34.2: Continuous Bodies

To find the center of mass of a continuous body, just imagine dividing the body up into little infinitesimal pieces, each of which has mass dm; then treat each of these infinitesimal masses as a point mass, and add together the products of dm and its position using an integral. In one dimension:

$$x_{\mathrm{cm}}=\frac{\int x d m}{\int d m} \tag{34.2.1}$$

where the integrals are taken over the entire length of the body. But there's a problem here. How are we going to integrate x with respect to m ? We need to write both the integrand and the variable of integration with respect to the same variable. If we have a rod in one dimension, for example, then we would want to integrate over the entire length of the rod, so it's natural to want the variable of integration to be x. Somehow, then, we need to change the variable of integration from m to x.

We do this through the density. In the case of a one-dimensional problem, we'll use the linear mass density (mass per unit length) λ :

$$\lambda=\frac{d m}{d x} \tag{34.2.2}$$

where λ has units of kg / m. In general, the density λ can be variable across the body, so it will be a function of x, so we can write it as $\lambda(x)$. In terms of x, we can therefore write the mass dm as

$$d m=\lambda(x) d x \tag{34.2.3}$$

Making this substitution into Eq. $34.2.1$, we have the one-dimensional formula

$$x_{\mathrm{cm}}=\frac{\int x \lambda(x) d x}{\int \lambda(x) d x} \tag{34.2.4}$$

The denominator $\int \lambda(x) d x$ is the total mass of the body M.

Example $34.2.1$

Example. Suppose we have a rod of length $5 \mathrm{~m}$, whose density is given by $\lambda(x)=2 x+3 \mathrm{~kg} / \mathrm{m}$, where x is in meters from the left end of the rod. Where is the center of mass of the rod?

Solution

Let's first solve the more general problem: where is the center of mass of a rod of length L, when the density is given by $\lambda(x)=a x+b$. The center of mass is given by Eq. $34.2.5$:

$$x_{\mathrm{cm}}=\frac{\int_0^L x \lambda(x) d x}{\int_0^L \lambda(x) d x} \tag{34.2.5}$$

$$=\frac{\int_0^L x(a x+b) d x}{\int_0^L(a x+b) d x} \tag{34.2.6}$$

$$=\frac{\int_0^L\left(a x^2+b x\right) d x}{\int_0^L(a x+b) d x} \tag{34.2.7}$$

$$=\frac{\left.\left(\frac{1}{3} a x^3+\frac{1}{2} b x^2\right)\right|_0^L}{\left.\left(\frac{1}{2} a x^2+b x\right)\right|_0^L} \tag{34.2.8}$$

$$= \frac{\frac{1}{3}aL^3 + \frac{1}{2}bL^2}{\frac{1}{2}aL^2 + bL} \tag{34.2.9}$$

$$= \frac{2aL^3 + 3bL^2}{3aL^2 + 6bL} \tag{34.2.10}$$

Now substitute $a = 2\ \mathrm{kg/m^2}$, $b = 3\ \mathrm{kg/m}$, and $L = 5\ \mathrm{m}$, and we get

$$x_{\mathrm{cm}} = \frac{2\left(2\ \mathrm{kg/m^2}\right)(5\ \mathrm{m})^3 + 3(3\ \mathrm{kg/m})(5\ \mathrm{m})^2}{3\left(2\ \mathrm{kg/m^2}\right)(5\ \mathrm{m})^2 + 6(3\ \mathrm{kg/m})(5\ \mathrm{m})} \tag{34.2.11}$$

$$= \frac{725\ \mathrm{kg\ m}}{240\ \mathrm{kg}} \tag{34.2.12}$$

$$= 3.021\ \mathrm{m}. \tag{34.2.13}$$

(The denominator is the total mass, $M = 240\ \mathrm{kg}$.)

We can take a similar approach with a two-dimensional continuous object. The position vector $\mathbf{r}_{\mathrm{cm}}$ of the center of mass in two dimensions is

$$\mathbf{r}_{\mathrm{cm}} = \frac{\int \mathbf{r}\sigma(\mathbf{r})dA}{\int \sigma(\mathbf{r})dA} \tag{34.2.14}$$

$$= \frac{\iint \mathbf{r}\sigma(\mathbf{r})dxdy}{\iint \sigma(\mathbf{r})dxdy} \tag{34.2.15}$$

where $\sigma(\mathbf{r})$ is the area mass density of the body (mass per unit area), in units of $\mathrm{kg/m^2}$. Here we imagine dividing the body up into infinitesimal squares of area $dA = dxdy$, and treat each square as a point mass. The integrals in Eq. (31.25) are called double integrals, which you will learn more about when you study the calculus of several variables in a calculus course. Briefly, though, a double integral is interpreted as

$$\iint f(x,y)dxdy = \int\left[\int f(x,y)dx\right]dy \tag{34.2.16}$$

To evaluate this, you first evaluate the integral inside the square brackets, treating x as the variable of integration and treating y as a constant. You then use the result as the integrand of the outer integral, this time treating y as the variable of integration.

Similarly, in three dimensions, the position vector $\mathbf{r}_{\mathrm{cm}}$ of the center of mass is

$$\mathbf{r}_{\mathrm{cm}} = \frac{\int \mathbf{r}\rho(\mathbf{r})dV}{\int \rho(\mathbf{r})dV} \tag{34.2.17}$$

$$= \frac{\iiint \mathbf{r}\rho(\mathbf{r})dxdydz}{\iiint \rho(\mathbf{r})dxdydz} \tag{34.2.18}$$

where $\rho(\mathbf{r})$ is the familiar volume mass density of the body (mass per unit volume), in units of $\mathrm{kg/m^3}$. In this case we imagine dividing the body into infinitesimal cubes of volume $dV = dxdydz$, and treat each cube as a point mass. The integrals in Eq. (31.28) are called a triple integrals. Such an integral is interpreted as

$$\iiint f(x, y, z) dx dy dz = \int \left\{ \int \left[\int f(x, y, z) dx \right] dy \right\} dz \tag{34.2.19}$$

Here you evaluate the innermost integral (in square brackets) first, treating x as the variable of integration, treating y and z as constants. You then use this result as the integrand for the next integral (curly braces), treating y as the variable of integration, with z constant. Finally, you use that result as the integrand for the outermost integral, treating z as the variable of integration.
