58.3: Elliptical Orbits

Several of the results we found earlier for a circular orbit can be generalized for an elliptical orbit. Suppose an elliptical orbit has semi-major axis a and eccentricity e. The the semi-minor axis $b=a\sqrt{1-e^2}$, and the distance from the center of the ellipse to either of the two foci is $c=ae=\sqrt{a^2-b^2}$. Then the distance from the center of the Earth (located at one focus) to the perigee point is

$$r_p=a-c=a(1-e), \tag{58.3.1}$$

and the distance from the center of the Earth to the apogee point is

$$r_a=a+c=a(1+e). \tag{58.3.2}$$

A little algebra gives an expression for the semi-major axis a in terms of the perigee and apogee distances:

$$a=\frac{r_p+r_a}{2} \tag{58.3.3}$$

and similarly we can get an expression for the eccentricity e :

$$e=\frac{r_a-r_p}{r_a+r_p}. \tag{58.3.4}$$

Energy

The total orbit energy E of a spacecraft in an elliptical orbit turns out to be

$$E=-G\frac{M_\oplus m}{2a} \tag{58.3.5}$$

The potential and kinetic energies vary with r around the orbit. The potential energy at r is given by Eq. 58.1.1. The orbit velocity at r is found from the vis viva equation, Eq. 57.10.6,

$$v=\sqrt{GM_\oplus\left(\frac{2}{r}-\frac{1}{a}\right)} \tag{58.3.6}$$

from which we find the kinetic energy at r to be

$$K=GM_\oplus m\left(\frac{1}{r}-\frac{1}{2a}\right) \tag{58.3.7}$$

At perigee, $r=r_p=a(1-e)$, and so

$$v_p=\sqrt{\frac{GM_\oplus}{a}\frac{1+e}{1-e}} \tag{58.3.8}$$

$$K_p=\frac{GM_\oplus m}{2a}\frac{1+e}{1-e} \tag{58.3.9}$$

At apogee, $r=r_a=a(1+e)$, and so

$$v_a=\sqrt{\frac{GM_\oplus}{a}\frac{1-e}{1+e}} \tag{58.3.10}$$

$$K_a=\frac{GM_\oplus m}{2a}\frac{1-e}{1+e} \tag{58.3.11}$$

Angular Momentum

The angular momentum also varies with r, and is given by

$$L=mrv\cos\phi. \tag{58.3.12}$$

Here ϕ is called the elevation angle, and is the angle between the tangent to the ellipse at the spacecraft and the spacecraft velocity vector.

At either perigee or apogee, $\phi = 0$, so $L = mrv$. At perigee, $r_p = a(1-e)$, and so the angular momentum is

$$L_p = mv_p a(1-e). \tag{58.3.13}$$

At apogee, $r_a = a(1+e)$, and so

$$L_a = mv_a a(1+e) \tag{58.3.14}$$

Since angular momentum is conserved, then $L_p = L_a$; if the orbit parameters a and e are known and the velocity at either the apogee or perigee point is known, then the velocity at the other point is known:

$$L_p = L_a \tag{58.3.15}$$

$$mv_p a(1-e) = mv_a a(1+e) \tag{58.3.16}$$

$$v_p(1-e) = v_a(1+e) \tag{58.3.17}$$

Thus the perigee velocity v_p is related to the apogee velocity v_a through

$$\frac{v_p}{v_a} = \frac{1+e}{1-e} \tag{58.3.18}$$

In terms of the apogee and perigee distances r_a and r_p,

$$\frac{v_p}{v_a} = \frac{r_a}{r_p} \tag{58.3.19}$$

Example. Suppose a spacecraft is in an Earth-orbiting elliptical orbit with a semi-major axis $a = 8000$ km and eccentricity $e = 0.1500$. What are its velocities at perigee and apogee?

Solution. From Eq. $58.3.8$, the perigee velocity is

$$\begin{aligned} v_p &= \sqrt{\frac{GM_\oplus}{a}\frac{1+e}{1-e}} \\ &= \sqrt{\frac{3.986005 \times 10^{14}\ \mathrm{m^3\ s^{-2}}}{8000 \times 10^3\ \mathrm{m}}\frac{1+0.1500}{1-0.1500}} \\ &= \underline{8210\ \mathrm{km/s}.} \end{aligned}$$

The apogee velocity can be found using Eq. |9|pageIndex{18}\):

$$v_a = v_p\frac{1-e}{1+e} = 8210\ \mathrm{km/s}\frac{1-0.1500}{1+0.1500} = \underline{6069\ \mathrm{km/s}} \tag{58.3.20}$$

Circularizing an Orbit

An elliptical orbit may be circularized by changing the spacecraft velocity appropriately. One can change the spacecraft velocity at perigee to create a circular orbit whose radius is equal to the perigee distance, or one can change the spacecraft velocity at apogee to create a circular orbit whose radius is equal to the apogee distance. To calculate the change in spacecraft velocity (called the Delta v, or Δv), one uses the principle of conservation of energy.

Suppose a spacecraft is in an elliptical orbit with semi-major axis a and eccentricity e, and we wish to circularize it at perigee. The spacecraft velocity at perigee is given by Eq. $58.3.8$, and the circular velocity at $r = r_p$ is given by Eq. 58.1.4. The required change in spacecraft velocity at perigee is their difference. Using these equations along with Eq. $58.3.1$ gives, after a little algebra,

$$\Delta v = v_c - v_p = \sqrt{\frac{GM_\oplus}{a(1-e)}}(1-\sqrt{1+e}). \tag{58.3.21}$$

Similarly, if we wanted to circularize the orbit at apogee, the required change in spacecraft velocity at apogee is found by finding the difference of Eqs. $58.3.2$ and $58.3.10$, using these equations along with Eq. $58.3.2$, we get

$$\Delta v = v_c - v_a = \sqrt{\frac{GM_\oplus}{a(1+e)}}(1-\sqrt{1-e}). \tag{58.3.22}$$

If the spacecraft velocity vector is perpendicular to the radius vector $\mathbf{r}$ at some instant in time, then the magnitude of the velocity determines what kind of orbit the spacecraft is in:

- If $v = v_c$ (Eq. 58.2.4), then the spacecraft is in a circular orbit.
- If $v > v_c$, then the spacecraft is at perigee in an elliptical orbit.
- If $v < v_c$, then the spacecraft is at apogee in an elliptical orbit.

Example 58.3.1

Suppose we have an Earth-orbiting spacecraft in an elliptical orbit, with perigee distance 8000 km and apogee distance $12000\ \mathrm{km}$. We wish to circularize the orbit at apogee to create a circular orbit with radius 12000 km.

Solution

From Eqs. 58.3.1 and 58.3.2, we have

$$a = \frac{r_p + r_a}{2} = \frac{8000\ \mathrm{km} + 12000\ \mathrm{km}}{2} = 10000\ \mathrm{km} \tag{58.3.23}$$

The eccentricity is

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.200 \tag{58.3.24}$$

Circularizing the elliptical orbit to the apogee distance will require a single engine burn at the apogee point that results in a change in spacecraft velocity given by Eq. 58.3.23:

$$\begin{aligned}
\Delta v &= \sqrt{\frac{GM_\oplus}{a(1+e)}}(1-\sqrt{1-e}) \\
&= \sqrt{\frac{3.986005 \times 10^{14}\ \mathrm{m^3\ s^{-2}}}{(10000 \times 10^3\ \mathrm{m})\ (1+0.200)}}(1-\sqrt{1-0.200}) \\
&= \underline{608\ \mathrm{m/s}}
\end{aligned}$$