28.1: Energy Conversion of a Falling Body

As an example, let's look at a body of mass m (near the surface of the Earth) released from height h and falling under the influence of gravity. As the body falls, we've noted that the initial potential energy is gradually converted to kinetic energy, until at impact, when the energy is all kinetic. At what rate are these energies changing with time? (These rates of change will be powers, in watts.)

Let's start with the kinetic energy: the time rate of change of kinetic energy is

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) \tag{28.1.1}$$

$$= mv\frac{dv}{dt} \tag{28.1.2}$$

But $dv/dt = a = g$, the acceleration due to gravity; hence

$$\frac{dK}{dt} = mgv \tag{28.1.3}$$

Since the gravitational force $F = mg$, this gives a general expression for power:

$$\mathscr{P} = Fv \tag{28.1.4}$$

Now what is the time rate of change of potential energy?

$$\frac{dU}{dt} = -\frac{d}{dt}(mgy) \tag{28.1.5}$$

$$= -mg\frac{dy}{dt}. \tag{28.1.6}$$

But dy/dt is the velocity v, so we get

$$\frac{dU}{dt} = -mgv \tag{28.1.7}$$

So as the body falls, its kinetic energy K increases at the rate $dK/dt = mgv$, while the potential energy U decreases at the rate $dU/dt = -mgv$. Therefore the total energy $E = K + U$ remains constant, which is consistent with the conservation of energy.

Since v increases as the body falls, the rate of change of the kinetic and potential energies increases as the body falls. At any height y, the potential energy is $U = mgy$. Since the total energy is $E = mgh$, the kinetic energy at height y must be $K = E - U = mg(h - y)$. Therefore the velocity v at height y is given by

$$K = \frac{1}{2}mv^2 = mg(h - y) \tag{28.1.8}$$

$$v = \sqrt{2g(h - y)}. \tag{28.1.9}$$

So the time rates of change of the energies as a function of y is

$$\frac{dK}{dt} = \frac{dU}{dt} = Fv \tag{28.1.10}$$

$$= mg\sqrt{2g(h - y)} \tag{28.1.11}$$

$$= m\sqrt{2g^3(h-y)}. \tag{28.1.12}$$

Right after the body is released, $dK/dt = -dU/dt = 0$; after the body falls through a height h, the rates of change have increased to $dK/dt = -dU/dt = m\sqrt{2g^3h}$.