11.E: Conservative two-body Central Forces (Exercises)

1. Listed below are several statements concerning central force motion. For each statement, give the reason for why the statement is true. If a statement is only true in certain situations, then explain when it holds and when it doesn't. The system referred to below consists of mass m_1 located at r_1 and mass m_2 located at r_2.
   1. The potential energy of the system depends only on the difference $r_1 - r_2$, not on r_1 and r_2 separately.
   2. The potential energy of the system depends only on the magnitude of $r_1 - r_2$, not the direction.
   3. It is possible to choose an inertial reference frame in which the center of mass of the system is at rest.
   4. The total energy of the system is conserved.
   5. The total angular momentum of the system is conserved.
2. A particle of mass m moves in a potential $U(r) = -U_0 e^{-\lambda^2 r^2}$.
   1. Given the constant l, find an implicit equation for the radius of the circular orbit. A circular orbit at $r=\rho$ is possible if
   $$\left(\frac{\partial V}{\partial r}\right)\bigg|_{r=\rho} = 0$$
   where V is the effective potential.
   2. What is the largest value of l for which a circular orbit exists? What is the value of the effective potential at this critical orbit?
3. A particle of mass m is observed to move in a spiral orbit given by the equation $r = k\theta$, where k is a constant. Is it possible to have such an orbit in a central force field? If so, determine the form of the force function.
4. The interaction energy between two atoms of mass m is given by the Lennard-Jones potential, $U(r) = \epsilon[(r_0/r)^{12} - 2(r_0/r)^6]$
   1. Determine the Lagrangian of the system where r_1 and r_2 are the positions of the first and second mass, respectively.
   2. Rewrite the Lagrangian as a one-body problem in which the center-of-mass is stationary.
   3. Determine the equilibrium point and show that it is stable.
   4. Determine the frequency of small oscillations about the stable point.
5. Consider two bodies of mass m in circular orbit of radius $r_0/2$, attracted to each other by a force $F(r)$, where r is the distance between the masses.
   1. Determine the Lagrangian of the system in the center-of-mass frame (Hint: a one-body problem subject to a central force).
   2. Determine the angular momentum. Is it conserved?
   3. Determine the equation of motion in r in terms of the angular momentum and $|\mathbf{F}(r)|$.
   4. Expand your result in (c) about an equilibrium radius r_0 and show that the condition for stability is, $\frac{F'(r_0)}{F(r_0)} + \frac{3}{r_0} > 0$
6. Consider two charges of equal magnitude q connected by a spring of spring constant k' in circular orbit. Can the charges oscillate about some equilibrium? If so, what condition must be satisfied?
7. Consider a mass m in orbit around a mass M, which is subject to a force $F = -\frac{k}{r^2}\hat{r}$, where r is the distance between the masses. Show that the eccentricity vector $A = p \times L - \mu k \hat{r}$ is conserved.
8. Show that the areal velocity is constant for a particle moving under the influence of an attractive force given by $F(r) = -kr$. Calculate the time averages of the kinetic and potential energies and compare with the the results of the virial theorem.
9. Assume that the Earth's orbit is circular and that the Sun's mass suddenly decreases by a factor of two.
   1. What orbit will the earth then have?
   2. Will the Earth escape the solar system?
10. Discuss the motion of a particle in a central inverse-square-law force field for a superimposed force whose magnitude is inversely proportional to the cube of the distance from the particle to force center; that is
    $$F(r) = -\frac{k}{r^2} - \frac{\lambda}{r^3} \qquad (\mathrm{k}, \lambda > 0)$$
    Show that the motion is described by a precessing ellipse. Consider the cases a) $\lambda < \frac{l^2}{\mu}$, b) $\lambda = \frac{l^2}{\mu}$, c) $\lambda > \frac{l^2}{\mu}$ where l is the angular momentum and μ the reduced mass.
11. A communications satellite is in a circular orbit around the earth at a radius R and velocity v. A rocket accidentally fires quite suddenly, giving the rocket an outward velocity v in addition to its original tangential velocity v.

1. Calculate the ratio of the new energy and angular momentum to the old.
2. Describe the subsequent motion of the satellite and plot $T(R)$, $U(r)$, the net effective potential, and $E(r)$ after the rocket fires.

12. Two identical point objects, each of mass m are bound by a linear two-body force $F = -kr$ where r is the vector distance between the two point objects. The two point objects each slide on a horizontal frictionless plane subject to a vertical gravitational field g. The two-body system is free to translate, rotate and oscillate on the surface of the frictionless plane.
    1. Derive the Lagrangian for the complete system including translation and relative motion.
    2. Use Noether's theorem to identify all constants of motion.
    3. Use the Lagrangian to derive the equations of motion for the system.
    4. Derive the generalized momenta and the corresponding Hamiltonian.
    5. Derive the period for small amplitude oscillations of the relative motion of the two masses.
13. A bound binary star system comprises two spherical stars of mass m_1 and m_2 bound by their mutual gravitational attraction. Assume that the only force acting on the stars is their mutual gravitation attraction and let r be the instantaneous separation distance between the centers of the two stars where r is much larger than the sum of the radii of the stars.
    1. Show that the two-body motion of the binary star system can be represented by an equivalent one-body system and derive the Lagrangian for this system.
    2. Show that the motion for the equivalent one-body system in the center of mass frame lies entirely in a plane and derive the angle between the normal to the plane and the angular momentum vector.
    3. Show whether H_{cm} is a constant of motion and whether it equals the total energy.
    4. It is known that a solution to the equation of motion for the equivalent one-body orbit for this gravitational force has the form

    $$\frac{1}{r} = -\frac{\mu k}{l^2}[1 + \epsilon \cos\theta]$$

    and that the angular momentum is a constant of motion $L = l$. Use these to prove that the attractive force leading to this bound orbit is

    $$\mathbf{F} = \frac{k}{r^2}\hat{\mathbf{r}}$$

    where k must be negative.
14. When performing the Rutherford experiment, Gieger and Marsden scattered $7.7\ MeV$ ^{4}He particles (alpha particles) from ^{238}U at a scattering angle in the laboratory frame of $\theta = 90^\circ$. Derive the following observables as measured in the laboratory frame.
    1. The recoil scattering angle of the ^{238}U in the laboratory frame.
    2. The scattering angles of the ^{4}He and ^{238}U in the center-of-mass frame
    3. The kinetic energies of the ^{4}He and ^{238}U in the laboratory frame
    4. The impact parameter
    5. The distance of closest approach r_{min}