10.3: Properties

Commutativity

Let's look at a few properties of the dot product. First of all, the dot product is commutative:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{10.3.1}$$

The proof of this property should be obvious from Eqs. (7.1.1) and (7.2.6). This isn't a trivial property; in fact, the other two types of vector multiplication are non-commutative.

Projections

The dot product is defined as it is because it gives the projection of one vector onto the direction of another. For example, dotting a vector $\mathbf{A}$ with any of the cartesian unit vectors gives the projection of $\mathbf{A}$ in that direction:

$$\mathbf{A} \cdot \mathbf{i} = A_x \tag{10.3.2}$$

$$\mathbf{A} \cdot \mathbf{j} = A_y \tag{10.3.3}$$

$$\mathbf{A} \cdot \mathbf{k} = A_z \tag{10.3.4}$$

In general, the projection of vector $\mathbf{A}$ in the direction of unit vector $\hat{\mathbf{u}}$ is $\mathbf{A} \cdot \hat{\mathbf{u}}$.

Magnitude

From Eq. (7.2.6), it follows that $\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2 = A^2$; so the magnitude of a vector $\mathbf{A}$ is given in terms of the dot product by

$$A^2 = \mathbf{A} \cdot \mathbf{A} \tag{10.3.5}$$

$$A = \sqrt{\mathbf{A} \cdot \mathbf{A}} \tag{10.3.6}$$

Angle between Two Vectors

The dot product is also useful for computing the separation angle between two vectors. From Eq. (7.1.1),

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} \tag{10.3.7}$$

Example 10.3.1

We wish to find the angle between the two vectors $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$.

Solution

We first find the dot product of the two vectors:

$$\mathbf{A} \cdot \mathbf{B} = (3)(1) + (4)(-5) + (-2)(2) = -21$$

The magnitudes of the two vectors are

$$A = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{29}$$

$$B = \sqrt{1^2 + (-5)^2 + 2^2} = \sqrt{30}$$

Therefore

$$\cos\theta = \frac{-21}{\sqrt{29}\sqrt{30}} = -0.711967$$

and so the angle between $\mathbf{A}$ and $\mathbf{B}$ is

$$\theta = 135.40^\circ$$

> You do not need to worry about getting the angle θ in the correct quadrant, because θ will necessarily always be between 0° and 180°, and the inverse cosine function will always return its result in this range.

Orthogonality

Another useful property of the dot product is: if two vectors are orthogonal, then their dot product is zero. For example, for the cartesian unit vectors:

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{i}\cdot\mathbf{k}=0 \tag{10.3.8}$$

The converse is also true: if the dot product is zero, then the two vectors are orthogonal.

The cartesian unit vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are orthonormal, so that

$$\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1 \tag{10.3.9}$$

Derivative

The derivative of the dot product is similar to the familiar product rule for scalars:

$$\frac{d(\mathbf{A}\cdot\mathbf{B})}{dt}=\mathbf{A}\cdot\frac{d\mathbf{B}}{dt}+\frac{d\mathbf{A}}{dt}\cdot\mathbf{B} \tag{10.3.10}$$