17.4: Interpretation of Gauss' Law and vector calculus

17.4.1: In this section, we provide a little more theoretical background and intuition on Gauss' Law, as well as its connection to vector calculus (which is beyond the scope of this textbook, but interesting to have a feeling for). Very generally, Gauss' Law is a statement that connects a property of a vector field to the "source" of that field. We think of mass as the source for the gravitational field, and we think of charge as the source for the electric field. The property of the field that we considered in this case was its "flux out of a closed surface".

Recall that determining the flux of a field out of a closed surface is equivalent to counting the net number of field lines that exit that closed surface. Field lines must start on a positive charge and must end on a negative charge. Thus, if there is a net number of field lines exiting the surface, there must be a positive charge in the volume defined by the surface (a "source" of field lines). If there is a net number of field lines entering the surface, then the volume defined by the surface must enclose a negative charge (a "sink" of field lines). Gauss' Law is simply a statement that the number of field lines entering/exiting a closed surface is proportional to the amount of charge enclosed in that volume.

The flux out of a closed surface is tightly connected to the vector calculus concept of "divergence", which describes whether field lines are diverging (spreading out or getting closer together). When a point charge is present, field lines will emanate radially from that point charge; in other words, they will diverge. We say that the electric field has non-zero divergence if there is a source of the electric field in that position of space. The key difference between the concept of divergence and that of "flux out of a closed surface", is that divergence is a local property of the field (it is true at a point), whereas the flux out of a surface must be calculated using a finite volume and makes it challenging to define the field at a specific position. Gauss's Law defined using flux is thus not as useful for describing how the field changes at specific positions, and is usually limited to situations with a high degree of symmetry.

The divergence, $\nabla\cdot\vec E$, of a vector field, $\vec E$, at some position is defined as:

$$\nabla\cdot\vec E=\frac{\partial E}{\partial x}+\frac{\partial E}{\partial y}+\frac{\partial E}{\partial z}$$

and corresponds to the sum of three partial derivatives evaluated at that position in space. Gauss' Theorem (also called the Divergence Theorem) states that:

$$\int_V\nabla\cdot\vec E=\oint_S\vec E\cdot d\vec A$$

where the V (S) on the integral indicates whether the sum (integral) should be carried out over a volume, V, or over a closed surface, S, as we have practised in this chapter. While it is not important at this level to understand the theorem in detail, the point is that one can convert a "flux over a closed surface" into an integral of the divergence of the field. In other words, we can convert a global property (flux) to a local property (divergence). Gauss' Law in terms of divergence can be written as:

$$\nabla\cdot\vec E=\frac{\rho}{\epsilon_0}\quad\text{(Local version of Gauss' Law)}\tag{17.4.1}$$

where ρ is the charge per unit volume at a specific position in space. This is the version of Gauss' Law that is usually seen in advanced textbooks and in Maxwell's unified theory of electromagnetism. This version of Gauss's Law relates a local property of the field (its divergence) to a local property of charge at that position in space (the charge per unit volume at that position in space). If we integrate both sides of the equation over volume, we recover the original formulation of Gauss' Law: the left hand side, by the Divergence Theorem, leads to flux when integrated over volume, whereas on the right hand side, the integral over volume of charge per unit volume, ρ, will give the total charge enclosed in that volume, Q^{enc}:

$$\begin{aligned}\int_V\left(\nabla\cdot\vec E\right)dV&=\int_V\left(\frac{\rho}{\epsilon_0}\right)dV\\ \oint_S\vec E\cdot d\vec A&=\frac{Q^{enc}}{\epsilon_0}\end{aligned}$$