# 23.6: Maxwell’s equations and electromagnetic waves

This section is meant to be informative, as the material is beyond the scope of this textbook. Nonetheless, it is worth summarizing what we have learned about electricity and magnetism, as Maxwell did. We can summarize the main laws from electromagnetism as follows:

$$\begin{aligned}
\oint \vec{E}\cdot d\vec{A} &= \frac{Q}{\epsilon_0} && \text{(Gauss' Law)}\\
\oint \vec{B}\cdot d\vec{A} &= 0 && \text{(No magnetic monopoles)}\\
\oint \vec{B}\cdot d\vec{l} &= \mu_0 I^{enc} && \text{(Ampere's Law)}\\
\oint \vec{E}\cdot d\vec{l} &= -\frac{d}{dt}\int \vec{B}\cdot d\vec{A} && \text{(Faraday's Law)}
\end{aligned}$$

where we wrote the magnetic flux in Faraday’s Law using the integral explicitly. As you recall, Gauss’ Law is equivalent to Coulomb’s Law, relating the electric field to electric charges that produce the electric field. Although we did not explicitly use the second equation, it is the equivalent to Gauss’ Law for the magnetic field. The flux of the magnetic field out of a closed surface must always be zero, since there are no magnetic monopoles, so that magnetic field lines never end.

When we covered ’s Law, we only considered a static current as the source of the magnetic field. However, if there is an electric field present, that is created by charges that are moving, then those can also contribute a current to ’s Law:

$$\begin{aligned}
\oint \vec{E}\cdot d\vec{A} &= \frac{Q}{\epsilon_0} \quad \text{(Gauss' Law)}\\
\therefore Q &= \epsilon_0 \oint \vec{E}\cdot d\vec{A}\\
\therefore I &= \frac{dQ}{dt} = \epsilon_0 \frac{d}{dt}\oint \vec{E}\cdot d\vec{A}
\end{aligned}$$

so that ’s Law, in its most general form, is written:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\left(I^{enc} + \epsilon_0 \frac{d}{dt}\oint \vec{E}\cdot d\vec{A}\right) \quad \text{(Ampere's Law)}$$

Writing out the four equations again:

$$\begin{aligned}
\oint \vec{E}\cdot d\vec{A} &= \frac{Q}{\epsilon_0} && \text{(Gauss' Law)}\\
\oint \vec{B}\cdot d\vec{A} &= 0 && \text{(No magnetic monopoles)}\\
\oint \vec{B}\cdot d\vec{l} &= \mu_0\left(I^{enc} + \epsilon_0 \frac{d}{dt}\oint \vec{E}\cdot d\vec{A}\right) && \text{(Ampere's Law)}\\
\oint \vec{E}\cdot d\vec{l} &= -\frac{d}{dt}\int \vec{B}\cdot d\vec{A} && \text{(Faraday's Law)}
\end{aligned}$$

These four equations are known as Maxwell’s equation, and form our most complete theory of classical electromagnetism. It is quite interesting to note the similarities and relations between the electric and magnetic field. Maxwell’s equations contain equations for the circulation and the total flux out of a closed surface for both fields. ’s Law implies that a changing electric field will produce a magnetic field. Faraday’s Law implies that a changing magnetic field produces an electric field. If a point charge oscillates up and down, it will produce a changing electric field, which will produce a changing magnetic field, which will induce a changing magnetic field, etc. This is precisely what an electromagnetic wave is! The light that we see, the wifi signals for our precious phones, the highly penetrating radiation from nuclear reactors are all examples of electromagnetic waves (of different wavelengths).

In fact, as Maxwell did, we can obtain the wave equation (Section 14.2) from Maxwell's equations. We sketch out the derivation here, but it is definitely beyond the scope of this textbook. However, you're so close to seeing one of the most exciting revelations of physics that it would be a shame to skip!

We first write out Maxwell's equations in differential form, as we have already shown for Gauss' Law (Section 17.4) and 's Law (Section 22.3)

$$\nabla\cdot\vec{E}=\frac{\rho}{\epsilon_0} \qquad \text{(Gauss' Law)}$$

$$\nabla\cdot\vec{B}=0 \qquad \text{(No magnetic monopoles)}$$

$$\nabla\times\vec{B}=\mu_0\left(\vec{j}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right) \qquad \text{(Ampere's Law)}$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad \text{(Faraday's Law)}$$

If we consider a vacuum region in space, with no charges and no currents, these equations reduce to:

$$\nabla\cdot\vec{E}=0 \qquad \nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{B}=\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

We will make use of the following identity from vector calculus:

$$\nabla\times(\nabla\times\vec{E})=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}$$

where:

$$\begin{aligned}\nabla^2\vec{E}&=\frac{\partial^2\vec{E}}{\partial x^2}+\frac{\partial^2\vec{E}}{\partial y^2}+\frac{\partial^2\vec{E}}{\partial z^2}\\&=\left(\frac{\partial^2E_x}{\partial x^2}+\frac{\partial^2E_x}{\partial y^2}+\frac{\partial^2E_x}{\partial z^2}\right)\hat{x}+\left(\frac{\partial^2E_y}{\partial x^2}+\frac{\partial^2E_y}{\partial y^2}+\frac{\partial^2E_y}{\partial z^2}\right)\hat{y}\\&+\left(\frac{\partial^2E_z}{\partial x^2}+\frac{\partial^2E_z}{\partial y^2}+\frac{\partial^2E_z}{\partial z^2}\right)\hat{z}\end{aligned}$$

is called the "vector Laplacian".

Consider taking the curl ($\nabla\times$) of the equation that has the curl of the electric field (Faraday's Law):

$$\begin{aligned}\nabla\times\left(\nabla\times\vec{E}\right.&=\left.-\frac{\partial\vec{B}}{\partial t}\right)\\\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}&=-\nabla\times\frac{\partial\vec{B}}{\partial t}\\-\nabla^2\vec{E}&=-\frac{\partial}{\partial t}\nabla\times\vec{B}\\-\nabla^2\vec{E}&=-\frac{\partial}{\partial t}\mu_0\epsilon_0\frac{\partial E}{\partial t}\\-\nabla^2\vec{E}&=-\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}\end{aligned}$$

where, in the third line, we made use of Gauss' Law ($\nabla\cdot\vec{E}=0$), and, in the fourth line, Ampere's Law ($\nabla\times\vec{B}=\mu_0\epsilon_0\frac{\partial E}{\partial t}$). The last equation that we obtained is a vector equation (the vector Laplacian has three components, as does the time-derivative of $\vec{E}$ on the right-hand side). Consider the $x$ component of this equation:

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}=\mu_0\epsilon_0\frac{\partial^2 E_x}{\partial t^2}$$

If we define the quantity:

$$c=\frac{1}{\sqrt{\epsilon_0\mu_0}}$$

then, the $x$ component of the equation can be written as:

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

which is exactly the wave equation for the component, $E_x$, of the electric field, propagating with a speed, $c$, the speed of light! Thus, the speed of light is directly related to the constant $\epsilon_0$, and $\mu_0$. You can write out similar equations for the $y$ and $z$ components of the electric field, and find the similar equations for the magnetic field if you start by taking the curl of 's Law instead of Faraday's Law.

We have just shown that electric and magnetic fields can behave as waves, which we now understand to be the waves that are responsible for light, radio waves, gamma rays, infra-red radiation, etc. All of these are types of electromagnetic waves, with different frequencies. Although we did not demonstrate this, the electromagnetic waves that propagate are such that the magnetic and electric field vectors are always perpendicular to each other. Electromagnetic waves also carry energy. Thus, a charge that is oscillating (say on a spring) and creating an electromagnetic wave must necessarily be losing energy (or work must be done to keep the charge oscillating with the same amplitude). Finally, it is worth noting that, according to Quantum Mechanics, light (and the other frequencies of radiation), are really carried by particles called "photons". Those particles are strange, since their propagation is described by a wave equation.

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