18: Atwood’s Machine

Atwood's machine is a device invented in 1784 by the English physicist Rev. George Atwood. (See Fig. 18.1) The purpose of the device is
to permit an accurate measurement the acceleration due to gravity g. In the 18th century, without accurate timepieces or photogate timers, this was a difficult measurement to make with good accuracy. Atwood's machine has the effect of essentially scaling g to a smaller value, so the masses accelerate more slowly and allow g to be determined more easily.

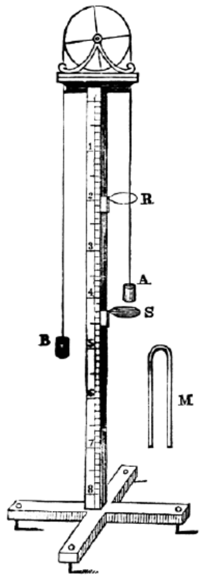

Figure 18.1: Atwood’s machine (Ref. [3]).

Let's see how the machine works. There are two identical masses (labeled A and B in the figure) connected by a light string that is strung over a pulley. Since the masses are identical, they will not move, regardless of whether one is higher than the other. The tall ($8 \mathrm{ft}$) vertical pole has a distance scale marked off in inches.

To use the machine, we move mass A to the top of the scale, and place a small U-shaped bar on top of the mass. (The bar is labeled M in the figure, but is shown somewhat enlarged; the actual bar would be just a little longer than the diameter of the ring.) Now mass A will begin accelerating downward until it reaches ring R. The mass will then pass through ring R, but the ring will lift the bar off the mass, so that the bar is left behind, sitting on the ring - in effect, the ring "locks in" the final post-acceleration velocity. After A passes through the ring, the masses on both ends of the string will be the same, so the acceleration will be zero and mass A will continue moving with a constant velocity until it lands on stage S. Both ring R and stage S are movable, and can be moved up and down the scale as needed.

To collect data, we use a pendulum as a timing device. Move ring R up and down until it takes one second for mass A (with bar M on it) to fall from the beginning of the scale to the ring. Then move stage S up and down until it takes one second for mass A (now missing bar M) to move from ring R to the stage S. The distance between the ring and the stage (divided by one second) gives the speed of mass A after it has accelerated for one second. Now repeat the experiment for the case where mass A takes two seconds to fall from the top of the scale to ring R, then again for three and four seconds; in each case move stage S so it is one second's falling time below the ring. In each case, the distance between the ring and the stage gives the velocity v of mass A after A has accelerated by the given number of seconds. Now since the acceleration is constant,

$$v=a t+v_{0} \tag{18.1}$$

and $v_{0}=0$, so the acceleration is $a=v / t$; for each experiment, we can then determine the acceleration a. In theory, a should be the same for each experiment, so we just take the average of the results.

Now that we've determined the acceleration of the masses a, how do we determine the actual acceleration due to gravity g ? To begin the analysis, let's first define some coordinate systems. For mass A, let $+x$ be downward, and for mass B, let $+x$ be upward; that way, as mass A is accelerating downward and B is accelerating upward, both will be accelerating in the $+x$ direction; obviously both masses must have the same acceleration a. Let the masses of A and B each be m, let the mass of the bar be m_{bar}, and let the tension in the string be T, which is the same throughout the length of the string. Now let's apply Newton's second law to both masses:

$$A: \quad \sum F_x=\left(m+m_{\text{bar}}\right) g-T=\left(m+m_{\text{bar}}\right) a \tag{18.2}$$

$$B: \quad \sum F_x=-m g+T=m a \tag{18.3}$$

Adding these two equations together, we get

$$m_{\text{bar}}\, g=\left(2 m+m_{\text{bar}}\right) a \tag{18.4}$$

and so the acceleration due to gravity g is determined from

$$g=\frac{2 m+m_{\text{bar}}}{m_{\text{bar}}} a, \tag{18.5}$$

where the acceleration a is determined as described earlier.

Conversely, if you already know g and wish to predict the acceleration of the masses in the machine,

$$a=\frac{m_{\text{bar}}}{2 m+m_{\text{bar}}} g. \tag{18.6}$$

More generally, if we refer to the two masses by their total mass and call them $m_A=m+m_{\text{bar}}$ and $m_B=m$, then the two masses accelerate with acceleration

$$a=\frac{m_A-m_B}{m_A+m_B} g. \tag{18.7}$$

The above equations may also be solved for the string tension T :

$$T=2 m\left(\frac{m+m_{\text{bar}}}{2 m+m_{\text{bar}}}\right) g=\frac{2 m_A m_B}{m_A+m_B} g \tag{18.8}$$

Complete Solution

Just for fun, let's work out the complete general solution for the Atwood's machine shown in Fig. 18.1. Suppose the mass with the bar is released from near the top of the scale at time $t=0$ and position x_0 on the scale; that the ring at position x_r lifts the bar off of the mass at time t_r; and that the the mass hits the stage at position x_s at time t_s. What is the acceleration due to gravity g in terms of x_0, t_r, x_r, t_s, and x_s ?

Let's begin with the motion between the release at time $t=0$ and reaching the ring at time $t=t_r$. The mass is accelerating with constant acceleration a, so from the equations of one-dimensional kinematics (Eq. (8.7.11)), we have at $t=t_r$

$$x_r=\frac{1}{2} a t_r^2+v_0 t_r+x_0, \tag{18.9}$$

where $v_0=0$ since the mass is released from rest. Assuming the scale increases going downward, the acceleration will be positive and given by Eq. 18.6; we then have

$$x_r=\frac{1}{2}\left(\frac{m_{\text{bar}}}{2 m+m_{\text{bar}}} g\right) t_r^2+x_0 \tag{18.10}$$

If we knew the time t_r at which the mass reaches the ring, then we would be finished, just by solving Eq. 18.10 for g-there would be no need for the stage later on. So apparently the time t_r was not measured directly in practice. Let's continue the analysis, incorporating information about the motion of the mass between the ring and the stage.

After the ring lifts the bar off of the mass, the mass will be moving at a constant velocity v_r given by

$$v_r=a t_r+v_0 \tag{18.11}$$

$$=\left(\frac{m_{\text{bar}}}{2 m+m_{\text{bar}}} g\right) t_r. \tag{18.12}$$

Solving for t_r,

$$t_r=\left(\frac{2m+m_{\text{bar}}}{m_{\text{bar}}g}\right)v_r \tag{18.13}$$

Substituting this for t_r into Eq. 18.10, we have

$$x_r=\frac{1}{2}\left(\frac{m_{\text{bar}}g}{2m+m_{\text{bar}}}\right)\left(\frac{2m+m_{\text{bar}}}{m_{\text{bar}}g}\right)^2 v_r^2+x_0 \tag{18.14}$$

$$=\frac{1}{2}\left(\frac{2m+m_{\text{bar}}}{m_{\text{bar}}g}\right)v_r^2+x_0 \tag{18.15}$$

We don't know the velocity v_r, so we'll need to eliminate it. At velocity v_r, the mass will move from the ring at $x=x_r$ to the stage at $x=x_s$ in time

$$\Delta t=\frac{x_s-x_r}{v_r} \tag{18.16}$$

where $\Delta t\equiv t_s-t_r$. Solving for v_r,

$$v_r=\frac{x_s-x_r}{\Delta t} \tag{18.17}$$

Using this expression to substitute for v_r in Eq. 18.15, we have

$$x_r=\frac{1}{2}\left(\frac{2m+m_{\text{bar}}}{m_{\text{bar}}g}\right)\left(\frac{x_s-x_r}{\Delta t}\right)^2+x_0. \tag{18.18}$$

Solving for g,

$$2\left(x_r-x_0\right)=\left(\frac{2m+m_{\text{bar}}}{m_{\text{bar}}g}\right)\left(\frac{x_s-x_r}{\Delta t}\right)^2 \tag{18.19}$$

$$2\left(x_r-x_0\right)\left(\frac{\Delta t}{x_s-x_r}\right)^2=\frac{2m+m_{\text{bar}}}{m_{\text{bar}}g} \tag{18.20}$$

or

$$g=\left[\frac{2m+m_{\text{bar}}}{2m_{\text{bar}}\left(x_r-x_0\right)}\right]\left(\frac{x_s-x_r}{\Delta t}\right)^2 \tag{18.21}$$

Simplifying somewhat, we have

$$g=\left[\frac{(m/m_{\text{bar}})+\frac{1}{2}}{x_r-x_0}\right]\left(\frac{x_s-x_r}{\Delta t}\right)^2 \tag{18.22}$$

Apparently in operating Atwood's machine, one needed to measure the masses (m and m_{bar}), the positions of the ring and stage (x_r and x_s), and the amount of time Δt it takes the mass to move from the ring to the stage. Then the acceleration due to gravity g would be given by Eq. 18.22.