14.2: Two Coupled Linear Oscillators

Consider the two-coupled linear oscillator, shown in Figure 14.2.1, which comprises two identical masses each connected to fixed locations by identical springs having a force constant κ. A spring with force constant κ' couples the two oscillators. The equilibrium lengths of the outer two springs are l while that of the coupling spring is l'. The problem is simplified by restricting the motion to be along the line connecting the masses and assuming fixed endpoints. The small displacements of m_1 and m_2 are taken to be x_1 and x_2 with respect to the equilibrium positions l and $l+l'$ respectively. The restoring force on m_1 is $-\kappa x_1 - \kappa'(x_1 - x_2)$ while the restoring force on m_2 is $-\kappa x_2 - \kappa'(x_2 - x_1)$. This coupled double-oscillator system exhibits basic features of coupled linear oscillator systems.

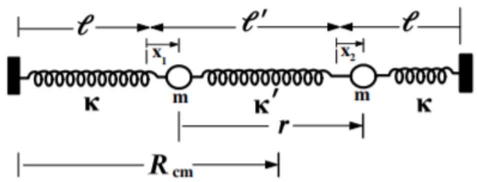

Figure 14.2.1: Two coupled linear oscillators. The equilibrium spring-lengths are l for the outer springs and l' for the coupling spring. The displacement from the stable locations are given by x_1 and x_2. The separation between the two masses is r and the location of the center-of-mass is R_{cm}.

Assuming $m_1 = m_2 = m$, then the equations of motion are

$$\begin{aligned} m\ddot{x}_1 + (\kappa+\kappa')x_1 - \kappa' x_2 = 0 \\ m\ddot{x}_2 + (\kappa+\kappa')x_2 - \kappa' x_1 = 0 \end{aligned} \tag{14.2.1}$$

Assume that the motion for these coupled equations is oscillatory with a solution of the form

$$\begin{aligned} x_1 = B_1 e^{i\omega t} \\ x_2 = B_2 e^{i\omega t} \end{aligned} \tag{14.2.2}$$

where the constants B may be complex to take into account both the magnitude and phase. Substituting these possible solutions into the equations of motion gives

$$\begin{aligned} -m\omega^2 B_1 e^{i\omega t} + (\kappa+\kappa')B_1 e^{i\omega t} - \kappa' B_2 e^{i\omega t} = 0 \\ -m\omega^2 B_2 e^{i\omega t} + (\kappa+\kappa')B_2 e^{i\omega t} - \kappa' B_1 e^{i\omega t} = 0 \end{aligned} \tag{14.2.3}$$

Collecting terms, and cancelling the common exponential factor, gives

$$\begin{aligned} (\kappa+\kappa' - m\omega^2)B_1 - \kappa' B_2 = 0 \\ (\kappa+\kappa' - m\omega^2)B_2 - \kappa' B_1 = 0 \end{aligned} \tag{14.2.4}$$

The existence of a non-trivial solution of these two simultaneous equations requires that the determinant of the coefficients of B_1 and B_2 must vanish, that is

$$\begin{vmatrix} \kappa+\kappa' - m\omega^2 & -\kappa' \\ -\kappa' & \kappa+\kappa' - m\omega^2 \end{vmatrix} = 0 \tag{14.2.5}$$

The expansion of this secular determinant yields

$$(\kappa+\kappa' - m\omega^2)^2 - \kappa'^2 = 0 \tag{14.2.6}$$

Solving for ω gives

$$\omega = \sqrt{\frac{\kappa+\kappa' \pm \kappa'}{m}} \tag{14.2.7}$$

That is, there are two characteristic frequencies (or eigenfrequencies) for the system

$$\omega_1 = \sqrt{\frac{\kappa+2\kappa'}{m}} \tag{14.2.8}$$

$$\omega_2 = \sqrt{\frac{\kappa}{m}} \tag{14.2.9}$$

Since superposition applies for these linear equations, then the general solution can be written as a sum of the terms that account for the two possible values of ω.

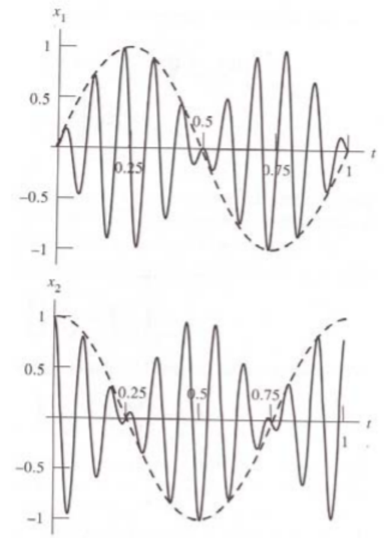

Figure 14.2.2: Displacement of each of two coupled linear harmonic oscillators with $\kappa = 4$ and $\kappa' = 1$ in relative units.

Figure 14.2.2 shows the solutions for a case where $\kappa = 4$ and $\kappa' = 1$, in arbitrary units, with the initial condition that $x_2 = D$, and $x_1 = \dot{x}_1 = \dot{x}_2 = 0$. The two characteristic frequencies are $\omega_1 = \sqrt{\frac{6}{m}}$ and $\omega_2 = \sqrt{\frac{4}{m}}$. The characteristic beats phenomenon is exhibited where the envelope over one complete cycle of the low frequency encompasses several higher frequency oscillations. That is, the solution is

$$x_2(t) = \frac{D}{4}\left[e^{i\omega_1 t} + e^{-i\omega_1 t} + e^{i\omega_2 t} + e^{-i\omega_2 t}\right] = D\cos\left[\left(\frac{\omega_1 + \omega_2}{2}\right)t\right]\cos\left[\left(\frac{\omega_1 - \omega_2}{2}\right)t\right] \tag{14.2.10}$$

while

$$x_1(t) = \frac{D}{4}\left[e^{i\omega_1 t} + e^{-i\omega_1 t} - e^{i\omega_2 t} - e^{-i\omega_2 t}\right] = D\sin\left[\left(\frac{\omega_1 + \omega_2}{2}\right)t\right]\sin\left[\left(\frac{\omega_1 - \omega_2}{2}\right)t\right] \tag{14.2.11}$$

The energy in the two-coupled oscillators flows back and forth between the coupled oscillators as illustrated in Figure 14.2.2.

A better understanding of the energy flow occurring between the two coupled oscillators is given by using a (x_1, x_2) configuration-space plot, shown in Figure 14.3.1. The flow of energy occurring between the two coupled oscillators can be represented by choosing normal-mode coordinates η_1 and η_2 that are rotated by 45° with respect to the spatial coordinates (x_1, x_2). These normal-mode coordinates (η_1, η_2) correspond to the two normal modes of the coupled double-oscillator system.