8.7: Constant Acceleration

The definitions of velocity and acceleration we've seen so far ($v = dx/dt, a = dv/dt$) are always true. But now let's look at an important special case: constant acceleration. First, assume that the acceleration a is a constant. Then by Eq. (5.6.4),

$$v(t) = \int a dt \tag{8.7.1}$$

$$= a \int dt \tag{8.7.2}$$

$$= at + C, \tag{8.7.3}$$

where C is a constant. The assumption of constant acceleration comes in Eq. 8.7.2, where we use that assumption to bring a outside the integral.

What is the physical significance of the integration constant C ? Let's look at what Eq. 8.7.3 gives us when $t = 0$:

$$v(0) = a \cdot 0 + C = C. \tag{8.7.4}$$

So C is just the velocity of the particle at time $t = 0$ (the initial velocity), which we'll write [3] as $v(0) = v_0$. Then Eq. 8.7.3 is written

$$v(t) = at + v_0 \tag{8.7.5}$$

Now let's substitute Eq. 8.7.5 for $v(t)$ into Eq. (5.6.2) to get an expression for $x(t)$ for constant acceleration:

$$x(t) = \int (at + v_0)\, dt \tag{8.7.6}$$

$$= \int atdt + \int v_0 dt \tag{8.7.7}$$

$$= a \int tdt + v_0 \int dt \tag{8.7.8}$$

$$= \frac{1}{2}at^2 + v_0 t + C'. \tag{8.7.9}$$

What is the physical significance of the integration constant C' ? We do the same trick we did before, and look at what happens when $t = 0$:

$$x(0) = \frac{1}{2}a \cdot 0^2 + v_0 \cdot 0 + C' = C'. \tag{8.7.10}$$

So C' is the position of the particle at time $t = 0$ (the initial position), which we'll write as $x(0) = x_0$. Then Eq. 8.7.9 becomes

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0 \tag{8.7.11}$$

Example 8.7.1

Suppose you're standing on a bridge, and want to know how high you are above the river below. You can do this by dropping a rock from the bridge and counting how many seconds it takes to hit the river.

Solution

We begin solving this problem by defining a coordinate system with $+x$ pointing downward, and the origin at the bridge. This is an arbitrary choice; we could just as easily define the x axis pointing up instead of down, and in either case we could put the origin at the bridge or at the river (or anywhere else, for that matter), and you'll get the same answer at the end. But putting the origin at the bridge simplifies the equations somewhat, and pointing the $+x$ axis downward makes the acceleration and velocity positive. The coordinate system is an artificial mathematical construction that we introduce into the problem; the choice of origin and direction will not affect the physics or the final answer, so we're free to choose whatever is convenient.

The acceleration is constant in this case (and equal to the acceleration due to gravity), so we can use the constant-acceleration expression for x, Eq. (5.23). Since the acceleration is always downward and we've defined $+x$ downward, we have $a=+g$. We'll define time $t=0$ as the instant the rock is released; then $v_0=0$ since the rock is released from rest, and $x_0=0$ because we defined the origin to be at the point of release. Then Eq. (5.23) becomes

$$x=\frac{1}{2}gt^2. \tag{8.7.12}$$

Let's say it takes 4 seconds for the rock to hit the water. Then the height of the bridge above the river is $x=gt^2/2=\left(9.80\ \mathrm{m/s^2}\right)(4\ \mathrm{s})^2/2=78.4\ \mathrm{m}$.

Sometimes we'll find a problem involving position and velocity, but not time. For such problems with constant acceleration, it is useful to have an expression for velocity v in terms of position x, i.e. $v(x)$. We begin by solving Eq. $8.7.5$ for time t:

$$t=\frac{v-v_0}{a} \tag{8.7.13}$$

Now substitute this expression for t into Eq. $8.7.11$:

$$x=\frac{1}{2}a\left(\frac{v-v_0}{a}\right)^2+v_0\left(\frac{v-v_0}{a}\right)+x_0. \tag{8.7.14}$$

We've eliminated time t; now we just need to solve this for v:

$$x=\frac{1}{2}\frac{(v-v_0)^2}{a}+\frac{vv_0-v_0^2}{a}+x_0 \tag{8.7.15}$$

$$2ax=\left(v^2-2vv_0+v_0^2\right)+2\left(vv_0-v_0^2\right)+2ax_0 \tag{8.7.16}$$

$$2a\left(x-x_0\right)=v^2-2vv_0+v_0^2+2vv_0-2v_0^2 \tag{8.7.17}$$

$$2a\left(x-x_0\right)=v^2-v_0^2 \tag{8.7.18}$$

and so

$$v^2=v_0^2+2a\left(x-x_0\right) \tag{8.7.19}$$

This says that under constant acceleration, if the particle has velocity v_0 at position x_0, then it will have velocity v at position x.

Example $8.7.2$ Impact Velocity

Suppose we drop a rock from a bridge that we know to be $h=125\ \mathrm{m}$ above water. What is the impact velocity of the rock, i.e. the velocity of the rock just before it hits the water?

Solution

Notice that there is no time involved in this problem: only a distance and a velocity. This suggests using Eq. $8.7.19$ to find the impact velocity. As in the previous example, we begin by defining a coordinate system, and we'll use the same system as before: with the origin at the bridge, and $+x$ pointing downward. Then $x_0=0$ (because of where we defined the origin), $v_0=0$ (because the rock is released from rest), and $a=+g$ (because we defined $+x$ as downward). Then Eq. $8.7.19$ becomes

$$v^2=2gh \tag{8.7.20}$$

Solving for v gives the velocity at position $x=h$ (at the water). We'll use only the positive square root of this equation, which gives the magnitude of the velocity, i.e. the speed:

$$v=\sqrt{2gh} \tag{8.7.21}$$

$$=\sqrt{2\left(9.8\ \mathrm{m/s^2}\right)(125\ \mathrm{m})} \tag{8.7.22}$$

$$=49.5\ \mathrm{m/s} \tag{8.7.23}$$

Notice that there is no time involved in this problem: only a distance and a velocity. This suggests using Eq. $8.7.19$ to find the impact velocity. As in the previous example, we begin by defining a coordinate system, and we'll use the same system as before: with the origin at the bridge, and $+x$ pointing downward. Then $x_0 = 0$ (because of where we defined the origin), $v_0 = 0$ (because the rock is released from rest), and $a = +g$ (because we defined $+x$ as downward). Then Eq. $8.7.19$ becomes

$$v^2 = 2gh \tag{8.7.24}$$

Solving for v gives the velocity at position $x = h$ (at the water). We'll use only the positive square root of this equation, which gives the magnitude of the velocity, i.e. the speed:

$$v = \sqrt{2gh} \tag{8.7.25}$$

$$= \sqrt{2\left(9.8 \mathrm{~m/s}^2\right)(125 \mathrm{~m})} \tag{8.7.26}$$

$$= 49.5 \mathrm{~m/s} \tag{8.7.27}$$

Just to show that the definition of coordinate system doesn't affect the final answer, let's re-work the problem using a coordinate system that has the origin at the water instead of at the bridge, and let's construct the x axis so that $+x$ points upward. In this case the rock will have velocity $v_0 = 0$ at position $x_0 = h$, $a = -g$ (because the x axis now points upward), and we wish to find the velocity v at $x = 0$. Then Eq. ($8.7.19$) becomes

$$v^2 = 2(-g)(0-h) \tag{8.7.28}$$

$$v = \sqrt{2gh}, \tag{8.7.29}$$

where we have again used only the positive square root, and we get the same result as before-the result is independent of how we define the coordinate system.

3. The quantity v_0 is customarily pronounced " v-nought", nought being an old-fashioned term for zero. Similarly, x_0 is pronounced “ x-nought”.