6.6: RC Circuits

Learning Objectives

By the end of the section, you will be able to:

- Describe the charging process of a capacitor
- Describe the discharging process of a capacitor
- List some applications of RC circuits

When you use a flash camera, it takes a few seconds to charge the capacitor that powers the flash. The light flash discharges the capacitor in a tiny fraction of a second. Why does charging take longer than discharging? This question and several other phenomena that involve charging and discharging capacitors are discussed in this module.

Circuits with Resistance and Capacitance

An **RC circuit** is a circuit containing resistance and capacitance. As presented in Capacitance, the capacitor is an electrical component that stores electric charge, storing energy in an electric field.

Figure $6.6.1a$ shows a simple **RC** circuit that employs a dc (direct current) voltage source e, a resistor R, a capacitor C, and a two-position switch. The circuit allows the capacitor to be charged or discharged, depending on the position of the switch. When the switch is moved to position \(**A**\), the capacitor charges, resulting in the circuit in Figure $6.6.1b$. When the switch is moved to position **B**, the capacitor discharges through the resistor.

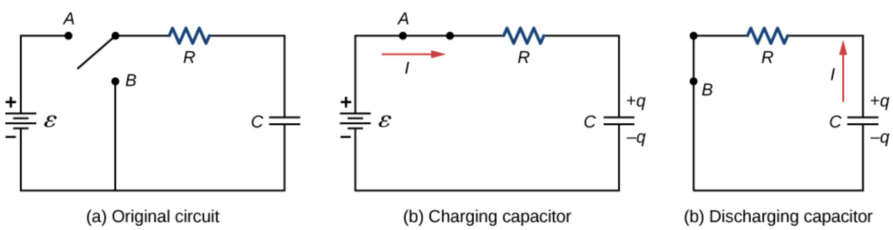


Figure 6.6.1: (a) An **RC** circuit with a two-pole switch that can be used to charge and discharge a capacitor. (b) When the switch is moved to position **A**, the circuit reduces to a simple series connection of the voltage source, the resistor, the capacitor, and the switch. (c) When the switch is moved to position **B**, the circuit reduces to a simple series connection of the resistor, the capacitor, and the switch. The voltage source is removed from the circuit.

Charging a Capacitor

We can use Kirchhoff's loop rule to understand the charging of the capacitor. This results in the equation $\epsilon - V_R - V_C = 0$. This equation can be used to model the charge as a function of time as the capacitor charges. Capacitance is defined as $C = q/V$, so the voltage across the capacitor is $V_C = \frac{q}{C}$. Using Ohm's law, the potential drop across the resistor is $V_R = IR$, and the current is defined as $I = dq/dt$.

$$\epsilon - V_R - V_C = 0, \tag{6.6.1}$$

$$\epsilon - IR - \frac{q}{C} = 0, \tag{6.6.2}$$

$$\epsilon - R\frac{dq}{dt} - \frac{q}{C} = 0. \tag{6.6.3}$$

This differential equation can be integrated to find an equation for the charge on the capacitor as a function of time.

$$\epsilon - R\frac{dq}{dt} - \frac{q}{C} = 0. \tag{6.6.4}$$

$$\frac{dq}{dt} = \frac{\epsilon C - q}{RC}, \tag{6.6.5}$$

$$\int_0^q \frac{dq}{\epsilon C - q} = \frac{1}{RC}\int_0^t dt. \tag{6.6.6}$$

Let $u = \epsilon C - q$, then $du = -dq$. The result is

$$-\int_0^q \frac{du}{u} = \frac{1}{RC}\int_0^t dt, \tag{6.6.7}$$

$$\ln\left(\frac{\epsilon C - q}{\epsilon C}\right) = -\frac{1}{RC}t. \tag{6.6.8}$$

$$\frac{\epsilon C - q}{\epsilon C} = e^{-t/RC}. \tag{6.6.9}$$

Simplifying results in an equation for the charge on the charging capacitor as a function of time:

$$q(t) = C\epsilon\left(1 - e^{-\frac{t}{RC}}\right) = Q\left(1 - e^{-\frac{t}{\tau}}\right). \tag{6.6.10}$$

A graph of the charge on the capacitor versus time is shown in Figure $6.6.2a$. First note that as time approaches infinity, the exponential goes to zero, so the charge approaches the maximum charge $Q = C\epsilon$ and has units of coulombs. The units of **RC** are seconds, units of time. This quantity is known as the **time constant**:

$$\tau = RC. \tag{6.6.11}$$

At time $t = \tau = RC$, the charge equal to $1 - e^{-1} = 1 - 0.368 = 0.632$ of the maximum charge $Q = C\epsilon$. Notice that the time rate change of the charge is the slope at a point of the charge versus time plot. The slope of the graph is large at time $t - 0.0\ s$ and approaches zero as time increases.

As the charge on the capacitor increases, the current through the resistor decreases, as shown in Figure $6.6.2b$. The current through the resistor can be found by taking the time derivative of the charge.

$$\begin{aligned} I(t) &= \frac{dq}{dt} \\ &= \frac{d}{dt}\left[C\epsilon\left(1 - e^{-\frac{t}{RC}}\right)\right], \\ &= C\epsilon\left(\frac{1}{RC}\right)e^{-\frac{t}{RC}} \\ &= \frac{\epsilon}{R}e^{-\frac{t}{TC}} \\ &= I_0 e^{-\frac{t}{RC}}, \end{aligned}$$

$$I(t) = I_0 e^{-t/\tau}. \tag{6.6.12}$$

At time $t = 0.0\ s$, the current through the resistor is $I_0 = \frac{\epsilon}{R}$. As time approaches infinity, the current approaches zero. At time $t = \tau$, the current through the resistor is $I(t = \tau) = I_0 e^{-1} = 0.368 I_0$.

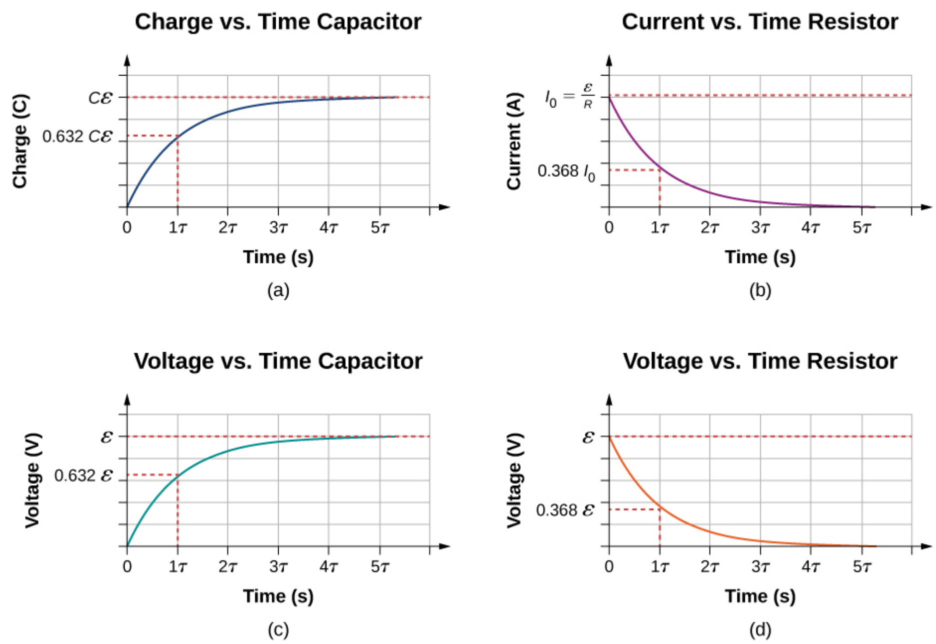


Figure 6.6.2: (a) Charge on the capacitor versus time as the capacitor charges. (b) Current through the resistor versus time. (c) Voltage difference across the capacitor. (d) Voltage difference across the resistor.

Figures $6.6.2c$ and Figure $6.6.2d$ show the voltage differences across the capacitor and the resistor, respectively. As the charge on the capacitor increases, the current decreases, as does the voltage difference across the resistor $V_R(t) = (I_0 R)e^{-t/\tau} = \epsilon e^{-t/\tau}$. The voltage difference across the capacitor increases as $V_C(t) = \epsilon(1 - e^{-t/\tau})$.

Discharging a Capacitor

When the switch in Figure $6.6.3a$ is moved to position **B**, the circuit reduces to the circuit in part (c), and the charged capacitor is allowed to discharge through the resistor. A graph of the charge on the capacitor as a function of time is shown in Figure $6.6.3a$. Using Kirchhoff's loop rule to analyze the circuit as the capacitor discharges results in the equation $-V_R - V_C = 0$, which simplifies to $IR + \frac{q}{C} = 0$. Using the definition of current $\frac{dq}{dt}R = -\frac{q}{C}$ and integrating the loop equation yields an equation for the charge on the capacitor as a function of time:

$$q(t) = Qe^{-t/\tau}. \tag{6.6.13}$$

Here, **Q** is the initial charge on the capacitor and $\tau = RC$ is the time constant of the circuit. As shown in the graph, the charge decreases exponentially from the initial charge, approaching zero as time approaches infinity.

The current as a function of time can be found by taking the time derivative of the charge:

$$I(t) = -\frac{Q}{RC}e^{-t/\tau}. \tag{6.6.14}$$

The negative sign shows that the current flows in the opposite direction of the current found when the capacitor is charging. Figure $6.6.3b$ shows an example of a plot of charge versus time and current versus time. A plot of the voltage difference across the capacitor and the voltage difference across the resistor as a function of time are shown in Figures $6.6.3c$ and $6.6.3d$. Note that the magnitudes of the charge, current, and voltage all decrease exponentially, approaching zero as time increases.

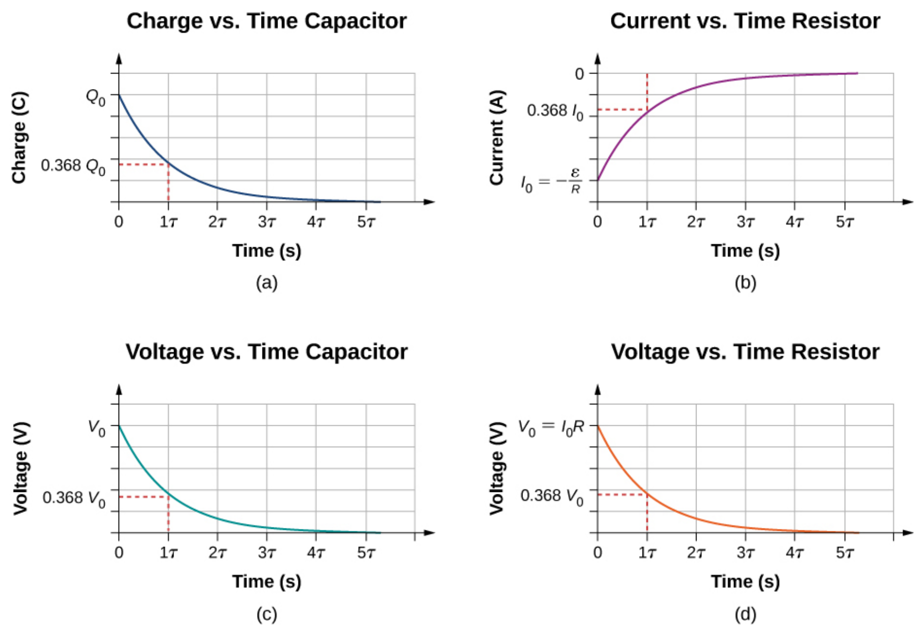


Figure 6.6.3: (a) Charge on the capacitor versus time as the capacitor discharges. (b) Current through the resistor versus time. (c) Voltage difference across the capacitor. (d) Voltage difference across the resistor.

Now we can explain why the **flash camera** mentioned at the beginning of this section takes so much longer to charge than discharge: The resistance while charging is significantly greater than while discharging. The internal resistance of the battery accounts for most of the resistance while charging. As the battery ages, the increasing internal resistance makes the charging process even slower.

Examples

Example 6.6.1 : Integrated Concept Problem: Calculating Capacitor Size—Strobe Lights

High-speed flash photography was pioneered by Doc Edgerton in the 1930s, while he was a professor of electrical engineering at MIT. You might have seen examples of his work in the amazing shots of hummingbirds in motion, a drop of milk splattering on a table, or a bullet penetrating an apple (see Figure). To stop the motion and capture these pictures, one needs a high-intensity, very short pulsed flash, as mentioned earlier in this module.

Suppose one wished to capture the picture of a bullet (moving at $5 \times 10^2\, m/s$) that was passing through an apple. The duration of the flash is related to the RC time constant τ. What size capacitor would one need in the RC circuit to succeed, if the resistance of the flash tube was $10\,\Omega$? Assume the apple is a sphere with a diameter of $8 \times 10^{-2}\, m$.

Strategy

We begin by identifying the physical principles involved. This example deals with the strobe light, as discussed above. Figure shows the circuit for this probe. The characteristic time τ of the strobe is given as $\tau = RC$.

Solution

We wish to find C, but we don't know τ. We want the flash to be on only while the bullet traverses the apple. So we need to use the kinematic equations that describe the relationship between distance x, velocity v, and time t:

$$x = vt \; or \; t - \frac{x}{v}. \tag{6.6.15}$$

The bullet's velocity is given as $5 \times 10^2\, m$, and the distance x is $8 \times 10^{-2}\, m$. The traverse time, then, is

$$t = \frac{x}{v} = \frac{8.0 \times 10^{-2}\, m}{5.0 \times 10^2\, m/s} = 1.6 \times 10^{-4}\, s. \tag{6.6.16}$$

We set this value for the crossing time t equal to τ. Therefore,

$$C = \frac{t}{R} = \frac{1.6 \times 10^{-4}\, s}{10.0\,\Omega} = 16\,\mu F. \tag{6.6.17}$$

(Note: Capacitance C is typically measured in farads, F, defined as Coulombs per volt. From the equation, we see that C can also be stated in units of seconds per ohm.)

Discussion

The flash interval of $160 \mu s$ (the traverse time of the bullet) is relatively easy to obtain today. Strobe lights have opened up new worlds from science to entertainment. The information from the picture of the apple and bullet was used in the Warren Commission Report on the assassination of President John F. Kennedy in 1963 to confirm that only one bullet was fired.

Example 6.6.2 : Calculating Time: *RC* Circuit in a Heart Defibrillator

A heart defibrillator is used to resuscitate an accident victim by discharging a capacitor through the trunk of her body. A simplified version of the circuit is seen in Figure. (a) What is the time constant if an $8.00\ \mu F$ capacitor is used and the path resistance through her body is $1 \times 10^3\ \Omega$? (b) If the initial voltage is 10.0 kV, how long does it take to decline to $5 \times 10^2\ V$?

Strategy

Since the resistance and capacitance are given, it is straightforward to multiply them to give the time constant asked for in part (a). To find the time for the voltage to decline to $5 \times 10^2\ V$, we repeatedly multiply the initial voltage by 0.368 until a voltage less than or equal to $5 \times 10^2\ V$ is obtained. Each multiplication corresponds to a time of τ seconds.

Solution

The time constant τ is given by the equation $\tau = RC$. Entering the given values for resistance and capacitance (and remembering that units for a farad can be expressed as s/Ω) gives

$$\tau = RC = (1.00 \times 10^3\ \Omega)(8.00\ \mu F) = 8.00\ ms. \tag{6.6.18}$$

Solution for (b)

In the first 8.00 ms, the voltage (10.0 kV) declines to 0.368 of its initial value. That is:

$$V = 0.368 V_0 = 3.680 \times 10^3\ V\ at\ t = 8.00\ ms. \tag{6.6.19}$$

(Notice that we carry an extra digit for each intermediate calculation.) After another 8.00 ms, we multiply by 0.368 again, and the voltage is

$$V' = 0.368\ V \tag{6.6.20}$$

$$= (0.368)(3.680 \times 10^3\ V) \tag{6.6.21}$$

$$= 1.354 \times 10^3\ V\ at\ t = 16.0\ ms. \tag{6.6.22}$$

Similarly, after another 8.00 ms, the voltage is

$$V" = 0.368\ V' = (0.368)(1.354 \times 10^3\ V) \tag{6.6.23}$$

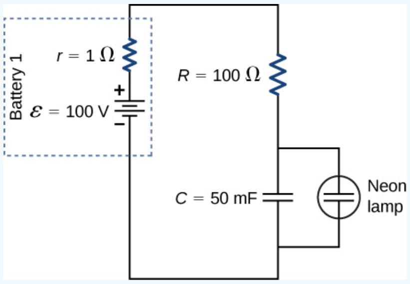
$$= 498\ V\ at\ t = 24.0\ ms. \tag{6.6.24}$$

Discussion

So after only 24.0 ms, the voltage is down to 498 V, or 4.98% of its original value.Such brief times are useful in heart defibrillation, because the brief but intense current causes a brief but effective contraction of the heart. The actual circuit in a heart defibrillator is slightly more complex than the one in Figure, to compensate for magnetic and AC effects that will be covered in Magnetism.

Example 6.6.3 : The Relaxation Oscillator

One application of an **RC** circuit is the relaxation oscillator, as shown below. The relaxation oscillator consists of a voltage source, a resistor, a capacitor, and a neon lamp. The neon lamp acts like an open circuit (infinite resistance) until the potential difference across the neon lamp reaches a specific voltage. At that voltage, the lamp acts like a short circuit (zero resistance), and the capacitor discharges through the neon lamp and produces light. In the relaxation oscillator shown, the voltage source charges the capacitor until the voltage across the capacitor is 80 V. When this happens, the neon in the lamp breaks down and allows the capacitor to discharge through the lamp, producing a bright flash. After the capacitor fully discharges through the neon lamp, it begins to charge again, and the process repeats. Assuming that the time it takes the capacitor to discharge is negligible, what is the time interval between flashes?



Strategy

The time period can be found from considering the equation $V_C(t) = \epsilon(1 - e^{=t/\tau})$. where $\tau = (R + r)C$.

Solution

The neon lamp flashes when the voltage across the capacitor reaches 80 V. The **RC** time constant is equal to $\tau = (R + r) = (101\,\Omega)(50 \times 10^{-3}\,F) = 5.05\,s$. We can solve the voltage equation for the time it takes the capacitor to reach 80 V:

$$V_C(t) = \epsilon(1 - e^{-t/\tau}), \tag{6.6.25}$$

$$e^{-t/\tau} = 1 - \frac{V_C(t)}{\epsilon}, \tag{6.6.26}$$

$$ln(e^{-t/\tau}) = ln\left(1 - \frac{V_C(t)}{\epsilon}\right), \tag{6.6.27}$$

$$t = -\tau ln\left(1 - \frac{V_C(t)}{\epsilon}\right) = -5.05\,s \cdot ln\left(1 - \frac{80\,V}{100\,V}\right) = 8.13\,s. \tag{6.6.28}$$

Significance

One application of the relaxation oscillator is for controlling indicator lights that flash at a frequency determined by the values for **R** and **C**. In this example, the neon lamp will flash every 8.13 seconds, a frequency of $f = \frac{1}{T} = \frac{1}{8.13\,s} = 0.55\,Hz$. The relaxation oscillator has many other practical uses. It is often used in electronic circuits, where the neon lamp is replaced by a transistor or a device known as a tunnel diode. The description of the transistor and tunnel diode is beyond the scope of this chapter, but you can think of them as voltage controlled switches. They are normally open switches, but when the right voltage is applied, the switch closes and conducts. The "switch" can be used to turn on another circuit, turn on a light, or run a small motor. A relaxation oscillator can be used to make the turn signals of your car blink or your cell phone to vibrate.

RC circuits have many applications. They can be used effectively as timers for applications such as intermittent windshield wipers, pace makers, and strobe lights. Some models of intermittent windshield wipers use a variable resistor to adjust the interval between sweeps of the wiper. Increasing the resistance increases the **RC** time constant, which increases the time between the operation of the wipers.

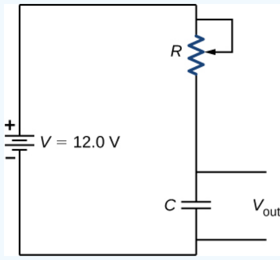
Another application is the **pacemaker**. The heart rate is normally controlled by electrical signals, which cause the muscles of the heart to contract and pump blood. When the heart rhythm is abnormal (the heartbeat is too high or too low), pace makers can be used to correct this abnormality. Pacemakers have sensors that detect body motion and breathing to increase the heart rate during physical activities, thus meeting the increased need for blood and oxygen, and an **RC** timing circuit can be used to control the time between voltage signals to the heart.

Looking ahead to the study of ac circuits (Alternating-Current Circuits), ac voltages vary as sine functions with specific frequencies. Periodic variations in voltage, or electric signals, are often recorded by scientists. These voltage signals could come from music recorded by a microphone or atmospheric data collected by radar. Occasionally, these signals can contain unwanted frequencies known as "noise." **RC** filters can be used to filter out the unwanted frequencies.

In the study of electronics, a popular device known as a 555 timer provides timed voltage pulses. The time between pulses is controlled by an **RC** circuit. These are just a few of the countless applications of **RC** circuits.

Example 6.6.4 : Intermittent Windshield Wipers

A relaxation oscillator is used to control a pair of windshield wipers. The relaxation oscillator consists of a 10.00-mF capacitor and a $10.00\,k\Omega$ variable resistor known as a rheostat. A knob connected to the variable resistor allows the resistance to be adjusted from $0.00\,\Omega$ to $10.00\,k\Omega$. The output of the capacitor is used to control a voltage-controlled switch. The switch is normally open, but when the output voltage reaches 10.00 V, the switch closes, energizing an electric motor and discharging the capacitor. The motor causes the windshield wipers to sweep once across the windshield and the capacitor begins to charge again. To what resistance should the rheostat be adjusted for the period of the wiper blades be 10.00 seconds?



Strategy

The resistance considers the equation $V_{out}(t) = V(1 - e^{-t/\tau})$, where $\tau = RC$. The capacitance, output voltage, and voltage of the battery are given. We need to solve this equation for the resistance.

Solution

The output voltage will be 10.00 V and the voltage of the battery is 12.00 V. The capacitance is given as 10.00 mF. Solving for the resistance yields

$$V_{out}(t) = V(1 - e^{-t/\tau}) \tag{6.6.29}$$

$$e^{-t/RC} = 1 - \frac{V_{out}(t)}{V}, \tag{6.6.30}$$

$$ln(e^{-t/RC}) = ln\left(1 - \frac{V_{out}(t)}{V}\right), \tag{6.6.31}$$

$$-\frac{t}{RC} = ln\left(1 - \frac{V_{out}(t)}{V}\right), \tag{6.6.32}$$

$$R = \frac{-t}{C\,ln\left(1 - \frac{V_C(t)}{V}\right)} = \frac{-10.00\,s}{10 \times 10^{-3} F\,ln\left(1 - \frac{10\,V}{12\,V}\right)} = 558.11\,\Omega. \tag{6.6.33}$$

Significance

Increasing the resistance increases the time delay between operations of the windshield wipers. When the resistance is zero, the windshield wipers run continuously. At the maximum resistance, the period of the operation of the wipers is:

$$t = -RC ln\left(1 - \frac{V_{out}(t)}{V}\right) = -(10 \times 10^{-3} F)(10 \times 10^{3}\,\Omega) ln\left(1 - \frac{10\,V}{12\,V}\right) = 179.18\,s = 2.98\,min. \tag{6.6.34}$$

Example 6.6.5

Given the circuit of Figure 6.6.3 , assume the switch is closed at time $t = 0$. Determine the charging time constant, the amount of time after the switch is closed before the circuit reaches steady-state, and the capacitor voltage at $t = 0$, $t = 50$ milliseconds and $t = 1$ second. Assume the capacitor is initially uncharged.

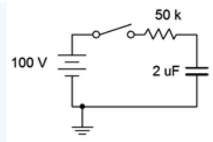


Figure 6.6.3 : Circuit for Example 6.6.1 .

First, the time constant:

$$\tau = RC$$

$$\tau = 50k\Omega 2\mu F$$

$$\tau = 100ms$$

Steady-state will be reached in five time constants, or 500 milliseconds. Therefore we know that $V_C(0) = 0$ volts and $V_C(1) = 100$ volts. To find $V_C(50ms)$ we simply solve Equation ??? .

$$V_C(t) = E\left(1 - \epsilon^{-\frac{t}{\tau}}\right)$$

$$V_C(50ms) = 100V\left(1 - \epsilon^{-\frac{50ms}{100ms}}\right)$$

$$V_C(50ms) \approx 39.35V$$

This value can also be determined graphically from Figure 6.6.2 . The time of 50 milliseconds represents one-half time constant. Find this value on the horizontal axis and then track straight up to the solid red curve that represents the charging capacitor voltage. The point of intersection is at approximately 40% of the maximum value on the vertical axis. The maximum value here is the source voltage of 100 volts. Therefore the capacitor will have reached approximately 40% of 100 volts, or just about 40 volts.

Example 6.6.6

For the circuit of Figure 6.6.6 , assume the capacitor is initially uncharged. At time $t = 0$ the switch contacts position 1. The switch is thrown to position 2 at time $t = 50$ milliseconds.

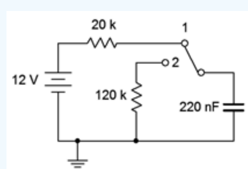


Figure 6.6.6 : Circuit for Example 6.6.2 .

Determine the charging time constant, the amount of time after the switch is closed before the circuit reaches steady-state, the maximum charging and discharging currents, and the capacitor voltage at $t = 0$, $t = 50$ milliseconds, $t = 90$ milliseconds, and $t = 1$ second.

We begin with the charge time constant:

$$\tau_{charge} = RC$$

$$\tau_{charge} = 20k\Omega 220nF$$

$$\tau_{charge} = 4.4ms$$

Steady-state will be reached in 5 times 4.4 milliseconds, or 22 milliseconds. The capacitor is initially uncharged, so $V_C(0) = 0$ volts. As the capacitor will have reached steady-state in 22 milliseconds, $V_C(50ms) = 12$ volts. The maximum charging current will occur at $t = 0$ when all of the 12 volt source drops across the 20 kΩ resistor, or 600 μamps, flowing left to right.

At 50 milliseconds the switch is thrown to position 2. The 12 volt source and 20 kΩ resistor are no longer engaged. At this point the capacitor has 12 volts across it, positive to negative, top to bottom. As the capacitor voltage cannot change instantaneously, the capacitor now acts as a voltage source and discharges through the 120 kΩ resistor. Note that the discharge current is flowing counterclockwise, the opposite of the charging current. The discharge time constant is:

$$\tau_{discharge} = RC$$

$$\tau_{discharge} = 120k\Omega 220nF$$

$$\tau_{discharge} = 26.4ms$$

The capacitor will fully discharge down to 0 volts in 5 time constants, or some 132 milliseconds after the switch is thrown to position 2. Thus steady-state occurs at $t = 182$ milliseconds. The maximum discharge current occurs the instant the switch is thrown to position 2 when all of the capacitor's 12 volts drops across the 120 kΩ resistor, yielding 100 μ amps, flowing top to bottom.

Clearly, at $t = 90$ milliseconds the capacitor is in the discharge phase. The capacitor's voltage and current during the discharge phase follow the solid blue curve of Figure 6.6.2 . The elapsed time for discharge is 90 milliseconds minus 50 milliseconds, or 40 milliseconds net. We can use a slight variation on Equation ??? to find the capacitor voltage at this time.

$$V_C(t) = E\epsilon^{-\frac{t}{\tau}}$$

$$V_C(40ms) = 12V - \epsilon^{-\frac{40ms}{26.4ms}}$$

$$V_C(40ms) \approx 2.637V$$

The shape of the capacitor's voltage will appear somewhat like a rounded pulse, rising with a curve and then falling back to zero with a complementary curve (the red and then blue curves of Figure 6.6.2).

Example 6.6.7

For this example we shall revisit the circuit of Example 8.3.1. The circuit is redrawn in Figure 6.6.7 for convenience. Assume the capacitor is initially uncharged.

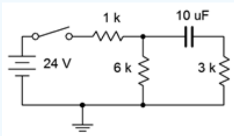


Figure 6.6.7 : Circuit for Example 6.6.3 .

Determine the charging time constant, the amount of time after the switch is closed before the circuit reaches steady-state, and the capacitor voltage at $t = 0$, 100 milliseconds, and 200 milliseconds. At 200 milliseconds, the switch is opened. Determine how long it takes for the capacitor to fully discharge and the voltage across the 6 kΩ resistor at $t = 275$ milliseconds (i.e., 75 milliseconds after the switch is opened).

The first step is to determine the Thévenin equivalent driving the capacitor. If we remove the capacitor and determine the open circuit voltage at those points, we see that it is just a voltage divider between the 24 volt source, the 6 kΩ resistor and the 1 kΩ resistor (the 3 kΩ resistor has no current through it and thus produces no voltage drop). This works out to 20.57 volts. The Thévenin resistance will be 3 kΩ in series with 1 kΩ || 6 kΩ, or roughly 3.857 kΩ. The equivalent charging circuit is drawn in Figure 6.6.8 .

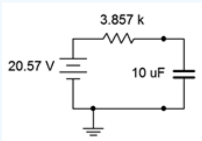


Figure 6.6.8 : Thévenin equivalent for the circuit of Figure 6.6.7 driving the capacitor.

We can now determine the charging time constant:

$$\tau_{charge} = RC$$

$$\tau_{charge} = 3.857k\Omega 10\mu F$$

$$\tau_{charge} = 38.57ms$$

Steady-state will be reached in 5 time constants, or 192.8 ms. Thus, we know that $V_C(0) = 0$ volts and $V_C(200ms) = 20.57$ volts. For the capacitor voltage at 100 milliseconds, we simply use the charge equation.

$$V_C(t) = E\left(1 - \epsilon^{-\frac{t}{\tau}}\right)$$

$$V_C(100ms) = 20.57V\left(1 - \epsilon^{-\frac{100ms}{38.57ms}}\right)$$

$$V_C(100ms) \approx 19.03V$$

For the discharge phase, we need to determine the time constant. With the voltage source removed, the capacitor will discharge through the now series combination of the 3 kΩ resistor and 6 kΩ resistor.

$$\tau_{discharge} = RC$$

$$\tau_{discharge} = 9k\Omega 10\mu F$$

$$\tau_{discharge} = 90ms$$

Steady-state will be reached 450 milliseconds later at $t = 650$ milliseconds. To find V_{6k} at $t = 275$ milliseconds, we can find the voltage across the capacitor and then perform a voltage divider between the 6 kΩ and 3 kΩ resistors. Remembering that $t = 275$ milliseconds is 75 milliseconds into the discharge phase, we have:

$$V_C(t) = E\epsilon^{-\frac{t}{\tau}}$$

$$V_C(75ms) = 20.57V\left(1 - \epsilon^{-\frac{75ms}{90ms}}\right)$$

$$V_C(75ms) \approx 8.94V$$

Finally, this voltage splits between the 6 kΩ and 3 kΩ resistors. Using the voltage divider rule, we find:

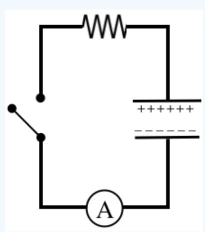
$$V_{6k} = V_C \frac{R_x}{R_x + R_y}$$

$$V_{6k} = 8.94V \frac{6k\Omega}{6k\Omega + 3k\Omega}$$

$$V_{6k} = 5.96V$$

Example 6.6.8 : Changing the time constant

The parallel-plate capacitor is charged and then the switch is closed. At the instant the switch is closed, the current measured through the ammeter is I_o. After a time of $2.4s$ elapses, the current through the ammeter is measured to be $0.60I_o$, and the switch is opened. A substance with a dielectric constant of 1.5 is then inserted between the plates of the capacitor, and the switch is once again closed and not reopened until the ammeter reads zero current.



a. *Find the period of time that elapses between when the switch is closed the second time and when the ammeter reads a current of $0.20I$.*
b. *At the end, all of the electrical potential energy is gone from the capacitor. Find the fraction of this energy that was converted into thermal energy by the resistor.*

Solution

a. We can calculate the time constant from the period of time that the current takes to drop to 0.6 of its original value:

$$0.60I_o = I_o e^{-\frac{t}{\tau}} \quad \Rightarrow \quad \tau = \frac{-t}{\ln 0.60} = \frac{-2.4s}{\ln 0.60} = 4.7s$$

When the dielectric is inserted, the time constant changes. The time constant is proportional to the capacitance, so since inserting the dielectric increases the capacitance by a factor of 1.5, that is the factor by which the time constant changes as well, giving a new time constant of:

$$\tau = RC \quad \Rightarrow \quad \tau_{new} = R(\kappa C) = \kappa \tau_{old} = (1.5)(4.7s) = 7.0s$$

The current is driven by the potential difference across the capacitor, and this is proportional to the charge on the capacitor, so when the current gets down to 60% of its initial value, that means that the charge on the capacitor has dropped by the same factor. To find the time for the current to drop to $0.20I_o$, *we need to know not only the new time constant, but also the new starting current. We can get this from the new starting voltage, which comes from the new starting charge and capacitance:*

$$V_{o(new)} = \frac{Q_{o(new)}}{C_{new}} \quad \Rightarrow \quad I_{o(new)} \frac{V_{o(new)}}{R} = \frac{Q_{o(new)}}{RC_{new}} = \frac{0.60Q_o}{R(1.5C)} = 0.40I_o$$

With the new starting current equal to $0.4I_o$, *we are looking for the time it takes to get down to* $0.2I_o$, *so:*

$$0.20I_o = 0.40I_o e^{\frac{t}{\tau}} \quad \Rightarrow \quad t = \tau \ln 2 = (7.0s)\ln 2 = 4.8s$$

b. We already determined that in the first stage of this process, the charge on the capacitor went down to 60% of its initial amount. This allows us to calculate the energy lost by the capacitor, which is what is converted to thermal:

$$\Delta U_1 = U_o - \frac{(0.60Q_o)^2}{2C} = U_o - 0.36\frac{Q_o^2}{2C} = 0.64U_o$$

So 64% of the energy on the capacitor is converted to thermal energy in the first stage. In the second stage, all of the internal energy in the capacitor is converted, but this amount of energy must be calculated in terms of the new capacitance:

$$\Delta U_2 = \frac{(0.60Q_o)^2}{2(1.5C)} = 0.24U_o$$

So of the original energy stored in the capacitor, 88% of the energy is converted to thermal. Where is the remaining 12%, if all of it is now gone from the capacitor? The field of the capacitor did work drawing the dielectric into it.

The **RC** circuit has thousands of uses and is a very important circuit to study. Not only can it be used to time circuits, it can also be used to filter out unwanted frequencies in a circuit and used in power supplies, like the one for your computer, to help turn ac voltage to dc voltage.

? Exercise 6.6.1

A simple RC circuit as shown in Figure 6.6.1 contains a charged capacitor of unknown capacitance, C, in series with a resistor, $R = 2\Omega$. When charged, the potential difference across the terminals of the capacitor is 9V.

At time $t = 0\text{s}$, the switch, S, is closed, allowing the capacitor to discharge through the resistor. The current is then measured to be $I = 0.05\text{A}$ at $t = 5\text{s}$ after opening the switch.

a. What is the capacitance of the capacitor?
b. What charge did the capacitor hold at $t = 2\text{s}$?

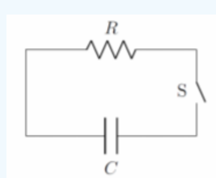


Figure 6.6.1: A simple circuit with a resistor and a capacitor.

Answer

a. In this case, the capacitor is discharging as a function of time. At time $t = 0$, the voltage across the capacitor is $\Delta V = 9\text{V}$. We can model this discharging circuit in a similar way as we modeled the charging circuit.

We start with Kirchhoff's junction rule, which leads to a differential equation for the charge stored on the capacitor, $Q(t)$, as function of time:

$$\begin{aligned}\Delta V-IR&=0\\ \frac{Q}{C}-IR&=0\\ \frac{Q}{C}-\frac{dQ}{dt}R&=0\\ \therefore\frac{dQ}{dt}&=-\frac{1}{RC}Q\end{aligned}$$

This differential equation is straightforward to solve, since it says that the derivative of $Q(t)$ is equal to a constant multiplied by $Q(t)$. Thus, $Q(t)$, must be an exponential function:

$$Q(t)=Q_0e^{-\frac{t}{RC}}$$

where, Q_0, is the (unknown) charge on the capacitor at $t=0$. You can easily verify that taking the derivative of this equation will result in the differential equation being satisfied.

The current, $I(t)$, as a function of time is given by:

$$I=\frac{dQ}{dt}=-\frac{1}{RC}Q=\frac{Q_0}{RC}e^{-\frac{t}{RC}}=I_0e^{-\frac{t}{RC}}$$

where $I_0=\frac{Q_0}{RC}$ is the current at $t=0$.

We also know that the current through the resistor at $t=0$ is given by Ohm's Law, since, at that time, the voltage, $\frac{Q_0}{C}=9\text{V}$:

$$I_0=\frac{Q_0}{RC}=\frac{(9\text{V})}{(2\Omega)}=4.5\text{A}$$

We then know that the current, at time $t=5\text{s}$, is equal to $I(5)=0.05\text{A}$, allowing us to determine the capacitance:

$$\begin{aligned}I(5)&=I_0e^{-\frac{t}{RC}}\\ \ln\left(\frac{I(5)}{I_0}\right)&=-\frac{t}{RC}\\ \therefore C&=\frac{t}{R\ln\left(\frac{I_0}{I(5)}\right)}=\frac{(5\text{s})}{(2\Omega)\ln\left(\frac{(4.5\text{A})}{(0.05\text{A})}\right)}=0.56\text{F}\end{aligned}$$

b. To find the charge stored in the capacitor at $t=2\text{s}$, we can use the function $Q(t)$ that we determined before:

$$Q(t=2\text{s})=Q_0e^{-\frac{t}{RC}}$$

where we can determine, Q_0, now that we know the capacitance. Q_0 is the charge on the capacitor at time $t=0$, when the voltage across the capacitor is 9V:

$$Q_0=C\Delta V=(0.56\text{F})(9\text{V})=5.0\text{C}$$

At $t=2\text{s}$, the charge on the capacitor is thus:

$$Q(t=2\text{s})=(5.0\text{C})e^{-\frac{(2\text{s})}{(2\Omega)(0.56\text{F})}}=0.84\text{C}$$

Exercise 6.6.2

A voltmeter with a resistance of $R_V=20\text{k}\Omega$ is attached to a circuit with a battery of unknown voltage and two resistors with a resistance of $R=2.5\text{k}\Omega$ as shown in Figure 6.6.2. The voltmeter reads that the voltage drop over one of the resistors is $\Delta V_{vm}=5.647\text{V}$. What is the voltage drop, V_R, over each resistor when the voltmeter is removed from the circuit?

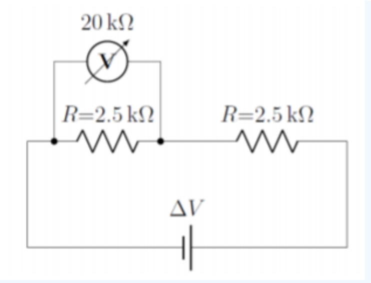


Figure 6.6.2: A circuit with a battery of unknown voltage.

Answer

In order to know the voltage across one of the resistors, we need to determine the voltage that is across the battery. Once we have determined the voltage across the battery, the voltage across one of the resistors will just be half of that across the battery, since the two resistors have the same resistance.

We can model the circuit with the voltmeter in place, since we know the voltage across the parallel combination of the voltmeter and resistor (that voltage which is readout by the voltmeter). We can combine the voltmeter and one of the resistors into a an equivalent resistor, R_{eff}:

$$R_{eff} = \frac{1}{R_V^{-1} + R^{-1}}$$

$$R_{eff} = \frac{1}{(20\text{k}\Omega)^{-1} + (2.5\text{k}\Omega)^{-1}}$$

$$R_{eff} = 2.22\text{k}\Omega$$

Now that we have the effective resistance as well as the voltage drop across that effective resistor, we can solve for current through the circuit:

$$I = \frac{\Delta V_{vm}}{R_{eff}}$$

$$I = \frac{5.647\text{V}}{2.22\text{k}\Omega}$$

$$I = 2.541\text{mA}$$

Now that we have the current through the circuit, we can determine the voltage drop across the second resistor. By adding that voltage drop to the known voltage across the effective resistor, we can determine the battery voltage:

$$\Delta V_{battery} = I(R_{eff} + R)$$

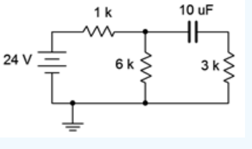
$$\Delta V_{battery} = (2.541\text{mA})(2.222\text{k}\Omega + 2.5\text{k}\Omega)$$

$$\Delta V_{battery} = 12\text{V}$$

Thus, with no voltmeter present, the voltage across each resistor is 6V.

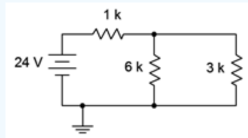
Example 6.6.9 :

Given the circuit shown, find the voltage across the 6 kΩ resistor for both the initial and steady-state conditions assuming the capacitor is initially uncharged.



Circuit for Example.

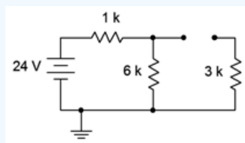
For the initial state the capacitor is treated as a short. The initial state equivalent circuit is drawn below. Immediately apparent is the parallel connection between the 6 kΩ and 3 kΩ resistors. This combination is equivalent to 2 kΩ. Therefore, we can perform a voltage divider to find the potential across the 6 kΩ (i.e., the 2 kΩ combo).



Circuit, initial state.

$$V_{6k} = E\frac{R_x}{R_x + R_y}$$
$$V_{6k} = 24V\frac{2\text{k}\Omega}{2\text{k}\Omega + 1\text{k}\Omega} = 16\text{V}$$

For the steady-state condition the capacitor will be fully charged, its current will be zero, and we treat it as an open. The steady-state equivalent circuit is drawn below.



Circuit, steady-state.

The 3 kΩ resistor is now out of the picture, leaving us with the 6 kΩ in series with the 1 kΩ resistor. Once again, a voltage divider may be used to determine the voltage across the 6 kΩ.

$$V_{6k} = E\frac{R_x}{R_x + R_y}$$
$$V_{6k} = 24V\frac{6\text{k}\Omega}{6\text{k}\Omega + 1\text{k}\Omega} = 20.57\text{V}$$