66.16: Vector Arithmetic

A vector A may be written in cartesian (rectangular) form as

$$\mathbf{A}=A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k} \tag{66.16.1}$$

where $\mathbf{i}$ is a unit vector (a vector of magnitude 1) in the x direction, $\mathbf{j}$ is a unit vector in the y direction, and $\mathbf{k}$ is a unit vector in the z direction. A_x, A_y, and A_z are called the x, y, and z components (respectively) of vector $\mathbf{A}$, and are the projections of the vector onto those axes.

The magnitude ("length") of vector $\mathbf{A}$ is

$$mathbf{A}|=A=\sqrt{A_x^2+A_y^2+A_z^2} \tag{66.16.2}$$

For example, if $\mathbf{A}=3\mathbf{i}+5\mathbf{j}+2\mathbf{k}$, then $|\mathbf{A}|=A=\sqrt{3^2+5^2+2^2}\ =\sqrt{38}$.

In two dimensions, a vector has no $\mathbf{k}$ component: $\mathbf{A}=A_x\mathbf{i}+A_y\mathbf{j}$.

Addition and Subtraction

To add two vectors, you add their components. Writing a second vector as $\mathbf{B}=B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}$, we have

$$\mathbf{A}+\mathbf{B}=(A_x+B_x)\,\mathbf{i}+(A_y+B_y)\,\mathbf{j}+(A_z+B_z)\,\mathbf{k}. \tag{66.16.3}$$

For example, if $\mathbf{A}=3\mathbf{i}+5\mathbf{j}+2\mathbf{k}$ and $\mathbf{B}=2\mathbf{i}-\mathbf{j}+4\mathbf{k}$, then $\mathbf{A}+\mathbf{B}=5\mathbf{i}+4\mathbf{j}+6\mathbf{k}$.

Subtraction of vectors is defined similarly:

$$\mathbf{A}-\mathbf{B}=(A_x-B_x)\,\mathbf{i}+(A_y-B_y)\,\mathbf{j}+(A_z-B_z)\,\mathbf{k}. \tag{66.16.4}$$

For example, if $\mathbf{A}=3\mathbf{i}+5\mathbf{j}+2\mathbf{k}$ and $\mathbf{B}=2\mathbf{i}-\mathbf{j}+4\mathbf{k}$, then $\mathbf{A}-\mathbf{B}=\mathbf{i}+6\mathbf{j}-2\mathbf{k}$.

Scalar Multiplication

To multiply a vector by a scalar, just multiply each component by the scalar. Thus if c is a scalar, then

$$c\mathbf{A}=cA_x\mathbf{i}+cA_y\mathbf{j}+cA_z\mathbf{k}. \tag{66.16.5}$$

For example, if $\mathbf{A}=3\mathbf{i}+5\mathbf{j}+2\mathbf{k}$, then $7\mathbf{A}=21\mathbf{i}+35\mathbf{j}+14\mathbf{k}$.

Dot Product

It is possible to multiply a vector by another vector, but there is more than one kind of multiplication between vectors. One type of vector multiplication is called the dot product, in which a vector is multiplied by another vector to give a scalar result. The dot product (written with a dot operator, as in $\mathbf{A}\cdot\mathbf{B}$) is

$$\mathbf{A}\cdot\mathbf{B}=AB\cos\theta=A_xB_x+A_yB_y+A_zB_z, \tag{66.16.6}$$

where θ is the angle between vectors $\mathbf{A}$ and $\mathbf{B}$. For example, if $\mathbf{A}=3\mathbf{i}+5\mathbf{j}+2\mathbf{k}$ and $\mathbf{B}=2\mathbf{i}-\mathbf{j}+4\mathbf{k}$, then $\mathbf{A}\cdot\mathbf{B}=6-5+8=9$.

The dot product can be used to find the angle between two vectors. To do this, we solve Eq. (P.6) for θ and find $\cos\theta=\mathbf{A}\cdot\mathbf{B}/(AB)$. Applying this to the previous example, we get $A=\sqrt{38}$ and $B=\sqrt{21}$, so $\cos\theta=9/(\sqrt{38}\sqrt{21})$, and thus $\theta=71.4^\circ$.

An immediate consequence of Eq. (P.6) is that two vectors are perpendicular if and only if their dot product is zero.

Cross Product

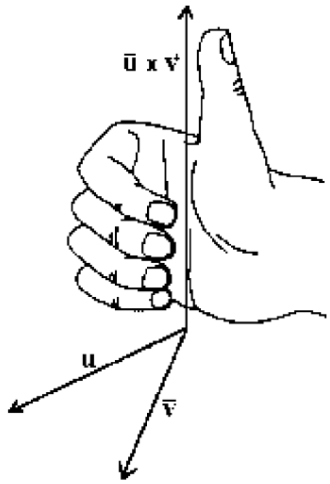

Figure 66.16.1: The vector cross product $\mathbf{A}\times\mathbf{B}$ is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$, and in the right-hand sense. (Credit: "Connected Curriculum Project", Duke University.)

Another kind of multiplication between vectors, called the cross product, involves multiplying one vector by another and giving another vector as a result. The cross product is written with a cross operator, as in $\mathbf{A}\times\mathbf{B}$. It is defined by

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (AB\sin\theta)\mathbf{u} \\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z\end{vmatrix} \\ &= (A_yB_z - A_zB_y)\,\mathbf{i} - (A_xB_z - A_zB_x)\,\mathbf{j} + (A_xB_y - A_yB_x)\,\mathbf{k},\end{aligned}$$

where again θ is the angle between the vectors, and $\mathbf{u}$ is a unit vector pointing in a direction perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$, in a right-hand sense: if you curl the fingers of your right hand from $\mathbf{A}$ into $\mathbf{B}$, then the thumb of your right hand points in the direction of $\mathbf{A}\times\mathbf{B}$ (Fig. P.1). As an example, if $\mathbf{A}=3\mathbf{i}+5\mathbf{j}+2\mathbf{k}$ and $\mathbf{B}=2\mathbf{i}-\mathbf{j}+4\mathbf{k}$, then $\mathbf{A}\times\mathbf{B}=(20-(-2))\mathbf{i}-(12-4)\mathbf{j}+(-3-10)\mathbf{k}=22\mathbf{i}-8\mathbf{j}-13\mathbf{k}$.

Rectangular and Polar Forms

A two-dimensional vector may be written in either rectangular form $\mathbf{A}=A_x\mathbf{i}+A_y\mathbf{j}$ described earlier, or in polar form $\mathbf{A}=A\angle\theta$, where A is the vector magnitude, and θ is the direction measured counterclockwise from the $+x$ axis. To convert from polar form to rectangular form, one finds

$$\begin{aligned}A_x &= A\cos\theta \\ A_y &= A\sin\theta\end{aligned}$$

Inverting these equations gives the expressions for converting from rectangular form to polar form:

$$\begin{aligned}A &= \sqrt{A_x^2+A_y^2} \\ \tan\theta &= \frac{A_y}{A_x}\end{aligned}$$