13.1: Introduction to Rigid-body Rotation

Rigid-body rotation features prominently in science, engineering, and sports. Prior chapters have focussed primarily on motion of point particles. This chapter extends the discussion to motion of finite-sized rigid bodies. A rigid body is a collection of particles where the relative separations remain rigidly fixed. In real life, there is always some motion between individual atoms, but usually this microscopic motion can be neglected when describing macroscopic properties. Note that the concept of perfect rigidity has limitations in the theory of relativity since information cannot travel faster than the velocity of light, and thus signals cannot be transmitted instantaneously between the ends of a rigid body which is implied if the body had perfect rigidity.

The description of rigid-body rotation is most easily handled by specifying the properties of the body in the rotating body-fixed coordinate frame whereas the observables are measured in the stationary inertial laboratory coordinate frame. In the body-fixed coordinate frame, the primary observable for classical mechanics is the inertia tensor of the rigid body which is well defined and independent of the rotational motion. By contrast, in the stationary inertial frame the observables depend sensitively on the details of the rotational motion. For example, when observed in the stationary fixed frame, rapid rotation of a long thin cylindrical pencil about the longitudinal symmetry axis gives a time-averaged shape of the pencil that looks like a thin cylinder, whereas the time-averaged shape is a flat disk for rotation about an axis perpendicular to the symmetry axis of the pencil. In spite of this, the pencil always has the same unique inertia tensor in the body-fixed frame. Thus the best solution for describing rotation of a rigid body is to use a rotation matrix that transforms from the stationary fixed frame to the instantaneous body-fixed frame for which the moment of inertia tensor can be evaluated. Moreover, the problem can be greatly simplified by transforming to a body-fixed coordinate frame that is aligned with any symmetry axes of the body since then the inertia tensor can be diagonal; this is called a principal axis system.

Rigid-body rotation can be broken into the following two classifications.

1) Rotation about a fixed axis:

A body can be constrained to rotate about an axis that has a fixed location and orientation relative to the body. The hinged door is a typical example. Rotation about a fixed axis is straightforward since the axis of rotation, plus the moment of inertia about this axis, are well defined and this case was discussed in chapter (2.12).

2) Rotation about a point

A body can be constrained to rotate about a fixed point of the body but the orientation of this rotation axis about this point is unconstrained. One example is rotation of an object flying freely in space which can rotate about the center of mass with any orientation. Another example is a child's spinning top which has one point constrained to touch the ground but the orientation of the rotation axis is undefined.

The prior discussion in chapter (2.12) showed that rigid-body rotation is more complicated than assumed in introductory treatments of rigid-body rotation. It is necessary to expand the concept of moment of inertia to the concept of the inertia tensor, plus the fact that the angular momentum may not point along the rotation axis. The most general case requires consideration of rotation about a body-fixed point where the orientation of the axis of rotation is unconstrained. The concept of the inertia tensor of a rotating body is crucial for describing rigid-body motion. It will be shown that working in the body-fixed coordinate frame of a rotating body allows a description of the equations of motion in terms of the inertia tensor for a given point of the body, and that it is possible to rotate the body-fixed coordinate system into a principal axis system where the inertia tensor is diagonal. For any principal axis, the angular momentum is parallel to the angular velocity if it is aligned with a principal axis. The use of a **principal axis** system greatly simplifies treatment of rigid-body rotation and exploits the powerful and elegant matrix algebra mentioned in appendix 19.1.

The following discussion of rigid-body rotation is broken into three topics, (1) the inertia tensor of the rigid body, (2) the transformation between the rotating body-fixed coordinate system and the laboratory frame, i.e., the Euler angles specifying the orientation of the body-fixed coordinate frame with respect to the laboratory frame, and (3) Lagrange and Euler's equations of motion for rigid-bodies. This is followed by a discussion of practical applications.