16.6: Electromagnetic Field Theory

Maxwell stress tensor

Analytical formulations for continuous systems, developed for describing elasticity, are generally applicable when applied to other fields, such as the electromagnetic field. The use of the Maxwell's stress tensor $\mathbf{T}$, to describe momentum in the electromagnetic field, is an important example of the application of continuum mechanics in field theory.

The Lorentz force can be written as

$$\mathbf{F}=\int\rho(\mathbf{E}+\mathbf{v}\times\mathbf{B})d\tau=\int(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})d\tau=\int\mathbf{f}\,d\tau \tag{16.6.1}$$

where the force density $\mathbf{f}$ is defined to be

$$\mathbf{f}=(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}) \tag{16.6.2}$$

Maxwell's equations

$$\rho=\epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E}\qquad\mathbf{J}=\frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{B}-\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{16.6.3}$$

can be used to eliminate the charge and current densities in Equation 16.6.1

$$\mathbf{f}=\epsilon_0(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{E}+\left(\frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{B}-\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B} \tag{16.6.4}$$

Vector calculus gives that

$$\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})=\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}+\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t} \tag{16.6.5}$$

while Faraday's law gives

$$\frac{\partial\mathbf{B}}{\partial t}=-\boldsymbol{\nabla}\times\mathbf{E} \tag{16.6.6}$$

Equation 16.6.6allows Equation 16.6.5to be rewritten as

$$\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}=+\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})-\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}=+\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})+\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) \tag{16.6.7}$$

Equation 16.6.7 can be inserted into Equation 16.6.4. In addition, a term $\frac{1}{\mu_0}(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{B}$ can be added since $\boldsymbol{\nabla}\cdot\mathbf{B}=0$ which allows equation 16.60 to be written in the symmetric form

$$\mathbf{f}=\epsilon_0(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{E}+\frac{1}{\mu_0}(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{B}+\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}-\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} \tag{16.6.8}$$

$$=\epsilon_0(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{E}+\frac{1}{\mu_0}(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{B}+\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}-\epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})-\epsilon_0\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) \tag{16.6.9}$$

Using the vector identity

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})+\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})+(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}+(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} \tag{16.6.10}$$

Let $\mathbf{A}=\mathbf{B}=\mathbf{E}$, then

$$\boldsymbol{\nabla}(E^2)=2\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})+2(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E} \tag{16.6.11}$$

That is

$$\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})=\frac{1}{2}\boldsymbol{\nabla}(E^2)-(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E} \tag{16.6.12}$$

Similarly

$$\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})=\frac{1}{2}\boldsymbol{\nabla}(B^2)-(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B} \tag{16.6.13}$$

Inserting equations 16.6.12and 16.6.13into Equation 16.6.9gives

$$\mathbf{f}=\epsilon_0\left[(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{E}+(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E}-\frac{1}{2}\boldsymbol{\nabla}E^2\right]+\frac{1}{\mu_0}\left[(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{B}+(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}-\frac{1}{2}\boldsymbol{\nabla}B^2\right]-\epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) \tag{16.6.14}$$

This complicated formula can be simplified by defining the rank-2 **Maxwell stress tensor** $\mathbf{T}$ which has components

$$T_{ij}\equiv\epsilon_0\left(E_iE_j-\frac{1}{2}\delta_{ij}E^2\right)+\frac{1}{\mu_0}\left(B_iB_j-\frac{1}{2}\delta_{ij}B^2\right) \tag{16.6.15}$$

The inner product of the del operator and the Maxwell stress tensor is a vector with j components of

$$(\boldsymbol{\nabla}\cdot\mathbf{T})_j=\epsilon_0\left[(\boldsymbol{\nabla}\cdot\mathbf{E})E_j+(\mathbf{E}\cdot\boldsymbol{\nabla})E_j-\frac{1}{2}\boldsymbol{\nabla}_j^2E^2\right]+\frac{1}{\mu_0}\left[(\boldsymbol{\nabla}\cdot\mathbf{B})B_j+(\mathbf{B}\cdot\boldsymbol{\nabla})B_j-\frac{1}{2}\boldsymbol{\nabla}_j^2B^2\right] \tag{16.6.16}$$

The above definition of the Maxwell stress tensor, plus the Poynting vector $\mathbf{S}=\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$, allows the force density Equation 16.6.2to be written in the form

$$\mathbf{f}=\boldsymbol{\nabla}\cdot\mathbf{T}-\epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t} \tag{16.6.17}$$

The divergence theorem allows the total force, acting of the volume τ, to be written in the form

$$\mathbf{F}=\int\left(\boldsymbol{\nabla}\cdot\mathbf{T}-\epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}\right)d\tau \tag{16.6.18}$$

$$=\oint\mathbf{T}\cdot d\mathbf{a}-\epsilon_0\mu_0\frac{d}{dt}\int\mathbf{S}d\boldsymbol{\tau} \tag{16.6.19}$$

Note that, if the Poynting vector is time independent, then the second term in Equation 16.6.19is zero and the Maxwell stress tensor $\mathbf{T}$ is the force per unit area, (stress) acting on the surface. The fact that $\mathbf{T}$ is a rank-2 tensor is apparent since the stress represents the ratio of the force-density vector $d\mathbf{f}$ and the infinitessimal area vector $d\mathbf{a}$, which do not necessarily point in the same directions.

Momentum in the electromagnetic field

Chapter 7.2 showed that the electromagnetic field carries a linear momentum $q\mathbf{A}$ where q is the charge on a body and $\mathbf{A}$ is the electromagnetic vector potential. It is useful to use the Maxwell stress tensor to express the momentum density directly in terms of the electric and magnetic fields.

Newton's law of motion can be used to write equation Equation 16.6.19as

$$\mathbf{F}=\frac{d\mathbf{p}_{mech}}{dt}=\oint\mathbf{T}\cdot d\mathbf{a}-\epsilon_0\mu_0\frac{d}{dt}\int\mathbf{S}d\boldsymbol{\tau} \tag{16.6.20}$$

where $\mathbf{p}$ is the total mechanical linear momentum of the volume τ. Equation 16.6.20implies that the electromagnetic field carries a linear momentum

$$\mathbf{p}_{field}=\epsilon_0\mu_0\int\mathbf{S}d\boldsymbol{\tau} \tag{16.6.21}$$

The $\oint\mathbf{T}\cdot d\mathbf{a}$ term in Equation 16.6.20is the momentum per unit time flowing into the closed surface. In field theory it can be useful to describe the behavior in terms of the momentum flux density $\boldsymbol{\pi}$. Thus the momentum flux density $\boldsymbol{\pi}_{field}$ in the electromagnetic field is

$$\boldsymbol{\pi}_{field}=\epsilon_0\mu_0\mathbf{S} \tag{16.6.22}$$

Then Equation 16.6.20implies that the total momentum flux density $\boldsymbol{\pi}=\boldsymbol{\pi}_{mech}+\boldsymbol{\pi}_{field}$ is related to Maxwell's stress tensor by

$$\frac{\partial}{\partial t}(\boldsymbol{\pi}_{mech}+\boldsymbol{\pi}_{field})=\boldsymbol{\nabla}\cdot\mathbf{T} \tag{16.6.23}$$

That is, like the elasticity stress tensor, the divergence of Maxwell's stress tensor $\mathbf{T}$ equals the rate of change of the total momentum density, that is, $-\mathbf{T}$ is the momentum flux density.

This discussion of the Maxwell stress tensor and its relation to momentum in the electromagnetic field illustrates the role that analytical formulations of classical mechanics can play in field theory