16.8: Viscous Fluid Dynamics

Viscous fluid dynamics is a branch of classical mechanics that plays a pivotal role in a wide range of aspects of life, such as blood flow in human anatomy, weather, hydraulic engineering, and transportation by land, sea, and air. Viscous fluid flow provides natures most common manifestation of nonlinearity and turbulence in classical mechanics, and provides an excellent illustration of possible solutions of non-linear equations of motion introduced in chapter 4. A detailed description of turbulence remains a challenging problem and this subject has the reputation of being the last great unsolved problem in classical mechanics. There is an apocryphal story that Werner Heisenberg was asked, if given the opportunity, what would he like to ask God. His reply was "When I meet God, I am going to ask him two questions: Why relativity? and why turbulence?, I really believe he will only have an answer to the first".

In contrast to solids, fluids do not have elastic restoring forces to support shear stress because the fluid flows. Shear stresses in fluids are balance by viscous forces which are velocity dependent. There are two mechanisms that lead to shear stress acting between adjacent fluid layers in relative motion. The first mechanism involves laminar flow where the viscous forces produce shear stress between adjacent layers of the fluid which are moving parallel along adjacent streamlines at differing velocities. Viscous forces typically dominate laminar flow. High viscosity fluids like honey exhibit laminar flow and are more difficult to stir or pour compared with low-viscosity fluids like water. The second mechanism involves turbulent flow where shear stress is due to momentum transfer between adjacent layers when the flow breaks up into large-scale coherent vortex structures which carry most of the kinetic energy. These eddies lead to transverse motion that transfers momentum plus heat between adjacent layers and leads to higher drag. The wing-tip vortex produced by the wing tip of an aircraft is an example of a dynamically-distinct, large-scale, coherent vortex structure which has considerable angular momentum and decays by fragmentation into a cascade of smaller scale structures.

Navier-Stokes equation

Viscous forces acting on the small-scale coherent structures eventually dissipate the energy in turbulent motion. The viscous drag can be handled in terms of a stress tensor $\mathbf{T}$ analogous to its use when accounting for the elastic restoring forces in elasticity as discussed in chapter $16.5.3$. That is, the viscous force density is related to the deceleration of the volume element by

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) = -\boldsymbol{\nabla}\cdot\mathbf{T} \tag{16.8.1}$$

where the components of the stress tensor are

$$T_{ki} = T_{ik} = P\delta_{ik} + \rho v_i v_k \tag{16.8.2}$$

Note that the stress tensor gives the momentum flux density tensor, which involves a diagonal term proportional to pressure P, plus a viscous drag term that is is proportional to the product of two velocities.

The Navier-Stokes equations are the fundamental equations characterizing fluid flow. They are based on application of Newton's second law of motion to fluids together with the assumption that the fluid stress is the sum of a diffusing viscous term plus a pressure term. Combining Euler's equation, $(16.7.11)$, with 16.8.1 gives the Navier-Stokes equation

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\right] = -\boldsymbol{\nabla}P + \boldsymbol{\nabla}\cdot T + \mathbf{f} \tag{16.8.3}$$

where ρ is the fluid density, $\mathbf{v}$ is the flow velocity vector, P the pressure, $\mathbf{T}$ is the shear stress tensor viscous drag term, and $\mathbf{f}$ represents external body forces per unit volume such as gravity acting on the fluid. For incompressible flow the stress tensor term simplifies to $\boldsymbol{\nabla}\cdot\mathbf{T} = \mu\boldsymbol{\nabla}^2\mathbf{v}$. Then the Navier-Stokes equation simplifies to

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\right] = -\boldsymbol{\nabla}P + \mu\boldsymbol{\nabla}^2\mathbf{v} + \mathbf{f} \tag{16.8.4}$$

where $\mu\boldsymbol{\nabla}^2\mathbf{v}$ is the viscosity drag term. The left-hand side of Equation 16.8.4 represents the rate of change of momentum per unit volume while the right-hand side represents the summation of the forces per unit volume that are acting.

The Navier-Stokes equations are nonlinear due to the $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ term as well as being a function of velocity. This non-linearity leads to a wide spectrum of dynamic behavior ranging from ordered laminar flow to chaotic turbulence. Numerical solution of the Navier-Stokes equations is extremely difficult because of the wide dynamic range of the dimensions of the coherent structures involved in turbulent motion. For example, simulation calculations require use of a high resolution mesh which is a challenge to the capabilities of current generation computers.

The microscopic boundary condition at the interface of the solid and fluid is that the fluid molecules have zero average tangential velocity relative to the normal to the solid-fluid interface. This implies that there is a boundary layer for which there is a gradient in the tangential velocity of the fluid between the solid-fluid interface and the free-steam velocity. This velocity gradient produces vorticity in the fluid. When the viscous forces are negligible then the angular momentum in any coherent vortex structure is conserved leading to the vortex motion being preserved as it propagates.

Reynolds number

Fluid flow can be characterized by the Reynolds number Re which is a dimensionless number that is a measure of the ratio of the inertial forces $\rho v^2/L$ to viscous forces $\mu v/L^2$. That is,

$$\mathrm{Re} \equiv \frac{\text{Inertial forces}}{\text{Viscous forces}} = \frac{\rho v L}{\mu} = \frac{vL}{\eta} \tag{16.8.5}$$

where v is the relative velocity between the free fluid flow and the solid surface, L is a characteristic linear dimension, μ is the dynamic viscosity of the fluid, η is the kinematic viscosity $(\eta = \frac{\mu}{\rho})$, and ρ is the density of the fluid. The Law of Similarity implies that at a given Reynolds number, for a specific shaped solid body, the fluid flow behaves identically independent of the size of the body. Thus one can use small models in wind tunnels, or water-flow tanks, to accurately model fluid flow that can be scaled up to a full-sized aircraft or boats by scaling v and L to give the same Reynolds number.

Laminar and turbulent fluid flow

Fluid flow over a cylinder illustrates the general features of fluid flow. The drag force F_D acting on a cylinder of diameter D and length l, with the cylindrical axis perpendicular to the fluid flow, is given by

$$F_D = \frac{1}{2}\rho v^2 C_D D l \tag{16.8.6}$$

where C_D is the coefficient of drag. Figure $16.8.1 upper$ shows the dependence of the drag coefficient C_D as a function of the Reynolds number, for fluid flow that is transverse to a smooth circular cylinder. The lower part of Figure $16.8.1$ shows the streamlines for flow around the cylinder at various Reynolds numbers for the points identified by the letters A, B, C, D, and E on the plot of the drag coefficient versus Reynolds number for a smooth cylinder.

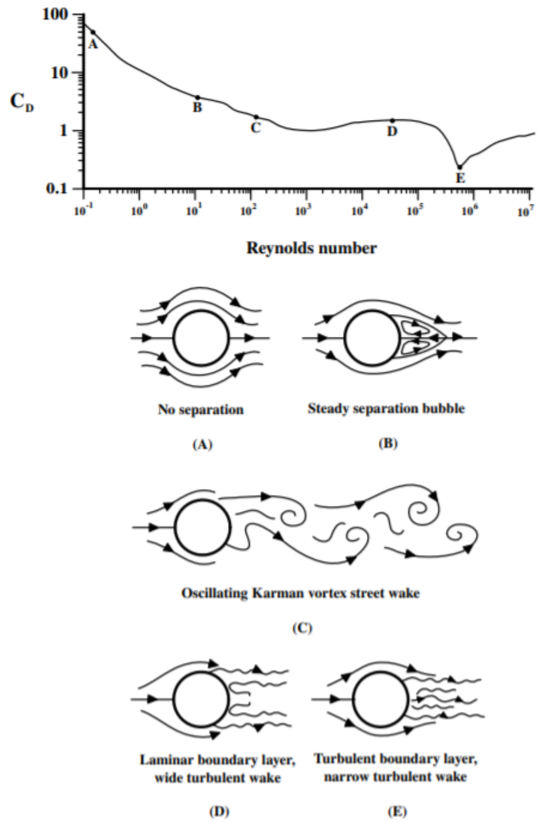

Figure 16.8.1: Upper: The dependence of the coefficient of drag C_D on Reynolds number Re for fluid flow perpendicular to a smooth circular cylinder of diameter D and length l. Lower: Typical flow patterns for flow past a circular cylinder at various Reynolds numbers as indicated in the upper figure.

A) At low velocities, where $\mathrm{Re} \leq 1$, the flow is laminar around the cylinder in that the low vorticity is damped by the viscous forces and the $\frac{\partial \mathbf{v}}{\partial t}$ term in Equation 16.8.4 can be ignored. The coefficient of drag C_D varies inversely with Re leading to the drag forces that are roughly linear with velocity as described in chapter $2.10.5$. The size and velocities of raindrops in a light rain shower correspond to such Reynolds numbers.

B) For $10 < \mathrm{Re} < 30$ the flow has two turbulent vortices immediately behind the body in the wake of the cylinder, but the flow still is primarily laminar as illustrated.

C) For $40 < \mathrm{Re} < 250$ the pair of vortices peel off alternately producing a regular periodic sequence of vortices although the flow still is laminar. This vortex sheet is called a von Kármán vortex sheet for which the velocity at a given position, relative to the cylinder, is time dependent in contrast to the situation at lower Reynolds numbers.

D) For $10^3 < \mathrm{Re} < 10^5$ viscous forces are negligible relative to the inertial effects of the vortices and boundary-layer vortices have less time to diffuse into the larger region of the fluid, thus the boundary layer is thinner. The boundary-layer flow exhibits a small scale chaotic turbulence in three dimensions superimposed on regular alternating vortex structures. In this range C_D is roughly constant and thus the drag forces are proportional to the square of the velocity. This regime of Reynold numbers corresponds to typical velocities of moving automobiles.

E) For $\mathrm{Re} \approx 10^6$, which is typical of a flying aircraft, the inertial effects dominate except in the narrow boundary layer close to the solid-fluid interface. The chaotic region works its way further forward on the cylinder reducing the volume of the chaotic turbulent boundary layer which results in a significant decreases in C_D. For a sailplane wing flying at about $50\ knots$, the boundary layer at the leading edge of the cylinder reduces to the order of a millimeter in thickness at the leading edge and a centimeter at the trailing edge. At these Reynold's numbers the airflow comprises a thin boundary layer, where viscous effects are important, plus fluid flow in the bulk of the fluid where the vortex inertial terms dominate and viscous forces can be ignored. That is, the viscous stress tensor term $\boldsymbol{\nabla} \cdot \mathbf{T}$, on the right-hand side of Equation 16.8.3, can be ignored, and the Navier-Stokes equation reduces to the simpler Euler equation for such inviscid fluid flow.

The importance of the inertia of the vortices is illustrated by the persistence of the vortex structure and turbulence over a wide range of length scales characteristic of turbulent flow. The dynamic range of the dimension of coherent vortex structures is enormous. For example, in the atmosphere the vortex size ranges from $10^5\ m$ in diameter for hurricanes down to $10^{-3}\ m$ in thin boundary layers adjacent to an aircraft wing. The transition from laminar to turbulent flow is illustrated by water flow over the hull of a ship which involves laminar flow at the bow followed by turbulent flow behind the bow wave and at the stern of the ship. The broad extent of the white foam of seawater along the side and the stern of a ship illustrates the considerable energy dissipation produced by the turbulence. The boundary layer of a stalled aircraft wing is another example. At a high angle of attack, the airflow on the lower surface of the wing remains laminar, that is, the stream velocity profile, relative to the wing, increases smoothly from zero at the wing surface outwards until it meets the ambient air velocity on the outer surface of the boundary layer which is the order of a millimeter thick. The flow on the top surface of the wing initially is laminar before becoming turbulent at which point the boundary layer rapidly increases in thickness. Further back the airflow detaches from the wing surface and large-scale vortex structures lead to a wide boundary layer comparable in thickness to the chord of the wing with vortex motion that leads to the airflow reversing its direction adjacent to the upper surface of the wing which greatly increases drag. When the vortices begin to shed off the bounded surface they do so at a certain frequency which can cause vibrations that can lead to structural failure if the frequency of the shedding vortices is close to the resonance frequency of the structure.

Considerable time and effort are expended by aerodynamicists and hydrodynamicists designing aircraft wings and ship hulls to maximize the length of laminar region of the boundary layer to minimize drag. When the Reynolds number is large the slightest imperfections in the shape of wing, such as a speck of dust, can trigger the transition from laminar to turbulent flow. The boundaries between adjacent large-scale coherent structures are sensitively identified in computer simulations by large divergence of the streamlines at any separatrix. A large positive, finite-time, Lyapunov exponent identifies divergence of the streamlines which occurs at a separatrix between adjacent large-scale coherent vortex structures, whereas the Lyapunov exponents are negative for converging streamlines within any coherent structure. Computations of turbulent flow often combine the use of finite-time Lyapunov exponents to identify coherent structures, plus Lagrangian mechanics for the equations of motion since the Lagrangian is a scalar function, it is frame independent, and it gives far better results for fluid motion than using Newtonian mechanics. Thus the Lagrangian approach in the continua is used extensively for calculations in aerodynamics, hydrodynamics, and studies of atmospheric phenomena such as convection, hurricanes, tornadoes, etc.
