5.4: Selection of the Independent Variable

A wide selection of variables can be chosen as the independent variable for variational calculus. The derivation of Euler's equation and example $(5.3.1)$ both assumed that the independent variable is x, whereas example $(5.3.2)$ used y as the independent variable, example $(5.3.3)$ used z, and Lagrange mechanics uses time t as the independent variable. Selection of which variable to use as the independent variable does not change the physics of a problem, but some selections can simplify the mathematics for obtaining an analytic solution. The following example of a cylindrically-symmetric soap-bubble surface formed by blowing a soap bubble that stretches between two circular hoops, illustrates the importance when selecting the independent variable.

Example $5.4.1$: Surface area of a cylindrically-symmetric soap bubble

Consider a cylindrically-symmetric soap-bubble surface formed by blowing a soap bubble that stretches between two circular hoops. The surface energy, that results from the surface tension of the soap bubble, is minimized when the surface area of the bubble is minimized. Assume that the axes of the two hoops lie along the z axis as shown in the adjacent figure. It is intuitively obvious that the soap bubble having the minimum surface area that is bounded by the two hoops will have a circular cross section that is concentric with the symmetry axis, and the radius will be smaller between the two hoops. Therefore, intuition can be used to simplify the problem to finding the shape of the contour of revolution around the axis of symmetry that defines the shape of the surface of minimum surface area. Use cylindrical coordinates (ρ, θ, z) and assume that hoop 1 at z_1 has radius ρ_1 and hoop 2 at z_2 has radius ρ_2. Consider the cases where either ρ, or z, are selected to be the independent variable.

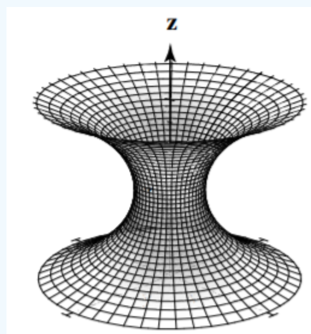

Figure $5.4.1$: Cylindrically-symmetric surface formed by rotation about the z axis of a soap bubble suspended between two identical hoops centred on the z axis.

The differential arc-length element of the circular annulus at constant θ between z and $z+dz$ is given by $ds = \sqrt{dz^2+d\rho^2}$. Therefore the area of the infinitessimal circular annulus is $dS = 2\pi\rho ds$ which can be integrated to give the area of the surface S of the soap bubble bounded by the two circular hoops as

$$S = 2\pi\int_1^2 \rho\sqrt{dz^2+d\rho^2}$$

Independent variable z

$$\frac{d}{dz}\left(\frac{\rho\rho'}{\sqrt{1+(\rho')^2}}\right) - \sqrt{1+\rho'^2} = 0$$

This is not an easy equation to solve.

Independent variable ρ

$$\rho = a\cosh\frac{z-b}{a}$$

which is the equation of a catenary. The catenary is the shape of a uniform flexible cable hung in a uniform gravitational field. The constants a and b are given by the end points. The physics of the solution must be identical for either choice of independent variable. However, mathematically one case is easier to solve than the other because, in the latter case, one term in Euler's equation is zero.