13.17: Euler’s equations of motion for rigid-body rotation

Rigid-body rotation can be confusing in that two coordinate frames are involved and, in general, the angular velocity and angular momentum are not aligned. The motion of the rigid body is observed in the space-fixed inertial frame whereas it is simpler to calculate the equations of motion in the body-fixed principal axis frame, for which the inertia tensor is known and is constant. The rigid body is rotating with angular velocity vector $\boldsymbol{\omega}$, which is not aligned with the angular momentum $\mathbf{L}$. For torque-free angular momentum, $\mathbf{L}$ is conserved and has a fixed orientation in the space-fixed axis system. Euler’s equations of motion, presented below, are given in the body-fixed frame for which the inertial tensor is known since this simplifies solution of the equations of motion. However, this solution has to be rotated back into the space-fixed frame to describe the rotational motion as seen by an observer in the inertial frame.

This chapter has introduced the inertial properties of a rigid body, as well as the Euler angles for transforming between the body-fixed and inertial frames of reference. This has prepared the stage for solving the equations of motion for rigid-body motion, namely, the dynamics of rotational motion about a body-fixed point under the action of external forces. The Euler angles are used to specify the instantaneous orientation of the rigid body.

In Newtonian mechanics, the rotational motion is governed by the equivalent Newton’s second law given in terms of the external torque $\mathbf{N}$ and angular momentum $\mathbf{L}$

$$\mathbf{N}=\left(\frac{d\mathbf{L}}{dt}\right)_{space} \tag{13.17.1}$$

Note that this relation is expressed in the inertial space-fixed frame of reference, not the non-inertial body-fixed frame. The subscript $space$ is added to emphasize that this equation is written in the inertial space-fixed frame of reference. However, as already discussed, it is much more convenient to transform from the space-fixed inertial frame to the body-fixed frame for which the inertia tensor of the rigid body is known. Thus the next stage is to express the rotational motion in terms of the body-fixed frame of reference. For simplicity, translational motion will be ignored.

The rate of change of angular momentum can be written in terms of the body-fixed value, using the transformation from the space-fixed inertial frame $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ to the rotating frame $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ as given in chapter 13.13,

$$\mathbf{N}=\left(\frac{d\mathbf{L}}{dt}\right)_{space}=\left(\frac{d\mathbf{L}}{dt}\right)_{body}+\boldsymbol{\omega}\times\mathbf{L} \tag{13.17.2}$$

However, the body axis $\hat{\mathbf{e}}_i$ is chosen to be the principal axis such that

$$L_i = I_i\omega_i \tag{13.17.3}$$

where the principal moments of inertia are written as I_i. Thus the equation of motion can be written using the body-fixed coordinate system as

$$\mathbf{N} = I_1\dot{\omega}_1\hat{\mathbf{e}}_1 + I_2\dot{\omega}_2\hat{\mathbf{e}}_2 + I_3\dot{\omega}_3\hat{\mathbf{e}}_3 + \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix} \tag{13.17.4}$$

$$= (I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3)\hat{\mathbf{e}}_1 + (I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1)\hat{\mathbf{e}}_2 + (I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2)\hat{\mathbf{e}}_3 \tag{13.17.5}$$

where the components in the body-fixed axes are given by

$$\begin{aligned} N_1 &= I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 \\ N_2 &= I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \\ N_3 &= I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 \end{aligned} \tag{13.17.6}$$

These are the **Euler equations for rigid body in a force field** expressed in the *body-fixed coordinate* frame. They are applicable for any applied external torque $\mathbf{N}$.

The motion of a rigid body depends on the structure of the body only via the three principal moments of inertia I_1, I_2, and I_3. Thus all bodies having the same principal moments of inertia will behave exactly the same even though the bodies may have very different shapes. As discussed earlier, the simplest geometrical shape of a body having three different principal moments is a homogeneous ellipsoid. Thus, the rigid-body motion often is described in terms of the equivalent ellipsoid that has the same principal moments.

A deficiency of Euler's equations is that the solutions yield the time variation of $\boldsymbol{\omega}$ as seen from the body-fixed reference frame axes, and not in the observers fixed inertial coordinate frame. Similarly the components of the external torques in the Euler equations are given with respect to the body-fixed axis system which implies that the orientation of the body is already known. Thus for non-zero external torques the problem cannot be solved until the the orientation is known in order to determine the components N_i^{ext}. However, these difficulties disappear when the external torques are zero, or if the motion of the body is known and it is required to compute the applied torques necessary to produce such motion.
