16.5: Linear Elastic Solids

Elasticity is a property of matter where the atomic forces in matter act to restore the shape of a solid when distorted due to the application of external forces. A perfectly elastic material returns to its original shape if the external force producing the deformation is removed. Materials are elastic when the external forces do not exceed the elastic limit. Above the elastic limit, solids can exhibit plastic flow and concomitant heat dissipation. Such non-elastic behavior in solids occurs when they are subject to strong external forces.

The discussion of linear systems, in chapters 3 and 14, focussed on one dimensional systems, such as the linear chain, where the transverse rigidity of the chain was ignored. An extension of the one-dimensional linear chain to two-dimensional membranes, such as a drum skin, is straightforward if the membrane is thin enough so that the rigidity of the membrane can be ignored. Elasticity for three-dimensional solids requires accounting for the strong elastic forces exerted against any change in shape in addition to elastic forces opposing change in volume. The stiffness of solids to changes in shape, or volume, is best represented using the concepts of stress and strain.

Forces in matter can be divided into two classes;

1. body forces, such as gravity, which act on each volume element, and
2. surface forces which are the forces that act on both sides of any infinitessimal surface element inside the solid.

Surface forces can have components along the normal to the infinitessimal surface, as well as shear components in the plane of the surface element. Typically solids are elastic to both normal and shear components of the surface forces whereas shear forces in liquids and gases lead to fluid flow plus viscous forces due to energy dissipation. As described below, the forces acting on an infinitessimal surface element are best expressed in terms of the stress tensor, while the relative distortion of the shape, or volume, of the body are best expressed in terms of the strain tensor. The moduli of elasticity relate the ratio of the corresponding stress and strain tensors. The moduli of elasticity are constant in linear elastic solids and thus the stress is proportional to the strain providing that the strains do not exceed the elastic limit.

Stress tensor

Consider an infinitessimal surface area $d\mathbf{A}$ of an arbitrary closed volume element dV inside the medium. The surface area element is defined as a vector $d\mathbf{A} = \hat{\mathbf{n}}dA$ where $\hat{\mathbf{n}}$ is the outward normal to the closed surface that encloses the volume element. Assume that $d\mathbf{F}$ is the force element exerted by the outside on the material inside the volume element. The stress tensor $\mathbf{T}$ is defined as the ratio of $d\mathbf{F}$ and $d\mathbf{A}$ where the force vector $d\mathbf{F}$ is given by the inner product of the stress tensor $\mathbf{T}$ and the surface element vector $d\mathbf{A}$. That is,

$$d\mathbf{F} = \mathbf{T}\cdot d\mathbf{A} \tag{16.5.1}$$

Since both $d\mathbf{F}$ and $d\mathbf{A}$ are vectors, then Equation 16.5.1 implies that the stress tensor must be a second-rank tensor as described in appendix 19.5, that is, the stress tensor is analogous to the rotation matrix or the inertia tensor. Note that if $d\mathbf{F}$ and $\hat{\mathbf{n}}d\mathbf{A}$ are colinear, then the stress tensor $\mathbf{T}$ reduces to the conventional pressure P. The general stress tensor equals the momentum flux density and has the dimensions of pressure.

Strain tensor

Forces applied to a solid body can lead to translational, or rotational acceleration, in addition to changing the shape or volume of the body. Elastic forces do not act when an overall displacement $\boldsymbol{\xi}$ of an infinitessimal volume occurs, such as is involved in translational or rotational motion. Elastic forces act to oppose position-dependent differences in the displacement vector $\boldsymbol{\xi}$, that is, the strain depends on the tensor product $\boldsymbol{\nabla}\otimes\boldsymbol{\xi}$. For an elastic medium, the strain depends only on the applied stress and not on the prior loading history.

Consider that the matter at the location $\mathbf{r}$ is subject to an elastic displacement $\boldsymbol{\xi}$, and similarly at a displaced location $\mathbf{r}' = \mathbf{r} + \sum_i \frac{\partial\boldsymbol{\xi}}{\partial x_i}dx_i$ where x_i are cartesian coordinates. The net relative displacement between $\mathbf{r}$ and $\mathbf{r}'$ is given by

$$d\xi^2 = \sum_i (dx_i + d\xi_i)^2 - \sum_i (dx_i)^2 = \sum_{ik}\left[2\left(\frac{d\xi_i}{dx_k} + \frac{d\xi_k}{dx_i}\right) + \frac{d\xi_m}{dx_i}\frac{d\xi_m}{dx_k}\right]dx_i dx_k \tag{16.5.2}$$

Ignoring the second order term $\frac{d\xi_m}{dx_i}\frac{d\xi_m}{dx_k}$ equation gives that the i^{th} component of $d\xi_i$ is

$$d\xi_i = \sum_k \frac{1}{2}\left(\frac{d\xi_i}{dx_k} + \frac{d\xi_k}{dx_i}\right) dx_i dx_k \tag{16.5.3}$$

Define the elements of the strain tensor to be given by

$$\sigma_{ik} = \frac{1}{2}\left(\frac{d\xi_i}{dx_k} + \frac{d\xi_k}{dx_i}\right) \tag{16.5.4}$$

then

$$d\xi_i = \sum_k \sigma_{ik} dx_i dx_k \tag{16.5.5}$$

Thus the strain tensor $\boldsymbol{\sigma}$ is a rank-2 tensor defined as the ratio of the strain vector $\boldsymbol{\xi}$ and the infinitessimal area vector $d\mathbf{A}$.

$$d\boldsymbol{\xi} = \boldsymbol{\sigma} \cdot d\mathbf{A} \tag{16.5.6}$$

where the component form of the rank -2 strain tensor is

$$\boldsymbol{\sigma} = \frac{1}{2}\begin{vmatrix} \frac{d\xi_1}{dx_1} & \frac{d\xi_1}{dx_2} & \frac{d\xi_1}{dx_3} \\ \frac{d\xi_2}{dx_1} & \frac{d\xi_2}{dx_2} & \frac{d\xi_2}{dx_3} \\ \frac{d\xi_3}{dx_1} & \frac{d\xi_3}{dx_2} & \frac{d\xi_3}{dx_3} \end{vmatrix} \tag{16.5.7}$$

The potential-energy density for linear elastic forces is quadratic in the strain components. That is, it is of the form

$$U = \sum_{ijkl} \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} \tag{16.5.8}$$

where C_{ijkl} is a rank-4 tensor. No preferential directions remain for a homogeneous isotropic elastic body which allows for two contractions, thereby reducing the potential energy density to the inner product

$$U = \sum_{ik} \frac{1}{2} D_{ik} (\sigma_{ik})^2 \tag{16.5.9}$$

Moduli of elasticity

The **modulus of elasticity** of a body is defined to be the slope of the stress-strain curve and thus, in principle, it is a complicated rank-4 tensor that characterizes the elastic properties of a material. Thus the general theory of elasticity is complicated because the elastic properties depend on the orientation of the microscopic composition of the elastic matter. The theory simplifies considerably for homogeneous, isotropic linear materials below the elastic limit, where the strain is proportional to the applied stress. That is, the modulus of elasticity then reduces by contractions to a constant scalar value that depends on the properties of the matter involved.

The potential energy density for homogeneous, isotropic, linear material, Equation 16.5.9, can be separated into diagonal and off-diagonal components of the strain tensor. That is,

$$U = \frac{1}{2}\left[\lambda \sum_i (\sigma_{ii})^2 + 2\mu \sum_{ik} (\sigma_{ik})^2\right] \tag{16.5.10}$$

The diagonal first term is the dilation term which corresponds to changes in the volume with no changes in shape. The off-diagonal second term involves the shear terms that correspond to changes of the shape of the body that also changes the volume. The constants λ and μ are Lamé's moduli of elasticity which are positive. The various moduli of elasticity, corresponding to different distortions in the shape and volume of any solid body, can be derived from Lamé's moduli for the material.

The components of the elastic forces can be derived from the gradient of the elastic potential energy, Equation 16.5.10 by use of Gauss' law plus vector differential calculus. The components of the elastic force, derived from the strain tensor $\boldsymbol{\sigma}$, can be associated with the corresponding components of the stress tensor $\mathbf{T}$. Thus, for homogeneous isotropic linear materials, the components of the stress tensor are related to the strain tensor by the relation

$$T_{ij} = \lambda\delta_{ij}\sum_k \frac{\partial\xi_k}{\partial x_k} + \mu\left(\frac{d\xi_i}{dx_j} + \frac{d\xi_j}{dx_i}\right) = \lambda\delta_{ij}\sum_k \sigma_{kk} + 2\mu\sigma_{ij} \tag{16.5.11}$$

where it has been assumed that $\sigma_{ij} = \sigma_{ji}$. The two moduli of elasticity λ and μ are material-dependent constants. Equation 16.5.11 can be written in tensor notation as

$$\mathbf{T} = \lambda tr(\boldsymbol{\sigma})\mathbf{I} + 2\mu\boldsymbol{\sigma} \tag{16.5.12}$$

where $tr(\sigma)$ is the trace of the strain tensor and I is the identity matrix.

Equation 16.5.12can be inverted to give the strain tensor components in terms of the stress tensor components.

$$\sigma_{ij} = \frac{1}{2\mu}\left[T_{ij} - \frac{\lambda}{(3\lambda + 2\mu)}\sum_k T_{kk}\delta_{ij}\right] \tag{16.5.13}$$

The various moduli of elasticity relate combinations of different stress and strain tensor components. The following five elastic moduli are used frequently to describe elasticity in homogeneous isotropic media, and all are related to Lamé's two moduli of elasticity.

1) *Young's modulus* E describes tensile elasticity which is axial stiffness of the length of a body to deformation along the axis of the applied tensile force.

$$E \equiv \frac{T_{11}}{\sigma_{11}} = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)} \tag{16.5.14}$$

2) *Bulk modulus* $B = \frac{\Delta V}{V}$ defines the relative dilation or compression of a bodies volume to pressure applied uniformly in all directions.

$$B = \lambda + \frac{2}{3}\mu \tag{16.5.15}$$

The bulk modulus is an extension of Young's modulus to three dimensions and typically is larger than E. The inverse of the bulk modulus is called the compressibility of the material.

3) *Shear modulus* G describes the shear stiffness of a body to volume-preserving shear deformations. The shear strain σ becomes a deformation angle given by the ratio of the displacement along the axis of the shear force and the perpendicular moment arm. The shear modulus G equals Lamé's constant μ. That is,

$$G = \mu \tag{16.5.16}$$

4) *Poisson's ratio* ν is the negative ratio of the transverse to axial strain. It is a measure of the volume conserving tendency of a body to contract in the directions perpendicular to the axis along which it is stretched. In terms of Lamé's constants, Poisson's ratio equals

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \tag{16.5.17}$$

Note that for a stable, isotropic elastic material, Poisson's ratio is bounded between $-1.0 \leq \nu \leq 0.5$ to ensure that the B, μ and λ moduli have positive values. At the incompressible limit, $\nu = 0.5$, and the bulk modulus and Lame parameter λ are infinite, that is, the compressibility is zero. Typical solids have Poisson's ratios of $\nu \approx 0.05$ if hard and $\nu = 0.25$ if soft.

The stiffness of elastic solids in terms of the elastic moduli of solids can be complicated due to the geometry and composition of solid bodies. Often it is more convenient to express the stiffness in terms of the **spring constant** κ where

$$\kappa = \frac{dF}{dx} \tag{16.5.18}$$

The spring constant is inversely proportional to the length of the spring because the strain of the material is defined to be the *fractional* deformation, not the *absolute* deformation.

16.5.4 Equations of motion in a uniform elastic media

The divergence theorem $(H.8)$ relates the volume integral of the divergence of $\mathbf{T}$ to the vector force density $\mathbf{F}$ acting on the closed surface.

$$\mathbf{F} = \oint \mathbf{T} \cdot d\mathbf{A} = \int \boldsymbol{\nabla} \cdot \mathbf{T} d\tau = \int \mathbf{f} d\tau \tag{16.5.19}$$

That is, the inner product of the del operator, $\boldsymbol{\nabla}$, and the rank-2 stress tensor $\mathbf{T}$, give the vector force density $\mathbf{f}$. This force acting on the enclosed mass $\oint \rho d\tau$, for the closed volume, leads to an acceleration $\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}$. Thus

$$\mathbf{F} = \oint \mathbf{T} \cdot d\mathbf{A} = \int \boldsymbol{\nabla} \cdot \mathbf{T} d\tau = \oint \rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} d\tau \tag{16.5.20}$$

Use Equation 16.5.12to relate the stress tensor $\mathbf{T}$ to the moduli of elasticity gives

$$\rho \frac{\partial^2 \boldsymbol{\xi}_i}{\partial t^2} = \sum_j \left[(\lambda + \mu) \frac{\partial^2 \boldsymbol{\xi}_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 \boldsymbol{\xi}_i}{\partial x_j^2} \right] \tag{16.5.21}$$

where $i = 1, 2, 3$. In general this equation is difficult to solve. However, for the simple case of a plane wave in the $i = 1$ direction, the problem reduces to the following three equations

$$\rho \frac{\partial^2 \boldsymbol{\xi}_1}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 \boldsymbol{\xi}_1}{\partial x_1^2} \tag{16.5.22}$$

$$\rho \frac{\partial^2 \boldsymbol{\xi}_2}{\partial t^2} = \mu \frac{\partial^2 \boldsymbol{\xi}_2}{\partial x_1^2} \tag{16.5.23}$$

$$\rho \frac{\partial^2 \boldsymbol{\xi}_3}{\partial t^2} = \mu \frac{\partial^2 \boldsymbol{\xi}_3}{\partial x_1^2} \tag{16.5.24}$$

Equation 16.5.22 corresponds to a longitudinal wave travelling with velocity $v = \sqrt{\frac{(\lambda+2\mu)}{\rho}}$. Equations 16.5.23, 16.5.24 correspond to two perpendicular transverse waves travelling with velocity $v = \sqrt{\frac{\mu}{\rho}}$. This illustrates the important fact that longitudinal waves travel faster than transverse waves in an elastic solid. Seismic waves in the Earth, generated by earthquakes, exhibit this property. Note that shearing stresses do not exist in ideal liquids and gases since they cannot maintain shear forces and thus $\mu = 0$.
