42.6: More on the Spring Constant

It is often not appreciated that the spring constant k depends not only on the rigidity of the spring, but also on the diameter of the spring and the total number of turns of wire in the spring. Consider a vertical spring with spring constant k, and a mass m hanging on one end. Assume the system is in its equilibrium position, and in this position it has length L_0 and consists of N turns of wire. Now if you apply an additional downward force F to the mass, the string will stretch by an additional amount x given by Hooke's law: $x = F/k$. This stretching will manifest itself as an additional spacing of x/N between adjacent turns of the spring. It is this additional spacing per turn that is the true measure of the inherent "stiffness" of the spring.

Now suppose this spring is cut in half and put in its equilibrium position. Its new length will be $L_0/2$, and will consist of $N/2$ turns of wire. When the same additional force F is applied to the mass m, the additional spacing between adjacent turns of the spring will be the same as before, x/N, because the spring still has

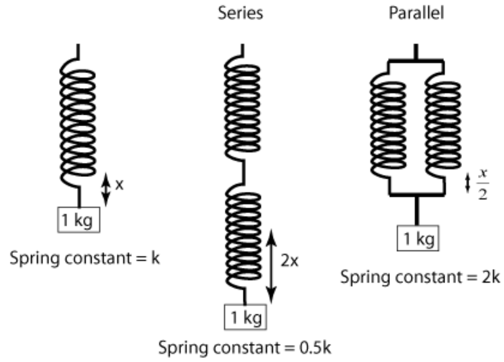

Figure 42.6.1: Springs in series and parallel (Credit: SPM Physics Forum).

the same stiffness. Since the number of turns is now $N/2$, this means that the additional total stretching of the spring is $x/2$, so it will stretch by only half as much as before. By Hooke's law, the spring constant is now $k' = F/(x/2) = 2F/x = 2k$, so the spring constant is now twice what it was before. In other words, cutting the spring in half will double the spring constant. Likewise, doubling the length (number of turns) of the spring will halve its spring constant.

Another way to think of this is to consider two springs connected in series or in parallel (Fig. $42.6.1$). If several springs are connected end-to-end (i.e. in series), then the equivalent spring constant k_s of the system will be given by

$$\frac{1}{k_s} = \sum_i \frac{1}{k_i} \tag{42.6.1}$$

$$= \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \cdots \tag{42.6.2}$$

If the springs are connected in parallel, then the equivalent spring constant k_p of the system will be

$$k_p = \sum_i k_i \tag{42.6.3}$$

$$= k_1 + k_2 + k_3 + \ldots \tag{42.6.4}$$

For example, if two identical springs, each of spring constant k, are connected in series, then the combination will have an equivalent spring constant of $k/2$. If the two identical springs were instead connected in parallel, then the combination would have an equivalent spring constant of $2k$, as shown in Figure $42.6.1$.

Now imagine you have a long spring of spring constant k. You can imagine it as being two identical springs connected in series, each having spring constant $2k$, so that the combination has a total equivalent spring constant of $[(1/2k) + (1/2k)]^{-1} = k$. If the long spring is cut in half, then you are left with only one of those smaller springs of spring constant $2k$, so again we reach the conclusion that cutting the spring in half will double the spring constant.

It's possible to calculate the spring constant from the geometry of the spring. The formula is [1]

$$k=\frac{Gd^4}{8ND^3} \tag{42.6.5}$$

where d is the wire diameter, N is the number of active turns in the spring, D is the coil diameter (measured from the center of the wire), and G is called the modulus of rigidity of the spring material; G is given by

$$G=\frac{Y}{2(1+v)} \tag{42.6.6}$$

where Y is the Young's modulus of the material (a measure of how much it stretches when pulled or compressed), and v is the material's Poisson ratio (a measure of how much it squeezes sideways when compressed). These are properties that are characteristic of the material, and can be looked up in a handbook of material properties. Values for a few materials are shown in the table below.

Table $42.6.1$: Young's Moduli and Poisson Ratios.

Material	Young's Modulus $Y\ (\mathrm{N/m^2})$	Poisson Ratio v
Aluminum	69×10^9	0.334
Bronze	100×10^9	0.34
Copper	117×10^9	0.355
Lead	14×10^9	0.431
Magnesium	45×10^9	0.35
Stainless steel	180×10^9	0.305
Titanium	110×10^9	0.32
Wrought iron	200×10^9	0.278

Notice from Eq. $42.6.5$ that if the spring is cut in half, N will be half its original value, and so the spring constant k will be doubled, in agreement with what we've found earlier.

Exercise $42.6.1$

Example. Suppose we make a spring of 1 mm diameter copper wire, the diameter of the spring is 1 cm, and there are 50 turns of wire in the spring. What is the spring constant?

Answer

From the above table, for copper, $Y=117\times10^9\ \mathrm{N/m^2}$ and $v=0.355$. From Eq. $42.6.6$ we have

$$G=\frac{Y}{2(1+v)}=\frac{117\times10^9\ \mathrm{N/m^2}}{2(1+0.355)}=43.2\times10^9\ \mathrm{N/m^2} \tag{42.6.7}$$

And the spring constant is found from Eq. $42.6.5$

$$k=\frac{Gd^4}{8ND^3}=\frac{(43.2\times10^9\ \mathrm{N/m^2})\left(10^{-3}\ \mathrm{m}\right)^4}{8(50)\left(10^{-2}\ \mathrm{m}\right)^3}=108\ \mathrm{N/m} \tag{42.6.8}$$

[1] See e.g. The Engineer's Edge