# 16.7: Ideal Fluid Dynamics

The distinction between a solid and a fluid is that a fluid flows under shear stress whereas the elasticity of solids oppose distortion and flow. Shear stress in a fluid is opposed by dissipative viscous forces, which depend on velocity, as opposed to elastic solids where the shear stress is opposed by the elastic forces which depend on the displacement. An ideal fluid is one where the viscous forces are negligible, and thus the shear stress **Lamé parameter** $\mu = 0$.

## Continuity Equation

Fluid dynamics requires a different philosophical approach than that used to describe the motion of an ensemble of known solid bodies. The prior discussions of classical mechanics used, as variables, the coordinates of each member of an ensemble of particles with known masses. This approach is not viable for fluids which involve an enormous number of individual atoms as the fundamental bodies of the fluid. The best philosophical approach for describing fluid dynamics is to employ continuum mechanics using definite fixed volume elements $d\tau$ and describe the fluid in terms of macroscopic variables of the fluid such as mass density $\rho$, pressure $P$, and fluid velocity $\mathbf{v}$.

Conservation of fluid mass requires that the rate of change of mass in a fixed volume must equal the net inflow of mass.

$$\frac{d}{dt}\int_{\tau}\rho d\tau + \oint \rho\mathbf{v}\cdot d\mathbf{a} = \mathbf{0} \tag{16.7.1}$$

Using the divergence theorem $(H2)$ allows this to be written as

$$\int_{\tau}\left(\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\right)d\tau = 0 \tag{16.7.2}$$

Mass conservation must hold for any arbitrary volume, therefore the *continuity equation* can be written in the differential form

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0 \tag{16.7.3}$$

## Euler's hydrodynamic equation

The fluid surrounding a volume $\tau$ exerts a net force $\mathbf{F}$ that equals the surface integral of the pressure $\mathbf{P}$. This force can be transformed to a volume integral of $\boldsymbol{\nabla}P$. The net force then will lead to an acceleration of the volume element. That is

$$\mathbf{F} = -\oint Pd\mathbf{a} = -\int \boldsymbol{\nabla}Pd\tau = \int \rho\frac{d\mathbf{v}}{dt}d\tau \tag{16.7.4}$$

Thus the force density $\mathbf{f}$ is given by

$$\mathbf{f} = -\boldsymbol{\nabla}\mathbf{P} = \rho\frac{d\mathbf{v}}{dt} \tag{16.7.5}$$

Note that the acceleration $\frac{d\mathbf{v}}{dt}$ in Equation 16.7.4 refers to the rate of change of velocity for *individual atoms in the fluid*, not the rate of change of fluid velocity at a *fixed point in space*. These two accelerations are related by noting that, during the time $dt$, the change in velocity $d\mathbf{v}$ of a given fluid particle is composed of two parts, namely

1. the change during $dt$ in the velocity at a fixed point in space, and
2. the difference between the velocities at that same instant in time at two points displaced a distance $d\mathbf{r}$ apart, where $d\mathbf{r}$ is the distance moved by a given fluid particle during the time $dt$.

The first part is given by $\frac{\partial\mathbf{v}}{\partial t}dt$ at a given point $(x, y, z)$ in space. The second part equals

$$dx\frac{\partial\mathbf{v}}{\partial x} + dy\frac{\partial\mathbf{v}}{\partial y} + dz\frac{\partial\mathbf{v}}{\partial z} = (d\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{v} \tag{16.7.6}$$

Thus

$$d\mathbf{v} = \frac{\partial\mathbf{v}}{\partial t}dt + (d\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{v} \tag{16.7.7}$$

Divide both sides by $dt$ gives that the acceleration of the atoms in the fluid equals

$$\frac{d\mathbf{v}}{dt}=\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} \tag{16.7.8}$$

Substitute Equation 16.7.8 into 16.7.5 gives

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}=-\frac{1}{\rho}\boldsymbol{\nabla}P \tag{16.7.9}$$

This is Euler's equation for hydrodynamics. The two terms on the left represent the acceleration in the individual fluid components while the right-hand side lists the force density producing the acceleration.

Additional forces can be added to the right-hand side. For example, the gravitational force density $\rho\mathbf{g}$ can be expressed in terms of the gravitational scalar potential $V$ to be

$$\rho\mathbf{g}=-\boldsymbol{\rho}\boldsymbol{\nabla}V \tag{16.7.10}$$

Inclusion of the gravitational field force density in Euler's equation gives

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}=-\frac{1}{\rho}\boldsymbol{\nabla}(P+\rho V) \tag{16.7.11}$$

## Irrotational flow and Bernoulli's equation

Streamlined flow corresponds to **irrotational flow**, that is, $\boldsymbol{\nabla}\times\mathbf{v}=\mathbf{0}$ . Since irrotational flow is curl free, the velocity streamlines can be represented by a scalar potential field $\phi$. That is

$$\mathbf{v}=-\boldsymbol{\nabla}\phi \tag{16.7.12}$$

This scalar potential field $\phi$ can be used to derive the vector velocity field for irrotational flow.

Note that the $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ term in Euler's Equation 16.7.11 can be rewritten using the vector identity

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}=\frac{1}{2}\boldsymbol{\nabla}(v^2)-\mathbf{v}\times\boldsymbol{\nabla}\times\mathbf{v} \tag{16.7.13}$$

Inserting Equation 16.7.13 into Euler's Equation 16.7.11 then gives.

$$\frac{\partial\mathbf{v}}{\partial t}=\mathbf{v}\times\boldsymbol{\nabla}\times\mathbf{v}-\frac{1}{\rho}\boldsymbol{\nabla}\left(\frac{1}{2}\rho v^2+P+\rho V\right) \tag{16.7.14}$$

Potential flow corresponds to time independent irrotational flow, that is, both $\frac{\partial\mathbf{v}}{\partial t}=0$ and $\boldsymbol{\nabla}\times\mathbf{v}=0$ . For potential flow Equation 16.7.14 reduces to

$$\boldsymbol{\nabla}\left(\frac{1}{2}\rho v^2+P+\rho V\right)=0$$

which implies that

$$\left(\frac{1}{2}\rho v^2+P+\rho V\right)=\text{ constant} \tag{16.7.15}$$

This is the famous Bernoulli's equation that relates the interplay of the fluid velocity, pressure and gravitational energy. Bernoulli's equation plays important roles in both hydrodynamics and aerodynamics.

## Gas flow

Fluid dynamics applied to gases is a straightforward extension of fluid dynamics that employs standard thermodynamical concepts. The following example illustrates the application of fluid mechanics for calculating the velocity of sound in a gas.

### Example 16.7.1: Acoustic Waves in a Gas

Propagation of acoustic waves in a gas provides an example of using the three-dimensional Lagrangian density. Only longitudinal waves occur in a gas and the velocity is given by thermodynamics of the gas. Let the displacement of each gas molecule be designated by the general coordinate $\mathbf{q}$ with corresponding velocity $\dot{\mathbf{q}}$. Let the gas density be $\rho$, then the kinetic energy density $(KED)$ of an infinitessimal volume of gas $\Delta\tau$ is given by

$$\Delta(KED) = \frac{1}{2}\rho_0 \dot{\mathbf{q}}^2$$

The rapid contractions and expansions of the gas in an acoustic wave occur adiabatically such that the product $PV^{\gamma}$ is a constant, where

$$\gamma = \frac{\text{specific heat at constant pressure}}{\text{specific heat at constant volume}}.$$

Therefore the change in potential energy density $\Delta(PED)$ is given to second order by

$$\Delta(PED) = \frac{1}{\tau_0}\int_{V_0}^{V_0+\Delta V} P d\tau = \frac{P_0}{\tau_0}\Delta\tau + \frac{1}{2\tau_0}\left(\frac{\partial P}{\partial \tau}\right)_0 (\Delta\tau)^2 = \frac{P_0}{\tau_0}\Delta\tau - \frac{1}{2\tau_0}\left(\gamma\frac{P_0}{\tau_0}\right)(\Delta\tau)^2$$

Since the volume and density are related by

$$\tau_o = \frac{M}{\rho_0}$$

then the fractional change in the density $\sigma$ is related to the density by

$$\rho = \rho_0(1+\sigma)$$

This implies that the potential energy density $(PED)$ is given by

$$\Delta(PED) = \left[P_0\sigma + \gamma\frac{P_0}{2}\sigma^2\right]$$

The mass flowing out of the volume $V_0$ must equal the fractional change in density of the volume, that is

$$\rho_0 \int \mathbf{q}\cdot\mathbf{dS} = -\boldsymbol{\rho}_0 \int \sigma d\tau$$

The divergence theorem gives that

$$\int \mathbf{q}\cdot\mathbf{dS} = \int \nabla\cdot\mathbf{q} d\tau = -\int \sigma d\tau$$

Thus the density $\sigma$ is given by minus the divergence of $\mathbf{q}$

$$\sigma = -\nabla\cdot\mathbf{q}$$

This allows the potential energy density to be written as

$$\Delta(PED) = -P_0\nabla\cdot\mathbf{q} + \frac{\gamma P_0}{2}(\nabla\cdot\mathbf{q})^2$$

Combining the kinetic energy density and the potential energy density gives the complete Lagrangian density for an acoustic wave in a gas to be

$$\mathfrak{L} = \frac{1}{2}\rho_0\dot{\mathbf{q}}^2 + P_0\nabla\cdot\mathbf{q} - \frac{\gamma P_0}{2}(\nabla\cdot\mathbf{q})^2$$

Inserting this Lagrangian density in the corresponding equations of motion, equation $(16.3.16)$, gives that

$$\nabla^2\mathbf{q} - \frac{\rho_0}{\gamma P_0}\frac{d^2\mathbf{q}}{dt^2} = 0$$

where $P_0$ and $\rho_0$ are the ambient pressure and density of the gas. This is the wave equation where the phase velocity of sound is given by

$$v_{phase} = \sqrt{\frac{\gamma P_0}{\rho_0}}$$