16.4: The Hamiltonian density formulation for continuous systems

Chapter 16.3 illustrated, in general terms, how field theory can be expressed in a Lagrangian formulation via use of the Lagrange density. It is equally possible to obtain a Hamiltonian formulation for continuous systems analogous to that obtained for discrete systems. As summarized in chapter 8, the Hamiltonian and Hamilton's canonical equations of motion are related directly to the Lagrangian by use of a Legendre transformation. The Hamiltonian is defined as being

$$H \equiv \sum_i \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L \tag{16.4.1}$$

The generalized momentum is defined to be

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{16.4.2}$$

Equation 16.4.2 allows the Hamiltonian 16.4.1 to be written in terms of the conjugate momenta as

$$H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t) = \sum_i (p_i \dot{q}_i - L_i(q_i, \dot{q}_i, t)) \tag{16.4.3}$$

where the Lagrangian has been partitioned into the terms for each of the individual coordinates, that is, $L(q_i, \dot{q}_i, t) = \sum_i L_i(q_i, \dot{q}_i, t)$.

In the limit that the coordinates q, p are continuous, then the summation in Equation 16.4.3 can be transformed into a volume integral over the Lagrangian density $\mathfrak{L}$. In addition, a momentum density can be represented by the vector field $\boldsymbol{\pi}$ where

$$\boldsymbol{\pi} \equiv \frac{\partial \mathfrak{L}}{\partial \dot{\mathbf{q}}} \tag{16.4.4}$$

Then the obvious definition of the Hamiltonian density $\mathfrak{H}$ is

$$H = \int \mathfrak{H} d\tau = \int (\boldsymbol{\pi} \cdot \dot{\mathbf{q}} - \mathfrak{L}) d\tau \tag{16.4.5}$$

where the Hamiltonian density is defined to be

$$\mathfrak{H} = \boldsymbol{\pi} \cdot \dot{\mathbf{q}} - \mathfrak{L} \tag{16.4.6}$$

Unfortunately the Hamiltonian density formulation does not treat space and time symmetrically making it more difficult to develop relativistically covariant descriptions of fields. Hamilton's principle can be used to derive the Hamilton equations of motion in terms of the Hamiltonian density analogous to the approach used to derive the Lagrangian density equations of motion. As described in Classical Mechanics 2^{nd} edition by Goldstein, the resultant Hamilton equations of motion for one dimension are

$$\frac{\partial \mathfrak{H}}{\partial \pi} = \dot{q} \tag{16.4.7}$$

$$\frac{\partial \mathfrak{H}}{\partial q} - \frac{d}{dx} \frac{\partial \mathfrak{H}}{\partial q'} = -\dot{\pi} \tag{16.4.8}$$

$$\frac{\partial \mathfrak{H}}{\partial t} = -\frac{\partial \mathfrak{L}}{\partial t} \tag{16.4.9}$$

Note that Equation 16.4.8 differs from that for discontinuous systems.