15.7: Symplectic Representation

The Hamilton's first-order equations of motion are symmetric if the generalized and constraint force terms, in equation $(15.1.9)$, are excluded.

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \qquad -\dot{\mathbf{p}} = \frac{\partial H}{\partial \mathbf{q}}$$

This stimulated attempts to treat the canonical variables $(\mathbf{q}, \mathbf{p})$ in a symmetric form using group theory. Some graduate textbooks in classical mechanics have adopted use of symplectic symmetry in order to unify the presentation of Hamiltonian mechanics. For a system of n degrees of freedom, a column matrix $\boldsymbol{\eta}$ is constructed that has $2n$ elements where

$$\eta_j = q_j \quad \eta_{n+j} = p_j \quad j \leq n \tag{15.7.1}$$

Therefore the column matrix

$$\left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)_j = \frac{\partial H}{\partial q_j} \quad \left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)_{n+j} = \frac{\partial H}{\partial p_j} \quad j \leq n \tag{15.7.2}$$

The symplectic matrix $\mathbf{J}$ is defined as being a $2n$ by $2n$ skew-symmetric, orthogonal matrix that is broken into four $n \times n$ null or unit matrices according to the scheme

$$\mathbf{J} = \begin{pmatrix} [\mathbf{0}] & +[\mathbf{1}] \\ -[\mathbf{1}] & [\mathbf{0}] \end{pmatrix} \tag{15.7.3}$$

where $[\mathbf{0}]$ is the n-dimension null matrix, for which all elements are zero. Also $[\mathbf{1}]$ is the n-dimensional unit matrix, for which the diagonal matrix elements are unity and all off-diagonal matrix elements are zero. The $\mathbf{J}$ matrix accounts for the opposite signs used in the equations for $\dot{\mathbf{q}}$ and $\dot{\mathbf{p}}$. The symplectic representation allows the Hamilton's equations of motion to be written in the compact form

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}} \tag{15.7.4}$$

This textbook does not use the elegant symplectic representation since this representation ignores the important generalized forces and Lagrange multiplier forces.