57.12: Differential Equation for an Orbit

It can be shown (Ref. [8]) that a central force $F(r)$ satisfies the differential equation

$$F\left(\frac{1}{u}\right)=-\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2}+u\right) \tag{57.12.1}$$

where the equation is in polar coordinates, l is the angular momentum of the orbit, m is the mass, and $u\equiv 1/r$. This equation has an interesting application: given the orbit function in polar coordinates $r(\theta)$, you can solve for the force law $F(r)$ that gives that orbit. In theory, you could, for example, use Eq. $57.12.1$to find what force law would be necessary to produce a square orbit.

Example 57.12.1

As a simple example, suppose we observe a mass m in circular orbit of radius R around a parent mass M, so that the orbit equation is $r(\theta)=R$ (a constant), and so $u=1/R$. If the force present is gravity, then the orbital angular momentum of m will be $l=m\sqrt{GMR}$. Eq. $57.12.1$then gives

$$F=-\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2}+u\right) \tag{57.12.2}$$

$$=-\frac{m^2GMR}{mR^2}\left(\frac{1}{R}\right) \tag{57.12.3}$$

$$=-\frac{GMm}{R^2} \tag{57.12.4}$$

and we recover Newton's law of gravity. This is not by any means a derivation of Newton's law of gravity-in order to get the result in this example, we had to assume Newton's law of gravity to get the expression for
angular momentum l. This example is really just an illustration of how you can derive the force law if you're given the orbit and its angular momentum.

Example 57.12.2

Suppose a particle orbits in a circle that passes through the center of force. Show that the force law must be an inverse-fifth law force $\left(F\propto 1/r^5\right)$.

Solution

The polar equation of a circle passing through the origin is $r=2a\cos\theta$, where a is the radius of the circle. From Eq. $57.12.1$, we can find the force law. Since $r=2a\cos\theta$, we have

$$u=\frac{1}{r}=\frac{1}{2a\cos\theta} \tag{57.12.5}$$

We'll need the second derivative of u with respect to θ :

$$\frac{du}{d\theta} = \frac{\sin\theta}{2a\cos^2\theta} \tag{57.12.6}$$

$$\frac{d^2u}{d\theta^2} = \frac{2a\cos^3\theta + 4a\cos\theta\sin^2\theta}{4a^2\cos^4\theta} \tag{57.12.7}$$

$$= \frac{2a\cos^3\theta + 4a\cos\theta\left(1-\cos^2\theta\right)}{4a^2\cos^4\theta} \tag{57.12.8}$$

$$= \frac{2a\cos^3\theta + 4a\cos\theta - 4a\cos^3\theta}{4a^2\cos^4\theta} \tag{57.12.9}$$

$$= \frac{1}{2a\cos\theta} + \frac{1}{a\cos^3\theta} - \frac{2}{2a\cos\theta} \tag{57.12.10}$$

$$= \frac{1}{a\cos^3\theta} - \frac{1}{2a\cos\theta} \tag{57.12.11}$$

$$= \frac{8a^2}{8a^3\cos^3\theta} - \frac{1}{2a\cos\theta} \tag{57.12.12}$$

$$= 8a^2u^3 - u \tag{57.12.13}$$

Using this result, Eq. \(\PageIndex{1}\ becomes

$$\begin{aligned} F &= -\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) \\ &= -\frac{l^2u^2}{m}\left(8a^2u^3 - u + u\right) \\ &= -\frac{8a^2l^2}{m}u^5 \\ &= -\frac{8a^2l^2}{m}\frac{1}{r^5}. \quad \text{Q.E.D.} \end{aligned}$$