66.15: Angular Measure

Plane Angle

The most common unit of measure for plane angle is the degree $(^\circ)$, which is $1/360$ of a full circle. Therefore a circle is 360°, a semicircle is 180°, and a right angle is 90°.

A similar unit (seldom used nowadays) is a sort of "metric" angle called the grad, defined so that a right angle is 100 grads, and so a full circle is 400 grads.

The SI unit of plane angle is the radian (rad), which is defined to be the angle that subtends an arc length equal to the radius of the circle. By this definition, a full circle subtends an angle equal to the arc length of a full circle $(2\pi r)$ divided by its radius r - and so a full circle is 2π radians.

Since a hemisphere is 180° or π radians, the conversion factors are:

$$\text{rad} = \frac{\pi}{180} \times \text{deg}$$
$$\text{deg} = \frac{180}{\pi} \times \text{rad}$$

Subunits of the Degree

For small angles, a degree may be subdivided into 60 minutes ($'$), and a minute into 60 seconds ("). Thus a minute is $1/60$ degree, and a second is $1/3600$degree. [1] Angles smaller than 1 second are sometimes expressed as milli-arcseconds (1/1000 arcsecond). [2]

Solid Angle

A solid angle is the three-dimensional version of a plane angle, and is subtended by the vertex of a cone. The SI unit of solid angle is the steradian (sr), which is defined to be the solid angle that subtends an area equal to the square of the radius of a circle. By this definition, a full sphere subtends an area equal to the area of a sphere $\left(4\pi r^2\right)$ divided by the square of its radius $\left(r^2\right)$ - so a full sphere is 4π steradians, and a hemisphere is 2π steradians.
\footnotetext{[1] Sometimes these units are called the minute of arc or arcminute, and the second of arc or arcsecond to distinguish them from the units of time that have the same name.

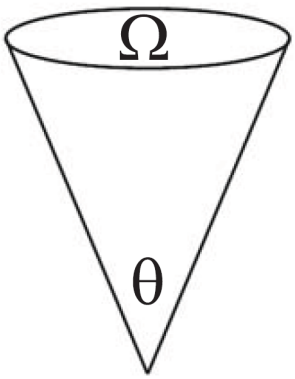

Figure 66.15.1: Relation between plane angle θ and solid angle Ω for a right circular cone.

There is a simple relation between plane angle and solid angle for a right circular cone. If the vertex of the cone subtends an angle θ (the aperture angle of the cone), then the corresponding solid angle Ω is (Fig. 66.15.1)

$$\Omega = 2\pi\left(1 - \cos\frac{\theta}{2}\right) \tag{66.15.1}$$

Another unit of solid angle is the square degree $\left(\text{deg}^2\right)$:

$$\text{sq. deg.} = \text{sr} \times \left(\frac{180}{\pi}\right)^2. \tag{66.15.2}$$

In these units, a hemisphere is $20,626.48\text{deg}^2$, and a complete sphere is $41,252.96\text{deg}^2$.

[1] Sometimes these units are called the minute of arc or arcminute, and the second of arc or arcsecond to distinguish them from the units of time that have the same name.

[2] In an old system (Ref. [10]), the second was further subdivided into 60 thirds ("'I'), the third into 60 fourths ($''''$), etc. Under this system, 1 milli-arcsecond is 3.6 fourths of arc. This system is no longer used, though; today the second of arc is simply subdivided into decimals (e.g. $32.86473''$).