63.3: Example- Simple Harmonic Oscillator

As an example of the use of the Schrödinger equation, consider a one-dimensional simple harmonic oscillator. We wish to find the wave function $\psi(x,t)$ of the oscillator at any position x and time t.

The potential energy U of a simple harmonic oscillator is given by

$$U(x)=\frac{1}{2}kx^2, \tag{63.3.1}$$

where k is the spring constant. With this potential energy function, the time-independent Schrödinger equation (Eq. 63.2.2) becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2}+\frac{1}{2}kx^2\varphi(x)=E\varphi(x) \tag{63.3.2}$$

This is a second-order differential equation whose solution can be worked out using the theory of differential equations. The solution turns out to be

$$\varphi_n(x)=\sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}H_n(\alpha x)e^{-\frac{1}{2}\alpha^2x^2} \quad (n=0,1,2,3,\ldots) \tag{63.3.3}$$

Here α is defined by $\alpha^4\equiv mk/\hbar^2$ and the H_n are special functions called Hermite polynomials, the first few of which are shown in Table 60-1. Notice that the solution is quantized: only certain discrete solutions are allowed, which we find by substituting the integers $0,1,2,3,\ldots$for n.

The solutions to the time-dependent Schrödinger equation are then found by multiplying Eq. (60.7) by $e^{-iEt/\hbar}$:

$$\psi_n(x,t)=\sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}H_n(\alpha x)e^{-\frac{1}{2}\alpha^2x^2}e^{-iE_nt/\hbar} \quad (n=0,1,2,3,\ldots) \tag{63.3.4}$$

The physical significance of the wave function is that its square, $|\psi|^2=\psi^*\psi$, gives the probability of finding the particle at position x.[1] Squaring Eq. (60.8), we find this probability function for the harmonic oscillator is

$$|\psi_n(x)|^2=\frac{\alpha}{\sqrt{\pi}2^n n!}[H_n(\alpha x)]^2e^{-\alpha^2x^2} \quad (n=0,1,2,3,\ldots) \tag{63.3.5}$$

It turns out that the energy, like the wave function, is also quantized; the allowed values of E are

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega \quad (n=0,1,2,3,\ldots) \tag{63.3.6}$$

where $\omega=\sqrt{k/m}$ is the angular frequency of a classical simple harmonic oscillator. This is in contrast to the classical harmonic oscillator, which can have any value of energy, $E=kA^2/2$.

Table $63.3.1$ Hermite polynomials.

$H_0(x)=1$

$H_1(x)=2x$

$H_2(x)=4x^2-2$

$H_3(x)=8x^3-12x$

$H_4(x)=16x^4-48x^2+12$

$H_5(x)=32x^5-160x^3+120x$

$H_6(x)=64x^6-480x^4+720x^2-120$

Notice that the quantum simple harmonic oscillator has a minimum energy, called the zero-point energy, when $n=0: E_0=\hbar\omega/2$. The classical harmonic oscillator can have zero energy, but the not quantum harmonic oscillator-in quantum mechanics, there is always a minimum non-zero energy that the particle must have. The same is true of the atom: an electron can be in the lowest-energy K shell of the atom, but cannot have any lower energy. This is fortunate: if the electron energy were not quantized, it would have no minimum energy, and could spiral all the way in to the nucleus. Quantization of energy is what keeps the atom from collapsing.