51.4: Change in Fluid Pressure with Depth

The pressure in a fluid in a gravitational field increases in the downward direction. A common example is the pressure of the Earth's atmosphere: atmospheric pressure is highest at the surface of the Earth, and decreases as you go up in altitude. Above a certain altitude (about 8000 feet above sea level), passengers in aircraft and mountain climbers need extra oxygen to be able to breathe properly. Another common example is well known to divers: water pressure increases with depth.

We can compute the change in pressure with depth using Archimedes' principle. Suppose we have a fluid like water, and we want to find how the pressure P increases with depth h from the surface. Imagine a slab of fluid (inside the bulk fluid) of area A and thickness Δh. We'll call the pressure on the top surface P_1, and the pressure on the bottom surface P_2. The net buoyant force on the slab of fluid will be $(P_2 - P_1)\,A = \triangle PA$. But by Archimedes' principle,

$$\Delta PA = \rho g A \Delta h, \tag{51.4.1}$$

and so

$$\frac{\Delta P}{\Delta h} = \rho g. \tag{51.4.2}$$

Taking the limit as $\Delta h \to 0$, we have

$$\frac{dP}{dh} = \rho g \tag{51.4.3}$$

Constant Density

Let's consider a special case where the density ρ is constant (as with water, for example). From Eq. $51.4.3$, we have

$$dP = \rho g dh. \tag{51.4.4}$$

Integrating both sides gives

$$P = P_0 + \rho g h \tag{51.4.5}$$

where P_0 is the pressure at depth $h = 0$.

Variable Density

Now consider another special case, where the density ρ is not constant. For a gas like the Earth's atmosphere, we typically have the density proportional to the pressure, so let's let the density $\rho = KP$, where K is a constant with units of $\mathrm{s}^2\ \mathrm{m}^{-2}$. Also, for the atmosphere, it will be convenient to use the upward-pointing altitude $y = -h$ rather than the downward-pointing depth. Eq. $51.4.3$ then becomes

$$\frac{dP}{dy} = -KPg \tag{51.4.6}$$

Now re-write this as

$$\frac{dP}{P} = -Kgdy \tag{51.4.7}$$

and integrate both sides; the result is

$$\ln P = -Kgy + C \tag{51.4.8}$$

where C is a constant. Taking e to the power of both sides, we get

$$P = e^C e^{-Kgy} \tag{51.4.9}$$

If $y = 0$, then this reduces to $P = e^C$, so e^C is the pressure at $y = 0$, which we'll write as P_0. Then the pressure P at altitude y is

$$P = P_0 e^{-Kgy}. \tag{51.4.10}$$

The quantity $H = 1/(Kg)$ has units of length, and is called the scale height. When the altitude y is equal to the scale height H, the pressure will be $1/e \approx 0.368$ of its value at $y = 0$. For the the lowest layer of the Earth's atmosphere (called the troposphere), the scale height is about $8 \mathrm{~km}$.

In terms of the scale height H, Eq. $51.4.10$ may be written

$$P=P_{0} e^{-y / H} \tag{51.4.11}$$

Equation $51.4.11$ assumes an isothermal (constant temperature) atmosphere. In reality, temperature decreases with increasing height at the rate of $0.0065^{\circ} \mathrm{C} / \mathrm{m}$ in the troposphere. This fact can be used with Eq. $51.4.3$ to show that

$$P=P_{0}\left(1-\frac{y}{\mathscr{H}}\right)^{n} \tag{51.4.12}$$

where for the Earth's troposphere $\mathscr{H}=44,329 \mathrm{~m}$ and $n=5.255876$. This expression for pressure vs. altitude is part of a numerical model of the atmospheric pressure, density, and temperature of the Earth's atmosphere called the U.S. Standard Atmosphere (Ref. [15]).