# 12.6: Lagrangian Mechanics in a Non-Inertial Frame

The above derivation of the equations of motion in the rotating frame is based on Newtonian mechanics. Lagrangian mechanics provides another derivation of these equations of motion for a rotating frame of reference by exploiting the fact that the Lagrangian is a scalar which is frame independent, that is, it is invariant to rotation of the frame of reference.

The Lagrangian in any frame is given by

$$L=\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}-U(r) \tag{12.6.1}$$

The scalar product $\mathbf{v}\cdot\mathbf{v}$ is the same in any rotated frame and can be evaluated in terms of the rotating frame variables using the same decomposition of the translational plus rotational motion as used previously and given in equation $(12.4.4)$.

Equation $(12.4.4)$ decomposes the velocity in the fixed inertial frame $\mathbf{v}_{fix}$ into four vector terms, the translational velocity $\mathbf{V}_{fix}$ of the translating frame, the velocity in the rotating-translating frame $\mathbf{v}''_{rot}$, and rotational velocity $(\omega\times\mathbf{r}'_{mov})$ . Using equations 12.6.1 and $(12.4.4)$, plus appendix equation $19.2.21$ for the triple products, gives that the Lagrangian evaluated using $\mathbf{v}_{fix}\cdot\mathbf{v}_{fix}$ equals

$$L=\frac{1}{2}m\left[\mathbf{V}_{fix}\cdot\mathbf{V}_{fix}+\mathbf{v}''_{rot}\cdot\mathbf{v}''_{rot}+2\mathbf{V}_{fix}\cdot\mathbf{v}''_{rot}+2\mathbf{V}_{fix}\cdot(\omega\times\mathbf{r}'_{mov})+2\mathbf{v}''_{rot}\cdot(\omega\times\mathbf{r}'_{mov})+(\omega\times\mathbf{r}'_{mov})^2\right]-U(r) \tag{12.6.2}$$

This can be used to derive the canonical momentum in the rotating frame

$$\mathbf{p}''_{rot}=\frac{\partial L}{\partial\mathbf{v}''_{rot}}=m[\mathbf{V}_{fix}+\mathbf{v}''_{rot}+\omega\times\mathbf{r}'_{mov}] \tag{12.6.3}$$

The Lagrange equations can be used to derive the equations of motion in terms of the variables evaluated in the rotating reference frame. The required Lagrange derivatives are

$$\frac{d}{dr}\frac{\partial L}{\partial\mathbf{v}''_{rot}}=m[\mathbf{A}_{fix}+\mathbf{a}''_{rot}+(\omega\times\mathbf{v}''_{rot})+(\dot{\omega}\times\mathbf{r}'_{mov})]_{rot} \tag{12.6.4}$$

and

$$\frac{\partial L}{\partial\mathbf{r}'}=-m[(\omega\times\mathbf{V}_{fix})-(\omega\times\mathbf{v}''_{rot})-\omega\times(\omega\times\mathbf{r}'_{mov})]_{rot}-\nabla U \tag{12.6.5}$$

where the scalar triple product, equation $19.2.21$, has been used. Thus the Lagrange equations give for the rotating frame basis that

$$m\mathbf{a}''_{rot}=-\nabla U-m[\mathbf{A}_{fix}+(\omega\times\mathbf{V}_{fix})+2(\omega\times\mathbf{v}''_{rot})+\omega\times(\omega\times\mathbf{r}'_{mov})+(\dot{\omega}\times\mathbf{r}'_{mov})]_{rot} \tag{12.6.6}$$

The external force is identified as $\mathbf{F}_{fixed}=-\nabla U$ . Equation $(12.3.7)$ can be used to transform between the fixed and the rotating bases.

$$\mathbf{A}_{fix}=\left[\mathbf{A}_{fix}+(\omega\times\mathbf{V})_{fix}\right]_{rot} \tag{12.6.7}$$

This leads to an effective force in the non-inertial translating plus rotating frame that corresponds to an effective Newtonian force of

$$\mathbf{F}^{eff}_{rot}=m\mathbf{a}''_{rot}=\mathbf{F}-m[\mathbf{A}_{fix}+2\omega\times\mathbf{v}''_{rot}+\omega\times(\omega\times\mathbf{r}'_{mov})+(\dot{\omega}\times\mathbf{r}'_{mov})] \tag{12.6.8}$$

where $\mathbf{A}_{fix}$ is expressed in the fixed frame. The derivation of Equation 12.6.8 using Lagrangian mechanics, confirms the identical formula 12.6.1 derived using Newtonian mechanics.

The four correction terms for the non-inertial frame basis correspond to the following effective forces.

- **Translational acceleration**: $\mathbf{F}^{eff}_{mov}=-m\mathbf{A}_{fix}$ is the usual inertial force experienced in a linearly accelerating frame of reference, and where $\mathbf{A}_{fix}$ is with respect to the fixed frame.
- **Coriolis force:** $\mathbf{F}^{eff}_{cor}=-2m\omega\times\mathbf{v}''_{rot}$ This is a new type of inertial force that is present only when a particle is moving in the rotating frame. This force is proportional to the velocity in the rotating frame and is independent of the position in the rotating frame
- **Centrifugal force**: $\mathbf{F}^{eff}_{ef}=-m\omega\times(\omega\times\mathbf{r}'_{mov})$ This is due to the centripetal acceleration of the particle owing to the rotation of the moving axis about the axis of rotation.
- **Transverse (azimuthal) force**: $\mathbf{F}^{eff}_{az}=-m\dot{\omega}\times\mathbf{r}'_{mov}$ This is a straightforward term due to acceleration of the particle due to the angular acceleration of the rotating axes.

The above inertial forces are correction terms arising from trying to extend Newton's laws of motion to a non-inertial frame involving both translation and rotation. These correction forces are often referred to as "fictitious" forces. However, these non-inertial forces are very real when located in the non-inertial frame. Since the centrifugal and Coriolis terms are unusual they are discussed below.

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