19.8: Appendix - Vector Differential Calculus

This appendix reviews vector differential calculus which is used extensively in both classical mechanics and electromagnetism.

Scalar differential operators

Scalar field

Differential operators like time $\left(\frac{d}{dt}\right)$ do not change the rotational properties of scalars or proper vectors. A scalar operator $\frac{d}{ds}$ acting on a scalar field $\phi(xyz)$, in a rotated coordinated frame $\phi'(x'y'z')$ is unchanged.

$$\frac{d\phi'}{ds} = \frac{d\phi}{ds} \tag{19.8.1}$$

Vector field

Similarly for a proper vector field

$$\frac{dA_i'}{ds} = \sum_j \lambda_{ij} \frac{dA_j}{ds} \tag{19.8.2}$$

That is, differentiation of scalar or vector fields with respect to a scalar operator does not change the rotational behavior. In particular, the scalar differentials of vectors continue to obey the rules of ordinary proper vectors. The scalar operator $\frac{\partial}{\partial t}$ is used for calculation of velocity or acceleration.

Vector differential operators in cartesian coordinates

Vector differential operators, such as the gradient operator, are important in physics. The action of vector operators differ along different orthogonal axes.

Scalar field

Consider a continuous, single-valued scalar function $\phi(x_i, x_j, x_k)$. Since

$$\phi' = \phi \tag{19.8.3}$$

then the partial differential with respect to one component x_i of the vector $\mathbf{x}'$ gives

$$\frac{\partial \phi'}{\partial x_i'} = \sum_j \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial x_i'} \tag{19.8.4}$$

The inverse rotation gives that

$$x_j = \sum_k \lambda_{kj} x_k' \tag{19.8.5}$$

Therefore

$$\frac{\partial x_j}{\partial x_i'} = \sum_k \lambda_{kj} \frac{\partial x_k'}{\partial x_i'} = \sum_k \lambda_{kj} \delta_{ik} = \lambda_{ij} \tag{19.8.6}$$

Thus

$$\frac{\partial \phi'}{\partial x_i'} = \sum_j \lambda_{ij} \frac{\partial \phi}{\partial x_j} \tag{19.8.7}$$

That is the vector derivative acting of a scalar field transforms like a proper vector.

Define the gradient, or $\boldsymbol{\nabla}$ operator, as

$$\boldsymbol{\nabla} \equiv \sum_i \widehat{\mathbf{e}_i} \frac{\partial}{\partial x_i} \tag{19.8.8}$$

where $\widehat{\mathbf{e}_i}$ is the unit vector along the x_i axis. In cartesian coordinates, the del vector operator is,

$$\boldsymbol{\nabla} \equiv \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \widehat{\mathbf{k}} \frac{\partial}{\partial z} \tag{19.8.9}$$

The gradient was applied to the gravitational and electrostatic potential to derive the corresponding field. For example, for electrostatics it was shown that the gradient of the scalar electrostatic potential field V can be written in cartesian coordinates as

$$\mathbf{E} = -\boldsymbol{\nabla} V \tag{19.8.10}$$

Note that the gradient of a scalar field produces a vector field. You are familiar with this if you are a skier in that the gravitational force pulls you down the line of steepest descent for the ski slope.

Vector field

Another possible operation for the del operator is the scalar product with a vector. Using the definition of a scalar product in cartesian coordinates gives

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} \frac{\partial A_x}{\partial x} + \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} \frac{\partial A_y}{\partial y} + \widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}} \frac{\partial A_z}{\partial z} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{19.8.11}$$

This scalar derivative of a vector field is called the divergence. Note that the scalar product produces a scalar field which is invariant to rotation of the coordinate axes.

The vector product of the del operator with another vector, is called the curl which is used extensively in physics. It can be written in the determinant form

$$\boldsymbol{\nabla} \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \widehat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{19.8.12}$$

By contrast to the scalar product, both the gradient of a scalar field, and the vector product, are vector fields for which the components along the coordinate axes transform in a specific manner, such as to keep the length of the vector constant, as the coordinate frame is rotated. The gradient, scalar and vector products with the $\boldsymbol{\nabla}$ operator are the first order derivatives of fields that occur most frequently in physics.

Second derivatives of fields also are used. Let us consider some possible combinations of the product of two del operators.

1) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} V) = \boldsymbol{\nabla}^2 V$

The scalar product of two del operators is a scalar under rotation. Evaluating the scalar product in cartesian coordinates gives

$$\left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \widehat{\mathbf{k}} \frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{i}} \frac{\partial V}{\partial x} + \hat{\mathbf{j}} \frac{\partial V}{\partial y} + \widehat{\mathbf{k}} \frac{\partial V}{\partial z}\right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{19.8.13}$$

This also can be obtained without confusion by writing this product as;

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} V) = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} V = (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}) V \tag{19.8.14}$$

where the scalar product of the del operator is a scalar, called the Laplacian ∇^2, given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{19.8.15}$$

The Laplacian operator is encountered frequently in physics.

2) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} V) = \mathbf{0}$

Note that the vector product of two identical vectors

$$\mathbf{A} \times \mathbf{A} = 0 \tag{19.8.16}$$

Therefore

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} V) = 0 \tag{19.8.17}$$

This can be confirmed by evaluating the separate components along each axis.

3) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = \mathbf{0}$

This is zero because the cross-product is perpendicular to $\boldsymbol{\nabla} \times \mathbf{A}$ and thus the dot product is zero.

4) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^{\mathbf{2}} \mathbf{A}$

The identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{19.8.18}$$

can be used to give

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{19.8.19}$$

since $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \nabla^2$.

There are pitfalls in the discussion of second derivatives in that it is assumed that both del operators operate on the same variable, otherwise the results are different.

Vector differential operators in curvilinear coordinates

As discussed in Appendix 19.3 there are many situations where the symmetries make it more convenient to use orthogonal curvilinear coordinate systems rather than cartesian coordinates. Thus it is necessary to extend vector derivatives from cartesian to curvilinear coordinates. Table $19.3.1$ can be used for expressing vector derivatives in curvilinear coordinate systems.

Gradient

The gradient in curvilinear coordinates is

$$\boldsymbol{\nabla} f = \frac{1}{h_1} \frac{\partial f}{\partial q_1} \hat{\mathbf{q}}_1 + \frac{1}{h_2} \frac{\partial f}{\partial q_2} \hat{\mathbf{q}}_2 + \frac{1}{h_3} \frac{\partial f}{\partial q_3} \hat{\mathbf{q}}_3 \tag{19.8.20}$$

where the coefficients h_i are listed in table $19.3.1$. For cylindrical coordinates this becomes

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial f}{\partial \varphi} \hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}} \tag{19.8.21}$$

In spherical coordinates

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \varphi} \hat{\boldsymbol{\varphi}} \tag{19.8.22}$$

Divergence

The divergence can be expressed as

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_3 h_1) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right] \tag{19.8.23}$$

In cylindrical coordinates the divergence is

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} = \frac{A_\rho}{\rho} + \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} \tag{19.8.24}$$

In spherical coordinates the divergence is

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} \left(A_r r^2 \sin\theta \right) + \frac{\partial}{\partial \theta} (A_\theta r \sin\theta) + \frac{\partial}{\partial \varphi} (A_\varphi r) \right] \tag{19.8.25}$$

Curl

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{q}}_1 & h_2 \hat{\mathbf{q}}_2 & h_3 \hat{\mathbf{q}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \tag{19.8.26}$$

In cylindrical coordinates the curl is

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix} \tag{19.8.27}$$

In spherical coordinates the curl is

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & r\rho A_\theta & r\sin\theta A_\varphi \end{vmatrix} \tag{19.8.28}$$

Laplacian

Taking the divergence of the gradient of a scalar gives

$$\nabla^2 f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial f}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial q_3} \right) \right] \tag{19.8.29}$$

The Laplacian of a scalar function f in cylindrical coordinates is

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \tag{19.8.30}$$

The Laplacian of a scalar function f in spherical coordinates is

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial^2 f}{\partial \varphi^2} \tag{19.8.31}$$

The gradient, divergence, curl and Laplacian are used extensively in curvilinear coordinate systems when dealing with vector fields in Newtonian mechanics, electromagnetism, and fluid flow.