15.E: Advanced Hamiltonian Mechanics (Exercises)

1. Poisson brackets are a powerful means of elucidating when observables are constant of motion and whether two observables can be simultaneously measured with unlimited precision. Consider a spherically symmetric Hamiltonian

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r)$$

for a mass m where $U(r$ is a central potential. Use the Poisson bracket plus the time dependence to determine the following:

a. Does p_ϕ commute with H and is it a constant of motion?
b. Does $p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$ commute with H and is it a constant of motion?
c. Does p_r commute with H and is it a constant of motion?
d. Does p_ϕ commute with p_θ and what does the result imply?

2. Consider the Poisson brackets for angular momentum L

a. Show $\{L_i, r_j\} = \epsilon_{ijk} r_k$, where the Levi-Cevita tensor is,

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ are cyclically permuted} \\ -1 & \text{if } ijk \text{ are anti-cyclically permuted} \\ 0 & \text{if } i = j \text{ or } i = k \text{ or } j = k \end{cases}$$

b. Show $\{L_i, p_j\} = \epsilon_{ijk} p_k$.
c. Show $\{L_i, L_j\} = \epsilon_{ijk} L_k$. The following identity may be useful: $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.
d. Show $\{L_i, L^2\} = 0$.

3. Consider the Hamiltonian of a two-dimensional harmonic oscillator,

$$H = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m(\omega_1^2 r_1^2 + \omega_2^2 r_2^2)$$

What condition is satisfied if L^2 a conserved quantity?

4. Consider the motion of a particle of mass m in an isotropic harmonic oscillator potential $U = \frac{1}{2}kr^2$ and take the orbital plane to be the $x - y$ plane. The Hamiltonian is then

$$H \equiv S_0 = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}k(x^2 + y^2)$$

Introduce the three quantities

$$S_1 = \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}k(x^2 - y^2)$$

$$S_2 = \frac{1}{m}p_x p_y + kxy$$

$$S_3 = \omega(xp_y - yp_x)$$

with $\omega = \sqrt{\frac{k}{m}}$. Use Poisson brackets to solve the following:

a. Show that $\{S_0, S_i\} = 0$ for $i = 1, 2, 3$ proving that (S_1, S_2, S_3) are constants of motion.
b. Show that

$$\{S_1, S_2\} = 2\omega S_3$$

$$\{S_2, S_3\} = 2\omega S_1$$

$$\{S_3, S_1\} = 2\omega S_2$$

so that $(2\omega)^{-1}(S_1, S_2, S_3)$ have the same Poisson bracket relations as the components of a 3-dimensional angular momentum.

$$S_0^2 = S_1^2 + S_2^2 + S_3^2$$

5. Assume that the transformation equations between the two sets of coordinates (q, p) and (Q, P) are

$$Q = \ln(1 + q^{\frac{1}{2}} \cos p)$$

$$P = 2(1 + q^{\frac{1}{2}} \cos p)q^{\frac{1}{2}} \sin p)$$

a. Assuming that q, p are canonical variables, i.e. $[q, p] = 1$, show directly from the above transformation equations that Q, P are canonical variables.
b. Show that the generating function that generates this transformation between the two sets of canonical variables is

$$F_3 = -[e^Q - 1]^2 \tan p$$

6. Consider a bound two-body system comprising a mass m in an orbit at a distance r from a mass M. The attractive central force binding the two-body system is

$$\mathbf{F} = \frac{k}{r^2} \hat{\mathbf{r}}$$

where k is negative. Use Poisson brackets to prove that the eccentricity vector $A = p \times L + \mu k \hat{r}$ is a conserved quantity.

7. Consider the case of a single mass m where the Hamiltonian $H = \frac{1}{2} p^2$.

a. Use the generating function $S(q, P, t)$ to solve the Hamilton-Jacobi equation with the canonical transformation $q = q(Q, P)$ and $p = p(Q, P)$ and determine the equations relating the (q, p) variables to the transformed coordinate and momentum (Q, P).
b. If there is a perturbing Hamiltonian $\Delta H = \frac{1}{2} q^2$, then P will not be constant. Express the transformed Hamiltonian H (using the transformation given above in terms of P, Q, and t). Solve for $Q(t)$ and $P(t)$ and show that the perturbed solution $q[Q(t), P(t)], p[Q(t), P(t)]$ is simple harmonic.
