37.1: Introduction to the Moment of Inertia

The moment of inertia is the rotational counterpart of mass. It takes into account not only the total mass of the body, but also how far the mass is distributed from the axis of rotation: a body will have a higher moment of inertia if it has a higher mass, or if more of the mass is distributed farther from the rotation axis. Two bodies can have the same mass, but different moments of inertia, if their mass is distributed through the bodies differently.

To introduce the concept of moment of inertia, let's first look at a point mass m moving in a circle of radius r (Fig. $37.1.1$).

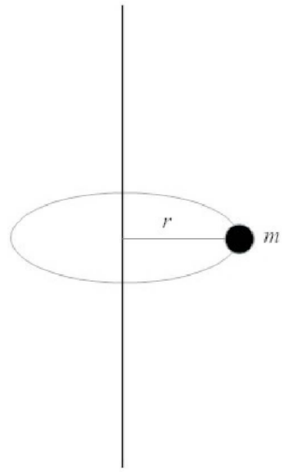

Figure $37.1.1$: The moment of inertia of a point mass m moving in a circle of radius r is $I = mr^2$.

The moment of inertia of the point mass is defined to be the mass times the square of its rotation radius:

$$I = mr^2 \tag{37.1.1}$$

In SI units, moment of inertia has units of kgm^2.

Knowing the definition of the moment of inertia of a single point mass, we may make use of the calculus to find the moment of inertia of any extended body. Imagine that we have some solid body that is rotating about some axis. Now imagine dividing the body into many infinitesimal cubes of mass dm, and treat each of these cubes as a point mass. If r is the perpendicular distance of dm from the rotation axis, then the moment of inertia of the body is found by adding up all the contributions $r^2 dm$ over the entire body by means of an integral:

$$I = \int r^2 dm \tag{37.1.2}$$

Note that, unlike with mass, it makes no sense to refer simply to the moment of inertia of a body—you must also specify the axis about which the body is rotated.

? Exercise $37.1.1$

As a simple example, let's find the moment of inertia of a uniform rod of length L and mass M when rotated about its center of mass (Fig. 34.2).

Answer

To set up the problem, we'll define an x axis running along the axis of the rod, and define the origin at the center of mass, as shown in the figure.

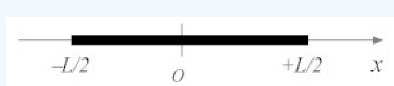

Figure $37.1.2$: Coordinate system for a rod of length L.

Now imagine dividing the rod into many infinitesimal segments of length dx. Each of these segments then has mass λdx, where λ is the density of the rod. Therefore, the moment of inertia of the rod is given by

Eq. 37.1.2 Since the distance r of dm from the rotation axis is $r = |x|$, we have

$$I = \int r^2 dm \tag{37.1.3}$$

$$= \int_{-L/2}^{L/2} x^2 \lambda dx \tag{37.1.4}$$

$$= \frac{\lambda}{3} x^3 \Big|_{-L/2}^{L/2} \tag{37.1.5}$$

$$= \frac{\lambda}{3} \left(\frac{L^3}{8} + \frac{L^3}{8} \right) \tag{37.1.6}$$

$$= \frac{1}{12} \lambda L^3. \tag{37.1.7}$$

Since the rod is uniform, its density is a constant $\lambda = M/L$; hence

$$I = \frac{1}{12} \frac{M}{L} L^3 \tag{37.1.8}$$

So the moment of an inertia of a uniform rod of length L and mass M when rotated about an axis perpendicular rod and passing through the center is

$$I = \frac{1}{12} ML^2 \tag{37.1.9}$$

Example 37.1.2

Let's repeat the previous example, but find the moment of inertia of the rod of length L and mass M when rotated about one end.

Solution

We move origin of the coordinate system to the left end; in this case $r = x$, we integrate from 0 to L, and we have

$$I = \int r^2 dm \tag{37.1.10}$$

$$= \int_0^L x^2 \lambda dx \tag{37.1.11}$$

$$= \frac{\lambda}{3} x^3 \Big|_0^L \tag{37.1.12}$$

$$= \frac{\lambda}{3} \left(L^3 - 0 \right) \tag{37.1.13}$$

$$=\frac{1}{3}\lambda L^3. \tag{37.1.14}$$

Since the rod is uniform, its density is a constant $\lambda = M/L$; hence

$$I=\frac{1}{3}\frac{M}{L}L^3 \tag{37.1.15}$$

So the moment of an inertia of a uniform rod of length L and mass M when rotated about an axis perpendicular rod and passing through one end is

$$I=\frac{1}{3}ML^2. \tag{37.1.16}$$

Example 37.1.3

As a third example, let's find the moment of inertia of a uniform thin hoop of mass M and radius R, when rotated about an axis passing through the center of the hoop and perpendicular to the plane of the hoop.

Solution

We imagine dividing the hoop into many infinitesimal segments of length ds. If the (constant) linear mass density of the hoop is λ, then the mass of each such segment is $dm = \lambda ds$. But the arc length $ds = Rd\theta$, so the mass of each segment becomes $dm = \lambda Rd\theta$. Since the distance of each segment from the rotation axis $r = R$, the moment of inertia I is then

$$I=\int r^2 dm \tag{37.1.17}$$

$$=\int_0^{2\pi} R^2\lambda Rd\theta \tag{37.1.18}$$

$$=\lambda R^3\int_0^{2\pi} d\theta \tag{37.1.19}$$

$$=2\pi\lambda R^3. \tag{37.1.20}$$

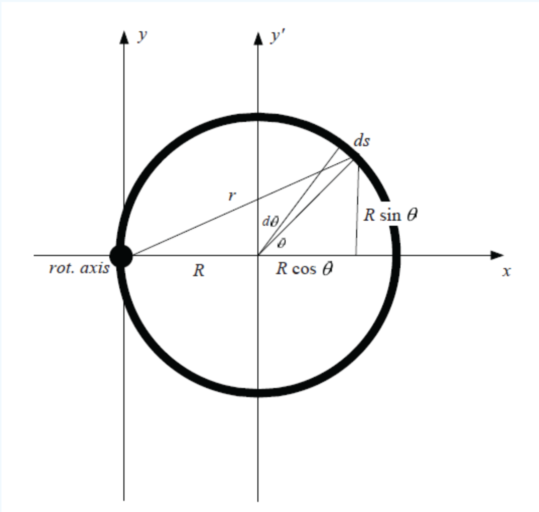

Figure 37.1.3: Calculation of the moment of inertia of a hoop when rotated about an axis passing through the hoop.

The linear mass density of the hoop λ is the total mass M divided by the total length $2\pi R$, so

$$I=2\pi\left(\frac{M}{2\pi R}\right)R^3 \tag{37.1.21}$$

so the moment of inertia of the hoop is

$$I=MR^2 \tag{37.1.22}$$

Example 37.1.4

As a fourth example, consider the same uniform thin hoop of mass M and radius R from the previous example — but this time, let's rotate it about an axis passing through the rim of the hoop, and perpendicular to the plane of the hoop (Figure $37.1.3$).

Solution

Then the moment of inertia is calculated as

$$I=\int r^2 dm \tag{37.1.23}$$

$$=\int_0^{2\pi} r^2\lambda R d\theta \tag{37.1.24}$$

as before. But this time, the distance from the infinitesimal piece of hoop at angle θ to the rotation axis is not R, but some more complicated function of θ. We'll need to derive a formula $r(\theta)$ for the distance to the rotation axis.

Looking at Figure $37.1.3$, this distance must be (from the Pythagorean theorem)

$$r^2=(R+R\cos\theta)^2+(R\sin\theta)^2 \tag{37.1.25}$$

$$=R^2\left[(1+\cos\theta)^2+\sin^2\theta\right] \tag{37.1.26}$$

$$=R^2\left(1+2\cos\theta+\cos^2\theta+\sin^2\theta\right) \tag{37.1.27}$$

$$=R^2(2+2\cos\theta) \tag{37.1.28}$$

$$=2R^2(1+\cos\theta) \tag{37.1.29}$$

This is the desired function $r(\theta)$ for the distance to the rotation axis. Putting this expression into the integral for the moment of inertia I, we have

$$I=\int_0^{2\pi}\left[2R^2(1+\cos\theta)\right]\lambda R d\theta \tag{37.1.30}$$

$$=2\lambda R^3\int_0^{2\pi}(1+\cos\theta)d\theta \tag{37.1.31}$$

$$=2\lambda R^3\left(\int_0^{2\pi}d\theta+\int_0^{2\pi}\cos\theta d\theta\right) \tag{37.1.32}$$

$$= 2\lambda R^3 \left(2\pi + \sin\theta\Big|_0^{2\pi}\right) \tag{37.1.33}$$

$$= 2\lambda R^3 (2\pi) \tag{37.1.34}$$

$$= 4\pi\lambda R^3 \tag{37.1.35}$$

Since the hoop is uniform, its density is the total mass divided by the total length: $\lambda = M/(2\pi R)$. The moment of inertia is then

$$I = 4\pi \left(\frac{M}{2\pi R}\right) R^3 \tag{37.1.36}$$

$$= 2MR^2 \tag{37.1.37}$$

In this same way, we can work out the moments of inertia of a number of common geometries. The results of such calculations are shown in Figure 37.2.1.