22.1: Model I F ∝ v

In this first model, we model the resistive force F_R through a fluid as being proportional to the first power of the velocity v :

$$F_R = -bv \tag{22.1.1}$$

where b is the constant of proportionality; the minus sign shows that the resistive force is always opposite the direction of motion.

Under this model, the net downward force on the falling body is $mg + F_R = mg - bv$. Then by Newton's second law,

$$F = ma \tag{22.1.2}$$

$$mg - bv = m\frac{dv}{dt} \tag{22.1.3}$$

Dividing through by m, we have

$$\frac{dv}{dt} + \frac{b}{m}v = g \tag{22.1.4}$$

This is a first-order differential equation, which you will learn to solve for $v(t)$ in a course on differential equations. But briefly, for a differential equation of the form

$$\frac{dy}{dt} + p(t)y = q(t), \tag{22.1.5}$$

the solution $y(t)$ is found to be (Ref. [2])

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + C\right] \tag{22.1.6}$$

where C is a constant that depends on the initial conditions, and $\mu(t)$ is an integrating factor, given by

$$\mu(t) = \exp\left[\int p(t)dt\right] \tag{22.1.7}$$

Since this is a first-order differential equation, there will be one arbitrary constant of integration, and it is the constant C in Eq. $22.1.6$.

Comparing Eq. $22.1.4$ with Eq. $22.1.5$, we have

$$y(t) = v(t) \tag{22.1.8}$$

$$p(t) = b/m \tag{22.1.9}$$

$$q(t) = g. \tag{22.1.10}$$

Then the integrating factor $\mu(t)$ is, from Eq. $22.1.7$,

$$\mu(t) = \exp\left[\int p(t)dt\right] \tag{22.1.11}$$

$$= \exp\left[\int \frac{b}{m}dt\right] \tag{22.1.12}$$

$$= Ae^{bt/m}, \tag{22.1.13}$$

where A is a constant of integration. The solution to Eq. $22.1.4$ is then given by Eq. $22.1.6$:

$$v = \frac{e^{-bt/m}}{A}\left[\int Ae^{bt/m}g dt + C\right] \tag{22.1.14}$$

$$= e^{-bt/m}\left[\frac{mg}{b}e^{bt/m} + C'\right] \tag{22.1.15}$$

$$= \frac{mg}{b} + C'e^{-bt/m}. \tag{22.1.16}$$

To find the constant C', we use the initial condition: if we release the body at time zero, then $v = 0$ when $t = 0$; Eq. $22.1.16$ then becomes at $t = 0$

$$0=\frac{mg}{b}+C' \tag{22.1.17}$$

and so

$$C'=-\frac{mg}{b} \tag{22.1.18}$$

Therefore, from Eq. $22.1.16$, the solution is

$$v=\frac{mg}{b}-\frac{mg}{b}e^{-bt/m}, \tag{22.1.19}$$

or

$$v=\frac{mg}{b}\left(1-e^{-bt/m}\right) \tag{22.1.20}$$

This is the solution we're after: it gives the falling object's velocity v at any time t, assuming that it's dropped from rest at time $t=0$.

Now let's examine what happens to the solution (Eq. $22.1.20$) as $t\to\infty$. In this case, the exponential term approaches zero, and the falling object's velocity approaches the limiting value

$$v_\infty=\frac{mg}{b} \tag{22.1.21}$$

This is called the terminal velocity, and is a general feature of bodies falling through resistive fluids: at some point the resistive force balances the downward gravitational force, and the body stops accelerating and moves at a constant velocity. [2] Sky divers experience this phenomenon: some time after jumping out of an airplane, a sky diver will reach a terminal velocity of roughly 100 miles per hour, and will not change speed further until the parachute is deployed.

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[2] A simpler way to arrive at Eq. $22.1.21$ is to simply set the acceleration $dv/dt=0$ in Eq. (19.5), which immediately gives $v_\infty=mg/b$.
