63.2: Quantum Mechanics

The quantum world at very small distance scales (atomic sizes and smaller) is very alien and strange, and completely beyond our everyday experience. Here are a few of the key concepts in quantum mechanics:

In quantum mechanics, it generally makes no sense to talk about the the exact position x of a particle at a time t. Instead, a particle is thought of as being in many different places at the same time. Only when we go measure the position of the particle does it appear at a precise location. When we're not measuring its position, it is, in a sense, in many places at once.

- The concept of the position x of a particle is replaced by the concept of a wave function $\psi(x,t)$. The physical interpretation of the wave function is that its square, $|\psi(x,t)|^2$, gives the probability that when we measure the particle's position at time t, it will appear at position x. This idea of probability is a central concept of quantum mechanics: when we go to measure the position of a particle, it is fundamentally impossible to predict where it will appear, no matter how much information we have. It is only possible to predict the probability that it will be found at a given location.
- This idea of a wave function is closely connected the the idea of wave-particle duality: matter fundamentally behaves like both a wave and a particle at the same time. For example, both photons (particles of light) and electrons show both particle-like behavior and wave-like behavior.
- It is fundamentally impossible to know both a particle's exact position and its velocity at the same time. (This is in contrast to Newtonian mechanics, where a particle's position and velocity can both be measured to arbitrary accuracy.) This idea is called the Heisenberg uncertainty principle, and is described in more detail below.
- In bound systems, we generally find that a particle cannot have just any value of energy. Instead, we find that the particle can have only certain discrete values of energy; we thus say that the energy is quantized. The particle cannot have an energy that lies in between the allowed discrete values. We also often find that quantities like position and angular momentum are also quantized. For example, an electron in orbit around an atom has its orbital position quantized: it can only be at certain allowed positions with respect to the nucleus, and other positions are now allowed

You may wonder: how can it be that a particle is in many places at once, or that the place where it appears is completely unpredictable, or that it is in an unknown state unless we're measuring it, or that it can be both a particle and a wave at the same time? The truth is that nobody really understands how it can be this way—it just is. We can write down the equations to describe it, and predict the outcomes of experiments to high accuracy, but nobody has a good intuitive picture of how things can possibly be this way. Nature is far stranger than we can imagine.

Now for a mathematical description of quantum mechanics. Recall how we work with Newtonian mechanics: we write down Newton's second law, substitute a specific force for $F(x)$, and solve the resulting differential equation for $x(t)$. Quantum mechanics does not use the concept of a force; rather, everything is formulated in terms of energy. In place of Newton's second law, we use the time-dependent Schrödinger equation, which is a partial differential equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+U(x)\psi(x,t)=i\hbar\frac{\partial\psi}{\partial t} \tag{63.2.1}$$

where m is the mass of the particle, $U(x)$ is the potential energy function, and $\psi(x,t)$ is the wave function we wish to solve for. The constant $\hbar$ (pronounced "h-bar") is an abbreviation for Planck's constant h divided by 2π, and has the value $\hbar \equiv h/2\pi = 1.054571726\times 10^{-34}$ J s. Notice the presence of the factor $i=\sqrt{-1}$ on the right-hand side: in general, quantum mechanical wave functions are complex, but the physically meaningful quantity is the square of the wave function, which is real.

Solving a problem in quantum mechanics consists of the following steps (analogous to the steps described earlier for Newtonian mechanics):

1. Write down the Schrödinger equation $63.2.1$;
2. Substitute for $U(x)$ the specific potential energy present in the problem;
3. Solve the resulting differential equation for $\psi(x,t)$.

It turns out that it is possible to separate the solution $\psi(x,y)$ into the product of two parts: a part that depends only on x and a part that depends only on t. The solution is $\psi(x,t)=\varphi(x)e^{-iEt/\hbar}$, where $\varphi(x)$ is the solution to the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2}+U(x)\varphi(x)=E\varphi(x) \tag{63.2.2}$$

and where E is the total energy of the particle. So to solve the time-dependent Schrödinger equation for $\psi(x,t)$, we first solve the time-independent Schrödinger equation for $\varphi(x)$, then multiply that solution by $e^{-iEt/\hbar}$

[1] Technically, it's the probability of finding the particle between positions x and $x+dx$.