12.5: Newton's Law of Motion in a Non-Inertial Frame

The acceleration of the system in the rotating inertial frame can be derived by differentiating the general velocity relation for $\mathbf{v}$, Equation 12.4.4, in the fixed frame basis which gives

$$\mathbf{a}_{fix} = \left(\frac{d\mathbf{v}_{fix}}{dt}\right)_{fixed} \tag{12.5.1}$$

$$= \left(\frac{d\mathbf{V}_{fix}}{dt}\right)_{fixed} + \left(\frac{d\mathbf{v}''_{rot}}{dt}\right)_{fixed} + \left(\frac{d\omega}{dt}\right)_{fixed} \times \mathbf{r}'_{mov} + \omega \times \left(\frac{d\mathbf{r}'_{mov}}{dt}\right)_{fixed} \tag{12.5.2}$$

Now we wish to use the general transformation to a rotating frame basis which requires inclusion of the time dependence of the unit vectors in the rotating frame, that is,

$$\left(\frac{d\mathbf{v}''_{rot}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{v}''_{rot}}{dt}\right)_{rotating} + \omega \times \mathbf{v}''_{rot} \tag{12.5.3}$$

$$\left(\frac{d\omega}{dt}\right)_{fixed} \times \mathbf{r}'_{mov} = \left(\frac{d\omega}{dt}\right)_{rot} \times \mathbf{r}'_{mov} \tag{12.5.4}$$

$$\omega \times \left(\frac{d\mathbf{r}'_{mov}}{dt}\right)_{fixed} = \omega \times \mathbf{v}''_{rot} + \omega \times (\omega \times \mathbf{r}'_{mov}) \tag{12.5.5}$$

Using Equations 12.5.3, 12.5.4, 12.5.5 gives

$$\mathbf{a}_{fix} = \mathbf{A}_{fix} + \mathbf{a}''_{rot} + 2\omega \times \mathbf{v}''_{rot} + \omega \times (\omega \times \mathbf{r}'_{mov}) + \dot{\omega} \times \mathbf{r}'_{mov} \tag{12.5.6}$$

where the acceleration in the rotating frame is $\mathbf{a}''_{rot} = \left(\frac{d\mathbf{v}''_{rot}}{dt}\right)_{rot}$ while the velocity is $\mathbf{v}''_{rot} = \left(\frac{\mathbf{r}''_{rot}}{dt}\right)_{rot}$ and $\mathbf{A}_{fix}$ is with respect to the fixed frame.

Newton's laws of motion are obeyed in the inertial frame, that is

$$\mathbf{F}_{fix} = m\mathbf{a}_{fix} \tag{12.5.7}$$

$$= m(\mathbf{A}_{fix} + \mathbf{a}''_{rot} + 2\omega \times \mathbf{v}''_{rot} + \omega \times (\omega \times \mathbf{r}'_{mov}) + \dot{\omega} \times \mathbf{r}'_{mov}) \tag{12.5.8}$$

In the double-primed frame, which may be both rotating and accelerating in translation, one can ascribe an effective force $\mathbf{F}^{eff}_{rot}$ that obeys an effective Newton's law for the acceleration $\mathbf{a}''_{rot}$ in the rotating frame

$$\mathbf{F}^{eff}_{rot} = m\mathbf{a}''_{rot} \tag{12.5.9}$$

$$= \mathbf{F}_{fix} - m(\mathbf{A}_{fix} + 2\omega \times \mathbf{v}''_{rot} + \omega \times (\omega \times \mathbf{r}'_{mov}) + \dot{\omega} \times \mathbf{r}'_{mov}) \tag{12.5.10}$$

Note that the effective force $\mathbf{F}^{eff}_{rot}$ comprises the physical force $\mathbf{F}_{fixed}$ minus four non-inertial forces that are introduced to correct for the fact that the rotating reference frame is a non-inertial frame.