

7.2: Torque

We are finally in a position to answer the question, when is angular momentum conserved? To do this, we will simply take the derivative of \vec{L} with respect to time, and use Newton's laws to find out under what circumstances it is equal to zero.

Let us start with a particle and calculate

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(m\vec{r} \times \vec{v}) = m \frac{d\vec{r}}{dt} \times \vec{v} + m\vec{r} \times \frac{d\vec{v}}{dt}. \quad (7.2.1)$$

The first term on the right-hand side goes as $\vec{v} \times \vec{v}$, which is zero. The second term can be rewritten as $m\vec{r} \times \vec{a}$. But, according to Newton's second law, $m\vec{a} = \vec{F}_{net}$. So, we conclude that

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}_{net}. \quad (7.2.2)$$

So the angular momentum, like the ordinary momentum, will be conserved if the net force on the particle is zero, but also, and this is an important difference, when the net force is parallel (or antiparallel) to the position vector.

The quantity $\vec{r} \times \vec{F}$ is called the *torque* of a force around a point (the origin from which \vec{r} is calculated, typically a pivot point or center of rotation). It is denoted with the Greek letter τ , "tau":

$$\vec{\tau} = \vec{r} \times \vec{F}. \quad (7.2.3)$$

For an extended object or system, the rate of change of the angular momentum vector would be given by the sum of the torques of all the forces acting on all the particles. For each torque one needs to use the position vector of the particle on which the force is acting.

The torque of a force around a point is basically a measure of how effective the force would be at causing a rotation around that point. Since $|\vec{r} \times \vec{F}| = rF \sin \theta$, you can see that it depends on three things: the magnitude of the force, the distance from the center of rotation to the point where the force is applied, and the angle at which the force is applied. All of this can be understood pretty well from Figure 7.2.1 below, especially if you have ever had to use a wrench to tighten or loosen a bolt:

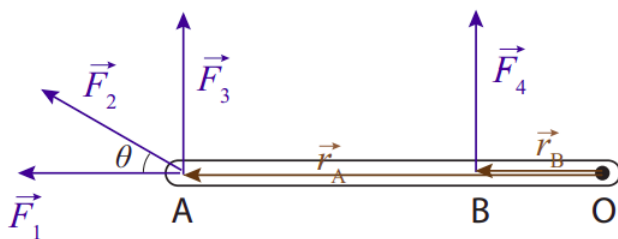


Figure 7.2.1: The torque around the point O of each of the forces shown is a measure of how effective it is at causing the rod to turn around that point.

Clearly, the force \vec{F}_1 will not cause a rotation at all, and accordingly its torque is zero (since it is parallel to \vec{r}_A). On the other hand, of all the forces shown, the most effective one is \vec{F}_3 : it is applied the farthest away from O, for the greatest leverage (again, think of your experiences with wrenches). It is also perpendicular to the rod, for maximum effect ($\sin \theta = 1$). The force \vec{F}_2 , by contrast, although also applied at the point A is at a disadvantage because of the relatively small angle it makes with \vec{r}_A . If you imagine breaking it up into components, parallel and perpendicular to the rod, only the perpendicular component (whose magnitude is $F_2 \sin \theta$) would be effective at causing a rotation; the other component, the one parallel to the rod, would be wasted, like \vec{F}_1 .

In order to calculate torques, then, we basically need to find, for every force, the component that is perpendicular to the position vector of its point of application. Clearly, for this purpose we can no longer represent an extended body as a mere dot, as we have done previously. What we need is a more careful sketch of the object, just detailed enough that we can tell how far from the center of rotation and at what angle each force is applied. That kind of diagram is called an *extended free-body diagram*.

Figure 7.2.1 could be an example of an extended free-body diagram, for an object being acted on by four forces. Typically, though, instead of drawing the vectors \vec{r}_A and \vec{r}_B we would just indicate their lengths on the diagram (or maybe even leave them out

altogether, if we do not want to overload the diagram with detail). I will show a couple of examples of extended free-body diagrams in the next couple of sections.

Coming back to Equation (7.2.3), the main message of this section (other, of course, than the definition of torque itself), is that the rate of change of an object or system's angular momentum is equal to the net torque due to the external forces. Two special results follow from this one. First, if the net external torque is zero, angular momentum will be conserved. For example, consider the collision of a particle and a rod pivoting around one end. The only external force is the force exerted on the rod, at the pivot point, by the pivot itself, but the torque of that force around that point is obviously zero, since $\vec{r} = 0$, so our assumption that the total angular momentum around that point was conserved is legitimate.

Finally, note that situations where the moment of inertia of a system, I , changes with time are relatively easy to arrange for any deformable system. Especially interesting is the case when the external torque is zero, so L is constant, and a change in I therefore brings about a change in $\omega = L/I$: this is how, for instance, an ice-skater can make herself spin faster by bringing her arms closer to the axis of rotation (reducing her I), and, conversely, slow down her spin by stretching out her arms. This can be done even in the absence of a contact point with the ground: high-board divers, for instance, also spin up in this way when they curl their bodies into a ball. Note that, throughout the dive, the diver's angular momentum around its center of mass is constant, since the only force acting on him (gravity, neglecting air resistance) has zero torque about that point.

Resources

Unfortunately, we will not really have enough time this semester to explore further the many interesting effects that follow from the vector nature of Equation (7.2.2), but you are at least subconsciously familiar with some of them if you have ever learned to ride a bicycle! A few interesting Internet references (some of which could perhaps inspire a good Honors project!) are the following:

- Walter Lewin's lecture on gyroscopic motion (and rolling motion):
<https://www.youtube.com/watch?v=N92FYHHT1qM>
 - A "Veritasium" video on "antigravity":
<https://www.youtube.com/watch?v=GeyDf4ooPdo>
<https://www.youtube.com/watch?v=tLMpdBjA2SU>
 - And the old trick of putting a gyroscope (flywheel) in a suitcase:
<https://www.youtube.com/watch?v=zdN6zhZSJkw>
- If any of the above links are dead, try googling them. (You may want to let me know, too!)

³The additional assumption is that the force between any two particles lies along the line connecting the two particles (which means it is parallel or antiparallel to the vector $\vec{r}_1 - \vec{r}_2$). In that case, $\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} = 0$. Most forces in nature satisfy this condition.

⁴Actually, friction forces and normal forces may be "spread out" over a whole surface, but, if the object has enough symmetry, it is usually OK to have them "act" at the midpoint of that surface. This can be proved along the lines of the derivation for gravity that follows.

This page titled 7.2: Torque is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Christopher Duston, Merrimack College (University of Arkansas Libraries) via source content that was edited to the style and standards of the LibreTexts platform.

- **9.4: Torque** by Julio Gea-Banacloche is licensed CC BY-SA 4.0. Original source: <https://scholarworks.uark.edu/oer/3>.
- **9.6: Rolling Motion** by Julio Gea-Banacloche is licensed CC BY-SA 4.0. Original source: <https://scholarworks.uark.edu/oer/3>.