19.1: Mass Suspended by Two Ropes

As a typical example of a problem in statics, consider the situation shown in Fig. 19.1.1(a). A block of mass m is suspended by a wire, and the upper end of the wire is attached to two more ropes or wires that connect to the ceiling. Each of the three ropes is under tension; the tensions are labeled T_1, T_2, and T_3.

To begin the analysis of this situation, it is often helpful to draw a free-body diagram for each body in the problem. A free-body diagram shows all the forces acting on the body, and helps clarify your thinking when doing the analysis. For this problem, there are two bodies present: the block and the knot. Fig. 19.1.1(b) is a free-body diagram for the block, and Fig. 19.1.1(c) is a free-body diagram for the knot.

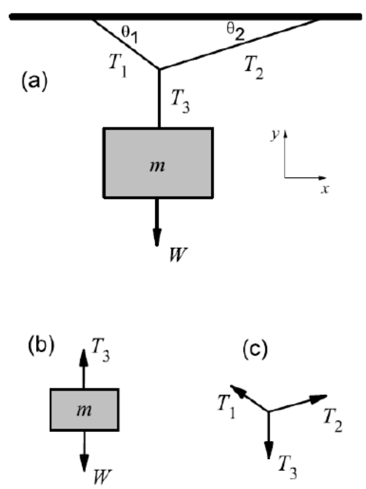

Figure 19.1.1: A block suspended from the ceiling by ropes. (a) Diagram of the situation. The block of mass m is suspended by a rope; the upper end of the rope is attached to two other ropes that are attached to the ceiling. (b) Free-body diagram for the block. (c) Free-body diagram for the knot.

Now let's begin the analysis; our goal will be to determine the three tensions T_1, T_2, and T_3, given the mass m and two angles θ_1 and θ_2. First, let's look at the free-body diagram for the block (Fig. 19.1.1(b)). For the block, the tension and weight vectors are given by

$$\mathbf{T}_3 = T_3\mathbf{j} \tag{19.1.1}$$

$$\mathbf{W} = -mg\mathbf{j} \tag{19.1.2}$$

(Note the x and y directions indicated in Fig. 19.1.1(a).) Now let's apply Newton's second law in both the x and y directions, noting that $F = ma = 0$ in this case:

$$x: \quad \sum F_x = ma_x \Rightarrow \qquad 0 = 0 \tag{19.1.3}$$

$$y: \quad \sum F_y = ma_y \Rightarrow \qquad T_3 - mg = 0 \tag{19.1.4}$$

Both right-hand sides are zero because the acceleration of the block is zero. The x equation (Eq. 19.1.3) yields a tautology $0 = 0$, which gives us no information. The y equation (Eq. 19.1.4) tells us $T_3 = mg$, so we've just found tension T_3.

We can find the other two tensions (T_1 and T_2) by analyzing the other body: the knot (Fig. 19.1.1(c)). For the knot, the three tension vectors are given by

$$\mathbf{T}_1 = -T_1\cos\theta_1\mathbf{i} + T_1\sin\theta_1\mathbf{j} \tag{19.1.5}$$

$$\mathbf{T}_2 = T_2\cos\theta_2\mathbf{i} + T_2\sin\theta_2\mathbf{j} \tag{19.1.6}$$

$$\mathbf{T}_3 = -T_3\mathbf{j} \tag{19.1.7}$$

Now let's apply Newton's second law ($F = ma = 0$) individually to the x and y components:

$$x: \quad \sum F_x = ma_x \Rightarrow \qquad -T_1\cos\theta_1 + T_2\cos\theta_2 = 0 \tag{19.1.8}$$

$$y: \quad \sum F_y = ma_y \Rightarrow \qquad T_1\sin\theta_1 + T_2\sin\theta_2 - T_3 = 0 \tag{19.1.9}$$

Again both right-hand sides are zero because the knot is not accelerating. Since T_3 is already known, this gives two simultaneous equations in the two unknown tensions T_1 and T_2. One method for solving this system of equations is to write the equations in matrix form:

$$\begin{pmatrix} -\cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 0 \\ T_3 \end{pmatrix} \tag{19.1.10}$$

Now multiplying both sides on the left by the inverse of the 2×2 matrix, we have

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} -\cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ T_3 \end{pmatrix} \tag{19.1.11}$$

Since the tension T_3 and the angles θ_1 and θ_2 are all known, this gives the two unknown tensions T_1 and T_2.

We can further simplify this by computing the matrix inverse explicitly. The determinant of the 2×2 matrix is (Appendix 63.17)

$$\det\begin{pmatrix} -\cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix} = -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \tag{19.1.12}$$

$$= -\sin(\theta_1+\theta_2) \tag{19.1.13}$$

and the matrix of cofactors is

$$\operatorname{cof}\begin{pmatrix} -\cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix} = \begin{pmatrix} \sin\theta_2 & -\sin\theta_1 \\ -\cos\theta_2 & -\cos\theta_1 \end{pmatrix} \tag{19.1.14}$$

Hence the matrix inverse, which is the transposed matrix of cofactors divided by the determinant, is

$$\begin{pmatrix} -\cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix}^{-1} = -\frac{1}{\sin(\theta_1+\theta_2)} \begin{pmatrix} \sin\theta_2 & -\cos\theta_2 \\ -\sin\theta_1 & -\cos\theta_1 \end{pmatrix} \tag{19.1.15}$$

$$= \frac{1}{\sin(\theta_1+\theta_2)} \begin{pmatrix} -\sin\theta_2 & \cos\theta_2 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}. \tag{19.1.16}$$

The tensions T_1 and T_2 are therefore

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \frac{1}{\sin(\theta_1+\theta_2)} \begin{pmatrix} -\sin\theta_2 & \cos\theta_2 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} 0 \\ T_3 \end{pmatrix} \tag{19.1.17}$$

$$= \frac{T_3}{\sin(\theta_1+\theta_2)} \begin{pmatrix} \cos\theta_2 \\ \cos\theta_1 \end{pmatrix}. \tag{19.1.18}$$

Recall that we've already found $T_3 = mg$; then the final results are

$$T_1 = \frac{mg\cos\theta_2}{\sin(\theta_1+\theta_2)}, \tag{19.1.19}$$

$$T_2 = \frac{mg\cos\theta_1}{\sin(\theta_1+\theta_2)}, \tag{19.1.20}$$

$$T_3 = mg. \tag{19.1.21}$$