10.S: Nonconservative systems (Summary)

Dissipative drag forces are non-conservative and usually are velocity dependent. Chapter 4 showed that the motion of non-linear dissipative dynamical systems can be highly sensitive to the initial conditions and can lead to chaotic motion.

Algebraic mechanics for nonconservative systems

Since Lagrangian and Hamiltonian formulations are invalid for the nonconservative degrees of freedom, the following three approaches are used to include nonconservative degrees of freedom directly in the Lagrangian and Hamiltonian formulations of mechanics.

1. Expand the number of degrees of freedom used to include all active degrees of freedom for the system, so that the expanded system is conservative. This is the preferred approach when it is viable. Unfortunately this approach typically is impractical for handling dissipated processes because of the large number of degrees of freedom that are involved in thermal dissipation.
2. Nonconservative forces can be introduced directly at the equations of motion stage as generalized forces Q_j^{EXC}. This approach is used extensively. For the case of linear velocity dependence, the Rayleigh's dissipation function provides an elegant and powerful way to express the generalized forces in terms of scalar potential energies.
3. New degrees of freedom or effective forces can be postulated that are then incorporated into the Lagrangian or the Hamiltonian in order to mimic the effects of the nonconservative forces.

Rayleigh's Dissipation Function

Generalized dissipative forces that have a linear velocity dependence can be easily handled in Lagrangian or Hamiltonian mechanics by introducing the powerful Rayleigh's dissipation function $\mathcal{R}(\dot{\mathbf{q}})$ where

$$\mathcal{R}(\dot{\mathbf{q}}) \equiv \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\dot{q}_i\dot{q}_j \tag{10.S.1}$$

This approach is used extensively in physics. This approach has been generalized by defining a linear velocity dependent Rayleigh dissipation function

$$\mathbf{F}_i^f = -\frac{\partial R(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} \tag{10.S.2}$$

where the generalized Rayleigh dissipation function $\mathcal{R}(\mathbf{q},\dot{\mathbf{q}})$ satisfies the general Lagrange mechanics relation

$$\frac{\delta L}{\delta q} - \frac{\partial R}{\partial \dot{q}} = 0 \tag{10.S.3}$$

This generalized Rayleigh's dissipation function eliminates the prior restriction to linear dissipation processes, which greatly expands the range of validity for using Rayleigh's dissipation function.

Rayleigh dissipation in Lagrange equations of motion

Linear dissipative forces can be directly, and elegantly, included in Lagrangian mechanics by using Rayleigh's dissipation function as a generalized force Q_j^f. Inserting Rayleigh dissipation function $(10.4.12)$ in the generalized Lagrange equations of motion $(6.5.12)$ gives

$$\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right\} = \left[\sum_{k=1}^{m}\lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q},t) + Q_j^{EXC}\right] - \frac{\partial \mathcal{R}(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{q}_j} \tag{10.S.4}$$

Where Q_j^{EXC} corresponds to the generalized forces remaining after removal of the generalized linear, velocity-dependent, frictional force Q_j^f. The holonomic forces of constraint are absorbed into the Lagrange multiplier term.

Rayleigh dissipation in Hamiltonian mechanics

If the nonconservative forces depend linearly on velocity, and are derivable from Rayleigh's dissipation function according to equation $(10.4.12)$, then using the definition of generalized momentum gives

$$\dot{p}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_i} + \left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t) + Q_j^{EXC}\right] - \frac{\partial \mathcal{R}(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{q}_j} \tag{10.S.5}$$

$$\dot{p}_i = -\frac{\partial H(\mathbf{p}, \mathbf{q}, t)}{\partial q_i} + \left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t) + Q_j^{EXC}\right] - \frac{\partial \mathcal{R}(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{q}_j} \tag{10.S.6}$$

Thus Hamilton’s equations become

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{10.S.7}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + \left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t) + Q_j^{EXC}\right] - \frac{\partial \mathcal{R}(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{q}_j} \tag{10.S.8}$$

The Rayleigh dissipation function $\mathcal{R}(\mathbf{q}, \dot{\mathbf{q}})$ provides an elegant and convenient way to account for dissipative forces in both Lagrangian and Hamiltonian mechanics.

Dissipative Lagrangians or Hamiltonians

New degrees of freedom or effective forces can be postulated that are then incorporated into the Lagrangian or the Hamiltonian in order to mimic the effects of the nonconservative forces. This approach has been used for special cases.
