12.3: Rotating Reference Frame

Consider a rotating frame of reference which will be designated as the double-primed (rotating) frame to differentiate it from the non-rotating primed (moving) frame, since both of which may be undergoing translational acceleration relative to the inertial fixed unprimed frame as described in Figure $12.2.1$.

Spatial time derivatives in a rotating, non-translating, reference frame

For simplicity assume that $\mathbf{R}_{fix} = \mathbf{V}_{fix} = 0$, that is, the primed reference frame is stationary and identical to the fixed stationary unprimed frame. The double-primed (rotating) frame is a non-inertial frame rotating with respect to the origin of the fixed primed frame.

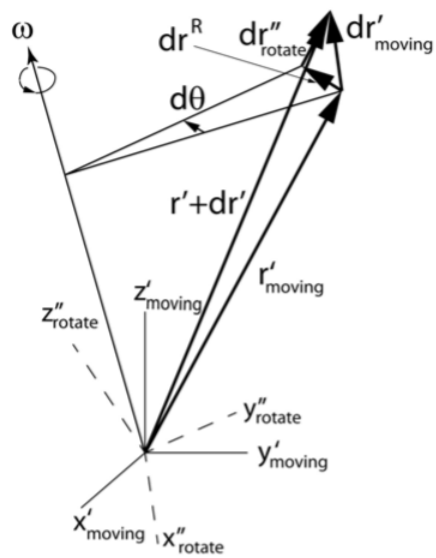

Figure 12.3.1: Infinitessimal displacement in the non rotating primed frame and in the rotating double-primed reference frame frame.

Appendix $19.4.2C$ shows that an infinitessimal rotation $d\theta$ about an instantaneous axis of rotation leads to an infinitessimal displacement $d\mathbf{r}^R$ where

$$d\mathbf{r}^R = d\theta \times \mathbf{r}'_{mov} \tag{12.3.1}$$

Consider that during a time dt, the position vector in the fixed primed reference frame moves by an arbitrary infinitessimal distance $d\mathbf{r}'_{mov}$. As illustrated in Figure $12.3.1$, this infinitessimal distance in the primed non-rotating frame can be split into two parts:

a. $d\mathbf{r}^R = d\theta \times \mathbf{r}'_{mov}$ which is due to rotation of the rotating frame with respect to the translating primed frame.
b. $(d\mathbf{r}''_{rot})$ which is the motion *with respect to the rotating (double-primed) frame*.

That is, the motion has been arbitrarily divided into a part that is due to the rotation of the double-primed frame, plus the vector displacement measured in this rotating (double-primed) frame. It is always possible to make such a decomposition of the displacement as long as the vector sum can be written as

$$d\mathbf{r}'_{mov} = d\mathbf{r}''_{rot} + d\theta \times \mathbf{r}'_{mov} \tag{12.3.2}$$

Since $d\theta = \omega dt$ then the time differential of the displacement, Equation 12.3.2, can be written as

$$\left(\frac{d\mathbf{r}'}{dt}\right)_{mov} = \left(\frac{d\mathbf{r}''}{dt}\right)_{rot} + \omega \times \mathbf{r}'_{mov} \tag{12.3.3}$$

The important conclusion is that a velocity measured in a non-rotating reference frame $\left(\frac{d\mathbf{r}'}{dt}\right)_{mov}$ can be expressed as the sum of the velocity $\left(\frac{d\mathbf{r}''}{dt}\right)_{rot}$, measured relative to a rotating frame, plus the term $\omega \times \mathbf{r}'_{mov}$ which accounts for the rotation of the frame.

The division of the $d\mathbf{r}'_{rot}$ vector into two parts, a part due to rotation of the frame plus a part with respect to the rotating frame, is valid for any vector as shown below.

General vector in a rotating, non-translating, reference frame

Consider an arbitrary vector $\mathbf{G}$ which can be expressed in terms of components along the three unit vector basis $\hat{\mathbf{e}}_i^{fix}$ in the fixed inertial frame as

$$\mathbf{G} = \sum_{i=1}^{3} G_i^{fix} \hat{\mathbf{e}}_i^{fix} \tag{12.3.4}$$

Neglecting translational motion, then it can be expressed in terms of the three unit vectors in the non-inertial rotating frame unit vector basis $\hat{\mathbf{e}}_i^{rot}$ as

$$\mathbf{G} = \sum_{i=1}^{3} (G_i)_{rot} \hat{\mathbf{e}}_i^{rot} \tag{12.3.5}$$

Since the unit basis vectors $\hat{\mathbf{e}}_i^{rot}$ are constant in the rotating frame, that is,

$$\left(\frac{d\hat{\mathbf{e}}_i^{rot}}{dt}\right)_{rot} = 0 \tag{12.3.6}$$

then the time derivatives of $\mathbf{G}$ in the rotating coordinate system $\hat{\mathbf{e}}_i^{rot}$ can be written as

$$\left(\frac{d\mathbf{G}}{dt}\right)_{rot} = \sum_{i-1}^{3} \left(\frac{dG_i}{dt}\right)_{rot} \hat{\mathbf{e}}_i^{rot} \tag{12.3.7}$$

The inertial-frame time derivative taken with components along the rotating coordinate basis $\hat{\mathbf{e}}_i^{rot}$, Equation 12.3.5, is

$$\left(\frac{d\mathbf{G}}{dt}\right)_{fix} = \sum_{i-1}^{3} \left(\frac{dG_i}{dt}\right)_{rot} \hat{\mathbf{e}}_i^{rot} + (G_i)_{rot} \frac{d\hat{\mathbf{e}}_i^{rot}}{dt} \tag{12.3.8}$$

Substitute the unit vector $\hat{\mathbf{e}}^{rot}$ for $\mathbf{r}'_{mov}$ in Equation 12.3.3, plus using Equation 12.3.6, gives that

$$\left(\frac{d\hat{\mathbf{e}}^{rot}}{dt}\right)_{fix} = \omega \times \hat{\mathbf{e}}^{rot} \tag{12.3.9}$$

Substitute this into the second term of Equation 12.3.8 gives

$$\left(\frac{d\mathbf{G}}{dt}\right)_{fix} = \left(\frac{d\mathbf{G}}{dt}\right)_{rot} + \omega \times \mathbf{G} \tag{12.3.10}$$

This important identity relates the time derivatives of any vector expressed in both the inertial frame and the rotating non-inertial frame bases. Note that the $\omega \times \mathbf{G}$ term originates from the fact that the unit basis vectors of the rotating reference frame are time dependent with respect to the non-rotating frame basis vectors as given by Equation 12.3.9. Equation 12.3.10 is used extensively for problems involving rotating frames. For example, for the special case where $\mathbf{G} = \mathbf{r}'$, then Equation 12.3.10 relates the velocity vectors in the fixed and rotating frames as given in Equation 12.3.3.

Another example is the vector $\dot{\omega}$

$$\dot{\omega} = \left(\frac{d\omega}{dt}\right)_{fix} = \left(\frac{d\omega}{dt}\right)_{rot} + \omega \times \omega = \left(\frac{d\omega}{dt}\right)_{rot} = \dot{\omega} \tag{12.3.11}$$

That is, the angular acceleration $\dot{\omega}$ has the same value in both the fixed and rotating frames of reference.