33.1: The Rocket Equation

Let's now derive the rocket equation. Given a rocket of mass m, we will wish to find an equation that tells us how much fuel (propellant) is required to change the rocket's speed by an amount Δv. The complication here is that the rocket loses mass as it expels propellant, so we need to allow for that.

Suppose that at an initial time $t=0$, a rocket has velocity v and total mass m, including propellant mass. The total momentum of the rocket and propellant at time $t=0$ is therefore mv.

Now let's look at the situation an instant later, at time $t=dt$. Let dm be the (negative) change in mass of the rocket due to the expulsion of propellant, and let dv be the corresponding (positive) change in the velocity of the rocket. Then at time $t=dt$, a mass of propellant $-dm$ is expelled at velocity $v-v_p$. (The rocket is moving at velocity v with respect to the Earth, the propellant is moving at speed $-v_p$ relative to the rocket, and so the velocity of the propellant relative to the Earth is $v-v_p$.) This expulsion of propellant will cause the rocket to then have mass $m+dm$ and velocity $v+dv$. The total momentum of the system at $t=dt$ is then the sum of the rocket and propellant momenta, $(m+dm)(v+dv)+(v-v_p)(-dm)$. By conservation of momentum, the momentum of the system at time $t=0$ must equal the momentum at time $t=dt$:

$$mv=(m+dm)(v+dv)+(v-v_p)(-dm) \tag{33.1.1}$$

$$=mv+vdm+mdv+dmdv-vdm+v_p dm \tag{33.1.2}$$

Now the two mv terms cancel, the two vdm terms cancel, and the term $dmdv$ is a second-order differential, which can also be cancelled. We're then left with

$$0=mdv+v_p dm \tag{33.1.3}$$

$$mdv=-v_p dm \tag{33.1.4}$$

$$dv=-v_p \frac{dm}{m} \tag{33.1.5}$$

Now let the rocket burn all its propellant. The rocket's velocity will change by a total amount Δv and its mass will change from m to its empty mass m_e. Integrating Eq. $33.1.5$ over the entire propellant burn, we find

$$\int_v^{v+\Delta v} dv=-v_p \int_m^{m_e} \frac{dm}{m} \tag{33.1.6}$$

Or, evaluating the integrals,

$$\Delta v=-v_p \ln \frac{m_e}{m} \tag{33.1.7}$$

or

$$\Delta v=v_p \ln \frac{m}{m_e} \tag{33.1.8}$$

Eq. $33.1.8$ is called the rocket equation. It relates the fueled and empty masses of the rocket and the velocity of the propellant to the total change in velocity of the rocket.