35.3: Inverse

Suppose we have vectors $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ such that $\mathbf{A} \times \mathbf{B} = \mathbf{C}$. If vectors $\mathbf{B}$ and $\mathbf{C}$ are known, can we solve for vector $\mathbf{A}$?

There is no such thing as a "cross division" operation, so we can't do anything similar to $A = C/B$. In fact, there is no unique solution for vector $\mathbf{A}$. There are an infinite number of vectors that can be crossed with $\mathbf{B}$ to yield vector $\mathbf{C}$; the smaller the angle between $\mathbf{A}$ and $\mathbf{B}$, the larger the magnitude $\mathbf{A}$ must have to yield a given vector $\mathbf{C}$.

To solve $\mathbf{A} \times \mathbf{B} = \mathbf{C}$ for vector $\mathbf{A}$, we will need to know vectors $\mathbf{B}$ and $\mathbf{C}$, along with one other piece of information, such as the magnitude of vector $\mathbf{A}$ or the angle θ between $\mathbf{A}$ and $\mathbf{B}$. Suppose the magnitude A of vector $\mathbf{A}$ is known; then since $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta = C$, we have

$$\sin\theta = \frac{C}{AB} \tag{35.3.1}$$

On the other hand, if θ is known, then

$$A = \frac{C}{B\sin\theta}. \tag{35.3.2}$$

In either case, we now know both the magnitude A and the angle θ. Then since $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$, we can now find the dot product $\mathbf{A} \cdot \mathbf{B}$. Now let's take

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}. \tag{35.3.3}$$

Crossing both sides on the right with vector $\mathbf{B}$, we get

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{B} = \mathbf{C} \times \mathbf{B}. \tag{35.3.4}$$

The left-hand side is a vector triple product; applying Eq. (32.3.8), we get

$$\mathbf{B}(\mathbf{A} \cdot \mathbf{B}) - B^2\mathbf{A} = \mathbf{C} \times \mathbf{B} \tag{35.3.5}$$

Solving for vector $\mathbf{A}$, we find

$$\mathbf{A} = \frac{1}{B^2}[(\mathbf{B} \times \mathbf{C}) + (\mathbf{A} \cdot \mathbf{B})\mathbf{B}] \tag{35.3.6}$$

So if either A or θ is known, then we can find $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$; knowing this and $\mathbf{B}$ and $\mathbf{C}$, Eq. (32.25) lets us solve for vector $\mathbf{A}$ (provided $\mathbf{B}, \mathbf{C} \neq \mathbf{0}$).