10.3: The center of mass

In this section, we show how to generalize Newton's Second Law so that it may describe the motion of an object that is not a point particle. Any object can be described as being made up of point particles; for example, those particles could be the atoms that make up regular matter. We can thus use the same terminology as in the previous sections to describe a complicated object as a "system" comprised of many point particles, themselves described by Newton's Second Law. A system could be a rigid object where the point particles cannot move relative to each other, such as atoms in a solid[1]. Or, the system could be a gas, made of many atoms moving around, or it could be a combination of many solid objects moving around.

In the previous section, we saw how the total momentum and the total mechanical energy of the system could be used to describe the system as a whole. In this section, we will define the center of mass which will allow us to describe the position of the system as a whole.

Consider a system comprised of N point particles. Each point particle i, of mass m_i, can be described by a position vector, $\vec r_i$, a velocity vector, $\vec v_i$, and an acceleration vector, $\vec a_i$, relative to some coordinate system in an inertial frame of reference. Newton's Second Law can be applied to any one of the particles in the system:

$$\sum_k \vec F_{ik} = m_i \vec a_i$$

where $\vec F_{ik}$ is the k-th force exerted on particle i. We can write Newton's Second Law once for each of the N particles, and we can sum those N equations together:

$$\begin{aligned} \sum_k \vec F_{1k} + \sum_k \vec F_{2k} + \sum_k \vec F_{3k} + \dots &= m_1\vec a_1 + m_2\vec a_2 + m_3\vec a_3 + \dots \\ \sum \vec F &= \sum_i m_i \vec a_i \end{aligned}$$

where the sum on the left is the sum of all of the forces exerted on all of the particles in the system[2] and the sum over i on the right is over all of the N particles in the system. As we have already seen, the sum of all of the forces exerted on the system can be divided into separate sums over external and internal forces:

$$\sum \vec F = \sum \vec F^{ext} + \sum \vec F^{int}$$

and the sum over the internal forces is zero[3]. We can thus write that the sum of the external forces exerted on the system is given by:

$$\sum \vec F^{ext} = \sum_i m_i \vec a_i \tag{10.3.1}$$

We would like this equation to resemble Newton's Second Law, but for the system as a whole. Suppose that the system has a total mass, M:

$$M = m_1 + m_2 + m_3 + \dots = \sum_i m_i$$

we would like to have an equation of the form:

$$\sum \vec F^{ext} = M \vec a_{CM} \tag{10.3.2}$$

to describe the system as a whole. However, it is not (yet) clear what is accelerating with acceleration, $\vec a_{CM}$, since the particles in the system could all be moving in different directions. Suppose that there is a point in the system, whose position is given by the vector, $\vec r_{CM}$, in such a way that the acceleration above is the second time-derivative of that position vector:

$$\vec a_{CM} = \frac{d^2}{dt^2}\vec r_{CM}$$

We can compare *Equations 10.3.1* and *10.3.2* to determine what the position vector $\vec r_{CM}$ corresponds to:

$$\begin{aligned}\sum \vec{F}^{ext} &= \sum_i m_i \vec{a}_i = \sum_i m_i \frac{d^2}{dt^2}\vec{r}_i \\ \sum \vec{F}^{ext} &= M\vec{a}_{CM} = M\frac{d^2}{dt^2}\vec{r}_{CM} \\ \therefore M\frac{d^2}{dt^2}\vec{r}_{CM} &= \sum_i m_i \frac{d^2}{dt^2}\vec{r}_i\end{aligned}$$

Re-arranging, and noting that the masses are constant in time, and so they can be factored into the derivatives:

$$\begin{aligned}\frac{d^2}{dt^2}\vec{r}_{CM} &= \frac{1}{M}\sum_i m_i \frac{d^2}{dt^2}\vec{r}_i \\ \frac{d^2}{dt^2}\vec{r}_{CM} &= \frac{d^2}{dt^2}\left(\frac{1}{M}\sum_i m_i\vec{r}_i\right) \\ \therefore \vec{r}_{CM} &= \frac{1}{M}\sum_i m_i\vec{r}_i\end{aligned}$$

where in the last line we set the quantities that have the same time derivative equal to each other[4]. $\vec{r}_{CM}$ is the vector that describes the position of the "center of mass" (CM). The position of the center of mass is described by Newton's Second Law applied to the system as a whole:

$$\sum \vec{F}^{ext} = M\vec{a}_{CM} \tag{10.3.3}$$

where M is the total mass of the system, and the sum of the forces is the sum over only external forces on the system.

Although we have formally derived Newton's Second Law for a system of particles, we really have been using this result throughout the text. For example, when we modeled a block sliding down an incline, we never worried that the block was made of many atoms all interacting with each other and the surroundings. Instead, we only considered the external forces on the block, namely, the normal force from the incline, any frictional forces, and the total weight of the object (the force exerted by gravity). Technically, the force of gravity is not exerted on the block as a whole, but on each of the atoms. However, when we sum the force of gravity exerted on each atom:

$$m_1\vec{g} + m_2\vec{g} + m_3\vec{g} + \dots = (m_1 + m_2 + m_3 + \dots)\vec{g} = M\vec{g}$$

we find that it can be modeled by considering the block as a single particle of mass M upon which gravity is exerted. The center of mass is sometimes described as the "center of gravity", because it **corresponds to the location where we can model the total force of gravity, $M\vec{g}$, as being exerted**. When we applied Newton's Second Law to the block, we then described the motion of the block as a whole (and not the motion of the individual atoms). Specifically, we modeled the motion of the center of mass of the block.

The position of the center of mass is a vector equation that is true for each coordinate:

$$\begin{aligned}\vec{r}_{CM} &= \frac{1}{M}\sum_i m_i\vec{r}_i \\ \therefore x_{CM} &= \frac{1}{M}\sum_i m_i x_i \\ \therefore y_{CM} &= \frac{1}{M}\sum_i m_i y_i \\ \therefore z_{CM} &= \frac{1}{M}\sum_i m_i z_i\end{aligned}$$

The center of mass is that **position in a system that is described by Newton's Second Law when it is applied to the system as a whole**. The center of mass can be thought of as an average position for the system (it is the average of the positions of the particles in the system, weighted by their mass). By describing the position of the center of mass, we are not worried about the detailed positions of the all of the particles in the system, but rather only the average position of the system as a whole. In other words, this is equivalent to viewing the whole system as a single particle of mass M located at the position of the center of mass.

Consider, for example, a person throwing a dumbbell that is made from two spherical masses connected by a rod, as illustrated in Figure 10.3.1. The dumbbell will rotate in a complex manner as it moves through the air. However, the center of mass of the dumbbell will travel along a parabolic trajectory (projectile motion), because the only external force exerted on the dumbbell during its trajectory is gravity.

Figure 10.3.1: The motion of the center of mass of a dumbbell is described by Newton's Second Law, even if the motion of the rotating dumbbell is more complex.

If we take the derivative with respect to time of the center of mass position, we obtain the velocity of the center of mass, and its components, which allow us to describe how the system is moving as a whole:

$$\vec{v}_{CM} = \frac{d}{dt}\vec{r}_{CM} = \frac{1}{M}\sum_i m_i \frac{d}{dt}\vec{r}_i = \frac{1}{m}\sum_i m_i \vec{v}_i$$

$$\therefore v_{CM_x} = \frac{1}{M}\sum_i m_i v_{i_x} \tag{10.3.4}$$

$$\therefore v_{CM_y} = \frac{1}{M}\sum_i m_i v_{i_y} \tag{10.3.5}$$

$$\therefore v_{CM_z} = \frac{1}{M}\sum_i m_i v_{i_z} \tag{10.3.6}$$

Note that this is the same velocity that we found earlier for the velocity of the center of mass frame of reference. In the center of mass frame of reference, the total momentum of the system is zero. This makes sense, because the center of mass represents the average position of the system; if we move "with the system", then the system appears to have zero momentum.

We can also define the total momentum of the system, $\vec{P}$, in terms of the total mass, M, of the system and the velocity of the center of mass:

$$\begin{aligned} \vec{P} &= \sum m_i \vec{v}_i = \frac{M}{M}\sum m_i \vec{v}_i \\ &= M\vec{v}_{CM} \end{aligned}$$

which we can also use in Newton's Second Law:

$$\frac{d}{dt}\vec{P} = \sum \vec{F}^{ext}$$

and again, we see that the total momentum of the system is conserved if the net external force on the system is zero. In other words, the center of mass of the system will move with constant velocity when momentum is conserved.

Finally, we can also define the acceleration of the center of mass by taking the time derivative of the velocity:

$$\vec{a}_{CM} = \frac{d}{dt}\vec{v}_{CM} = \frac{1}{M}\sum_i m_i \frac{d}{dt}\vec{v}_i = \frac{1}{M}\sum_i m_i \vec{a}_i$$

$$\therefore a_{CM_x} = \frac{1}{M}\sum_i m_i a_{i_x} \tag{10.3.7}$$

$$\therefore a_{CM_y} = \frac{1}{M}\sum_i m_i a_{i_y} \tag{10.3.8}$$

$$\therefore a_{CM_z} = \frac{1}{M}\sum_i m_i a_{i_z} \tag{10.3.9}$$

Example 10.3.1

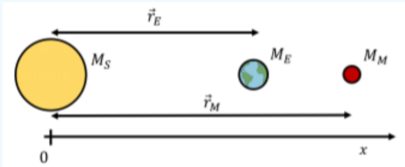

Figure 10.3.2: A syzygy between the Sun, Earth, and Mars.

In astronomy, a syzygy is defined as the event in which three bodies are all lined up along a straight line. For example, a syzygy occurs when the Sun (mass $M_S = 2.00\times10^{30}\text{kg}$), Earth (mass $M_E = 5.97\times10^{24}\text{kg}$), and Mars (mass $M_M = 6.39\times10^{23}\text{kg}$) are all lined up, as in Figure 10.3.2. How far from the center of the Sun is the center of mass of the Sun, Earth, Mars system during a syzygy?

Solution

Since this is a one-dimensional problem, we can define an x axis that is co-linear with the three bodies, and find only the x coordinate of the position of the center of mass. We are free to choose the origin of the coordinate system, so we choose the origin to be located at the center of the Sun. This way, the position of the center of mass along the x axis will directly correspond to its distance from the center of the Sun.

The Sun, Earth, and Mars are not point particles. However, because they are spherically symmetric, their centers of mass correspond to their geometric centers. We can thus model them as point particles with the mass of the body located at the corresponding geometric center. If $r_E = 1.50\times10^{11}\text{m}$ ($r_M = 2.28\times10^{11}\text{m}$) is the distance from the center of the Earth (Mars) to the center of the Sun, then the position of the center of mass is given by:

$$\begin{aligned} x_{CM} &= \frac{1}{M}\sum_i m_i x_i \\ &= \frac{M_S(0) + M_E r_E + M_M r_M}{M_S + M_E + M_M} \\ &= \frac{(2.00\times10^{30}\text{kg})(0) + (5.97\times10^{24}\text{kg})(1.50\times10^{11}\text{m}) + (6.39\times10^{23}\text{kg})(2.28\times10^{11}\text{m})}{(2.00\times10^{30}\text{kg}) + (5.97\times10^{24}\text{kg}) + (6.39\times10^{23}\text{kg})} \\ &= 5.21\times10^{5}\text{m} \end{aligned}$$

The center of mass of the Sun-Earth-Mars system during a syzygy is located approximately 500km from the center of the Sun.

Discussion

The radius of the Sun is approximately 700000km, so the center of mass of the system is well inside of the Sun. The Sun is so much more massive than either of the Earth or Mars, that the two planets hardly contribute to shifting the center of mass away from the center of the Sun. We would generally consider the masses of the two planets to be negligible if one wanted to model how the solar system itself moves around the Milky Way galaxy.

Example 10.3.2

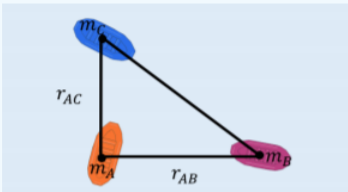

Figure 10.3.3: Three people on rafts on a lake.

Alice (mass m_A), Brice (mass m_B), and (mass m_C) are stranded on individual rafts of negligible mass on a lake, off of the coast of Nyon. The rafts are located at the corners of a right-angle triangle, as illustrated in Figure $10.3.3$, and are connected by ropes. The distance between Alice and Brice is r_{AB} and the distance between Alice and is r_{AC}, as illustrated. Alice decides to pull on the rope that connects her to , while Brice decide to pull on the rope that connects him to Alice. Where will the three rafts meet?

Solution

We consider the system comprised of the three people and their rafts and model each person and their raft as a point particle with the mass concentrated at the center of the raft. The forces exerted by pulling on the ropes are internal forces (one particle on the other), and will thus have no impact on the motion of the center of mass of the system. There are no net external forces exerted on the system (the forces of gravity are balanced out by the forces of buoyancy from the rafts). The center of mass of the system does not move when the people are pulling on the ropes, so they must ultimately meet at the center of mass.

We can define a coordinate system such that the origin is located where Alice is initially located, the x axis is in the direction from Alice to Brice, and the y axis is in the direction from Alice to Chloë. The initial positions of Alice, Brice, and are thus:

$$\begin{aligned}\vec r_A &= 0\hat x + 0\hat y\\ \vec r_B &= r_{AB}\hat x + 0\hat y\\ \vec r_C &= 0\hat x + r_{AC}\hat y\end{aligned}$$

respectively. The x and y coordinates of the center of mass are thus:

$$\begin{aligned}x_{CM} &= \frac{1}{M}\sum_i m_i x_i = \frac{m_A(0)+m_B r_{AB}+m_C(0)}{m_A+m_B+m_C} = \left(\frac{m_B}{m_A+m_B+m_C}\right) r_{AB}\\ y_{CM} &= \frac{1}{M}\sum_i m_i y_i = \frac{m_A(0)+m_B(0)+m_C r_{AC}}{m_A+m_B+m_C} = \left(\frac{m_C}{m_A+m_B+m_C}\right) r_{AC}\end{aligned}$$

which corresponds to the position where the three rafts will meet, relative to the initial position of Alice.

Discussion

By using the center of mass, we easily found where the three rafts would meet. If we had used Newton's Second Law on the three rafts individually, the model would have been complicated by the fact that the forces exerted by Alice and Brice on the ropes change direction as the rafts begin to move, which would have required the use of integrals to determine the motion of each person.

10.3.1: The center of mass for a continuous object

So far, we have considered the center of mass for a system made of point particles. In this section, we show how one can determine the center of mass for a "continuous object"[5]. We previously argued that if an object is uniform and symmetric, its center of mass will be located at the center of the object. Let us show this explicitly for a uniform rod of total mass M and length L, as depicted in Figure $10.3.4$

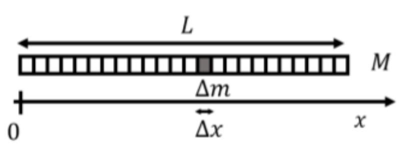

Figure 10.3.4: A rod of length L and mass M.

In order to determine the center of mass of the rod, we first model the rod as being made of N small "mass elements" each of equal mass, Δm, and of length Δx, as shown in Figure $10.3.4$ If we choose those mass elements to be small enough, we can model them as point particles, and use the same formulas as above to determine the center of mass of the rod.

We define the x axis to be co-linear with the rod, such that the origin is located at one end of the rod. We can define the "linear mass density" of the rod, λ, as the mass per unit length of the rod:

$$\lambda = \frac{M}{L}.$$

A small mass element of length Δx, will thus have a mass, Δm, given by:

$$\Delta m = \lambda \Delta x$$

If there are N mass elements that make up the rod, the x position of the center of mass of the rod is given by:

$$\begin{aligned} x_{CM} &= \frac{1}{M}\sum_i^N m_i x_i = \frac{1}{M}\sum_i^N \Delta m x_i \\ &= \frac{1}{M}\sum_i^N \lambda \Delta x x_i \end{aligned}$$

where x_i is the x coordinate of the i-th mass element. Of course, we can take the limit over which the length, Δx, of each mass element goes to zero to obtain an integral:

$$x_{CM} = \lim_{\Delta x \to 0} \frac{1}{M}\sum_i^N \lambda \Delta x x_i = \frac{1}{M}\int_0^L \lambda x dx$$

where the discrete variable x_i became the continuous variable x, and Δx was replaced by dx (which is the same, but indicates that we are taking the limit of $\Delta x \to 0$). The integral is easily found:

$$\begin{aligned} x_{CM} &= \frac{1}{M}\int_0^L \lambda x dx = \frac{1}{M}\lambda\left[\frac{1}{2}x^2\right]_0^L \\ &= \frac{1}{M}\lambda\frac{1}{2}L^2 = \frac{1}{M}\left(\frac{M}{L}\right)\frac{1}{2}L^2 \\ &= \frac{1}{2}L \end{aligned}$$

where we substituted the definition of λ back in to find, as expected, that the center of mass of the rod is half its length away from one of the ends.

Suppose that the rod was instead not uniform and that its linear density depended on the position x along the rod:

$$\lambda(x) = 2a + 3bx$$

We can still find the center of mass by considering an infinitesimally small mass element of mass dm, and length dx. In terms of the linear mass density and length of the mass element, dx, the mass dm is given by:

$$dm = \lambda(x)dx$$

The x position of the center of mass is thus found the same way as before, except that the linear mass density is now a function of x:

$$\begin{aligned} x_{CM} &= \frac{1}{M}\int_0^L \lambda(x) x dx = \frac{1}{M}\int_0^L (2a+3bx)x dx = \frac{1}{M}\int_0^L (2ax+3bx^2)dx \\ &= \frac{1}{M}\left[ax^2+bx^3\right]_0^L \\ &= \frac{1}{M}(aL^2+bL^3) \end{aligned}$$

In general, for a continuous object, the position of the center of mass is given by:

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r} dm$$

$$\therefore x_{CM} = \frac{1}{M}\int x dm \tag{10.3.10}$$

$$\therefore y_{CM} = \frac{1}{M}\int y dm \tag{10.3.11}$$

$$\therefore z_{CM} = \frac{1}{M}\int z dm \tag{10.3.12}$$

where in general, one will need to write dm in terms of something that depends on position (or a constant) so that the integrals can be evaluated over the spatial coordinates (x,y,z) over the range that describe the object. In the above, we wrote $dm = \lambda dx$ to express the mass element in terms of spatial coordinates.

Example 10.3.3

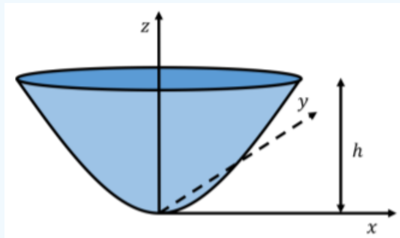

Figure 10.3.5: A symmetric bowl with parabolic sides is completely filled with water. The bowl has a height h.

A bowl of height h has parabolic sides and a circular cross-section, as illustrated in Figure 10.3.5. The bowl is filled with water. The bowl itself has a negligible mass and thickness, so that the mass of the full bowl is dominated by the mass of the water. Where is the center of mass of the full bowl?

Solution

We can define a coordinate system such that the origin is located at the bottom of the bowl and the z axis corresponds to the axis of symmetry of the bowl. Because the bowl is full of water, and the bowl itself has negligible mass, we can model the full bowl as a uniform body of water with the same shape as the bowl and (volume) mass density ρ equal to the density of water. Furthermore, by symmetry, the center of mass of the bowl will be on the z axis.

Because the bowl has a circular cross-section, we can divide it up into disk-shaped mass elements, dm, that have an infinitesimally small height dz, and a radius $r(z)$, that depends on their z coordinate (Figure 10.3.5).

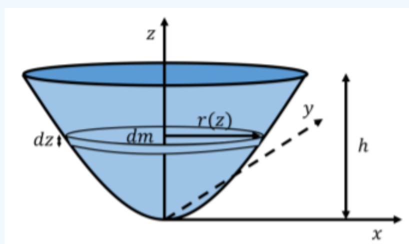

Figure 10.3.6: The parabolic bowl divided up into disk-shaped mass elements, dm, that have an infinitesimally small height dz, and a radius $r(z)$, that depends on their z coordinate.

The center of mass of each disk-shaped mass element will be located where the corresponding disk intersects the z axis. The mass of one disk element is given by:

$$dm = \rho dV = \rho\pi r^2(z)dz$$

where $dV = \pi r(z)^2 dz$ is the volume of the disk with radius $r(z)$ and thickness dz. The radius of the infinitesimal disk depends on its z position, since the radii of the different disks must describe a parabola:

$$\begin{aligned} z(r) &= \frac{1}{a^2}r^2 \\ r(z) &= a\sqrt{z} \\ \therefore dm &= \rho\pi r^2(z)dz = \rho\pi a^2 z dz \end{aligned}$$

where we introduced the constant a so that the dimensions are correct. The constant a determines how “steep” the parabolic sides are. The z coordinate of the center of mass is thus given by:

$$
\begin{aligned}
z_{CM} &= \frac{1}{M}\int z dm = \frac{1}{M}\int_0^h z(\rho\pi a^2 z dz) = \frac{\rho\pi a^2}{M}\int_0^h z^2 dz \\
&= \frac{\rho\pi a^2}{M}\left[\frac{1}{3}z^3\right]_0^h \\
&= \frac{\rho\pi a^2}{3M}h^3
\end{aligned}
$$

However, we are not quite done, since we do not know the total mass, M, of the water. To find the total mass of water, M, we proceed in an analogous way, and determine the value of the sum (integral) of all of the mass elements:

$$M = \int dm = \int_0^h \rho\pi a^2 z dz = \rho\pi a^2\left[\frac{1}{2}z^2\right]_0^h = \frac{1}{2}\rho\pi a^2 h^2$$

Substituting this value for M, we can determine the z coordinate of the center of mass of the full bowl:

$$z_{CM} = \frac{\rho\pi a^2}{3M}h^3 = \frac{2\rho\pi a^2}{3\rho\pi a^2 h^2}h^3 = \frac{2}{3}h$$

Regardless of the actual shape of the parabola (the parameter a), the center of mass will always be two thirds of the way up from the bottom of the bowl.

Discussion

In determining the center of mass of a three dimensional object, we used symmetry to argue that the x and y coordinates would be zero. We then found the z position of the center of mass by dividing up the bowl into infinitesimally small mass elements (disks) along the direction in which we needed to find the center of mass coordinate.

? Exercise 10.3.1

True or False: The center of mass of a continuous object is always located within the object.

A. True
B. False

Answer

10.3.2: Footnotes

1. In reality, even atoms in a solid can move relative to each other, but they do not move by large amounts compared to the object.

2. Again, we are summing together forces that are acting on **different** particles.

3. Recall, the internal forces are those forces that particles in the system are exerting on one another. Because of Newton's Third Law, these will sum to zero.

4. Technically, the terms in the derivatives are only equal to within two constants of integration, $\vec{r}_{CM} = \frac{1}{M}\sum_i m_i\vec{r}_i + at + b$, which we can set to zero.

5. In reality, there are of course no continuous objects since, at the atomic level, everything is made of particles.