44.1: Velocity

Let's imagine the following scenario: suppose we have an inclined plane, inclined at an angle θ to the horizontal. Now place a round body of mass M and radius R at a height h above the base of the incline. If we release the body from rest, what will be its speed v at the bottom of the incline?

Let's look at the problem from a point of view of energy. At any given instant, the rolling body will be pivoting about the point of contact with the incline (we'll call this point P). Its total kinetic energy is therefore the rotational kinetic energy

$$K=\frac{1}{2} I_P \omega^2 \tag{44.1.1}$$

where I_P is the moment of inertia about P and ω is the rotational angular velocity of the body. Now by the parallel axis theorem, we know

$$I_P=I_{\mathrm{cm}}+M R^2 \tag{44.1.2}$$

where I_{cm} is the moment of inertia of the body about its center of mass. Substituting into Eq. 44.1.1, we get

$$K=\frac{1}{2}\left(I_{\mathrm{cm}}+M R^2\right) \omega^2 \tag{44.1.3}$$

$$=\frac{1}{2} I_{\mathrm{cm}} \omega^2+\frac{1}{2} M R^2 \omega^2 . \tag{44.1.4}$$

Now using $v=R \omega$ in the second term on the right, we have

$$K=\frac{1}{2} I_{\mathrm{cm}} \omega^2+\frac{1}{2} M v^2 \tag{44.1.5}$$

This says that the total kinetic energy of the body is the sum of the rotational kinetic energy (the first term on the right) and the translational kinetic energy (the second term on the right).

Now let's use the conservation of energy to solve for the speed v at the bottom of the incline. At the top of the incline, the body is at rest, and its energy is all potential and equal to Mgh. At the bottom of the incline, the energy is all kinetic, and is given by Eq. 44.1.5). Then by conservation of energy,

$$M g h=\frac{1}{2} I_{\mathrm{cm}} \omega^2+\frac{1}{2} M v^2 . \tag{44.1.6}$$

Substituting $\omega=v / R$ into the first term on the left,

$$M g h=\frac{1}{2} I_{\mathrm{cm}}\left(\frac{v}{R}\right)^2+\frac{1}{2} M v^2 . \tag{44.1.7}$$

Now out factor $v^2 / 2$ on the right-hand side to get

$$M g h=\left(I_{\mathrm{cm}} \frac{1}{R^2}+M\right) \frac{v^2}{2} \tag{44.1.8}$$

Now dividing through by M,

$$g h=\left(\frac{I_{\mathrm{cm}}}{M R^2}+1\right) \frac{v^2}{2} . \tag{44.1.9}$$

The dimensionless combination $I_{\mathrm{cm}} /\left(M R^2\right)$ occurs often enough that it's convenient to introduce the abbreviation

$$\beta \equiv \frac{I_{\mathrm{cm}}}{M R^2} \tag{44.1.10}$$

(Values of β for several common geometries are shown in 44.3.1.) With this definition, Eq. 44.1.9 becomes

$$g h=(\beta+1) \frac{v^2}{2} . \tag{44.1.11}$$

Solving for v, we finally have the speed at the bottom of the incline given by

$$v=\sqrt{\frac{2 g h}{\beta+1}} \tag{44.1.12}$$