2.8: Total Linear Momentum of a Many-body System

Center-of-mass decomposition

The total linear momentum $\mathbf{P}$ for a system of n particles is given by

$$\mathbf{P}=\sum_{i}^{n}\mathbf{p}_i=\frac{d}{dt}\sum_{i}^{n}m_i\mathbf{r}_i \tag{2.8.1}$$

It is convenient to describe a many-body system by a position vector $\mathbf{r}'_i$ with respect to the center of mass.

$$\mathbf{r}_i=\mathbf{R}+\mathbf{r}'_i \tag{2.8.2}$$

That is,

$$\mathbf{P}=\sum_{i}^{n}\mathbf{p}_i=\frac{d}{dt}\sum_{i}^{n}m_i\mathbf{r}_i=\frac{d}{dt}M\mathbf{R}+\frac{d}{dt}\sum_{i}^{n}m_i\mathbf{r}'_i=\frac{d}{dt}M\mathbf{R}+0=M\dot{\mathbf{R}} \tag{2.8.3}$$

since $\sum_i^n m_i\mathbf{r}'_i=0$ as given by th definition of the center of mass. That is,

$$\mathbf{P}=M\dot{\mathbf{R}} \tag{2.8.4}$$

Thus the total linear momentum for a system is the same as the momentum of a single particle of mass $M=\sum_i^n m_i$ located at the center of mass of the system.

Equations of motion

The force acting on particle i in an n-particle many-body system, can be separated into an external force $\mathbf{F}_i^{Ext}$ plus internal forces $\mathbf{f}_{ij}$ between the n particles of the system

$$\mathbf{F}_i=\mathbf{F}_i^E+\sum_{\substack{j\\ i\neq j}}^{n}\mathbf{f}_{ij} \tag{2.8.5}$$

The origin of the external force is from outside of the system while the internal force is due to the mutual interaction between the n particles in the system. Newton’s Law tells us that

$$\dot{\mathbf{p}}_i=\mathbf{F}_i=\mathbf{F}_i^E+\sum_{\substack{j\\ i\neq j}}^{n}\mathbf{f}_{ij} \tag{2.8.6}$$

Thus the rate of change of total momentum is

$$\dot{\mathbf{P}}=\sum_{i}^{n}\dot{\mathbf{p}}_i=\sum_{i}^{n}\mathbf{F}_i^E+\sum_{i}^{n}\sum_{\substack{j\\ i\neq j}}^{n}\mathbf{f}_{ij} \tag{2.8.7}$$

Note that since the indices are dummy then

$$\sum_{i}\sum_{\substack{j\\ i\neq j}}^{n}\mathbf{f}_{ij}=\sum_{j}\sum_{\substack{i\\ i\neq j}}^{n}\mathbf{f}_{ji} \tag{2.8.8}$$

Substituting Newton’s third law $\mathbf{f}_{ij}=-\mathbf{f}_{ji}$ into Equation 2.8.8 implies that

$$\sum_{i}\sum_{\substack{j\\ i\neq j}}^{n}\mathbf{f}_{ij}=\sum_{j}\sum_{\substack{i\\ i\neq j}}^{n}\mathbf{f}_{ji}=-\sum_{i}^{n}\sum_{\substack{j\\ i\neq j}}^{n}\mathbf{f}_{ij}=0 \tag{2.8.9}$$

which is satisfied only for the case where the summations equal zero. That is, for every internal force, there is an equal and opposite reaction force that cancels that internal force.

Therefore the first-order integral for linear momentum can be written in differential and integral forms as

$$\dot{\mathbf{P}} = \sum_i^n \mathbf{F}_I^E \qquad \int_1^2 \sum_i^n \mathbf{F}_i^E dt = \mathbf{P}_2 - \mathbf{P}_1 \tag{2.8.10}$$

The reaction of a body to an external force is equivalent to a single particle of mass M located at the center of mass assuming that the internal forces cancel due to Newton's third law.

Note that the total linear momentum $\mathbf{P}$ is conserved if the net external force $\mathbf{F}^E$ is zero, that is

$$\mathbf{F}^E = \frac{d\mathbf{P}}{dt} = 0 \tag{2.8.11}$$

Therefore the total linear momentum $\mathbf{P}$ of the center of mass is a constant. Moreover, if the component of the force along any direction $\hat{\mathbf{e}}$ is zero, that is,

$$\mathbf{F}^E \cdot \hat{\mathbf{e}} = \frac{d\mathbf{P}\cdot\hat{\mathbf{e}}}{dt} = 0 \tag{2.8.12}$$

then $\mathbf{P}\cdot\hat{\mathbf{e}}$ is a constant. This fact is used frequently to solve problems involving motion in a constant force field. For example, in the earth's gravitational field, the momentum of an object moving in vacuum in the vertical direction is time dependent because of the gravitational force, whereas the horizontal component of momentum is constant if no forces act in the horizontal direction.

Example 2.8.1: Exploding cannon shell

Consider a cannon shell of mass M moves along a parabolic trajectory in the earths gravitational field. An internal explosion, generating an amount E of mechanical energy, blows the shell into two parts. One part of mass kM, where $k < 1$, continues moving along the same trajectory with velocity v' while the other part is reduced to rest. Find the velocity of the mass kM immediately after the explosion.

It is important to remember that the energy release E is given in the center of mass. If the velocity of the shell immediately before the explosion is v and v' is the velocity of the kM part immediately after the explosion, then energy conservation gives that $\frac{1}{2}Mv^2 + E = \frac{1}{2}kMv'^2T$. The conservation of linear momentum gives $Mv = kMv'$. Eliminating v from these equations gives

$$v' = \sqrt{\frac{2E}{[k(1-k)M]}}$$

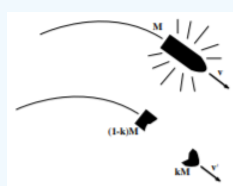

Figure 2.8.1: Exploding cannon shell

Example 2.8.2: Billiard-ball collisions

A billiard ball with mass m and incident velocity v collides with an identical stationary ball. Assume that the balls bounce off each other elastically in such a way that the incident ball is deflected at a scattering angle θ to the incident direction. Calculate the final velocities v_f and V_f of the two balls and the scattering angle ϕ of the target ball. The conservation of linear momentum in the incident direction x, and the perpendicular direction give

$$mv = mv_f \cos\theta + mV_f \cos\phi$$
$$0 = mv_f \sin\theta - mV_f \sin\phi$$

Energy conservation gives.

$$\frac{m}{2}v^2 = \frac{m}{2}v_f^2 + \frac{m}{2}V_f^2$$

Solving these three equations gives $\phi = 90^{\circ} - \theta$, that is, the balls bounce off perpendicular to each other in the laboratory frame. The final velocities are

$$v_f = v \cos \theta$$
$$V_f = v \sin \theta$$