$

□

Example 2.9.2.3: Applying the Power Rule

Find the derivative of the function $f(x)=x^{10}$ by applying the power rule.

Solution

Using the power rule with $n=10$, we obtain

$$f'(x)=10x^{10-1}=10x^9.$$

Exercise 2.9.2.3

Find the derivative of $f(x)=x^7$.

Hint

Use the power rule with $n=7$.

Answer

$f'(x)=7x^6$

2.9.2.4 The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x)+g(x))=\frac{d}{dx}(f(x))+\frac{d}{dx}(g(x));$$

that is,

$$\text{for } s(x)=f(x)+g(x),\quad s'(x)=f'(x)+g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g :

$$\frac{d}{dx}(f(x)-g(x))=\frac{d}{dx}(f(x))-\frac{d}{dx}(g(x));$$

that is,

$$\text{for } d(x)=f(x)-g(x),\quad d'(x)=f'(x)-g'(x).$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}(kf(x))=k\frac{d}{dx}(f(x));$$

that is,

$$\text{for } m(x)=kf(x),\quad m'(x)=kf'(x).$$

□

Example 2.9.2.4: Applying the Constant Multiple Rule

Find the derivative of $g(x)=3x^2$ and compare it to the derivative of $f(x)=x^2$.

Solution

We use the power rule directly:

$$g'(x)=\frac{d}{dx}(3x^2)=3\frac{d}{dx}(x^2)=3(2x)=6x.$$

Since $f(x)=x^2$ has derivative $f'(x)=2x$, we see that the derivative of $g(x)$ is 3 times the derivative of $f(x)$. This relationship is illustrated in Figure 2.9.2.1.

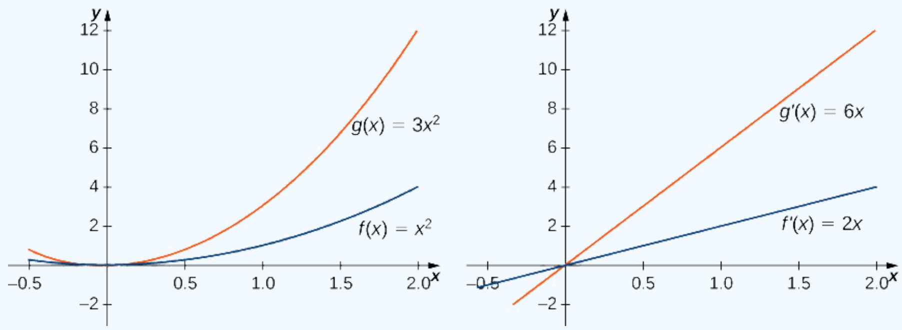

Figure 2.9.2.1: The derivative of $g(x)$ is 3 times the derivative of $f(x)$.

Example 2.9.2.5: Applying Basic Derivative Rules

Find the derivative of $f(x)=2x^5+7$.

Solution

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x^5+7) \\ &= \frac{d}{dx}(2x^5)+\frac{d}{dx}(7) && \text{Apply the sum rule.} \\ &= 2\frac{d}{dx}(x^5)+\frac{d}{dx}(7) && \text{Apply the constant multiple rule.} \\ &= 2(5x^4)+0 && \text{Apply the power rule and the constant rule.} \\ &= 10x^4 && \text{Simplify.} \end{aligned}$$

Exercise 2.9.2.4

Find the derivative of $f(x)=2x^3-6x^2+3$.

Hint

Use the preceding example as a guide.

Answer

$f'(x)=6x^2-12x$.

Example 2.9.2.6: Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x)=x^2-4x+6$ at $x=1$

Solution

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

$$f(1)=1^2-4(1)+6=3.$$

This gives us the point $(1,3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have

$$f'(x) = 2x - 4$$

so the slope of the tangent line is $f'(1) = -2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y - 3 = -2(x - 1).$$

Putting the equation of the line in slope-intercept form, we obtain

$$y = -2x + 5.$$

? Exercise 2.9.2.5

Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

Hint

Use the preceding example as a guide.

Answer

$y = 12x - 23$

2.9.2.5 The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).$$

That is,

$$\text{if } p(x) = f(x)g(x), \quad \text{then } p'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

✓ Example 2.9.2.7: Applying the Product Rule to Constant Functions

For $p(x) = f(x)g(x)$, use the product rule to find $p'(2)$ if $f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, and $g'(2) = 6$.

Solution

Since $p(x) = f(x)g(x)$, $p'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

$p'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14$.

✓ Example 2.9.2.8: Applying the Product Rule to Binomials

For $p(x) = (x^2 + 2)(3x^3 - 5x)$, find $p'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$. Thus,

$p'(x) = f'(x)g(x) + g'(x)f(x) = (2x)(3x^3 - 5x) + (9x^2 - 5)(x^2 + 2)$.

Simplifying, we have

$$p'(x) = 15x^4 + 3x^2 - 10.$$

To check, we see that $p(x) = 3x^5 + x^3 - 10x$ and, consequently, $p'(x) = 15x^4 + 3x^2 - 10$.

> **Exercise 2.9.2.6**
>
> Use the product rule to obtain the derivative of $p(x)=2x^5(4x^2+x)$.
>
> **Hint**
>
> Set $f(x)=2x^5$ and $g(x)=4x^2+x$ and use the preceding example as a guide.
>
> **Answer**
>
> $p'(x)=10x^4(4x^2+x)+(8x+1)(2x^5)=56x^6+12x^5$.

2.9.2.6 The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}(x^2)=2x,\ \text{not}\ \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)}=\frac{3x^2}{1}=3x^2.$$

> **The Quotient Rule**
>
> Let $f(x)$ and $g(x)$ be differentiable functions. Then
>
> $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}(f(x))\cdot g(x)-\frac{d}{dx}(g(x))\cdot f(x)}{\left(g(x)\right)^2}.$$
>
> That is, if
>
> $$q(x)=\frac{f(x)}{g(x)}$$
>
> then
>
> $$q'(x)=\frac{f'(x)g(x)-g'(x)f(x)}{\left(g(x)\right)^2}.$$

The proof of the quotient rule is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

> **Example 2.9.2.9: Applying the Quotient Rule**
>
> Use the quotient rule to find the derivative of $q(x)=\frac{5x^2}{4x+3}$.
>
> **Solution**
>
> Let $f(x)=5x^2$ and $g(x)=4x+3$. Thus, $f'(x)=10x$ and $g'(x)=4$.
>
> Substituting into the quotient rule, we have
>
> $$q'(x)=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}=\frac{10x(4x+3)-4(5x^2)}{(4x+3)^2}.$$
>
> Simplifying, we obtain
>
> $$q'(x)=\frac{20x^2+30x}{(4x+3)^2}$$

> **Exercise 2.9.2.7**
>
> Find the derivative of $h(x)=\frac{3x+1}{4x-3}$.
>
> **Hint**

Apply the quotient rule with $f(x)=3x+1$ and $g(x)=4x-3$.

Answer

$h'(x)=-\frac{13}{(4x-3)^2}$.

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

Extended Power Rule

If k is a negative integer, then

$$\frac{d}{dx}(x^k)=kx^{k-1}.$$

Example 2.9.2.10: Using the Extended Power Rule

Find $\frac{d}{dx}(x^{-4})$.

Solution

By applying the extended power rule with $k=-4$, we obtain

$$\frac{d}{dx}(x^{-4})=-4x^{-4-1}=-4x^{-5}.$$

Example 2.9.2.11: Using the Extended Power Rule and the Constant Multiple Rule

Use the extended power rule and the constant multiple rule to find $f(x)=\frac{6}{x^2}$.

Solution

It may seem tempting to use the quotient rule to find this derivative, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this function by first rewriting it as $f(x)=6x^{-2}$.

$$\begin{aligned} f'(x)&=\frac{d}{dx}\left(\frac{6}{x^2}\right)=\frac{d}{dx}(6x^{-2}) && \text{Rewrite } \frac{6}{x^2} \text{ as } 6x^{-2}. \\ &=6\frac{d}{dx}(x^{-2}) && \text{Apply the constant multiple rule.} \\ &=6(-2x^{-3}) && \text{Use the extended power rule to differentiate } x^{-2}. \\ &=-12x^{-3} && \text{Simplify.} \end{aligned}$$

Exercise 2.9.2.8

Find the derivative of $g(x)=\frac{1}{x^7}$ using the extended power rule.

Hint

Rewrite $g(x)=\frac{1}{x^7}=x^{-7}$. Use the extended power rule with $k=-7$.

Answer

$g'(x)=-7x^{-8}$.

2.9.2.7 Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example 2.9.2.12: Combining Differentiation Rules

For $k(x) = 3h(x) + x^2 g(x)$, find $k'(x)$.

Solution: Finding this derivative requires the sum rule, the constant multiple rule, and the product rule.

$k'(x) = \frac{d}{dx}\big(3h(x) + x^2 g(x)\big) = \frac{d}{dx}\big(3h(x)\big) + \frac{d}{dx}\big(x^2 g(x)\big)$	Apply the sum rule.
$= 3\frac{d}{dx}\big(h(x)\big) + \left(\frac{d}{dx}(x^2)g(x) + \frac{d}{dx}(g(x))x^2\right)$	Apply the constant multiple rule to differentiate $3h(x)$ and the product rule to differentiate $x^2 g(x)$.
$= 3h'(x) + 2xg(x) + g'(x)x^2$	

Example 2.9.2.13: Extending the Product Rule

For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x)$, $g(x)$, $h(x)$, and their derivatives.

Solution

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x) = (f(x)g(x)) \cdot h(x)$. Thus,

$$
\begin{aligned}
k'(x) &= \frac{d}{dx}\big(f(x)g(x)\big) \cdot h(x) + \frac{d}{dx}\big(h(x)\big) \cdot \big(f(x)g(x)\big). && \text{Apply the product rule to the product of } f(x)g(x) \text{ and } h(x). \\
&= \big(f'(x)g(x) + g'(x)f(x)\big)h(x) + h'(x)f(x)g(x) && \text{Apply the product rule to } f(x)g(x) \\
&= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x). && \text{Simplify.}
\end{aligned}
$$

Example 2.9.2.14: Combining the Quotient Rule and the Product Rule

For $h(x) = \frac{2x^3 k(x)}{3x+2}$, find $h'(x)$.

Solution

This procedure is typical for finding the derivative of a rational function.

$$
\begin{aligned}
h'(x) &= \frac{\frac{d}{dx}(2x^3k(x)) \cdot (3x+2) - \frac{d}{dx}(3x+2) \cdot (2x^3k(x))}{(3x+2)^2} && \text{Apply the quotient rule.} \\
&= \frac{(6x^2k(x) + k'(x) \cdot 2x^3)(3x+2) - 3(2x^3k(x))}{(3x+2)^2} && \text{Apply the product rule to find } \frac{d}{dx}(2x^3k(x)). \text{ Use } \frac{d}{dx}(3x+2) = 3. \\
&= \frac{-6x^3k(x) + 18x^3k(x) + 12x^2k(x) + 6x^4k'(x) + 4x^3k'(x)}{(3x+2)^2} && \text{Simplify}
\end{aligned}
$$

Exercise 2.9.2.9

Find $\frac{d}{dx}(3f(x) - 2g(x))$.

Hint

Apply the difference rule and the constant multiple rule.

Answer

$3f'(x) - 2g'(x)$.

Example 2.9.2.15: Determining Where a Function Has a Horizontal Tangent

Determine the values of x for which $f(x) = x^3 - 7x^2 + 8x + 1$ has a horizontal tangent line.

Solution

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'(x) = 0$.

Since $f'(x) = 3x^2 - 14x + 8 = (3x-2)(x-4)$,

we must solve $(3x-2)(x-4) = 0$. Thus we see that the function has horizontal tangent lines at $x = \frac{2}{3}$ and $x = 4$ as shown in the following graph.

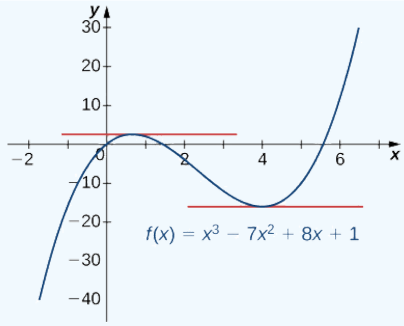

Figure 2.9.2.2: This function has horizontal tangent lines at $x = 2/3$ and $x = 4$.

Example 2.9.2.16: Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s(t) = \frac{t}{t^2+1}$. What is the initial velocity of the object?

Solution

Since the initial velocity is $v(0) = s'(0)$, begin by finding $s'(t)$ by applying the quotient rule:

$$s'(t) = \frac{1(t^2+1) - 2t(t)}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}.$$

After evaluating, we see that $v(0) = 1$.

Exercise 2.9.2.10

Find the values of x for which the line tangent to the graph of $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to the line $y = 2x + 3$.

Hint

Solve $f'(x) = 2$.

Answer

$\frac{5}{8}$

Formula One Grandstands

Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (Figure $2.9.2.3$).

Figure 2.9.2.3: The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ (Figure $2.9.2.4$). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9, 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

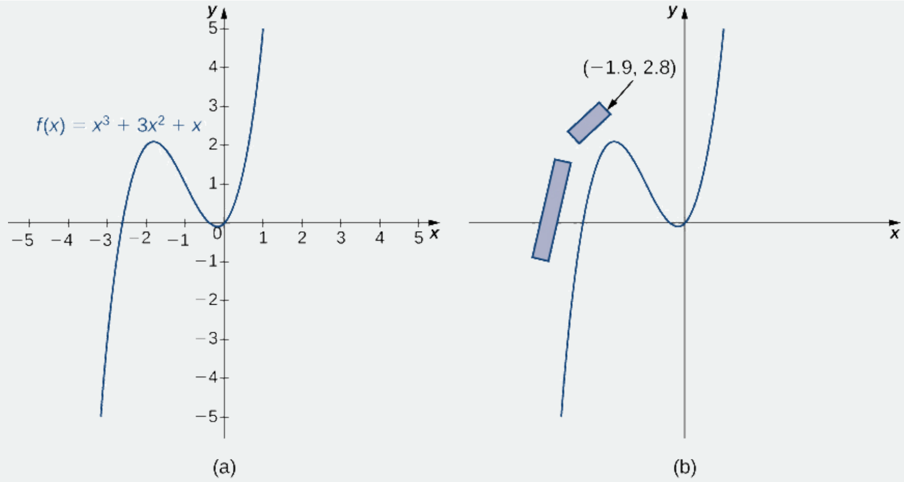

Figure 2.9.2.4: (a) One section of the racetrack can be modeled by the function $f(x)=x^3+3x^2+x$. (b) The front corner of the grandstand is located at $(-1.9, 2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x, y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the x-coordinate of the point where the tangent line crosses the line $y=2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?
4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

Learning Objectives

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We have seen the techniques for differentiating basic functions ($x^n, \sin x, \cos x$, etc.) as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x)=\sin(x^3)$ or $k(x)=\sqrt{3x^2+1}$. In this section, we study the rule for finding the derivative of the composition of two or more functions.

2.9.2.8 Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the *chain rule*, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x)=\sin(x^3)$. We can think of the derivative of this function with respect to x as the rate of change of $\sin(x^3)$ relative to the change in x. Consequently, we want to know how $\sin(x^3)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin(x^3)$. This chain reaction gives us hints as to what is involved in computing the derivative of $\sin(x^3)$. First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin(x^3)$ suggests that the derivative of $\sin(u)$ with respect to u, where $u=x^3$, is also part of the final derivative.

We can take a more formal look at the derivative of $h(x)=\sin(x^3)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x)=\sin(x^3)$.

$$h'(a)=\lim_{x\to a}\frac{\sin(x^3)-\sin(a^3)}{x-a}$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression x^3-a^3 to obtain

$$h'(a)=\lim_{x\to a}\frac{\sin(x^3)-\sin(a^3)}{x^3-a^3}\cdot\frac{x^3-a^3}{x-a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x=a$. That is,

$$\lim_{x\to a}\frac{x^3-a^3}{x-a}=\frac{d}{dx}(x^3)\Big|_{x=a}=3a^2.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u=x^3$ and observing that as $x\to a, u\to a^3$:

$$\begin{aligned}\lim_{x\to a}\frac{\sin(x^3)-\sin(a^3)}{x^3-a^3}&=\lim_{u\to a^3}\frac{\sin u-\sin(a^3)}{u-a^3}\\&=\frac{d}{du}(\sin u)\Big|_{u=a^3}\\&=\cos(a^3)\end{aligned}.$$

Thus, $h'(a)=\cos(a^3)\cdot 3a^2$.

In other words, if $h(x)=\sin(x^3)$, then $h'(x)=\cos(x^3)\cdot 3x^2$. Thus, if we think of $h(x)=\sin(x^3)$ as the composition $(f\circ g)(x)=f\big(g(x)\big)$ where $f(x)=\sin x$ and $g(x)=x^3$, then the derivative of $h(x)=\sin(x^3)$ is the product of the derivative of $g(x)=x^3$ and the derivative of the function $f(x)=\sin x$ evaluated at the function $g(x)=x^3$. At this point, we anticipate that for $h(x)=\sin\big(g(x)\big)$, it is quite likely that $h'(x)=\cos\big(g(x)\big)g'(x)$. As we determined above, this is the case for $h(x)=\sin(x^3)$.

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x)=(f\circ g)(x)=f\big(g(x)\big) \tag{2.9.2.1}$$

is given by

$$h'(x)=f'\big(g(x)\big)\cdot g'(x). \tag{2.9.2.2}$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}. \tag{2.9.2.3}$$

Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h(x)=f\big(g(x)\big)$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'\big(g(x)\big)$.
3. Find $g'(x)$.
4. Write $h'(x)=f'\big(g(x)\big)\cdot g'(x)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

2.9.2.9 The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x)=\big(g(x)\big)^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x)=\big(g(x)\big)^n$ as $f\big(g(x)\big)$ where $f(x)=x^n$. Then $f'(x)=nx^{n-1}$. Thus, $f'\big(g(x)\big)=n\big(g(x)\big)^{n-1}$. This leads us to the derivative of a power function using the chain rule,

$$h'(x)=n\big(g(x)\big)^{n-1}\cdot g'(x)$$

Rule: Power Rule for Composition of Functions (General Power Rule)

For all values of x for which the derivative is defined, if

$$h(x)=\big(g(x)\big)^n, \tag{2.9.2.4}$$

Then

$$h'(x)=n\big(g(x)\big)^{n-1}\cdot g'(x). \tag{2.9.2.5}$$

Example 2.9.2.1: Using the Chain and Power Rules

Find the derivative of $h(x) = \frac{1}{(3x^2+1)^2}$.

Solution

First, rewrite $h(x) = \frac{1}{(3x^2+1)^2} = (3x^2+1)^{-2}$.

Applying the power rule with $g(x) = 3x^2+1$, we have

$$h'(x) = -2(3x^2+1)^{-3} \cdot 6x.$$

Rewriting back to the original form gives us

$$h'(x) = \frac{-12x}{(3x^2+1)^3}$$

Exercise 2.9.2.1

Find the derivative of $h(x) = (2x^3+2x-1)^4$.

Hint

Use the General Power Rule (Equation 2.9.2.5) with $g(x) = 2x^3+2x-1$.

Answer

$h'(x) = 4(2x^3+2x-1)^3(6x^2+2) = 8(3x^2+1)(2x^3+2x-1)^3$

Example 2.9.2.2: Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

Solution

First recall that $\sin^3 x = (\sin x)^3$, so we can rewrite $h(x) = \sin^3 x$ as $h(x) = (\sin x)^3$.

Applying the power rule with $g(x) = \sin x$, we obtain

$$h'(x) = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x.$$

Example 2.9.2.3: Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \frac{1}{(3x-5)^2}$ at $x = 2$.

Solution

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2) = \frac{1}{(3(2)-5)^2} = 1$, the point is $(2, 1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x) = (3x-5)^{-2}$ and apply the power rule to obtain

$$h'(x) = -2(3x-5)^{-3}(3) = -6(3x-5)^{-3}.$$

By substituting, we have $h'(2) = -6(3(2)-5)^{-3} = -6$.

Therefore, the line has equation $y-1 = -6(x-2)$. Rewriting, the equation of the line is $y = -6x+13$.

Exercise 2.9.2.2

Find the equation of the line tangent to the graph of $f(x) = (x^2-2)^3$ at $x = -2$.

Hint

Use the preceding example as a guide.

Answer

$y = -48x-88$

2.9.2.10 Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the product rule.

Example 2.9.2.4: Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x)=\cos\big(g(x)\big)$.

Solution

Think of $h(x)=\cos\big(g(x)\big)$ as $f\big(g(x)\big)$ where $f(x)=\cos x$. Since $f'(x)=-\sin x$, we have $f'\big(g(x)\big)=-\sin\big(g(x)\big)$. Then we do the following calculation.

$$\begin{aligned} h'(x) &= f'\big(g(x)\big)\cdot g'(x) && \text{Apply the chain rule.} \\ &= -\sin\big(g(x)\big)\cdot g'(x) && \text{Substitute } f'\big(g(x)\big)=-\sin\big(g(x)\big). \end{aligned}$$

Thus, the derivative of $h(x)=\cos\big(g(x)\big)$ is given by $h'(x)=-\sin\big(g(x)\big)\cdot g'(x)$.

In the following example we apply the rule that we have just derived.

Example 2.9.2.5: Using the Chain Rule on a Cosine Function

Find the derivative of $h(x)=\cos(5x^2)$.

Solution

Let $g(x)=5x^2$. Then $g'(x)=10x$. Using the result from the previous example,

$$h'(x)=-\sin(5x^2)\cdot 10x=-10x\sin(5x^2)$$

Example 2.9.2.6: Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x)=\sec(4x^5+2x)$.

Solution

Apply the chain rule to $h(x)=\sec\big(g(x)\big)$ to obtain

$$h'(x)=\sec(g(x))\tan\big(g(x)\big)\cdot g'(x).$$

In this problem, $g(x)=4x^5+2x$, so we have $g'(x)=20x^4+2$. Therefore, we obtain

$$h'(x)=\sec(4x^5+2x)\tan(4x^5+2x)(20x^4+2)=(20x^4+2)\sec(4x^5+2x)\tan(4x^5+2x).$$

Exercise 2.9.2.3

Find the derivative of $h(x)=\sin(7x+2)$.

Hint

Apply the chain rule to $h(x)=\sin\big(g(x)\big)$ first and then use $g(x)=7x+2$.

Answer

$h'(x)=7\cos(7x+2)$

At this point we provide a list of derivative formulas that may be obtained by applying the chain rule in conjunction with the formulas for derivatives of trigonometric functions. Their derivations are similar to those used in the examples above. For convenience, formulas are also given in Leibniz's notation, which some students find easier to remember. (We discuss the chain rule using Leibniz's notation at the end of this section.) It is not absolutely necessary to memorize these as separate formulas as they are all applications of the chain rule to previously learned formulas.

Using the Chain Rule with Trigonometric Functions

For all values of x for which the derivative is defined,

$\frac{d}{dx}\Big(\sin(g(x))\Big)=\cos(g(x))\cdot g'(x)$	$\frac{d}{dx}\Big(\sin u\Big)=\cos u\cdot\frac{du}{dx}$
$\frac{d}{dx}\Big(\cos(g(x))\Big)=-\sin(g(x))\cdot g'(x)$	$\frac{d}{dx}\Big(\cos u\Big)=-\sin u\cdot\frac{du}{dx}$

$\frac{d}{dx}\Big(\tan(g(x))\Big) = \sec^2(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\tan u\Big) = \sec^2 u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\cot(g(x))\Big) = -\csc^2(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\cot u\Big) = -\csc^2 u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\sec(g(x))\Big) = \sec(g(x))\tan(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\sec u\Big) = \sec u \tan u \cdot \frac{du}{dx}$
$\frac{d}{dx}\Big(\csc(g(x))\Big) = -\csc(g(x))\cot(g(x)) \cdot g'(x)$	$\frac{d}{dx}\Big(\csc u\Big) = -\csc u \cot u \cdot \frac{du}{dx}.$

Example 2.9.2.7: Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

Solution

First apply the product rule, then apply the chain rule to each term of the product.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left((2x+1)^5\right) \cdot (3x-2)^7 + \frac{d}{dx}\left((3x-2)^7\right) \cdot (2x+1)^5 && \text{Apply the product rule.} \\ &= 5(2x+1)^4 \cdot 2 \cdot (3x-2)^7 + 7(3x-2)^6 \cdot 3 \cdot (2x+1)^5 && \text{Apply the chain rule.} \\ &= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{Simplify.} \\ &= (2x+1)^4(3x-2)^6(10(3x-2)+21(2x+1)) && \text{Factor out } (2x+1)^4(3x-2)^6 \\ &= (2x+1)^4(3x-2)^6(72x+1) && \text{Simplify.} \end{aligned}$$

Exercise 2.9.2.4

Find the derivative of $h(x) = \frac{x}{(2x+3)^3}$.

Hint

Start out by applying the quotient rule. Remember to use the chain rule to differentiate the denominator.

Answer

$$h'(x) = \frac{3-4x}{(2x+3)^4}$$

2.9.2.11 Composites of Three or More Functions

We can now combine the chain rule with other rules for differentiating functions, but when we are differentiating the composition of three or more functions, we need to apply the chain rule more than once. If we look at this situation in general terms, we can generate a formula, but we do not need to remember it, as we can simply apply the chain rule multiple times.

In general terms, first we let

$$k(x) = h\Big(f\big(g(x)\big)\Big).$$

Then, applying the chain rule once we obtain

$$k'(x) = \frac{d}{dx}\Big(h\big(f(g(x))\big)\Big) = h'\Big(f\big(g(x)\big)\Big) \cdot \frac{d}{dx}\Big(f\big(g(x)\big)\Big).$$

Applying the chain rule again, we obtain

$$k'(x) = h'\Big(f\big(g(x)\big)\Big) \cdot f'\big(g(x)\big) \cdot g'(x).$$

Rule: Chain Rule for a Composition of Three Functions

For all values of x for which the function is differentiable, if

$$k(x) = h\Big(f\big(g(x)\big)\Big),$$

then

$$k'(x) = h'\Big(f\big(g(x)\big)\Big) \cdot f'\big(g(x)\big) \cdot g'(x).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

Example 2.9.2.8: Differentiating a Composite of Three Functions

Find the derivative of $k(x)=\cos^4(7x^2+1)$.

Solution

First, rewrite $k(x)$ as

$$k(x)=\left(\cos(7x^2+1)\right)^4.$$

Then apply the chain rule several times.

$$\begin{aligned} k'(x) &= 4(\cos(7x^2+1))^3\cdot\frac{d}{dx}\left(\cos(7x^2+1)\right) && \text{Apply the chain rule.}\\ &= 4(\cos(7x^2+1))^3(-\sin(7x^2+1))\cdot\frac{d}{dx}(7x^2+1) && \text{Apply the chain rule.}\\ &= 4(\cos(7x^2+1))^3(-\sin(7x^2+1))(14x) && \text{Apply the chain rule.}\\ &= -56x\sin(7x^2+1)\cos^3(7x^2+1) && \text{Simplify} \end{aligned}$$

Exercise 2.9.2.5

Find the derivative of $h(x)=\sin^6(x^3)$.

Hint

Rewrite $h(x)=\sin^6(x^3)=\left(\sin(x^3)\right)^6$ and use Example 2.9.2.8as a guide.

Answer

$h'(x)=18x^2\sin^5(x^3)\cos(x^3)$

Example 2.9.2.9: Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time t is given by $s(t)=\sin(2t)+\cos(3t)$. What is the velocity of the particle at time $t=\frac{\pi}{6}$?

Solution

To find $v(t)$, the velocity of the particle at time t, we must differentiate $s(t)$. Thus,

$$v(t)=s'(t)=2\cos(2t)-3\sin(3t).$$

To find the velocity at $t=\frac{\pi}{6}$, calculate

$$\begin{aligned} v\left(\frac{\pi}{6}\right) &= 2\cos\left(2\times\frac{\pi}{6}\right)-3\sin\left(3\times\frac{\pi}{6}\right)\\ &= 2\cos\left(\frac{\pi}{3}\right)-3\sin\left(\frac{\pi}{2}\right)\\ &= 1-3\\ &= -2 \end{aligned}$$

Proof of Chain Rule

At this point, we present a very informal proof of the chain rule. For simplicity's sake we ignore certain issues: For example, we assume that $g(x)\neq g(a)$ for $x\neq a$ in some open interval containing a. We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:

$$h'(a)=\lim_{x\to a}\frac{f\left(g(x)\right)-f\left(g(a)\right)}{x-a}. \tag{2.9.2.6}$$

Rewriting, we obtain

$$h'(a)=\lim_{x\to a}\frac{f\left(g(x)\right)-f\left(g(a)\right)}{g(x)-g(a)}\cdot\frac{g(x)-g(a)}{x-a}. \tag{2.9.2.7}$$

Although it is clear that

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a), \tag{2.9.2.8}$$

it is not obvious that

$$\lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} = f'\big(g(a)\big). \tag{2.9.2.9}$$

To see that this is true, first recall that since g is differentiable at a, g is also continuous at a. Thus,

$$\lim_{x \to a} g(x) = g(a). \tag{2.9.2.10}$$

Next, make the substitution $y = g(x)$ and $b = g(a)$ and use change of variables in the limit to obtain

$$\lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'\big(g(a)\big). \tag{2.9.2.11}$$

Finally,

$$h'(a) = \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} = f'\big(g(a)\big) \cdot g'(a). \tag{2.9.2.12}$$

□

Example 2.9.2.10: Using the Chain Rule with Functional Values

Let $h(x) = f\big(g(x)\big)$. If $g(1) = 4, g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

Solution

Use the chain rule, then substitute.

$$\begin{aligned} h'(1) &= f'\big(g(1)\big) \cdot g'(1) && \text{Apply the chain rule.} \\ &= f'(4) \cdot 3 && \text{Substitute } g(1) = 4 \text{ and } g'(1) = 3. \\ &= 7 \cdot 3 && \text{Substitute } f'(4) = 7. \\ &= 21 && \text{Simplify.} \end{aligned}$$

Exercise 2.9.2.6

Given $h(x) = f(g(x))$. If $g(2) = -3, g'(2) = 4,$ and $f'(-3) = 7$, find $h'(2)$.

Hint

Follow Example 2.9.2.10

Answer

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2.9.2.12 The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This notation for the chain rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Thus,

$$h'(x) = \frac{dy}{dx}$$

$$f'(g(x)) = f'(u) = \frac{dy}{du}$$

and

$$g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'\big(g(x)\big) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \tag{2.9.2.13}$$

Example 2.9.2.11: Taking a Derivative Using Leibniz's Notation I

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the quotient rule,

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and

$$\frac{dy}{du} = 5u^4.$$

Finally, we put it all together.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= 5u^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } \frac{dy}{du} = 5u^4 \text{ and } \frac{du}{dx} = \frac{2}{(3x+2)^2}. \\ &= 5\left(\frac{x}{3x+2}\right)^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } u = \frac{x}{3x+2}. \\ &= \frac{10x^4}{(3x+2)^6} && \text{Simplify.} \end{aligned}$$

It is important to remember that, when using the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Example 2.9.2.12: Taking a Derivative Using Leibniz's Notation II

Find the derivative of $y = \tan(4x^2 - 3x + 1)$.

Solution

First, let $u = 4x^2 - 3x + 1$. Then $y = \tan u$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$:

$$\frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u.$$

Finally, we put it all together.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= \sec^2 u \cdot (8x-3) && \text{Use } \frac{du}{dx} = 8x-3 \text{ and } \frac{dy}{du} = \sec^2 u. \\ &= \sec^2(4x^2-3x+1) \cdot (8x-3) && \text{Substitute } u = 4x^2-3x+1. \end{aligned}$$

Exercise 2.9.2.7

Use Leibniz's notation to find the derivative of $y = \cos(x^3)$. Make sure that the final answer is expressed entirely in terms of the variable x.

Hint

Let $u = x^3$.

Answer

$$\frac{dy}{dx} = -3x^2 \sin(x^3).$$

2.9.2.13 Key Concepts

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.
- The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g.
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function.
- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.
- The chain rule allows us to differentiate compositions of two or more functions. It states that for $h(x) = f\big(g(x)\big)$,

$$h'(x) = f'\big(g(x)\big) \cdot g'(x).$$

In Leibniz's notation this rule takes the form

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- We can use the chain rule with other rules that we have learned, and we can derive formulas for some of them.
- The chain rule combines with the power rule to form a new rule:

$$\text{If } h(x) = \big(g(x)\big)^n \text{, then } h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x).$$

- When applied to the composition of three functions, the chain rule can be expressed as follows: If $h(x) = f\Big(g\big(k(x)\big)\Big)$, then $h'(x) = f'\Big(g\big(k(x)\big)\Big) \cdot g'\big(k(x)\big) \cdot k'(x)$.

2.9.2.14 Key Equations

- **The chain rule**

  $h'(x) = f'\big(g(x)\big) \cdot g'(x)$

- **The power rule for functions**

  $h'(x) = n\big(g(x)\big)^{n-1} \cdot g'(x)$

2.9.2.15 Glossary

constant multiple rule

the derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative: $\frac{d}{dx}\big(cf(x)\big) = cf'(x)$

constant rule

the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0$, where c is a constant

difference rule

the derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x)$

power rule

the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

product rule

the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + g'(x)f(x)$

quotient rule

the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-g'(x)f(x)}{\big(g(x)\big)^2}$

sum rule

the derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g: $\frac{d}{dx}\big(f(x)+g(x)\big)=f'(x)+g'(x)$

chain rule

the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function