16.9: Summary and Implications

The goal of this chapter is to provide a glimpse into the classical mechanics of the continua which introduces the Lagrangian density and Hamiltonian density formulations of classical mechanics.

Lagrangian density formulation

In three dimensional Lagrangian density $\mathfrak{L}(\mathbf{q}, \frac{d\mathbf{q}}{dt}, \boldsymbol{\nabla}\cdot\mathbf{q}, x, y, z, t)$ is related to the Lagrangian L by taking the volume integral of the Lagrangian density.

$$L = \int \mathfrak{L}(\mathbf{q}, \frac{d\mathbf{q}}{dt}, \boldsymbol{\nabla}\cdot\mathbf{q}, x, y, z, t) d\tau \tag{16.9.1}$$

Applying Hamilton's Principle to the three-dimensional Lagrangian density leads to the following set of differential equations of motion

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}}{\frac{\partial \mathbf{q}}{\partial t}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathfrak{L}}{\frac{\partial \mathbf{q}}{\partial x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathfrak{L}}{\frac{\partial \mathbf{q}}{\partial y}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathfrak{L}}{\frac{\partial \mathbf{q}}{\partial z}}\right) - \frac{\partial \mathfrak{L}}{\partial \mathbf{q}} = 0 \tag{16.9.2}$$

Hamiltonian density formulation

In the limit that the coordinates q, p are continuous, then the Hamiltonian density can be expressed in terms of a volume integral over the momentum density π and the Lagrangian density $\mathfrak{L}$ where

$$\boldsymbol{\pi} \equiv \frac{\partial \mathfrak{L}}{\partial \dot{\mathbf{q}}} \tag{16.9.3}$$

Then the obvious definition of the Hamiltonian density $\mathfrak{H}$ is

$$H = \int \mathfrak{H} dV = \int (\boldsymbol{\pi}\cdot\dot{\mathbf{q}} - \mathfrak{L}) d\tau \tag{16.9.4}$$

where the Hamiltonian density is given by

$$\mathfrak{H} = \boldsymbol{\pi}\cdot\dot{\mathbf{q}} - \mathfrak{L} \tag{16.9.5}$$

These Lagrangian and Hamiltonian density formulations are of considerable importance to field theory and fluid mechanics.

Linear elastic solids

The theory of continuous systems was applied to the case of linear elastic solids. The **stress tensor** $\mathbf{T}$ is a rank 2 tensor defined as the ratio of the force vector $d\mathbf{F}$ and the surface element vector $d\mathbf{A}$. That is, the force vector is given by the inner product of the stress tensor $\mathbf{T}$ and the surface element vector $d\mathbf{A}$.

$$d\mathbf{F} = \mathbf{T}\cdot d\mathbf{A} \tag{16.9.6}$$

The **strain tensor** $\boldsymbol{\sigma}$ also is a rank 2 tensor defined as the ratio of the strain vector $\boldsymbol{\xi}$ and infinitessimal area $d\mathbf{A}$.

$$d\boldsymbol{\xi} = \boldsymbol{\sigma}\cdot d\mathbf{A} \tag{16.9.7}$$

where the component form of the rank 2 strain tensor is

$$\boldsymbol{\sigma} = \frac{1}{2}\begin{vmatrix} \frac{d\xi_1}{dx_1} & \frac{d\xi_1}{dx_2} & \frac{d\xi_1}{dx_3} \\ \frac{d\xi_2}{dx_1} & \frac{d\xi_2}{dx_2} & \frac{d\xi_2}{dx_3} \\ \frac{d\xi_3}{dx_1} & \frac{d\xi_3}{dx_2} & \frac{d\xi_3}{dx_3} \end{vmatrix} \tag{16.9.8}$$

The modulus of elasticity is defined as the slope of the stress-strain curve. For linear, homogeneous, elastic matter, the potential energy density U separates into diagonal and off-diagonal components of the strain tensor

$$U=\frac{1}{2}\left[\lambda \sum_{i}(\sigma_{ii})^{2}+2\mu \sum_{ik}(\sigma_{ik})^{2}\right] \tag{16.9.9}$$

where the constants λ and μ are Lamé's moduli of elasticity which are positive. The stress tensor is related to the strain tensor by

$$T_{ij}=\lambda \delta_{ij} \sum_{k} \frac{\partial \xi_{k}}{\partial x_{k}}+\mu\left(\frac{d \xi_{i}}{d x_{j}}+\frac{d \xi_{j}}{d x_{i}}\right)=\lambda \delta_{ij} \sum_{k} \sigma_{kk}+2 \mu \sigma_{ij} \tag{16.9.10}$$

Electromagnetic field theory

The rank 2 Maxwell stress tensor $\mathbf{T}$ has components

$$T_{ij} \equiv \epsilon_{0}\left(E_{i} E_{j}-\frac{1}{2} \delta_{ij} E^{2}\right)+\frac{1}{\mu_{0}}\left(B_{i} B_{j}-\frac{1}{2} \delta_{ij} B^{2}\right) \tag{16.9.11}$$

The divergence theorem allows the total electromagnetic force, acting of the volume τ, to be written as

$$\mathbf{F}=\int\left(\boldsymbol{\nabla} \cdot \mathbf{T}-\epsilon_{0} \mu_{0} \frac{\partial \mathbf{S}}{\partial t}\right) d \tau=\oint \mathbf{T} \cdot d \mathbf{a}-\epsilon_{0} \mu_{0} \frac{d}{d t} \int \mathbf{S} \mathbf{d} \boldsymbol{\tau} \tag{16.9.12}$$

The total momentum flux density is given by

$$\frac{\partial}{\partial t}(\boldsymbol{\pi}_{mech}+\boldsymbol{\pi}_{field})=\boldsymbol{\nabla} \cdot \mathbf{T} \tag{16.9.13}$$

where the electromagnetic field momentum density is given by the Poynting vector $\mathbf{S}$ as $\boldsymbol{\pi}_{field}=\epsilon_{0} \mu_{0} \mathbf{S}$.

Ideal fluid dynamics

Mass conservation leads to the continuity equation

$$\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla} \cdot(\rho \mathbf{v})=0 \tag{16.9.14}$$

Euler's hydrodynamic equation gives

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v}=-\frac{1}{\rho} \boldsymbol{\nabla}(P+\rho V) \tag{16.9.15}$$

where V is the scalar gravitational potential. If the flow is irrotational and time independent then

$$\left(\frac{1}{2} \rho v^{2}+P+\rho V\right)=\text { constant } \tag{16.9.16}$$

Viscous fluid dynamics

For incompressible flow the stress tensor term simplifies to $\boldsymbol{\nabla} \cdot \mathbf{T}=\mu \boldsymbol{\nabla}^{2} \mathbf{v}$. Then the Navier-Stokes equation becomes

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v}\right]=-\boldsymbol{\nabla} P+\mu \boldsymbol{\nabla}^{2} \mathbf{v}+\mathbf{f} \tag{16.9.17}$$

where $\mu \boldsymbol{\nabla}^{2} \mathbf{v}$ is the viscosity drag term. The left-hand side of Equation 16.9.17represents the rate of change of momentum per unit volume while the right-hand side represents the summation of the forces per unit volume that are acting.

The Reynolds number is a dimensionless number that characterizes the ratio of inertial forces to viscous forces in a viscous medium. The evolution of flow from laminar flow to turbulent flow, with increase of Reynolds number, was discussed.

The classical mechanics of continuous fields encompasses a remarkably broad range of phenomena with important applications to laminar and turbulent fluid flow, gravitation, electromagnetism, relativity, and quantum fields.