16.1: Vector Calculus

As we have seen, the study of physics is all about creating a mathematical abstraction of the world, and what kinds of mathematics are required depends on what we want to describe about the world. A basic feature of how the Universe works is "smoothly" - objects move gradually from one point to the next, without stopping. (The alternative to this might be some kind of pixilated version of the universe, in which objects can only exist on a grid, and they move by jumping from point to point. Like a video game or something?) That means that the mathematics we use to describe the universe must similarly be "smooth" - and the mathematics of gradual change mathematics is **calculus**. In addition to gradual change, we've also seen that some quantities in physics can be **vectors** - that is, have both magnitudes and directions. So, clearly, we are going to have to combine these two ideas if we are going to more fully understand how to describe the physical world, into a field of mathematics called **vector calculus**.

In truth, vector calculus can be enormously complicated (as well as enourmously rich and interesting!), but fortunately for mechanics we only need to know the basics of how calculus and vectors interact with each other. We will simply need to know how to take derivatives and integrals of vectors, *e.g.*

$$\vec{a}(t)=\frac{d}{dt}\vec{v}(t), \qquad \vec{v}(t)=\int\vec{r}(t')dt'. \tag{16.1.1}$$

(The two specific examples here are the acceleration as a derivative of velocity, and the velocity as an integral of positive.) At first glance, it might not be obvious how to proceed, but with a little reflection we can see the answer: rewrite the vectors using unit vectors,

$$\vec{v}(t)=v_x(t)\hat{x}+v_y(t)\hat{y}, \qquad \vec{r}(t)=x(t)\hat{x}+y(t)\hat{y}. \tag{16.1.2}$$

Now, if we just replace these two quantities in the expressions above, we can use the additive nature of integrals and derivatives to rewrite them in expressions we understand from usual calculus:

$$\vec{a}(t)=\frac{d}{dt}\vec{v}(t)=\frac{d}{dt}(v_x(t)\hat{x}+v_y(t)\hat{y})=\frac{dv_x(t)}{dt}\hat{x}+\frac{dv_y(t)}{dt}\hat{y} \tag{16.1.3}$$

$$\vec{v}(t)=\int\vec{r}(t')dt'=\int(x(t')\hat{x}+y(t')\hat{y})dt'=\left(\int x(t')dt'\right)\hat{x}+\left(\int y(t')dt'\right)\hat{y} \tag{16.1.4}$$

Notice the important thing that happened - *the unit vector is constant* (in time, in this case). In more detail, the derivative could be written as a chain rule,

$$\frac{d}{dt}(v_x(t)\hat{x})=\frac{dv_x(t)}{dt}\hat{x}+v_x(t)\frac{d\hat{x}}{dt}, \tag{16.1.5}$$

but since $\hat{x}$ is constant this second term is just zero. So, **the derivatives and integrals just pass right through the vector onto the components individually**, and we can do all our usual calculus operations on them without changing what anything means.

There is one slight complication that is probably worth mentioning - what happens to vector products, like dot products and cross products? For example, the definition of work is

$$W=\int\vec{F}(\vec{r})\cdot d\vec{r}. \tag{16.1.6}$$

The basic trick to understanding this expression is the same - use unit vectors. In cartesian coordinates, the force will simply be

$$\vec{F}=F_x(\vec{r})\hat{x}+F_y(\vec{r})\hat{y}, \tag{16.1.7}$$

while the infinitesimal element will be

$$d\vec{r}=dx\hat{x}+dy\hat{y} \tag{16.1.8}$$

(this expression looks a little strange, but it's simply dx in the x-direction and dy in the y-direction). Now we can take the dot product and follow the additive rules for integrals that we followed above:

$$\int\vec{F}(\vec{r})\cdot d\vec{r}=\int(F_x(\vec{r})dx+F_y(\vec{r})dy)=\int F_x(\vec{r})dx+\int F_y(\vec{r})dy. \tag{16.1.9}$$

Thus, the integrals break into an integral of dx and an integral of dy, which you can perform as you normally would. Now we are still not quite done - the force could be some complicated function of either the vector itself, like $\vec{F}(\vec{r}) = r^2\hat{r}$, or the coordinates like $\vec{F}(\vec{r}) = xy\hat{x} + y^2\hat{y}$. In this case we would have to have a relationship between x and y to perform the integrals (this is generally called a line integral, and you can find more information about those at Wikipedia or Khan Academy). We will have to treat these particular cases with a little bit of care!