3.3: Linearity and Superposition

An important aspect of linear systems is that the solutions obey the *Principle of Superposition*, that is, for the superposition of different oscillatory modes, the amplitudes add linearly. The linearly-damped linear oscillator is an example of a linear system in that it involves only linear operators, that is, it can be written in the operator form (appendix $19.6.2$)

$$\left(\frac{d^2}{dt^2}+\Gamma\frac{d}{dt}+\omega_0^2\right)x(t)=A\cos\omega t \tag{3.3.1}$$

The quantity in the brackets on the left hand side is a linear operator that can be designated by $\mathbb{L}$ where

$$\mathbb{L}x(t)=F(t) \tag{3.3.2}$$

An important feature of linear operators is that they obey the principle of superposition. This property results from the fact that linear operators are distributive, that is

$$\mathbb{L}(x_1+x_2)=\mathbb{L}(x_1)+\mathbb{L}(x_2) \tag{3.3.3}$$

Therefore if there are two solutions $x_1(t)$ and $x_2(t)$ for two different forcing functions $F_1(t)$ and $F_2(t)$

$$\begin{aligned} \mathbb{L}x_1(t)= &\quad F_1(t) \\ \mathbb{L}x_1(t)= &\quad F_2(t) \end{aligned}$$

then the addition of these two solutions, with arbitrary constants, also is a solution for linear operators.

$$\mathbb{L}(\alpha_1 x_! + \alpha_2 x_2)=\alpha_1 F_1(t)+\alpha_2 F_2(t) \tag{3.3.4}$$

In general then

$$\mathbb{L}\left(\sum_{n=1}^{N}\alpha_n x_n(t)\right)=\left(\sum_{n=1}^{N}\alpha_n F_n(t)\right) \tag{3.3.5}$$

The left hand bracket can be identified as the linear combination of solutions

$$x(t)=\sum_{n=1}^{N}\alpha_n x_n(t) \tag{3.3.6}$$

while the driving force is a linear superposition of harmonic forces

$$F(t)=\sum_{n=1}^{N}\alpha_n F_n(t) \tag{3.3.7}$$

Thus these linear combinations also satisfy the general linear equation

$$\mathbb{L}x(t)=F(t) \tag{3.3.8}$$

Applicability of the Principle of Superposition to a system provides a tremendous advantage for handling and solving the equations of motion of oscillatory systems.