11.S: Conservative two-body Central Forces (Summary)

This chapter has focussed on the classical mechanics of bodies interacting via conservative, two-body, central interactions. The following are the main topics presented in this chapter.

Equivalent one-body representation for two bodies interacting via a central interaction

The equivalent one-body representation of the motion of two bodies interacting via a two-body central interaction greatly simplifies solution of the equations of motion. The position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are expressed in terms of the center-of-mass vector $\mathbf{R}$ plus total mass $M = m_1 + m_2$ while the position vector $\mathbf{r}$, plus associated reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$, describe the relative motion of the two bodies in the center of mass. The total Lagrangian then separates into two independent parts

$$L = \frac{1}{2} M \left|\dot{\mathbf{R}}\right|^2 + L_{cm} \tag{11.S.1}$$

where the center-of-mass Lagrangian is

$$L_{cm} = \frac{1}{2}\mu |\dot{\mathbf{r}}|^2 - U(r) \tag{11.S.2}$$

Equations $(11.2.8)$, and $(11.2.9)$ can be used to derive the actual spatial trajectories of the two bodies expressed in terms of $\mathbf{r}_1$ and $\mathbf{r}_2$, from the relative equations of motion, written in terms of $\mathbf{R}$ and $\mathbf{r}$, for the equivalent one-body solution..

Angular momentum

Noether's theorem shows that the angular momentum is conserved if only a spherically-symmetric two-body central force acts between the interacting two bodies. The plane of motion is perpendicular to the angular momentum vector and thus the Lagrangian can be expressed in polar coordinates as

$$L_{cm} = \frac{1}{2}\mu \left(\dot{r}^2 + r^2\dot{\psi}^2\right) - U(r) \tag{11.S.3}$$

Differential orbit equation of motion

The Binet transformation $u = \frac{1}{r}$ allows the center-of-mass Lagrangian L_{cm} for a central force $\mathbf{F} = f(r)\hat{\mathbf{r}}$ to be used to express the differential orbit equation for the radial motion as

$$\frac{d^2u}{d\psi^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(\frac{1}{u}) \tag{11.S.4}$$

The Lagrangian, and the Hamiltonian all were used to derive the equations of motion for two bodies interacting via a two-body, conservative, central interaction. The general features of the conservation of angular momentum and conservation of energy for a two-body, central potential were presented.

Inverse-square, two-body, central force

The inverse-square, two-body, central force is of pivotal importance in nature since it is applies to both the gravitational force and the Coulomb force. The underlying symmetries of the inverse-square, two-body, central interaction, lead to conservation of angular momentum, conservation of energy, Gauss's law, and that the two-body orbits follow closed, degenerate, orbits that are conic sections, for which the eccentricity vector is conserved. The radial dependence, relative to the force center lying at one focus of the conic section, is given by

$$\frac{1}{r} = -\frac{\mu k}{l^2}[1 + \epsilon \cos(\psi - \psi_0)] \tag{11.S.5}$$

where the orbit eccentricity ϵ equals

$$\epsilon = \sqrt{1 + \frac{2E_{cm}l^2}{\mu k^2}} \tag{11.S.6}$$

These lead to Kepler's three laws of motion for two bodies in a bound orbit due to the attractive gravitational force for which $k = -Gm_1m_2$. The inverse-square law is special in that the eccentricity vector $\mathbf{A}$ is a third invariant of the motion, where

$$\mathbf{A} \equiv (\mathbf{p} \times \mathbf{L}) + (\mu k \hat{\mathbf{r}}) \tag{11.S.7}$$

The eccentricity vector unambiguously defines the orientation and direction of the major axis of the elliptical orbit. The invariance of the eccentricity vector, and the existence of stable closed orbits, are manifestations of the dynamical 04 symmetry.

Isotropic, harmonic, two-body, central force

The isotropic, harmonic, two-body, central interaction is of interest since, like the inverse-square law force, it leads to closed elliptical orbits described by

$$\frac{1}{r^2} = \frac{E\mu}{p_\psi^2}\left(1 + \left(1 + \frac{kp_\psi^2}{E^2\mu}\right)^{\frac{1}{2}} \cos 2(\psi - \psi_0)\right) \tag{11.S.8}$$

where the eccentricity ϵ is given by

$$\left(1 + \frac{kp_\psi^2}{E^2\mu}\right)^{\frac{1}{2}} = \frac{\epsilon^2}{2 - \epsilon^2} \tag{11.S.9}$$

The harmonic force orbits are distinctly different from those for the inverse-square law in that the force center is at the center of the ellipse, rather than at the focus for the inverse-square law force. This elliptical orbit is reflection symmetric for the harmonic force, but not for the inverse square force. The isotropic harmonic two-body force leads to invariance of the symmetry tensor, $\mathbf{A}'$ which is an invariant of the motion analogous to the eccentricity vector $\mathbf{A}$. This leads to stable closed orbits, which are manifestations of the dynamical $SU3$ symmetry.

Orbit stability

Bertrand's theorem states that only the inverse square law and the linear radial dependences of the central forces lead to stable closed bound orbits that do not precess. These are manifestation of the dynamical symmetries that occur for these two specific radial forms of two-body forces.

The three-body problem

The difficulties encountered in solving the equations of motion for three bodies, that are interacting via two-body central forces, was discussed. The three-body motion can include the existence of chaotic motion. It was shown that solution of the three-body problem is simplified if either the planar approximation, or the restricted three-body approximation, are applicable.

Two-body scattering

The total and differential two-body scattering cross sections were introduced. It was shown that for the inverse-square law force there is a simple relation between the impact parameter b and scattering angle θ given by

$$b = \frac{k}{2E_{cm}} \cot \frac{\theta}{2} \tag{11.S.10}$$

This led to the solution for the differential scattering cross-section for Rutherford scattering due to the Coulomb interaction.

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{k}{2E_{cm}}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} \tag{11.S.11}$$

This cross section assumes elastic scattering by a repulsive two-body inverse-square central force. For scattering of nuclei in the Coulomb potential the constant k is given to be

$$k = \frac{Z_p Z_T e^2}{4\pi\varepsilon_o} \tag{11.S.12}$$

Two-body kinematics

The transformation from the center-of-momentum frame to laboratory frames of reference was introduced. Such transformations are used extensively in many fields of physics for theoretical modelling of scattering, and for analysis of experiment data.
