

15.5: Action-angle Variables

Canonical transformation

Systems possessing periodic solutions are a ubiquitous feature in physics. The periodic motion can be either an oscillation, for which the trajectory in phase space is a closed loop (libration), or rolling (rotational) motion as discussed in chapter 3.4. For many problems involving periodic motion, the interest often lies in the frequencies of motion rather than the detailed shape of the trajectories in phase space. The action-angle variable approach uses a canonical transformation to action and angle variables which provide a powerful, and elegant method to exploit Hamiltonian mechanics. In particular, it can determine the frequencies of periodic motion without having to calculate the exact trajectories for the motion. This method was introduced by the French astronomer Ch. E. Delaunay (1816 – 1872) for applications to orbits in celestial mechanics, but it has equally important applications beyond celestial mechanics such as to bound solutions of the atom in quantum mechanics.

The action-angle method replaces the momenta in the Hamilton-Jacobi procedure by the **action phase integral** for the closed loop (libration) trajectory in phase space defined by

$$J_i \equiv \oint p_i dq_i \quad (15.5.1)$$

where for each cyclic variable the integral is taken over one complete period of oscillation. The cyclic variable I_i is called the **action variable** where

$$I_i \equiv \frac{1}{2\pi} J_i = \frac{1}{2\pi} \oint p_i dq_i \quad (15.5.2)$$

The canonical variable to the action variable \mathbf{I} is the angle variable ϕ . Note that the name “action variable” is used to differentiate \mathbf{I} from the action functional $S = \int L dt$ which has the same units; i.e. angular momentum.

The general principle underlying the use of action-angle variables is illustrated by considering one body, of mass m , subject to a one-dimensional bound conservative potential energy $U(q)$. The Hamiltonian is given by

$$H(p, q) = \frac{p^2}{2m} + U(q) \quad (15.5.3)$$

This bound system has a (q, p) phase space contour for each energy $H = E$.

$$p(q, E) = \pm \sqrt{2m(E - U(q))} \quad (15.5.4)$$

For an oscillatory system the two-valued momentum of Equation 15.5.4 is non-trivial to handle. By contrast, the area $J \equiv \oint p dq$ of the closed loop in phase space is a single-valued scalar quantity that depends on E and $U(q)$. Moreover, Liouville’s theorem states that the area of the closed contour in phase space $J \equiv \oint p dq$ is invariant to canonical transformations. These facts suggest the use of a new pair of conjugate variables, (ϕ, I) , where $I(E)$ uniquely labels the trajectory, and corresponding area, of a closed loop in phase space for each value of E , and the single-valued function ϕ is a corresponding angle that specifies the exact point along the phase-space contour as illustrated in Fig 15.5.1.

For simplicity consider the linear harmonic oscillator where

$$U(q) = \frac{1}{2} m \omega^2 q^2 \quad (15.5.5)$$

Then the Hamiltonian, 15.5.3 equals

$$H(p, q) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 \quad (15.5.6)$$

Hamilton’s equations of motion give that

$$\dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q \quad (15.5.7)$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad (15.5.8)$$

The solution of equations 15.5.7 and 15.5.8 is of the form

$$q = C \cos(\omega(t - t_0)) \quad (15.5.9)$$

$$p = -m\omega C \sin \omega(t - t_0) \quad (15.5.10)$$

where C , and t_0 are integration constants. For the harmonic oscillator, equations 15.5.9 and 15.5.10 correspond to the usual elliptical contours in phase space, as illustrated in Figure 15.5.1.

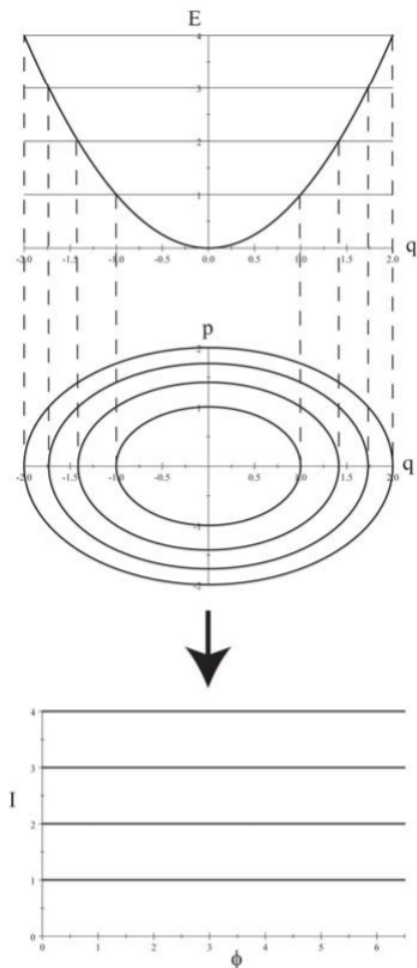


Figure 15.5.1: The potential energy $V(q)$, (upper) and corresponding phase space (p, q) (middle) for the harmonic oscillator at four equally spaced total energies E . The corresponding action-angles (I, ϕ) resulting from a canonical transformation of this system are shown in the lower plot.

The action-angle canonical transformation involves making the transform

$$(q, p) \rightarrow (\phi, I) \quad (15.5.11)$$

where I is defined by Equation 15.5.2 and the angle ϕ being the corresponding canonical angle. The logical approach to this canonical transformation for the harmonic oscillator is to define q and p in terms of ϕ and I

$$q = \sqrt{\frac{2I}{m\omega}} \cos \phi \quad (15.5.12)$$

$$p = \sqrt{2mI\omega} \sin \phi \quad (15.5.13)$$

Note that the Poisson bracket is unity

$$[q, p]_{(\phi, I)} = 1$$

which implies that the above transformation is canonical, and thus the phase space area $I(E) \equiv \frac{1}{2\pi} \oint p dq$ is conserved.

For this canonical transformation the transformed Hamiltonian $\mathcal{H}(\phi, I)$ is

$$\mathcal{H}(\phi, I) = \frac{1}{2m}(2m\omega I) \sin^2 \phi + \frac{1}{2}m\omega^2 \frac{2I}{m\omega} \cos^2 \phi = \omega I \quad (15.5.14)$$

Note that this Hamiltonian is a constant that is independent of the angle ϕ , and thus Hamilton's equations of motion give

$$\dot{I} = -\frac{\partial \mathcal{H}(\phi, I)}{\partial \phi} = 0 \quad (15.5.15)$$

$$\dot{\phi} = \frac{\partial \mathcal{H}(\phi, I)}{\partial I} = \omega \quad (15.5.16)$$

Thus we have mapped the harmonic oscillator to new coordinates (ϕ, I) where

$$I = \frac{\mathcal{H}(\phi, I)}{\omega} = \frac{E}{\omega} \quad (15.5.17)$$

$$\phi = \omega(t - t_0) \quad (15.5.18)$$

That is, the phase space has been mapped from ellipses, with area proportional to E in the (q, p) phase space, to a cylindrical (ϕ, I) phase space where $I = \frac{E}{\omega}$ are constant values that are independent of the angle, while ϕ increases linearly with time. Thus the variables (q, p) are periodic with modulus $\Delta\phi = 2\pi$.

$$q(\phi + 2\pi, I) = q(\phi, I) \quad (15.5.19)$$

$$p(\phi + 2\pi, I) = p(\phi, I) \quad (15.5.20)$$

The period τ of the periodic oscillatory motion is given simply by $\Delta\phi = 2\pi = \omega\tau$ which is the well known result for the harmonic oscillator. Note that the action-angle variable canonical transformation has determined the frequency of the periodic motion without solving the detailed trajectory of the motion.

The above example of the harmonic oscillator has shown that, for integrable periodic systems, it is possible to identify a canonical transformation to (ϕ, I) such that the Hamiltonian is independent of the angle ϕ which specifies the instantaneous location on the constant energy contour I . If the phase space contour is a separatrix, then it divides phase space into invariant regions containing phase-space contours with differing behavior. The action-angle variables are not useful for separatrix contours. For rolling motion, the system rotates with continuously increasing, or decreasing angle, and there is no natural boundary for the action angle variable since the phase space trajectory is continuous and not closed. However, the action-angle approach still is valid if the motion involves periodic as well as rolling motion.

The example of the one-dimensional, one-body, harmonic oscillator can be expanded to the more general case for many bodies in three dimensions. This is illustrated by considering multiple periodic systems for which the Hamiltonian is conservative and where the equations of the canonical transformation are separable. The generalized momenta then can be written as

$$p_i = \frac{\partial W_i(q_i; \alpha_1, \alpha_2, \dots, \alpha_n)}{\partial q_i} \quad (15.5.21)$$

for which each p_i is a function of q_i and the n integration constants α_j

$$p_i = p_i(q_i, \alpha_1, \alpha_2, \dots, \alpha_n) \quad (15.5.22)$$

The momentum $p_i(q_i, \alpha_1, \alpha_2, \dots, \alpha_n)$ represents the trajectory of the system in the (q_i, p_i) phase space that is characterized by Hamilton's characteristic function $W(q, J)$. Combining equations 15.5.1, 15.5.21 gives

$$J_i \equiv \oint \frac{\partial W_i(q_i; \alpha_1, \alpha_2, \dots, \alpha_n)}{\partial q_i} dq_i \quad (15.5.23)$$

Since q_i is merely a variable of integration, each active action variable J_i is a function of the n constants of integration in the Hamilton-Jacobi equation. Because of the independence of the separable-variable pairs (q_i, p_i) , the J_i form n independent functions of the α_i , and hence are suitable for use as a new set of constant momenta. Thus the characteristic function W can be written as

$$W(q_1, \dots, q_n; J_1, \dots, J_n) = \sum_j W_j(q_j; J_1, \dots, J_n) \quad (15.5.24)$$

while the Hamiltonian is only a function of the momenta $H(J_1, \dots, J_n)$

The generalized coordinate, conjugate to J , is known as the **angle variable** ϕ_i which is defined by the transformation equation

$$\phi_i = \frac{\partial W}{\partial J_i} = \sum_{j=1}^n \frac{\partial W_j(q_j; J_1, \dots, J_n)}{\partial J_i} \quad (15.5.25)$$

The corresponding equation of motion for ϕ is given by

$$\dot{\phi}_i = \frac{\partial H(J)}{\partial J_i} = 2\pi\omega_i(J_1, \dots, J_n) \quad (15.5.26)$$

where $\omega_i(J)$ are constant functions of the action variables J_j with a solution

$$\phi_i = 2\pi\omega_i t + \beta_i \quad (15.5.27)$$

that is, they are linear functions of time. The constants ω_i can be identified with the frequencies of the multiple periodic motions.

The action-angle variables appear to be no different than a particular set of transformed coordinates. Their merit appears when the physical interpretation is assigned to ω_i . Consider the change $\delta\phi_i$ as the q_j are changed infinitesimally

$$\delta\phi_i = \sum_j \frac{\partial \phi_i}{\partial q_j} \delta q_j = \sum_j \frac{\partial^2 W}{\partial J_i \partial q_j} \delta q_j \quad (15.5.28)$$

The derivative with respect to q_i vanishes except for the W_j component of W . Thus Equation 15.5.28 reduces to

$$\delta\phi_i = \frac{\partial}{\partial J_i} \sum_j p_j(q_j, J) \delta q_j \quad (15.5.29)$$

Therefore, the total change in ϕ , as the system goes through one complete cycle is

$$\Delta\phi_i = \sum_j \frac{\partial}{\partial J_i} \oint p_j(q_j, J) dq_j = 2\pi\delta_{ij} \quad (15.5.30)$$

where $\frac{\partial}{\partial J_i}$ is outside the integral since the J_i are constants for cyclic motion. Thus $\Delta\phi_i = 2\pi = \omega_i\tau_i$ where τ_i is the period for one cycle of oscillation, where the angular frequency ω_i is given by

$$\frac{\omega_i}{2\pi} = \nu_i = \frac{1}{\tau_i} \quad (15.5.31)$$

Thus the frequency ν associated with the periodic motion is the reciprocal of the period τ . The secret here is that the derivative of H with respect to the action variable J given by Equation 15.5.26 directly determines the frequency of the periodic motion without the need to solve the complete equations of motion. Note that multiple periodic motion can be represented by a Fourier expansion of the form

$$q_k = \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} \dots \sum_{j_n=-\infty}^{\infty} a_{j_1, \dots, j_n}^k e^{2\pi i(j_1\omega_1 + j_2\omega_2 + j_3\omega_3 + \dots + j_n\omega_n)} \quad (15.5.32)$$

Although the action-angle approach to Hamilton-Jacobi theory does not produce complete equations of motion, it does provide the frequency decomposition that often is the physics of interest. The reason that the powerful action-angle variable approach has been introduced here is that it is used extensively in celestial mechanics. The action-angle concept also played a key role in the development of quantum mechanics, in that Sommerfeld recognized that Bohr's ad hoc assumption that angular momentum is quantized, could be expressed in terms of quantization of the angle variable as is mentioned in chapter 18.

Adiabatic invariance of the action variables

When the Hamiltonian depends on time it can be quite difficult to solve for the motion because it is difficult to find constants of motion for time-dependent systems. However, if the time dependence is sufficiently slow, that is, if the motion is adiabatic, then there exist dynamical variables that are almost constant which can be used to solve for the motion. In particular, such approximate

constants are the familiar action-angle integrals. The adiabatic invariance of the action variables played an important role in the development of quantum mechanics during the 1911 Solvay Conference. This was a time when physicists were grappling with the concepts of quantum mechanics. Einstein used the following classical mechanics example of adiabatic invariance, applied to the simple pendulum, in order to illustrate the concept of adiabatic invariance of the action. This example demonstrates the power of using action-angle variables.

Example 15.5.1: Adiabatic invariance for the simple pendulum

Consider that the pendulum is made up of a point mass M suspended from a pivot by a light string of length L that is swinging freely in a vertical plane. Derive the dependence of the amplitude of the oscillations θ , assuming θ is small, if the string is very slowly shortened by a factor of 2, that is, assume that the change in length during one period of the oscillation is very small. The tension in the string T is given by

$$T = Mg\langle\cos\theta\rangle + \left\langle \frac{ML^2\dot{\theta}^2}{L} \right\rangle$$

Let the pendulum angle be oscillatory

$$\theta = \theta_0 \cos(\omega t + \varphi_0)$$

Then the average mean square amplitude and velocity over one period are

$$\begin{aligned}\langle\theta^2\rangle &= \langle[\theta_0 \cos(\omega t + \varphi_0)]^2\rangle = \frac{\theta_0^2}{2} \\ \langle\dot{\theta}^2\rangle &= \langle[-\theta_0\omega \sin(\omega t + \varphi_0)]^2\rangle = \frac{\omega^2\theta_0^2}{2}\end{aligned}$$

Since, for the simple pendulum, $\omega^2 = \frac{g}{L}$, then the tension in the string

$$T = Mg\left(1 - \frac{\langle\theta^2\rangle}{2}\right) + ML\langle\dot{\theta}^2\rangle = Mg\left(1 + \frac{\theta_0^2}{4}\right)$$

Assuming that θ_0 is a small angle, and that the change in length $-\Delta L$ is very small during one period τ , then the work done is

$$\Delta W = T\Delta L = -Mg\Delta L - Mg\frac{\theta_0^2}{4}\Delta L \quad (\text{a})$$

while the change in internal oscillator energy is

$$\Delta(-MgL \cos\theta_0) = \Delta\left[-MgL\left(1 - \frac{\theta_0^2}{2}\right)\right] = -Mg\Delta L + \frac{1}{2}Mg\Delta(L\theta_0^2) = -Mg\Delta L + \frac{1}{2}Mg\theta_0^2\Delta L + MgL\theta_0\Delta\theta_0 \quad (\text{b})$$

The work done must balance the increment in internal energy therefore

$$L\theta_0\Delta\theta_0 + \frac{3\theta_0^2\Delta L}{4} = 0$$

or

$$L\theta_0^2\Delta \ln(\theta_0 L^{\frac{3}{4}}) = 0$$

Therefore it follows that

$$(\theta_0 L^{\frac{3}{4}}) = \text{constant} \quad (\text{c})$$

or

$$\theta_0 \propto L^{-\frac{3}{4}}$$

Thus shortening the length of the pendulum string from L to $\frac{L}{2}$ adiabatically corresponds to the amplitude increasing by a factor 1.68.

Consider the action-angle integral for one closed period $\tau = \frac{2\pi}{\omega}$ for this problem

$$\begin{aligned}
 J &= \oint P_{\theta} d\theta \\
 &= \oint ML^2 \dot{\theta} \cdot \dot{\theta} dt \\
 &= ML^2 \langle \dot{\theta}^2 \rangle \frac{2\pi}{\omega} \\
 &= \pi ML^2 \theta_0^2 \omega \\
 &= \pi M g^{\frac{1}{2}} \theta_0^2 L^{\frac{3}{2}} = \text{constant}
 \end{aligned}$$

where that last step is due to Equation c.

The above example shows that the action integral $J = \text{constant}$, that is, it is invariant to an adiabatic change. In retrospect this result is as expected in that the action integral should be minimized.

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