56.5: Vincenty's Formulæ- Direct Problem

In the direct problem, we're given the latitude ϕ_1 (north positive) and longitude L_1 (east positive) of one point on the Earth's surface; a distance s; and a direction α_1 (measured clockwise from north). The goal of the direct problem is to find the latitude ϕ_2 and longitude L_2 of the point you would reach by starting at (ϕ_1, L_1) and traveling a distance s in the direction α_1.

We're also given the following constants that define the size and shape of the Earth ellipsoid:[1]

- Earth ellipsoid semi-major axis (i.e. equatorial radius): $a = 6378137.0$ meters.
- Earth flattening factor $f = 1/298.257223563$ This is defined as the difference between semi-major and semi-minor axes, divided by the semi-major axis: $f = (a-b)/a$.

We begin by finding the semi-minor axis b of the Earth's ellipsoid:

$$b = (1-f)a. \tag{56.5.1}$$

Then calculate the following, step by step, working with all angles in radians:

$$\tan U_1 = (1-f)\tan\phi_1 \tag{56.5.2}$$

$$U_1 = \tan^{-1}(\tan U_1) \tag{56.5.3}$$

$$\sigma_1 = \arctan\left(\frac{\tan U_1}{\cos\alpha_1}\right) \tag{56.5.4}$$

$$\sin\alpha = \cos U_1 \sin\alpha_1 \tag{56.5.5}$$

$$\cos^2\alpha = (1-\sin\alpha)(1+\sin\alpha) \tag{56.5.6}$$

$$u^2 = \left(\cos^2\alpha\right)\left(\frac{a^2-b^2}{b^2}\right) \tag{56.5.7}$$

$$A = 1 + \frac{u^2}{16384}\left\{4096 + u^2\left[-768 + u^2\left(320 - 175u^2\right)\right]\right\} \tag{56.5.8}$$

$$B = \frac{u^2}{1024}\left\{256 + u^2\left[-128 + u^2\left(74 - 47u^2\right)\right]\right\} \tag{56.5.9}$$

Then, using an initial value $\sigma = s/bA$, iterate Eqs. 56.5.10through 56.5.12until there is no significant change in σ :

$$2\sigma_m = 2\sigma_1 + \sigma \tag{56.5.10}$$

$$\Delta\sigma = B\sin\sigma\left\{\cos(2\sigma_m) + \frac{1}{4}B\left[\cos\sigma\left(-1+2\cos^2(2\sigma_m)\right) - \frac{1}{6}B\cos(2\sigma_m)\left(-3+4\sin^2\sigma\right)\left(-3+4\cos^2(2\sigma_m)\right)\right]\right\} \tag{56.5.11}$$

$$\sigma = \frac{s}{bA} + \Delta\sigma \tag{56.5.12}$$

Once σ is obtained to sufficient accuracy, calculate:

$$\phi_2 = \arctan\left(\frac{\sin U_1\cos\sigma + \cos U_1\sin\sigma\cos\alpha_1}{(1-f)\sqrt{\sin^2\alpha + \left(\sin U_1\sin\sigma - \cos U_1\cos\sigma\cos\alpha_1\right)^2}}\right) \tag{56.5.13}$$

$$\lambda = \arctan\left(\frac{\sin\sigma\sin\alpha_1}{\cos U_1\cos\sigma - \sin U_1\sin\sigma\cos\alpha_1}\right) \tag{56.5.14}$$

$$C = \frac{f}{16}\cos^2\alpha\left[4 + f\left(4 - 3\cos^2\alpha\right)\right] \tag{56.5.15}$$

$$L = \lambda - (1-C)f\sin\alpha\left\{\sigma + C\sin\sigma\left[\cos(2\sigma_m) + C\cos\sigma\left(-1+2\cos^2(2\sigma_m)\right)\right]\right\} \tag{56.5.16}$$

$$L_2 = L_1 + L \tag{56.5.17}$$

$$\alpha_2 = \arctan\left(\frac{\sin\alpha}{-\sin U_1\sin\sigma + \cos U_1\cos\sigma\cos\alpha_1}\right) \tag{56.5.18}$$

Then (ϕ_2, L_2) are the latitude and longitude of the ending point (in radians).

Example 56.5.1

If you travel exactly 1000 miles northwest of the sounding rocket in Chesapeake Hall at Prince George's Community College ($38^\circ 53' 17.62'' \text{N}, 76^\circ 49' 23.40'' \text{W}$), where do you end up? (Give the answer as latitude, longitude, and describe the location.)

Solution

The coordinates of Chesapeake Hall are: $\phi_1 = +38.888228^\circ, L_1 = -76.823167^\circ$. The given distance is 1000 miles $= 1609.344 \text{ km}$, and the given azimuth $\alpha_1 = 315^\circ$. Employing Vincenty's formulæ (direct method), we find:

$$\begin{aligned} b &= 6356752.3 \text{ meters} \\ U_1 &= 38.794230^\circ \\ \sigma_1 &= 48.663693^\circ \\ \cos^2\alpha &= 0.696266995365 \\ u^2 &= 0.0046924891470 \\ A &= 1.0011720921377 \\ B &= 0.0011703772996 \\ \sigma &= 14.482402^\circ \\ \phi_2 &= 48.206878^\circ \\ \lambda &= -15.357896^\circ \\ C &= 5.84547783404 \times 10^{-4} \\ L &= -15.331156^\circ \\ L_2 &= -92.154324^\circ \end{aligned}$$

Hence the ending point is at latitude $48^\circ 12' 24.76'' \text{N}$, longitude $92^\circ 09' 15.56'' \text{W}$. This is in northern Minnesota (St. Louis county), within Superior National Forest, just a few miles south of the Canadian border.

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[1] These are the values used for the WGS-84 ellipsoid, used by GPS receivers.