42.1: Introduction to Simple Harmonic Motion

The small-angle approximation of the simple plane pendulum is an example of what is called simple harmonic motion. Simple harmonic motion is the motion that a particle exhibits when under the influence of a force of the form given by Hooke's law (named for the 17th century English scientist Robert Hooke):

$$F=-kx. \tag{42.1.1}$$

A force of this form describes, for example, the force on a mass attached to a spring with spring constant k, where k is a measure of the stiffness of the spring. In this case F is the force exerted by the spring, and x is the distance of the mass from its equilibrium position - that is, the "resting" position at which the mass can be left where it will not oscillate.

Substituting Hooke's law as the force in Newton's second law $F=ma$ (and recalling the acceleration $a=d^2x/dt^2$) gives the equation

$$-kx=m\frac{d^2x}{dt^2} \tag{42.1.2}$$

This is a second-order linear differential equation with constant coefficients, and can be solved for $x(t)$ using standard methods from the theory of differential equations. We won't go into the theory of differential equations here, but just present the result. The solution is

$$x(t)=A\cos(\omega t+\delta). \tag{42.1.3}$$

Here ω is called the angular frequency of the motion, and measures how fast the particle oscillates back and forth. The constant A is called the amplitude of the motion, and is the maximum distance the particle travels from its equilibrium position, $x=0$. The constant δ called the phase constant, and determines where in its cycle the particle is at time $t=0$. A plot of $x(t)$ is shown in Fig. 42.1.1.

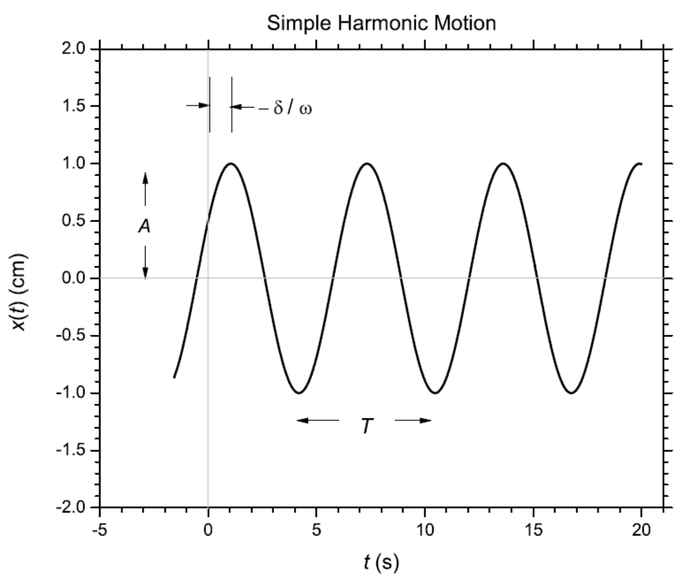

Figure 42.1.1: Simple harmonic motion. Shown are the amplitude A, period T, and phase constant δ. The horizontal line $x(t)=0$ is the equilibrium position.

Since the sine and cosine function differ only by a phase shift ($\sin\theta\equiv\cos(\theta-\pi/2)$), we could replace the cosine function in Eq. 42.1.3 with a sine by simply adding an extra $\pi/2$ to the phase constant δ. So either the sine or the cosine can be used equally well to describe simple harmonic motion; here we will choose to use the cosine function.

The calculus may also be used to find the velocity of the particle at any time t; the result is

$$v(t)=-A\omega\sin(\omega t+\delta) \tag{42.1.4}$$

so that the maximum speed of the simple harmonic oscillator is

$$|v_{\max}|=A\omega \tag{42.1.5}$$

Further, it can be shown that the acceleration at any time t is

$$a(t) = -A\omega^2 \cos(\omega t + \delta) \tag{42.1.6}$$

$$= -\omega^2 x(t). \tag{42.1.7}$$

Multiplying Eq. $42.1.7$ by the particle mass m, we find

$$ma(t) = F(t) = -m\omega^2 x(t). \tag{42.1.8}$$

Comparing this with Eq. $42.1.1$ we see that

$$k = m\omega^2, \tag{42.1.9}$$

or

$$\omega = \sqrt{\frac{k}{m}}. \tag{42.1.10}$$

In Eq. $42.1.3$, the amplitude A depends on how far the particle was displaced from equilibrium before being released; the phase constant δ just depends on when we choose time $t = 0$; but the angular frequency ω depends on the physical parameters of the system: the stiffness of the spring k and the mass of the particle m.