13.2: Example- Square Roots

When the first electronic calculators became available in the mid-1970s, many of them were simple "fourfunction" calculators that could only add, subtract, multiply, and divide. The author's father, L.L. Simpson (Ref. [11]), showed him how he could calculate square roots on one of these calculators using Newton's method, as described here.

To calculate the square root of a number k, we wish to find the number x in the equation

$$x=\sqrt{k}. \tag{13.2.1}$$

Squaring both sides then subtracting k from both sides, we get a function of the form of Eq. (13.1.1):

$$f(x)=x^{2}-k=0. \tag{13.2.2}$$

The values of x that satisfy this equation are the desired square roots of k. Newton's method for finding square roots is then Eq. (10.2.2) with this $f(x)$ (and with $f'(x)=2x$):

$$x_{n+1}=x_{n}-\frac{x_{n}^{2}-k}{2x_{n}} \tag{13.2.3}$$

For example, to calculate $\sqrt{5}$, set $k=5$. Make an initial estimate of the answer-say $x_0=2$. Then we calculate several iterations of Newton's method (Eq. $13.2.3$to get better and better estimates of $\sqrt{5}$:

$$x_0=2 \tag{13.2.4}$$

$$x_1=x_0-\frac{x_0^2-5}{2x_0}=2-\frac{2^2-5}{2\times 2}=2.2500 \tag{13.2.5}$$

$$x_2=x_1-\frac{x_1^2-5}{2x_1}=2.2500-\frac{2.2500^2-5}{2\times 2.2500}=2.2361 \tag{13.2.6}$$

$$x_3=x_2-\frac{x_2^2-5}{2x_2}=2.2361-\frac{2.2361^2-5}{2\times 2.2361}=2.2361 \tag{13.2.7}$$

After just a few iterations, the solution has converged to four decimal places: we have $\sqrt{5}=2.2361$.

There are actually two square roots of 5 . To find the other solution, we choose a different initial estimate, one that is closer to the other root. If we take the initial estimate $x_0=-2$, we get

$$x_0=-2 \tag{13.2.8}$$

$$x_1=x_0-\frac{x_0^2-5}{2x_0}=-2-\frac{(-2)^2-5}{2\times(-2)}=-2.2500 \tag{13.2.9}$$

$$x_2=x_1-\frac{x_1^2-5}{2x_1}=-2.2500-\frac{(-2.2500)^2-5}{2\times(-2.2500)}=-2.2361 \tag{13.2.10}$$

$$x_3=x_2-\frac{x_2^2-5}{2x_2}=-2.2361-\frac{(-2.2361)^2-5}{2\times(-2.2361)}=-2.2361 \tag{13.2.11}$$

So to four decimals, the other square root of 5 is -2.2361 .

L.L. Simpson notes that Eq. $13.2.3$for computing square roots was typically used in the equivalent form

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{k}{x_n}\right), \tag{13.2.12}$$

so that you repeatedly find the average of x_n and k/x_n. For the above example of finding $\sqrt{5}$, this gives:

Initial est		= 2
1st iteration:	Average of 2 and 5/2	= 2.25
2nd iteration:	Average of 2.25 and 5/2.25	= 2.2361

3rd iteration:	Average of 2.2361 and 5/2.2361	= 2.2361 (converged)