11.2: Constant Acceleration

As was done in one-dimensional kinematics, we may derive a set of equations for the motion of a particle under a constant acceleration. In two or three dimensions, though, it's a constant acceleration vector a. If the acceleration vector a is constant, we can bring it outside the integral sign of Eq. (11.1.4) just as we do with constant scalars. We get

$$\mathbf{v}(t)=\int \mathbf{a} d t=\mathbf{a} \int d t \tag{11.2.1}$$

or

$$\mathbf{v}(t)=\mathbf{a} t+\mathbf{C} \tag{11.2.2}$$

where $\mathbf{C}$ is the constant of integration. By setting $t=0$, we can see that physically, just as in one-dimensional kinematics, $\mathbf{C}=\mathbf{v}_0=\mathbf{v}(0)$ represents the velocity vector at time $t=0$, so

$$\mathbf{v}(t)=\mathbf{a} t+\mathbf{v}_0 \tag{11.2.3}$$

Substituting this result into Eq. (11.1.3), we have

$$\mathbf{r}(t)=\int\left(\mathbf{a} t+\mathbf{v}_0\right) d t \tag{11.2.4}$$

$$=\int \mathbf{a} t d t+\int \mathbf{v}_0 d t \tag{11.2.5}$$

$$=\mathbf{a} \int t d t+\mathbf{v}_0 \int d t \tag{11.2.6}$$

or

$$\mathbf{r}(t)=\frac{1}{2} \mathbf{a} t^2+\mathbf{v}_0 t+\mathbf{r}_0 \tag{11.2.7}$$

where $\mathbf{r}_0=\mathbf{r}(0)$ is the position vector at time $t=0$.

The remaining constant-acceleration formula is a formula for $\mathbf{v}(\mathbf{r})$, in which we eliminate time t to get an expression for velocity in terms of position. We did this in one dimension by solving the equation for $v(t)$ for t, then substituting into the equation for $x(t)$ and solving for v. Unfortunately, that technique won't work with vectors, because it would require dividing by a vector, which is not defined. Instead, being guided by the knowledge that the vector formula must reduce to the known scalar formula when the vectors are one-dimensional, we proceed as follows. Start with Eq. 11.2.7 for $\mathbf{r}(t)$ for constant acceleration:

$$\mathbf{r}(t)=\frac{1}{2} \mathbf{a} t^2+\mathbf{v}_0 t+\mathbf{r}_0 \tag{11.2.8}$$

$$\mathbf{r}-\mathbf{r}_0=\frac{1}{2} \mathbf{a} t^2+\mathbf{v}_0 t \tag{11.2.9}$$

Now take the dot product of both sides with the acceleration a:

$$\mathbf{a} \cdot\left(\mathbf{r}-\mathbf{r}_0\right)=\mathbf{a} \cdot\left(\frac{1}{2} \mathbf{a} t^2+\mathbf{v}_0 t\right) \tag{11.2.10}$$

$$=\frac{1}{2} a^2 t^2+\mathbf{a} \cdot \mathbf{v}_0 t, \tag{11.2.11}$$

and multiply both sides by 2 :

$$2 \mathbf{a} \cdot\left(\mathbf{r}-\mathbf{r}_0\right)=a^2 t^2+2 \mathbf{a} \cdot \mathbf{v}_0 t \tag{11.2.12}$$

The left-hand side looks similar to the second term on the right-hand side of the one-dimensional Eq.11.2.7, but we still need to eliminate t on the right-hand side. To do that, let's start by working on the first term on the right-hand side of Eq. 11.2.12 Starting with Eq. 11.2.3, we have

$$\mathbf{v}(t) = \mathbf{a}t + \mathbf{v}_0 \tag{11.2.13}$$

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{a}t \tag{11.2.14}$$

Now take the dot product of the left-hand side of Eq. $11.2.10$with itself, and dot the right-hand side with itself:

$$(\mathbf{v} - \mathbf{v}_0) \cdot (\mathbf{v} - \mathbf{v}_0) = (\mathbf{a}t) \cdot (\mathbf{a}t) \tag{11.2.15}$$

$$v^2 - 2\mathbf{v} \cdot \mathbf{v}_0 + v_0^2 = a^2 t^2 \tag{11.2.16}$$

Next, let's work on the second term on the right-hand side of Eq. $11.2.12$ To do this, let's take the dot product of both sides of Eq. $11.2.14$with $\mathbf{v}_0$:

$$\mathbf{v}_0 \cdot (\mathbf{v} - \mathbf{v}_0) = \mathbf{v}_0 \cdot \mathbf{a}t \tag{11.2.17}$$

$$2\mathbf{v} \cdot \mathbf{v}_0 - 2v_0^2 = 2\mathbf{a} \cdot \mathbf{v}_0 t \tag{11.2.18}$$

Now we have all the pieces we need to eliminate t. In Eq. $11.2.12$, we use Eq. $11.2.16$to replace a^2t^2, and we use Eq. $11.2.18$to replace $2\mathbf{a} \cdot \mathbf{v}_0 t$:

$$2\mathbf{a} \cdot (\mathbf{r} - \mathbf{r}_0) = \left(v^2 - 2\mathbf{v} \cdot \mathbf{v}_0 + v_0^2\right) + \left(2\mathbf{v} \cdot \mathbf{v}_0 - 2v_0^2\right) \tag{11.2.19}$$

$$= v^2 - v_0^2 \tag{11.2.20}$$

or

$$v^2 = v_0^2 + 2\mathbf{a} \cdot (\mathbf{r} - \mathbf{r}_0) \tag{11.2.21}$$