6.6: Applying the Euler-Lagrange equations to classical mechanics

d'Alembert's principle of virtual work has been used to derive the Euler-Lagrange equations, which also satisfy Hamilton's Principle, and the Newtonian plausibility argument. These imply that the actual path taken in configuration space $(q_i, \dot{q}_i, t)$ is the one that minimizes the action integral $\int_{t_1}^{t_2} L(q_j, \dot{q}_j; t)dt$. As a consequence, the Euler equations for the calculus of variations lead to the Lagrange equations of motion.

$$\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right\} \equiv \Lambda_j L = \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t) + Q_j^{EXC} \tag{6.6.1}$$

for n variables, with m equations of constraint. The generalized forces Q_j^{EXC} are not included in the conservative, potential energy U, or the Lagrange multipliers approach for holonomic equations of constraint.[1]

The following is a logical procedure for applying the Euler-Lagrange equations to classical mechanics.

1) Select a set of independent generalized coordinates:

Select an optimum set of independent generalized coordinates as described in chapter $6.5.1$. Use of generalized coordinates is always advantageous since they incorporate the constraints, and can reduce the number of unknowns, both of which simplify use of Lagrangian mechanics

2) Partition of the active forces:

The active forces should be partitioned into the following three groups:

i. **Conservative one-body forces plus the velocity-dependent electromagnetic force** which can be characterized by the scalar potential U, that is absorbed into the Lagrangian. The gravitational forces plus the velocity-dependent electromagnetic force can be absorbed into the potential U as discussed in chapter 6.10. This approach is by far the easiest way to account for such forces in Lagrangian mechanics.
ii. **Holonomic constraint forces** provide algebraic relations that couple some of the generalized coordinates. This coupling can be used either to reduce the number of generalized coordinates used, or to determine these holonomic constraint forces using the Lagrange multiplier approach.
iii. **Generalized forces** provide a mechanism for introducing non-conservative and non-holonomic constraint forces into Lagrangian mechanics. Typically general forces are used to introduce dissipative forces.

Typical systems can involve a mixture of all three categories of active forces. For example, mechanical systems often include gravity, introduced as a potential, holonomic constraint forces are determined using Lagrange multipliers, and dissipative forces are included as generalized forces.

3) Minimal set of generalized coordinates:

The ability to embed constraint forces directly into the generalized coordinates is a tremendous advantage enjoyed by the Lagrangian and Hamiltonian variational approaches to classical mechanics. If the constraint forces are not required, then choice of a minimal set of generalized coordinates significantly reduces the number of equations of motion that need to be solved.

4) Derive the Lagrangian:

The Lagrangian is derived in terms of the generalized coordinates and including the conservative forces that are buried into the scalar potential U.

5) Derive the equations of motion:

Equation 6.6.1 is solved to determine the n generalized coordinates, plus the m Lagrange multipliers characterizing the holonomic constraint forces, plus any generalized forces that were included. The holonomic constraint forces then are given by evaluating the $\lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t)$ terms for the m holonomic forces.

In summary, in Lagrangian mechanics is based on energies which are scalars in contrast to Newtonian mechanics which is based on vector forces and momentum. As a consequence, Lagrange mechanics allows use of any set of independent generalized coordinates, which do not have to be orthogonal, and they can have very different units for different variables. The generalized coordinates can incorporate the correlations introduced by constraint forces.

The active forces are split into the following three categories;

1. Velocity-independent conservative forces are taken into account using scalar potentials U_i.
2. Holonomic constraint forces can be determined using Lagrange multipliers.
3. Non-holonomic constraints require use of generalized forces Q_j^{EXC}.

Use of the concept of scalar potentials is a trivial and powerful way to incorporate conservative forces in Lagrangian mechanics. The Lagrange multipliers approach requires using the Euler-Lagrange equations for $n+m$ coordinates but determines both holonomic constraint forces and equations of motion simultaneously. Non-holonomic constraints and dissipative forces can be incorporated into Lagrangian mechanics via use of generalized forces which broadens the scope of Lagrangian mechanics.

Note that the equations of motion resulting from the Lagrange-Euler algebraic approach are the same equations of motion as obtained using Newtonian mechanics. However, the Lagrangian is a scalar which facilitates rotation into the most convenient frame of reference. This can greatly simplify determination of the equations of motion when constraint forces apply. As discussed in chapter 17, the Lagrangian and the Hamiltonian variational approaches to mechanics are the only viable way to handle relativistic, statistical, and quantum mechanics.

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[2]Euler's differential equation is ubiquitous in Lagrangian mechanics. Thus, for brevity, it is convenient to define the concept of the **Lagrange linear operator** Λ_j, as described in table $19.6.1$.

$$\Lambda_j \equiv \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j} \tag{6.6.2}$$

where Λ_j operates on the Lagrangian L. Then Euler's equations can be written compactly in the form $\Lambda_j L = 0$.
