5.E: Calculus of Variations (Exercises)

1. Find the extremal of the functional

$$J(x) = \int_1^2 \frac{\dot{x}^2}{t^3} dt$$

that satisfies $x(1) = 3$ and $x(2) = 18$. Show that this extremal provides the global minimum of J.

2. Consider the use of equations of constraint.

a. A particle is constrained to move on the surface of a sphere. What are the equations of constraint for this system?
b. A disk of mass m and radius R rolls without slipping on the outside surface of a half-cylinder of radius $5R$. What are the equations of constraint for this system?
c. What are holonomic constraints? Which of the equations of constraint that you found above are holonomic?
d. Equations of constraint that do not explicitly contain time are said to be scleronomic. Moving constraints are rheonomic. Are the equations of constraint that you found above scleronomic or rheonomic?

3. For each of the following systems, describe the generalized coordinates that would work best. There may be more than one answer for each system.

a. An inclined plane of mass M is sliding on a smooth horizontal surface, while a particle of mass m is sliding on the smooth inclined surface.
b. A disk rolls without slipping across a horizontal plane. The plane of the disk remains vertical, but it is free to rotate about a vertical axis.
c. A double pendulum consisting of two simple pendula, with one pendulum suspended from the bob of the other. The two pendula have equal lengths and have bobs of equal mass. Both pendula are confined to move in the same plane.
d. A particle of mass m is constrained to move on a circle of radius R. The circle rotates in space about one point on the circle, which is fixed. The rotation takes place in the plane of the circle, with constant angular speed ω, in the absence of a gravitational force.
e. A particle of mass m is attracted toward a given point by a force of magnitude k/r^2, where k is a constant.

4. Looking back at the systems in problem 3, which ones could have equations of constraint? How would you classify the equations of constraint (holonomic, scleronomic, rheonomic, etc.)?

5. Find the extremal of the functional

$$J(x) = \int_0^\pi (2x \sin t - \dot{x}^2) dt$$

that satisfies $x(o) = x(\pi) = 0$. Show that this extremal provides the global maximum of J.

6. Find and describe the path $y = y(x)$ for which the the integral $\int_{x_1}^{x_2} \sqrt{x}\sqrt{1 + (y')^2}\, dx$ is stationary.

7. Find the dimensions of the parallelepiped of maximum volume circumscribed by a sphere of radius R.

8. Consider a single loop of the cycloid having a fixed value of a as shown in the figure. A car released from rest at any point P_0 anywhere on the track between O and the lowest point P, that is, P_0 has a parameter $0 < \theta_0 < \pi$.

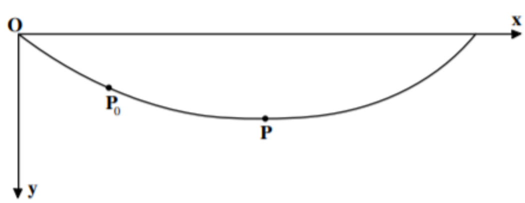

Figure $5.E.1$

a. Show that the time T for the cart to slide from P_0 to P is given by the integral

$$T(P_0 \to P) = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1-\cos\theta}{\cos\theta_0 - \cos\theta}} d\theta$$

b. Prove that this time T is equal to $\pi\sqrt{a/g}$ which is independent of the position P_0.
c. Explain qualitatively how this surprising result can possibly be true.

9. Consider a medium for which the refractive index $n = \frac{a}{r^2}$ where a is a constant and r is the distance from the origin. Use Fermat's Principle to find the path of a ray of light travelling in a plane containing the origin. Hint, use two-dimensional polar coordinates with $\phi = \phi(r)$. Show that the resulting path is a circle through the origin.

10. Find the shortest path between the (x, y, z) points $(0, -1, 0)$ and $(0, 1, 0)$ on a conical surface

$$z = 1 - \sqrt{x^2 + y^2}$$

What is the length of this path? Note that this is the shortest mountain path around a volcano.

11. Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.