4.5: Incompatible Measurements

Plane Waves

We have spent a lot of time talking about particles associated plane waves, because they are easy to conceive – they have a single, specific wavelength, and therefore a definite momentum. These are particles free from any forces, moving at a constant, well-defined speed. Easy, right? Well, let's take a look at this wave function in terms of locating the position of the particle. Choosing a cosine function to describe this wave function moving in the $+x$-direction, we have:

$$\psi\left(x,t\right)=A\cos\left(\frac{2\pi}{\lambda}x-2\pi ft\right)\tag{4.5.1}$$

Let's simplify this discussion by looking at the wave only at time $t=0$:

$$\psi\left(x\right)=A\cos\left(\frac{2\pi}{\lambda}x\right)\tag{4.5.2}$$

If we form a probability density from this probability amplitude, we get:

$$\mathcal{P}\left(x\right)=\left|\psi\left(x\right)\right|^2=A^2\cos^2\left(\frac{2\pi}{\lambda}x\right)\tag{4.5.3}$$

Okay, let's normalize our probability density (i.e. find the value of A):

$$1=\int\limits_{\text{all }x}\mathcal{P}\left(x\right)dx=\int\limits_{-\infty}^{+\infty}A^2\cos^2\left(\frac{2\pi}{\lambda}x\right)dx\tag{4.5.4}$$

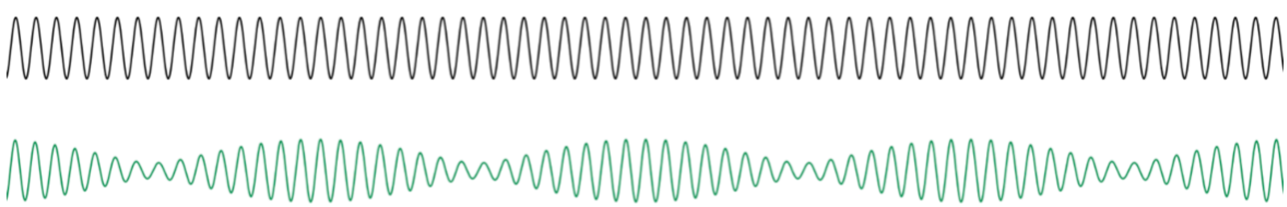
Uh-oh. We have a problem. This integral blows up, making $A=0$, and $\mathcal{P}\equiv0$. This makes no sense, what is going on here?

Actually, it does make sense – with a wave function that has the same amplitude along the entire x-axis, the particle must be equally-probable to be found anywhere, so if we look for it in any finite interval, the measure of that interval is zero compared with the measure of the remaining infinite space where it can be found. A simple way to express this is that *we have no idea where the particle is*. This complete lack of knowledge about the particle's position was the small price we had to pay for knowing the particle's momentum precisely. We will see here that this price cuts both ways.

Localizing Free Particles

We have a hint that the precise knowledge of a particle's momentum goes together with a complete lack of knowledge of its location, so let's see if reducing what we know about momentum (or equivalently, wavelength) has the effect of improving our ability to discern the particle's position. We'll start simple: Let's see what happens when we superpose two plane waves with different wavelengths.

Figure 4.5.1 – Superposition of Two Plane Waves



The top graph depicts a plane wave. The one below it is what happens when another plane wave with a slightly shorter wavelength (the wave number $k=\frac{2\pi}{\lambda}$ is larger) is superposed with it. The contribution of this second wave to the superposition is slightly less as well (i.e. its amplitude is smaller than that of the original wave):

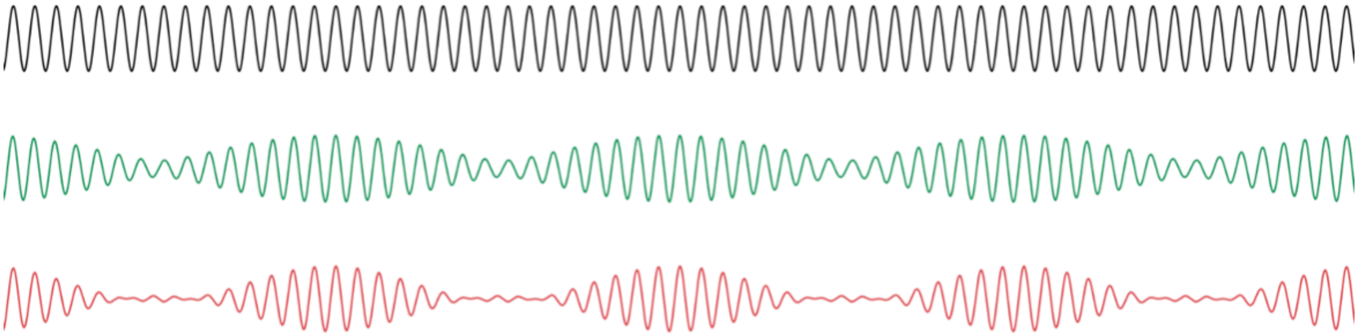
$$\psi\left(x\right)=A\cos kx$$

$$\psi\left(x\right)=A\cos kx+0.6A\cos(k+4\delta k)x$$

The first thing that jumps out is the way that the amplitude (not the displacement!) varies when you evaluate it at different places on the x-axis. In probability terms, this means that the probability of finding the particle in a region near the maximum bulges ("antinodes") is quite high compared to finding it near the narrower regions ("nodes"). While this wave is still infinitely-long, and our knowledge of the position of the particle is still zero, in a *relative* sense, we have a better sense of where the particle is than when we were dealing with a single harmonic wave function.

If we can do this much just by using two wavelengths, perhaps we can improve things even more by adding some additional harmonic waves. If we choose a third plane wave appropriately, we find that the bulges become more defined:

Figure 4.5.2 – Superposition of Three Plane Waves

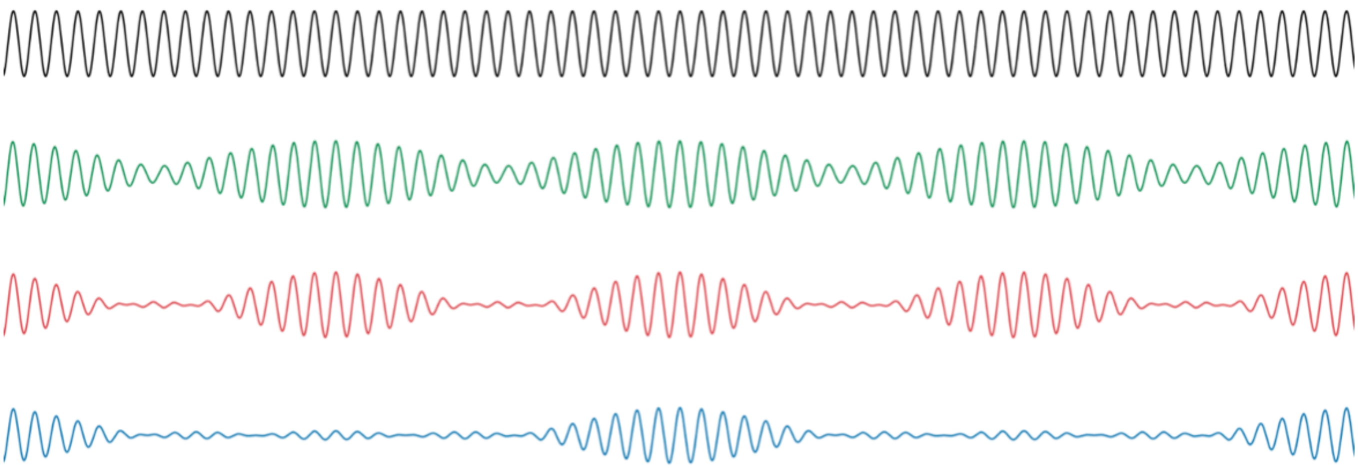


All we did here was add a third plane wave with a wave number *below* that of the original wave by the same amount as the second wave's wave number was larger:

$$\psi\left(x\right)=A\cos kx+0.6A\cos(k+4\delta k)x+0.6A\cos(k-4\delta k)x$$

Let's see what happens if we "fill in" a couple of the wave number gaps. That is, to get the above result, we added & subtracted $4\delta k$ to the wavenumber of the original wave for the added waves. Let's add & subtract $2\delta k$, and since it is closer to the original frequency, we'll weight its amplitude more as well ($0.8A$). The result is:

Figure 4.5.3 – Superposition of Five Plane Waves



The wave function for this fourth case is:

$$\psi\left(x\right)=A\cos kx+0.8A\cos(k+2\delta k)x+0.8A\cos(k-2\delta k)x+0.6A\cos(k+4\delta k)x+0.6A\cos(k-4\delta k)x$$

It should be clear what happens if we continue this program indefinitely – we are left with a single *wave packet*, as all the other bulges get pushed out to infinity. This localizes our free particle, giving it a finite probability of being found within a given region. This localization improved as we added the number of possible wavelengths (momenta) that could be measured. So we can improve our knowledge of the particle's position at the cost of knowing about its momentum, and vice-versa.

Spectral Content and Fourier (Again!)

Adding together lots of harmonic functions is exactly what we did when we did Fourier decomposition of periodic functions, but this is slightly different. Here we are creating a *non-periodic* function by adding together harmonic functions. The wave packet (in the limit of adding every wave number) is completely isolated – its bulge brethren are long gone – so it doesn't repeat and is therefore not periodic.

Notice that the bulges separated when we inserted plane waves with wave numbers *between* the ones we already had in place. In order to get the bulges separated to infinity we need to fill in *all* of those in-between wave numbers. Of course, there is a whole continuum of these available, so rather than use a sum of harmonics as we did with Fourier series for periodic waves, we need to use an integral to capture all of the harmonics. So to replace the Fourier series for periodic functions, we now have what is called the *Fourier transform* for non-periodic functions. We can see how to make the extension from the Fourier series to the Fourier transform by looking again at the Fourier series (Equation 1.7.5). The series is over the integer n, which ranges from $-\infty$ to $+\infty$, and this integer increments the wave number $k_n = \frac{2\pi n}{\lambda}$. If we now add over the continuum of wave numbers, this sum becomes an integral. The coefficients that are the amplitudes of the harmonic waves (the "recipe" of the wave being decomposed) depend upon the value of n in each case (or equivalently, the wave number), so the same is true in the continuous case. Even the sine and cosine are represented in the transform, though in a way that is somewhat different than in the series. The end result is:

$$\psi(x) = \int_{-\infty}^{+\infty} \left[A(k)\cos kx + iA(k)\sin kx\right] dk \tag{4.5.5}$$

The function $A(k)$ is the "recipe" for the transform, and yes it is the same function multiplying both the cosine and sine functions. Also, yes, the imaginary 'i' makes an appearance here. This gives us a more compact way to express the transform, using the Euler identity:

$$\psi(x) = \int_{-\infty}^{+\infty} A(k)\, e^{ikx} dk \tag{4.5.6}$$

In the case of the Fourier series, we had a means for computing the a_n and b_n coefficients, and the same is true for $A(k)$:

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi(x)\, e^{-ikx} dx \tag{4.5.7}$$

This is usually referred to as the *inverse Fourier transform*.

In Equation 4.4.10 we introduced the probability amplitude for measuring a particle's momentum (sometimes referred to as the wave function in "momentum space"). Given that the wave number is proportional to the momentum ($p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k$), the "recipe" that gives the amount of each plane wave of a given wave number should be very closely related to the probability amplitude for momentum. Well, it turns out that these are simply proportional:

$$\phi(k) = \sqrt{2\pi}\, A(k) \tag{4.5.8}$$

This results in a nicely-symmetric relationship between the position and momentum probability densities:

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \psi(x)\, e^{-ikx} dx \tag{4.5.9}$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \phi(k)\, e^{+ikx} dk \tag{4.5.10}$$

We will not show it here, but it is not difficult to prove that with this relationship, if either $\psi(x)$ or $\phi(k)$ is normalized, then so is its counterpart.

Digression: Dirac Brackets Again

There is a nice way to express all of this in terms of bras and kets. Thinking of them once again as vectors, we can put a general vector into any "basis" (set of unit vectors), by dotting the vector with those unit vectors to get its components, then multiplying by the unit vectors. For example:

$$\vec v=\hat i v_x+\hat j v_y+\hat k v_z=\hat i\left(\vec v\cdot\hat i\right)+\hat j\left(\vec v\cdot\hat j\right)+\hat k\left(\vec v\cdot\hat k\right)$$

In the bracket notation, there are an infinite number of these unit vectors, so we have to integrate to add them all up. Using the position "unit vectors" $|x\rangle$ and a state vector $|\psi\rangle$, we express the same vector decomposition as above this way:

$$|\psi\rangle=\int\limits_{\text{all x}}|x\rangle\langle x|\psi\rangle\,dx$$

Now we can get the momentum wave function, which are components of the state vector using the momentum "unit vectors":

$$\phi(p)=\langle p|\psi\rangle=\int\limits_{\text{all x}}\langle p|x\rangle\langle x|\psi\rangle\,dx$$

And now we get back the same relation as above if we have the dot product of the momentum and position unit vectors equal to:

$$\langle p|x\rangle=\frac{1}{\sqrt{2\pi}}e^{ikx}$$

The Heisenberg Uncertainty Principle

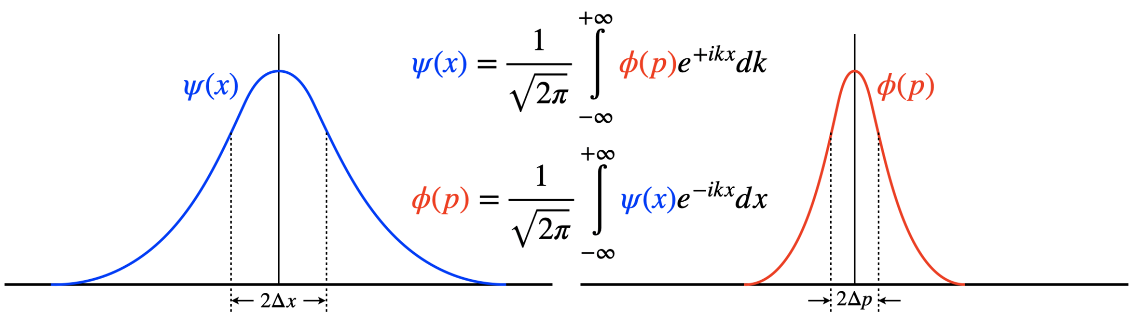
Now that it's quite clear that there is an inverse relationship between the uncertainty of measurements of position and momentum, we can state it formally as was first done by Werner Heisenberg. The specific predicament of a particular particle will define what the uncertainty will be in measuring either the position or the momentum, and we can change the conditions of our experiment to improve the certainty of our measurement of either of these quantities, but as we improve one, we necessarily worsen the other. According to Heisenberg's math, we get the following inequality expressing the principle that bears his name:

$$\Delta x\Delta p\geq\frac{\hbar}{2}\tag{4.5.11}$$

No matter how we devise our experiment to measure x and p, when we compute their uncertainties statistically, we will find that the product of these uncertainties will always come out to a number no less than $\frac{\hbar}{2}$.

This principle is really well demonstrated through the Fourier transform. If we consider a localized (in position) wave packet (by "localized," we mean that the probability amplitude for measuring various positions drops off very rapidly far away from the center), and then Fourier-transform this function, we get the wave function of the same particle expressed in terms of its spectral content (probability amplitude for measuring various momenta).

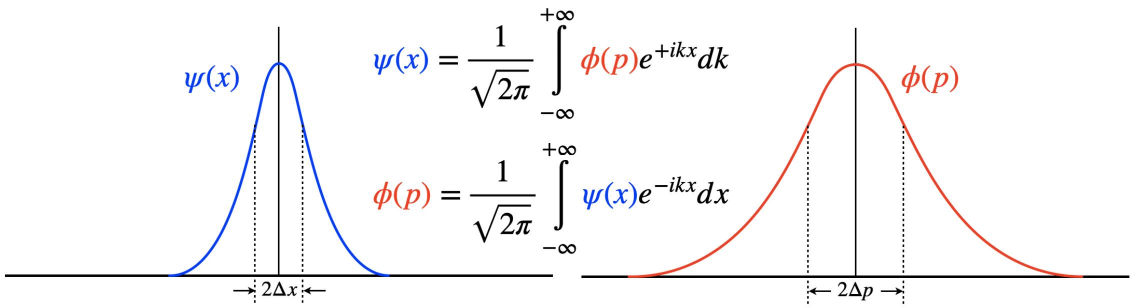
Figure 4.5.4 – Particle Wave Function Expressed in Terms of Position and Momentum



The uncertainties in position and momentum can be calculated in the usual way from the probability densities that come from these wave functions, and these are expressed in the diagram above.

Now suppose we change the physical conditions that brought about this quantum state. For example, suppose we change the way we take measurements so that we better-confine our knowledge of the position of the particle. This will serve to "tighten" its wave packet. When we take the Fourier transform of this new position wave packet, the momentum wave packet broadens.

Figure 4.5.5 – Same Particle with Position Measured More Precisely



Heisenberg's principle states that even if one provides the ideal conditions for the particle, this inverse relationship between the position and momentum uncertainties results in a limit to the minimum value that the product of these uncertainties can attain.

One last comment here: Notice that above we used the phrase "change the physical conditions" and "measure differently" interchangeably. This is a very important aspect of quantum theory. The probabilities we measure are dependent upon physical conditions, and the process of making measurements *necessarily affects these conditions*. We have encountered this once before, when we discussed watching electrons pass through a double slit, back in Section 3.5.