60.1: Examples

Example 60.1.1 Simple Harmonic Oscillator

As an example of the use of Lagrange's equation, consider a one-dimensional simple harmonic oscillator. We wish to find the position x of the oscillator at any time \(t\

Solution

We begin by writing the usual expression for the kinetic energy K :

$$K=\frac{1}{2} m v^2 \tag{60.1.1}$$

The potential energy U of a simple harmonic oscillator is given by

$$U=\frac{1}{2} k x^2 \tag{60.1.2}$$

The Lagrangian in this case is then

$$\begin{aligned} L(x, v) &= K-U \\ &= \frac{1}{2} m v^2-\frac{1}{2} k x^2 \end{aligned}$$

Lagrange's equation in one dimension is

$$\frac{d}{d t}\left(\frac{\partial L}{\partial v}\right)-\frac{\partial L}{\partial x}=0 \tag{60.1.3}$$

Substituting for L from Eq. $60.1.4$, we find

$$\frac{d}{d t}\left[\frac{\partial}{\partial v}\left(\frac{1}{2} m v^2-\frac{1}{2} k x^2\right)\right]-\frac{\partial}{\partial x}\left(\frac{1}{2} m v^2-\frac{1}{2} k x^2\right)=0 \tag{60.1.4}$$

Evaluating the partial derivatives, we get

$$\frac{d}{d t}(m v)+k x=0 \tag{60.1.5}$$

or, since $v=d x / d t$,

$$m \frac{d^2 x}{d t^2}+k x=0 \tag{60.1.6}$$

which is a second-order ordinary differential equation that one can solve for $x(t)$. Note that the first term on the left is $m a=F$, so this equation is equivalent to $F=-k x$ (Hooke's Law). The solution to the differential equation (57.10) turns out to be

$$x(t)=A \cos (\omega t+\delta) \tag{60.1.7}$$

where A is the amplitude of the motion, $\omega=\sqrt{k / m}$ is the angular frequency of the oscillator, and δ is a phase constant that depends on where the oscillator is at $t=0$.

Example 60.1.2 Plane Pendulum

Part of the power of the Lagrangian formulation of mechanics is that one may define any coordinates that are convenient for solving the problem; those coordinates and their corresponding velocities are then used in place of x and v in Lagrange's equation.

For example, consider a simple plane pendulum of length ℓ with a bob of mass m, where the pendulum makes an angle θ with the vertical. The goal is to find the angle θ at any time t.

Solution

In this case we replace x with the angle θ, and we replace v with the pendulum's angular velocity ω. The kinetic energy K of the pendulum is the rotational kinetic energy

$$K=\frac{1}{2}I\omega^2=\frac{1}{2}m\ell^2\omega^2, \tag{60.1.8}$$

where I is the moment of inertia of the pendulum, $I=m\ell^2$. The potential energy of the pendulum is the gravitational potential energy

$$U=mg\ell(1-\cos\theta) \tag{60.1.9}$$

The Lagrangian in this case is then

$$\begin{aligned} L(\theta,\omega) &= K-U \\ &= \frac{1}{2}m\ell^2\omega^2-mg\ell(1-\cos\theta) \end{aligned}$$

Lagrange's equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\omega}\right)-\frac{\partial L}{\partial\theta}=0 \tag{60.1.10}$$

Substituting for L,

$$\frac{d}{dt}\left\{\frac{\partial}{\partial\omega}\left[\frac{1}{2}m\ell^2\omega^2-mg\ell(1-\cos\theta)\right]\right\}-\frac{\partial}{\partial\theta}\left[\frac{1}{2}m\ell^2\omega^2-mg\ell(1-\cos\theta)\right]=0 \tag{60.1.11}$$

Computing the partial derivatives, we find

$$\frac{d}{dt}\left(m\ell^2\omega\right)+mg\ell\sin\theta=0 \tag{60.1.12}$$

Since $\omega=d\theta/dt$, this gives

$$m\ell^2\frac{d^2\theta}{dt^2}+mg\ell\sin\theta=0, \tag{60.1.13}$$

which is a second-order ordinary differential equation that one may solve for the motion $\theta(t)$. The first term on the left-hand side is the torque τ on the pendulum, so this equation is equivalent to $\tau=-mg\ell\sin\theta$.

The solution to the differential equation ($60.1.17$) is quite complicated, but we can simplify it if the pendulum only makes small oscillations. In that case, we can approximate $\sin\theta\approx\theta$, and the differential equation ($60.1.17$ becomes a simple harmonic oscillator equation with solution

$$\theta(t)\approx\theta_0\cos(\omega t+\delta) \tag{60.1.14}$$

where θ_0 is the (angular) amplitude of the pendulum, $\omega=\sqrt{g/\ell}$ is the angular frequency, and δ is a phase constant that depends on where the pendulum is at $t=0$.