2.S: Newtonian Mechanics (Summary)

Newton's Laws of Motion

A cursory review of Newtonian mechanics has been presented. The concept of inertial frames of reference was introduced since Newton's laws of motion apply only to inertial frames of reference.

Newton's Law of motion

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{2.2.2}$$

leads to second-order equations of motion which can be difficult to handle for many-body systems.

Solution of Newton's second-order equations of motion can be simplified using the three first-order integrals coupled with corresponding conservation laws. The first-order time integral for linear momentum is

$$\int_1^2 \mathbf{F}_i dt = \int_1^2 \frac{d\mathbf{p}_i}{dt} dt = (\mathbf{p}_2 - \mathbf{p}_1)_i \tag{2.4.1}$$

The first-order time integral for angular momentum is

$$\frac{d\mathbf{L}_i}{dt} = \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} = \mathbf{N}_i \qquad \int_1^2 \mathbf{N}_i dt = \int_1^2 \frac{d\mathbf{L}_i}{dt} dt = (\mathbf{L}_2 - \mathbf{L}_1)_i \tag{2.4.7}$$

The first-order spatial integral is related to kinetic energy and the concept of work. That is

$$\mathbf{F}_i = \frac{dT_i}{d\mathbf{r}_i} \qquad \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = (T_2 - T_1)_i \tag{2.4.12}$$

The conditions that lead to conservation of linear and angular momentum and total mechanical energy were discussed for many-body systems. The important class of conservative forces was shown to apply if the position-dependent force do not depend on time or velocity, and if the work done by a force $\int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$ is independent of the path taken between the initial and final locations. The total mechanical energy is a constant of motion when the forces are conservative.

It was shown that the concept of center of mass of a many-body or finite sized body separates naturally for all three first-order integrals. The center of mass is that point about which

$$\sum_i^n m_i \mathbf{r}_i' = \int \mathbf{r}' \rho dV = 0 \tag{Centre of mass definition}$$

where $\mathbf{r}_i'$ is the vector defining the location of mass m_i with respect to the center of mass. The concept of center of mass greatly simplifies the description of the motion of finite-sized bodies and many-body systems by separating out the important internal interactions and corresponding underlying physics, from the trivial overall translational motion of a many-body system..

The Virial theorem states that the time-averaged properties are related by

$$\langle T \rangle = -\frac{1}{2} \left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle \tag{2.11.7}$$

It was shown that the Virial theorem is useful for relating the time-averaged kinetic and potential energies, especially for cases involving either linear or inverse-square forces.

Typical examples were presented of application of Newton's equations of motion to solving systems involving constant, linear, position-dependent, velocity-dependent, and time-dependent forces, to constrained and unconstrained systems, as well as systems with variable mass. Rigid-body rotation about a body-fixed rotation axis also was discussed.

It is important to be cognizant of the following limitations that apply to Newton's laws of motion:

1) Newtonian mechanics assumes that all observables are measured to unlimited precision, that is $t, E, \mathbf{P}, \mathbf{r}$ are known exactly. Quantum physics introduces limits to measurement due to wave-particle duality.

2) The Newtonian view is that time and position are absolute concepts. The Theory of Relativity shows that this is not true. Fortunately for most problems $v << c$ and thus Newtonian mechanics is an excellent approximation.

3) Another limitation, to be discussed later, is that it is impractical to solve the equations of motion for many interacting bodies such as molecules in a gas. Then it is necessary to resort to using statistical averages, this approach is called statistical mechanics.

Newton's work constitutes a theory of motion in the universe that introduces the concept of causality. Causality is that there is a one-to-one correspondence between cause of effect. Each force causes a known effect that can be calculated. Thus the causal universe is pictured by philosophers to be a giant machine whose parts move like clockwork in a predictable and predetermined way according to the laws of nature. This is a deterministic view of nature. There are philosophical problems in that such a deterministic viewpoint appears to be contrary to free will. That is, taken to the extreme it implies that you were predestined to read this book because it is a natural consequence of this mechanical universe!

Newton's Laws of Gravitation

Newton's Laws of Gravitation and the Laws of Electrostatics are essentially identical since they both involve a central inverse square-law dependence of the forces. The important difference is that the gravitational force is attractive whereas the electrostatic force between identical charges is repulsive. That is, the gravitational constant G is replaced by $\frac{1}{4\pi\epsilon_0}$, and the mass density ρ becomes the charge density for the case of electrostatics. As a consequence it is unnecessary to make a detailed study of Newton's law of gravitation since it is identical to what has already been studied in your accompanying electrostatic courses. Table $2.S.1$ summarizes and compares the laws of gravitation and electrostatics. For both gravitation and electrostatics the field is central and conservative and depends as $\frac{1}{r^2}\hat{\mathbf{r}}$.

The laws of gravitation and electrostatics can be expressed in a more useful form in terms of the flux and circulation of the gravitational field as given either in the vector integral or vector differential forms. The radial independence of the flux, and corresponding divergence, is a statement that the fields are radial and have a $\frac{1}{r^2}\hat{\mathbf{r}}$ dependence. The statement that the circulation, and corresponding curl, are zero is a statement that the fields are radial and conservative.

Table $2.S.1$: Comparison of Newton's law of gravitation and electrostatics.

	Gravitation	Electrostatics
Force field	$\mathbf{g} \equiv \frac{\mathbf{F}_G}{m}$	$\mathbf{E} \equiv \frac{\mathbf{F}_E}{g}$
Density	Mass density $\rho(\mathbf{r}')$	Charge density $\rho(\mathbf{r}')$
Conservative central field	$\mathbf{g}(\mathbf{r}) = -G\int_V \frac{\rho(\mathbf{r}')(\hat{\mathbf{r}}-\hat{\mathbf{r}'})}{(\mathbf{r}-\mathbf{r}')^2}dv'$	$\mathbf{E}(\bar{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')(\hat{\mathbf{r}}-\hat{\mathbf{r}'})}{(\mathbf{r}-\mathbf{r}')^2}dv'$
Flux	$\Phi \equiv \int_S \mathbf{g}\cdot d\mathbf{S} = -4\pi G\int_{\substack{enclosed\\volume}} \rho dv$	$\Phi \equiv \int_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\epsilon_0}\int_{\substack{enclosed\\volume}} \rho dv$
Circulation	$\oint \mathbf{g}_{net}\cdot d\mathbf{l} = 0$	$\oint \mathbf{E}_{net}\cdot d\mathbf{l} = 0$
Divergence	$\nabla\cdot\mathbf{g} = -4\pi G\rho$	$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho$
Curl	$\nabla\times\mathbf{g} = 0$	$\nabla\times\mathbf{E} = 0$
Potential	$\Delta\phi_{\infty\to p} = -G\int_v \frac{\rho(p')dv'}{r_{p'p}}$	$\Delta\phi_{\infty\to p} = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho(p')dv'}{r_{p'p}}$
Poisson's equation	$\nabla^2\phi = 4\pi G\rho$	$\nabla^2\phi = -\frac{1}{\epsilon_0}\rho$

Both the gravitational and electrostatic central fields are conservative making it possible to use the concept of the scalar potential field ϕ. This concept is especially useful for solving some problems since the potential can be evaluated using a scalar integral. An alternate approach is to solve Poisson's equation if the boundary values and mass distributions are known. The methods of solution of Newton's law of gravitation are identical to those used in electrostatics and are readily accessible in the literature.