6.1: Introduction to Lagrangian Dynamics

Newtonian mechanics is based on vector observables such as momentum and force, and Newton's equations of motion can be derived if the forces are known. However, Newtonian mechanics becomes difficult for many-body systems when constraint forces apply. The alternative algebraic Lagrangian mechanics approach is based on the concept of scalar energies which circumvent many of the difficulties in handling constraint forces and many-body systems.

The Lagrangian approach to classical dynamics is based on the calculus of variations introduced in chapter 5. It was shown that the calculus of variations determines the function $y_i(x)$ such that the scalar functional

$$F=\int_{x_1}^{x_2}\sum_i^n f[y_i(x),y_i'(x);x]dx \tag{6.1.1}$$

is an extremum, that is, a maximum or minimum. Here x is the independent variable, $y_i(x)$ are the n dependent variables, and their derivatives $y_i'\equiv\frac{dy_i}{dx}$, where $i=1,2,3,..n$. The function $f[y_i(x),y_i'(x);x]$ has an assumed dependence on y_i, y_i' and x. The calculus of variations determines the functional dependence of the dependent variables $y_i(x)$ on the independent variable x, that is needed to ensure that F is an extremum. For n *independent variables*, F has a stationary point, which is presumed to be an extremum, that is determined by solution of **Euler's differential equations**

$$\frac{d}{dx}\frac{\partial f}{\partial y_i'}-\frac{\partial f}{\partial y_i}=0 \tag{6.1.2}$$

If the coordinates $y_i(x)$ are independent, then the Euler equations, 6.1.2, for each coordinate i are independent. However, for constrained motion, the constraints lead to auxiliary conditions that correlate the coordinates. As shown in chapter 5, a transformation to *independent generalized coordinates* can be made such that the correlations induced by the constraint forces are embedded into the choice of the independent generalized coordinates. The use of generalized coordinates in Lagrangian mechanics simplifies derivation of the equations of motion for constrained systems. For example, for a system of n coordinates, that involves m holonomic constraints, there are $s=n-m$ independent generalized coordinates. For such holonomic constrained motion, it will be shown that the Euler equations can be solved using either of the following three alternative ways.

1) The **minimal set of generalized coordinates** approach involves finding a set of $s=n-m$ independent generalized coordinates q_i that satisfy the assumptions underlying 6.1.2. These generalized coordinates can be determined if the m equations of constraint are holonomic, that is, related by algebraic equations of constraint

$$g_k(q_i;x)=0 \tag{6.1.3}$$

where $k=1,2,3,\dots.m$. These equations uniquely determine the relationship between the n correlated coordinates. This method has the advantage that it reduces the system of n coordinates, subject to m constraints, to $s=n-m$ independent generalized coordinates which reduces the dimension of the problem to be solved. However, it does not explicitly determine the forces of constraint which are effectively swept under the rug.

2) The **Lagrange multipliers** approach takes account of the correlation between the n coordinates and m holonomic constraints by introducing the Lagrange multipliers $\lambda_k(x)$. These n generalized coordinates q_i are correlated by the m holonomic constraints.

$$\frac{d}{dx}\frac{\partial f}{\partial q_i'}-\frac{\partial f}{\partial q_i}=\sum_k^m\lambda_k(x)\frac{\partial g_k}{\partial q_i} \tag{6.1.4}$$

where $i=1,2,3,\dots n$. The Lagrange multiplier approach has the advantage that Euler's calculus of variations automatically use the n Lagrange equations, plus the m equations of constraint, to explicitly determine both the n coordinates q_i plus the m forces of constraint which are related to the Lagrange multipliers λ_k as given in Equation 6.1.4. Chapter 6.2 shows that the $\sum_k^m\lambda_k(x)\frac{\partial g_k}{\partial y_i}$ terms are directly related to the holonomic forces of constraint.

3) The **generalized force** approach incorporates the forces of constraint explicitly as will be shown in chapter $6.5.4$. Incorporating the constraint forces explicitly allows use of holonomic, non-holonomic, and non-conservative constraint forces.

Understanding the Lagrange formulation of classical mechanics is facilitated by use of a simple non-rigorous plausibility approach that is based on Newton's laws of motion. This introductory plausibility approach will be followed by two more rigorous derivations of the Lagrangian formulation developed using either d'Alembert Principle or Hamiltons Principle. These better elucidate the physics underlying the Lagrange and Hamiltonian analytic representations of classical mechanics. In 1788 Lagrange derived his equations of motion using the differential *d'Alembert Principle,* that extends to dynamical systems the Bernoulli Principle of infinitessimal virtual displacements and virtual work. The other approach, developed in 1834, uses the integral *Hamilton's Principle* to derive the Lagrange equations. Hamilton's Principle is discussed in more detail in chapter 9. Euler's variational calculus underlies d'Alembert's Principle and Hamilton's Principle since both are based on the philosophical belief that the laws of nature prefer economy of motion. Chapters $6.2 - 6.5$ show that both d'Alembert's Principle and Hamilton's Principle lead to the Euler-Lagrange equations. This will be followed by a series of examples that illustrate the use of Lagrangian mechanics in classical mechanics.