12.6: Hitting a Target on a Hill

In the previous section, we looked at how to aim a projectile so that it hits a target on the ground. Now let's look at a more general case: suppose the target is not necessarily on the ground, but on a hill, so that it's located at coordinates (x_t, y_t). How do we aim the projectile to hit the target in this case?

Fixed Launch Angle

Let's first look at the case where the launch angle is fixed and we can vary the muzzle velocity. We require that the projectile's parabolic path pass through both the origin and the target's position (x_t, y_t), so let's begin by substituting the point (x_t, y_t) into Eq. (12.4.3):

$$y_t = \left(-\frac{g}{2v_0^2\cos^2\theta}\right)x_t^2 + (\tan\theta)x_t. \tag{12.6.1}$$

We just need to solve this for the muzzle velocity v_0 :

$$(\tan\theta)x_t - y_t = \left(\frac{g}{2v_0^2\cos^2\theta}\right)x_t^2 \tag{12.6.2}$$

$$\tan\theta - \frac{y_t}{x_t} = \frac{gx_t}{2v_0^2\cos^2\theta} \tag{12.6.3}$$

$$v_0^2 = \frac{gx_t}{2\left(\tan\theta - \frac{y_t}{x_t}\right)\cos^2\theta} \tag{12.6.4}$$

or

$$v_0 = \sqrt{\frac{gx_t}{2\left(\tan\theta - \frac{y_t}{x_t}\right)\cos^2\theta}} \tag{12.6.5}$$

Note that y_t/x_t is tangent of the angle that the target makes with the horizontal, as seen from the origin; we'll call this angle θ_t. Then Eq. $12.6.5$ becomes

$$v_0 = \sqrt{\frac{gx_t}{2(\tan\theta - \tan\theta_t)\cos^2\theta}}. \tag{12.6.6}$$

If we aim directly at the target, then $\theta = \theta_t$, the denominator becomes zero, and we get $v_0 = \infty$: this says that the muzzle velocity would have to be infinite to get it to follow a straight-line path directly to the target.

Fixed Muzzle Velocity

Now let's look at the more common problem, where the muzzle velocity fixed and we're allowed to vary the launch angle. As before, we substitute the point x_t, y_t into Eq. (12.4.3):

$$y_t = \left(-\frac{g}{2v_0^2\cos^2\theta}\right)x_t^2 + (\tan\theta)x_t. \tag{12.6.7}$$

Now we solve this for the launch angle θ. Multiplying both sides by $2\cos^2\theta$,

$$2y_t\cos^2\theta = \left(-\frac{g}{v_0^2}\right)x_t^2 + (2\sin\theta\cos\theta)x_t \tag{12.6.8}$$

Now using the identity $\sin 2\theta \equiv 2\sin\theta\cos\theta$,

$$x_t\sin 2\theta - 2y_t\cos^2\theta = \frac{gx_t^2}{v_0^2} \tag{12.6.9}$$

It turns out that this is about the best we can do-we just can't solve this equation for θ in closed form. To find θ, we must resort to a numerical method such as Newton's method, as described in the following chapter.

---

[2] Actually, it's complex; in this case, $\sin^{-1} 1.225 = 90^\circ - 37.75^\circ i$.
