6.11: Time-dependent forces

All examples discussed in this chapter have assumed Lagrangians that are time independent. Mathematical systems where the ordinary differential equations do not depend explicitly on the independent variable, which in this case is time t, are called *autonomous* systems. Systems having differential equations governing the dynamical behavior that have time-dependent coefficients are called *non-autonomous* systems.

In principle it is trivial to incorporate time-dependent behavior into the equations of motion by introducing either a time dependent generalized force $Q(r, t)$, or allowing the Lagrangian to be time dependent. For example, in the rocket problem the mass is time dependent. In some cases the time dependent forces can be represented by a time-dependent potential energy rather than using a generalized force. Solutions for non-autonomous systems can be considerably more difficult to obtain, and can involve regions where the motion is stable and other regions where the motion is unstable or chaotic similar to the behavior discussed in chapter 4. The following case of a simple pendulum, whose support is undergoing vertical oscillatory motion, illustrates the complexities that can occur for systems involving time-dependent forces.

Example 6.11.1: Plane pendulum hanging from a vertically-oscillating support

Consider a plane pendulum having a mass M fastened to a massless rigid rod of length L that is at an angle $\theta(t)$ to the vertical gravitational field g. The pendulum is attached to a support that is subject to a vertical oscillatory force F such that the vertical position y of the support is

$$\begin{aligned}\ddot{\theta}+\left(\frac{g}{L}+\frac{\ddot{y}}{L}\right)\theta = & \quad 0 \\ \ddot{y}+g = & \quad \frac{F}{M}\end{aligned}$$

Substitute $\ddot{y}=-A\omega^2\cos\omega t$ into these equations gives

$$\begin{aligned}\ddot{\theta}+\left(\frac{g}{L}-\frac{A\omega^2}{L}\cos\omega t\right)\theta = & \quad 0 \\ M\left(g-A\omega^2\cos\omega t\right) = & \quad F\end{aligned}$$

These correspond to stable harmonic oscillations about $\theta\approx 0$ if the bracket term is positive, and to unstable motion if the bracket is negative. Thus, for small amplitude oscillation about $\theta\approx 0$ the motion of the system can be unstable whenever the bracket is negative, that is, when the acceleration $A\omega^2\cos\omega t>g$ and resonance behavior can occur coupling the pendulum period and the forcing frequency ω.

$$\begin{aligned}\ddot{\theta}-\left(\frac{g}{L}-\frac{A\omega^2}{L}\cos\omega t\right)\theta = & \quad 0 \\ m\left(g-A\omega^2\cos\omega t\right) = & \quad F\end{aligned}$$

The inverted pendulum has stable oscillations about $\theta\approx\pi$ if the bracket is negative, that is, if $A\omega^2\cos\omega t>g$. This illustrates that nonautonomous dynamical systems can involve either stable or unstable motion.