6.4: Tunneling

The Potential Step

In our examination of square wells, we noted that inside the well the wave function is a superposition of left- and right-moving plane waves. In these cases, these plane waves had equal wavelengths, because the energy was confined to a region. Here we will consider the possibility that when a plane wave strikes a vertical potential change, only *some* of the wave is reflected, while some of it is transmitted. Indeed, in our study of light, we found that this occurs. The upshot is that we cannot write our stationary-state solutions as sines and cosines as we did in the case where the left- and right-going plane waves had equal amplitudes.

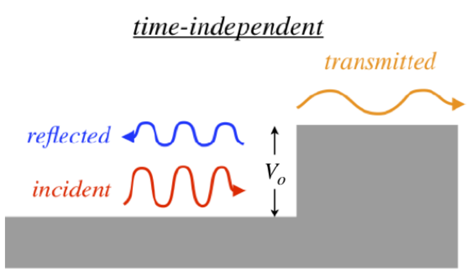
We will remain with the stationary-state solution, meaning that the probabilities don't change over time, and the total energy remains fixed. The picture is a little more confusing here, as we will first consider a "before/after" scenario (which is hardly time-independent!), but eventually we will turn this into a steady-state situation, where "before" and "after" are occurring simultaneously and continuously. We will do this little-by-little, starting with the most basic cases and building our way up. The first case involves a totally-free particle that "encounters" a sudden change in the constant potential (a "step"), from 0 to V_o..When the wave encounters this step, it's reasonable from our understanding of the behavior of waves to assume that some of the wave will be reflected backward, while some is transmitted forward. A picture you might have in your head for this physical situation is:

Figure 6.4.1 – Step Potential Before/After Picture

While this makes sense from a classical standpoint – moving objects "encounter" things in their travels all the time – it's a bit troublesome for quantum mechanics, as a plane wave occupies all values of x at once, so how can it be moving along the x direction? Put another way, we are seeking a *stationary state* solution, and this picture clearly shows a system evolving over time.

We therefore are looking for a solution where the wave function is unchanging over all values of x, but in practical terms we know that particles (which travel in wave packets of many energies rather than perfect plane waves) are localized, and therefore do really encounter changing potentials. What compromise can we reach in this regard? As stated above, any combination of plane waves going in both directions with the same momentum (for a given potential) will be an energy eigenstate, so how about if we restrict ourselves to only right-moving plane waves on the "transmitted" side of the step, and plane waves in both directions on the incident/reflected side? We can interpret this in our "real world" setting as a steady-state circumstance, where many particles are coming in, and some are reflected while others are transmitted, where the fractions of each are determined (mainly) by the relative amplitudes of the reflected and transmitted waves. The correct picture then becomes:

Figure 6.4.2 – Step Potential Steady-State Picture



Keep in mind that we are assuming the particle remains free, so even after the increase in potential energy, it has kinetic energy (albeit less). Clearly three is the minimum number of wave functions that are possible. For example, it is neither possible to have all of the wave to be reflected, nor have all of the wave to be transmitted, thanks to the finite value of V_o that is smaller than the particle's total energy. It *is* possible to get a solution that includes a left-moving wave on the right of the step, but we have discarded these solutions to fit with our narrative that there is a steady source of incoming particles from the left.

Okay, with all that out of the way, we follow our usual procedure of writing the wave function in each of the regions as superpositions of plane waves of appropriate energy, with unknown amplitudes. We'll define the position of the step as $x=0$, which gives:

$$\psi\left(x\right)=\begin{cases}\psi_{inc}\left(x\right)+\psi_{refl}\left(x\right) & x<0\\ \psi_{trans}\left(x\right) & x>0\end{cases} \tag{6.4.1}$$

The incident and reflected plane waves experience the same potential, so they have the same wave number (k), while the wave number of the transmitted plane wave is different (k'). Also, they all have different amplitudes in general, so:

$$\left.\begin{array}{l}\psi_{inc}(x)=Ae^{+ikx}\\ \psi_{refl}(x)=Be^{-ikx}\end{array}\right\} \quad k=\frac{\sqrt{2mE}}{\hbar}$$
$$\psi_{trans}(x)=Ce^{+ik'x} \quad k'=\frac{\sqrt{2m(E-V_o)}}{\hbar} \tag{6.4.2}$$

Let's stop a moment to interpret the quantities A, B and C. Suppose we were asked the probability density of finding an incident particle at a position between x and $x+dx$. The answer would be the magnitude-squared of its wave function. The same is true for a reflected or transmitted particle:

$$\text{probability of finding between } x \text{ and } x+dx: \begin{cases}\text{incident particle:} & |\psi_{inc}(x)|^2dx & = & |A|^2dx\\ \text{reflected particle:} & |\psi_{refl}(x)|^2dx & = & |B|^2dx\\ \text{transmitted particle:} & |\psi_{trans}(x)|^2dx & = & |C|^2dx\end{cases} \tag{6.4.3}$$

We would like to compare these quantities to determine the relative probabilities of the reflected and transmitted waves. In our steady-state model where many particles are coming in, this would tell us what fraction of the incoming particles are reflected, and what fraction is transmitted. The trouble is, these cannot be compared directly, because they have different wavelengths. Why is this a problem, and how do we resolve it?

Huygens's principle states that a wave propagates by having a lead crest generate more crests which propagate by generating more crests, and so on. It therefore stands to reason that one cycle of the incident incoming wave is responsible for one cycle of the reflected and one cycle of the transmitted wave. Suppose that the incident and reflected waves (which have the same wavelength) have a wavelength one half as long as the transmitted wave (which must be longer, as it has less kinetic energy and therefore less momentum). That means that two full cycles of the incident and reflected waves fit into the same space as the transmitted wave, so the particle is *twice as likely* to be found to the left of the barrier as in an equal space to the right. So to compare probabilities, we need to divide each of the probability densities by their associated wavelengths (or equivalently, multiply them by their associated wave numbers) in order to make a proper comparison. [In our steady-state many-particle model, this is the equivalent of accounting for the speed at which the particles are moving into and out of the step.]

We therefore define the *transmission and reflection probabilities* as the ratios of the relevant (adjusted) probability densities:

$$T=\frac{|C|^2k'}{|A|^2k} \qquad R=\frac{|B|^2k}{|A|^2k}=\frac{|B|^2}{|A|^2} \tag{6.4.4}$$

All we need now are A, B and C. We get these by matching the boundary conditions of the wave functions in the two regions – continuous (equal value) and smooth (equal derivative) – at $x=0$.

Sparing you the algebra, we get:

$$T=\frac{4\sqrt{E(E-V_o)}}{\left(\sqrt{E}+\sqrt{E-V_o}\right)^2} \qquad R=\frac{\left(\sqrt{E}-\sqrt{E-V_o}\right)^2}{\left(\sqrt{E}+\sqrt{E-V_o}\right)^2} \tag{6.4.5}$$

It is easy to show that the sum of these two probabilities comes out to one, which is consistent with the requirement that the incident particle must either reflect or be transmitted.

It should be pointed out that a similar solution results from a step *down*. Intuitively, it might seem like the wave only has a partial reflection when it steps up, because that seems like an obstacle, while stepping down is "easy." But even in a classical study of waves, one finds that waves reflect off any interface between media that result in different wave speeds, whether the speed changes to a slower speed or a faster one.

Speed Bumps and Potholes

We can extend this analysis to the case of a square potential bump (which isn't higher than the particle's energy) or dip (of any depth), where the potential before and after the obstruction is the same. Unfortunately, the stakes are raised, in that we now have *two* transitions – one at the front and one at the back of the obstruction. Following the reasoning above, we throw out the left-moving part of the wave function on the side opposite the right-moving incident wave. But we *cannot* throw out the reflected wave at the back surface of the obstruction. This leaves us with 5 parts of the wave function: The incident wave, the reflected wave, the two oppositely-moving waves in the region of the obstruction, and the transmitted wave.

This gives us 5 amplitudes to deal with, and two conditions at each boundary. Also, there is another parameter involved – the width of the obstruction. We are not interested in the specifics of what is going on in the region of the obstruction, only the transmission and reflection probabilities, which means there is one element of this problem that is simpler than the potential step: We don't have to account for different wavelengths of the incoming and transmitted waves, since we are assuming the starting and ending potential energies are the same.

Once again skipping the rather daunting amount of algebra, we get the following probabilities. Calling the width of the obstruction L and noting that the solutions involving a difference $(E-V_o)$ are for potential bumps of height V_o, while those involving the sum $(E+V_o)$ are for potential dips of depth V_o:

$$T=\frac{4\left(\frac{E}{V_o}\right)\left(\frac{E}{V_o}\pm1\right)}{\sin^2\left(\sqrt{2m\left(E\pm V_o\right)}\frac{L}{\hbar}\right)+4\left(\frac{E}{V_o}\right)\left(\frac{E}{V_o}\pm1\right)}\qquad R=\frac{\sin^2\left(\sqrt{2m\left(E\pm V_o\right)}\frac{L}{\hbar}\right)}{\sin^2\left(\sqrt{2m\left(E\pm V_o\right)}\frac{L}{\hbar}\right)+4\left(\frac{E}{V_o}\right)\left(\frac{E}{V_o}\pm1\right)}\tag{6.4.6}$$

Once again, the sum of these two probabilities is one, which means the particle doesn't have any chance of being trapped within the obstruction indefinitely. Or, put in terms of our steady stream of particles, none are left in the obstruction, so every particle that reaches the obstruction comes out in one direction or the other.

There is a fascinating special case that comes from this solution. It is possible for the numerator of the reflection probability to be zero, which means that *all the particles are transmitted*. This occurs when the argument of the sine function is a multiple of π:

$$\sqrt{2m\left(E\pm V_o\right)}\frac{L}{\hbar}=n\pi\quad\Rightarrow\quad E\pm V_o=\frac{n^2\pi^2\hbar^2}{2mL^2}=\frac{n^2h^2}{8mL^2}\tag{6.4.7}$$

Notice that $E\pm V_o$ is the kinetic energy of the particle within the obstruction – it increases in the case of a dip ($E+V_o$), and decreases in the case of a bump ($E-V_o$). It is a plane wave, so the kinetic energy can be written in terms of the wave number, which can then be written in terms of the wavelength, giving:

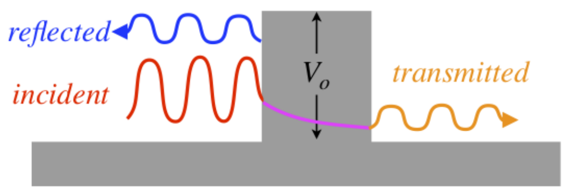
$$E\pm V_o=\frac{\hbar^2k'^2}{2m}=\frac{4\hbar^2\pi^2}{2m\lambda'^2}=\frac{n^2\pi^2\hbar^2}{2mL^2}\quad\Rightarrow\quad L=\frac{n\lambda'}{2}\tag{6.4.8}$$

Obstructions with thicknesses that are an integer number of half-wavelengths of the wave function measured within the obstruction are totally transparent to the beam of particles. We see this phenomenon with light, in the topic of thin films, which includes applications such as camera lens coatings.

Tunneling

We at last come to the quantum-mechanically iconic phenomenon of tunneling. In the case of a bump above, we assumed that the height of the bump was lower than than the energy of the particle. We now assume that the potential increase of the barrier, while finite, is greater than the energy of the incoming particle. As we already know, with a wall of finite height, some of the wave function "leaks" into the wall, exponentially decaying with respect to the penetration distance. Although it decays, it doesn't go to zero in the finite distance that is the thickness of the wall. Matching the boundary conditions on the other side of the wall results in a non-vanishing free particle wave function on the opposite side.

Figure 6.4.3 – Tunneling Through a Barrier



We have a remarkable shortcut to get us to the transmission and reflection probabilities. The math for this case is identical to the math used to derive T and R for the classically-surmountable bump above, with the exception that the kinetic energy $E-V_o$ within the barrier is negative. This appears within a square root, so we can introduce an imaginary number thus:

$$\sin^2\left(\sqrt{2m\left(E-V_o\right)}\frac{L}{\hbar}\right)=\sin^2\left(i\sqrt{2m\left(V_o-E\right)}\frac{L}{\hbar}\right)\tag{6.4.9}$$

Sine functions with imaginary arguments can be converted to hyperbolic sine functions multiplied by imaginary i. The proof is quick:

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i}\quad\Rightarrow\quad\sin ix=\frac{e^{-x}-e^{x}}{2i}=i\frac{e^{x}-e^{-x}}{2}=i\sinh x\tag{6.4.10}$$

This converts the square of the sine function into the negative of the square of the sinh function, and since the sign of $\frac{E}{V_o}-1$ also flips when $\frac{E}{V_o}<1$, the negative sign that appears in both numerator and denominator cancel, leaving:

$$T=\frac{4\left(\frac{E}{V_o}\right)\left(1-\frac{E}{V_o}\right)}{\sinh^2\left(\sqrt{2m\left(V_o-E\right)}\frac{L}{\hbar}\right)+4\left(\frac{E}{V_o}\right)\left(1-\frac{E}{V_o}\right)}\qquad R=\frac{\sinh^2\left(\sqrt{2m\left(V_o-E\right)}\frac{L}{\hbar}\right)}{\sinh^2\left(\sqrt{2m\left(V_o-E\right)}\frac{L}{\hbar}\right)+4\left(\frac{E}{V_o}\right)\left(1-\frac{E}{V_o}\right)}\tag{6.4.11}$$

Unlike the case of a small potential bump, there are no special wavelengths that allow the particle to pass through without any reflecting. Not surprisingly, the transmission rate rises as the energy of the particle rises, and drops as the barrier's height or width increases.
