2.10: Work and Kinetic Energy for a Many-Body System

Center-of-mass kinetic energy

For a many-body system the position vector $\mathbf{r}'_i$ with respect to the center of mass is given by.

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i \tag{2.10.1}$$

The location of the center of mass is uniquely defined as being at the location where $\int \rho \mathbf{r}'_i dV = 0$. The velocity of the i^{th} particle can be expressed in terms of the velocity of the center of mass $\dot{\mathbf{R}}$ plus the velocity of the particle with respect to the center of mass $\dot{\mathbf{r}}'_i$. That is,

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{r}}'_i \tag{2.10.2}$$

The total kinetic energy T is

$$T = \sum_i^n \frac{1}{2} m_i v_i^2 = \sum_i^n \frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \sum_i^n \frac{1}{2} m_i \dot{\mathbf{r}}'_i \cdot \dot{\mathbf{r}}'_i + \left(\frac{d}{dt} \sum_i m_I \dot{\mathbf{r}}'_i \right) \cdot \dot{\mathbf{R}} + \sum_i \frac{1}{2} m_i \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} \tag{2.10.3}$$

For the special case of the center of mass, the middle term is zero since, by definition of the center of mass, $\sum_i m_i \dot{\mathbf{r}}'_i$.Therefore

$$T = \sum_i^n \frac{1}{2} m_i v_i'^2 + \frac{1}{2} M V^2 \tag{2.10.4}$$

Thus the total kinetic energy of the system is equal to the sum of the kinetic energy of a mass M moving with the center of mass velocity plus the kinetic energy of motion of the individual particles relative to the center of mass. This is called Samuel König's second theorem.

Note that for a fixed center-of-mass energy, the total kinetic energy T has a minimum value of $\sum_i^n \frac{1}{2} m_i v_i'^2$ when the velocity of the center of mass V = 0. For a given internal excitation energy, the minimum energy required to accelerate colliding bodies occurs when the colliding bodies have identical, but opposite, linear momenta. That is, when the center-of-mass velocity V = 0.

Conservative forces and Potential Energy

In general, the line integral of a force field $\mathbf{F}$, that is, $\int_1^2 \mathbf{F} \cdot d\mathbf{r}$ is both path and time dependent. However, an important class of forces, called conservative forces, exist for which the following two facts are obeyed.

1. **Time independence**: *The force depends only on the particle position* $\mathbf{r}$, *that is, it does not depend on velocity or time.*
2. **Path independence:** *For any two points 1 and 2 , the work done by* $\mathbf{F}$ *is independent of the path taken between 1 and 2.*

If forces are path independent, then it is possible to define a scalar field, called potential energy and denoted by $U(\mathbf{r})$ that is only a function of position. The path independence can be expressed by noting that the integral around a closed loop is zero. That is

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \tag{2.10.5}$$

Applying Stokes theorem for a path-independent force leads to the alternate statement that the curl is zero.

See appendix $19.7.3C$.

$$\nabla \times \mathbf{F} = 0 \tag{2.10.6}$$

Note that the vector product of two del operators ∇ acting on a scalar field U equals

$$\nabla \times \nabla U = 0 \tag{2.10.7}$$

Thus it is possible to express a path-independent force field as the gradient of a scalar field, U, that is

$$\mathbf{F} = -\nabla U \tag{2.10.8}$$

Then the spatial integral

$$\int_1^2 \mathbf{F}\cdot d\mathbf{r} = -\int_1^2 (\nabla U)\cdot d\mathbf{r} = U_1 - U_2 \tag{2.10.9}$$

Thus for a path-independent force, the work done on the particle is given by the change in potential energy if there is no change in kinetic energy. For example, if an object is lifted against the gravitational field, then work is done on the particle and the final potential energy U_2 exceeds the initial potential energy, U_1.

Total Mechanical Energy

The **total mechanical energy** E of a particle is defined as the sum of the kinetic and potential energies.

$$E = T + U \tag{2.10.10}$$

Note that the potential energy is defined only to within an additive constant since the force $\mathbf{F} = -\nabla U$ depends only on difference in potential energy. Similarly, the kinetic energy is not absolute since any inertial frame of reference can be used to describe the motion and the velocity of a particle depends on the relative velocities of inertial frames. Thus the total mechanical energy $E = T + U$ is not absolute.

If a single particle is subject to several path-independent forces, such as gravity, linear restoring forces, etc., then a potential energy U_i can be ascribed to each of the m forces where for each force $\mathbf{F}_i = -\nabla U_i$. In contrast to the forces, which add vectorially, these scalar potential energies are additive, $U = \sum_i^m U_i$. Thus the total mechanical energy for m potential energies equals

$$E = T + U(\mathbf{r}) = T + \sum_i^m U_i(\mathbf{r}) \tag{2.10.11}$$

The time derivative of the total mechanical energy $E = T + U$ equals

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} \tag{2.10.12}$$

Equation (2.4.9) gave that $dT = \mathbf{F}\cdot d\mathbf{r}$. Thus, the first term in Equation 2.10.12equals

$$\frac{dT}{dt} = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} \tag{2.10.13}$$

The potential energy can be a function of both position and time. Thus the time difference in potential energy due to change in both time and position is given as

$$\frac{dU}{dt} = \sum_i \frac{\partial U}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial U}{\partial t} = (\nabla U)\cdot\frac{d\mathbf{r}}{dt} + \frac{\partial U}{\partial t} \tag{2.10.14}$$

The time derivative of the total mechanical energy is given using Equations 2.10.13and 2.10.14in Equation 2.10.12

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} + (\nabla U)\cdot\frac{d\mathbf{r}}{dt} + \frac{\partial U}{\partial t} = [\mathbf{F} + (\nabla U)]\cdot\frac{d\mathbf{r}}{dt} + \frac{\partial U}{\partial t} \tag{2.10.15}$$

Note that if the field is path independent, that is $\nabla\times\mathbf{F} = 0$ then the force and potential are related by

$$\mathbf{F} = -\nabla U \tag{2.10.16}$$

Therefore, for *path independent forces,* the first term in the time derivative of the total energy in Equation 2.10.15is zero. That is,

$$\frac{dE}{dt} = \frac{\partial U}{\partial t} \tag{2.10.17}$$

In addition, when the potential energy U is not an explicit function of time, then $\frac{\partial U}{\partial t} = 0$ and thus the total energy is conserved. That is, for the combination of (a) path independence plus (b) time independence, then the *total energy of a conservative field is conserved.*

Note that there are cases where the concept of potential still is useful even when it is time dependent. That is, if path independence applies, i.e. $\mathbf{F} = -\nabla U$ at any instant. For example, a Coulomb field problem where charges are slowly changing due to leakage etc., or during a peripheral collision between two charged bodies such as nuclei.

Example 2.10.1: Central force

A particle of mass m moves along a trajectory given by $x = x_o \cos \omega_1 t \; and \; y_0 \sin \omega_2 t$.

a) Find the x and y components of the force and determine the condition for which the force is a central force.

Differentiating with respect to time gives

$$\begin{array}{ll} \dot{x} = -x_0\omega_1 \sin(\omega_1 t) & \ddot{x} = -x_0\omega_1^2 \cos(\omega_1 t) \\ \dot{y} = -y_0\omega_1 \cos(\omega_2 t) & \ddot{y} = -y_0\omega_2^2 \sin(\omega_1 t) \end{array}$$

Newton's second law gives

$$\mathbf{F} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) = -m\left[x_0\omega_1^2 \cos(\omega_1 t)\hat{i} + y_0\omega_2^2 \sin(\omega_1 t)\hat{j}\right] = -m\left[\omega_1^2 x\hat{i} + \omega_2^2 y\hat{j}\right]$$

Note that if $\omega_1 = \omega_2 = \omega$ then

$$\mathbf{F} = -m\omega^2[x\hat{i} + y\hat{j}] = -m\omega^2\mathbf{r}$$

That is, it is a central force if $\omega_1 = \omega_2 = \omega$.

b) Find the potential energy as a function of x and y.

Since

$$\mathbf{F} = -\nabla U = -\left[\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j}\right]$$

then

$$U = \frac{1}{2}m(\omega_1^2 x^2 + \omega_2^2 y^2)$$

assuming that $U = 0$ at the origin.

c) Determine the kinetic energy of the particle and show that it is conserved.

The total energy

$$E = T + U = \frac{1}{2}m(\dot{x}^2 + \dot{y}_2) + \frac{1}{2}m(\omega_1^2 x^2 + \omega_2^2 y^2) = \frac{1}{2}m(x_0^2\omega_1^2 + y_0^2\omega_2^2)$$

since $\cos^2\theta + \sin^2\theta = 1$. Thus the total energy E is a constant and is conserved.

Total mechanical energy for conservative systems

Equation (2.4.11) showed that, using Newton's second law, $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, the first-order spatial integral gives that the work done W_{12} is related to the change in the kinetic energy. That is,

$$W_{12} \equiv \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}mv_2^2 - \frac{1}{2}v_1^2 = T_2 - T_1 \tag{2.10.18}$$

The work done W_{12} also can be evaluated in terms of the known forces $\mathbf{F}_i$ in the spatial integral. Consider that the resultant force acting on particle i in this n-particle system can be separated into an external force $\mathbf{F}_i^{Ext}$ plus internal forces between the n particles of the system

$$\mathbf{F}_i = \mathbf{F}_i^E + \sum_{\substack{j \\ i\neq j}}^{n} \mathbf{f}_{ij} \tag{2.10.19}$$

The origin of the external force is from outside of the system while the internal force is due to the interaction with the other $n-1$ particles in the system. Newton’s Law tells us that

$$\dot{\mathbf{p}}_i = \mathbf{F}_i = \mathbf{F}_i^E + \sum_{\substack{j \\ i \neq j}}^{n} \mathbf{f}_{ij} \tag{2.10.20}$$

The work done on the system by a force moving from configuration $1 \rightarrow 2$ is given by

$$W_{1\rightarrow 2} = \sum_i^n \int_1^2 \mathbf{F}_i^E \cdot d\mathbf{r}_i + \sum_i^n \sum_{\substack{j \\ i \neq j}}^{n} \int_1^2 \mathbf{f}_{ij} \cdot d\mathbf{r}_i \tag{2.10.21}$$

Since $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$ then

$$W_{1\rightarrow 2} = \sum_i^n \int_1^2 \mathbf{F}_i^E \cdot d\mathbf{r}_i + \sum_i^n \sum_{\substack{j \\ i < j}}^{n} \int_1^2 \mathbf{f}_{ij} \cdot (d\mathbf{r}_i - d\mathbf{r}_j) \tag{2.10.22}$$

Where $d\mathbf{r}_i - d\mathbf{r}_j = d\mathbf{r}_{ij}$ is the vector from j to i.

Assume that both the external and internal forces are conservative, and thus can be derived from time independent potentials, that is

$$\mathbf{F}_i^E = -\nabla_i U_i^{Ext} \tag{2.10.23}$$

$$\mathbf{f}_{ij} = -\nabla_i U_{ij}^{Int} \tag{2.10.24}$$

Then

$$\begin{aligned} W_{1\rightarrow 2} &= -\sum_i^n \int_1^2 -\nabla_i U_i^{Ext} \cdot d\mathbf{r}_i + \sum_i^n \sum_{\substack{j \\ i<j}}^{n} \int_1^2 -\nabla_i U_{ij}^{Int} \cdot d\mathbf{r}_i \\ &= \sum_i^n U_i^{Ext}(1) - \sum_i^n U_i^{Ext}(2) + \sum_i^n U_i^{Int}(1) - \sum_i^n U_i^{Int}(2) \\ &= U^{Ext}(1) - U^{Ext}(2) + U^{Int}(1) - U^{Int}(2) \end{aligned} \tag{2.10.25}$$

Define the total external potential energy,

$$U^{Ext} = \sum_i^n U_i^{Ext} \tag{2.10.26}$$

and the total internal energy

$$U^{Int} = \sum_i^n U_i^{Int} \tag{2.10.27}$$

Equating the two equivalent equations for $W_{1\rightarrow 2}$, that is, Equations 2.10.18and 2.10.25gives that

$$W_{1\rightarrow 2} = T_2 - T_1 = U^{Ext}(1) - U^{Ext}(2) + U^{Int}(1) - U^{Int}(2) \tag{2.10.28}$$

Regroup these terms in Equation 2.10.28gives

$$T_1 + U^{Ext}(1) + U^{Int}(1) = T_2 + U^{Ext}(2) + U^{Int}(2)$$

This shows that, for *conservative forces*, the total energy is conserved and is given by

$$E = T + U^{Ext} + U^{Int} \tag{2.10.29}$$

The three first-order integrals for linear momentum, angular momentum, and energy provide powerful approaches for solving the motion of Newtonian systems due to the applicability of conservation laws for the corresponding linear and angular momentum, plus energy conservation for conservative forces. In addition, the important concept of center-of-mass motion naturally separates out for these three first-order integrals. Although these conservation laws were derived assuming Newton's Laws of motion, these conservation laws are more generally applicable, and *these conservation laws surpass the range of validity of Newton's Laws of motion*. For example, in 1930 Pauli and Fermi postulated the existence of the neutrino in order to account for non-conservation of energy and momentum in β -decay because they did not wish to relinquish the concepts of energy and momentum conservation. The neutrino was first detected in 1956 confirming the correctness of this hypothesis.