5.10: Geodesic

The geodesic is defined as the shortest path between two fixed points for motion that is constrained to lie on a surface. Variational calculus provides a powerful approach for determining the equations of motion constrained to follow a geodesic.

The use of variational calculus is illustrated by considering the geodesic constrained to follow the surface of a sphere of radius R. As discussed in appendix $19.3.2C$, the element of path length on the surface of the sphere is given in spherical coordinates as $ds = R\sqrt{d\theta^2 + (\sin\theta d\phi)^2}$. Therefore the distance s between two points 1 and 2 is

$$s = R\int_1^2 \left[\sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}\right] d\phi \tag{5.10.1}$$

The function f for ensuring that s be an extremum value uses

$$f = \sqrt{\theta'^2 + \sin^2\theta} \tag{5.10.2}$$

where $\theta' = \frac{d\theta}{d\phi}$. This is a case where $\frac{\partial f}{\partial\phi} = 0$ and thus the integral form of Euler's equation can be used leading to the result that

$$\sqrt{\theta'^2 + \sin^2\theta} - \theta'\frac{\partial}{\partial\theta'}\sqrt{\theta'^2 + \sin^2\theta} = \text{ constant} = a \tag{5.10.3}$$

This gives that

$$\sin^2\theta = a\sqrt{\theta'^2 + \sin^2\theta} \tag{5.10.4}$$

This can be rewritten as

$$\frac{d\phi}{d\theta} = \frac{1}{\theta'} = \frac{a\csc^2\theta}{\sqrt{1 - a^2\csc^2\theta}} \tag{5.10.5}$$

Solving for ϕ gives

$$\phi = \sin^{-1}\left(\frac{\cot\theta}{\beta}\right) + \alpha \tag{5.10.6}$$

where

$$\beta \equiv \frac{1 - a^2}{a^2} \tag{5.10.7}$$

That is

$$\cot\theta = \beta\sin(\phi - \alpha) \tag{5.10.8}$$

Expanding the sine and cotangent gives

$$(\beta\cos\alpha)\, R\sin\theta\sin\phi - (\beta\sin\alpha)\, R\sin\theta\cos\phi = R\cos\theta \tag{5.10.9}$$

Since the brackets are constants, this can be written as

$$A\,(R\sin\theta\sin\phi) - B\,(R\sin\theta\cos\phi) = (R\cos\theta) \tag{5.10.10}$$

The terms in the brackets are just expressions for the rectangular coordinates x, y, z. That is,

$$Ay - Bx = z \tag{5.10.11}$$

This is the equation of a plane passing through the center of the sphere. Thus the geodesic on a sphere is the path where a plane through the center intersects the sphere as well as the initial and final locations. This geodesic is called a great circle. Euler's equation gives both the maximum and minimum extremum path lengths for motion on this great circle.

Chapter 17 discusses the geodesic in the four-dimensional space-time coordinates that underlie the General Theory of Relativity. As a consequence, the use of the calculus of variations to determine the equations of motion for geodesics plays a pivotal role in the General Theory of Relativity.