8.1: Measuring Angular Momentum

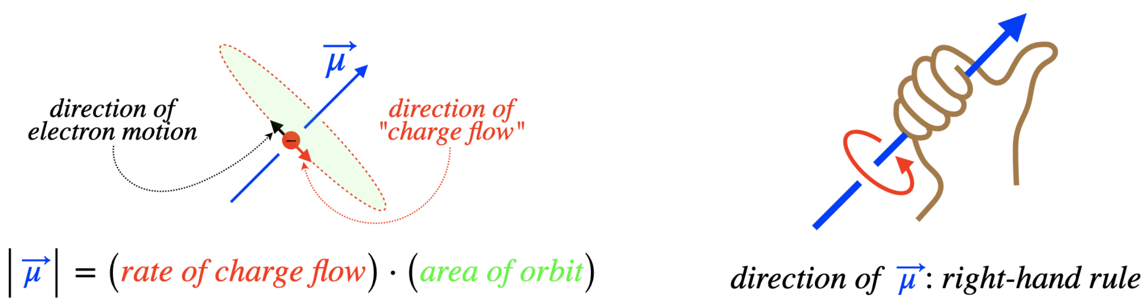
Linking Angular Momentum to Magnetism

While we have some idea now how to handle angular momentum in the quantum realm mathematically, it isn't immediately clear how we are supposed to make measurements of this quantity. For energy it was easy – we looked at emission spectra and used Planck's constant along with the frequency of the light we measured using a diffraction grating. For linear momentum, we looked at the results of collisions. But measuring angular momentum is a tougher nut to crack. We in fact need to look at a different property of particles and derive from what we observe about this property the implications for angular momentum. This other property is called *magnetic dipole moment*, and we will have a quick look at the basics of magnetic forces so that we can understand how this property is used to measure angular momentum.

We have already seen in our discussion of the hydrogen atom that electric charges pull directly toward ot push directly away from each other through the electrical force. Well it turns out that electrical charges exert another type of force (magnetism) on each other, which results from their relative motion. That is, while the mere existence of charge results in an electric field, which in turn results in a force on another particle that possesses charge, more is needed for a magnetic force to occur. The first charge must be moving to produce a magnetic field, and the second charge must be moving to experience a force from that magnetic field. At this point, in light of your Physics 9HB training, alarms are probably going off in your head, and you are wondering, "What does he mean by *moving*?! Relative to what?" Well, I'm afraid the answer to these questions of relativity applied to magnetism will have to wait for Physics 9HD.

We are not even going to look into the details of this magnetic force, as we are interested in one specific case – a charge that is moving in a closed loop while in the presence of a magnetic field. Macroscopically, this could be an electrical current circulating in a conducting loop of wire. Naturally we will be interested in the microscopic version of this – an electron orbiting a nucleus. Rather than looking at the effect of the magnetic field on the electron at any given instant, we simplify things by treating the orbiting electron as a full system (with the location of the electron in its orbit left undefined), and call this system a *magnetic dipole*. The aforementioned magnetic dipole moment is a vector property of such a system. [Interestingly, the idea of treating a magnetic dipole as charge indeterminately smeared-out around its orbit was a classical idea, but actually fits pretty well with what we know of quantum theory!] Here is how the magnetic dipole moment is defined:

Figure 8.1.1 – Magnetic Dipole Moment Definition



The rate of charge flow is the amount of charge (e) divided by the time it takes the charge to complete a full (circular) orbit. This time is the circumference of the orbit divided by the speed of the electron. So the magnitude of the magnetic moment is:

$$|\vec{\mu}| = \left(\frac{e}{\frac{2\pi r}{v}}\right)\left(\pi r^2\right) = \frac{evr}{2} \tag{8.1.1}$$

The direction of the magnetic dipole moment is determined by the right-hand-rule applied to the direction of "charge flow". As we are talking about an electron (a negatively-charged particle), the direction of charge flow is opposite to the direction of the electron motion.

Okay, so now we can relate this back to the angular momentum. For a particle of mass m moving in a circle of radius r with speed v, the angular momentum is:

$$\vec{L} = \vec{r} \times \vec{p} \quad \Rightarrow \quad \left|\vec{L}\right| = mvr \tag{8.1.2}$$

We can therefore write the magnetic moment in terms of the angular momentum. Note that the mass of the particle is what matters in the angular momentum, while the charge flow (which is in the opposite direction) is what matters for the magnetic moment. This means that $\vec{\mu}$ and $\vec{L}$ point in opposite directions for the electron, and we have:

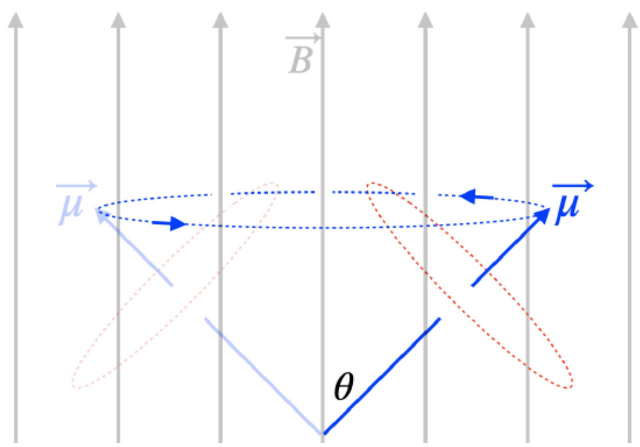
$$\vec{\mu}=-\frac{e}{2m_e}\vec{L} \tag{8.1.3}$$

The magnetic moment and the angular momentum for this particle only differ by a constant, which means that whatever we can do to measure magnetic dipole moment will also tell us the angular momentum.

Precession of Magnetic Dipoles

For our purposes, it will be sufficient to know about two properties of magnetic dipoles. The first of these we have already seen in a different context. In Physics 9HA, we saw what happens when we apply a torque to an object with angular momentum – gyroscopic precession. It turns out that a magnetic field will exert a torque on a magnetic dipole, in the same way that gravity and normal force combined to exert a torque on our gyroscope.

Figure 8.1.2 – Magnetic Dipole Precession (for Electron) Around a Magnetic Field



The torque exerted on a magnetic dipole by a magnetic field $\vec{B}$ turns out to be:

$$\vec{\tau}=\vec{\mu}\times\vec{B} \quad\Rightarrow\quad |\vec{\tau}|=|\mu|\left|\vec{B}\right|\sin\theta \tag{8.1.4}$$

When we studied the gyroscope in Physics 9HA, we determined (Equation 6.2.4) the precession rate in terms of the torque and angular momentum. It's important to keep in mind that in the case of the gyroscope, we had the angular momentum vector pointing perpendicular to the gravitational force. In this case, there is a component of the dipole's angular momentum along the magnetic field. This component will play no role in the precession rate. So defining the $+z$-direction as the direction of the magnetic field, the equation we obtained for gyroscopic precession rate needs to be adjusted so that the angular momentum in the denominator is only the component perpendicular to the z-axis:

$$\Omega=\frac{\tau}{L_{\perp z}}=\frac{\tau}{L\sin\theta} \tag{8.1.5}$$

Plugging in for the torque and the angular momentum gives the precession rate, known as the *Larmor frequency*:

$$\Omega=\frac{\mu B\sin\theta}{\frac{2m_e}{e}\mu\sin\theta}=\frac{eB}{2m_e} \tag{8.1.6}$$

> **Alert**
>
> *In the absence of any external constraints, there is no preferred direction in space, and when we seek to describe a spherically-symmetric physical system with spherical coordinates, we are forced to choose an arbitrary z-axis. However, when there is a constraint, such as the applied magnetic field in this case, it is incredibly convenient to define our z-axis in a specific manner. It is universally assumed in quantum theory that the z-direction is chosen to be in the direction of an applied magnetic field.*

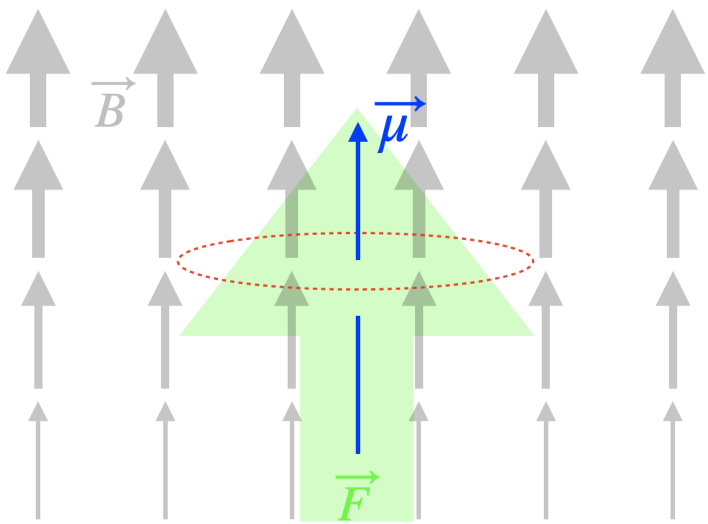
A couple items of note here:

- The Larmor frequency has no dependence on θ, which means it doesn't depend upon the specific angular momentum state of the particle, though the *direction* of the precession does.
- The cone swept-out by the precession bears a striking resemblance to the "uncertainty cone" for angular momentum in Figure 7.4.1. Indeed, while we can measure the rate of precession of the angular momentum vector, we cannot determine its actual direction at any moment in time.
- A comment on the direction of precession indicated in the figure above: The direction of the torque is determined by using the right-hand-rule for $\vec{\mu} \times \vec{B}$, and for the position of the dipole in the diagram, this comes out to be out of the page. But the figure has the dipole precessing *into* the page at that position. The reason is that we use the torque to find the change in the *angular momentum vector*, not the magnetic dipole moment vector. In this case (a negative-charged electron), the dipole moment and angular momentum point in opposite directions, giving the direction of reaction to the torque that is opposite to what it would be for the same diagram depicting a magnetic moment from a positive charge.

Forces on Magnetic Dipoles

While the uniform magnetic field exerts a force at all times on the moving charge, when averaged over a full orbit, there is no net force – only a net torque. But as we have all at some point in our lives stuck a magnet to a refrigerator (or another magnet), we know that it is possible for magnetic fields to exert net forces. The reason the magnetic field above exerts no net force is that it is *uniform*. If we instead introduce a magnetic field that grows (or diminishes) in strength along the direction that it points, then a magnetic dipole pointing parallel to the direction of the magnetic field will experience a net force.

Figure 8.1.3 – Force on a Magnetic Dipole by a Non-Uniform Magnetic Field



The force exerted on the dipole in terms of the gradient of the magnetic field along its direction is:

$$F = \mu \frac{d}{dz} B \tag{8.1.7}$$

Of course, the magnetic dipole moment is not always aligned with the field, so really it is just the z-component of the dipole moment that comes into play. Also, it turns out that it is impossible to have a magnetic field like that described above (you'll have to wait until Physics 9HD to find out why!), where the field is in a single direction everywhere, while getting stronger (or weaker) along that direction. Instead, the direction of the field has to converge or diverge in order for it to get stronger or weaker (respectively) along a given direction. The upshot is that our equation for force on a dipole needs a bit more labeling to be accurate:

$$F = \mu_z \frac{\partial}{\partial z} B_z \tag{8.1.8}$$

As indicated by the diagram, the direction of the force is along the common direction of the field and dipole moment when the field strength is growing in that direction (the derivative is positive).

A Measuring Device

We started this section talking about how to measure angular momentum, and we finally have the means: If we pass a beam of particles through a non-uniform field, then each particle in the beam will be deflected according to its component of magnetic dipole moment along the direction of the field. A device for shooting beams of atoms through a non-uniform field was conceived and constructed in 1922 by Stern and Gerlach. This Stern-Gerlach ("SG") device is as important for demonstrating quantization as the double-slit experiment is for demonstrating wave-particle duality. To see this, consider the results of this experiment...

Randomly-prepared hydrogen atoms have no particular preference for a direction of angular momentum – they can be oriented in any possible direction. So when a beam of these atoms are sent through a non-uniform field, it would seem that each atom should be deflected by whatever force results from that particular atom's z-component of magnetic dipole moment. If its magnetic moment vector happens to be pointing in the direction of increasing field strength, then it will feel a strong force that way. If it is pointing in the direction opposite to the increasing field strength, it will feel a strong force in the opposite direction. If the z-component happens to be perpendicular to the magnetic field, then it will not be deflected at all. And of course, all the z-components between these three extremes should result in a whole continuum of deflections. But this is not what is seen!

We found already that the z-component of angular momentum must be quantized (Equation 7.4.12), and since angular momentum is proportional to magnetic dipole moment, we should *not* see a whole continuum of forces on the particles. Only a few forces are possible – one for each quantum number m_l. It should now be clear why this quantum number is referred-to as the "magnetic quantum number" – it is the one responsible for interactions with magnetic fields.

Similarity to Polaroids

An important lesson learned from these SG devices is what happens when they are used in succession. That is, starting with a randomly-prepared beam of particles, pass it through a SG device, splitting the beam according to the quantized states of angular momentum. Then send *one of the resulting split beams* through a second SG device. What should we expect to happen?

First of all, the original beam was in a mixed-state of z-component of angular momentum, and the SG device had the effect of separating the eigenstates of this operator. Now with all of the beams are eigenstates of that operator, if we send them through a second filter they will not split again, *provided it is the same operator*. Recall that the $\widehat{L_z}$ operator depends upon which way we have defined as our $+z$-direction. If our second SG device is oriented so that its magnetic field points along the same z-axis as the magnetic field in the first SG device, then the second beam will deflect the same as the first did.

If, however, we rotate the z-axis of the second SG device relative to the first one, then the new $\widehat{L_z}$ operator is different from the old one, and the beam that was an eigenstate of the first operator is not in an eigenstate of the new one – the beam splits again! If we pass one of these split beams through a third SG device oriented the same way as the first SG device, we might think that a single beam should emerge – after all, those particles were all eigenstates of the original $\widehat{L_z}$ operator. But this doesn't happen! The particles don't "remember" that they were once in a eigenstate of the original $\widehat{L_z}$ operator – once they are measured using a new z-axis, the emerging beams are eigenstates of *that* $\widehat{L_z}$ operator. As we have seen before, observing the state disturbs the state.

We saw this same sort of behavior for light passing through polaroids, with the exception that the "splitting" of the light results in one of two "beams" being absorbed, while both are passed in the SG device. There is another difference as well, with respect to the fraction of a polarized beam that passes through a second SG device whose orientation differs from the original by some arbitrary angle. We will discuss this difference in the next section.