13.10: General Properties of the Inertia Tensor

Inertial Equivalence

The elements of the inertia tensor, the values of the principal moments of inertia, and the orientation of the principal axes for a rigid body, all depend on the choice of origin for the system. Recall that for the kinetic energy to be separable into translational and rotational portions, the origin of the body coordinate system must coincide with the center of mass of the body. However, for *any* choice of the origin of *any* body, there always exists an orientation of the axes that diagonalizes the inertia tensor.

The inertial properties of a body for rotation about a specific body-fixed location is defined completely by only three principal moments of inertia irrespective of the detailed shape of the body. As a result, the inertial properties of any body about a body-fixed point are equivalent to that of an ellipsoid that has the same three principal moments of inertia. The symmetry properties of this equivalent ellipsoidal body define the symmetry of the inertial properties of the body. If a body has some simple symmetry then usually it is obvious as to what will be the principal axes of the body.

Spherical Top: $I_1 = I_2 = I_3$

A spherical top is a body having three degenerate principal moments of inertia. Such a body has the same symmetry as the inertia tensor about the center of a uniform sphere. For a sphere it is obvious from the symmetry that any orientation of three mutually orthogonal axes about the center of the uniform sphere are equally good principal axes. For a uniform cube the principal axes of the inertia tensor about the center of mass were shown to be aligned such that they pass through the center of each face, and the three principal moments are identical; that is, inertially it is equivalent to a spherical top. A less obvious consequence of the spherical symmetry is that any orientation of three mutually perpendicular axes about the center of mass of a uniform cube is an equally good principal axis system.

Symmetric Top: $I_1 = I_2 \neq I_3$

The equivalent ellipsoid for a body with two degenerate principal moments of inertia is a spheroid which has cylindrical symmetry with the cylindrical axis aligned along the third axis. A body with $I_3 < I_1 = I_2$ is a prolate spheroid while a body with $I_3 > I_1 = I_2$ is an oblate spheroid. Examples with a prolate spheroidal equivalent inertial shape are a rugby ball, pencil, or a baseball bat. Examples of an oblate spheroid are an orange, or a frisbee. A uniform sphere, or a uniform cube, rotating about a point displaced from the center-of-mass also behave inertially like a symmetric top. The cylindrical symmetry of the equivalent spheroid makes it obvious that any mutually perpendicular axes that are normal to the axis of cylindrical symmetry are equally good principal axes even when the cross section in the $1-2$ plane is square as opposed to circular.

A **rotor** is a diatomic-molecule shaped body which is a special case of a symmetric top where $I_1 = 0$, and $I_2 = I_3$. The rotation of a rotor is perpendicular to the symmetry axis since the rotational energy and angular momentum about the symmetry axis are zero because the principal moment of inertia about the symmetry axis is zero.

Asymmetric Top: $I_1 \neq I_2 \neq I_3$

A body where all three principal moments of inertia are distinct, $I_1 \neq I_2 \neq I_3$, is called an **asymmetric top**. Some molecules, and nuclei have asymmetric, triaxially-deformed, shapes.

Orthogonality of principal axes

The body-fixed principal axes comprise an orthogonal set, for which the vectors $\mathbf{L}$ and $\boldsymbol{\omega}$ are simply related. Components of $\mathbf{L}$ and $\boldsymbol{\omega}$ can be taken along the three body-fixed axes denoted by i. Thus for the m^{th} principal moment I_m

$$L_{im} = I_m \omega_{im} \tag{13.10.1}$$

Written in terms of the inertia tensor

$$L_{im} = \sum_k^3 I_{ik} \omega_{km} = I_m \omega_{im} \tag{13.10.2}$$

Similarly the n^{th} principal moment can be written as

$$L_{kn} = \sum_i^3 I_{ki}\omega_{in} = I_n\omega_{kn} \tag{13.10.3}$$

Multiply the Equation 13.10.2by ω_{in} and sum over i gives

$$\sum_{i,k} I_{ik}\omega_{km}\omega_{in} = \sum_i I_{mm}\omega_{im}\omega_{in} \tag{13.10.4}$$

Similarly multiplying Equation 13.10.3by ω_{km} and summing over k gives

$$\sum_{i,k} I_{ki}\omega_{km}\omega_{in} = \sum_i I_{nn}\omega_{km}\omega_{kn} \tag{13.10.5}$$

The left-hand sides of these equations are identical since the inertia tensor is symmetric, that is $I_{ik} = I_{ki}$. Therefore subtracting these equations gives

$$\sum_i I_{mm}\omega_{im}\omega_{in} - \sum_k I_{nn}\omega_{km}\omega_{kn} = 0 \tag{13.10.6}$$

That is

$$(I_{mm} - I_{nn})\sum_k \omega_{km}\omega_{kn} = 0 \tag{13.10.7}$$

or

$$(I_{mm} - I_{nn})\boldsymbol{\omega}_m \cdot \boldsymbol{\omega}_n = 0 \tag{13.10.8}$$

If $I_m \neq I_n$ then

$$\boldsymbol{\omega}_m \cdot \boldsymbol{\omega}_n = 0 \tag{13.10.9}$$

which implies that the m and n principal axes are perpendicular. However, if $I_{mm} = I_{nn}$ then Equation 13.10.8does not require that $\boldsymbol{\omega}_m \cdot \boldsymbol{\omega}_n = 0$, that is, these axes are not necessarily perpendicular, but, with no loss of generality, these two axes can be chosen to be perpendicular with any orientation in the plane perpendicular to the symmetry axis.

1. Summarizing the above discussion, the inertia tensor has the following properties.
2. Diagonalization may be accomplished by an appropriate rotation of the axes in the body.
3. The principal moments (eigenvalues) and principal axes (eigenvectors) are obtained as roots of the secular determinant and are real.
4. The principal axes (eigenvectors) are real and orthogonal.
5. For a symmetric top with two identical principal moments of inertia, any orientation of two orthogonal axes perpendicular to the symmetry axis are satisfactory eigenvectors.
6. For a spherical top with three identical principal moment of inertia, the principal axes system can have any orientation with respect to the origin.