# 6.6: The Bohr Model of the Hydrogen Atom

## The Classical Hydrogen Atom

Every quantum mechanical model we have discussed so far has been built from a classical system that we are familiar with. A square well is a pair of walls between which a ball is bouncing, and a harmonic oscillator is a mass on a spring. Here we will once again start with a classical model originated by Ernest Rutherford and later modified for quantum physics by Niels Bohr. It should be noted before continuing that Bohr did this work over a decade before Schrödinger introduced the equation that bears his name, so appropriate treatment of "matter waves" was not yet known.

The classical model of Rutherford perfectly parallels what we see in our solar system. Heavy, electrically positively-charged protons occupy a small nucleus, and they attract the much lighter negatively-charged electrons, and the combination of these form an atom. Just as gravity doesn't cause the solar system to collapse, the electrical attraction between the protons do not cause atoms to collapse – the tangential orbital motion prevents it.

The orbital model of the atom is even close to that of the solar system mathematically. It had been known for over a century that the electrical force between point-like charges obeys an inverse-square law, just like that of Newton's law of universal gravitation (see Equation 7.1.3 in the 9HA textbook):

$$\overrightarrow{F}_{electric} = \frac{kq_1q_2}{r^2}\,\hat{r} \tag{6.6.1}$$

The $q$'s are electric charge, which can be either positive or negative. When the two charges have opposite signs (as in the case of protons and electrons), the direction is $-\hat{r}$, which means the force is attractive. This is known as *Coulomb's Law*, and the SI units of electric charge ($C$) bear Coulomb's name. While the constant $k$ that plays the role of $G$ from gravitation is pretty common to see, this is the one and only time we will use it, as we don't want to confuse it with wave number. Instead, we will exchange it for another constant that is even more ubiquitous in the subject of electricity. This advantage of using this particular constant will have to remain a mystery for now (you'll see it in Physics 9HD!):

$$k \equiv \frac{1}{4\pi\epsilon_o} \quad \Rightarrow \quad \overrightarrow{F}_{electric} = \frac{q_1q_2}{4\pi\epsilon_o r^2}\,\hat{r}\;, \qquad \epsilon_o = 8.85\times10^{-12}\,\frac{C^2}{Nm^2} \tag{6.6.2}$$

Of course, if we are going to somehow link this to quantum mechanics, we will be better off with an energy approach, and as in the case of gravitation (Equation 7.3.6, Physics 9HA textbook), this force can be expressed as a potential energy that is only a function of the particle separation:

$$V(r) = \frac{q_1q_2}{4\pi\epsilon_o r} \tag{6.6.3}$$

Note that this potential does not have an overall minus sign as the gravitational potential does, because the charges carry signs with them. When they are opposite (giving an overall negative value), the force is attractive, as it is in the case of gravity where both masses are always positive.

This force is a central force, so it is conservative, and we can apply conservation of mechanical energy to this system. Before we do, there is one more simplification to make. It turns out that while the mass of the proton and electron are far apart, they have precisely the same magnitude of charge, usually referred to as "$e$". Rutherford's nucleus allowed for many protons, and this integer is typically represented with a "$Z$". We are interested in the effect this nucleus has on a single electron (at least as a start), so the atomic potential for a single electron is written as:

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_o r} \tag{6.6.4}$$

Now at last we can express the total energy of the electron as it orbits the nucleus:

$$E_{tot} = \frac{1}{2}mv^2 - \frac{Ze^2}{4\pi\epsilon_o r} \tag{6.6.5}$$

As with the case of gravitation, in the absence of outside forces, the closer the electron gets to the nucleus the faster it moves, as its potential energy goes down (it becomes more negative at smaller values of $r$).

Lastly, it should be noted that we don't have to worry too much about treating the nucleus as a fixed point. Usually we solve such two-body problems using something called "reduced mass" (which we first encounters in Physics 9HA in Equation 8.2.12), but in this case the proton is nearly 2000 times as massive as the electron, so our approximation of the proton being a fixed central source of the electrical force is a good one.

## Two Puzzles

As nice and tidy as this classical picture of the atom is, it runs into two major flaws when the quantum theory is taken into account. It was known since the time of Maxwell that light (and all EM radiation) is produced when electric charges accelerate. Rutherford also showed that atoms had a localized, hard nucleus, which meant that atoms consisted of negatively-charged electrons bound by the electric force to a positively-charged nucleus, like satellites are bound in their orbits by gravitational forces. But this posed a conundrum – electric charges in orbits are accelerating centripetally, so they should constantly be radiating, and since light carries away energy, the electrons should spiral down into the nucleus, which it clearly does not.

The second puzzle is characteristically quantum-mechanical: There is no reason to believe that any light frequencies released by atoms that lose energy should be preferred over any others. But it was known for some time that viewing light emitted by various elements through a diffraction grating reveals distinct sets of *spectral lines* – lines that are separated according to frequency.

With what little quantum theory we have already learned, we have insight into both of these phenomena, since an electron is clearly in a bound state due to the electrical potential. The first puzzle is "solved" by remembering that the ground state of a particle in a bound state can never be at the bottom of the well. It's spiral down forever until it has no energy because there is no zero-energy eigenstate available. The second puzzle is just another example of what we have seen about bound states generally being quantized.

But as noted above, Bohr did not have Schrödinger's equation, and he set out to stitch together the ideas of Rutherford's classical picture of the orbiting electron atom with the new paradigm of matter waves exhibiting de Broglie wavelengths.

## Bohr's Model

Bohr reasoned that a particle whose location is extended in space in the form of a wave and is orbiting a central point would have to interfere with itself when it gets all the way around. For the particle to remain in such an orbit, it would have to (as we call it now) "match boundary conditions" – its wave function would have to be in phase with itself when a full orbit is completed. He then had to address the issue of orbits of inverse-square forces being elliptical. While he was not able to motivate *why*, he postulated that the orbits can only be circular. While this assumption makes little sense, its results are striking enough to let it slide, and maybe take it up later.

The in-phase and circular orbit requirements, along with de Broglie's formula for the wavelength of a matter wave, and the coulomb potential for two point charges is all that was required to complete this model. Let's see what this semi-classical approach predicts for the simplest of atoms – hydrogen...

The assumption that the electron follows a circular orbit requires that this force causes a centripetal acceleration, so calling the mass of the electron "$m_e$", we have:

$$m_e\frac{v^2}{r}=\frac{e^2}{4\pi\epsilon_o r^2} \quad\Rightarrow\quad p^2=\left(m_e v\right)^2=\frac{m_e e^2}{4\pi\epsilon_o r} \tag{6.6.6}$$

We put this in terms of the magnitude of the electron's momentum, so that we can use the deBroglie relation. Applying Bohr's assumption that an integer number of full wavelengths fit within the circular orbit (so that it meets itself in phase) gives:

$$\left.\begin{array}{ll} deBroglie: & p=\dfrac{h}{\lambda} \\ Bohr: & 2\pi r=n\lambda \end{array}\right\} \quad\Rightarrow\quad p^2=\left(\frac{nh}{2\pi r}\right)^2=\frac{n^2\hbar^2}{r^2} \tag{6.6.7}$$

Plugging this back into Equation 6.6.6 and solving for the radius of the orbit gives:

$$r_n=n^2\left(\frac{4\pi\epsilon_o\hbar^2}{m_e e^2}\right)\equiv n^2 a_o \tag{6.6.8}$$

The orbit radii are thus quantized! The quantity $a_o$, derived purely from physical constants, is called the *Bohr radius*, which is a useful unit of length measurement, even in the more enlightened quantum mechanical model that came later.

Of course, quantization of the orbital radii was not what we were after – we are interested in the energy levels of the hydrogen atom, which are somehow involved in providing energy to emitted photons, and we want to know why the hydrogen atom doesn't radiate away all its energy. To this end, we note that the assumption of a circular orbit results in a very simple relationship between the kinetic, potential, and total energy. Dividing Equation 6.6.6 by twice the mass of the electron gives us its kinetic energy, and comparing the result with Equation 6.6.4 (with $Z=1$ for the hydrogen atom) gives:

$$\frac{p^2}{2m}=\frac{r}{2}\left(\frac{e^2}{4\pi\epsilon_o r^2}\right) \quad \Rightarrow \quad KE=-\frac{1}{2}PE \quad \Rightarrow \quad E_{tot}=KE+PE=\frac{1}{2}PE=-\frac{e^2}{8\pi\epsilon_o r} \tag{6.6.9}$$

But the radii are quantized, so plugging this in gives quantized energy levels:

$$E_n=-\frac{e^2}{8\pi\epsilon_o r_n}=-\frac{1}{n^2}\left(\frac{m_e e^4}{2(4\pi\epsilon_o)^2\hbar^2}\right) \tag{6.6.10}$$

The energy spectrum is also quantized. The quantity in parentheses is a unit of energy known as a *Rydberg*, which $\approx 13.6eV$.

Bohr reasoned that the hydrogen atom doesn't radiate away all of its energy, because the lowest energy level (which we would now call the ground state) corresponds to exactly one wavelength fitting in the orbit, so $n=1$ is the lowest it can go.

## Emission/Absorption Spectrum

The problem of only seeing certain spectral lines is also solved in this model, if one insists that the atom can only exist in one of these quantized states, and therefore the only energy transitions it can make are between the allowed energy levels. A transition that lowers the energy level of the hydrogen atom from \(n_1) to \(n_2) frees up the amount of energy equal to the difference, which then goes into an emitted photon according to Planck's relation:

$$hf=\Delta E=E_{n_1}-E_{n_2}=-13.6eV\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) \tag{6.6.11}$$

This matched perfectly with experiment! So although there are many problems with this model, a more evolved version will need to agree with this energy spectrum. Note that this quantized energy change works both ways – when a hydrogen atom absorbs a photon, it must absorb an amount of energy that carries the atom from one of its quantized energy levels to another (higher) one.

## So Many Flaws!

The final result is so striking that to this very day we memorialize the result in textbooks like this one, but this euphoria only lasts so long, and eventually we have to address the obvious flaws of the model. The most obvious is probably the arbitrary assumption of circular orbits. If we allow for elliptical orbits as we see in gravitation, we once again get a continuum of energies possible, even with the requirement that the orbiting matter wave is in phase with itself. But there are other flaws as well, some of which come from disagreement with experiment, and others just don't agree with what we already know about quantum theory.

Orbital motions of planets and electrons share the property that their orbits remain in a plane. This, along with the circular orbits assumption, is why this model fits into the "One-Dimensional Models" chapter. But this causes a very large problem with the uncertainty principle. If we call the plane of orbit of the electron the $x-y$ plane, then this model confines the electron to a single, precise value of $z$. Its position remains undetermined in its orbital plane, but we exactly know its $z$-position. This means that we know *nothing* about its $z$-component of momentum, which really messes-up what we thought we know about its kinetic energy.

Another problem comes from experiments. There are many of these, but the most obvious comes from what happens when hydrogen atoms are sent through magnetic fields. A very basic result from electromagnetism is that charges that go in circles behave like magnets – they can be deflected by non-uniform magnetic fields as they pass through them (we will discuss this in a future chapter). Well, there is no way for a Bohr atom to *not* have this magnetic property, but we do see hydrogen atoms (most notably, all of them in their ground states) pass through magnetic fields without deflecting. Well, okay, they still do, but that cannot be explained from the orbital motion of the electron – again, this is coming attractions.