12.11: Free Motion on the Earth

The calculation of trajectories for objects as they move near the surface of the earth is frequently required for many applications. Such calculations require inclusion of the noninertial Coriolis force.

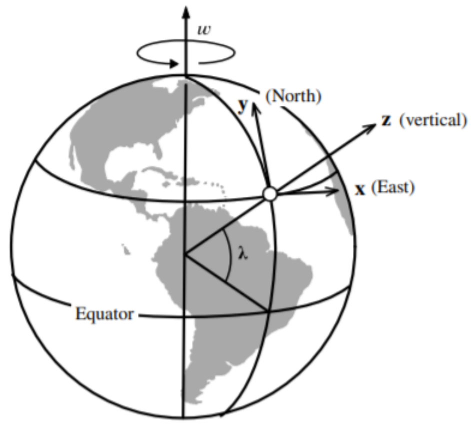

Figure 12.11.1: Rotating frame fixed on the surface of the Earth.

In the frame of reference fixed to the earth's surface, assuming that air resistance and other forces can be neglected, then the acceleration equals

$$\mathbf{a}' = \mathbf{g}_{eff} - 2\boldsymbol{\omega}\times\mathbf{v}' \tag{12.11.1}$$

Neglect the centrifugal correction term since it is very small, that is, let $\mathbf{g}_{eff} = \mathbf{g}$. Using the coordinate axis shown in Figure $12.11.1$, the surface-frame vectors have components

$$\boldsymbol{\omega} = 0\widehat{\mathbf{i}'} + \omega\cos\lambda\widehat{\mathbf{j}'} + \omega\sin\lambda\widehat{\mathbf{k}'} \tag{12.11.2}$$

and

$$\mathbf{g}_{eff} = -g\widehat{\mathbf{k}'} \tag{12.11.3}$$

Thus the Coriolis term is

$$\begin{aligned} 2\boldsymbol{\omega}\times\mathbf{v}' &= 2\begin{vmatrix} \widehat{\mathbf{i}'} & \widehat{\mathbf{j}'} & \widehat{\mathbf{k}'} \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix} \\ &= 2\left[\left(\omega\dot{z}'\cos\lambda - \omega\dot{y}'\sin\lambda\right)\widehat{\mathbf{i}'} + \left(\omega\dot{x}'\sin\lambda\right)\widehat{\mathbf{j}'} - \left(\omega\dot{x}'\cos\lambda\right)\widehat{\mathbf{k}'}\right] \end{aligned}$$

Therefore the equations of motion are

$$m\ddot{\mathbf{r}}' = -mg\widehat{\mathbf{k}'} - 2m[\widehat{\mathbf{i}'}(\dot{z}'\omega\cos\lambda - \dot{y}'\omega\sin\lambda) + \widehat{\mathbf{j}'}\dot{x}'\omega\sin\lambda - \widehat{\mathbf{k}'}\dot{x}'\omega\cos\lambda] \tag{12.11.4}$$

That is, the components of this equation of motion are

$$\begin{aligned} \ddot{x}' &= -2\omega(\dot{z}'\cos\lambda - \dot{y}'\sin\lambda) \\ \ddot{y}' &= -2\omega\dot{x}'\sin\lambda \\ \ddot{z}' &= -g + 2\omega\dot{x}'\cos\lambda \end{aligned}$$

Integrating these differential equations gives

$$\begin{aligned} \dot{x}' &= -2\omega(z'\cos\lambda - y'\sin\lambda) + \dot{x}'_0 \\ \dot{y}' &= -2\omega x'\sin\lambda + \dot{y}'_0 \\ \dot{z}' &= -gt + 2\omega x'\cos\lambda + \dot{z}'_0 \end{aligned}$$

where $\dot{x}_0', \dot{y}_0', \dot{z}_0'$ are the initial velocities. Substituting the above velocity relations into the equation of motion for $\ddot{x}$ gives

$$\ddot{x}' = 2\omega gt\cos\lambda - 2\omega(\dot{z}_0'\cos\lambda - \dot{y}_0'\sin\lambda) - 4\omega^2 x' \tag{12.11.5}$$

The last term $4\omega^2 x$ is small and can be neglected leading to a simple uncoupled second-order differential equation in x. Integrating this twice assuming that $x_0' = y_0' = z_0' = 0$, plus the fact that $2\omega gt\cos\lambda$ and $2\omega(\dot{z}_0'\cos\lambda - \dot{y}_0'\sin\lambda)$ are constant, gives

$$x' = \frac{1}{3}\omega gt^3\cos\lambda - \omega t^2(\dot{z}_0'\cos\lambda - \dot{y}_0'\sin\lambda) + \dot{x}_0' t \tag{12.11.6}$$

Similarly,

$$\begin{aligned} y' &= (\dot{y}_0' t - \omega\dot{x}_0' t^2\sin\lambda) \\ z' &= -\frac{1}{2}gt^2 + \dot{z}_0' t + \omega\dot{x}_0' t^2\cos\lambda \end{aligned}$$

Consider the following special cases;

Example 12.11.1: Free fall from rest

Assume that an object falls a height h starting from rest at $t=0$, $x=0$, $y=0$, $z=h$. Then

$$x' = \frac{1}{3}\omega gt^3\cos\lambda$$

$$y' = 0$$

$$z' = h - \frac{1}{2}gt^2$$

Substituting for t gives

$$x' = \frac{1}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}}$$

Thus the object drifts eastward as a consequence of the earth's rotation. Note that relative to the fixed frame it is obvious that the angular velocity of the body must increase as it falls to compensate for the reduced distance from the axis of rotation in order to ensure that the angular momentum is conserved.

Example 12.11.2: Projectile fired vertically upwards

An upward fired projectile with initial velocities $\dot{x}_0' = \dot{y}_0' = 0$ and $\dot{z}_0' = v_0$ leads to the relations

$$x' = \frac{1}{3}\omega gt^3\cos\lambda - \omega t^2 v_0\cos\lambda$$

$$y' = 0$$

$$z' = -\frac{1}{2}gt^2 + v_0 t$$

Solving for t when $z' = 0$ gives $t = 0$, and $t = \frac{2v_0}{g}$. Also since the maximum height h that the projectile reaches is related by

$$v_0 = \sqrt{2gh}$$

then the final deflection is

$$x' = -\frac{4}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}}$$

Thus the body drifts westwards.

Example 12.11.3: Motion parallel to Earth's surface

For motion in the horizontal $x' - y'$ plane the deflection is always to the right in the northern hemisphere of the Earth since the vertical component of ω is upwards and thus $-2\overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{v}'}$ points to the right. In the southern hemisphere the vertical component of ω is downward and thus $-2\overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{v}'}$ points to the left. This is also shown using the above relations for the case of a projectile fired upwards in an easterly direction with components $\dot{x}'_0, 0, \dot{z}'_0$. The resultant displacements are

$$x' = \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 \dot{z}'_0 \cos\lambda + \dot{x}'_0 t$$

Similarly,

$$y' = -\omega \dot{x}'_0 t^2 \sin\lambda$$

$$z' = -\frac{1}{2} g t^2 + \dot{z}'_0 t + \omega \dot{x}'_0 t^2 \cos\lambda$$

The trajectory is non-planar and, in the northern hemisphere, the projectile drifts to the right, that is southerly.

In the battle of the River de la Plata, during World War 2, the gunners on the British light cruisers Exeter, Ajax and Achilles found that their accurately aimed salvos against the German pocket battleship Graf Spee were falling 100 yards to the left. The designers of the gun sighting mechanisms had corrected for the Coriolis effect assuming the ships would fight at latitudes near 50° north, not 50° south.