35.2: Properties

Anti-Commutativity

The cross product is anti-commutative:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \tag{35.2.1}$$

as should be clear by applying the right-hand rule.

Orthogonality

If two vectors are parallel or anti-parallel, their cross product will be zero. For example, for the cartesian unit vectors,

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \tag{35.2.2}$$

Notice that the result is the zero vector, encountered earlier in Chapter 6: a vector whose components are all zero. The zero vector has magnitude zero, and no defined direction.

Also, the products of any two different cartesian unit vectors permute cyclically:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}; \mathbf{j} \times \mathbf{i} = -\mathbf{k} \tag{35.2.3}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}; \mathbf{k} \times \mathbf{j} = -\mathbf{i} \tag{35.2.4}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}; \mathbf{i} \times \mathbf{k} = -\mathbf{j} \tag{35.2.5}$$

Derivative

The derivative of the cross product is similar to the familiar product rule for scalars:

$$\frac{d(\mathbf{A} \times \mathbf{B})}{dt} = \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B} \tag{35.2.6}$$

Note, though, that since the cross product is not commutative, you must keep the order of multiplications as they're shown here.

The Triple Vector Product

Unlike normal scalar multiplication, the cross product is non-associative: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$. The cross products of three vectors may be expanded like so:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{35.2.7}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \tag{35.2.8}$$

Eq. $35.2.9$ is sometimes remembered as the "back cab" rule (from the letters "BAC CAB" on the righthand side), but this requires remembering where the parentheses are on the left-hand side. A better way to remember both products in Eqs. (32.12) and (32.13) is: "The middle vector times the dot product of the two on the ends, minus the dot product of the two vectors straddling the parenthesis times the remaining one.”

Products of Two Cross Products

The dot product of two cross products can be expanded as

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}), \tag{35.2.9}$$

while the cross product of two cross products can be expanded as

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})\mathbf{D}. \tag{35.2.10}$$

The Triple Scalar Product

An interesting vector product is the so-called triple scalar product, $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$, involving one dot product and one scalar product. No parentheses are needed here: the cross product must be done before the dot product. (Attempting to do the dot product first results in the cross product of a scalar with a vector, which is not defined.) The result is a scalar.

The triple scalar product has a number of interesting properties:

- The dot and cross operators can be exchanged without changing the result: $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$. (Because of this property, the triple scalar product is sometimes written simply as $[\mathbf{A}, \mathbf{B}, \mathbf{C}]$.)
- Vectors A, B, and $\mathbf{C}$ can be permuted cyclically without changing the result: $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$.
- The absolute value of the triple scalar product is equal to the volume of the parallelepiped whose edges are formed by the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.
- In terms of cartesian components, the triple scalar product can be written as a determinant:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{35.2.11}$$

$$= A_x B_y C_z - A_x B_z C_y - A_y B_x C_z + A_y B_z C_x + A_z B_x C_y - A_z B_y C_x \tag{35.2.12}$$