8.2: It's Not Rotating!

Yet Another Surprising Experimental Result

Armed with what we know about quantized angular momentum, suppose we launch into the following simple experiment: We send a beam of hydrogen atoms in their ground state though a SG apparatus. We know that for hydrogen atoms in their $n=1$ state, they must have an orbital quantum number $l=n-1=0$, and a magnetic quantum number $m_l=l=0$. That is, it has no magnetic moment at all, and we expect to see the beam pass undeflected through the magnetic field. Strangely, we don't see this at all!

Maybe we made a mistake somewhere, and we accidentally sent $n=2,\ l=1,\ m_l=0,\ \pm 1$ atoms through? No, this can't be it either, because we don't see three beams emerge (one for each value of m_l – one deflected up, one down, and one undeflected) – we see *two beams* emerge! Okay, things are getting weird. Maybe there are weird complications with the hydrogen atom that we don't understand. Let's remove these complications by taking the proton out of the picture – we'll send just the electrons through the field. To our amazement, we find that there are *still* two beams emerging. We can draw no other conclusion than this: Electrons have a magnetic moment of their own.

Let's Call it "Spin"

It made sense that a hydrogen atom can have magnetic moment – the electron orbits the nucleus, and the radius of this orbit comes into the calculation of what we have defined as magnetic moment in Figure 8.1.1 – the rate of charge flow times the area of the orbit. But an elementary particle like an electron, while it does obviously have the charge needed, *has no radius*, so it is unclear how it can have a magnetic moment.

Okay, so given how little we know about magnetism, maybe there is something we are missing. But there is a more fundamental quantity that we can't forget got us started down this road of using magnetism – angular momentum. If a single point particle has a magnetic dipole moment, it must also possess angular momentum. This is fine when it is orbiting, but our experiment shows that it is somehow intrinsic to the particle. We have actually encountered these two faces of angular momentum before, in 9HA, in Equation 6.1.13. In that case, we found that the total angular momentum of an object breaks down into two terms – one that accounts for the motion of the object through space relative to some reference point (the "orbital" part), and one that accounts for the rotation of the object around its center of mass (the angular momentum "intrinsic" to the object). The problem is that in the case of classical mechanics, even the rotating portion of the angular momentum is at its core orbital, as the particles that make up the rigid object are all "orbiting" the object's center of mass as the object spins, but this is not the case here – the electron has no extension in space, no moment of inertia – *it isn't spinning*!

Despite the conceptual pitfalls of doing so, we nevertheless call this property of the electron (and indeed every elementary particle) *spin*. While it does not have the attributes we normally assume must be present for angular momentum and magnetic dipole moment to exist, electrons do possess these properties. These are distilled down to their very essence, and are fundamentally intrinsic to the particle. An alternative name for spin that is not quite so fraught with faulty intuition is *intrinsic angular momentum*.

Quantization of Spin

Intrinsic angular momentum is perhaps the most purely quantum-mechanical phenomenon we have seen. It really can't be measured in the macroscopic realm like the other quantities we have discussed. When we go to the "classical limit", kinetic energy, momentum, and even orbital angular momentum converge to values we measure easily. But this cannot be done with spin. We can see the *effects* of spin (magnetic materials exhibit magnetic fields at least in part due to spins of a large fraction of particles aligning), but this is really a property we have to look carefully in the microscopic realm to see directly, such as with a SG apparatus.

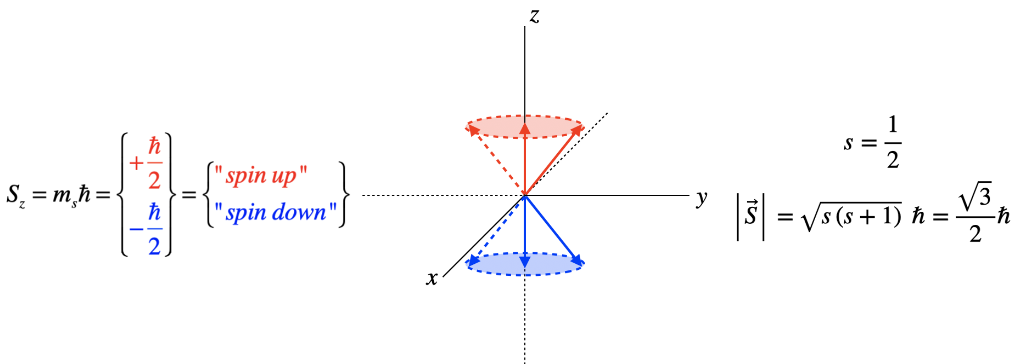
Spin is a measure of angular momentum, and we already know that angular momentum is quantized. Our SG experiment where we see two beams emerge from a single beam of electrons confirms this. What this experiment also tells us is that the spin angular momentum of electrons comes in only two varieties, with neither of them being zero. We find that no matter what we do to the electrons, we cannot get them to ever split differently in a SG device than into two beams – electrons can only ever have two possible values of the z-component of their intrinsic angular momentum. We generally refer to these two states as *spin up* and *spin down*.

We can do further SG experiments to look into the magnitudes of angular momentum for these two states. For the $l=1$ orbital case, we found from the deflection of the $m_l=\pm 1$ cases that those z components have a magnitude of $\hbar$. If we compare this to the results for individual electrons, we find the deflection of individual electrons to be half as great, which means that the z component of angular momentum for an electron is $\pm\frac{\hbar}{2}$. The electron is therefore said to have an intrinsic angular momentum of *spin-½*.

We can now see why we put a subscript "l" on the quantum number "m_l" – the orbital angular momentum is separate from the spin angular momentum. We therefore define "s" to be the quantum number for spin analogous to l for the orbital case, and "m_s" to be the quantum number for the z-component of spin, analogous to the orbital version m_l. As with the orbital case, the possible values for m_s run from $-s$ to $+s$, and change by integer amounts. So for the electron, $s=\frac{1}{2}$, and $m_s=\pm\frac{1}{2}$.

As in the case of orbital angular momentum, the magnitude of the total spin of the electron remains fixed (and the x and y components remain indeterminate), and the resulting picture is similar to Figure 7.4.2:

Figure 8.2.1 – States of Spin-½



Gyromagnetic Ratio

We have come to this description of the intrinsic angular momentum of an electron through the discovery of it having a magnetic moment, so we should revisit the calculation that relates magnetic moment to angular momentum. In our previous calculation that related $\vec{\mu}$ to $\vec{L}$, the radii of the circular orbit divided-out, resulting in Equation 8.1.3. But there was still a remnant of the orbital nature of this calculation – the factor of 2 in the denominator. Given that there is no orbital element to intrinsic spin, it's reasonable to wonder if this factor remains for this case when we replace $\vec{L}$ with $\vec{S}$. The answer is that it does not, and the way this is typically quantified for spin is with a dimensionless constant g, called the *gyromagnetic ratio*:

$$\vec{\mu}=\pm\frac{ge}{2m}\vec{S} \tag{8.2.1}$$

[Note that we have not defined the sign of the charge and left m in the denominator without a subscript, so that this can be applied to any particle, not just electrons.] For electrons, of course the sign is negative, and we find that the value of g is very nearly equal to 2.

Revisiting the Degeneracy of Hydrogen Atom States

Another consequence of electrons having two varieties of spin (up or down) is that the number of degrees of freedom for their wave functions is doubled. We have described an electron in a hydrogen atom with quantum numbers n, l, m_l, but now we see that it doesn't tell us everything – it can either be spin up or spin down, without changing n, l, or m_l. The spin up/down status of the electron does not play a role in the energy levels of a hydrogen atom, so there are now twice as many states as we thought for the same energy level, bringing the degeneracy of the hydrogen atom with principle quantum number n up to $2n^2$.

Fitting Spin into the "Big Picture" of Quantum States

The following question now naturally arises about spin: How does the spin quantum number fit into our previous discussion of the number of quantum numbers matching the degrees of freedom? We already have three quantum numbers for the three spatial dimensions, so where did this additional quantum number come from?

The answer must be that particles have an additional "internal" degree of freedom. Up to now, we have been declaring that the wave function "contains all of the information about the particle," and that we can extract it using operators for physical quantities. But the wave function's information is based on *external* physics – position of the particle in space, potential energy as a function of position, etc. We can't "derive" the spin property, or come up with an operator that acts on our usual wave function that extracts the information about spin. We need to fundamentally change the wave function itself. We do this by taking-on another component to the wave function, called a *spinor*. This "tacking-on" can be expressed mathematically as follows:

$$\Psi(x,t) \rightarrow \Psi(x,t) \otimes \chi_s \tag{8.2.2}$$

The spinor part of the quantum state "lives" in an entirely different mathematical space than the rest of the wave function (this is emphasized by the "⊗" symbol). Operations like the derivative in the momentum operator have no effect on this part of the wave function.

In the case of an electron (which has only two eigenvalues of spin, "up" and "down"), this part of the state is completely describable with two complex numbers, which are essentially the probability amplitudes of measuring each of these two states. As was the case for the spatial part of the wave function, a general spin state is a linear combination of the eigenstates, and since there are only two of these for the electron, we have:

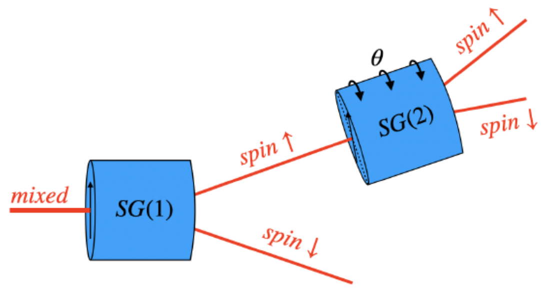
$$\chi_s = \alpha\,\chi_\uparrow + \beta\,\chi_\downarrow\,, \qquad |\alpha|^2 + |\beta|^2 = 1 \tag{8.2.3}$$

The complex-valued numbers α and β are just the spin equivalents of the "C_n's" we used to expand the spatial wave function in terms of its eigenstates. Naturally the value $|\alpha|^2$ is the probability that a SG device will measure the electron's spin to be "up", and $|\beta|^2$ is the probability that a SG device will measure the electron's spin to be "down".

Successive Spin Measurements

In the previous section, we noted that using a SG device as a "spin filter" bears similarities to our discussion of passing light through polaroids. In the latter case, we derived a relation called Malus's law, which gives the intensity of light that emerges from a polaroid after incoming polarized light passes through it, in terms of the angle formed between the incoming light polarization and the orientation of the polaroid. We know that particles that are eigenstates of spin "up" through one SG device are not in an eigenstate of a second SG device that does not have its magnetic field pointing in the same direction, so there must be a relation that is equivalent to Malus's law for spin-½. Equivalent, yes, but not identical, as it turns out...

Figure 8.2.2 – Successive SG Filters at a Relative Angle



If the orientation of the second SG device's magnetic field differs from that of the first's by an angle θ, and the spin "up" beam that comes from the first SG device is passed through the second, then what started as an eigenstate of spin becomes a mixed state (purely spin "up" becomes a linear combination of spin "up" and spin "down"), with the mixing coefficients (choosing real-values only) being:

$$\chi_s\,[\text{after } SG(2)] = \alpha\,\chi_\uparrow + \beta\,\chi_\downarrow = \cos\left(\frac{\theta}{2}\right)\chi_\uparrow + \sin\left(\frac{\theta}{2}\right)\chi_\downarrow \tag{8.2.4}$$

The difference with Malus's law becomes clear when we create a "polaroid" out of an SG device by simply absorbing all beams that come out spin "down", so that it only permits a fraction of the particles to pass through, as a polaroid does. When you flip $SG(2)$ over, the beam that was previously measured as spin "up" is now measured as spin "down", which means that as a "polaroid" turning it 180 degrees blocks all of the particles. In the case of light through a polaroid, a 180 degree rotation allows all of the light to pass.

The probability of a single spin-½ particle getting through $SG(2)$ as spin "up" after passing through $SG(1)$ as spin "up" when the fields of $SG(1)$ and $SG(2)$ are rotated relative to each other by an angle θ is:

$$P(\uparrow) = |\alpha|^2 = \cos^2\left(\frac{\theta}{2}\right) \tag{8.2.5}$$

The half-angle argument of the cosine function differs from the full-angle result for Malus's law.