26.6: The Virial Theorem

The virial theorem relates the time-average kinetic energy of a system to the time-average potential energy. In the common situation that the force is proportional to some power of the distance,

$$F \propto r^n, \tag{26.6.1}$$

then the virial theorem states that the time-average kinetic energy $\langle K\rangle$ is related to the time-average potential energy $\langle U\rangle$ by

$$\langle K\rangle = \frac{n+1}{2}\langle U\rangle \tag{26.6.2}$$

Since the total energy $E=\langle K\rangle+\langle U\rangle$, we can use the virial theorem (Eq. 26.6.2) to derive a useful expression for the total energy in terms of the time-average energies:

$$E = \frac{n+3}{n+1}\langle K\rangle = \frac{n+3}{2}\langle U\rangle \tag{26.6.3}$$

Example 26.6.1

For the spring (Hooke's law) force $F=-kx$, we have $n=1$. So by the virial theorem (Eq. 26.6.2),

$$\langle K\rangle = \langle U\rangle. \tag{26.6.4}$$

It turns out in this case that $\langle K\rangle=\langle U\rangle=kA^2/4$, where A is the amplitude of the motion. By Eq. 26.6.3,

$$E = 2\langle K\rangle = 2\langle U\rangle = \frac{1}{2}kA^2. \tag{26.6.5}$$

Example 26.6.2

For the gravitational force given by Newton's law of gravity ($F=-Gm_1m_2/r^2$), and so $n=-2$. Then by the virial theorem,

$$\langle K\rangle = -\frac{1}{2}\langle U\rangle \tag{26.6.6}$$

In this case, $\langle K\rangle=Gm_1m_2/(2r)$. By Eq. 26.6.3,

$$E = -\langle K\rangle = \frac{1}{2}\langle U\rangle = -\frac{Gm_1m_2}{2r} \tag{26.6.7}$$

This second example has some interesting consequences. Suppose we have a body orbiting the earth with orbital radius r. Its velocity is then given by Eq. (23.2.4):

$$v = \sqrt{\frac{GM_\oplus}{r}} \tag{26.6.8}$$

where G is the gravitational constant and $M_\oplus$ is the mass of the Earth. Now suppose we increase r, putting the body into a higher orbit. What happens to the energy? Since the potential energy is $U=-GM_\oplus m/r$, boosting the body to a higher orbit increases its potential energy. By Eq. (23.43), its velocity will decrease, thereby decreasing its kinetic energy. What happens to the total energy? By the virial theorem, the second example shows that the increase in potential energy is twice the decrease in potential energy, so overall, the total energy is increased for higher orbits.

Now suppose you're in a spacecraft, trying to dock with the International Space Station, which is in orbit ahead of you (looking in the direction of motion), at the same orbital radius. To reach the Space Station, you must do something counterintuitive: fire your spacecraft jets toward the station, so that there's a force pushing you away from the Station. This will slow the spacecraft down, decreasing its total energy, thereby dropping it into a lower orbit, where its velocity will increase-causing the spacecraft to move toward the Space Station.