6.5: Constrained Systems

The motion for systems subject to constraints is difficult to calculate using Newtonian mechanics because all the unknown constraint forces must be included explicitly with the active forces in order to determine the equations of motion. Lagrangian mechanics avoids these difficulties by allowing selection of independent generalized coordinates that incorporate the correlated motion induced by the constraint forces. This allows the constraint forces acting on the system to be ignored by reducing the system to a minimal set of generalized coordinates. The holonomic constraint forces can be determined using the Lagrange multiplier approach, or all constraint forces can be determined by including them as generalized forces, as described below.

Choice of generalized coordinates

As discussed in chapter 5.8, the flexibility and freedom for selection of generalized coordinates is a considerable advantage of Lagrangian mechanics when handling constrained systems. The generalized coordinates can be any set of *independent* variables that completely specify the scalar action functional, equation $(6.4.1)$. The generalized coordinates are not required to be orthogonal as is required when using the vectorial Newtonian approach. The secret to using generalized coordinates is to select coordinates that are perpendicular to the constraint forces so that the constraint forces do no work. Moreover, if the constraints are rigid, then the constraint forces do no work in the direction of the constraint force. As a consequence, the constraint forces do not contribute to the action integral and thus the $\sum_i^n \mathbf{f}_i^C \cdot \delta \mathbf{r}_i$ term in equation $(6.3.2)$ can be omitted from the action integral. Generalized coordinates allow reducing the number of unknowns from n to $s = n - m$ when the system has m holonomic constraints. In addition, generalized coordinates facilitate using both the Lagrange multipliers, and the generalized forces, approaches for determining the constraint forces.

Minimal set of generalized coordinates

The set of n generalized coordinates q_i are used to describe the motion of the system. No restrictions have been placed on the nature of the constraints other than they are workless for a virtual displacement. *If the m constraints are holonomic,* then it is possible to find sets of $s = n - m$ *independent generalized coordinates* q_j that contain the m constraint conditions implicitly in the transformation equations

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, q_3 \dots, q_s, t) \tag{6.5.1}$$

For the case of $s = n - m$ unknowns, *any virtual displacement δq_j is independent of δq_k*, therefore the only way for $(6.3.27)$ to hold is for the term in brackets to vanish for each value of j, that is

$$\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right\} = Q_j^{EX} \tag{6.5.2}$$

where $j = 1, 2, 3, .. s$. These are the **Lagrange equations** for the minimal set of s *independent* generalized coordinates**.**

If all the generalized forces are conservative plus velocity independent, and are included in the potential U, and $Q_j^{EX} = 0$, then $6.5.2$ simplifies to

$$\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right\} = 0 \tag{6.5.3}$$

This is Euler's differential equation, derived earlier using the calculus of variations. Thus d'Alembert's Principle leads to a solution that minimizes the action integral $\delta \int_{t_1}^{t_2} L dt = 0$ as stated by Hamilton's Principle.

Lagrange multipliers approach

Equation $(6.3.27)$ sums over all n coordinates for N particles, providing n equations of motion. If the m constraints are holonomic they can be expressed by m algebraic equations of constraint

$$g_k(q_1, q_2, .. q_n, t) = 0 \tag{6.5.4}$$

where $k = 1, 2, 3, \dots m$. Kinematic constraints can be expressed in terms of the infinitessimal displacements of the form

$$\sum_{j=1}^{n} \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t) dq_j + \frac{\partial g_k}{\partial t} dt = 0 \tag{6.5.5}$$

where $k=1,2,3,\ldots m$, $j=1,2,3,\ldots n$, and where the $\frac{\partial g_k}{\partial q_j}$, and $\frac{\partial g_k}{\partial t}$ are functions of the generalized coordinates q_j, described by the vector $\mathbf{q}$, that are derived from the equations of constraint. As discussed in chapter 5.7, if 6.5.5 represents the total differential of a function, then it can be integrated to give a holonomic relation of the form of Equation 6.5.4. However, if 6.5.5 is not the total differential, then it can be integrated only after having solved the full problem. If $\frac{\partial g_k}{\partial t}=0$ then the k^{th} constraint is scleronomic.

The discussion of Lagrange multipliers in chapter $5.9.1$, showed that, for virtual displacements δq_j, the correlation of the generalized coordinates, due to the constraint forces, can be taken into account by multiplying 6.5.5 by unknown Lagrange multipliers λ_k and summing over all m constraints. Generalized forces can be partitioned into a Lagrange multiplier term plus a remainder force. That is

$$Q_j^{EX}=\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)+Q_j^{EXC} \tag{6.5.6}$$

since by definition $\delta t=0$ for virtual displacements.

Chapter $5.9.1$ showed that holonomic forces of constraint can be taken into account by introducing the Lagrange undetermined multipliers approach, which is equivalent to defining an extended Lagrangian $L'(\mathbf{q},\dot{\mathbf{q}},\lambda,t)$ where

$$L'(\mathbf{q},\dot{\mathbf{q}},\lambda,t)=L(\mathbf{q},\dot{\mathbf{q}},t)+\sum_{k=1}^{m}\sum_{j=1}^{n}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t) \tag{6.5.7}$$

Finding the extremum for the extended Lagrangian $L'(\mathbf{q},\dot{\mathbf{q}},\lambda,t)$ using $(6.4.2)$ gives

$$\sum_{j}^{n}\left[\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right\}-\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)-Q_j^{EXC}\right]\delta q_j=0 \tag{6.5.8}$$

where Q_j^{EXC} is the remaining part of the generalized force Q_j after subtracting both the part of the force absorbed in the potential energy U, which is buried in the Lagrangian L, as well as the holonomic constraint forces which are included in the Lagrange multiplier terms $\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)$. The m Lagrange multipliers λ_k can be chosen arbitrarily in 6.5.8. Utilizing the free choice of the m Lagrange multipliers λ_k allows them to be determined in such a way that the coefficients of the first m infinitessimals, i.e. the square brackets vanish. Therefore the expression in the square bracket must vanish for each value of $1\leq j\leq m$. Thus it follows that

$$\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right\}-\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)-Q_j^{EXC}=0 \tag{6.5.9}$$

when $j=1,2,..m$. Thus 6.5.8 reduces to a sum over the remaining coordinates between $m+1\leq j\leq n$

$$\sum_{j=m+1}^{n}\left[\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right\}-\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)-Q_j^{EXC}\right]\delta q_j=0 \tag{6.5.10}$$

In Equation 6.5.10 the $s=n-m$ infinitessimals δq_j can be chosen freely since the $s=n-m$ degrees of freedom are *independent*. Therefore the expression in the square bracket must vanish for each value of $m+1\leq j\leq n$. Thus it follows that

$$\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right\}-\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)-Q_j^{EXC}=0 \tag{6.5.11}$$

where $j=m+1,m+2,..n$. Combining equations 6.5.9 and 6.5.11 then gives the important general relation that for $1\leq j\leq n$

$$\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right\}=\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)+Q_j^{EXC} \tag{6.5.12}$$

To summarize, the Lagrange multiplier approach 6.5.12 automatically solves the n equations plus the m holonomic equations of constraint, which determines the $n+m$ unknowns, that is, the n coordinates plus the m forces of constraint. The beauty of the Lagrange multipliers is that all n variables, plus the m constraint forces, are found simultaneously by using the calculus of variations to determine the extremum for the expanded Lagrangian $L'(\mathbf{q},\dot{\mathbf{q}},\lambda,t)$.

Generalized forces approach

The two right-hand terms in 6.5.12 can be understood to be those forces acting on the system that are not absorbed into the scalar potential U component of the Lagrangian L. The Lagrange multiplier terms $\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t)$ account for the holonomic forces of constraint that are not included in the conservative potential or in the generalized forces Q_j^{EXC}. The generalized force

$$Q_j^{EXC} = \sum_i^n \mathbf{F}_i^A \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{6.5.13}$$

is the sum of the components in the q_j direction for all external forces that have not been taken into account by the scalar potential or the Lagrange multipliers. Thus the non-conservative generalized force Q_j^{EXC} contains non-holonomic constraint forces, including dissipative forces such as drag or friction, that are not included in U, or used in the Lagrange multiplier terms to account for the holonomic constraint forces.

The concept of generalized forces is illustrated by the case of spherical coordinate systems. The attached table gives the displacement elements δq_i, (taken from table $C4$) and the generalized force for the three coordinates. Note that Q_i has the dimensions of force and $Q_i . \delta q_i$ has the units of energy. By contrast equation $(6.3.13)$ gives that $Q_\theta = F_\theta r$ and $Q_\phi = F_\phi r$ which have the dimensions of torque. However, $Q_\theta \delta\theta$ and $Q_\phi \delta\phi$ both have the dimensions of energy as is required in equation $(6.3.13)$. This illustrates that the units used for generalized forces depend on the units of the corresponding generalized coordinate.

Unit vectors	δq_i	Q_i	$Q_i \cdot \delta q_i$
$\hat{r}$	$\hat{\mathbf{r}} dr$	$\hat{\mathbf{r}} F_r$	$F_r dr$
$\hat{\theta}$	$\hat{\theta} r d\theta$	$\hat{\theta} F_\theta r$	$F_\theta r d\theta$
$\hat{\phi}$	$\hat{\phi} r \sin\theta d\phi$	$\hat{\phi} F_\phi r \sin\theta$	$F_\phi r \sin\theta d\phi$