14.S: Coupled linear oscillators (Summary)

This chapter has focussed on many—body coupled linear oscillator systems which are a ubiquitous feature in nature. A summary of the main conclusions are the following.

Normal modes

It was shown that coupled linear oscillators exhibit normal modes and normal coordinates that correspond to independent modes of oscillation with characteristic eigenfrequencies ω_i.

General analytic theory for coupled linear oscillators

Lagrangian mechanics was used to derive the general analytic procedure for solution of the many-body coupled oscillator problem which reduces to the conventional eigenvalue problem. A summary of the procedure for solving coupled oscillator problems is as follows:.

1) Choose generalized coordinates q_j and evaluate T and U.

$$T = \frac{1}{2}\sum_{j,k}^{n} T_{jk}\dot{q}_j\dot{q}_k \tag{14.S.1}$$

and

$$U' = \frac{1}{2}\sum_{j,k}^{n} V_{jk}q_j q_k \tag{14.S.2}$$

where the components of the $\mathbf{T}$ and $\mathbf{V}$ tensors are

$$T_{jk} \equiv \left(\sum_{\alpha}^{N} m_\alpha \sum_{i}^{3} \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}\right)_0 \tag{14.S.3}$$

and

$$V_{jk} \equiv \left(\frac{\partial^2 U}{\partial q_j \partial q_k}\right)_0 \tag{14.S.4}$$

2) Determine the eigenvalues ω_r using the secular determinant.

$$\begin{vmatrix} V_{11}-\omega^2 T_{11} & V_{12}-\omega^2 T_{12} & V_{13}-\omega^2 T_{13} & \dots \\ V_{12}-\omega^2 T_{12} & V_{22}-\omega^2 T_{22} & V_{23}-\omega^2 T_{23} & \dots \\ V_{13}-\omega^2 T_{13} & V_{23}-\omega^2 T_{23} & V_{33}-\omega^2 T_{33} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0 \tag{14.S.5}$$

3) The eigenvectors are obtained by inserting the eigenvalues ω_r into

$$\sum_{j}^{n} (V_{jk} - \omega_r^2 T_{jk})a_j = 0 \tag{14.S.6}$$

4) From the initial conditions determine the complex scale factors β_r where

$$\eta_r(t) \equiv \beta_r e^{i\omega_r t} \tag{14.S.7}$$

5) Determine the normal coordinates where each η_r is a normal mode. The normal coordinates can be expressed as

$$\boldsymbol{\eta} = \{\mathbf{a}\}^{-1}\mathbf{q} \tag{14.S.8}$$

Few-body coupled oscillator systems

The general analytic theory was used to determine the solutions for parallel and series couplings of two and three linear oscillators. The phenomena observed include degenerate and non-degenerate eigenvalues and spurious center-of-mass oscillatory modes. There are two broad classifications for three or more coupled oscillators, that is, either complete coupling of all oscillators, or coupling of the nearest-neighbor oscillators. It is observed that the eigenvalue corresponding to the most coherent motion of the coupled oscillators corresponds to the most collective motion and its eigenvalue is displaced the most in energy from the remaining eigenvalues. For some systems this coherent collective mode corresponded to a center-of-mass motion with no internal excitation of the other modes, while the other eigenvalues corresponded to modes with internal excitation of the oscillators such that the center of mass is stationary. The above procedure has been applied to two classification of coupling, complete coupling of many oscillators, and nearest neighbor coupling. Both degenerate and spurious center-of-mass modes were observed. Strong collective shape degrees of freedom in nuclei are examples of complete coupling due to the weak residual interactions between nucleons in the nucleus. It was seen that, for many coupled oscillators, one coherent state separates from the other states and this coherent state carries the bulk of the collective strength.

Discrete lattice chain

Transverse and longitudinal modes of motion on the discrete lattice chain were discussed because of the important role it plays in nature, such as in crystalline lattice structures. Both normal modes and travelling waves were discussed including the phenomena of dispersion and cut-off frequencies. Molecules and the crystalline lattice chains are examples where nearest neighbor coupling is manifest. It was shown that, for the $n-$oscillator discrete lattice chain, there are only n independent longitudinal modes plus n modes for the two transverse polarizations, and that the angular frequency $\omega_r \leq 2\omega_0$ that is, a cut-off frequency exists.

Damped coupled linear oscillators

It was shown that linearly-damped coupled oscillator systems can be solved analytically using the concept of the Rayleigh dissipation function.

Collective synchronization of coupled oscillators

The Kuramoto schematic phase model was used to illustrate how weak residual forces can cause collective synchronization of the motion of many coupled oscillators. This is applicable to many large coupled systems such as nuclei, molecules, and biological systems.