13.14: Angular Velocity

Angular velocity $\boldsymbol{\omega}$

It is useful to relate the rigid-body equations of motion in the space-fixed $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ coordinate system to those in the body-fixed $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ coordinate system where the principal axis inertia tensor is defined. It was shown in appendix 19.4 that an infinitessimal rotation can be represented by a vector. Thus the time derivatives of these rotation angles can be associated with the components of the angular velocity $\boldsymbol{\omega}$, where the *precession* $\omega_\phi = \dot{\phi}$, the *nutation* $\omega_\theta = \dot{\theta}$, and the *spin* $\omega_\psi = \dot{\psi}$. Unfortunately the coordinates (ϕ, θ, ψ) are with respect to mixed coordinate frames and thus are not orthogonal axes. That is, the Euler angular velocities are expressed in different coordinate frames, where the *precession* $\dot{\phi}$ is around the space-fixed $\hat{\mathbf{z}}$ axis measured relative to the $\hat{\mathbf{x}}$-axis, the *spin* $\dot{\psi}$ is around the body-fixed $\hat{\mathbf{e}}_3$ axis relative to the rotating line-of-nodes, and the *nutation* $\dot{\theta}$ is the angular velocity between the $\hat{\mathbf{z}}$ and $\hat{\mathbf{e}}_3$ axes and points along the instantaneous line-of-nodes in the $\hat{\mathbf{e}}_3 \times \hat{\mathbf{z}}$ direction. By reference to Figure $13.13.1$it can be seen that the components along the body-fixed axes are as given in Table $13.14.1$

Table 13.14.1: Euler angular velocity components in the body-fixed frame

Precession $\dot{\phi}$	Nutation $\dot{\theta}$	Spin $\dot{\psi}$
$\dot{\phi}_1 = \dot{\phi}\sin\theta\sin\psi$	$\dot{\theta}_1 = \dot{\theta}\cos\psi$	$\dot{\psi}_1 = 0$
$\dot{\phi}_2 = \dot{\phi}\sin\theta\cos\psi$	$\dot{\theta}_2 = -\dot{\theta}\sin\psi$	$\dot{\psi}_2 = 0$
$\dot{\phi}_3 = \dot{\phi}\cos\theta$	$\dot{\theta}_3 = 0$	$\dot{\psi}_3 = \dot{\psi}$

Note that the precession angular velocity $\dot{\phi}$ is the angular velocity that the body-fixed $\hat{\mathbf{e}}_3$ and $\hat{\mathbf{z}} \times \hat{\mathbf{3}}$ axes precess around the space-fixed $\hat{\mathbf{z}}$ axis. Table $13.14.1$gives the Euler angular velocities required to calculate the components of the angular velocity $\boldsymbol{\omega}$ for the body-fixed $(\mathbf{1}, \mathbf{2}, \mathbf{3})$ axis system. Collecting the individual components of $\boldsymbol{\omega}$, gives the components of the angular velocity of the body, relative to the space-fixed axes, in the *body-fixed axis system* $(1, 2, 3)$

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \tag{13.14.1}$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \tag{13.14.2}$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi} \tag{13.14.3}$$

The angular velocity of the body about the body-fixed $\mathbf{3}$-axis, ω_3, is the sum of the projection of the precession angular velocity of the line-of-nodes $\dot{\phi}$ with respect to the space-fixed $\mathbf{x}$-axis, plus the angular velocity $\dot{\psi}$ of the body-fixed 3-axis with respect to the rotating line-of-nodes.

Similarly, the components of the body angular velocity $\boldsymbol{\omega}$ for the *space-fixed axis system* (x, y, z) can be derived to be

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \tag{13.14.4}$$

$$\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \tag{13.14.5}$$

$$\omega_z = \dot{\phi} + \dot{\psi}\cos\theta \tag{13.14.6}$$

Note that when $\theta = 0$ then the Euler angles are singular in that the space-fixed z axis is parallel with the body-fixed 3 axis and there is no way of distinguishing between precession $\dot{\phi}$ and spin $\dot{\psi}$, leading to $\omega_z = \omega_3 = \dot{\phi} + \dot{\psi}$. When $\theta = \pi$ then the z axis and 3 axis are antiparallel and $\omega_z = \dot{\phi} - \dot{\psi} = -\omega_3$. The other special case is when $\cos\theta = 0$ for which the Euler angle system is orthogonal and the space-fixed $\omega_z = \dot{\phi}$, that is, it equals the precession, while the body-fixed $\omega_3 = \dot{\psi}$, that is, it equals the spin. When the Euler angle basis is not orthogonal then equations 13.14.1- 13.14.3and 13.14.4- 13.14.6are needed for expressing the Euler equations of motion in either the body-fixed frame or the space-fixed frame respectively.

Equations 13.14.1- 13.14.3for the components of the angular velocity in the body-fixed frame can be expressed in terms of the Euler angle velocities in a matrix form as

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \tag{13.14.7}$$

Note that the transformation matrix is not orthogonal which is to be expected since the Euler angular velocities are about axes that do not form a rectangular system of coordinates. Similarly equations 13.14.4- 13.14.6 for the angular velocity in the space-fixed frame can be expressed in terms of the Euler angle velocities in matrix form as

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 & \cos\phi & \sin\theta\sin\phi \\ 0 & \sin\phi & \sin\theta\cos\phi \\ 1 & 0 & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \tag{13.14.8}$$