5.6: Euler's Integral Equation

An integral form of the Euler differential equation can be written which is useful for cases when the function f does not depend explicitly on the independent variable x, that is, when $\frac{\partial f}{\partial x}=0$. Note that

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{5.6.1}$$

But

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)=\frac{\partial f}{\partial y'}\frac{dy'}{dx}+y'\frac{d}{dx}\frac{\partial f}{\partial y'} \tag{5.6.2}$$

Combining these two equations gives

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)=\frac{df}{dx}-\frac{\partial f}{\partial x}-y'\frac{\partial f}{\partial y}+y'\frac{d}{dx}\frac{\partial f}{\partial y'} \tag{5.6.3}$$

The last two terms can be rewritten as

$$y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'}-\frac{\partial f}{\partial y}\right) \tag{5.6.4}$$

which vanishes when the Euler equation is satisfied. Therefore the above equation simplifies to

$$\frac{\partial f}{\partial x}-\frac{d}{dx}\left(f-y'\frac{\partial f}{\partial y'}\right)=0 \tag{5.6.5}$$

This integral form of Euler's equation is especially useful *when* $\frac{\partial f}{\partial x}=0$, *that is, when* f *does not depend explicitly on the independent variable* x. Then the first integral of Equation 5.6.5 is a constant, i.e.

$$f-y'\frac{\partial f}{\partial y'}=\text{constant} \tag{5.6.6}$$

This is Euler's integral variational equation. Note that the shortest distance between two points, the minimum surface of rotation, and the brachistochrone, described earlier, all are examples where $\frac{\partial f}{\partial x}=0$ and thus the integral form of Euler's equation is useful for solving these cases.