7.S: Symmetries, Invariance and the Hamiltonian (Summary)

This chapter has explored the importance of symmetries and invariance in Lagrangian mechanics and has introduced the Hamiltonian. The following summarizes the important conclusions derived in this chapter.

Noether's theorem:

Noether's theorem explores the remarkable connection between symmetry, plus the invariance of a system under transformation, and related conservation laws which imply the existence of important physical principles, and constants of motion. Transformations where the equations of motion are invariant are called *invariant transformations*. Variables that are invariant to a transformation are called cyclic variables. It was shown that if the Lagrangian does not explicitly contain a particular coordinate of displacement, q_i then the corresponding conjugate momentum, $\dot{p}_i$ is conserved. This is Noether's theorem which states " *For each symmetry of the Lagrangian, there is a conserved quantity"* . In particular it was shown that translational invariance in a given direction leads to the conservation of linear momentum in that direction, and rotational invariance about an axis leads to conservation of angular momentum about that axis. These are the first-order spatial and angular integrals of the equations of motion. Noether's theorem also relates the properties of the Hamiltonian to time invariance of the Lagrangian, namely;

(1) H *is conserved if, and only if, the Lagrangian, and consequently the Hamiltonian, are not explicit functions of time.*

(2) The Hamiltonian gives the total energy if the constraints and coordinate transformations are time independent and the potential energy is velocity independent. This is equivalent to stating that $H=E$ *if the constraints, or generalized coordinates, for the system are time independent*.

Noether's theorem is of importance since it underlies the relation between symmetries, and invariance in all of physics; that is, its applicability extends beyond classical mechanics.

Generalized momentum:

The generalized momentum associated with the coordinate q_j is defined to be

$$\frac{\partial L}{\partial \dot{q}_j} \equiv p_j \tag{7.S.1}$$

where p_j is also called the **conjugate momentum (**or **canonical momentum)** to q_j where q_j, p_j are conjugate, or canonical, variables. Remember that the linear momentum p_j is the first-order time integral given by equation $(3.4.1)$. Note that if q_j is not a spatial coordinate, then p_j is not linear momentum, but is the conjugate momentum. For example, if q_j is an angle, then p_j will be angular momentum.

Kinetic energy in generalized coordinates:

It was shown that the kinetic energy can be expressed in terms of generalized coordinates by

$$\begin{aligned} T(\mathbf{q}, \dot{\mathbf{q}}, t) = & \sum_\alpha \sum_{i,j,k} \frac{1}{2} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial q_k} \dot{q}_j \dot{q}_k + \sum_\alpha \sum_{i,j} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial t} \dot{q}_j + \sum_\alpha \sum_i \frac{1}{2} m_\alpha \left(\frac{\partial x_{\alpha,i}}{\partial t} \right)^2 & (7.S.2) \\ = & \; T_2(\mathbf{q}, \dot{\mathbf{q}}, t) + T_1(\mathbf{q}, \dot{\mathbf{q}}, t) + T_0(\mathbf{q}, t) & (7.S.3) \end{aligned}$$

For scleronomic systems with a potential that is velocity independent, then the kinetic energy can be expressed as

$$T = T_2 = \frac{1}{2} \sum_l \dot{q}_l p_l = \frac{1}{2} \dot{\mathbf{q}} \cdot \mathbf{p} \tag{7.S.4}$$

Generalized energy

Jacobi's **Generalized Energy** $h(\mathbf{q}, \dot{q}, t)$ was defined as

$$h(\mathbf{q}, \dot{\mathbf{q}}, t) \equiv \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) - L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{7.S.5}$$

Hamiltonian function

The Hamiltonian $H(\mathbf{q},\mathbf{p},t)$ was defined in terms of the generalized energy $h(\mathbf{q},\dot{\mathbf{q}},t)$ and by introducing the generalized momentum. That is

$$H(\mathbf{q},\mathbf{p},t) \equiv h(\mathbf{q},\dot{\mathbf{q}},t) = \sum_j p_j \dot{q}_j - L(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{p}\cdot\dot{\mathbf{q}} - L(\mathbf{q},\dot{\mathbf{q}},t) \tag{7.S.6}$$

Generalized energy theorem

The equations of motion lead to the generalized energy theorem which states that the time dependence of the Hamiltonian is related to the time dependence of the Lagrangian.

$$\frac{dH(\mathbf{q},\mathbf{p},t)}{dt} = \sum_j \dot{q}_j \left[Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q},t) \right] - \frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial t} \tag{7.S.7}$$

Note that if all the generalized non-potential forces are zero, then the bracket in Equation 7.S.7 is zero, and if the Lagrangian is not an explicit function of time, then the Hamiltonian is a constant of motion.

Generalized energy and total energy:

The generalized energy, and corresponding Hamiltonian, equal the total energy if:

1) The kinetic energy has a homogeneous quadratic dependence on the generalized velocities and the transformation to generalized coordinates is independent of time, $\frac{\partial x_{\alpha,i}}{\partial t} = 0$.

2) The potential energy is not velocity dependent, thus the terms $\frac{\partial U}{\partial \dot{q}_i} = 0$.

Chapter 8 will introduce Hamiltonian mechanics that is built on the Hamiltonian, and chapter 15 will explore applications of Hamiltonian mechanics.
