2.2: Units and dimensions

In 1999, the NASA Mars Climate Orbiter disintegrated in the Martian atmosphere because of a mixup in the units used to calculate the thrust needed to slow the probe and place it in orbit about Mars. A computer program provided by a private manufacturer used units of pounds seconds to calculate the change in momentum of the probe instead of the Newton seconds expected by NASA. As a result, the probe was slowed down too much and disintegrated in the Martian atmosphere. This example illustrates the need for us to **use and specify units** when we describe the properties of a physical quantity, and it also demonstrates the difference between a dimension and a unit.

"Dimensions" can be thought of as types of measurements. For example, length and time are both dimensions. A unit is the standard that we choose to quantify a dimension. For example, meters and feet are both units for the dimension of length, whereas seconds and jiffys[1] are units for the dimension of time.

When we compare two numbers, for example a prediction from a model and a measurement, it is important that both quantities have the same dimension and be expressed in the same units.

? Exercise 2.2.1

The speed limit on a highway...

A. has the dimension of length over time and can be expressed in units of kilometers per hour.
B. has the dimension of length can and be expressed in units of kilometers per hour.
C. has the dimension of time over length and can be expressed in units of meters per second.
D. has the dimension of time and can be expressed in units of meters.

Answer

2.2.1: Base dimensions and their SI units

In order to facilitate communication of scientific information, the International System of units (SI for the french, Systeme International d'unites) was developed. This allows us to use a well-defined convention for which units to use when describing quantities. For example, the SI unit for the dimension of length is the meter and the SI unit for the dimension of time is the second.

In order to simplify the SI unit system, a fundamental (base) set of dimensions was chosen and the SI units were defined for those dimensions. Any other dimension can always be re-expressed in terms of the base dimensions shown in *Table 2.2.1* and its units in terms of the corresponding combination of the base SI units.

Dimension	SI Unit
Length $[L]$	meter $[m]$
Time $[T]$	second $[s]$
Mass $[M]$	kilogram $[kg]$
Temperature $[\theta]$	kelvin $[K]$
Electric Current $[I]$	ampere $[A]$
Amount of Substance $[N]$	mole $[mol]$
Luminous Intensity $[J]$	candela $[cd]$
Dimensionless $[l]$	unitless $[]$

Table 2.2.1*:* Base dimensions and their SI units with abbreviations.

From the base dimensions, one can obtain "derived" dimensions such as "speed" which is a measure of how fast an object is moving. The dimension of speed is L/T (length over time) and the corresponding SI unit is m/s (meters per second)[2] Many of the derived dimension have corresponding derived SI units which can be expressed in terms of the base SI units. *Table 2.2.2* shows a few derived dimensions and their corresponding SI units and how those SI units are obtained from the base SI units.

Dimension	SI unit	SI base units
Speed $[L/T]$	meter per second $[m/s]$	$[m/s]$
Frequency $[1/T]$	hertz $[Hz]$	$[1/s]$
Force $[M \cdot L \cdot T^{-2}]$	newton $[N]$	$[kg \cdot m \cdot s^{-2}]$
Energy $[M \cdot L^2 \cdot T^{-2}]$	joule $[J]$	$[N \cdot m = kg \cdot m^2 \cdot s^{-2}]$
Power $[M \cdot L^2 \cdot T^{-3}]$	watt $[W]$	$[J/s = kg \cdot m^2 \cdot s^{-3}]$
Electric Charge $[I \cdot T]$	coulomb $[C]$	$[A \cdot s]$
Voltage $[M \cdot L^2 \cdot T^{-3} \cdot I^{-1}]$	volt $[V]$	$[J/C = kg \cdot m^2 \cdot s^{-3} \cdot A^{-1}]$

Table 2.2.2: *Example of derived dimensions and their SI units with abbreviations.*

By convention, we can indicate the dimension of a quantity, X, by writing it in square brackets, $[X]$. For example, $[X] = I$, would mean that the quantity X has the dimension I, so it has the dimension of electric current. Similarly, we can indicate the SI units of X with $SI[X]$. Referring to *Table 2.2.1*, since X has the dimension of current, $SI[X] = A$.

2.2.2: Dimensional analysis

We call "dimensional analysis" the process of working out the dimensions of a quantity in terms of the base dimensions and a model prediction for that quantity. A few simple rules allow us to easily work out the dimensions of a derived quantity. Suppose that we have two quantities, X and Y, both with dimensions. We then have the following rules to find the dimension of a quantity that depends on X and Y:

1. Addition/Subtraction: You can only add or subtract two quantities if they have the same dimension: $[X+Y] = [X] = [Y]$
2. Multiplication: The dimension of the product, $[XY]$, is the product of the dimensions: $[XY] = [X] \cdot [Y]$
3. Division: The dimension of the ratio, $[X/Y]$, is the ratio of the dimensions: $[X/Y] = [X]/[Y]$

The next two examples show how to apply dimensional analysis to obtain the unit or dimension of a derived quantity.

Example 2.2.1

Acceleration has SI units of ms^{-2} and force has the dimension of mass multiplied by acceleration. What are the dimensions and SI units of force, expressed in terms of the base dimensions and units?

Solution

We can start by expressing the dimension of acceleration, since we know from its SI units that it must have the dimension of length over time squared.

$$[acceleration] = \frac{L}{T^2}$$

Since force has the dimension of mass times acceleration, we have:

$$[force] = [mass] \cdot [acceleration] = M\frac{L}{T^2}$$

and the SI units of force are thus:

$$SI[force] = kg \cdot m/s^2$$

Force is such a common dimension that it, like many other derived dimensions, has its own derived SI unit, the Newton $[N]$.

Example 2.2.2

Use *Table 2.2.2* to show that voltage has the same dimension as force multiplied by speed and divided by electric current.

Solution

According to *Table 2.2.2*, voltage has the dimension:

$$[voltage] = M \cdot L^2 \cdot T^{-3} \cdot I^{-1}$$

while force, speed and current have dimensions:

$$= M \cdot L \cdot T^{-2}$$
$$[speed] = L \cdot T^{-1}$$
$$[current] = I$$

The dimension of force multiplied by speed divided by electric charge

$$\left[\frac{force \cdot speed}{current}\right] = \frac{[force] \cdot [speed]}{[current]} = \frac{M]c \cdot L \cdot T^{-2} \cdot L \cdot T^{-1}}{I}$$
$$= M \cdot L^2 \cdot T^{-3} \cdot I^{-1}$$

where, in the last line, we combined the powers of the same dimensions. By inspection, this is the same dimension as voltage.

When you build a model to predict the value of a physical quantity, you should always use dimensional analysis to ensure that the dimension of the quantity your model predicts is correct.

Example 2.2.3

Your model predicts that the speed, v, of an object of mass m, after having fallen a distance h on the surface of a planet with mass M and radius R is given by:

$$v = \frac{mMh}{R}$$

Is this a reasonable prediction?

Solution

First, we can see that the speed will be larger if h is bigger, which makes sense, since we expect the speed to be greater if the object fell a greater distance. Similarly, we expect that the speed would be higher if the mass of the planet, M, is larger, as it would exert a larger gravitational force, as given by this model. We also expect that the object will have a greater speed if it has a larger mass, m, if the drag from the atmosphere on the planet is significant. Finally, if the radius of the planet R is larger, we would expect the speed to be smaller, as the planet would be less dense and exert less gravitational force at its surface. However, if we verify the dimensions for the prediction of v, we find the model does not predict dimensions of speed:

$$= \frac{[m][M][h]}{[R]}$$
$$= \frac{MML}{L} = M^2$$

and our model predicts a speed with dimensions of mass squared. By performing simple dimensional analysis, we can easily confirm that our model is definitely wrong. You should always check the dimensions of any model prediction, to make sure it is correct.

Olivia's Thoughts

In this section, we were given three rules for combining dimensions. You'll notice that these rules are the same as the rules for algebra, except you're using dimensions instead of x's and y's. So, you can really just approach dimensional analysis problems as you would algebra problems.

There are some basic steps you can follow when you are trying to find the SI units for a value/variable in your equation. I'll go through *Example 2.2.1* in a bit of a different way. Let's say that you have the equation $F = ma$ and this time, you know the dimensions of F and m, and you want to find the dimensions of a:

1. Rewrite the values/variables in your equation in terms of their dimensions, leaving all other operations (multiplication, exponents, etc.) as is: $F = m \cdot a \rightarrow [F] = [m] \cdot [a]$
2. Rearrange for your unknown dimension: $[a] = \frac{[F]}{[m]}$

3. Substitute in your known dimensions: $[a]=\frac{[F]}{[m]} \rightarrow [a]=\frac{MLT^{-2}}{M}=\frac{ML}{MT^2}$
4. Solve using the rules of algebra: $[a]=\frac{L}{T^2}$ (where we just canceled out the M's)
5. Replace the dimensions with their corresponding SI units: $[a]=\frac{L}{T^2} \rightarrow SI[a]=\frac{m}{s^2}$

Exercise 2.2.2

In Chloe's theory of falling objects from Chapter 1, the time, t, for an object to fall a distance, x, was given by $t=k\sqrt{x}$. What must the SI units of Chloe's constant, k, be?

A. $TL^{\frac{1}{2}}$
B. $TL^{-\frac{1}{2}}$
C. $sm^{\frac{1}{2}}$
D. $sm^{-\frac{1}{2}}$

Answer

Dimensional analysis can also be used to determine formulas (usually to within an order of magnitude). One famous example of this is when a British physicist named G.I. Taylor was able to determine a formula that showed how the blast radius of an atomic bomb scaled with time. Using pictures of the first atomic bomb explosion, he was able to determine the amount of energy released in the explosion, which was classified information at the time.

Example 2.2.4

Find a formula that shows how the blast radius, r, scales with the time since the explosion, t, where the radius also depends on the energy released in the explosion, E, and the density of the medium into which the bomb explodes, ρ.

Solution

We want to find out how the blast radius scales with time, so we want an expression that relates r to some combination of E, ρ, and t:

$$r \sim E^x \rho^y t^z$$

where x, y, and z are our unknown exponents, since we don't know yet how we will combine E, ρ, and t. However, we do know that when we combine these quantities, we have to get the correct dimension (length) for the radius:

$$[r]=[E]^x[\rho]^y[t]^z$$

We know the dimensions for radius and time, and the dimension for E can be found in *Table 2.1.2*. Density is mass divided by volume, so its dimension is M/L3 . Our equation then becomes:

$$L=(ML^2T^{-2})^x(ML^{-3})^y(T)^z$$

$$L=(M^xL^{2x}T^{-2x})(M^yL^{-3y})(T^z)$$

We have three unknowns, so we need three equations. We can recognize that the left hand side (with dimension of length, L) is equivalent to $L^1 \cdot M^0 \cdot T^0$. We can then separate the above expression into three equations, one for each of M, L, and T:

$$M^0=M^xM^y \rightarrow 0=x+y$$

$$L^1=L^{2x}L^{-3y} \rightarrow 1=2x-3y$$

$$T^0=T^{-2x}T^z \rightarrow 0=z-2x$$

Solving the sytem of equations, we find that $x=\frac{1}{5}, y=-\frac{1}{5}$, and $z=\frac{2}{5}$. So, the combination of E, ρ, and t that gives us the dimension of length is:

$$r \sim E^{1/5}\rho^{-1/5}t^{2/5}$$

$$\therefore r \propto t^{2/5}$$

You can also write this equation as:

$$r \sim \sqrt[5]{\frac{Et^2}{\rho}}$$

Thus, by measuring the blast radius at some time, and knowing the density of the air, you can estimate the energy that was released during the explosion.

2.2.3: Footnotes

1. A jiffy is a unit used in electronics and generally corresponds to either $\frac{1}{50}$ or $\frac{1}{60}$ seconds.

2. Note that we can also write meters per second as $\mathrm{m{\cdot}s^{-1}}$, but we often use a divide by sign if the power of the unit in the denominator is 1.
