

3.5: Linearly-damped Free Linear Oscillator

General solution

All simple harmonic oscillations are damped to some degree due to energy dissipation via friction, viscous forces, or electrical resistance etc. The motion of damped systems is not conservative since energy is dissipated as heat. As was discussed in chapter 2 the damping force can be expressed as

$$\mathbf{F}_D(v) = -f(v)\hat{\mathbf{v}} \quad (3.5.1)$$

where the velocity dependent function $f(v)$ can be complicated. Fortunately there is a very large class of problems in electricity and magnetism, classical mechanics, molecular, atomic, and nuclear physics, where the damping force depends linearly on velocity which greatly simplifies solution of the equations of motion. Therefore this chapter will discuss linear damping.

Consider the free simple harmonic oscillator, that is, assuming no oscillatory forcing function, with a linear damping term $\mathbf{F}_D(v) = -b\mathbf{v}$ where the parameter b is the damping factor. Then the equation of motion is

$$-kx - b\dot{x} = m\ddot{x} \quad (3.5.2)$$

This can be rewritten as

$$\ddot{x} + \dot{x} + \Gamma\dot{x} + \omega_0^2 x = 0 \quad (3.5.3)$$

where the damping parameter

$$\Gamma = \frac{b}{m} \quad (3.5.4)$$

and the characteristic angular frequency

$$\omega_0 = \sqrt{\frac{k}{m}} \quad (3.5.5)$$

The general solution to the linearly-damped free oscillator is obtained by inserting the complex trial solution $z = z_0 e^{i\omega t}$. Then

$$(i\omega)^2 z_0 e^{i\omega t} + i\omega\Gamma z_0 e^{i\omega t} + \omega_0^2 z_0 e^{i\omega t} = 0 \quad (3.5.6)$$

This implies that

$$\omega^2 - i\omega\Gamma - \omega_0^2 = 0 \quad (3.5.7)$$

The solution is

$$\omega_{\pm} = i\frac{\Gamma}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\Gamma}{2}\right)^2} \quad (3.5.8)$$

The two solutions ω_{\pm} are complex conjugates and thus the solutions of the damped free oscillator are

$$z = z_1 e^{i\left(\frac{\Gamma}{2} + \sqrt{\omega_0^2 - \left(\frac{\Gamma}{2}\right)^2}\right)t} + z_2 e^{i\left(\frac{\Gamma}{2} - \sqrt{\omega_0^2 - \left(\frac{\Gamma}{2}\right)^2}\right)t} \quad (3.5.9)$$

This can be written as

$$z = e^{-\left(\frac{\Gamma}{2}\right)t} \left[z_1 e^{i\omega_1 t} + z_2 e^{-i\omega_1 t} \right] \quad (3.5.10)$$

where

$$\omega_1 \equiv \sqrt{\omega_0^2 - \left(\frac{\Gamma}{2}\right)^2} \quad (3.5.11)$$

Underdamped motion $\omega_1^2 \equiv \omega_o^2 - \left(\frac{\Gamma}{2}\right)^2 > 0$

When $\omega_1^2 > 0$ then the square root is real so the solution can be written taking the real part of z which gives that Equation 3.5.10 equals

$$x(t) = Ae^{-\left(\frac{\Gamma}{2}\right)t} \cos(\omega_1 t - \beta) \quad (3.5.12)$$

Where A and β are adjustable constants fit to the initial conditions. Therefore the velocity is given by

$$\dot{x}(t) = -Ae^{-\frac{\Gamma}{2}t} \left[\omega_1 \sin(\omega_1 t - \beta) + \frac{\Gamma}{2} \cos(\omega_1 t - \beta) \right] \quad (3.5.13)$$

This is the damped sinusoidal oscillation illustrated in Figure 3.5.1-upper. The solution has the following characteristics:

- The oscillation amplitude decreases exponentially with a time constant $\tau_D = \frac{2}{\Gamma}$.
- There is a small reduction in the frequency of the oscillation due to the damping leading to $\omega_1 = \sqrt{\omega_o^2 - \left(\frac{\Gamma}{2}\right)^2}$

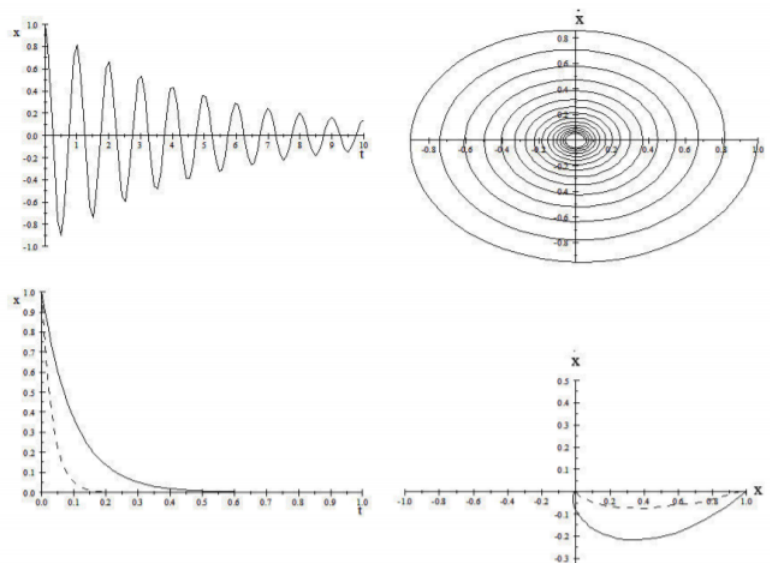


Figure 3.5.1: The amplitude-time dependence and state-space diagrams for the free linearly-damped harmonic oscillator. The upper row shows the underdamped system for the case with damping $\Gamma = \frac{\omega_0}{5\pi}$. The lower row shows the overdamped ($\frac{\Gamma}{2} > \omega_0$) [solid line] and critically damped ($\frac{\Gamma}{2} = \omega_0$) [dashed line] in both cases assuming that initially the system is at rest.

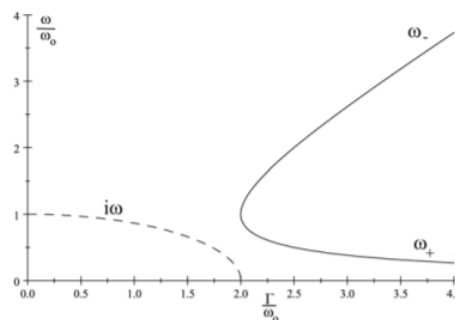


Figure 3.5.2: Real and imaginary solutions ω_{\pm} of the damped harmonic oscillator. A phase transition occurs at $\Gamma = 2\omega_0$. For $\Gamma < 2\omega_0$ (dashed) the two solutions are complex conjugates and imaginary. For $\Gamma > 2\omega_0$, (solid), there are two real solutions ω_+ and ω_- with widely different decay constants where ω_+ dominates the decay at long times.

Overdamped case $\omega_1^2 \equiv \omega_o^2 - \left(\frac{\Gamma}{2}\right)^2 < 0$

In this case the square root of ω_1^2 is imaginary and can be expressed as $\omega'_1 = i\sqrt{\left(\frac{\Gamma}{2}\right)^2 - \omega_o^2}$. Therefore the solution is obtained more naturally by using a real trial solution $z = z + 0e^{\omega t}$ in Equation 3.5.10 which leads to two roots

$$\omega_{\pm} = - \left[-\frac{\Gamma}{2} \pm \sqrt{\left(\frac{\Gamma}{2}\right)^2 - \omega_o^2} \right]$$

Thus the exponentially damped decay has two time constants ω_+ and ω_- .

$$x(t) = A_1 e^{-\omega_+ t} + A_2 e^{-\omega_- t} \quad (3.5.14)$$

The time constant $\frac{1}{\omega_-} < \frac{1}{\omega_+}$ thus the first term $A_1 e^{-\omega_+ t}$ in the bracket decays in a shorter time than the second term $A_2 e^{-\omega_- t}$. As illustrated in Figure 3.5.2 the decay rate, which is imaginary when underdamped, i.e. $\frac{\Gamma}{2} < \omega_o$ bifurcates into two real values ω_{\pm} for overdamped, i.e. $\frac{\Gamma}{2} > \omega_o$. At large times the dominant term when overdamped is for ω_+ which has the smallest decay rate, that is, the longest decay constant $\tau_+ = \frac{1}{\omega_+}$. There is no oscillatory motion for the overdamped case, it slowly moves monotonically to zero as shown in fig 3.5 lower. The amplitude decays away with a time constant that is longer than $\frac{2}{\Gamma}$.

Critically damped $\omega_1^2 \equiv \omega_o^2 - \left(\frac{\Gamma}{2}\right)^2 = 0$

This is the limiting case where $\frac{\Gamma}{2} = \omega_o$ For this case the solution is of the form

$$x(t) = (A + Bt)e^{-(\frac{\Gamma}{2})t} \quad (3.5.15)$$

This motion also is non-sinusoidal and evolves monotonically to zero. As shown in Figure 3.5.1 the critically-damped solution goes to zero with the shortest time constant, that is, largest ω . Thus analog electric meters are built almost critically damped so the needle moves to the new equilibrium value in the shortest time without oscillation.

It is useful to graphically represent the motion of the damped linear oscillator on either a state space (\dot{x}, x) diagram or phase space (p_x, x) diagram as discussed in chapter 3.4. The state space plots for the undamped, overdamped, and critically-damped solutions of the damped harmonic oscillator are shown in Figure 3.5.1. For underdamped motion the state space diagram spirals inwards to the origin in contrast to critical or overdamped motion where the state and phase space diagrams move monotonically to zero.

Energy dissipation

The instantaneous energy is the sum of the instantaneous kinetic and potential energies

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \quad (3.5.16)$$

where x and \dot{x} are given by the solution of the equation of motion. Consider the total energy of the underdamped system

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_o^2 x^2 \quad (3.5.17)$$

where $k = m\omega_o^2$. The average total energy is given by substitution for x and \dot{x} and taking the average over one cycle. Since

$$x(t) = A e^{-(\frac{\Gamma}{2})t} \cos(\omega_1 t - \beta) \quad (3.5.18)$$

Then the velocity is given by

$$\dot{x}(t) = -A e^{-\frac{\Gamma}{2}t} \left[\omega_1 \sin(\omega_1 t - \beta) + \frac{\Gamma}{2} \cos(\omega_1 t - \beta) \right] \quad (3.5.19)$$

Inserting equations 3.5.18 and 3.5.19 into 3.5.17 gives a small amplitude oscillation about an exponential decay for the energy E . Averaging over one cycle and using the fact that $\langle \sin \theta \cos \theta \rangle = 0$, and $\langle [\sin \theta]^2 \rangle = \langle [\cos \theta]^2 \rangle = \frac{1}{2}$, gives the time-averaged total energy as

$$\langle E \rangle = e^{-\Gamma t} \left(\frac{1}{4} m A^2 \omega_1^2 + \frac{1}{4} m A^2 \left(\frac{\Gamma}{2} \right)^2 + \frac{1}{4} m A^2 \omega_o^2 \right) \quad (3.5.20)$$

which can be written as

$$\langle E \rangle = E_0 e^{-\Gamma t} \quad (3.5.21)$$

Note that the *energy* of the linearly damped free oscillator decays away exponentially with a time constant $\tau = \frac{1}{\Gamma}$. That is, the *intensity* has a time constant that is half the time constant for the decay of the *amplitude* of the transient response. Note that the

average kinetic and potential energies are identical, as implied by the Virial theorem, and both decay away with the same time constant. This relation between the mean life τ for decay of the damped harmonic oscillator and the damping width term Γ occurs frequently in physics.

The damping of an oscillator usually is characterized by a single parameter Q called the **Quality Factor** where

$$Q \equiv \frac{\text{Energy stored in the oscillator}}{\text{Energy dissipated per radian}} \quad (3.5.22)$$

The energy loss per radian is given by

$$\Delta E = \frac{dE}{dt} \frac{1}{\omega_1} = \frac{E\Gamma}{\omega_1} = \frac{E\Gamma}{\sqrt{\omega_o^2 - \left(\frac{\Gamma}{2}\right)^2}} \quad (3.5.23)$$

where the numerator $\omega_1 = \sqrt{\omega_o^2 - \left(\frac{\Gamma}{2}\right)^2}$ is the frequency of the free damped linear oscillator.

Thus the Quality factor Q equals

$$Q = \frac{E}{\Delta E} = \frac{\omega_1}{\Gamma} \quad (3.5.24)$$

The larger the Q factor, the less damped is the system, and the greater is the number of cycles of the oscillation in the damped wave train. Chapter 3.11.3 shows that the longer the wave train, that is the higher is the Q factor, the narrower is the frequency distribution around the central value. The Mössbauer effect in nuclear physics provides a remarkably long wave train that can be used to make high precision measurements. The high- Q precision of the LIGO laser interferometer was used in the recent successful search for gravity waves.

Table 3.5.1: Typical Q factors in nature.

Oscillating system	Typical Q factors
Earth, for earthquake wave	250-1400
Piano string	3000
Crystal in digital watch	10^4
Microwave cavity	10^4
Excited atom	10^7
Neutron star	10^{12}
LIGO laser	10^{13}
Mössbauer effect in nucleus	10^{14}

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