7.4: Angular Momentum

The Dynamical Quantity We Have Ignored

We have spent all of our time in the quantum theory dealing with the three physical quantities of position, momentum, and energy, but there is one other that we studied at length in classical mechanics that has barely been mentioned so far – angular momentum. This is quite an omission, given that it, along with linear momentum and energy, satisfies a very useful conservation principle under the proper conditions. This apparent oversight is even more striking given that Planck's constant, the most fundamental of all physical constants in the realm of quantum mechanics, has units of angular momentum! But avoidance of angular momentum was not an oversight, it just took a lot of background to be able to handle it, and the time is now finally right.

Digression: Types of Vectors

Since Physics 9HA, we have treated angular momentum as though it is a vector. After all, it does have magnitude and direction, right? Well, yes on the magnitude, but we actually defined its direction ourselves with the right-hand-rule – if the majority of humanity had been left-handed, we might have defined it direction as exactly the opposite. It doesn't have a "natural" direction like velocity and momentum does. In fact, angular momentum is an example of a mathematical quantity known as an axial vector, or a pseudovector ("regular" vectors like momentum and velocity are referred to as polar vectors). What distinguishes axial vectors from polar vectors mathematically is how they transform when the coordinate system defining their components is reflected about the origin ($x \rightarrow -x$, $y \rightarrow -y$, $z \rightarrow -z$). Clearly polar vectors will change sign under such a transformation (all of their components flip sign), but see what happens to angular momentum:

$$\left.\begin{array}{l} x \rightarrow -x \\ y \rightarrow -y \\ z \rightarrow -z \end{array}\right\} \rightarrow \vec{L} = \vec{r} \times \vec{p} = (-\vec{r}) \times (-\vec{p}) = +\vec{L} \tag{7.4.1}$$

Building an Angular Momentum Operator

We know from the very origins of our work in quantum mechanics that all physically-observable quantities have quantum-mechanical operators associated with them. We also determined that the operators we needed could be built from a couple of fundamental ones. For example, we built the kinetic energy operator from the momentum operators for each of the three directions:

$$\widehat{KE} \equiv \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 +\right) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \tag{7.4.2}$$

Well, classically we can express the components of angular momentum in terms of components of position and momentum, so we should be able to do a similar construction from their operators. We have to use some care, however, because whenever we "mix" position and momentum, the uncertainty principle shows up. Let's start with the classical definition:

$$\vec{L} = \vec{r} \times \vec{p} = \left(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\right) \times \left(p_x\,\hat{i} + p_y\,\hat{j} + p_z\,\hat{k}\right) = (yp_z - zp_y)\,\hat{i} + (zp_x - xp_z)\,\hat{j} + (xp_y - yp_x)\,\hat{z} \tag{7.4.3}$$

Let's focus only on the z-component. If we want to create an operator for L_z, we must replace the classical values with operators:

$$\widehat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{7.4.4}$$

The operators for L_x and L_y come out similarly. Speaking of the operators of the other components of angular momentum, close examination of these reveals a brand-new property that we haven't seen before. back in Section 5.3, we found that the uncertainty principle for two "incompatible" physical quantities is manifested in their operators in that they do not produce the same result when they act on a wave function in different orders. Put another way, physical quantities that cannot simultaneously be measured with arbitrary precision have operators that don't commute with each other. We know that the operators representing the components of the position all commute with each other – we can simultaneously measure the x-position, the y-position, and the z-position without the uncertainty principle making them trip over each other. We can similarly do this with the components of linear momentum, since the order of partial derivatives can be freely swapped. But this is not true of the operators of the components of the angular momentum vector! Let's demonstrate this with the x and y components:

$$\begin{aligned}
\widehat{L}_x\widehat{L}_y\Psi &= \left[-i\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\right]\left[-i\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)\right]\Psi = \\
&-\hbar^2\left[y\frac{\partial}{\partial x}\Psi+yz\frac{\partial^2}{\partial x\partial z}\Psi+xz\frac{\partial^2}{\partial y\partial z}\Psi-z^2\frac{\partial^2}{\partial x\partial y}\Psi-xy\frac{\partial^2}{\partial z^2}\Psi\right] \\
\widehat{L}_y\widehat{L}_x\Psi &= \left[-i\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)\right]\left[-i\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\right]\Psi = \\
&-\hbar^2\left[x\frac{\partial}{\partial y}\Psi+yz\frac{\partial^2}{\partial x\partial z}\Psi+xz\frac{\partial^2}{\partial y\partial z}\Psi-z^2\frac{\partial^2}{\partial x\partial y}\Psi-xy\frac{\partial^2}{\partial z^2}\Psi\right]
\end{aligned} \tag{7.4.5}$$

We see that the derivatives of the second (leftmost) operator act on the position components present in the second (rightmost) operator. The derivatives make quite a mess in each case, but the point is that the two "messes" are not equal, as can be seen by comparing the two results above – the last 4 terms are all the same, but the first terms differ. If one measures the x-component of angular momentum of a particle precisely, knowledge of the y component of the same angular momentum vector is lost!

Another extremely interesting (and as it happens, useful!) result we get from above is that we can construct an operator of one component of the angular momentum from the other two. Start by subtracting the two results above. This gets rid of the last four common terms and leaves:

$$\left(\widehat{L}_x\widehat{L}_y-\widehat{L}_y\widehat{L}_x\right)\Psi=\hbar^2\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\Psi \tag{7.4.6}$$

The right side of this equation looks familiar – it is very nearly the operator of the z-component of angular momentum (Equation 7.4.4). Indeed we have:

$$\widehat{L}_x\widehat{L}_y-\widehat{L}_y\widehat{L}_x=i\hbar\widehat{L}_z \tag{7.4.7}$$

There is a shorthand notation frequently used in such situations, called the *commutator* of the operators $\widehat{A}$ and $\widehat{B}$:

$$\left[\widehat{A},\widehat{B}\right]\equiv\widehat{A}\widehat{B}-\widehat{B}\widehat{A} \tag{7.4.8}$$

Thanks to the cyclical nature of the cross-product, we can similarly construct all the components from the other two, so we have:

$$\left[\widehat{L}_x,\widehat{L}_y\right]=i\hbar\widehat{L}_z\,,\quad\left[\widehat{L}_y,\widehat{L}_z\right]=i\hbar\widehat{L}_x\,,\quad\left[\widehat{L}_z,\widehat{L}_x\right]=i\hbar\widehat{L}_y \tag{7.4.9}$$

These are known as the *commutation relations* of the angular momentum component operators.

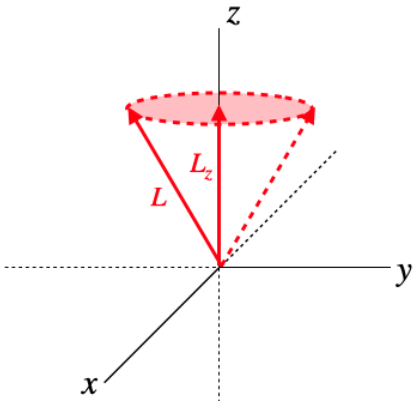
Magnitude of Angular Momentum

We now know how the components of angular momentum relate to each other, but what can be said about the *magnitude* of the angular momentum (or its magnitude-squared, which is more commonly used)? Interestingly, this measurable quantity commutes with all of the components:

$$\left[\widehat{L}^2,\widehat{L}_x\right]=\left[\widehat{L}^2,\widehat{L}_y\right]=\left[\widehat{L}^2,\widehat{L}_z\right]=0\quad\widehat{L}^2=\widehat{L}_x^2+\widehat{L}_y^2+\widehat{L}_z^2 \tag{7.4.10}$$

We will not do so here, but this fact can be proven using purely the commutation relations between the components. The physical meaning is that a particle can be in an eigenstate of total momentum, and one of its components can simultaneously be known to arbitrary precision. But because of the uncertainty relation between the components, the particle can be in an eigenstate of *only one* of those components at a time. We can depict this in a diagram as follows. Let's suppose that we measure both the total magnitude of angular momentum and the z-component precisely, then the uncertain *vector* angular momentum state of the particle looks like the figure below.

Figure 7.4.1 – Uncertain Angular Momentum Vector



Uncertainty in the vector $\vec{L}$ is characterized by the circle shown around the z-axis – every such vector with a tail at the origin and the head landing in the edge of that circle is possible. Note that the *length* of the angular momentum vector is known precisely, as is the z-component, but the x and y components are unknown.

Using Spherical Coordinates

In defining spherical coordinates, we have selected-out the z-direction as special (it is the axis from which the polar angle is measured, and around which the azimuthal angle is measured). By convention, we therefore define that to be the direction for which the single component of angular momentum is known. We can convert the operator $\widehat{L}_z$ into spherical coordinates to get an operator to use in cases where spherical coordinates are in play. The amount of math busywork requires to show this is substantial, but the end result is remarkably simple:

$$\widehat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \tag{7.4.11}$$

If the potential acting on a particle happens to be spherically symmetric, then the separated stationary-state wave function has an azimuthal piece given by Equation 7.3.12, and it's clear that this is an eigenfunction of the z-component of angular momentum:

$$\widehat{L}_z\Psi_{nlm_l}(r,\theta,\phi) = -i\hbar\frac{\partial}{\partial\phi}\left[R_{nl}(r)\,\Theta_{lm_l}(\theta)\,e^{im_l\phi}\right] = m_l\hbar\,\Psi_{nlm_l}(r,\theta,\phi) \tag{7.4.12}$$

The quantum number m_l therefore describes the quantized z component of angular momentum – the eigenvalue is this integer multiplied by $\hbar$.

The operator use in spherical coordinates for $\widehat{L^2}$ is considerably more complicated than the one for $\widehat{L}_z$, and we will not take the discussion this deep in this class. But it is important to know that the polar portion of the separated wave function (the Legendre polynomials, $\Theta_{lm_l}(\theta)$, for which we have also postponed examination to a future, more advanced class) are eigenfunctions of this operator. Specifically:

$$\widehat{L^2}\Psi_{nlm_l}(r,\theta,\phi) = \widehat{L^2}\left[R_{nl}(r)\,\Theta_{l,m_l}(\theta)\,e^{im_l\phi}\right] = l(l+1)\,\hbar^2\,\Psi_{nlm_l}(r,\theta,\phi) \tag{7.4.13}$$

The quantum number l therefore describes the quantized magnitude-squared of angular momentum – the eigenvalue is $l(l+1)$ multiplied by $\hbar^2$. Or, if you like, the eigenvalue for the magnitude of angular momentum is $\left|\vec{L}\right| = \sqrt{l(l+1)}\,\hbar$.

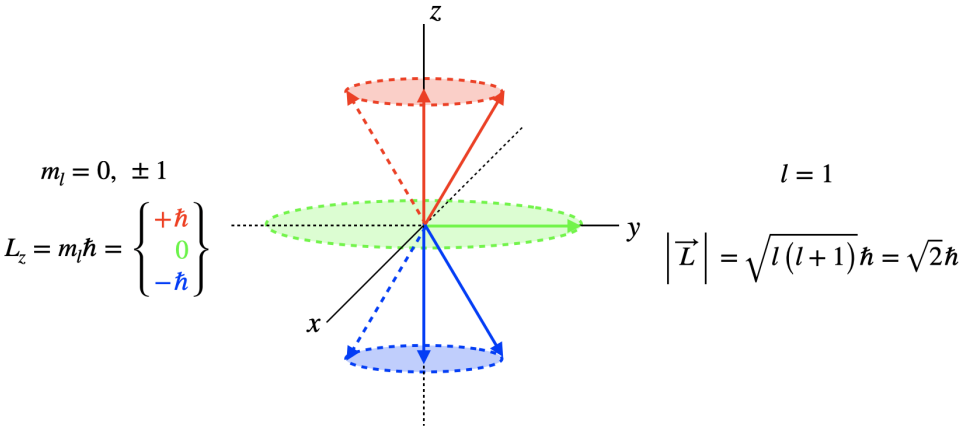
To see further justification for this eigenvalue for the square of the total angular momentum, take a look at the radial Schrödinger equation (Equation 7.3.16). The terms on the left side of the equation that do not include the potential energy must be the kinetic energy (since the right hand side is the total energy). The kinetic energy can be divided into a part that comes from radial motion and a part that comes from tangential motion. We did a similar thing in Physics 9HA when discussing gravitation, and wrote the tangential part of the kinetic energy in terms of the angular momentum in Equation 7.3.8. Comparing these immediately suggests that $L^2 = l(l+1)\,\hbar^2$.

Given that the z-component of the angular momentum must always be *strictly less than* the magnitude of the angular momentum vector (the x and y components can never be zero due to uncertainty), the relation given in Equation 7.3.15, limiting the value of m_l to extremes of $\pm l$, and this magnitude of $\vec{L}$ are perfectly consistent. This leads to the number of possible quantized angular momentum states for a given quantum number l to be:

$$\text{number of quantized angular momentum states} = 2l+1 \tag{7.4.14}$$

For example, for $l = 1$:

Figure 7.4.2 – Quantized Angular Momentum States ($l = 1$)



A Few Brief Words About the Hydrogen Atom

Clearly the radial symmetry of the potential energy of the electron in a hydrogen atom qualifies it for this analysis, but we will not be delving into the details of of the eigenfunctions for this system in this class (rest assured it will be a high priority for a future class in quantum physics!). However, there are some interesting/important facts about the hydrogen atom that are worth sharing.

The electron in a hydrogen atom comes with the three quantum numbers already mentioned: n, l, andm_l. From our study of one-dimensional wells, we are used to having a quantum number related directly to energy eigenvalues. In this case, the energy eigenvalue appears imbedded in the radial wave equation, the eigenfunction of which depends on both n and l. It's not surprising that the magnetic quantum number doesn't effect the energy of the electron, since m_l relates to the quantized z-component of angular momentum, and this depends upon our choice of z axis – the energy levels should not depend upon our choice of a z-axis. But certainly it makes sense that the energy of the electron in the Coulomb potential should depend upon the other two. Strangely, however, it does not! It *only* depends upon the principal quantum number, n. Indeed, Bohr's remarkable result of the energy spectrum of hydrogen with his simple model holds up, even when incorporating the angular momentum properly.

The mathematics (which we didn't cover in detail) of the two angular-momentum quantum numbers gave us a restriction on the possible values of m_l for a given value of l (Equation 7.3.15), and it turns out that still more mathematics (that we also won't cover) results in a restriction on the number of l values for a given n. The values of l are restricted to be only positive, and it turns out that they can never exceed the principle quantum number:

$$l = 0,\ 1,\ 2, \ldots,\ n-1 \tag{7.4.15}$$

When we first started with 3-dimensions, we noted that the extra degrees of freedom can, especially in cases involving symmetry, lead to degeneracies – multiple wave functions responsible for the same particle energy. Clearly this happens for the hydrogen atom. We already know about the $2l+1$ states that have the same l quantum number but different m_l quantum numbers, and now we see that there are multiple orbital quantum numbers for a given principle quantum number. The degeneracy associated with the multiple values of m_l are attributable to the spherical symmetry, but the degeneracy associated with the l-values is a famous example of an accidental degeneracy (recall these are degeneracies that come from coincidence, and not some symmetry (at least not an obvious one).

The degeneracy we get for a given principle quantum number can be counted. There are n possible values of l, and for each of these, there are $2l+1$ values of m_l. This leads to the following number of states for the same energy:

$$\begin{aligned}\text{degeneracy of } n^{th} \text{ energy eigenstate} = &\left[1\ \ (l=0,\ m=0)\right] + \left[3\ \ (l=1,\ m=0,\pm1)\right] + \cdots \\ &+ \left[2l+1\ \ (l=n-1,\ m=0,\pm1,\ldots,\pm l)\right] = n^2\end{aligned} \tag{7.4.16}$$