44.5: Ball Rolling in a Bowl

Suppose a ball of mass m and radius r is allowed to roll (without slipping) back and forth inside a hemispherical bowl of radius R. Does this constitute simple harmonic motion? And if so, what is the period of the motion?

To begin, let θ be the angle the ball makes with the vertical, so that $\theta=0$ when the ball is at the bottom of the bowl. Also let ω be the rotational angular velocity of the ball about its center of mass, $\Omega=d\theta/dt$ be the angular velocity of the ball's motion within the bowl, and $v=(R-r)\Omega$ the translational speed of the ball. It turns out that we can find the equation of motion by computing the time derivative of the total mechanical energy of the ball. The ball's total mechanical energy is the sum of three components: its translational kinetic energy, its rotational kinetic energy, and its gravitational potential energy.

The translational kinetic energy of the ball is

$$K_t=\frac{1}{2}mv^2 \tag{44.5.1}$$

$$=\frac{1}{2}m(R-r)^2\Omega^2 \tag{44.5.2}$$

The rotational kinetic energy of the ball about its axis is

$$K_r=\frac{1}{2}I\omega^2 \tag{44.5.3}$$

where I is the moment of inertia of the ball about its center of mass. If we take zero potential energy to be the point where the ball is at the bottom of the bowl, then the potential energy of the ball is

$$U=mg(R-r)(1-\cos\theta). \tag{44.5.4}$$

Therefore the total mechanical energy of the ball is

$$E=K_t+K_r+U \tag{44.5.5}$$

$$=\frac{1}{2}m(R-r)^2\Omega^2+\frac{1}{2}I\omega^2+mg(R-r)(1-\cos\theta) \tag{44.5.6}$$

We'll want to get all terms of this equation in terms of θ; to do this, we'll need to write ω in terms of Ω. Since the ball rolls without slipping, we know $v=r\omega$, and so

$$v=r\omega=(R-r)\Omega \tag{44.5.7}$$

$$\omega=\left(\frac{R-r}{r}\right)\Omega. \tag{44.5.8}$$

Substituting this into Eq. 44.5.5, we have

$$E=\frac{1}{2}m(R-r)^2\Omega^2+\frac{1}{2}I\left(\frac{R-r}{r}\right)^2\Omega^2+mg(R-r)(1-\cos\theta). \tag{44.5.9}$$

Writing the moment of inertia as $I=\beta mr^2$, the total energy may be written

$$E=\frac{1}{2}m(R-r)^2\Omega^2+\frac{1}{2}\beta mr^2\left(\frac{R-r}{r}\right)^2\Omega^2+mg(R-r)(1-\cos\theta) \tag{44.5.10}$$

$$=\frac{1}{2}m(R-r)^2\Omega^2+\frac{1}{2}\beta m(R-r)^2\Omega^2+mg(R-r)(1-\cos\theta) \tag{44.5.11}$$

$$= \frac{1}{2} m (R-r)^2 \Omega^2 (\beta+1) + mg(R-r)(1-\cos\theta) \tag{44.5.12}$$

$$= \frac{1}{2} m (R-r)^2 \left(\frac{d\theta}{dt}\right)^2 (\beta+1) + mg(R-r)(1-\cos\theta) \tag{44.5.13}$$

where in the last step we substituted the definition $\Omega = d\theta/dt$. We can find the equation of motion by taking the time derivative dE/dt, which must be zero, since E must be constant:

$$\frac{dE}{dt} = m(R-r)^2 \left(\frac{d\theta}{dt}\right) \left(\frac{d^2\theta}{dt^2}\right) (\beta+1) + mg(R-r)\sin\theta \left(\frac{d\theta}{dt}\right) = 0 \tag{44.5.14}$$

And so, cancelling a common $d\theta/dt$ on both sides, we get

$$m(R-r)^2 \left(\frac{d^2\theta}{dt^2}\right) (\beta+1) = -mg(R-r)\sin\theta \tag{44.5.15}$$

Cancelling a common $m(R-r)$ on both sides,

$$(R-r) \left(\frac{d^2\theta}{dt^2}\right) (\beta+1) = -g\sin\theta \tag{44.5.16}$$

Now solving for $d^2\theta/dt^2$, we get the equation of motion:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{(\beta+1)(R-r)}\sin\theta \tag{44.5.17}$$

This is the same as Eq. 41.1.3 for a simple plane pendulum, with effective length

$$L_{\text{eff}} = (\beta+1)(R-r). \tag{44.5.18}$$

The ball rolling in the hemispherical bowl is, like the simple plane pendulum, not exactly a simple harmonic oscillator; but it is approximately a simple harmonic oscillator for small oscillations.

For small oscillations, the period of oscillation T is given by Eq. 41.1.8, with L replaced by L eff in Eq. $44.5.19$:

$$T = 2\pi\sqrt{\frac{(\beta+1)(R-r)}{g}}. \tag{44.5.19}$$

For example, if the ball is a uniform solid sphere, then $\beta = 2/5$, and so $\beta+1 = 7/5$ and we have

$$T = 2\pi\sqrt{\frac{7(R-r)}{5g}} \tag{44.5.20}$$

A ball rolling in a hemispherical bowl will have a period greater than that of a simple plane pendulum of the same length, by a factor of $\sqrt{\beta+1} = \sqrt{7/5} \approx 1.1832$.