13.9: Perpendicular-axis Theorem for Plane Laminae

Rigid-body rotation of thin plane laminae objects is encountered frequently. Examples of such laminae bodies are a plane sheet of metal, a thin door, a bicycle wheel, a thin envelope or book. Deriving the inertia tensor for a plane lamina is relatively simple because there are limits on the possible relative magnitude of the principal moments of inertia. Consider that the principal axis are along the $x, y, z,$ coordinate axes. Then the sum of two principal moments of inertia about the center of mass are

$$\begin{aligned} I_x + I_y &= \int \rho(y^2+z^2)dV + \int \rho(x^2+z^2)dV \\ &= \int \rho(x^2+y^2)dV + 2\int \rho z^2 dV \geq \int \rho(x^2+y^2)dV = I_z \end{aligned}$$

Note that for any body the three principal moments of inertia must satisfy the triangle rule that the sum of any pair must exceed or equal the third. Moreover, if the body is a thin lamina with thickness $z=0$, that is, a thin plate in the $x-y$ plane, then

$$I_x + I_y = I_z \tag{13.9.1}$$

This **perpendicular-axis theorem** can be very useful for solving problems involving rotation of plane laminae.

The opposite of a plane laminae is a long thin cylindrical needle of mass m, length L, and radius r. Along the symmetry axis the principal moments are $I_z = \frac{1}{2}mr^2 \to 0$ as $r \to 0$, while perpendicular to the symmetry axis $I_x = I_y = \frac{1}{12}mL^2$. These satisfy the triangle rule.

Example 13.9.1: Inertia Tensor of a Hula Hoop

The hula hoop is a thin plane circular ring or radius R and mass M. Assume that the symmetry axis of the circular ring is the 3 axis.

a. The principal moments of inertia about the center of mass: The principal moment of inertia along the 3 axis is $I_{33} = MR^2$. Then Equation 13.9.1 plus symmetry tells us that the two principal moments of inertia in the plane of the hula hoop must be $I_{11} = I_{22} = \frac{1}{2}MR^2$.
b. The principal moments of inertia about the periphery of the ring: Using the Parallel-axis theorem tells us that the moment perpendicular to the plane of the hula hoop $I_{33} = 2MR^2$. In the plane of the hoop the moment tangential to the hoop is $I_{11} = \frac{3}{2}MR^2$ and the moment radial to the hoop $I_{22} = \frac{1}{2}MR^2$. The hula dancer often swings the hoop about the periphery and perpendicular to the plane by swinging their hips. Another movement is jumping through the hoop by rotating the hoop tangential to the periphery. Calculation of such maneuvers requires knowledge of these principal moments of inertia.

Example 13.9.2: Inertia Tensor of a Thin Book

Consider a thin rectangular book of mass M, width a and length b with thickness $t \ll a$ and $t \ll b$. About the center of mass the inertia tensor perpendicular to the plane of the book is $I_{33} = \frac{M}{12}(a^2+b^2)$. The other two moments are $I_{11} = \frac{M}{12}a^2$ and $I_{22} = \frac{M}{12}b^2$ which satisfy Equation 13.9.1.