6.7: Applications to unconstrained systems

Although most dynamical systems involve constrained motion, it is useful to consider examples of systems subject to conservative forces with no constraints . For no constraints, the Lagrange-Euler equations $(6.6.1)$ simplify to $\Lambda_j L = 0$ where $j = 1, 2, ..n$, and the transformation to generalized coordinates is of no consequence.

Example 6.7.1: Motion of a free particle, $U = 0$

The Lagrangian in cartesian coordinates is $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Then

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0$$

Insert these in the Lagrange equation gives

$$\begin{aligned} \Lambda_x L &= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \\ &= \frac{d}{dt} m\dot{x} - 0 = 0 \end{aligned}$$

Thus

$$\begin{aligned} p_x &= m\dot{x} = constant \\ p_y &= m\dot{y} = constant \\ p_z &= m\dot{z} = constant \end{aligned}$$

That is, this shows that the linear momentum is conserved if U is a constant, that is, no forces apply. Note that momentum conservation has been derived without any direct reference to forces.

Example 6.7.2: Motion in a uniform gravitational field

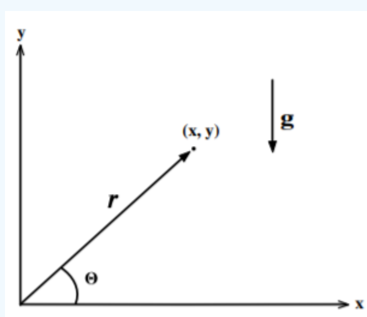

Figure 6.7.1: Motion in a gravitational field

Consider the motion is in the $x - y$ plane. The kinetic energy $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$ while the potential energy is $U = mgy$ where $U(y = 0) = 0$. Thus

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

Using the Lagrange equation for the x coordinate gives

$$\begin{aligned}\Lambda_x L &= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \\ &= \frac{d}{dt} m\dot{x} - 0 \\ &= 0\end{aligned}$$

Thus the horizontal momentum $m\dot{x}$ is conserved and $\ddot{x} = 0$. The y coordinate gives

$$\begin{aligned}\Lambda_y L &= \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} \\ &= \frac{d}{dt} m\dot{y} + mg \\ &= 0\end{aligned}$$

Thus the Lagrangian produces the same results as derived using Newton's Laws of Motion.

$$\ddot{x} = 0$$

$$y = -g$$

The importance of selecting the most convenient generalized coordinates is nicely illustrated by trying to solve this problem using polar coordinates r, θ, where r is radial distance and θ the elevation angle from the x axis as shown in the adjacent figure. Then

$$\begin{aligned}T &= \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m \left(r\dot{\theta}\right)^2 \\ U &= mgr \sin\theta\end{aligned}$$

Thus

$$L = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m \left(r\dot{\theta}\right)^2 - mgr\sin\theta$$

$\Lambda_r L = 0$ for the r coordinate

$$r\dot{\theta}^2 - g\sin\theta - \ddot{r} = 0$$

$\Lambda_\theta L = 0$ for the θ coordinate

$$-gr\cos\theta - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} = 0$$

These equations written in polar coordinates are more complicated than the result expressed in Cartesian coordinates. This is because the potential energy depends directly on the y coordinate, whereas it is a function of both r, θ. This illustrates the freedom for using different generalized coordinates, plus the importance of choosing a sensible set of generalized coordinates.

Example 6.7.3: Central forces

Consider a mass m moving under the influence of a spherically-symmetric, conservative, attractive, inverse-square force. The potential then is

$$U = -\frac{k}{r}$$

It is natural to express the Lagrangian in spherical coordinates for this system. That is,

$$L = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m \left(r\dot{\theta}\right)^2 + \frac{1}{2} m (r\sin\theta\dot{\phi})^2 + \frac{k}{r}$$

$\Lambda_r L = 0$ for the r coordinate gives

$$m\ddot{r} - mr[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2] = \frac{k}{r^2}$$

where the $mr\sin^2\theta\dot{\phi}^2$ term comes from the centripetal acceleration.

$\Lambda_\phi L = 0$ *for the* ϕ coordinate gives

$$\frac{d}{dt}\left(mr^2\sin^2\theta\dot{\phi}\right) = 0$$

This implies that the derivative of the angular momentum about the ϕ axis, $\dot{p}_\phi = 0$ and thus $p_\phi = mr^2\sin^2\theta\dot{\phi}$ is a **constant of motion**.

$\Lambda_\theta L = 0$ for the θ coordinate gives

$$\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\cos\theta\dot{\phi}^2 = 0$$

That is,

$$\dot{p}_\theta = mr^2\sin\theta\cos\theta\dot{\phi}^2 = \frac{p_\phi^2\cos\theta}{2mr^2\sin^3\theta}$$

Note that p_θ is a constant of motion if $p_\phi = 0$ and only the radial coordinate is influenced by the radial form of the central potential.