13.11: Angular Momentum and Angular Velocity Vectors

The angular momentum is a primary observable for rotation. As discussed in chapter 13.5, the angular momentum $\mathbf{L}$ is compactly and elegantly written in matrix form using the tensor algebra relation

$$\begin{aligned}\mathbf{L} &= \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &= \{\mathbf{I}\} \cdot \boldsymbol{\omega} \end{aligned} \tag{13.11.1}$$

where $\boldsymbol{\omega}$ is the angular velocity, $\{\mathbf{I}\}$ the inertia tensor, and $\mathbf{L}$ the corresponding angular momentum.

Two important consequences of Equation 13.11.1are that:

- The angular momentum $\mathbf{L}$ and angular velocity $\boldsymbol{\omega}$ are not necessarily colinear.
- In general the Principal axis system of the rotating rigid body is not aligned with either the angular momentum or angular velocity vectors.

An exception to these statements occurs when the angular velocity $\boldsymbol{\omega}$ is aligned along a principal axes for which the inertia tensor is diagonal, i.e. $I_{ij} = I_i\delta_{ij}$, and then both $\mathbf{L}$ and $\boldsymbol{\omega}$ point along this principal axis. In general the angular momentum $\mathbf{L}$ and angular velocity $\boldsymbol{\omega}$ precess around each other. An important special case is for torque-free systems where Noether's theorem implies that the angular momentum vector $\mathbf{L}$ is conserved both in magnitude and amplitude. In this case, the angular velocity $\boldsymbol{\omega}$, and the Principal axis system, both precesses around the angular momentum vector $\mathbf{L}$. That is, the body appears to tumble with respect to the laboratory fixed frame. Understanding rigid-body rotation requires care not to confuse the body-fixed Principal axis coordinate frame, used to determine the inertia tensor, and the fixed laboratory frame where the motion is observed.

Example 13.11.1: Rotation about the center of mass of a solid cube

It is illustrative to use the inertia tensors of a uniform cube to compute the angular momentum for any applied angular velocity vector ω using Equation 13.11.1. If the angular velocity is along the x axis, then using the inertia tensor for a solid cube, derived earlier, in Equation 13.11.1gives the angular momentum to be

$$\begin{aligned}\mathbf{L} &= \{\mathbf{I}\} \cdot \boldsymbol{\omega} \\ &= \frac{1}{6}Mb^2\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{6}Mb^2\omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\end{aligned}$$

This shows that $\mathbf{L}$ and $\boldsymbol{\omega}$ are colinear and thus the x axis is a principal axis. By symmetry, the y and z body fixed axis also must be principal axes.

Consider that the body is rotated about a diagonal of the cube for which the center of mass will be on the rotation axis. Then the angular velocity vector is written as $\boldsymbol{\omega} = \omega\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ where the components of $\omega_x = \omega_y = \omega_z = \omega\frac{1}{\sqrt{3}}$ with the angular velocity magnitude $\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = \omega$.

$$\begin{aligned}\mathbf{L} &= \{\mathbf{I}\}\cdot\boldsymbol{\omega} \\ &= \frac{1}{6}Mb^2\omega\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} \\ &= \frac{1}{6}Mb^2\omega\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \\ &= \frac{1}{6}Mb^2\boldsymbol{\omega}\end{aligned}$$

Note that $\mathbf{L}$ and $\boldsymbol{\omega}$ again are colinear showing it also is a principal axis. Moreover, the magnitude of $\mathbf{L}$ is identical for orientations of the rotation axes ω passing through the center of mass when centered on either one face, or the diagonal, of the cube implying that the principal moments of inertia about these axes are identical. This illustrates the important property that, when the three principal moments of inertia are identical, then any orientation of the coordinate system is an equally good principal axis system. That is, this corresponds to the spherical top where all orientations are principal axes, not just along the obvious symmetry axes.

Example 13.11.2: Rotation about the corner of the cube

Let us repeat the above exercise for rotation about one corner of the cube. Consider that the angular velocity is along the x axis. Then example $(13.8.2)$ gives the angular momentum to be

$$\begin{aligned}\mathbf{L} &= \{\mathbf{I}\}\cdot\boldsymbol{\omega} \\ &= \frac{1}{12}Mb^2\omega\begin{pmatrix}+8&-3&-3\\-3&+8&-3\\-3&-3&+8\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} \\ &= \frac{1}{12}Mb^2\boldsymbol{\omega}\begin{pmatrix}+8\\-3\\-3\end{pmatrix}\end{aligned}$$

The angular momentum is far from being aligned with the axis ω, that is, it is not a principal axis.

Consider that the body is rotated with the angular velocity aligned along a diagonal of the cube through the center of mass on this axis. Then the angular velocity is written as $\boldsymbol{\omega} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$ where the components of $\omega_x = \omega_y = \omega_z = \omega\frac{1}{\sqrt{3}}$ ensuring that the magnitude equals $\sqrt{\omega_x^2+\omega_y^2+\omega_z^2} = \omega$.

$$\begin{aligned}\mathbf{L} &= \{\mathbf{I}\}\cdot\boldsymbol{\omega} \\ &= \frac{1}{12}Mb^2\omega\frac{1}{\sqrt{3}}\begin{pmatrix}+8&-3&-3\\-3&+8&-3\\-3&-3&+8\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} \\ &= \frac{1}{12}Mb^2\omega\frac{1}{\sqrt{3}}\begin{pmatrix}2\\2\\2\end{pmatrix} \\ &= \frac{1}{6}Mb^2\boldsymbol{\omega}\end{aligned}$$

This is a principal axis since $\mathbf{L}$ and ω again are colinear and the angular momentum is the same as for any axis through the center of mass of a uniform solid cube due to the high symmetry of the cube. If the angular velocity is perpendicular to the diagonal of the cube, then, for either of these perpendicular axes, the relation between L and ω is given by

$$\mathbf{L} = \frac{1}{12}Mb^2\omega\frac{1}{\sqrt{2}}\begin{pmatrix} +8 & -3 & -3 \\ -3 & +8 & -3 \\ -3 & -3 & +8 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ +1 \\ 0 \end{pmatrix}$$
$$= \frac{1}{12}Mb^2\omega\frac{1}{\sqrt{2}}\begin{pmatrix} -11 \\ +11 \\ 0 \end{pmatrix}$$
$$= \frac{11}{12}Mb^2\omega\begin{pmatrix} -1 \\ +1 \\ 0 \end{pmatrix}$$

Note that this must be a principal axis for rotation about a corner of the cube since $\mathbf{L}$ and $\boldsymbol{\omega}$ are colinear. The angular momentum is the same for both possible orientations of ω that are perpendicular to the diagonal through the center of mass. Diagonalizing the inertia tensor in example $(13.8.2)$ also gave the above result with the symmetry axis along the diagonal of the cube.

This example illustrates that it is not necessary to diagonalize the inertia tensor matrix to obtain the principal axes. The corner of the cube has three mutually perpendicular principal axes independent of the choice of a body-fixed coordinate frame. The advantage of the principal axis coordinate frame is that the inertia tensor is diagonal making evaluation of the angular momentum trivial. That is, there is no physics associated with the orientation chosen for the body-fixed coordinate frame, this frame only determines the ratio of the components of the inertia tensor along the chosen coordinates. Note that, if a body has an obvious symmetry, then intuition is a powerful way to identify the principal axis frame.