14.1: Types of Collisions

Although momentum is conserved in all interactions, not all interactions (collisions or explosions) are the same. The possibilities include:

- A single object can explode into multiple objects (explosions).
- Multiple objects can collide and stick together, forming a single object (inelastic).
- Multiple objects can collide and bounce off of each other, remaining as multiple objects (elastic). If they do bounce off each other, then they may recoil at the same speeds with which they approached each other before the collision, or they may move off more slowly.

It's useful, therefore, to categorize different types of interactions, according to how the interacting objects move before and after the interaction.

Explosions

The first possibility is that a single object may break apart into two or more pieces. An example of this is a firecracker, or a bow and arrow, or a rocket rising through the air toward space. These can be difficult to analyze if the number of fragments after the collision is more than about three or four; but nevertheless, the total momentum of the system before and after the explosion is identical.

Note that if the object is initially motionless, then the system (which is just the object) has no momentum and no kinetic energy. After the explosion, the net momentum of all the pieces of the object must sum to zero (since the momentum of this closed system cannot change). However, the system **will** have a great deal of kinetic energy after the explosion, although it had none before. Thus, we see that, although the momentum of the system is conserved in an explosion, the kinetic energy of the system most definitely is not; it increases. This interaction—one object becoming many, with an increase of kinetic energy of the system—is called an **explosion**.

Where does the energy come from? Does conservation of energy still hold? Yes; some form of potential energy is converted to kinetic energy. In the case of gunpowder burning and pushing out a bullet, chemical potential energy is converted to kinetic energy of the bullet, and of the recoiling gun. For a bow and arrow, it is elastic potential energy in the bowstring.

Inelastic

The second possibility is the reverse: that two or more objects collide with each other and stick together, thus (after the collision) forming one single composite object. The total mass of this composite object is the sum of the masses of the original objects, and the new single object moves with a velocity dictated by the conservation of momentum. However, it turns out again that, although the total momentum of the system of objects remains constant, the kinetic energy doesn't; but this time, the kinetic energy decreases. This type of collision is called **inelastic**.

Any collision where the objects stick together will result in the maximum loss of kinetic energy (i.e., K_f will be a minimum). Such a collision is said to be **perfectly inelastic**. In the extreme case, multiple objects collide, stick together, and remain motionless after the collision. Since the objects are all motionless after the collision, the final kinetic energy is also zero; therefore, the loss of kinetic energy is a maximum.

- If $0 < K_f < K_i$, the collision is inelastic.
- If K_f is the lowest energy, or the energy lost by both objects is the most, the collision is perfectly inelastic (objects stick together).
- If $K_f = K_i$, the collision is elastic.

Elastic

The extreme case on the other end is if two or more objects approach each other, collide, and bounce off each other, moving away from each other at the same relative speed at which they approached each other. In this case, the total kinetic energy of the system is conserved. Such an interaction is called **elastic**.

In any interaction of a closed system of objects, the total momentum of the system is conserved ($\vec{p}_f = \vec{p}_i$) but the kinetic energy may not be:

- If $0 < K_f < K_i$, the collision is inelastic.

- If $K_f = 0$, the collision is perfectly inelastic.
- If $K_f = K_i$, the collision is elastic.
- If $K_f > K_i$, the interaction is an explosion.

The point of all this is that, in analyzing a collision or explosion, you can use both momentum and kinetic energy.

Dimensions and Equation Counting

It's worth pointing out how many equations and unknown variables we are dealing with when it comes to collision problems, because it is quite predictable and can give us some insight into how hard a particular problem might be before we get started on it. As discussed above, momentum is conserved in every collision, so

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = 0. \tag{14.1.1}$$

Since this is a vector equation, it actually contains a *number of linear independent equations equal to the dimension of the problem* (typically 1 or 2 for us, but generally 3). Since these linear equations can only be solved if there are an equal number of unknown variables and equations, we can only solve problems that have the same number of unknowns as dimensions (for example, a 1D problem can only ask one question - "what is the final velocity?" or "what was the mass of the first object?" - never "what was the final velocity AND the mass of the first object?"). This is the complete story for inelastic collisions - the number of unknowns has to match the dimension.

For elastic collisions, we have one more relationship, conservation of energy:

$$\Delta E = E_f - E_i = 0. \tag{14.1.2}$$

This is a scalar equation, and represents one further constraint on our system. However, that extra relationship means we can leave one further quantity unspecified - it is no longer free to be set, but must satisfy the extra equation from conservation of energy. This makes elastic collisions generally more complicated than inelastic problems, because we have an extra equation and unknown to deal with. To take the example from above, in 1D we have now two equations that govern our collision:

$$p_{f,x} - p_{i,x} = 0, \qquad E_f - E_i = 0. \tag{14.1.3}$$

So we can have two unknowns - the question "what was the final velocity AND the mass of the first object?" actually is well-posed and can be answered. We do this in the next example:

✔ Example 14.1.1: Inelastic vs Elastic collision in 1D

We want this example to be as simple as possible - a cart of mass m moving with an initial speed v_0 towards a cart of mass $3m$, which is initially stationary. They collide, and we want to consider two possible situations:

1. If the collision was inelastic, what was the final speed of the first cart? Here, we will assume the second cart moves off with a speed of $v_0/4$.

2. If the collision was elastic, what was the final speeds of *both carts*?

Solution

1. In the inelastic case, just momentum is conserved, so we have a fairly simple conservation of momentum problem:

$$\Delta p_x = p_{x,f} - p_{x,i} = 0 \rightarrow (mv_{1,f} + 3m\frac{v_0}{4}) - (mv_0) = 0. \tag{14.1.4}$$

The first step to solving this is recognizing that there is an m in every term, so we can divide by that. Physically, that means *the mass does not contribute to the physics at all* - the solution will be the same no matter what the mass is. Solving this for the final velocity gets us

$$v_{1,f} = \frac{v_0}{4}. \tag{14.1.5}$$

So, the first cart moves at one quarter the speed, no matter what it's initial mass is. *(Note that although the mass does not matter, the relative sizes of the masses do matter. If their ratio was anything besides $m_2/m_1 = 3$, we would get a different answer here. The same goes for the speeds - if we picked a final speed of something besides $v_0/4$, we would get a different final answer.)*

2. For the elastic case, we have the exact same conservation of momentum equation (now with the speed of the second cart not yet known!)

$$\Delta p_x = p_{x,f} - p_{x,i} = 0 \rightarrow (mv_{1,f} + 3mv_{2,f}) - (mv_0) = 0. \tag{14.1.6}$$

Further, we have the following conservation of energy equation,

$$\Delta E = E_f - E_i = 0 \rightarrow \left(\frac{1}{2}mv_{1,f}^2 + \frac{1}{2}(3m)v_{2,f}^2\right) - \frac{1}{2}mv_0^2 = 0. \tag{14.1.7}$$

These two expressions must be solving simultaneously, since we do not know what $v_{2,f}$ is! The first step is to eliminate the 1/2 and the m from the conservation of energy equation, and the mass from the momentum equation:

$$v_{1,f}^2 + 3v_{2,f}^2 - v_0^2 = 0 \qquad (v_{1,f} + 3v_{2,f}) - v_0 = 0 \tag{14.1.8}$$

We can proceed in several different ways - probably the easiest is to solve the momentum equation for the first speed:

$$v_{1,f} = v_0 - 3v_{2,f}, \tag{14.1.9}$$

and plug it into the energy equation:

$$(v_0 - 3v_{2,f})^2 + 3v_{2,f}^2 - v_0^2 = 0 \rightarrow v_0^2 - 6v_0v_{2,f} + 9v_{2,f}^2 + 3v_{2,f}^2 - v_0^2 = 0 \rightarrow 12v_{2,f}^2 - 6v_0v_{2,f} = 0. \tag{14.1.10}$$

Here we notice that we can divide by 6, as well as $v_{2,f}$, and find $v_{2,f} = v_0/2$. The final speed is different, but notice we have less freedom to pick the initial conditions - we can't choose how fast the second cart moves after the collision, it's always $v_0/2$.

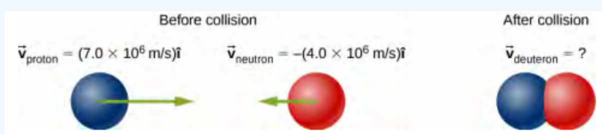
We still need to solve for the first cart - we can do that by going back to the solution for it's speed and plugging in our solution for the second:

$$v_{1,f} = v_0 - 3\frac{v_0}{2} = -\frac{v_0}{2}. \tag{14.1.11}$$

So in this case, the first cart bounced backwards, and moved at half the original speed.

Example 14.1.2: Formation of a deuteron

A proton (mass 1.67 x 10^{-27} kg) collides with a neutron (with essentially the same mass as the proton) to form a particle called a deuteron. What is the velocity of the deuteron if it is formed from a proton moving with velocity 7.0 x 10^6 m/s to the left and a neutron moving with velocity 4.0 x 10^6 m/s to the right?



Strategy

Define the system to be the two particles. This is a collision, so we should first identify what kind. Since we are told the two particles form a single particle after the collision, this means that the collision is perfectly inelastic. Thus, kinetic energy is not conserved, but momentum is. Thus, we use conservation of momentum to determine the final velocity of the system.

Solution

Treat the two particles as having identical masses M. Use the subscripts p, n, and d for proton, neutron, and deuteron, respectively. This is a one-dimensional problem, so we have

$$Mv_p - Mv_n = 2Mv_d. \tag{14.1.12}$$

The masses divide out:

$$\begin{aligned} v_p - v_n &= 2v_d \\ (7.0 \times 10^6\ m/s) - (4.0 \times 10^6\ m/s) &= 2v_d \\ v_d &= 1.5 \times 10^6\ m/s. \end{aligned}$$

The velocity is thus $\vec{v}_d = (1.5 \times 10^6\ m/s)\hat{i}$.

Significance

This is essentially how particle colliders like the Large Hadron Collider work: They accelerate particles up to very high speeds (large momenta), but in opposite directions. This maximizes the creation of so-called “daughter particles.”

Example 14.1.3: Ice hockey 2

(This is a variation of an earlier example.)

Two ice hockey pucks of different masses are on a flat, horizontal hockey rink. The red puck has a mass of 15 grams, and is motionless; the blue puck has a mass of 12 grams, and is moving at 2.5 m/s to the left. It collides with the motionless red puck (Figure 14.1.1). If the collision is perfectly elastic, what are the final velocities of the two pucks?

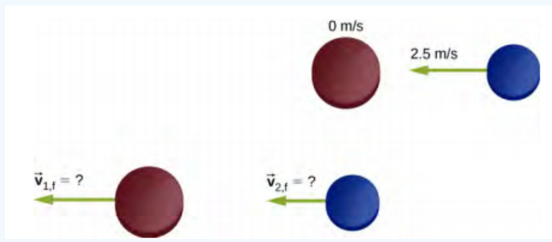


Figure 14.1.1: Two different hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

Strategy

We’re told that we have two colliding objects, and we’re told their masses and initial velocities, and one final velocity; we’re asked for both final velocities. Conservation of momentum seems like a good strategy; define the system to be the two pucks. There is no friction, so we have a closed system. We have two unknowns (the two final velocities), but only one equation. The comment about the collision being perfectly elastic is the clue; it suggests that kinetic energy is also conserved in this collision. That gives us our second equation.

The initial momentum and initial kinetic energy of the system resides entirely and only in the second puck (the blue one); the collision transfers some of this momentum and energy to the first puck.

Solution

Conservation of momentum, in this case, reads

$$\begin{aligned} p_i &= p_f \\ m_2 v_{2,i} &= m_1 v_{1,f} + m_2 v_{2,f}. \end{aligned}$$

Conservation of kinetic energy reads

$$\begin{aligned} K_i &= K_f \\ \frac{1}{2} m_2 v_{2,i}^2 &= \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2. \end{aligned}$$

There are our two equations in two unknowns. The algebra is tedious but not terribly difficult; you definitely should work it through. The solution is

$$v_{1,f} = \frac{(m_1 - m_2) v_{1,i} + 2 m_2 v_{2,i}}{m_1 + m_2} \tag{14.1.13}$$

$$v_{2,f} = \frac{(m_2 - m_1) v_{2,i} + 2 m_1 v_{1,i}}{m_1 + m_2} \tag{14.1.14}$$

Substituting the given numbers, we obtain

$$v_{1,f} = 2.22\ m/s \tag{14.1.15}$$

$$v_{2,f} = -0.28\ m/s. \tag{14.1.16}$$

Significance

Notice that after the collision, the blue puck is moving to the right; its direction of motion was reversed. The red puck is now moving to the left.

? Exercise 14.1.4

There is a second solution to the system of equations solved in this example (because the energy equation is quadratic): $v_{1,f}$ = −2.5 m/s, $v_{2,f}$ = 0. This solution is unacceptable on physical grounds; what's wrong with it?

✔ Example 14.1.5: Thor vs. iron man

The 2012 movie "The Avengers" has a scene where Iron Man and Thor fight. At the beginning of the fight, Thor throws his hammer at Iron Man, hitting him and throwing him slightly up into the air and against a small tree, which breaks. From the video, Iron Man is standing still when the hammer hits him. The distance between Thor and Iron Man is approximately 10 m, and the hammer takes about 1 s to reach Iron Man after Thor releases it. The tree is about 2 m behind Iron Man, which he hits in about 0.75 s. Also from the video, Iron Man's trajectory to the tree is very close to horizontal. Assuming Iron Man's total mass is 200 kg:

a. Estimate the mass of Thor's hammer
b. Estimate how much kinetic energy was lost in this collision

Strategy

After the collision, Thor's hammer is in contact with Iron Man for the entire time, so this is a perfectly inelastic collision. Thus, with the correct choice of a closed system, we expect momentum is conserved, but not kinetic energy. We use the given numbers to estimate the initial momentum, the initial kinetic energy, and the final kinetic energy. Because this is a one-dimensional problem, we can go directly to the scalar form of the equations.

Solution

a. First, we posit conservation of momentum. For that, we need a closed system. The choice here is the system (hammer + Iron Man), from the time of collision to the moment just before Iron Man and the hammer hit the tree. Let:
   - M_H = mass of the hammer
   - M_I = mass of Iron Man
   - v_H = velocity of the hammer before hitting Iron Man
   - v = combined velocity of Iron Man + hammer after the collision

Again, Iron Man's initial velocity was zero. Conservation of momentum here reads:

$$M_H v_H = (M_H + M_I)v. \tag{14.1.17}$$

We are asked to find the mass of the hammer, so we have

$$\begin{aligned} M_H v_H &= M_H v + M_1 v \\ M_H(v_H - v) &= M_I v \\ M_H &= \frac{M_I v}{v_H - v} \\ &= \frac{(200\ kg)\left(\frac{2\ m}{0.75\ s}\right)}{10\ m/s - \left(\frac{2\ m}{0.75\ s}\right)} \\ &= 73\ kg. \end{aligned}$$

Considering the uncertainties in our estimates, this should be expressed with just one significant figure; thus, $M_H = 7 \times 10^1$ kg.

b. The initial kinetic energy of the system, like the initial momentum, is all in the hammer: $

$$\begin{aligned} K_i &= \frac{1}{2} M_H v_H^2 \\ &= \frac{1}{2}(70\ kg)(10\ m/s)^2 \\ &= 3500\ J. \end{aligned}$$

$After the collision, $

$$\begin{aligned} K_f &= \frac{1}{2}(M_H + M_I)v^2 \\ &= \frac{1}{2}(70\ kg + 200\ kg)(2.67\ m/s)^2 \\ &= 960\ J. \end{aligned}$$

$Thus, there was a loss of 3500 J − 960 J = 2540 J.

Significance

From other scenes in the movie, Thor apparently can control the hammer's velocity with his mind. It is possible, therefore, that he mentally causes the hammer to maintain its initial velocity of 10 m/s while Iron Man is being driven backward toward the tree. If so, this would represent an external force on our system, so it would not be closed. Thor's mental control of his hammer is beyond the scope of this book, however.

Exercise 14.1.6

Suppose there had been no friction (the collision happened on ice); that would make μ_k zero, and thus $v_{c,f} = \sqrt{2\mu_k g d} = 0$, which is obviously wrong. What is the mistake in this conclusion?

Subatomic Collisions and Momentum

Conservation of momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

At the beginning of the twentieth century, there was considerable interest in, and debate about, the structure of the atom. It was known that atoms contain two types of electrically charged particles: negatively charged electrons and positively charged protons. (The existence of an electrically neutral particle was suspected, but would not be confirmed until 1932.) The question was, how were these particles arranged in the atom? Were they distributed uniformly throughout the volume of the atom (as J.J. Thomson proposed), or arranged at the corners of regular polygons (which was Gilbert Lewis' model), or rings of negative charge that surround the positively charged nucleus—rather like the planetary rings surrounding Saturn (as suggested by Hantaro Nagaoka), or something else?

The New Zealand physicist Ernest Rutherford (along with the German physicist Hans Geiger and the British physicist Ernest Marsden) performed the crucial experiment in 1909. They bombarded a thin sheet of gold foil with a beam of high energy (that is, high-speed) alpha-particles (the nucleus of a helium atom). The alpha-particles collided with the gold atoms, and their subsequent velocities were detected and analyzed, using conservation of momentum and conservation of energy.

If the charges of the gold atoms were distributed uniformly (per Thomson), then the alpha-particles should collide with them and nearly all would be deflected through many angles, all small; the Nagaoka model would produce a similar result. If the atoms were arranged as regular polygons (Lewis), the alpha-particles would deflect at a relatively small number of angles.

What **actually** happened is that nearly **none** of the alpha-particles were deflected. Those that were, were deflected at large angles, some close to 180° —those alpha-particles reversed direction completely (Figure 14.1.2). None of the existing atomic models could explain this. Eventually, Rutherford developed a model of the atom that was much closer to what we now have—again, using conservation of momentum and energy as his starting point.

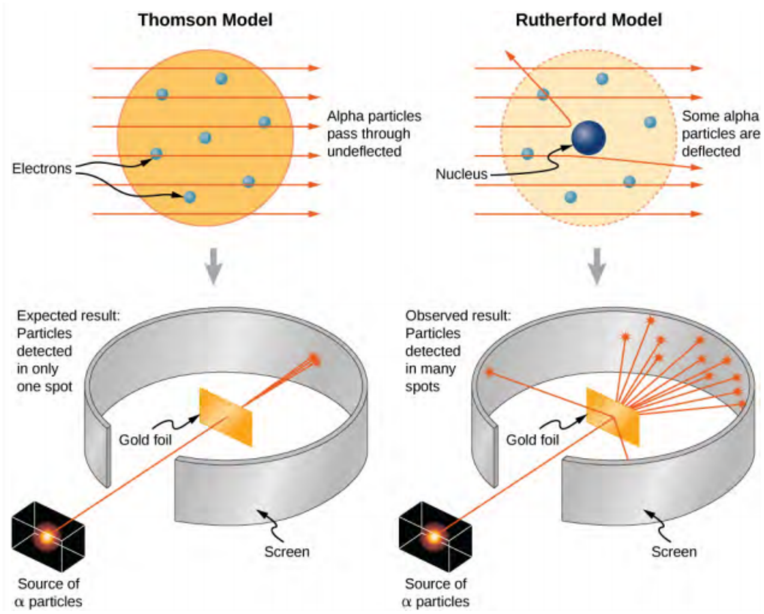


Figure 14.1.2: The Thomson and Rutherford models of the atom. The Thomson model predicted that nearly all of the incident alpha-particles would be scattered and at small angles. Rutherford and Geiger found that nearly none of the alpha particles were scattered, but those few that were deflected did so through very large angles. The results of Rutherford's experiments were inconsistent with the Thomson model. Rutherford used conservation of momentum and energy to develop a new, and better model of the atom—the nuclear model.