13.21: Torque-free rotation of an asymmetric rigid rotor

The Euler equations of motion for the case of torque-free rotation of an asymmetric (triaxial) rigid rotor about the center of mass, with principal moments of inertia $I_1 \neq I_2 \neq I_3$, lead to more complicated motion than for the symmetric rigid rotor.[3] The general features of the motion of the asymmetric rotor can be deduced using the conservation of angular momentum and rotational kinetic energy.

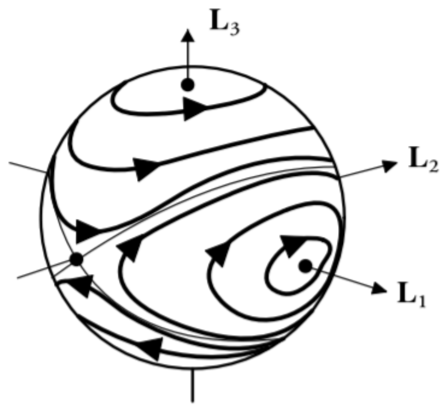

Figure 13.21.1: Rotation of an asymmetric rigid rotor. The dark lines correspond to contours of constant total rotational kinetic energy T, which has an ellipsoidal shape, projected onto the angular momentum L sphere in the body-fixed frame.

Assuming that the external torques are zero then the Euler equations of motion can be written as

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \end{aligned} \tag{13.21.1}$$

Since $L_i = I_i\omega_i$ for $i = 1, 2, 3$, then Equation 13.21.1gives

$$\begin{aligned} I_2I_3\dot{L}_1 &= (I_2 - I_3)L_2L_3 \\ I_1I_3\dot{L}_2 &= (I_3 - I_1)L_3L_1 \\ I_1I_2\dot{L}_3 &= (I_1 - I_2)L_1L_2 \end{aligned} \tag{13.21.2}$$

Multiply the first equation by I_1L_1, the second by I_2L_2 and the third by I_3L_3 and sum, which gives

$$I_1I_2I_3\left(L_1\dot{L}_1 + L_2\dot{L}_2 + L_3\dot{L}_3\right) = 0 \tag{13.21.3}$$

The bracket is equivalent to $\frac{d}{dt}(L_1^2 + L_2^2 + L_3^2) = 0$ which implies that *the total rotational angular momentum L is a constant of motion* as expected for this torque-free system, even though the individual components L_1, L_2, L_3 may vary. That is

$$L_1^2 + L_2^2 + L_3^2 = L^2 \tag{13.21.4}$$

Note that *equation* 13.21.4*is the equation of a sphere of radius L.*

Multiply the first equation of 13.21.2by L_1, the second by L_2, and the third by L_3, and sum gives

$$I_2I_3L_1\dot{L}_1 + I_1I_3L_2\dot{L}_2 + I_1I_2L_3\dot{L}_3 = 0 \tag{13.21.5}$$

Divide 13.21.5by $I_1I_2I_3$ gives $\frac{d}{dt}(\frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}) = 0$. This implies that the *total rotational kinetic energy T*, given by

$$\frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} = T \tag{13.21.6}$$

is a constant of motion as expected when there are no external torques and zero energy dissipation. Note that 13.21.6 *is the equation of an ellipsoid.*

Equations 13.21.4and 13.21.6both must be satisfied by the rotational motion for any value of the total angular momentum $\mathbf{L}$ and kinetic energy T. Fig 13.21.1 shows a graphical representation of the intersection of the L sphere and T ellipsoid as seen *in the* *body-fixed frame*. The angular momentum vector $\mathbf{L}$ must follow the constant-energy contours given by where the T-ellipsoids intersect the L-sphere, shown for the case where $I_3 > I_2 > I_1$. Note that the precession of the angular momentum vector $\mathbf{L}$ follows a trajectory that has closed paths that circle around the principal axis with the smallest I, that is, $\hat{\mathbf{e}}_1$, or the principal axis with the maximum I, that is, $\hat{\mathbf{e}}_3$. However, the angular momentum vector does not have a stable minimum for precession around the intermediate principal moment of inertia axis $\hat{\mathbf{e}}_2$. In addition to the precession, the angular momentum vector $\mathbf{L}$ executes nutation, that is a nodding of the angle θ. For any fixed value of L, the kinetic energy has upper and lower bounds given by

$$\frac{L^2}{2I_3} \leq T \leq \frac{L^2}{2I_1} \tag{13.21.7}$$

Thus, for a given value of L, when $T = T_{\min} = \frac{L^2}{2I_3}$, the orientation of $\mathbf{L}$ in the body-fixed frame is either $(0, 0, +L)$ or $(0, 0, -L)$, that is, aligned with the $\hat{\mathbf{e}}_3$ axis along which the principal moment of inertia is largest. For slightly higher kinetic energy the trajectory of L follows closed paths precessing around $\hat{\mathbf{e}}_3$. When the kinetic energy $T = \frac{L_2^2}{2I_2}$ the angular momentum vector L follows either of the two thin-line trajectories each of which are a separatrix. These do not have closed orbits around $\hat{\mathbf{e}}_2$ and they separate the closed solutions around either $\hat{\mathbf{e}}_3$ or $\hat{\mathbf{e}}_1$. For higher kinetic energy the precessing angular momentum vector follows closed trajectories around $\hat{\mathbf{e}}_1$ and becomes fully aligned with $\hat{\mathbf{e}}_1$ at the upper-bound kinetic energy.

Note that for the special case when $I_3 > I_2 = I_1$, then the asymmetric rigid rotor equals the symmetric rigid rotor for which the solutions of Euler's equations were solved exactly in chapter 13.19. For the symmetric rigid rotor the T-ellipsoid becomes a spheroid aligned with the symmetry axis and thus the intersections with the L-sphere lead to circular paths around the $\hat{\mathbf{e}}_3$ body-fixed principal axis, while the separatrix circles the equator corresponding to the $\hat{\mathbf{e}}_3$ axis separating clockwise and anticlockwise precession about $\mathbf{L}_3$. This discussion shows that energy, plus angular momentum conservation, provide the general features of the solution for the torque-free symmetric top that are in agreement with those derived using Euler's equations of motion.

[3] Similar discussions of the freely-rotating asymmetric top are given by Landau and Lifshitz [La60] and by Gregory [Gr06].