13.5: Matrix and Tensor Formulations of Rigid-Body Rotation

The prior notation is clumsy and can be streamlined by use of matrix methods. Write the inertia tensor in a matrix form as

$$\{\mathbb{I}\} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \tag{13.5.1}$$

The angular velocity and angular momentum both can be written as a column vectors, that is

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \quad \mathbf{L} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} \tag{13.5.2}$$

As discussed in appendix $19.5.2$, Equation $13.4.7$ now can be written in tensor notation as an inner product of the form

$$L = \{\mathbb{I}\} \cdot \boldsymbol{\omega} \tag{13.5.3}$$

Note that the above notation uses boldface for the inertia tensor $\mathbb{I}$, implying a rank-2 tensor representation, while the angular velocity $\boldsymbol{\omega}$ and the angular momentum $\mathbf{L}$ are written as column vectors. The inertia tensor is a 9-component rank-2 tensor defined as the ratio of the angular momentum vector $\mathbf{L}$ and the angular velocity $\boldsymbol{\omega}$.

$$\{\mathbb{I}\} = \frac{\mathbf{L}}{\boldsymbol{\omega}} \tag{13.5.4}$$

Note that, as described in appendix 19.5, the inner product of a vector $\boldsymbol{\omega}$, which is the rank 1 tensor, and a rank 2 tensor $\{\mathbb{I}\}$, leads to the vector $\mathbf{L}$. This compact notation exploits the fact that the matrix and tensor representation are completely equivalent, and are ideally suited to the description of rigid-body rotation.