8.S: Hamiltonian Mechanics (Summary)

Hamilton's equations of motion

Inserting the generalized momentum into Jacobi's generalized energy relation was used to define the Hamiltonian function to be

$$H(\mathbf{q},\mathbf{p},t)=\mathbf{p}\cdot\dot{\mathbf{q}}-L(\mathbf{q},\dot{\mathbf{q}},t) \tag{8.S.1}$$

The Legendre transform of the Lagrange-Euler equations, led to Hamilton's equations of motion.

$$\dot{q}_j=\frac{\partial H}{\partial p_j} \tag{8.S.2}$$

$$\dot{p}_j=-\frac{\partial H}{\partial q_j}+\left[\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}+Q_j^{EXC}\right] \tag{8.S.3}$$

The generalized energy equation $(8.8.1)$ gives the time dependence

$$\frac{dH(\mathbf{q},\mathbf{p},t)}{dt}=\sum_j\left(\left[\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}+Q_j^{EXC}\right]\dot{q}_j\right)-\frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial t} \tag{8.S.4}$$

where

$$\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t} \tag{8.S.5}$$

The p_k, q_k are treated as independent canonical variables. Lagrange was the first to derive the canonical equations but he did not recognize them as a basic set of equations of motion. Hamilton derived the canonical equations of motion from his fundamental variational principle and made them the basis for a far-reaching theory of dynamics. Hamilton's equations give $2s$ first-order differential equations for p_k, q_k for each of the s degrees of freedom. Lagrange's equations give s second-order differential equations for the variables $q_k, \dot{q}_k$.

Routhian reduction technique

The Routhian reduction technique is a hybrid of Lagrangian and Hamiltonian mechanics that exploits the advantages of both approaches for solving problems involving cyclic variables. It is especially useful for solving motion in rotating systems in science and engineering. Two Routhians are used frequently for solving the equations of motion of rotating systems. Assuming that the variables between $1\leq i\leq s$ are non-cyclic, while the m variables between $s+1\leq i\leq n$ are ignorable cyclic coordinates, then the two Routhians are:

$$R_{cyclic}(q_1,\dots,q_n;\dot{q}_1,\dots,\dot{q}_s;p_{s+1},\dots,p_n;t)=\sum_{cyclic}^{m}p_i\dot{q}_i-L=H-\sum_{noncyclic}^{s}p_i\dot{q}_i \tag{8.S.6}$$

$$R_{noncyclic}(q_1,\dots,q_n;p_1,\dots,p_s;\dot{q}_{s+1},\dots,\dot{q}_n;t)=\sum_{noncyclic}^{s}p_i\dot{q}_i-L=H-\sum_{cyclic}^{m}p_i\dot{q}_i \tag{8.S.7}$$

The Routhian R_{cyclic} is a negative Lagrangian for the non-cyclic variables between $1\leq i\leq s$, where $s=n-m$, and is a Hamiltonian for the m cyclic variables between $s+1\leq i\leq n$. Since the cyclic variables are constants of the Hamiltonian, their solution is trivial, and the number of variables included in the Lagrangian is reduced from n to $s=n-m$. The Routhian R_{cyclic} is useful for solving some problems in classical mechanics. The Routhian $R_{noncyclic}$ is a Hamiltonian for the non-cyclic variables between $1\leq i\leq s$, and is a negative Lagrangian for the m cyclic variables between $s+1\leq i\leq n$. Since the cyclic variables are constants of motion, the Routhian $R_{noncyclic}$ also is a constant of motion but it does not equal the total energy since the coordinate transformation is time dependent. The Routhian $R_{noncyclic}$ is especially valuable for solving rotating many-body systems such as galaxies, molecules, or nuclei, since the Routhian $R_{noncyclic}$ is the Hamiltonian in the rotating body-fixed coordinate frame.

Variable mass systems:

Two examples of heavy flexible chains falling in a uniform gravitational field were used to illustrate how variable mass systems can be handled using Lagrangian and Hamiltonian mechanics. The falling-mass system is conservative assuming that both the donor plus the receptor body systems are included.

Comparison of Lagrangian and Hamiltonian mechanics

Lagrangian and the Hamiltonian dynamics are two powerful and related variational algebraic formulations of mechanics that are based on Hamilton's action principle. They can be applied to any conservative degrees of freedom as discussed in chapters 7, 9, and 16. Lagrangian and Hamiltonian mechanics both concentrate solely on active forces and can ignore internal forces. They can handle many-body systems and allow convenient generalized coordinates of choice. This ability is impractical or impossible using Newtonian mechanics. Thus it is natural to compare the relative advantages of these two algebraic formalisms in order to decide which should be used for a specific problem.

For a system with n generalized coordinates, plus m constraint forces that are not required to be known, then the Lagrangian approach, using a minimal set of generalized coordinates, reduces to only $s = n - m$ *second-order* differential equations and unknowns compared to the Newtonian approach where there are $n + m$ unknowns. Alternatively, use of Lagrange multipliers allows determination of the constraint forces resulting in $n + m$ second order equations and unknowns. The Lagrangian potential function is limited to conservative forces, Lagrange multipliers can be used to handle holonomic forces of constraint, while generalized forces can be used to handle non-conservative and non-holonomic forces. The advantage of the Lagrange equations of motion is that they can deal with any type of force, conservative or non-conservative, and they directly determine $q, \dot{q}$ rather than q, p which then requires relating p to $\dot{q}$.

For a system with n generalized coordinates, the Hamiltonian approach determines $2n$ *first-order* differential equations which are easier to solve than second-order equations. However, the $2n$ solutions must be combined to determine the equations of motion. The Hamiltonian approach is superior to the Lagrange approach in its ability to obtain an analytical solution of the integrals of the motion. Hamiltonian dynamics also has a means of determining the unknown variables for which the solution assumes a soluble form. Important applications of Hamiltonian mechanics are to quantum mechanics and statistical mechanics, where quantum analogs of q_i and p_i, can be used to relate to the fundamental variables of Hamiltonian mechanics. This does not apply for the variables q_i and $\dot{q}_i$ of Lagrangian mechanics. The Hamiltonian approach is especially powerful when the system has m cyclic variables, then the m conjugate momenta p_i are constants. Thus the m conjugate variables (q_i, p_i) can be factored out of the Hamiltonian, which reduces the number of conjugate variables required to $n - m$. This is not possible using the Lagrangian approach since, even though the m coordinates q_i can be factored out, the velocities $\dot{q}_i$ still must be included, thus the n conjugate variables must be included. The Lagrange approach is advantageous for obtaining a numerical solution of systems in classical mechanics. However, Hamiltonian mechanics expresses the variables in terms of the fundamental canonical variables $(\mathbf{q}, \mathbf{p})$ which provides a more fundamental insight into the underlying physics.[2]