41.1: The Simple Plane Pendulum

A simple plane pendulum is a pendulum that consists of a point mass m at the end of a string of length L of negligible mass (Fig. 41.1.1). The pendulum is displaced from vertical by an angle θ_0 and released; after that, it swings back and forth under the influence of gravity. The pendulum is constrained to swing back and forth in a plane.

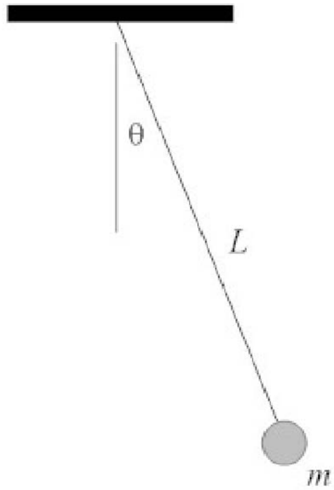

Figure 41.1.1: A simple plane pendulum.

When the pendulum makes an angle θ from the vertical, the torque acting to move it back toward vertical is $-mgL\sin\theta$. Then by the rotational version of Newton's second law of motion,

$$\tau = I\alpha \tag{41.1.1}$$

$$-mgL\sin\theta = mL^2\frac{d^2\theta}{dt^2} \tag{41.1.2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \tag{41.1.3}$$

This is a second-order differential equation that is fairly difficult to solve; the solution is shown in Appendix S. If we constrain the pendulum to small angles θ, then we can make the approximation

$$\sin\theta \approx \theta \quad (\theta \text{ in radians }). \tag{41.1.4}$$

Under this approximation, Eq. (38.3) becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \tag{41.1.5}$$

This is a second-order differential equation that's fairly easy to solve; you'll learn how to solve differential equations like this in a course on differential equations. The solution turns out to be

$$\theta(t) = \theta_0\cos(\omega t + \delta) \tag{41.1.6}$$

where θ_0 is the (angular) amplitude of the motion (in radians), $\omega = \sqrt{g/L}$ is the angular frequency of the motion (rad/s), and δ is an arbitrary integration constant (seconds). The solution can be verified by direct substitution into Eq. (38.5).

The period T of the motion (the time required for one complete back-and-forth cycle) is given by

$$T = \frac{2\pi}{\omega} \tag{41.1.7}$$

or

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{41.1.8}$$

Remember that this is an approximation, and is valid only for small θ. The period of motion for a large period is given by an infinite series, and is shown in Appendix S.