66.19: The Simple Plane Pendulum- Exact Solution

The solution to the simple plane pendulum problem described in Chapter 38 is only approximate; here we will examine the exact solution, which is surprisingly complicated. We will begin by deriving the differential equation of the motion, then find expressions for the angle θ from the vertical and the period T at any time t. We won't go through the derivations here-we'll just look at the results. Here we'll assume the amplitude of the motion $\theta_0 < \pi$, so that the pendulum does not spin in complete circles around the pivot, but simply oscillates back and forth.

The mathematics involved in the exact solution to the pendulum problem is somewhat advanced, but is included here so that you can see that even a very simple physical system can lead to some complicated mathematics.

Equation of Motion

To derive the differential equation of motion for the pendulum, we begin with Newton's second law in rotational form:

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} \tag{66.19.1}$$

where τ is the torque, I is the moment of inertia, α is the angular acceleration, and θ is the angle from the vertical. In the case of the pendulum, the torque is given by

$$\tau = -mgL\sin\theta \tag{66.19.2}$$

and the moment of inertia is

$$I = mL^2. \tag{66.19.3}$$

Substituting these expressions for τ and I into Eq. (S.1), we get the second-order differential equation

$$-mgL\sin\theta = mL^2\frac{d^2\theta}{dt^2} \tag{66.19.4}$$

which simplifies to give the differential equation of motion,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \tag{66.19.5}$$

Solution, $\theta(t)$

If the amplitude θ_0 is small, we can approximate $\sin\theta \approx \theta$, and find the position $\theta(t)$ at any time t is given by Eq. (38.6) in Chapter 38. But when the amplitude is not necessarily small, the angle θ from the vertical at any time t is found (by solving Eq. (S.5)) to be a more complicated function:

$$\theta(t) = 2\sin^{-1}\left\{k\,\mathrm{sn}\left[\sqrt{\frac{g}{L}}\,(t-t_0)\,;k\right]\right\} \tag{66.19.6}$$

where $\mathrm{sn}(x;k)$ is a Jacobian elliptic function with modulus $k = \sin(\theta_0/2)$. The time t_0 is a time at which the pendulum is vertical $(\theta = 0)$ and moving in the $+\theta$ direction.

The Jacobian elliptic function is one of a number of so-called "special functions" that often appear in mathematical physics. In this case, the function $\mathrm{sn}(x;k)$ is defined as a kind of inverse of an integral. Given the function

$$u(y;k) = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{66.19.7}$$

the Jacobian elliptic function is defined as:

$$\mathrm{sn}(u;k) = y. \tag{66.19.8}$$

Values of $\mathrm{sn}(x;k)$ may be found in tables of functions or computed by specialized mathematical software libraries.

Period

As found in Chapter 38, the approximate period of a pendulum for small amplitudes is given by

$$T_0=2\pi\sqrt{\frac{L}{g}} \tag{66.19.9}$$

This equation is really only an approximate expression for the period of a simple plane pendulum; the smaller the amplitude of the motion, the better the approximation. An exact expression for the period is given by

$$T=4\sqrt{\frac{L}{g}}\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{66.19.10}$$

which is a type of integral known as a complete elliptic integral of the first kind.

The integral in Eq. (S.10) cannot be evaluated in closed form, but it can be expanded into an infinite series. The result is

$$\begin{aligned}T&=2\pi\sqrt{\frac{L}{g}}\left\{1+\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\sin^{2n}\left(\frac{\theta_0}{2}\right)\right\}\\&=2\pi\sqrt{\frac{L}{g}}\left\{1+\sum_{n=1}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2\sin^{2n}\left(\frac{\theta_0}{2}\right)\right\}.\end{aligned}$$

We can explicitly write out the first few terms of this series; the result is

$$\begin{aligned}T=2\pi\sqrt{\frac{L}{g}}&\left[1+\frac{1}{4}\sin^2\left(\frac{\theta_0}{2}\right)+\frac{9}{64}\sin^4\left(\frac{\theta_0}{2}\right)+\frac{25}{256}\sin^6\left(\frac{\theta_0}{2}\right)\right.\\&+\frac{1225}{16384}\sin^8\left(\frac{\theta_0}{2}\right)+\frac{3969}{65536}\sin^{10}\left(\frac{\theta_0}{2}\right)+\frac{53361}{1048576}\sin^{12}\left(\frac{\theta_0}{2}\right)+\frac{184041}{4194304}\sin^{14}\left(\frac{\theta_0}{2}\right)\\&\left.+\frac{41409225}{1073741824}\sin^{16}\left(\frac{\theta_0}{2}\right)+\frac{147744025}{4294967296}\sin^{18}\left(\frac{\theta_0}{2}\right)+\frac{2133423721}{68719476736}\sin^{20}\left(\frac{\theta_0}{2}\right)+\cdots\right].\end{aligned}$$

If we wish, we can write out a series expansion for the period in another form—one which does not involve the sine function, but only involves powers of the amplitude θ_0. To do this, we expand $\sin(\theta_0/2)$ into a Taylor series:

$$\begin{aligned}\sin\frac{\theta_0}{2}&=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\theta_0^{2n-1}}{2^{2n-1}(2n-1)!}\\&=\frac{\theta_0}{2}-\frac{\theta_0^3}{48}+\frac{\theta_0^5}{3840}-\frac{\theta_0^7}{645120}+\frac{\theta_0^9}{185794560}-\frac{\theta_0^{11}}{81749606400}+\cdots\end{aligned}$$

Now substitute this series into the series of Eq. (S.11) and collect terms. The result is

$$\begin{aligned}T=2\pi\sqrt{\frac{L}{g}}&\left(1+\frac{1}{16}\theta_0^2+\frac{11}{3072}\theta_0^4+\frac{173}{737280}\theta_0^6+\frac{22931}{1321205760}\theta_0^8+\frac{1319183}{951268147200}\theta_0^{10}\right.\\&+\frac{233526463}{2009078326886400}\theta_0^{12}+\frac{2673857519}{265928913086054400}\theta_0^{14}\\&+\frac{39959591850371}{44931349155019751424000}\theta_0^{16}+\frac{8797116290975003}{109991942731488351485952000}\theta_0^{18}\\&\left.+\frac{4872532317019728133}{668751011807449177034588160000}\theta_0^{20}+\cdots\right).\end{aligned}$$

An entirely different formula for the exact period of a simple plane pendulum has appeared in a recent paper (Adlaj, 2012). According to Adlaj, the exact period of a pendulum may be calculated more efficiently using the arithmetic-geometric mean, by means of the formula

$$T=2\pi\sqrt{\frac{L}{g}}\times\frac{1}{\operatorname{agm}(1,\cos(\theta_0/2))} \tag{66.19.11}$$

where agm (x, y) denotes the arithmetic-geometric mean of x and y, which is found by computing the arithmetic and geometric means of x and y, then the arithmetic and geometric mean of those two means, then iterating this process over and over again until the two means converge:

$$a_{n+1}=\frac{a_n+g_n}{2}$$

$$g_{n+1}=\sqrt{a_n g_n}$$

Here a_n denotes an arithmetic mean, and g_n a geometric mean.

Shown in Fig. S. 1 is a plot of the ratio of the pendulum's true period T to its small-angle period T_0 $(T/(2\pi\sqrt{L/g}))$ vs. amplitude θ_0 for values of the amplitude between 0 and 180°, using Eq. (S.17). As you can see, the ratio is 1 for small amplitudes (as expected), and increasingly deviates from 1 for large amplitudes. The true period will always be longer than the small-angle period T_0.

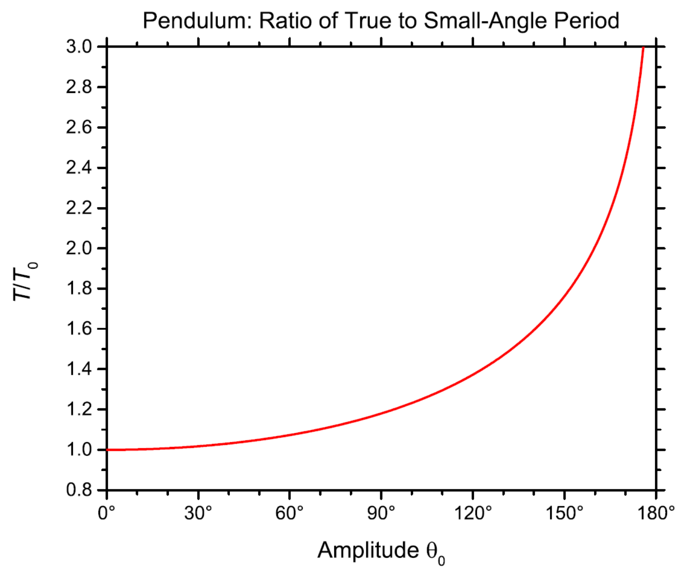

Figure 66.19.1: Ratio of a pendulum's true period T to its small-angle period $T_0=2\pi\sqrt{L/g}$, as a function of amplitude θ_0. For small amplitudes, this ratio is near 1 ; for larger amplitudes, the true period is longer than predicted by the small-angle approximation.

References

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