2.2.21: Probability

Probability is the likelihood that some event occurs. If the event occurs, we call that a favorable outcome. The set of all possible events (or outcomes) is called the sample space of the event. We will limit our focus to independent events, which do not influence each other. For example, if we roll a 5 on one die, that does not affect the probability of rolling a 5 on the other die. (We will not be studying *dependent* events, which do influence each other.)

If we are working with something simple like dice, cards, or coin flips where we know all of the possible outcomes, we can calculate the theoretical probability of an event occurring. To do this, we divide the number of ways the event can occur by the total number of possible outcomes. We may choose to write the probability as a fraction, a decimal, or a percent depending on what form seems most useful.

Theoretical probability of an event:

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

Suppose two six-sided dice, numbered 1 through 6, are rolled. There are $6 \cdot 6 = 36$ possible outcomes in the sample space. If we are playing a game where we take the sum of the dice, the only possible outcomes are 2 through 12. However, as the following table shows, these outcomes are not all equally likely. For example, there are two different ways to roll a 3, but only one way to roll a 2.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

? Exercises 2.2.21.1

Two six-sided dice numbered 1 through 6 are rolled. Find the probability of each event occurring.

1. The sum of the dice is 7.

2. The sum of the dice is 11.

3. The sum of the dice is 7 or 11.

4. The sum of the dice is greater than 1.

5. The sum of the dice is 13.

Answer

1. $\frac{6}{36} = \frac{1}{6}$

2. $\frac{2}{36}=\frac{1}{18}$
3. $\frac{8}{36}=\frac{2}{9}$
4. $\frac{36}{36}=1$
5. $\frac{0}{36}=0$

Some things to notice…

If an event is impossible, its probability is 0% or 0.

If an event is certain to happen, its probability is 100% or 1.

If it will be tedious to count up all of the favorable outcomes, it may be easier to count up the unfavorable outcomes and subtract from the total.

Exercises 2.2.21.1

Two six-sided dice numbered 1 through 6 are rolled. Find the probability of each event occuring.

6. The sum of the dice is 5.

7. The sum of the dice is not 5.

8. The sum of the dice is greater than 9.

9. The sum of the dice is 9 or lower.

Answer

6. $\frac{4}{36}=\frac{1}{9}$
7. $\frac{32}{36}=\frac{8}{9}$
8. $\frac{6}{36}=\frac{1}{6}$
9. $\frac{30}{36}=\frac{5}{6}$

The set of outcomes in which an event does not occur is called the ***complement*** of the event. The event "the sum is not 5" is the complement of "the sum is 5". Two complements *complete* the sample space.

If the probability of an event happening is p, the probability of the complement is $1-p$.

Exercises 2.2.21.1

A bowl of 60 Tootsie Rolls Fruit Chews contains the following: 15 cherry, 14 lemon, 13 lime, 11 orange, 7 vanilla.

10. If one Tootsie Roll is randomly selected from the bowl, what is the probability that it is cherry?

11. What is the probability that a randomly selected Tootsie Roll is either lemon or lime?

12. What is the probability that a randomly selected Tootsie Roll is not orange or vanilla?

Answer

10. $\frac{15}{60} = \frac{1}{4}$
11. $\frac{27}{60} = \frac{9}{20}$
12. $\frac{42}{60} = \frac{7}{10}$

Here's where we try to condense the basics of genetic crosses into one paragraph.

Each parent gives one allele to their child. The allele for brown eyes is *B*, and the allele for blue eyes is *b*. If two parents both have genotype *Bb*, the table below (which biologists call a Punnett square) shows that there are four equally-likely outcomes: *BB*, *Bb*, *Bb*, *bb*. The allele for brown eyes, *B*, is dominant over the gene for blue eyes, *b*, which means that if a child has any *B* alleles, they will have brown eyes. The only genotype for which the child will have blue eyes is *bb*.

	B	*b*
B	*BB*	*Bb*
b	*Bb*	*bb*

? Exercises 2.2.21.1

Two parents have genotypes *Bb* and *Bb*. (*B* = brown, *b* = blue)

13. What is the probability that their child will have blue eyes?

14. What is the probability that their child will have brown eyes?

Answer

13. $\frac{1}{4} = 25\%$
14. $\frac{3}{4} = 75\%$

Now suppose that one parent has genotype *Bb* but the other parent has genotype *bb*. The Punnett square will look like this.

	B	*b*
b	*Bb*	*bb*
b	*Bb*	*bb*

? Exercises 2.2.21.1

Two parents have genotypes *Bb* and *bb*. (*B* = brown, *b* = blue)

15. What is the probability that their child will have blue eyes?

16. What is the probability that their child will have brown eyes?

Answer

15. $\frac{2}{4} = 50\%$
16. $\frac{2}{4} = 50\%$

The previous methods work when we know the total number of outcomes and we can assume that they are all equally likely. (The dice aren't loaded, for example.) However, life is usually more complicated than a game of dice or a bowl of Tootsie Rolls. In many situations, we have to observe what has happened in the past and use that data to predict what might happen in the future. If someone predicts that an Alaska Airlines flight has a 95.5% of arriving on time, that is of course based on Alaska's past rate of success.[1] When we calculate the probability this way, by observation, we call it an empirical probability.

Empirical probability of an event:

$$P(\text{event}) = \frac{\text{number of favorable observations}}{\text{total number of observations}}$$

Although the wording may seem complicated, we are still just thinking about $\frac{\text{part}}{\text{whole}}$.

Exercises 2.2.21.1

A photocopier makes 250 copies, but 8 of them are unacceptable because they have toner smeared on them.

17. What is the empirical probability that a copy will be unacceptable?

18. What is the empirical probability that a copy will be acceptable?

19. Out of the next $1,000$ copies, how many should we expect to be acceptable?

An auditor examined 200 tax returns and found errors on 44 of them.

20. What percent of the tax returns contained errors?

21. How many of the next $1,000$ tax returns should we expect to contain errors?

22. What is the probability that a given tax return, chosen at random, will contain errors?

Answer

17. $\frac{8}{250} = 3.2\%$

18. $\frac{242}{250} = 96.8\%$

19. we should expect 968 copies to be acceptable

20. $\frac{44}{200} = 22\%$

21. we should expect 220 tax returns to have errors

22. $22\% = 0.22$

It was mentioned earlier in this module that independent events have no influence on each other. Some examples:

- Rolling two dice are independent events because the result of the first die does not affect the probability of what will happen with the second die.
- If we flip a coin ten times, each flip is independent of the previous flip because the coin doesn't remember how it landed before. The probability of heads or tails remains $\frac{1}{2}$ for each flip.
- Drawing marbles out of a bag are independent events only if we put the first marble back in the bag before drawing a second marble. If we draw two marbles at once, or we draw a second marble without replacing the first marble, these are

dependent events, which we are not studying in this course.

- Drawing two cards from a deck of 52 cards are independent events only if we put the first card back in the deck before drawing a second card. If we draw a second card without replacing the first card, these are *dependent events*; the probabilities change because there are only 51 cards available on the second draw.

If two events are independent, then the probability of both events happening can be found by multiplying the probability of each event happening separately.

If A and B are independent events, then $P(A \text{ and } B) = P(A) \cdot P(B)$.

Note: This can be extended to three or more events. Just multiply all of the probabilities together.

Exercises 2.2.21.1

An auditor examined 200 tax returns and found errors on 44 of them.

23. What is the probability that the next two tax returns both contain errors?

24. What is the probability that the next three tax returns all contain errors?

25. What is the probability that the next tax return contains errors but the one after it does not?

26. What is the probability that the next tax return does not contain errors but the one after it does?

27. What is the probability that neither of the next two tax returns contain errors?

28. What is the probability that none of the next three tax returns contain errors?

29. What is the probability that at least one of the next three tax returns contain errors? (This one is tricky!)

Answer

23. $(0.22)^2 \approx 4.8\%$
24. $(0.22)^3 \approx 1.1\%$
25. $0.22 \cdot 0.78 \approx 17.2\%$
26. $0.78 \cdot 0.22 \approx 17.2\%$
27. $(0.78)^2 \approx 60.8\%$
28. $(0.78)^3 \approx 47.5\%$
29. $1-(0.78)^3 \approx 52.5\%$

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1. pdf file: https://www.transportation.gov/sites/dot.gov/files/2020-08/July%202020%20ATCR.pdf↵