10.5: Dissipative Lagrangians

The prior discussion of nonconservative systems mentioned the following three ways to incorporate dissipative processes into Lagrangian or Hamiltonian mechanics.

1. Expand the number of degrees of freedom to include all the active dissipative active degrees of freedom as well as the conservative ones.
2. Use generalized forces to incorporate dissipative processes.
3. Add dissipative terms to the Lagrangian or Hamiltonian to mimic dissipation.

The following illustrates the use of dissipative Lagrangians.

Bateman pointed out that an isolated dissipative system is physically incomplete, that is, a complete system must comprise at least two coupled subsystems where energy is transferred from a dissipating subsystem to an absorbing subsystem. A complete system should comprise both the dissipating and absorbing systems to ensure that the total system Lagrangian and Hamiltonian are conserved, as is assumed in conventional Lagrangian and Hamiltonian mechanics. Both Bateman and Dekker have illustrated that the equations of motion for a linearly-damped, free, one-dimensional harmonic oscillator are derivable using the Hamilton variational principle via introduction of a fictitious complementary subsystem that mimics dissipative processes. The following example illustrate that deriving the equations of motion for the linearly-damped, linear oscillator may be handled by three alternative equivalent non-standard Lagrangians that assume either: (1) a multidimensional system, (2) explicit time dependent Lagrangians and Hamiltonians, or (3) complex non-standard Lagrangians.

Example 10.5.1: The linearly-damped, linear oscillator

Three toy dynamical models have been used to describe the linearly-damped, linear oscillator employing very different non-standard Lagrangians to generate the required Hamiltonians, and to derive the correct equations of motion.

1: Dual-component Lagrangian: L_{Dual}

Bateman proposed a dual system comprising a mass m subject to two coupled one-dimensional variables (x, y) where x is the observed variable and y is the mirror variable for the subsystem that absorbs the energy dissipated by the subsystem x.

Assume a non-standard Lagrangian of the form

$$L_{Dual} = \frac{m}{2}\left[\dot{x}\dot{y} - \frac{\Gamma}{2}[y\dot{x} - x\dot{y}] - \omega_0^2 xy\right] \tag{a}$$

where $\Gamma = \frac{\lambda}{m}$ is the damping coefficient. Minimizing by variation of the auxiliary variable y, that is, $\Lambda_y L = 0$, leads to the uncoupled equation of motion for x

$$\frac{m}{2}\left[\ddot{x} + \Gamma\dot{x} + \omega_0^2 x\right] = 0 \tag{b}$$

Similarly minimizing by variation of the primary variable x, that is $\Lambda_x L = 0$, leads to the uncoupled equation of motion for y

$$\frac{m}{2}\left[\ddot{y} - \Gamma\dot{y} + \omega_0^2 y\right] = 0 \tag{c}$$

Note that equation of motion b, which was obtained by variation of the auxiliary variable y, corresponds to that for the usual free, linearly-damped, one-dimensional harmonic oscillator for the x variable which dissipates energy as is discussed in chapter 3.5. The equation of motion c is obtained by variation of the primary variable x and corresponds to a free linear, one-dimensional, oscillator for the y variable that is absorbing the energy dissipated by the dissipating x system.

The generalized momenta,

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

can be used to derive the corresponding Hamiltonian

$$H_{Dual}(x,p_x,y,p_y) = [p_x\dot{x}+p_y\dot{y}-L]$$
$$= \frac{p_xp_y}{2m} - \frac{\Gamma}{2}[xp_x - yp_y] + \frac{m}{2}\left(\omega_0^2 - \left(\frac{\Gamma}{2}\right)^2\right)xy \qquad (d)$$

Note that this Hamiltonian is time independent, and thus is conserved for this complete dual-variable system. Using Hamilton's equations of motion gives the same two uncoupled equations of motion as obtained using the Lagrangian, i.e. b and c.

2: Time-dependent Lagrangian: L_{Damped}

The complementary subsystem of the above dual-component Lagrangian, that is added to the primary dissipative subsystem, is the adjoint to the equations for the primary subsystem of interest. In some cases, a set of the solutions of the complementary equations can be expressed in terms of the solutions of the primary subsystem allowing the equations of motion to be expressed solely in terms of the variables of the primary subsystem. Inspection of the solutions of the damped harmonic oscillator, presented in chapter 3.5, implies that x and y must be related by the function

$$y = xe^{\Gamma t} \qquad (e)$$

Therefore Bateman proposed a time-dependent, non-standard Lagrangian L_{Damped} of the form

$$L_{Damped} = \frac{m}{2}e^{\Gamma t}\left[\dot{x}^2 - \omega_0^2x^2\right] \qquad (f)$$

This Lagrangian L_{Dampes} corresponds to a harmonic oscillator for which the mass $m = m_0e^{\Gamma t}$ is accreting exponentially with time in order to mimic the exponential energy dissipation. Use of this Lagrangian in the Euler-Lagrange equations gives the solution

$$me^{\Gamma t}\left[\ddot{x}+\Gamma\dot{x}+\omega_0^2x\right] = 0 \qquad (g)$$

If the factor outside of the bracket is non-zero, then the equation in the bracket must be zero. The expression in the bracket is the required equation of motion for the linearly-damped linear oscillator. This Lagrangian generates a generalized momentum of

$$p_x = me^{\Gamma t}\dot{x}$$

and the Hamiltonian is

$$H_{Damped} = p_x\dot{x} - L_2 = \frac{p_x^2}{2m}e^{-\Gamma t} + \frac{m}{2}\omega_0^2e^{\Gamma t}x^2 \qquad (h)$$

The Hamiltonian is time dependent as expected. This leads to Hamilton's equations of motion

$$\dot{x} = \frac{\partial H_{Damped}}{\partial p_x} = \frac{p_x}{m}e^{-\Gamma t} \qquad (i)$$

$$-\dot{p}_x = \frac{\partial H_{Damped}}{\partial x} = m\omega_0^2e^{\Gamma t}x \qquad (j)$$

Take the total time derivative of equation h and use equation i to substitute for $\dot{p}_x$ gives

$$me^{\Gamma t}\left[\ddot{x}+\Gamma\dot{x}+\omega_0^2x\right] = 0 \qquad (k)$$

If the term $me^{\Gamma t}$ is non-zero, then the term in brackets is zero. The term in the bracket is the usual equation of motion for the linearly-damped harmonic oscillator.

3: Complex Lagrangian: $L_{Complex}$

Dekker proposed use of complex dynamical variables for solving the linearly-damped harmonic oscillator. It exploits the fact that, in principle, each second order differential equation can be expressed in terms of a set of first-order differential equations. This feature is the essential difference between Lagrangian and Hamiltonian mechanics. Let q be complex and assume it can be expressed in the form of a real variable x as

$$q = \dot{x} - \left(i\omega + \frac{\Gamma}{2} \right) x \tag{l}$$

Substituting this complex variable into the relation

$$\dot{q} + \left[i\omega + \frac{\Gamma}{2} \right] q = 0 \tag{m}$$

leads to the second-order equation for the real variable x of

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 = 0 \tag{n}$$

This is the desired equation of motion for the linearly-damped harmonic oscillator. This result also can be shown by taking the time derivative of Equation m and taking only the real part, i.e.

$$\ddot{q} + i\omega\dot{q} + \frac{\Gamma}{2}\dot{q} = \ddot{q} + \left(i\omega - \frac{\Gamma}{2} \right) \dot{q} + \Gamma\dot{q} = \ddot{q} + \Gamma\dot{q} + \omega_0^2 x = 0 \tag{o}$$

This feature is exploited using the following Lagrangian

$$L_{Complex} = \frac{i}{2}(q^*\dot{q} - q\dot{q}^*) - \left[\omega - i\frac{\Gamma}{2} \right] q^* q \tag{p}$$

where $\omega^2 \equiv \omega_0^2 - \left(\frac{\Gamma}{2}\right)^2$. The Lagrangian $L_{Complex}$ is real for a conservative system and complex for a dissipative system. Using the Lagrange-Euler equation for variation of q^*, that is, $\Lambda_{q^*} L_{Complex} = 0$, gives Equation m which leads to the required equation of motion n.

The canonical conjugate momenta are given by

$$p = \frac{\partial L_{Complex}}{\partial \dot{q}} \tilde{p} = \frac{\partial L_{Complex}}{\partial \dot{q}^*} \tag{q}$$

The above Lagrangian plus canonically conjugate momenta lead to the complimentary Hamiltonians

$$H_{Complex}(p, q, \tilde{p}, q^*) = \left(i\omega + \frac{\Gamma}{2} \right) (\tilde{p}_* q^* - pq) \tag{s}$$

$$\tilde{H}_{Complex}(p, q, \tilde{p}, q^*) = \left(i\omega - \frac{\Gamma}{2} \right) (\tilde{p}_* q^* - pq) \tag{r}$$

These Hamiltonians give Hamilton equations of motion that lead to the correct equations of motion for q and q^*

The above examples have shown that three very different, non-standard, Lagrangians, plus their corresponding Hamiltonians, all lead to the correct equation of motion for the linearly-damped harmonic oscillator. This illustrates the power of using non-standard Lagrangians to describe dissipative motion in classical mechanics. However, postulating non-standard Lagrangians to produce the required equations of motion appears to be of questionable usefulness. A fundamental approach is needed to build a firm foundation upon which non-standard Lagrangian mechanics can be based. Non-standard Lagrangian mechanics remains an active, albeit narrow, frontier of classical mechanics