43.1: The Half-Cylinder

To analyze rocking motion, let's consider a fairly simple body: a uniform right circular cylinder of radius r that has been cut in half by a plane passing through the axis of the cylinder to form a half-cylinder, whose cross-section is a semicircle (Figure $43.1.1$), and which is resting on a flat table. Let m be the mass of the half-cylinder.

It can be shown using the multivariate calculus that the the distance p between the axis S and the center of mass T is

$$p = \frac{4r}{3\pi} \tag{43.1.1}$$

We will need to find the moment of inertia I_H of the half-cylinder when rotated about an axis that lies along the line of contact between the half-cylinder and the table. We'll find this by first finding the moment of inertia I_T about an axis parallel to the half-cylinder axis and passing through the center of mass T; from that, we can then use the parallel axis theorem to find I_H. We'll find I_T by first finding I_S, the moment of inertia when rotated about the cylinder axis. In summary, we'll find I_S, then I_T, then I_H.

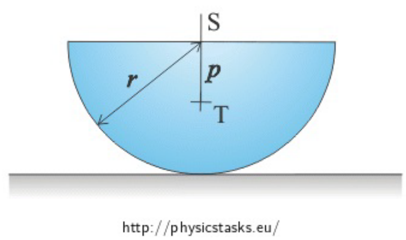

Figure 43.1.1: Rocking half-cylinder. The center of mass is at point T. From the Collection of Solved Problems in Physics

To find I_S, note that if we take two half-cylinders and place their flat ends together, we will have a full cylinder of mass $2m$ and moment of inertia equal to the sum of the moments of inertia of the two half cylinders, $I_S + I_S$. The moment of inertia of a solid cylinder when rotated about its axis is $1/2$ its mass times the square of its radius, so this full cylinder would have moment of inertia

$$I_S + I_S = \frac{1}{2}(2m)r^2 \tag{43.1.2}$$

so

$$I_S = \frac{1}{2}mr^2 \tag{43.1.3}$$

For the half-cylinder, the moments of inertia I_S and I_T are related by the parallel-axis theorem,

$$I_S = I_T + mp^2 \tag{43.1.4}$$

and the moments of inertia I_T and I_H are related by (again using the parallel-axis theorem),

$$I_H = I_T + m(r-p)^2 \tag{43.1.5}$$

and so

$$I_H = \left(I_S - mp^2\right) + m(r-p)^2 \tag{43.1.6}$$

$$= \left(\frac{1}{2}mr^2 - mp^2\right) + m(r-p)^2 \tag{43.1.7}$$

$$= \frac{1}{2}mr^2 - mp^2 + mr^2 - 2mrp + mp^2 \tag{43.1.8}$$

$$= \frac{3}{2}mr^2 - 2mrp \tag{43.1.9}$$

$$= \frac{3}{2}mr^2 - 2mr\left(\frac{4r}{3\pi}\right) \tag{43.1.10}$$

or

$$I_H = mr^2\left(\frac{3}{2} - \frac{8}{3\pi}\right) \tag{43.1.11}$$

Now we'll find the period of oscillation using conservation of energy. Let's rock the cylinder by some small angle α_0 (Figure 40.2). In this position, the cylinder is momentarily at rest, so it has zero kinetic energy. It does have a potential energy, though, equal to mgh, where h is the height of the center of mass above its height when in equilibrium. (Here we choose zero potential energy to be when the cylinder is in its equilibrium position.)

From the figure and using geometry, we see that

$$h = p - p\cos\alpha_0 = p\left(1 - \cos\alpha_0\right) \tag{43.1.12}$$

and so the potential energy is

$$mgh = mg\left(p - p\cos\alpha_0\right) = mgp\left(1 - \cos\alpha_0\right) \tag{43.1.13}$$

Now, starting from angle α_0, we release the half-cylinder. When the half-cylinder reaches its equilibrium position, its potential energy is zero, but it has kinetic energy $\frac{1}{2}I_H\omega_m^2$, where $\omega_m = \alpha_0\omega$ is the angular velocity at the equilibrium point (cf. Eq. 42.1.5). By conservation of energy, the total energy at angle α_0 must equal the total energy at the equilibrium position:

$$0 + mgp\left(1 - \cos\alpha_0\right) = \frac{1}{2}I_H\omega_m^2 + 0 \tag{43.1.14}$$

$$= \frac{1}{2}I_H\left(\omega\alpha_0\right)^2 + 0 \tag{43.1.15}$$

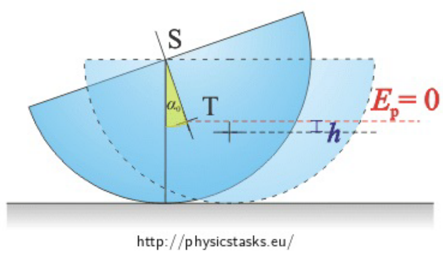

Figure 43.1.2: Rocking cylinder rocked by angle α_0.

We now use the small-angle approximation

$$\cos\alpha_0 \approx 1 - \frac{\alpha_0^2}{2} \quad \Rightarrow \quad 1 - \cos\alpha_0 \approx \frac{\alpha_0^2}{2} \tag{43.1.16}$$

we get (approximately)

$$mgp\left(\frac{\alpha_0^2}{2}\right) = \frac{1}{2}I_H\omega^2\alpha_0^2 \tag{43.1.17}$$

Solving for the angular frequency ω, we find

$$\omega = \sqrt{\frac{mgp\alpha_0^2}{I_H\alpha_0^2}} = \sqrt{\frac{mg\left(\frac{4r}{3\pi}\right)}{mr^2\left(\frac{3}{2} - \frac{8}{3\pi}\right)}} \tag{43.1.18}$$

We now do some simplifying:

$$\omega=\sqrt{\frac{g\left(\frac{4r}{3\pi}\right)}{r^2\left(\frac{9\pi-16}{6\pi}\right)}} \tag{43.1.19}$$

$$=\sqrt{\frac{g(4r)(6\pi)}{3\pi r^2(9\pi-16)}} \tag{43.1.20}$$

$$=\sqrt{\frac{8g}{r(9\pi-16)}} \tag{43.1.21}$$

and so the period of oscillation $T=2\pi/\omega$ is

$$T=2\pi\sqrt{\frac{r(9\pi-16)}{8g}} \tag{43.1.22}$$

Notice that $T\propto\sqrt{r}$, so the larger the radius, the longer the period of oscillation. Notice also that the period is independent of the mass m, so that all half-cylinders of the same radius will rock with the same period.