9.3: Lagrangian

Standard Lagrangian

Lagrangian mechanics, as introduced in chapter 6, was based on the concepts of kinetic energy and potential energy. d'Alembert's principle of virtual work was used to derive Lagrangian mechanics in chapter 6 and this led to the definition of the *standard Lagrangian*. That is, the *standard Lagrangian* was defined in chapter 6.2 to be the difference between the kinetic and potential energies.

$$L(\mathbf{q},\dot{\mathbf{q}},t) = T(\dot{\mathbf{q}},t) - U(\mathbf{q},t) \tag{9.3.1}$$

Hamilton extended Lagrangian mechanics by defining Hamilton's Principle, equation $(9.1.2)$, which states that *a dynamical system follows a path for which the action functional is stationary, that is, the time integral of the Lagrangian*. Chapter 6 showed that using the standard Lagrangian for defining the action functional leads to the Euler-Lagrange variational equations

$$\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right\} = Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q},t) \tag{9.3.2}$$

The Lagrange multiplier terms handle the holonomic constraint forces and Q_j^{EXC} handles the remaining excluded generalized forces. Chapters $6-8$ showed that the use of the standard Lagrangian, with the Euler-Lagrange equations 9.3.2, provides a remarkably powerful and flexible way to derive second-order equations of motion for dynamical systems in classical mechanics.

Note that the Euler-Lagrange equations, expressed solely in terms of the standard Lagrangian 9.3.2, that is, excluding the $Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q},t)$ terms, are valid only under the following conditions:

1. The forces acting on the system, apart from any forces of constraint, must be derivable from scalar potentials.
2. The equations of constraint must be relations that connect the coordinates of the particles and may be functions of time, that is, the constraints are holonomic.

The $Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q},t)$ terms extend the range of validity of using the standard Lagrangian in the Lagrange-Euler equations by introducing constraint and omitted forces explicitly.

Chapters $6-8$ exploited Lagrangian mechanics based on use of the standard definition of the Lagrangian. The present chapter will show that the powerful Lagrangian formulation, using the standard Lagrangian, can be extended to include alternative non-standard Lagrangians that may be applied to dynamical systems where use of the standard definition of the Lagrangian is inapplicable. If these non-standard Lagrangians satisfy Hamilton's Action Principle, $(9.1.2)$, then they can be used with the Euler-Lagrange equations to generate the correct equations of motion, even though the Lagrangian may not have the simple relation to the kinetic and potential energies adopted by the standard Lagrangian. Currently, the development and exploitation of non-standard Lagrangians is an active field of Lagrangian mechanics.

Gauge invariance of the standard Lagrangian

Note that the standard Lagrangian is not unique in that there is a continuous spectrum of equivalent standard Lagrangians that all lead to identical equations of motion. This is because the Lagrangian L is a scalar quantity that is invariant with respect to coordinate transformations. The following transformations change the standard Lagrangian, but leave the equations of motion unchanged.

1. The Lagrangian is indefinite with respect to addition of a constant to the scalar potential which cancels out when the derivatives in the Euler-Lagrange differential equations are applied.
2. The Lagrangian is indefinite with respect to addition of a constant kinetic energy.
3. The Lagrangian is indefinite with respect to addition of a total time derivative of the form $L_2 \rightarrow L_1 + \frac{d}{dt}[\Lambda(q_i,t)]$, for any differentiable function $\Lambda(q_i t)$ of the generalized coordinates plus time, that has continuous second derivatives.

This last statement can be proved by considering a transformation between two related standard Lagrangians of the form

$$L_2(\mathbf{q},\dot{q},t) = L_1(\mathbf{q},\dot{q},t) + \frac{d\Lambda(\mathbf{q},t)}{dt} = L_1(\mathbf{q},\dot{q},t) + \left(\frac{\partial\Lambda(\mathbf{q},t)}{\partial q_j}\dot{q}_j + \frac{\partial\Lambda(\mathbf{q},t)}{\partial t}\right) \tag{9.3.3}$$

This leads to a standard Lagrangian L_2 that has the same equations of motion as L_1 as is shown by substituting Equation 9.3.3 into the Euler-Lagrange equations. That is,

$$\frac{d}{dt}\left(\frac{\partial L_2}{\partial \dot{q}_j}\right)-\frac{\partial L_2}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}_j}\right)-\frac{\partial L_1}{\partial q_j}+\frac{\partial^2\Lambda(\mathbf{q},t)}{\partial t\partial q_j}-\frac{\partial^2\Lambda(\mathbf{q},t)}{\partial t\partial q_j} \tag{9.3.4}$$

$$=\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}_j}\right)-\frac{\partial L_1}{\partial q_j} \tag{9.3.5}$$

Thus even though the related Lagrangians L_1 and L_2 are different, they are completely equivalent in that they generate identical equations of motion.

There is an unlimited range of equivalent standard Lagrangians that all lead to the same equations of motion and satisfy the requirements of the Lagrangian. That is, there is no unique choice among the wide range of equivalent standard Lagrangians expressed in terms of generalized coordinates. This discussion is an example of gauge invariance in physics.

Modern theories in physics describe reality in terms of potential fields. Gauge invariance, which also is called gauge symmetry, is a property of field theory for which different underlying fields lead to identical observable quantities. Well-known examples are the static electric potential field and the gravitational potential field where any arbitrary constant can be added to these scalar potentials with zero impact on the observed static electric field or the observed gravitational field. Gauge theories constrain the laws of physics in that the impact of gauge transformations must cancel out when expressed in terms of the observables. Gauge symmetry plays a crucial role in both classical and quantal manifestations of field theory, e.g. it is the basis of the Standard Model of electroweak and strong interactions.

Equivalent Lagrangians are a clear manifestation of gauge invariance as illustrated by equations 9.3.3, 9.3.4 which show that adding any total time derivative of a scalar function $\Lambda(\mathbf{q},t)$ to the Lagrangian has no observable consequences on the equations of motion. That is, although addition of the total time derivative of the scalar function $\Lambda(\mathbf{q},t)$ changes the value of the Lagrangian, it does not change the equations of motion for the observables derived using equivalent standard Lagrangians.

For Lagrangian formulations of classical mechanics, the gauge invariance is readily apparent by direct inspection of the Lagrangian.

Example 9.3.1: Gauge invariance in electromagnetism

The scalar electric potential Φ and the vector potential A fields in electromagnetism are examples of gauge-invariant fields. These electromagnetic-potential fields are not directly observable, that is, the electromagnetic observable quantities are the electric field E and magnetic field B which can be derived from the scalar and vector potential fields Φ and A. An advantage of using the potential fields is that they reduce the problem from 6 components, 3 each for E and B, to 4 components, one for the scalar field Φ and 3 for the vector potential A. The Lagrangian for the velocity-dependent Lorentz force, given by equation $(6.10.7)$, provides an example of gauge invariance. Equations $(6.10.3)$ and $(6.10.5)$ showed that the electric and magnetic fields can be expressed in terms of scalar and vector potentials Φ and $\mathbf{A}$ by the relations

$$\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A} \tag{9.3.6}$$

$$\mathbf{E}=-\boldsymbol{\nabla}\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

The equations of motion for a charge q in an electromagnetic field can be obtained by using the Lagrangian

$$L=\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}-q(\Phi-\mathbf{A}\cdot\mathbf{v})$$

Consider the transformations $(\mathbf{A},\Phi)\rightarrow(\mathbf{A}',\Phi')$ in the transformed Lagrangian L' where

$$\mathbf{A}'=\mathbf{A}+\nabla\Lambda(\mathbf{r},t)$$

$$\Phi'=\Phi-\frac{\partial\Lambda(\mathbf{r},t)}{\partial t}$$

The transformed Lorentz-force Lagrangian L' is related to the original Lorentz-force Lagrangian L by

$$L' = L + q\left[\dot{\mathbf{r}}\cdot\nabla\Lambda(\mathbf{r},t) + \frac{\partial\Lambda(\mathbf{r},t)}{\partial t}\right] = L + q\frac{d}{dt}\Lambda(\mathbf{r},t)$$

Note that the additive term $q\frac{d}{dt}\Lambda(\mathbf{r},t)$ is an exact time differential. Thus the Lagrangian L' is gauge invariant implying identical equations of motion are obtained using either of these equivalent Lagrangians.

The force fields $\mathbf{E}$ and $\mathbf{B}$ can be used to show that the above transformation is gauge-invariant. That is,

$$\begin{aligned}\mathbf{E}' &= -\boldsymbol{\nabla}\Phi' - \frac{\partial\mathbf{A}'}{\partial t}\\ &= -\boldsymbol{\nabla}\Phi - \frac{\partial\mathbf{A}}{\partial t}\\ &= \mathbf{E}\\ \mathbf{B}' &= \boldsymbol{\nabla}\times\mathbf{A}'\\ &= \boldsymbol{\nabla}\times\mathbf{A}\\ &= \mathbf{B}\end{aligned}$$

That is, the additive terms due to the scalar field $\Lambda(\mathbf{r},t)$ cancel. Thus the electromagnetic force fields following a gauge-invariant transformation are shown to be identical in agreement with what is inferred directly by inspection of the Lagrangian.

Non-standard Lagrangians

The definition of the standard Lagrangian was based on d'Alembert's differential variational principle. The flexibility and power of Lagrangian mechanics can be extended to a broader range of dynamical systems by employing an extended definition of the Lagrangian that is based on Hamilton's Principle, equation $(9.1.2)$. Note that Hamilton's Principle was introduced 46 years after development of the standard formulation of Lagrangian mechanics. Hamilton's Principle provides a general definition of the Lagrangian that applies to standard Lagrangians, which are expressed as the difference between the kinetic and potential energies, as well as to non-standard Lagrangians where there may be no clear separation into kinetic and potential energy terms. These non-standard Lagrangians can be used with the Euler-Lagrange equations to generate the correct equations of motion, even though they may have no relation to the kinetic and potential energies. The extended definition of the Lagrangian based on Hamilton's action functional $(9.1.1)$ can be exploited for developing non-standard definitions of the Lagrangian that may be applied to dynamical systems where use of the standard definition is inapplicable. Non-standard Lagrangians can be equally as useful as the standard Lagrangian for deriving equations of motion for a system. Secondly, non-standard Lagrangians, that have no energy interpretation, are available for deriving the equations of motion for many nonconservative systems. Thirdly, Lagrangians are useful irrespective of how they were derived. For example, they can be used to derive conservation laws or the equations of motion. Coordinate transformations of the Lagrangian is much simpler than that required for transforming the equations of motion. The relativistic Lagrangian defined in chapter 17.6 is a well-known example of a non-standard Lagrangian.

Inverse variational calculus

Non-standard Lagrangians and Hamiltonians are not based on the concept of kinetic and potential energies. Therefore, development of non-standard Lagrangians and Hamiltonians require an alternative approach that ensures that they satisfy Hamilton's Principle, equation $(9.1.2)$, which underlies the Lagrangian and Hamiltonian formulations. One useful alternative approach is to derive the Lagrangian or Hamiltonian via an inverse variational process based on the assumption that the equations of motion are known. Helmholtz developed the field of inverse variational calculus which plays an important role in development of non-standard Lagrangians. An example of this approach is use of the well-known Lorentz force as the basis for deriving a corresponding Lagrangian to handle systems involving electromagnetic forces. Inverse variational calculus is a branch of mathematics that is beyond the scope of this textbook. The Douglas theorem states that, if the three Helmholtz conditions are satisfied, then there exists a Lagrangian that, when used with the Euler-Lagrange differential equations, leads to the given set of equations of motion. Thus, it will be assumed that the inverse variational calculus technique can be used to derive a Lagrangian from known equations of motion.