23.1: Introduction to Circular Motion

As we've already seen, the acceleration vector a is defined by Eq. (11.1.2):

$$\mathbf{a}=\frac{d\mathbf{v}}{dt} \tag{23.1.1}$$

This says that acceleration is the time rate of change of the velocity. We typically think of an acceleration as being a change in the magnitude of the velocity, but it can also be a change in the direction of the velocity. For example, if a particle is moving in a circle at constant speed, it is still accelerating: the velocity vector, while not changing its magnitude, is changing its direction with time.

So if we have a particle moving in a circle of radius r with a constant speed v, then it's accelerating; what are the magnitude and direction of the acceleration vector? The situation is shown in Fig. $23.1.1$.

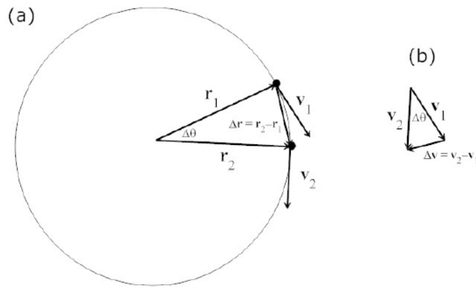

Figure $23.1.1$: (a) A body moving in a circular path of radius r with constant speed v. At time t_1, the body is at position $\mathbf{r}_1$ and has velocity $\mathbf{v}_1$; at a slightly later time t_2, it is at position $\mathbf{v}_2$ and has velocity $\mathbf{v}_2$. (b) The difference in velocity vectors $\mathbf{v}_2-\mathbf{v}_1$ is the direction of the acceleration vector.

Note that since the path is circular, the triangle in Fig. $23.1.1$(a) is an isosceles triangle with apex angle $\Delta\theta$. Since the speed v is constant, the triangle in Fig. $23.1.1$(b) is also isosceles, and also has apex angle $\Delta\theta$. Since the two isosceles triangles have the same vertex angle, it follows from geometry that the two triangles are similar. Therefore

$$\frac{|\Delta\mathbf{v}|}{v}=\frac{|\Delta\mathbf{r}|}{r} \tag{23.1.2}$$

Multiplying both sides by $v/\Delta t$,

$$\frac{|\Delta\mathbf{v}|}{\Delta t}=\frac{v}{r}\frac{|\Delta\mathbf{r}|}{\Delta t}. \tag{23.1.3}$$

Taking the limit as $\Delta t\to 0$,

$$\lim_{\Delta t\to 0}\frac{|\Delta\mathbf{v}|}{\Delta t}=\frac{v}{r}\lim_{\Delta t\to 0}\frac{|\Delta\mathbf{r}|}{\Delta t} \tag{23.1.4}$$

or

$$\frac{dv}{dt}=\frac{v}{r}\frac{dr}{dt} \tag{23.1.5}$$

The left-hand side dv/dt is just the acceleration a_c; the second factor on the right-hand side dr/dt is the velocity v. Therefore this equation becomes

$$a_c=\frac{v^2}{r} \tag{23.1.6}$$

We've now found the magnitude of the acceleration of a particle moving in a circle: it's the square of its speed divided by the radius of the circle. This acceleration is called the centripetal acceleration.

The direction of the centripetal acceleration vector can be seen by examining Fig. $23.1.1$(b): by inspection, you can see that the acceleration vector a points inward, toward the center of the circle. [1]

In summary, if a particle is moving in a circular path of radius r with constant speed v, its acceleration is:

- in magnitude: $a_c=v^2/r$;

- in direction: inward, toward the center of the circle.

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[1] Since $\mathbf{a} = \Delta\mathbf{v}/\Delta t$ points in the same direction as $\Delta\mathbf{v}$.
