8.3: Fermions and Bosons

Not All Spins Are "One-Half"

The only particle we have discussed with regard to spin is the electron, but we can talk about the intrinsic spin of any particle. Some particles are spinless ($s=0$), others have half-integer spins ($s=1/2,\ 3/2$), and still others have non-zero integer spins ($s=1,\ 2$). Note that this latter group (which includes the photon, $s=1$) has an integer *intrinsic* angular momentum, and this should not be confused with the integer-valued *orbital* angular momentum we have seen. That is, while the only possible orbital quantum numbers are integers, spin quantum numbers can be either integer or half-integer.

Two Categories of Particles

We will see shortly that particles with half-integer spin behave quite differently from those with integer spin, and this different behavior results in a specific "division of labor" in the universe. The exploration of the nature of forces in quantum theory has resulted in a model that describes forces as exchanges of particles. For example, we know that accelerated electrons emit light (photons). These accelerations result from electromagnetic forces on the electrons. This "exchange particle" model ties these together neatly, by explaining that the mechanism of force itself comes about due to photon exchanges between charged particles. The mathematics of this theory is much too involved to go into in this class, but the concept interests us for the following reason: It turns out that all the particles that mediate force (e.g. the photon for the EM force, and the graviton for the gravity force) have integer spin, while all the particles that comprise matter (which we think of as the stuff that feels the force) have half-integer spin (electrons, protons, and neutrons are all spin-½).

This stark division of the particle types, the tasks they perform, and as we will see, the properties they exhibit, inspires us to give these two categories specific names (named after physicists who studied them in the earliest years of quantum theory). Half-integer spin particles are known generically as *fermions* (named for Enrico Fermi), and integer spin particles are called *bosons* (named for Satyendra Nath Bose).

Identical Particles

When we first discussed the notion of a "quantum state" of a particle, we said that it is a collection of all the information available about that particle, and that this information can be organized into a wave function (with a spinor attached). Suppose now that we consider two particles in tandem. These now constitute a new system, for which we can define a new quantum state. The puzzle we need to solve is how to express this new state in terms of the states of the individual particles. We are not unfamiliar with this idea, thanks to our experience with statistical/thermal physics in 9HB. In kinetic theory, we describe the motions of the individual particles, but then as we step back and look at the collection as a whole, we define a "thermodynamic state". In that case, we bridge the gap between the collective state and individual states with Maxwell-Boltzmann statistics, which relates things like the distribution of energy of individual particles to the temperature of the collection as a while.

Alert

Throughout our discussion of multi-particle quantum states, we will assume (much like we do with ideal gases) that the particles involved do not interact with each other (such as through the coulomb potential). The particles can be individually affected by some external potential, but the effects we will be discussing come only from the fact that they are combined into a single system.

In our quantum mechanics version of linking multiple particles to individual ones, we have a few new problems. For one, we ultimately can only deal with probabilities. This was true for the collection in the Maxwell-Boltzmann case as well, but with quantum theory this probabilistic nature goes all the way down to individual particles. In thermodynamics, we assume that if we could track all of the particles, we can precisely predict the evolution of the system, and we only use probability because we cannot practically accomplish this particle-tracking. In quantum mechanics, we can't track the individual particles *even in principle*.

There is an even more important issue that comes about due to out inability to track particles in quantum theory. We assert that *all* of the information about a particle's quantum state is contained in its wave function. Wave functions track quantities like energy and angular momentum with their quantum numbers, but there is no quantum number that acts as a particle identifier. If two particles of the same type (say electrons) are commingling in a confined region (say in an infinite square well), their wave functions are overlapping, and when we measure the position or energy of one of the particles, we cannot tell which of the two particles we have looked at. In classical physics the particles don't have labels either, but the particles are distinguishable because we can just watch the particles closely and know which one we are measuring the energy of. This property of like particles being indistinguishable in quantum mechanics will turn out to have profound consequences.

So how do we treat a quantum state of multiple particles mathematically? Let's start by considering a simple case – two *distinguishable* particles, like an electron and a proton (remember, we are still assuming that these particles are not interacting, so perhaps a neutron is a better choice for the other particle). Each of these particles has their own wave function, but we want to form a single wave function, which stores all the information about this "system". The answer lies in how we incorporated spinors into our states. In that case, we asserted that the spinor "lives" in another space, and the intrinsic space of the spin is subject to operators that don't act on the "extrinsic" wave function. This combination of entities is known mathematically as a *direct product space*, and can also be applied to separate particles:

$$\Psi\left(\text{two-particle system}\right)=\Psi\left(\text{particle }A\right)\otimes\Psi\left(\text{particle }B\right)\tag{8.3.1}$$

The real key here is the probabilities. The probability density comes out to be just a product of probability densities:

$$P\left(\text{two-particle system}\right)=\left|\Psi\left(\text{two-particle system}\right)\right|^2=\left|\Psi\left(\text{particle }A\right)\right|^2\otimes\left|\Psi\left(\text{particle }B\right)\right|^2\tag{8.3.2}$$

We can now ask for the probability of particle A being found between x_1 and x_1+dx_1 *and at the same time* finding particle B between x_2 and x_2+dx_2 :

$$P\left(A\text{ near }x_1\text{ and }B\text{ near }x_2\right)=\left|\Psi_A\left(x_1\right)\right|^2dx_1\left|\Psi_B\left(x_2\right)\right|^2dx_2\tag{8.3.3}$$

Here we have dropped the direct product notation, because the variables x_1 and x_2 keep things straight. We want to express this as a single probability density of a system, and clearly there are now two inputs (x_1 and x_2), so this looks like:

$$\left|\Psi_{system}\left(x_1,x_2\right)\right|^2dx_1dx_2=\left|\Psi_A\left(x_1\right)\right|^2dx_1\left|\Psi_B\left(x_2\right)\right|^2dx_2\tag{8.3.4}$$

Okay, so this is all simple enough. It makes sense, because the two particles are not interacting, so the two events (measuring one particle at one position and the other at another) are independent, and the probability of two independent events occurring is just the product of the probabilities of each one occurring.

But now if the two particles in question are identical, we can't say "particle A near x_1 and particle B near x_2" anymore. We can only say, "one of the particles near x_1 and the other particle near x_2". The independence necessary for the simple multiplication of probabilities is lost, because in the first case, finding A near x_1 and B near x_2 is different from finding B near x_1 and A near x_2, but when the particles are indistinguishable, these events are the same.

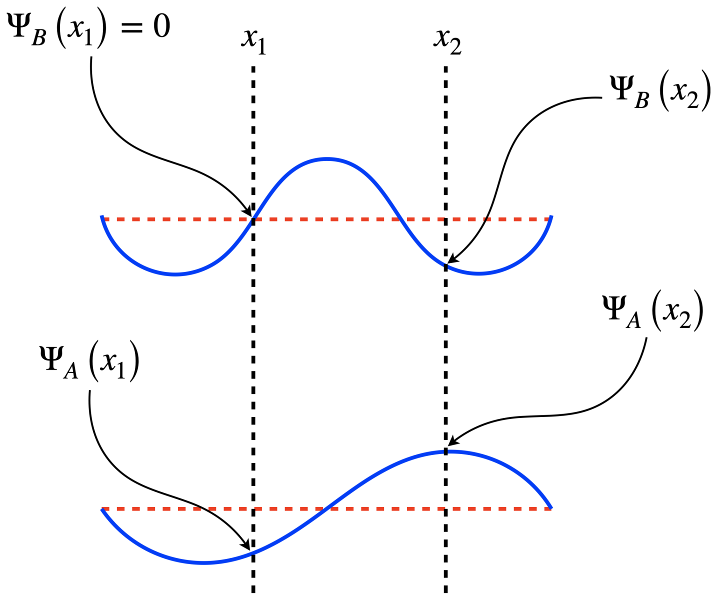
Exchange Symmetry

With the particles indistinguishable, the probabilities measured by the quantum state should not change if we swap the positions of the two particles. That is, if we swap the variables x_1 and x_2 in the two-particle wave function, the probability density should not change:

$$\left|\Psi_{system}\left(x_1,x_2\right)\right|^2dx_1dx_2=\left|\Psi_{system}\left(x_2,x_1\right)\right|^2dx_1dx_2\tag{8.3.5}$$

It should be clear that this is not the case for Equation 8.3.4, but in case it is not, consider the diagram below, which depicts two partial wave functions for a one-dimensional two-particle system, and two positions along the x-axis. You can think of particle A as being in the $n=2$ eigenstate of an infinite square well, and particle B being in the $n=3$ eigenstate. We don't actually know *which* particle is in each of these states, of course, and that's the point.

Figure 8.3.1 – Partial Wave Functions of a Two Particle State



If we plug in the values $\Psi_A(x_1)$ and $\Psi_B(x_2)$ into Equation 8.3.4, we get some non-zero probability density. But now suppose we swap the positions of the two identical particles – they are indistinguishable, so it should not change the probability density. But $\Psi_B(x_1)$ is *zero*, which means that the probabilities don't match when the particles switch positions.

So how do we fix the construction of the two-particle wave function from the partial wave functions? We do this by not linking the each specific particle with a particular position. In the original solution with separation of variables, we can acknowledge that although there are two quantum numbers (one for each degree of freedom, the degrees of freedom in this case coming from two particles being present), we have to leave it undetermined which one goes with each wave function. It's possible to show that there are two ways to construct the two-particle wave function from the partial wave functions, that both satisfies the stationary-state Schrödinger equation and yields the same probability when the positions are swapped. They are these two states:

$$\Psi_+(x_1, x_2) = \frac{1}{\sqrt{2}}[\Psi_A(x_1)\Psi_B(x_2) + \Psi_A(x_2)\Psi_B(x_1)] \tag{8.3.6}$$

$$\Psi_-(x_1, x_2) = \frac{1}{\sqrt{2}}[\Psi_A(x_1)\psi_B(x_2) - \Psi_A(x_2)\Psi_B(x_1)] \tag{8.3.7}$$

The subscripts "$+$" and "$-$" stand for "symmetric" and "antisymmetric," respectively, for obvious reasons. These two cases provide the two possibilities for *exchange symmetry* for quantum particles. The constant in front of these is there to normalize the two-particle wave function assuming that the partial wave functions are already normalized.

Pauli Exclusion Principle

So now we have asserted that there are two types of particles in the universe (fermions with half-integer spin, and bosons with integer spin), and that when it comes to two-particle states, the individual wave functions can be combined in either symmetric or antisymmetric fashion to make a system wave function. There is no reason to believe that these two elements of quantum theory should be linked in any way, and yet remarkably they are! Proving this is far from straightforward (so we will not do it in this class), but the link is this: If our system of two identical particles happens to consist of bosons, then their wave functions observe symmetric exchange symmetry, and if the two identical particles are fermions, then their wave functions observe antisymmetric exchange symmetry.

The most striking consequence of this fact becomes clear when we consider two identical fermions with the same quantum numbers. For example, suppose we have two electrons in the ground state of a one-dimensional infinite well, both with spin up. These particles have identical wave functions, and when we combine them with an antisymmetric exchange, we just get *zero* for the system's wave function. A zero wave function means zero probability, so this means that we can never witness such a thing. When we put two electrons into this box, and we measure the energies and spins of the particles, we will never find that the quantum numbers are all the same for both. This *exclusion principle* is attributed to Wolfgang Pauli, and has many far-reaching consequences.