13.22: Stability of torque-free rotation of an asymmetric body

It is of interest to extend the prior discussion to address the stability of an asymmetric rigid rotor undergoing force-free rotation close to a principal axes, that is, when subject to small perturbations. Consider the case of a general asymmetric rigid body with $I_3 > I_2 > I_1$. Let the system start with rotation about the $\hat{\mathbf{e}}_1$ axis, that is, the principal axis associated with the moment of inertia I_1. Then

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{e}}_1 \tag{13.22.1}$$

Consider that a small perturbation is applied causing the angular velocity vector to be

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{e}}_1 + \lambda \hat{\mathbf{e}}_2 + \mu \hat{\mathbf{e}}_3 \tag{13.22.2}$$

where λ, μ are very small. The Euler equations $(13.21.1)$become

$$\begin{aligned}(I_2 - I_3)\lambda\mu - I_1\dot{\omega}_1 = 0 \\ (I_3 - I_1)\mu\omega_1 - I_2\dot{\lambda} = 0 \\ (I_1 - I_2)\omega_1\lambda - I_3\dot{\mu} = 0\end{aligned}$$

Assuming that the product $\lambda\mu$ in the first equation is negligible, then $\dot{\omega}_1 = 0$, that is, ω_1 is constant.

The other two equations can be solved to give

$$\dot{\lambda} = \left(\frac{(I_3 - I_1)}{I_2}\omega_1\right)\mu \tag{13.22.3}$$

$$\dot{\mu} = \left(\frac{(I_1 - I_2)}{I_3}\omega_1\right)\lambda \tag{13.22.4}$$

Take the time derivative of the first equation

$$\ddot{\lambda} = \left(\frac{(I_3 - I_1)}{I_2}\omega_1\right)\dot{\mu} \tag{13.22.5}$$

and substitute for $\dot{\mu}$ gives

$$\ddot{\lambda} + \left(\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\omega_1^2\right)\lambda = 0 \tag{13.22.6}$$

The solution of this equation is

$$\lambda(t) = Ae^{i\Omega_{1\lambda}t} + Be^{-i\Omega_{1\lambda}t} \tag{13.22.7}$$

where

$$\Omega_{1\lambda} = \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}} \tag{13.22.8}$$

Note that since it was assumed that $I_3 > I_2 > I_1$, then $\Omega_{1\lambda}$ is real. The solution for $\lambda(t)$ therefore represents a stable oscillatory motion with precession frequency $\Omega_{1\lambda}$. The identical result is obtained for $\Omega_{1\mu} = \Omega_{1\lambda} = \Omega_1$. Thus the motion corresponds to a stable minimum about the $\hat{\mathbf{e}}_1$ axis with oscillations about the $\lambda = \mu = 0$ minimum with period.

$$\Omega_1 = \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}} \tag{13.22.9}$$

Permuting the indices gives that for perturbations applied to rotation about either the 2 or 3 axes give precession frequencies

$$\Omega_2 = \omega_2\sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}} \tag{13.22.10}$$

$$\Omega_3 = \omega_3 \sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}} \tag{13.22.11}$$

Since $I_3 > I_2 > I_1$ then Ω_1 and Ω_3 are real while Ω_2 is imaginary. Thus, whereas rotation about either the I_3 or the I_1 axes are stable, the imaginary solution about $\hat{\mathbf{e}}_2$ corresponds to a perturbation increasing with time. Thus, only rotation about the largest or smallest moments of inertia are stable. Moreover for the symmetric rigid rotor, with $I_1 = I_2 \neq I_3$, stability exists only about the symmetry axis $\hat{\mathbf{e}}_3$ independent on whether the body is prolate or oblate. This result was implied from the discussion of energy and angular momentum conservation in chapter 13.20. Friction was not included in the above discussion. In the presence of dissipative forces, such as friction or drag, only rotation about the principal axis corresponding to the maximum moment of inertia is stable.

Stability of rigid-body rotation has broad applications to rotation of satellites, molecules and nuclei. The first U.S. satellite, Explorer 1, was launched in 1958 with the rotation axis aligned with the cylindrical axis which was the minimum principal moment of inertia. After a few hours the satellite started tumbling with increasing amplitude due to a flexible antenna dissipating and transferring energy to the perpendicular axis which had the largest moment of inertia. Torque-free motion of a deformed rigid body is a ubiquitous phenomena in many branches of science, engineering, and sports as illustrated by the following examples.

Example 13.22.1: Tennis racquet dynamics

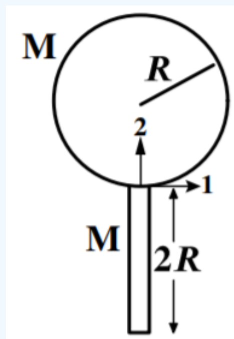

Figure 13.22.1: Principal rotation axes for the center of mass of a tennis racket. The 1 and 2 -axes are in the plane of the racket head and the 3 axis is perpendicular to the plane of the racket head.

A tennis racquet is an asymmetric body that exhibits the above rotational behavior. Assume that the head of a tennis racquet is a uniform thin circular disk of radius R and mass M which is attached to a cylindrical handle of diameter $r = \frac{R}{10}$, length $2R$, and mass M as shown in the figure. The principle moments of inertia about the three axes through the center-of-mass can be calculated by addition of the moments for the circular disk and the cylindrical handle and using both the parallel-axis and the perpendicular-axis theorems.

Table 13.22.1

Axis	Head	Handle	Racquet
1	$\frac{1}{4}MR^2 + MR^2 = \frac{5}{4}MR^2$	$\frac{4}{3}MR^2$	$\frac{31}{12}MR^2$
2	$\frac{1}{4}MR^2 + 0 = \frac{1}{4}MR^2$	$\frac{1}{200}MR^2$	$\frac{51}{200}MR^2$
3	$\frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$	$\frac{4}{3}MR^2$	$\frac{17}{6}MR^2$

Note that $I_{11} : I_{22} : I_{33} = 2.5833 : 0.2550 : 2.8333$. Inserting these principle moments of inertia into equations 13.22.9-13.22.11 gives the following precession frequencies

$$\Omega_1 = i0.8976\omega_1 \quad \Omega_2 = 0.9056\omega_2 \quad \Omega_3 = 0.9892\omega_3$$

The imaginary precession frequency Ω_1 about the 1 axis implies unstable rotation leading to tumbling whereas the minimum moment I_{22} and maximum moment I_{33} imply stable rotation about the 2 and 3 axes. This rotational behavior is easily demonstrated by throwing a tennis racquet and is called the tennis racquet theorem. The center of percussion, example $2.12.8$ is another important inertial property of a tennis racquet.

Example 13.22.2: Rotation of asymmetrically-deformed nuclei

Some nuclei and molecules have average shapes that have significant asymmetric deformation leading to interesting quantal analogs of the rotational properties of an asymmetrically-deformed rigid body. The major difference between a quantal and a classical rotor is that the energies, and angular momentum are quantized, rather than being continuously variable quantities. Otherwise, the quantal rotors exhibit general features similar to the classical analog. Studies [Cli86] of the rotational behavior of asymmetrically-deformed nuclei exploit three aspects of classical mechanics, namely classical Coulomb trajectories, rotational invariants, and the properties of ellipsoidal rigid-bodies.

Ellipsoidal deformation can be specified by the dimensions along each of the three principle axes. Bohr and Mottelson parameterized the ellipsoidal deformation in terms of three parameters, R_0 which is the radius of the equivalent sphere, β which is a measure of the magnitude of the ellipsoidal deformation from the sphere, and γ which specifies the deviation of the shape from axial symmetry. The ellipsoidal intrinsic shape can be expressed in terms of the deviation from the equivalent sphere by the equation

$$\delta R(\theta,\phi) = R(\theta,\phi) - R_0 = R_0 \sum_{\mu=-2}^{\mu+2} \alpha^*_{2\mu} Y_{2\mu}(\theta,\phi) \tag{a}$$

where $Y_{\lambda\mu}(\theta,\phi)$ is a Laplace spherical harmonic defined as

$$Y_{\lambda\mu}(\theta,\phi) = \sqrt{\frac{(2\lambda+1)}{4\pi}\frac{(\lambda-\mu)!}{(\lambda+\mu)!}} P_{\lambda\mu}(\cos\theta) e^{-i\mu\phi}$$

and $P_{\lambda\mu}(\cos\theta)$ is an associated Legendre function of $\cos\theta$. Spherical harmonics are the angular portion of a set of solutions to Laplace's equation. Represented in a system of spherical coordinates, Laplace's spherical harmonics $Y_{\lambda\mu}(\theta,\phi)$ are a specific set of spherical harmonics that form an orthogonal system. Spherical harmonics are important in many theoretical and practical applications.

In the principal axis frame of the body, there are three non-zero quadrupole deformation parameters which can be written in terms of the deformation parameters β,γ where $\alpha_{20} = \beta\cos\gamma$, $\alpha_{21} = \alpha_{2-1} = 0$, and $\alpha_{22} = \alpha_{2-2} = \frac{1}{\sqrt{2}}\beta\sin\gamma$. Using these in equations a give the three semi-axis dimensions in the principal axis frame, (primed frame),

$$\delta R_k = \sqrt{\frac{5}{4\pi}} R_0 \beta \cos(\gamma - \frac{2\pi k}{3}) \tag{b}$$

Note that for $\gamma = 0$, then $\delta R_1 = \delta R_2 = -\frac{1}{2}\sqrt{\frac{5}{4\pi}} R_0\beta$ while $\delta R_3 = +\sqrt{\frac{5}{4\pi}} R_0\beta$, that is the body has prolate deformation with the symmetry axis along the 3 axis. The same prolate shape is obtained for $\gamma = \frac{2\pi}{3}$ and $\gamma = \frac{4\pi}{3}$ with the prolate symmetry axes along the 1 and 2 axes respectively. For $\gamma = \frac{\pi}{3}$ then $\delta R_1 = \delta R_3 = +\frac{1}{2}\sqrt{\frac{5}{4\pi}} R_0\beta$ while $\delta R2 = -\sqrt{\frac{5}{4\pi}} R_0\beta$, that is the body has oblate deformation with the symmetry axis along the 2 axis. The same oblate shape is obtained for $\gamma = \pi$ and $\gamma = \frac{5\pi}{3}$ with the oblate symmetry axes along the 3 and 1 axes respectively. For other values of γ the shape is ellipsoidal.

For the asymmetric deformed rigid body, the rotational Hamiltonian can be expressed in the form[Dav58]

$$H = \sum_{k=1}^{3} \frac{|R|^2}{4B\beta^2 \sin^2(\gamma' - \frac{2\pi k}{3})}$$

where the rotational angular momentum is $\mathbf{R}$. The principal moments of inertia are related by the triaxiality parameter γ' which they assumed is identical to the shape parameter γ. For axial symmetry the moment of inertia about the symmetry axis is taken to be zero for a quantal system since rotation of the potential well about the symmetry axis corresponds to no change in the potential well, or corresponding rotation of the bound nucleons. That is, the nucleus is not a rigid body, the nucleons only rotate to the extent that the ellipsoidal potential well is cranked around such that the nucleons must follow the rotation of the potential well. In addition, vibrational modes coexist about the average asymmetric deformation, plus octupole deformation often coexists with the above quadrupole deformed modes.