8.7: Summary

8.7.1: Key Takeaways

A force is conservative if the work done by that force on a closed path is zero:

$$\oint \vec F(\vec r)\cdot d\vec l = 0$$

Equivalently, the force is conservative if the work done by the force on an object moving from position A to position B does not depend on the particular path between the two points. The conditions for a force to be conservative are given by:

$$\begin{aligned}\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}&=0\\ \frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}&=0\\ \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}&=0\end{aligned}$$

In particular, a force that is constant in magnitude and direction will be conservative. A force that depends on quantities other than position (e.g. speed, time) will not be conservative. The force exerted by gravity and the force exerted by a spring are conservative.

For any conservative force, $\vec F(\vec r)$, we can define a potential energy function, $U(\vec r)$, that can be used to calculate the work done by the force along any path between position A and position B:

$$-W = -\int_A^B \vec F(\vec r)\cdot d\vec l = U(\vec r_B) - U(\vec r_A) = \Delta U$$

where the change in potential energy function in going from A to B is equal to the negative of the work done in going from point A to point B. We can determine the function $U(\vec r)$ by calculating the work integral over an “easy” path (e.g. a straight line that is co-linear with the direction of the force).

It is important to note that an arbitrary constant can be added to the potential energy function, because only differences in potential energy are meaningful. In other words, we are free to choose the location in space where the potential energy function is defined to be zero.

We can break up the net work done on an object as the sum of the work done by conservative (W^C) and non-conservative forces (W^{NC}):

$$W^{net} = W^{NC} + W^C = W^{NC} - \Delta U$$

where ΔU is the difference in the total potential energy of the object (the sum of the potential energies for each conservative force acting on the object).

The Work-Energy Theorem states that the net work done on an object in going from position A to position B is equal to the object’s change in kinetic energy:

$$W^{net} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = \Delta K$$

We can thus write that the total work done by non conservative forces is equal to the change in potential and kinetic energies:

$$W^{NC} = \Delta K + \Delta U$$

In particular, if no non-conservative forces do work on an object, then the change in total potential energy is equal to the negative of the change in kinetic energy of the object:

$$-\Delta U = \Delta K$$

We can introduce the mechanical energy, E, of an object as:

$$E = U + K$$

The net work done by non-conservative forces is then equal to the change in the object's mechanical energy:

$$W^{NC} = \Delta E$$

In particular, if no net work is done on the object by non-conservative forces, then the mechanical energy of the object does not change ($\Delta E = 0$). In this case, we say that the mechanical energy of the object is conserved.

The Lagrangian description of classical mechanics is based on the Lagrangian, L:

$$L = K - U$$

which is the difference between the kinetic energy, K, and the potential energy, U, of the object. The equations of motion are given by the Principle of Least Action, which leads to the Euler-Lagrange equation (written here for the case of a particle moving in one dimension):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right) - \frac{\partial L}{\partial x} = 0$$

8.7.2: Important Equations

8.7.2.1: Conditions for a force to be conservative:

$$\oint \vec{F}(\vec{r}) \cdot d\vec{l} = 0$$

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0$$

$$\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = 0$$

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$

8.7.2.2: Potential energy for a conservative force:

$$\Delta U = -W$$

$$U(\vec{r}_B) - U(\vec{r}_A) = -\int_A^B \vec{F}(\vec{r}) \cdot d\vec{l}$$

8.7.2.3: Work-energy theorem:

$$W^{net} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = \Delta K$$

8.7.2.4: Work:

$$W^{net} = W^{NC} + W^{C} = W^{NC} - \Delta U$$

$$W^{NC} = \Delta K + \Delta U$$

8.7.2.5: Energy:

$$E = U + K$$

$$W^{NC} = \Delta E$$

8.7.2.6: Lagrange:

$$L = K - U$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right) - \frac{\partial L}{\partial x} = 0$$

8.7.3: Important Definitions

Definition

Conservative force: A force that does no net work when exerted over a closed path.

Definition

Potential energy: A form of energy that an object has by virtue of its position in space. The potential energy is associated with a conservative force, which is exerted in the direction that lowers the potential energy of the object. SI units: $[\mathrm{J}]$. Common variable(s): U.