22.2: Model II- F ∝ v²

Now let's consider a different model of resistive force in a fluid, in which the resistive force is proportional to the square of the velocity:

$$F_R \propto v^2 \tag{22.2.1}$$

This model is appropriate for most situations when Model I is not used. We find experimentally that the resistive force in this case is proportional to the area A normal to the flow direction, and to the density ρ of the fluid. We write Eq. $22.2.1$ as

$$F_R = \frac{1}{2} C_D \rho A v^2 \tag{22.2.2}$$

where C_D is called the drag coefficient, and the factor of $1/2$ is conventional. The drag coefficient C_D depends on things like the shape of the falling body, its smoothness, and how turbulent the flow of fluid around the body is.

Proceeding as with Model I, we have, starting with Newton's second law,

$$F = ma \tag{22.2.3}$$

$$mg - \frac{1}{2} C_D \rho A v^2 = m \frac{dv}{dt} \tag{22.2.4}$$

This is a nonlinear differential equation that is much more difficult to solve than the one we had for Model I (Eq. 19.2.4). Instead of trying to solve for $v(t)$, we'll simply note that we can find the terminal velocity v_∞ by setting the acceleration $dv/dt = 0$ in Eq. $22.2.3$. This immediately gives

$$v_\infty = \sqrt{\frac{2mg}{C_D \rho A}} \tag{22.2.5}$$

So Model II of the resistive force, like Model I, exhibits a terminal velocity: as time $t \rightarrow \infty$, the velocity of the falling body will approach a constant, v_∞.

Example 22.2.1

With what speed do raindrops hit the Earth?

Solution

Assume the following rough estimates:

- Cloud base height: $h = 1000 \text{ m}$
- Air density: $\rho_{\text{air}} = 1.29 \text{ kg/m}^3$
- Raindrop (spherical) diameter: $d = 2 \text{ mm}$
- Raindrop (water) density: $\rho_{\text{w}} = 1.00 \times 10^3 \text{ kg/m}^3$
- Raindrop coefficient of friction: $C_D = 0.5$

First, let's try a naïve approach, and neglect air resistance. As seen in Chapter 5, the velocity v of a raindrop falling under gravity through a height h is given by

$$v = \sqrt{2gh} \tag{22.2.6}$$

$$= \sqrt{2\left(9.8 \text{ m/s}^2\right)(1000 \text{ m})} \tag{22.2.7}$$

$$= 140 \text{ m/s} \tag{22.2.8}$$

$$= 313 \text{mph} \tag{22.2.9}$$

Clearly raindrops are not hitting the Earth with a speed of 313mph, or they would be lethal. The problem here is that it is very important to consider air resistance, or you will not get even close to the correct answer.

A more accurate analysis would be to allow for air resistance by computing the terminal velocity. After falling 1000 meters, a raindrop will have more than enough time to reach the terminal velocity, so the impact velocity will equal the terminal velocity, given by Eq. 22.2.5. We're given g, C_D, and ρ; the cross-sectional $A=\pi d^2/4$; and the raindrop mass $m=\rho_{\mathrm{w}} V=\rho_{\mathrm{w}}\left(\pi d^3/6\right)$. Then by Eq. 22.2.5, the impact velocity will be

$$\begin{aligned}
v_{\infty} &=\sqrt{\frac{2 m g}{C_D \rho A}} && (22.2.10)\\
&=\sqrt{\frac{2\left(\rho_{\mathrm{w}} \pi d^3/6\right) g}{C_D \pi \rho d^2/4}} && (22.2.11)\\
&=\sqrt{\frac{4 \rho_{\mathrm{w}} d g}{3 C_D \rho}} && (22.2.12)\\
&=\sqrt{\frac{4\left(1000 \mathrm{~kg}/\mathrm{m}^3\right)\left(2 \times 10^{-3} \mathrm{~m}\right)\left(9.8 \mathrm{~m}/\mathrm{s}^2\right)}{3(0.5)\left(1.29 \mathrm{~kg}/\mathrm{m}^3\right)}} && (22.2.13)\\
&=6.37 \mathrm{~m}/\mathrm{s} && (22.2.14)\\
&=14.2 \mathrm{mph} && (22.2.15)
\end{aligned}$$

Whether or not it's important to consider air resistance in a particular problem is a matter of judgment and experience. With practice you develop an intuition about when it's likely to be important to include these kinds of effects.

Example 22.2.2

Consider the following problem due to L.L. Simpson (Ref. [11]): if it is considered safe for an adult to jump off of a three-foot high ladder without injury, what is the maximum design load for a conical parachute that is 30 feet in diameter and has a drag coefficient of 1.5 ? The design air density is $0.08 \mathrm{lb}/\mathrm{ft}^3$.

Solution

First, let's convert everything to SI units:

- Height: $3 \mathrm{ft}=0.9144 \mathrm{~m}$.
- Parachute diameter: $30 \mathrm{ft}=9.144 \mathrm{~m}$.
- Air density: $0.08 \mathrm{lb}/\mathrm{ft}^3=1.281477 \mathrm{~kg}/\mathrm{m}^3$.

Here's the general approach to the solution: from the first part of the problem, we can calculate the impact velocity of an adult jumping from a three-foot ladder. That impact velocity is considered safe, so we'll use that as the terminal velocity (using Eq. 22.2.5, since Model II is applicable for parachutes). We then solve for the weight mg, which is the required maximum weight that can be attached to the parachute and still have it reach a safe terminal velocity.

Let's begin. The impact velocity is given by Eq. 22.2.11,

$$\begin{aligned}
v &=\sqrt{2 g h} && (22.2.16)\\
&=\sqrt{2\left(9.8 \mathrm{~m}/\mathrm{s}^2\right)(0.9144 \mathrm{~m})} && (22.2.17)\\
&=4.23347 \mathrm{~m}/\mathrm{s} && (22.2.18)
\end{aligned}$$

We'll set the parachute terminal velocity v_{∞} equal to this impact velocity. Now solve Eq. 22.2.5 for the weight mg :

$$m g=\frac{1}{2} C_D \rho A v_{\infty}^2 \tag{22.2.19}$$

Next replace the parachute area A with $\pi d^2/4$, where d is the parachute diameter:

$$m g=\frac{1}{8} \pi C_D \rho d^2 v_{\infty}^2 \tag{22.2.20}$$

and substitute the numbers we're given:

$$
\begin{aligned}
mg &= \frac{1}{8}\pi(1.5)\left(1.281447\ \mathrm{kg/m^3}\right)(9.144\ \mathrm{m})^2(4.23347\ \mathrm{m/s})^2 && (22.2.21)\\
&= 1131.1670\ \mathrm{N} && (22.2.22)\\
&= 254\mathrm{lbf} && (22.2.23)
\end{aligned}
$$

where in the final step we've converted back to British engineering units; the maximum design load is a weight of $254\mathrm{lbf}$.