4.2: Weak Nonlinearity

Most physical oscillators become non-linear with increase in amplitude of the oscillations. Consequences of non-linearity include breakdown of superposition, introduction of additional harmonics, and complicated chaotic motion that has great sensitivity to the initial conditions as illustrated in this chapter. Weak non-linearity is interesting since perturbation theory can be used to solve the non-linear equations of motion.

The potential energy function for a linear oscillator has a pure parabolic shape about the minimum location, that is, $U = \frac{1}{2}k(x-x_0)^2$ where x_0 is the location of the minimum. Weak non-linear systems have small amplitude oscillations Δx about the minimum allowing use of the Taylor expansion

$$U(\Delta x) = U(x_0) + \Delta x \frac{dU(x_0)}{dx} + \frac{\Delta x^2}{2!}\frac{d^2U(x_0)}{dx^2} + \frac{\Delta x^3}{3!}\frac{d^3U(x_0)}{dx^3} + \frac{\Delta x^4}{4!}\frac{d^4U(x_0)}{dx^4} + \dots \tag{4.2.1}$$

By definition, at the minimum $\frac{dU(x_0)}{dx} = 0$, and thus Equation 4.2.1 can be written as

$$\Delta U = U(\Delta x) - U(x_0) = \frac{\Delta x^2}{2!}\frac{d^2U(x_0)}{dx^2} + \frac{\Delta x^3}{3!}\frac{d^3U(x_0)}{dx^3} + \frac{\Delta x^4}{4!}\frac{d^4U(x_0)}{dx^4} + \dots \tag{4.2.2}$$

For small amplitude oscillations the system is linear when only the second-order $\frac{\Delta x^2}{2!}\frac{d^2U(x_0)}{dx^2}$ term in Equation 4.2.2 is significant. The linearity for small amplitude oscillations greatly simplifies description of the oscillatory motion in that superposition applies, and complicated chaotic motion is avoided. For slightly larger amplitude motion, where the higher-order terms in the expansion are still much smaller than the second-order term, then perturbation theory can be used as illustrated by the simple plane pendulum which is non linear since the restoring force equals

$$mg\sin\theta \simeq mg(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots) \tag{4.2.3}$$

This is linear only at very small angles where the higher-order terms in the expansion can be neglected. Consider the equation of motion at small amplitudes for the harmonically-driven, linearly-damped plane pendulum

$$\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\sin\theta = \ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2(\theta - \frac{\theta^3}{6}) = F_0\cos(\omega t) \tag{4.2.4}$$

where only the first two terms in the expansion 4.2.3 have been included. It was shown in chapter 3 that when $\sin\theta \approx \theta$ then the steady-state solution of Equation 4.2.4 is of the form

$$\theta(t) = A\cos(\omega t - \delta) \tag{4.2.5}$$

Insert this first-order solution into Equation 4.2.4, then the cubic term in the expansion gives a term $\cos^3\omega t = \frac{1}{4}(\cos 3\omega t + 3\cos\omega t)$. Thus the perturbation expansion to third order involves a solution of the form

$$\theta(t) = A\cos(\omega t - \delta) + B\cos 3(\omega t - \delta) \tag{4.2.6}$$

This perturbation solution shows that the non-linear term has distorted the signal by addition of the third harmonic of the driving frequency with an amplitude that depends sensitively on θ. This illustrates that the superposition principle is not obeyed for this non-linear system, but, if the non-linearity is weak, perturbation theory can be used to derive the solution of a non-linear equation of motion.

Figure $4.2.1$ illustrates that for a potential $U(x) = 2x^2 + x^4$, the x^4 non-linear term are greatest at the maximum amplitude x, which makes the total energy contours in state-space more rectangular than the elliptical shape for the harmonic oscillator as shown in figure (3.4.2). The solution is of the form given in Equation 4.2.6.

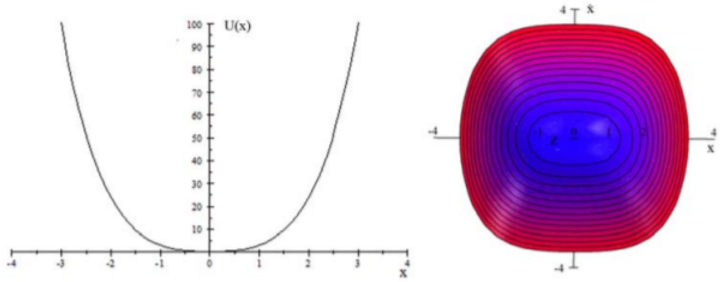

Figure 4.2.1: The left side shows the potential energy for a symmetric potential $U(x) = 2x^2 + x^4$. The right side shows the contours of constant total energy on a state-space diagram.

Example 4.2.1: Non-linear oscillator

Assume that a non-linear oscillator has a potential given by

$$U(x) = \frac{kx^2}{2} - \frac{m\lambda x^3}{3}$$

where λ is small. Find the solution of the equation of motion to first order in λ, assuming $x = 0$ at $t = 0$.

Solution

The equation of motion for the nonlinear oscillator is

$$m\ddot{x} = -\frac{dU}{dx} = -kx + m\lambda x^2$$

If the $m\lambda x^2$ term is neglected, then the second-order equation of motion reduces to a normal linear oscillator with

$$x_0 = A\sin(\omega_0 t + \varphi)$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Assume that the first-order solution has the form

$$x_1 = x_0 + \lambda x_1$$

Substituting this into the equation of motion, and neglecting terms of higher order than λ, gives

$$\ddot{x}_1 + \omega_0^2 x_1 = x_0^2 = \frac{A^2}{2}[1 - \cos(2\omega_0 t)]$$

To solve this try a particular integral

$$x_1 = B + C\cos(2\omega_0 t)$$

and substitute into the equation of motion gives

$$-3\omega_0^2 C\cos(2\omega_0 t) + \omega_0^2 B = \frac{A^2}{2} - \frac{A^2}{2}\cos(2\omega_0 t)$$

Comparison of the coefficients gives

$$\begin{aligned} B &= \frac{A^2}{2\omega_0^2} \\ C &= \frac{A^2}{6\omega_0^2} \end{aligned}$$

The homogeneous equation is

$$\ddot{x}_1 + \omega_0^2 x_1 = 0$$

which has a solution of the form

$$x_1 = D_1 \sin(\omega_0 t) + D_2 \cos(\omega_0 t)$$

Thus combining the particular and homogeneous solutions gives

$$x_1 = (A + \lambda D_1) \sin(\omega_0 t) + \lambda \left[\frac{A^2}{2\omega_0^2} + D_2 \cos(\omega_0 t) + \frac{A^2}{6\omega_0^2} \cos(2\omega_0 t) \right]$$

The initial condition $x = 0$ *at* $t = 0$ *then gives*

$$D_2 = -\frac{2A^2}{3\omega^2}$$

and

$$x_1 = (A + \lambda D_1) \sin(\omega_0 t) + \frac{\lambda A^2}{\omega_0^2} \left[\frac{1}{2} - \frac{2}{3} \cos(\omega_0 t) + \frac{1}{6} \cos(2\omega_0 t) \right]$$

The constant $(A + \lambda D_1)$ *is given by the initial amplitude and velocity.*

This system is nonlinear in that the output amplitude is not proportional to the input amplitude. Secondly, a large amplitude second harmonic component is introduced in the output waveform; that is, for a non-linear system the gain and frequency decomposition of the output differs from the input. Note that the frequency composition is amplitude dependent. This particular example of a nonlinear system does not exhibit chaos. The Laboratory for Laser Energetics uses nonlinear crystals to double the frequency of laser light.