7.6: Kinetic Energy in Generalized Coordinates

Application of Noether's theorem to the conservation of energy requires the kinetic energy to be expressed in generalized coordinates. In terms of fixed rectangular coordinates, the kinetic energy for N bodies, each having three degrees of freedom, is expressed as

$$T=\frac{1}{2}\sum_{\alpha=1}^{N}\sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha,i}^2 \tag{7.6.1}$$

These can be expressed in terms of generalized coordinates as $x_{\alpha,i}=x_{\alpha,i}(q_j,t)$ and in terms of generalized velocities

$$\dot{x}_{\alpha,i}=\sum_{j=1}^{s}\frac{\partial x_{\alpha,i}}{\partial q_j}\dot{q}_j+\frac{\partial x_{\alpha,i}}{\partial t} \tag{7.6.2}$$

Taking the square of $\dot{x}_{\alpha,i}$ and inserting into the kinetic energy relation gives

$$T(\mathbf{q},\dot{\mathbf{q}},t)=\sum_\alpha\sum_{i,j,k}\frac{1}{2}m_\alpha\frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}\dot{q}_j\dot{q}_k+\sum_\alpha\sum_{i,j}m_\alpha\frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial t}\dot{q}_j+\sum_\alpha\sum_i\frac{1}{2}m_\alpha\left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2 \tag{7.6.3}$$

This can be abbreviated as

$$T(\mathbf{q},\dot{\mathbf{q}},t)=T_2(\mathbf{q},\dot{\mathbf{q}},t)+T_1(\mathbf{q},\dot{\mathbf{q}},t)+T_0(\mathbf{q},t) \tag{7.6.4}$$

where

$$T_2(\mathbf{q},\dot{\mathbf{q}},t)= \sum_\alpha\sum_{i,j,k}\frac{1}{2}m_\alpha\frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}\dot{q}_j\dot{q}_k=\sum_{j,k}a_{jk}\dot{q}_j\dot{q}_k \tag{7.6.5}$$

$$T_1(\mathbf{q},\dot{\mathbf{q}},t)= \sum_\alpha\sum_{i,j}m_\alpha\frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial t}\dot{q}_j=\sum_{j,k}b_j\dot{q}_j \tag{7.6.6}$$

$$T_0(\mathbf{q},t)= \sum_\alpha\sum_i\frac{1}{2}m_\alpha\left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2 \tag{7.6.7}$$

where

$$a_{jk}\equiv\sum_{\alpha=1}^{n}\sum_{i,=1}^{3}\frac{1}{2}m_\alpha\frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k} \tag{7.6.8}$$

When the transformed system is scleronomic, time does not appear explicitly in the transformation equations to generalized coordinates since $\frac{\partial x_{\alpha,i}}{\partial t}=0$. Then $T_1=T_0=0$, and the kinetic energy reduces to a homogeneous quadratic function of the generalized velocities

$$T(\mathbf{q},\dot{\mathbf{q}},t)=T_2(\mathbf{q},\dot{\mathbf{q}},t) \tag{7.6.9}$$

A useful relation can be derived by taking the differential of Equation 7.6.5 with respect to $\dot{q}_l$. That is

$$\frac{\partial T_2(\mathbf{q},\dot{\mathbf{q}},t)}{\partial \dot{q}_l}=\sum_k a_{lk}\dot{q}_k+\sum_j a_{jl}\dot{q}_j \tag{7.6.10}$$

Multiply this by $\dot{q}_l$ and sum over l gives

$$\sum_l \dot{q}_l\frac{\partial T_2(\mathbf{q},\dot{\mathbf{q}},t)}{\partial \dot{q}_l}=\sum_{k,l}a_{lk}\dot{q}_k\dot{q}_l+\sum_{j,l}a_{jl}\dot{q}_j\dot{q}_l=2\sum_{j,k}a_{lk}\dot{q}_k\dot{q}_l=2T_2 \tag{7.6.11}$$

Similarly, the products of the generalized velocities $\dot{q}$, with the corresponding derivatives of T_1 and T_0 give

$$\sum_{l} \dot{q}_{l} \frac{\partial T_{2}}{\partial \dot{q}_{l}} = 2T_{2} \tag{7.6.12}$$

$$\sum_{l} \dot{q}_{l} \frac{\partial T_{1}(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}_{l}} = T_{1}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{7.6.13}$$

$$\sum_{l} \dot{q}_{l} \frac{\partial T_{0}(\mathbf{q}, t)}{\partial \dot{q}_{l}} = 0 \tag{7.6.14}$$

Equation 7.6.9 gives that $T = T_2$ when the transformed system is scleronomic, i.e. $\frac{\partial x_{\alpha,i}}{\partial t} = 0$, and then the kinetic energy is a quadratic function of the generalized velocities $\dot{q}_j$. Using the definition of the generalized momentum equation $(7.2.3)$, assuming $T = T_2$, and that the potential U is velocity independent, gives that

$$p_l \equiv \frac{\partial L}{\partial \dot{q}_l} = \frac{\partial T}{\partial \dot{q}_l} - \frac{\partial U}{\partial \dot{q}_l} = \frac{\partial T_2}{\partial \dot{q}_l} \tag{7.6.15}$$

Then Equation 7.6.12 reduces to the useful relation that

$$T_2 = \frac{1}{2} \sum_l \dot{q}_l p_l = \frac{1}{2} \dot{\mathbf{q}} \cdot \mathbf{p} \tag{7.6.16}$$

where, for compactness, the summation is abbreviated as a scalar product.