3.8: Travelling and standing wave solutions of the wave equation

The wave equation can have both travelling and standing-wave solutions. Consider a one-dimensional travelling wave with velocity v having a specific wavenumber $k \equiv \frac{2\pi}{\lambda}$. Then the travelling wave is best written in terms of the phase of the wave as

$$\Psi(x,t) = A(k)e^{i\frac{2\pi}{\lambda}(x\mp vt)} = A(k)e^{i(kx\mp\omega t)} \tag{3.8.1}$$

where the wave number $k \equiv \frac{2\pi}{\lambda}$, with λ being the wave length, and angular frequency $\omega \equiv kv$. This particular solution satisfies the wave equation and corresponds to a travelling wave with phase velocity $v = \frac{\omega_n}{k_n}$ in the positive or negative direction x depending on whether the sign is negative or positive. Assuming that the superposition principle applies, then the superposition of these two particular solutions of the wave equation can be written as

$$\Psi(x,t) = A(k)(e^{i(kx-\omega t)} + e^{i(kx+\omega t)}) = A(k)e^{ikx}(e^{-i\omega t} + e^{i\omega t}) = 2A(k)e^{ikx}\cos\omega t \tag{3.8.2}$$

Thus the superposition of two identical single wavelength travelling waves propagating in opposite directions can correspond to a standing wave solution. Note that a standing wave is identical to a stationary normal mode of the system discussed in chapter 14. This transformation between standing and travelling waves can be reversed, that is, the superposition of two standing waves, i.e. normal modes, can lead to a travelling wave solution of the wave equation. Discussion of waveforms is simplified when using either of the following two limits.

1) The time dependence of the waveform at a given location $x = x_0$ which can be expressed using a Fourier decomposition, appendix $19.9.2$, of the time dependence as a function of angular frequency $\omega = n\omega_0$.

$$\Psi(x_0,t) = \sum_{n=-\infty}^{\infty} A_n e^{in(k_0x_0-\omega_0 t)} = \sum_{n=-\infty}^{\infty} B_n(x_0)e^{-in\omega_0 t} \tag{3.8.3}$$

2) The spatial dependence of the waveform at a given instant $t = t_0$ which can be expressed using a Fourier decomposition of the spatial dependence as a function of wavenumber $k = nk_0$

$$\Psi(x,t_0) = \sum_{n=-\infty}^{\infty} A_n e^{in(k_0x-\omega_1 t_0)} = \sum_{n=-\infty}^{\infty} C_n(t_0)e^{ink_0 x} \tag{3.8.4}$$

The above is applicable both to discrete, or continuous linear oscillator systems, e.g. waves on a string. In summary, stationary normal modes of a system are obtained by a superposition of travelling waves travelling in opposite directions, or equivalently, travelling waves can result from a superposition of stationary normal modes.