9.3: Vector Arithmetic- Algebraic Methods

Although the graphical methods just described give a good intuitive picture of the mathematical operations, they can be a bit tedious to draw. A much more convenient and accurate alternative is the set of *algebraic* methods, which involve working with numbers instead of graphs. Before we can do that, though, we need to find a way to quantify a vector—to change it from a graph of an arrow to a set of numbers we can work with.

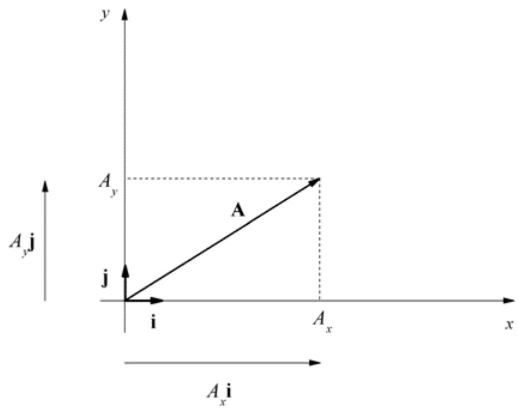

Figure 9.3.1: Cartesian components of a vector.

Rectangular Form

One idea would be to keep track of the coordinates of the head and tail of the vector. But remember that we are free to move a vector around whereever we want, as long as the direction and magnitude remain unchanged. So let's choose to always put the tail of the vector at the origin—that way, we only have to keep track of the head of the vector, and we cut our work in half. A vector can then be completely specified by just giving the coordinates of its head.

There's a little bit of a different way of writing this, though. We begin by defining two *unit vectors* (vectors with magnitude 1): **i** is a unit vector in the *x* direction, and **j** is a unit vector in the *y* direction. (In three dimensions, we add a third unit vector **k** in the ´ direction.)

Referring to Fig. 9.3.1, let A_x be the projection of vector **A** onto the *x*-axis, and let A_y by its projection onto the *y*-axis. Then, recalling the rules for the multiplication of a vector by a scalar, A_x**i** is a vector pointing in the *x*-direction, and whose length is equal to the projection A_x. Similarly, A_y **j** is a vector pointing in the *y*-direction, and whose length is equal to the projection A_y. Then by the parallelogram rule for adding two vectors, vector **A** is the sum of vectors A_x**i** and A_y**j** (Fig. 9.3.1). This means we can write a vector **A** as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} \tag{9.3.1}$$

or, if we're working in three dimensions,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}. \tag{9.3.2}$$

Eq. (9.3.1) or (9.3.2) is called the rectangular or cartesian [1] form of vector **A**.

Magnitude

The magnitude of a vector is a measure of its total "length." It is indicated with absolute value signs around the vector ($|\mathbf{A}|$ in type, or $|\vec{A}|$ in handwriting), or more simply by just writing the name of the vector in regular type (A; no boldface or arrow). In terms of rectangular components, the magnitude of a vector is simply given by the Pythagorean theorem:

$$|\mathbf{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{9.3.3}$$

Example. The magnitude of vector $\mathbf{A} = 2\mathbf{i} + 5\mathbf{j}$ is

$$|\mathbf{A}| = A = \sqrt{2^2 + 5^2} = \sqrt{29} = 5.3852$$

Polar Form

Instead of giving the x and y coordinates of the head of the vector, an alternative form is to give the magnitude and direction of the vector. This is called the polar form of a vector, and is indicated by the notation

$$\mathbf{A} = A\angle\theta \tag{9.3.4}$$

where A is the magnitude of the vector, and θ is the direction, measured counterclockwise from the $+x$ axis.

By convention, in polar form, we always take the magnitude of a vector as positive. If the magnitude comes out negative (as the result of a calculation, for example), then we can make it positive by changing its sign and adding 180° to the direction.

Converting between the rectangular and polar forms of a vector is fairly straightforward. To convert from polar to rectangular form, we use the definitions of the sine and cosine to get $\sin\theta = \text{opp}/\text{hyp} = A_y/A$, and $\cos\theta = \text{adj}/\text{hyp} = A_x/A$. Therefore to convert from polar to rectangular form, we use

$$A_x = A\cos\theta \tag{9.3.5}$$

$$A_y = A\sin\theta \tag{9.3.6}$$

To go the other way (rectangular to polar form), we just invert these equations to solve for A and θ. To solve for A, take the sum of the squares of both equations and add; to solve for θ, divide the A_y equation by the A_x equation. The results are

$$A = \sqrt{A_x^2 + A_y^2} \tag{9.3.7}$$

$$\tan\theta = \frac{A_y}{A_x} \tag{9.3.8}$$

To find θ, you must take the arctangent of the right-hand side of Eq. $9.3.8$. But be careful: to get the angle in the correct quadrant, you first compute the right-hand side of Eq. ($9.3.8$), then use the arctangent (TAN $^{-1}$) function on your calculator. If $A_x > 0$, then the calculator shows θ. But if $A_x < 0$, you must remember to add 180° (πrad) to the calculator's answer to get θ in the correct quadrant.

It is also possible to write three-dimensional vectors in polar form, but this requires a magnitude and two angles. We won't have any need to write three-dimensional vectors in polar form for this course.

Example 9.3.1

Polar to rectangular. Convert the vector $\mathbf{A} = 7\angle 40^\circ$ from polar form to rectangular form:

Solution

$$A_x = A\cos\theta = 7\cos 40^\circ = 5.3623 \tag{9.3.9}$$

$$A_y = A\sin\theta = 7\sin 40^\circ = 4.4995 \tag{9.3.10}$$

so the rectangular form is $\mathbf{A} = 5.3623\mathbf{i} + 4.4995\mathbf{j}$

Example 9.3.2

Rectangular to polar. Convert the vector $\mathbf{B} = -4\mathbf{i} + 8\mathbf{j}$ from rectangular form to polar form:

Solution

$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2} = \sqrt{(-4)^2 + 8^2} = \sqrt{16 + 64} = \sqrt{80} = 8.9443 \tag{9.3.11}$$

$$\tan\theta = \frac{B_y}{B_x} = \frac{8}{-4} = -2 \quad \Rightarrow \quad \theta = 116.565^\circ \tag{9.3.12}$$

so the polar form is $8.9443\angle 116.565^\circ$.

Notice that to find θ, we take the inverse tangent of -2 and the calculator returns -63.435°. But because the denominator (-4) is negative, we add 180° to the calculator's answer: $-63.435^\circ + 180^\circ = 116.565^\circ$. If the denominator had been positive, we would not have added this $180°$. For example, the rectangular vector $\mathbf{C}=4\mathbf{i}-8\mathbf{j}$ would be $8.9443\angle-64.435°$ in polar form.

Vector Equality

In order for two vectors to be equal, they must have the same magnitude and point in the same direction. This means that each of their components must be equal. For example, if $\mathbf{A}=\mathbf{B}$, then all of the following must be true:

$$A_x=B_x \tag{9.3.13}$$

$$A_y=B_y \tag{9.3.14}$$

$$A_z=B_z \tag{9.3.15}$$

Addition

Now we're ready to describe the algebraic method for the addition of two vectors. First, both vectors must be in rectangular (cartesian) form - you cannot add vectors in polar form. If you're given two vector in polar form and must add them, you must first convert them to rectangular form using Eq. (9.3.5-9.3.6).

Once the vectors are in rectangular form, you simply add the two vectors component by component: the x-component of the sum is the sum of the x components, etc.:

$$\begin{array}{rl}\mathbf{A} & =A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}\\ +\mathbf{B} & =B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}\\ \hline \mathbf{A}+\mathbf{B} & =(A_x+B_x)\,\mathbf{i}+(A_y+B_y)\,\mathbf{j}+(A_z+B_z)\,\mathbf{k}\end{array}$$

Subtraction

Just as with addition, vectors must be in rectangular (cartesian) form before they can be subtracted. Vector subtraction is similar to vector addition: you simply subtract the two vectors component by component:

$$\begin{array}{rl}\mathbf{A} & =A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}\\ -\mathbf{B} & =B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}\\ \hline \mathbf{A}-\mathbf{B} & =(A_x-B_x)\,\mathbf{i}+(A_y-B_y)\,\mathbf{j}+(A_z-B_z)\,\mathbf{k}\end{array}$$

Scalar Multiplication

Multiplication of a vector by a scalar may be done in either rectangular or polar form. In rectangular form, you multiply each component of the vector by the scalar. For example, given the vector $\mathbf{A}$ and scalar c :

$$c\mathbf{A}=c\,(A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}) \tag{9.3.16}$$

$$=cA_x\mathbf{i}+cA_y\mathbf{j}+cA_z\mathbf{k} \tag{9.3.17}$$

It's even simpler in polar form: if the vector $\mathbf{A}=A\angle\theta$, then

$$c\mathbf{A}=(cA)\angle\theta. \tag{9.3.18}$$

It's conventional to keep the vector magnitude positive, so if $cA<0$, you should change the sign of the magnitude cA, then add $180°$ (π radians) to the angle θ.

Example 9.3.3

Addition. Add the vectors $\mathbf{A}=6\mathbf{i}-9\mathbf{j}$ and $\mathbf{B}=2\mathbf{i}+12\mathbf{j}$:

Solution

$$\begin{aligned} \mathbf{A} &= 6\mathbf{i} - 9\mathbf{j} \\ +\mathbf{B} &= 2\mathbf{i} + 12\mathbf{j} \\ \hline \mathbf{A} + \mathbf{B} &= 8\mathbf{i} + 3\mathbf{j} \end{aligned}$$

Example 9.3.4

Subtraction. Subtract the vectors $\mathbf{A} = 6\mathbf{i} - 9\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} + 12\mathbf{j}$:

Solution

$$\begin{aligned} \mathbf{A} &= 6\mathbf{i} - 9\mathbf{j} \\ -\mathbf{B} &= 2\mathbf{i} + 12\mathbf{j} \\ \hline \mathbf{A} - \mathbf{B} &= 4\mathbf{i} - 21\mathbf{j} \end{aligned}$$

Example 9.3.5

Scalar multiplication. Multiply the vector $\mathbf{A} = 6\mathbf{i} - 9\mathbf{j}$ by 5 :

Solution

$$5 \times (6\mathbf{i} - 9\mathbf{j}) = 30\mathbf{i} - 45\mathbf{j}$$

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[1] The name cartesian is from Cartesius, the Latin form of the name of the French mathematician René Descartes, the founder of analytic geometry.
