15.1: Introduction to Advanced Hamiltonian Mechanics

This study of classical mechanics has involved climbing a vast mountain of knowledge, while the pathway to the top has led us to elegant and beautiful theories that underlie much of modern physics. Being so close to the summit provides the opportunity to take a few extra steps in order to provide a glimpse of applications to physics at the summit. These are described in chapters $15-18$.

Hamilton's development of Hamiltonian mechanics in 1834 is the crowning achievement for applying variational principles to classical mechanics. A fundamental advantage of Hamiltonian mechanics is that it uses the conjugate coordinates $\mathbf{q}, \mathbf{p}$, plus time t, which is a considerable advantage in most branches of physics and engineering. Compared to Lagrangian mechanics, Hamiltonian mechanics has a significantly broader arsenal of powerful techniques that can be exploited to obtain an analytical solution of the integrals of the motion for complicated systems. In addition, Hamiltonian dynamics provides a means of determining the unknown variables for which the solution assumes a soluble form, and is ideal for study of the fundamental underlying physics in applications to fields such as quantum or statistical physics. As a consequence, Hamiltonian mechanics has become the preeminent variational approach used in modern physics. This chapter introduces the following four techniques in Hamiltonian mechanics:

1. the elegant Poisson bracket representation of Hamiltonian mechanics, which played a pivotal role in the development of quantum theory;
2. the powerful Hamilton-Jacobi theory coupled with Jacobi's development of canonical transformation theory;
3. action-angle variable theory; and
4. canonical perturbation theory.

Prior to further development of the theory of Hamiltonian mechanics, it is useful to summarize the major formula relevant to Hamiltonian mechanics that have been presented in chapters 7, 8, and 9.

Action functional S:

As discussed in chapter 9.2, Hamiltonian mechanics is built upon Hamilton's action functional

$$S(\mathbf{q}, \mathbf{p}, t)=\int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) d t \tag{15.1.1}$$

Hamilton's Principle of least action states that

$$\delta S(\mathbf{q}, \mathbf{p}, t)=\delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) d t=0 \tag{15.1.2}$$

Generalized momentum p:

In chapter 7.2, the generalized (canonical) momentum was defined in terms of the Lagrangian L to be

$$p_i \equiv \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}_i} \tag{15.1.3}$$

Chapter 9.2 defined the generalized momentum in terms of the action functional S to be

$$p_j=\frac{\partial S(\mathbf{q}, \mathbf{p}, t)}{\partial q_j} \tag{15.1.4}$$

Generalized energy $h(\mathbf{q}, \dot{q}, t)$:

Jacobi's Generalized Energy $h(\mathbf{q}, \dot{q}, t)$ was defined in equation $(7.7.6)$ as

$$h(\mathbf{q}, \dot{\mathbf{q}}, t) \equiv \sum_j\left(\dot{q}_j \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}_j}\right)-L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{15.1.5}$$

Hamiltonian function:

The Hamiltonian $H(\mathbf{q}, \mathbf{p}, t)$ was defined in terms of the generalized energy $h(\mathbf{q}, \dot{\mathbf{q}}, t)$ plus the generalized momentum. That is

$$H(\mathbf{q}, \mathbf{p}, t) \equiv h(\mathbf{q}, \dot{\mathbf{q}}, t)=\sum_j p_j \dot{q}_j-L(\mathbf{q}, \dot{\mathbf{q}}, t)=\mathbf{p} \cdot \dot{\mathbf{q}}-L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{15.1.6}$$

where $\mathbf{q}, \mathbf{p}$ correspond to n-dimensional vectors, e.g. $\mathbf{q} \equiv (q_1, q_2, \dots, q_n)$ and the scalar product $\mathbf{p} \cdot \dot{\mathbf{q}} = \sum_i p_i \dot{q}_i$. Chapter 8.2 used a Legendre transformation to derive this relation between the Hamiltonian and Lagrangian functions. Note that whereas the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is expressed in terms of the coordinates $\mathbf{q}$, plus conjugate velocities $\dot{\mathbf{q}}$, the Hamiltonian $H(\mathbf{q}, \mathbf{p}, t)$ is expressed in terms of the coordinates $\mathbf{q}$ plus their conjugate momenta $\mathbf{p}$. For scleronomic systems, plus assuming the standard Lagrangian, then equations $(7.9.4)$ and $(7.6.13)$ give that the Hamiltonian simplifies to equal the total mechanical energy, that is, $H = T + U$.

Generalized energy theorem:

The equations of motion lead to the generalized energy theorem which states that the time dependence of the Hamiltonian is related to the time dependence of the Lagrangian.

$$\frac{dH(\mathbf{q}, \mathbf{p}, t)}{dt} = \sum_j \dot{q}_j \left[Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t) \right] - \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial t} \tag{15.1.7}$$

Note that if all the generalized non-potential forces and Lagrange multiplier terms are zero, and if the Lagrangian is not an explicit function of time, then the Hamiltonian is a constant of motion.

Hamilton's equations of motion:

Chapter 8.3 showed that a Legendre transform plus the Lagrange-Euler equations led to Hamilton's equations of motion. Hamilton derived these equations of motion directly from the action functional, as shown in chapter 9.2.

$$\dot{q}_j = \frac{\partial H(\mathbf{q}, \mathbf{p}, t)}{\partial p_j} \tag{15.1.8}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}(\mathbf{q}, \mathbf{p}, t) + \left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j} + Q_j^{EXC} \right] \tag{15.1.9}$$

$$\frac{\partial H(\mathbf{q}, \mathbf{p}, t)}{\partial t} = -\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial t} \tag{15.1.10}$$

Note the symmetry of Hamilton's two canonical equations. The canonical variables p_k, q_k are treated as independent canonical variables. Lagrange was the first to derive the canonical equations but he did not recognize them as a basic set of equations of motion. Hamilton derived the canonical equations of motion from his fundamental variational principle and made them the basis for a far-reaching theory of dynamics. Hamilton's equations give $2s$ first-order differential equations for p_k, q_k for each of the s degrees of freedom. Lagrange's equations give s second-order differential equations for the variables $q_k, \dot{q}_k$.

Hamilton-Jacobi equation:

Hamilton used Hamilton's Principle to derive the Hamilton-Jacobi equation $(9.2.17)$.

$$\frac{\partial S}{\partial t} + H(\mathbf{q}, \mathbf{p}, t) = 0 \tag{15.1.11}$$

The solution of Hamilton's equations is trivial if the Hamiltonian is a constant of motion, or when a set of generalized coordinates can be identified for which all the coordinates q_i are constant, or are cyclic (also called *ignorable* coordinates). Jacobi developed the mathematical framework of canonical transformations required to exploit the Hamilton-Jacobi equation.