13.3: Projectile Problem

Recall the problem from Section 12.6 of directing a projectile to hit a target on a hill at position (x_t, y_t), when the muzzle velocity is fixed and we're allowed to vary the angle. We found that in order to hit the target, the launch angle θ is the solution to Eq. 12.6.9,

$$x_t \sin 2\theta - 2y_t \cos^2 \theta = \frac{g x_t^2}{v_0^2} \tag{13.3.1}$$

which cannot be solved in closed form and must be solved numerically. To solve this for θ using Newton's method, we must first put it in the form $f(\theta) = 0$:

$$f(\theta) = x_t \sin 2\theta - 2y_t \cos^2 \theta - \frac{g x_t^2}{v_0^2} = 0 \tag{13.3.2}$$

Newton's method also requires the derivative of f :

$$\begin{aligned} f'(\theta) &= 2x_t \cos 2\theta + 4y_t \cos\theta \sin\theta \\ &= 2x_t \cos 2\theta + 2y_t \sin 2\theta \end{aligned}$$

Using these expressions for $f(\theta)$ and $f'(\theta)$ in Newton's method (Eq. (13.1.2)), we find an iterative expression that lets us solve numerically for the launch angle θ :

$$\theta_{n+1} = \theta_n - \frac{x_t \sin 2\theta_n - 2y_t \cos^2 \theta_n - g x_t^2 / v_0^2}{2x_t \cos 2\theta_n + 2y_t \sin 2\theta_n} \tag{13.3.3}$$

Here the target coordinates (x_t, y_t) are known, as are the muzzle velocity v_0 and acceleration due to gravity g, so the only variable on the right-hand side is θ_n. To use this expression, we begin with an initial guess for the launch angle, θ_0 (in radians). Then plug this θ_0 into the right-hand side, which returns θ_1; for the next iteration, plug this θ_1 into the right-hand side, which returns θ_2, etc. After a few iterations, you should get approximately the same angle over and over again on successive iterations. If the target is out of range, the method will "blow up" and not converge, typically by returning larger and larger values of θ_n for each iteration.

For this type of iterative calculation, it is handy to program the iteration formula into a programmable calculator, or write a computer program.