10.4: Summary

10.4.1: Key Takeaways

The momentum vector, $\vec{p}$, of a point particle of mass, m, with velocity, $\vec{v}$, is defined as:

$$\vec{p}=m\vec{v}$$

We can write Newton's Second Law for a point particle in term of its momentum:

$$\frac{d}{dt}\vec{p}=\sum\vec{F}=\vec{F}^{net}$$

where the net force on the particle determines the rate of change of its momentum. In particular, if there is no net force on a particle, its momentum will not change.

The net impulse vector, $\vec{J}^{net}$, is defined as the net force exerted on a particle integrated from a time t_A to a time t_B:

$$\vec{J}^{net}=\int_{t_A}^{tB}\vec{F}^{net}dt$$

The net impulse vector is also equal to the change in momentum of the particle in that same period of time:

$$\vec{J}^{net}=\Delta\vec{p}=\vec{p}_B-\vec{p}_A$$

When we define a system of particles, we can distinguish between internal and external forces. Internal forces are those forces exerted by the particles in the system on each other. External forces are those forces on the particles in the system that are not exerted by the particles on each other. The sum over all of the forces on all of the particles in the system will be equal to the sum over the external forces, because the sum over all internal forces on a system is always zero (Newton's Third Law).

The total momentum of a system, $\vec{P}$, is the sum of the momenta, $\vec{p}_i$, of all of the particles in the system:

$$\vec{P}=\sum\vec{p}_i$$

The rate of change of the momentum of a system is equal to the sum of the external forces exerted on the system:

$$\frac{d}{dt}\vec{P}=\sum\vec{F}^{ext}$$

which can be thought of as an equivalent description as Newton's Second Law, but for the system as a whole. If the net (external) force on a system is zero, then the total momentum of the system is conserved.

Collisions are those events when the particles in a system interact (e.g. by colliding) and change their momenta. When modeling collisions, it is usually beneficial to first define a system for which the total momentum is conserved before and after the collision.

Collisions can be elastic or inelastic. Elastic collisions are those where, in addition to the total momentum, the total mechanical energy of the system is conserved. The total mechanical energy can usually be taken as the sum of the kinetic energies of the particles in the system.

Inelastic collisions are those in which the total mechanical energy of the system is not conserved. One can usually identify if mechanical energy was introduced or removed from the system and determine if the collision is elastic. It is important to identify when momentum and mechanical energy are conserved. Momentum is conserved if no net force is exerted on the system, whereas mechanical energy is conserved if no net work was done on the system by non-conservative forces (internal or external) or by external conservative forces.

We can always choose in which frame of reference to model a collision. In some cases, it is convenient to use the frame of reference of the center of mass of the system, because in that frame of reference, the total momentum of the system is zero.

If a system has a total mass M, then one can use Newton's Second Law to describe its motion:

$$\sum \vec{F}^{ext} = M\vec{a}_{CM}$$
$$\sum \vec{F}^{ext} = \frac{d}{dt}\vec{P}$$

where the sum of the forces is over all of the external forces on the system. The acceleration vector, $\vec{a}_{CM}$, describes the motion of the "center of mass" of the system. $\vec{P} = M\vec{v}_{CM}$ is the total momentum of the system.

The center of mass of a system is a mass-weighted average of the positions of all of the particles of mass m_i and position $\vec{r}_i$ that comprise the system:

$$\vec{r}_{CM} = \frac{1}{M}\sum_i m_i\vec{r}_i$$

The vector equation can be broken into components to find the x, y, and z component of the position of the center of mass. Similarly, one can also define the velocity of the center of mass of the system, in terms of the individual velocities, $\vec{v}_i$, of the particles in the system:

$$\vec{v}_{CM} = \frac{1}{M}\sum_i m_i\vec{v}_i$$

Finally, one can define the acceleration of the center of mass of the system, in terms of the individual accelerations, $\vec{a}_i$, of the particles in the system:

$$\vec{a}_{CM} = \frac{1}{M}\sum_i m_i\vec{a}_i$$

If the system is a continuous object, we can find its center of mass using a sum (integral) of infinitesimally small mass elements, dm, weighted by their position:

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}dm$$
$$\therefore x_{CM} = \frac{1}{M}\int xdm$$
$$\therefore y_{CM} = \frac{1}{M}\int ydm$$
$$\therefore z_{CM} = \frac{1}{M}\int zdm$$

The strategy to set up the integrals above is usually to express the mass element, dm, in terms of the position and density of the material of which the object is made. One can then integrate over position in the range defined by the dimensions of the object.

10.4.2: Important Equations

10.4.2.1: Momentum of a point particle:

$$\vec{p} = m\vec{v}$$
$$\frac{d}{dt}\vec{p} = \sum \vec{F} = \vec{F}^{net}$$

10.4.2.2: Impulse:

$$\vec{J}^{net} = \int_{t_A}^{tB} \vec{F}^{net} dt$$
$$\vec{J}^{net} = \Delta\vec{p} = \vec{p}_B - \vec{p}_A$$

10.4.2.3: Momentum of a system:

$$\vec{P}=\sum\vec{p}_i$$

$$\frac{d}{dt}\vec{P}=\sum\vec{F}^{ext}$$

10.4.2.4: Newton's Second Law for a system:

$$\sum\vec{F}^{ext}=M\vec{a}_{CM}$$

$$\sum\vec{F}^{ext}=\frac{d}{dt}\vec{P}$$

10.4.2.5: Position of the Center of Mass of a system:

$$\vec{r}_{CM}=\frac{1}{M}\sum_i m_i\vec{r}_i$$

10.4.2.6: Velocity of the Center of Mass of a system:

$$\vec{v}_{CM}=\frac{1}{M}\sum_i m_i\vec{v}_i$$

10.4.2.7: Acceleration of the Center of Mass of a system:

$$\vec{a}_{CM}=\frac{1}{M}\sum_i m_i\vec{a}_i$$

10.4.2.8: Position of the Center of Mass for a continuous object:

$$\vec{r}_{CM}=\frac{1}{M}\int\vec{r}dm$$

$$\therefore x_{CM}=\frac{1}{M}\int xdm$$

$$\therefore y_{CM}=\frac{1}{M}\int ydm$$

$$\therefore z_{CM}=\frac{1}{M}\int zdm$$

10.4.3: Important Definitions

Definition

Momentum: The product of velocity and mass. SI units: $[\mathrm{kgms}^{-1}]$. Common variable(s): $\vec{p}$.

Definition

Impulse: A property of matter which describes an object’s resistance to rotational motion. SI units: $[\mathrm{Ns}]$. Common variable(s): $\vec{J}$.