11.6: Hamiltonian

Since the center-of-mass Lagrangian is not an explicit function of time, then

$$\frac{dH_{cm}}{dt} = -\frac{\partial L_{cm}}{\partial t} = 0 \tag{11.6.1}$$

Thus *the center-of mass Hamiltonian H_{cm} is a constant of motion*. However, since the transformation to center of mass can be time dependent, then $H_{cm} \neq E$, that is, it does not include the total energy because the kinetic energy of the center-of-mass motion has been omitted from H_{cm}. Also, since no transformation is involved, then

$$H_{cm} = T_{cm} + U = E_{cm} \tag{11.6.2}$$

That is, the center-of-mass Hamiltonian H_{cm} equals the center-of-mass total energy. The center-of-mass Hamiltonian then can be written using the effective potential $(11.4.6)$ in the form

$$H_{cm} = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + U(r) = \frac{p_r^2}{2\mu} + \frac{l^2}{2\mu r^2} + U(r) = \frac{p_r^2}{2\mu} + U_{eff}(r) = E_{cm} \tag{11.6.3}$$

It is convenient to express the center-of-mass Hamiltonian H_{cm} in terms of the energy equation for the orbit in a central field using the transformed variable $u = \frac{1}{r}$. Substituting equations $(11.4.6)$ and $(11.5.3)$ into the Hamiltonian Equation 11.6.3 gives the *energy equation of the orbit*

$$\frac{l^2}{2\mu}\left[\left(\frac{du}{d\psi}\right)^2 + u^2\right] + U\left(u^{-1}\right) = E_{cm} \tag{11.6.4}$$

Energy conservation allows the Hamiltonian to be used to solve problems directly. That is, since

$$H_{cm} = \frac{\mu \dot{r}^2}{2} + \frac{l^2}{2\mu r^2} + U(r) = E_{cm} \tag{11.6.5}$$

then

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E_{cm} - U - \frac{l^2}{2\mu r^2}\right)} \tag{11.6.6}$$

The time dependence can be obtained by integration

$$t = \int \frac{\pm dr}{\sqrt{\frac{2}{\mu}\left(E_{cm} - U - \frac{l^2}{2\mu r^2}\right)}} + \text{constant} \tag{11.6.7}$$

An inversion of this gives the solution in the standard form $r = r(t)$. However, it is more interesting to find the relation between r and θ. From relation 11.6.7 for $\frac{dr}{dt}$ then

$$dt = \frac{\pm dr}{\sqrt{\frac{2}{\mu}\left(E_{cm} - U - \frac{l^2}{2\mu r^2}\right)}} \tag{11.6.8}$$

while equation $(11.4.2)$ gives

$$d\psi = \frac{l dt}{\mu r^2} = \frac{\pm l dr}{r^2\sqrt{2\mu\left(E_{cm} - U - \frac{l^2}{2\mu r^2}\right)}} \tag{11.6.9}$$

Therefore

$$\psi = \int \frac{\pm l dr}{r^2\sqrt{2\mu\left(E_{cm} - U - \frac{l^2}{2\mu r^2}\right)}} + \text{constant} \tag{11.6.10}$$

which can be used to calculate the angular coordinate. This gives the relation between the radial and angular coordinates which specifies the trajectory.

Although equations 11.6.6 and 11.6.10 formally give the solution, the actual solution can be derived analytically only for certain specific forms of the force law and these solutions differ for attractive versus repulsive interactions.