13.7: Diagonalize the Inertia Tensor

Finding the three principal axes involves diagonalizing the inertia tensor, which is the classic eigenvalue problem discussed in appendix 19.1. Solution of the eigenvalue problem for rigid-body motion corresponds to a rotation of the coordinate frame to the principal axes resulting in the matrix

$$\{\mathbf{I}\}\cdot\boldsymbol{\omega}=I\boldsymbol{\omega} \tag{13.7.1}$$

where I comprises the three-valued eigenvalues, while the corresponding vector $\boldsymbol{\omega}$ is the eigenvector. Appendix 19.1 gives the solution of the matrix relation

$$\{\mathbf{I}\}\cdot\boldsymbol{\omega}=I\{\mathbb{I}\}\boldsymbol{\omega} \tag{13.7.2}$$

where I are three-valued eigen values for the principal axis moments of inertia, and $\{\mathbb{I}\}$ is the unity tensor, equation $(A.2.4)$.

$$\{\mathbb{I}\}\equiv\begin{Bmatrix}1&0&0\\0&1&0\\0&0&1\end{Bmatrix} \tag{13.7.3}$$

Rewriting 13.7.2 gives

$$(\{\mathbf{I}\}-I\{\mathbb{I}\})\cdot\boldsymbol{\omega}=0 \tag{13.7.4}$$

This is a matrix equation of the form $\mathbf{A}\cdot\boldsymbol{\omega}=0$ where $\mathbf{A}$ is a 3×3 matrix and $\boldsymbol{\omega}$ is a vector with values $\omega_x,\omega_y,\omega_z$. The matrix equation $\mathbf{A}\cdot\boldsymbol{\omega}=0$ really corresponds to three simultaneous equations for the three numbers $\omega_x,\omega_y,\omega_z$. It is a well-known property of equations like 13.7.4 that they have a non-zero solution if, and only if, the determinant $\det(\mathbf{A})$ is zero, that is

$$\det(\mathbf{I}-I\mathbb{I})=0 \tag{13.7.5}$$

This is called the **characteristic equation**, or **secular equation** for the matrix $\mathbf{I}$. The determinant involved is a cubic equation in the value of I that gives the three principal moments of inertia. Inserting one of the three values of I into equation $(13.4.6)$ gives the corresponding eigenvector ω. Applying the above eigenvalue problem to rigid-body rotation corresponds to requiring that some arbitrary set of body-fixed axes be the principal axes of inertia. This is obtained by rotating the body-fixed axis system such that

$$\begin{aligned}L_1=&\quad I_{11}\omega_1+I_{12}\omega_2+I_{13}\omega_3=I\omega_1\\L_2=&\quad I_{21}\omega_1+I_{22}\omega_2+I_{23}\omega_3=I\omega_2\\L_3=&\quad I_{31}\omega_1+I_{32}\omega_2+I_{33}\omega_3=I\omega_3\end{aligned} \tag{13.7.6}$$

or

$$\begin{aligned}(I_{11}-I)\omega_1+I_{12}\omega_2+I_{13}\omega_3=0\\I_{21}\omega_1+(I_{22}-I)\omega_2+I_{23}\omega_3=0\\I_{31}\omega_1+I_{32}\omega_2+(I_{33}-I)\omega_3=0\end{aligned} \tag{13.7.7}$$

These equations have a non-trivial solution for the ratios $\omega_1:\omega_2:\omega_3$ since the determinant vanishes, that is

$$\begin{vmatrix}(I_{11}-I)&I_{12}&I_{13}\\I_{21}&(I_{22}-I)&I_{23}\\I_{31}&I_{32}&(I_{33}-I)\end{vmatrix}=0 \tag{13.7.8}$$

The expansion of this determinant leads to a cubic equation with three roots for I. This is the **secular equation** for I whose eigenvalues are the **principal moments of inertia**.

The directions of the **principal axes**, that is the eigenvectors, can be found by substituting the corresponding solution for I into the prior equation. Thus for eigensolution I_1 the eigenvector is given by solving

$$\begin{aligned}(I_{11}-I_1)\omega_{11}+I_{12}\omega_{21}+I_{13}\omega_{31}=0\\I_{21}\omega_{11}+(I_{22}-I_1)\omega_{21}+I_{23}\omega_{31}=0\\I_{31}\omega_{11}+I_{32}\omega_{21}+(I_{33}-I_1)\omega_{31}=0\end{aligned} \tag{13.7.9}$$

These equations are solved for the ratios $\omega_{11}:\omega_{21}:\omega_{31}$ which are the direction numbers of the principle axis system corresponding to solution I_1. This principal axis system is defined relative to the original coordinate system. This procedure is repeated to find the orientation of the other two mutually perpendicular principal axes.