31.3: Perfectly Elastic Collisions

A slightly more difficult situation to analyze is the perfectly elastic collision. In this type of collision, none of the kinetic energy is lost, and so kinetic energy is conserved. [2]

Let's begin the analysis of a perfectly elastic collision in one dimension. We begin with two masses m_1 and m_2 with initial velocities v_{1i} and v_{2i}, respectively. After the collision, the two masses have velocities v_{1f} and v_{2f}. The typical problem is: given the masses and initial velocities, what are the final velocities?

We know the total momentum of the system is conserved, so the sum of the momenta before the collision equals the sum of the momenta after the collision:

$$p_i = p_f \tag{31.3.1}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{31.3.2}$$

But because the collision is perfectly elastic, we know that the kinetic energy is also conserved. This gives us a second equation:

$$K_i = K_f \tag{31.3.3}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \tag{31.3.4}$$

$$m_1 v_{1i}^2 + m_2 v_{2i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2 \tag{31.3.5}$$

Equations $31.3.2$ and $31.3.5$ give two simultaneous equations in the two unknown final velocities v_{1f} and v_{2f} :

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{31.3.6}$$

$$m_1 v_{1i}^2 + m_2 v_{2i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2 \tag{31.3.7}$$

To solve these equations simultaneously, let's rearrange to put the m_1 terms on the left and the m_2 terms on the right:

$$m_1 \left(v_{1i} - v_{1f}\right) = m_2 \left(v_{2f} - v_{2i}\right) \tag{31.3.8}$$

$$m_1 \left(v_{1i}^2 - v_{1f}^2\right) = m_2 \left(v_{2f}^2 - v_{2i}^2\right) \tag{31.3.9}$$

Expanding the difference of squares in Eq. $31.3.9$, we have

$$m_1 \left(v_{1i} - v_{1f}\right) = m_2 \left(v_{2f} - v_{2i}\right) \tag{31.3.10}$$

$$m_1 \left(v_{1i} + v_{1f}\right)\left(v_{1i} - v_{1f}\right) = m_2 \left(v_{2f} + v_{2i}\right)\left(v_{2f} - v_{2i}\right) \tag{31.3.11}$$

Now divide Eq. $31.3.11$ by Eq. $31.3.10$ to get

$$v_{1i} + v_{1f} = v_{2i} + v_{2f} \tag{31.3.12}$$

To solve for the final velocities v_{1f} and v_{2f}, we write Eqs. $31.3.6$ and $31.3.12$ in matrix form:

$$\begin{pmatrix} m_1 m_2 \\ 1 - 1 \end{pmatrix} \begin{pmatrix} v_{1f} \\ v_{2f} \end{pmatrix} = \begin{pmatrix} m_1 v_{1i} + m_2 v_{2i} \\ -v_{1i} + v_{2i} \end{pmatrix} \tag{31.3.13}$$

and solve for the final velocities:

$$\begin{pmatrix} v_{1f} \\ v_{2f} \end{pmatrix} = \begin{pmatrix} m_1 m_2 \\ 1-1 \end{pmatrix}^{-1} \begin{pmatrix} m_1 v_{1i} + m_2 v_{2i} \\ -v_{1i} + v_{2i} \end{pmatrix} \tag{31.3.14}$$

Let's now expand the matrix inverse as the transposed matrix of cofactors divided by the determinant (Appendix Q):

$$\begin{pmatrix} v_{1f} \\ v_{2f} \end{pmatrix} = \frac{1}{m_1+m_2} \begin{pmatrix} 1 m_2 \\ 1-m_1 \end{pmatrix} \begin{pmatrix} m_1 v_{1i} + m_2 v_{2i} \\ -v_{1i} + v_{2i} \end{pmatrix} \tag{31.3.15}$$

$$= \frac{1}{m_1+m_2} \begin{pmatrix} m_1 v_{1i} + m_2 v_{2i} - m_2 v_{1i} + m_2 v_{2i} \\ m_1 v_{1i} + m_2 v_{2i} + m_1 v_{1i} - m_1 v_{2i} \end{pmatrix} \tag{31.3.16}$$

$$= \frac{1}{m_1+m_2} \begin{pmatrix} (m_1 - m_2) v_{1i} + 2 m_2 v_{2i} \\ 2 m_1 v_{1i} + (m_2 - m_1) v_{2i} \end{pmatrix} \tag{31.3.17}$$

Thus

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2 m_2}{m_1 + m_2}\right) v_{2i} \tag{31.3.18}$$

$$v_{2f} = \left(\frac{2 m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i} \tag{31.3.19}$$

Eqs. $31.3.18$and $31.3.19$are the general solution for finding the final velocities in a one-dimensional perfectly elastic collision.

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[1] The 1961 Disney movie The Absent-Minded Professor is about a college professor who invents a material called flubber, whose coefficient of restitution is greater than 1 , so that it bounces higher and higher with each bounce. Among other uses, it is attached to the bottoms of the shoes of the college basketball team, giving the players a significant advantage.

[2] Note that in general, total energy is conserved, but kinetic energy is not. Kinetic energy is only conserved in perfectly elastic collisions.