26.2: Potential Energy

Potential energy is stored energy; energy is stored in the system in some fashion. Once the potential energy is released, it can do work.

Since potential energy U is the ability to do work, it can be expressed as

$$U=-W=-\int F(x) d x \tag{26.2.1}$$

where W is the work done by the force. Unlike the kinetic energy K, for which there is a single formula (Eq. 26.2.8), potential energy has many formulæ, depending on what force present.

Gravity

For example, suppose the force is gravity between two point masses, for which the force given by $F=-G m_1 m_2 / r^2$. Then the potential energy is

$$U(r)=-\int F(r) d r \tag{26.2.2}$$

$$=-\frac{G m_1 m_2}{r}+C. \tag{26.2.3}$$

We now have a formula for the potential energy for the gravitational force-but what do we do with the integration constant C ? It turns out that C is completely arbitrary; you can set it to any convenient value. Values of potential energy actually have no physical meaning; it is only differences in potential energy that are physically meaningful, and for differences in potential energy, the integration constants C cancel out. This is an important feature of potential energy that you should keep in mind.

By convention, one generally chooses the integration constant for gravity so that $U=0$ when $r=\infty$, and by inspection of Eq. $26.2.3$ this implies that we choose $C=0$. So for gravity we find the potential energy function $U(r)$ to be

$$U(r)=-\frac{G m_1 m_2}{r} \quad \text{(gravity)}. \tag{26.2.4}$$

Note that the gravitational potential energy is always negative; as the masses become increasingly separated, the potential energy increases, becoming less and less negative, finally reaching zero when the masses are infinitely far apart.

Eq. $26.2.3$ may be used to find an expression for the gravitational potential energy of a mass m due to the Earth, which has mass $M_{\oplus}$ and radius $R_{\oplus}$:

$$U(r)=-\frac{G M_{\oplus} m}{r}+C \tag{26.2.5}$$

Here one may choose $U=0$ at $r=\infty$ (for which $C=0$), or one may choose $U=0$ at the surface of the Earth $\left(r=R_{\oplus}\right)$. The choice is arbitrary, and just depends on what is most convenient for the problem at hand. In the second case ($U=0$ at $r=R_{\oplus}$), we can find C by noting that

$$U\left(R_{\oplus}\right)=-\frac{G M_{\oplus} m}{R_{\oplus}}+C=0 \tag{26.2.6}$$

so

$$C=\frac{G M_{\oplus} m}{R_{\oplus}} \tag{26.2.7}$$

Thus

$$U(r)=G M_{\oplus} m\left(\frac{1}{R_{\oplus}}-\frac{1}{r}\right) \tag{26.2.8}$$

or in terms of the altitude $h=r-R_{\oplus}$,

$$U(h) = GM_{\oplus} m \left(\frac{1}{R_{\oplus}} - \frac{1}{h + R_{\oplus}} \right) \quad \text{(gravity, Earth).} \tag{26.2.9}$$

An important special case of this is when a body of mass m is a short altitude h above the surface of the Earth. In this case, Eq. (23.18) may reduced to a much simpler form. First, expand $1/(h + R_{\oplus})$ into a binomal series:

$$\frac{1}{h + R_{\oplus}} = \frac{1}{R_{\oplus}} - \frac{h}{R_{\oplus}^2} + \frac{h^2}{R_{\oplus}^3} - \frac{h^3}{R_{\oplus}^4} + \cdots \tag{26.2.10}$$

Now substitute this result into Eq. 26.2.9:

$$U(h) = GM_{\oplus} m \left[\frac{1}{R_{\oplus}} - \left(\frac{1}{R_{\oplus}} - \frac{h}{R_{\oplus}^2} + \frac{h^2}{R_{\oplus}^3} - \frac{h^3}{R_{\oplus}^4} + \cdots \right) \right] \tag{26.2.11}$$

$$= GM_{\oplus} m \left(\frac{h}{R_{\oplus}^2} - \frac{h^2}{R_{\oplus}^3} + \frac{h^3}{R_{\oplus}^4} - \cdots \right) \tag{26.2.12}$$

If $h \ll R_{\oplus}$, we can neglect all but the first term in the series in parentheses; we then have

$$U(h) \approx GM_{\oplus} m \left(\frac{h}{R_{\oplus}^2} \right) \tag{26.2.13}$$

$$= \frac{GM_{\oplus}}{R_{\oplus}^2} mh \tag{26.2.14}$$

Or, since $GM_{\oplus}/R_{\oplus}^2 = g$,

$$U(h) = mgh \quad \text{(gravity, near Earth surface).} \tag{26.2.15}$$

In this case, h is the height above any convenient surface. Choose what height you want to use for the $U = 0$ level at the beginning of a problem, then stay with that choice throughout the solution to the problem. A typical choice for many problems is to choose $U = 0$ at the floor, ground, or a tabletop, so that h is the height above that surface. But remember: the choice of where you choose $U = 0$ is arbitrary, so you can use any choice that is convenient.

Electric Potential Energy

As a third example, consider the electrostatic force between two point charges, which is similar in form to the gravitational force between point masses. The electrostatic force is given by Coulomb's law,

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \tag{26.2.16}$$

where q_1 and q_2 are the two charges in coulombs, r is their separation, and ε_0 is the permittivity of free space. Proceeding as we did with gravity, we find

$$U(r) = \frac{q_1 q_2}{4\pi\varepsilon_0 r} \quad \text{(electric force),} \tag{26.2.17}$$

where again we choose, by convention, to have $U = 0$ at $r = \infty$. In this case $U(r)$ will be negative if the charges are attracted, and positive if they are repelled.

Elastic Potential Energy

As a fourth example, consider the elastic force on a mass attached to a spring. In this case, the force is given by Hooke's law, $F = -kx$. The potential energy function is

$$U(r) = -\int (-kx) dx \tag{26.2.18}$$

$$=\frac{1}{2}kx^2+C \tag{26.2.19}$$

Conventionally we choose $C=0$, so that

$$U(r)=\frac{1}{2}kx^2 \quad \text{(spring, Hooke's law)}. \tag{26.2.20}$$

Summary

In summary:

- There are many different formulæ for potential energy, depending on what force is present. A few such formulæ are shown in Table 23-1.
- You can always add an arbitrary constant to the potential energy; it is only differences in potential energy that are physically meaningful.

Table $26.2.1$: A few formulæ for potential energy.

Force	Formula
Gravity	$U=-\frac{Gm_1m_2}{r}$
Gravity (near Earth surface)	$U=mgh$
Electric	$U=\frac{q_1q_2}{4\pi\varepsilon_0 r}$
Elastic (spring)	$U=\frac{1}{2}kx^2$