8.4: Hamiltonian in Different Coordinate Systems

Prior to solving problems using Hamiltonian mechanics, it is useful to express the Hamiltonian in cylindrical and spherical coordinates for the special case of conservative forces since these are encountered frequently in physics.

Cylindrical Coordinates ρ, z, ϕ

Consider cylindrical coordinates ρ, z, ϕ. Expressed in Cartesian coordinate

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}$$

Using appendix table $19.3.3$, the Lagrangian can be written in cylindrical coordinates as

$$\begin{aligned} L &= T - U \qquad (8.4.1) \\ &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - U(\rho, z, \phi) \qquad (8.4.2) \end{aligned}$$

The conjugate momenta are

$$\begin{aligned} p_\rho &= \frac{\partial L}{\partial\dot{\rho}} = m\dot{\rho} \qquad (8.4.3) \\ p_\phi &= \frac{\partial L}{\partial\dot{\phi}} = m\rho^2\dot{\phi} \qquad (8.4.4) \\ p_z &= \frac{\partial L}{\partial\dot{z}} = m\dot{z} \qquad (8.4.5) \end{aligned}$$

Assume a conservative force, then H is conserved. Since the transformation from Cartesian to non-rotating generalized cylindrical coordinates is time independent, then $H = E$. Then using Equations 8.4.2-8.4.5 gives the Hamiltonian in cylindrical coordinates to be

$$\begin{aligned} H(\mathbf{q}, \mathbf{p}, t) &= \sum_i p_i\dot{q}_i - L(\mathbf{q}, \dot{\mathbf{q}}, t) \qquad (8.4.6) \\ &= \left(p_\rho\dot{\rho} + p_\phi\dot{\phi} + p_z\dot{z}\right) - \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + U(\rho, z, \phi) \\ &= \frac{1}{2m}\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2\right) + U(\rho, z, \phi) \qquad (8.4.7) \end{aligned}$$

The canonical equations of motion in cylindrical coordinates can be written as

$$\begin{aligned} \dot{p}_\rho &= -\frac{\partial H}{\partial\rho} = \frac{p_\phi^2}{m\rho^3} - \frac{\partial U}{\partial\rho} \qquad (8.4.8) \\ \dot{p}_\phi &= -\frac{\partial H}{\partial\phi} = -\frac{\partial U}{\partial\phi} \qquad (8.4.9) \\ \dot{p}_z &= -\frac{\partial H}{\partial z} = -\frac{\partial U}{\partial z} \qquad (8.4.10) \\ \dot{\rho} &= \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} \qquad (8.4.11) \\ \dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2} \qquad (8.4.12) \\ \dot{z} &= \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad (8.4.13) \end{aligned}$$

Note that if ϕ is cyclic, that is $\frac{\partial U}{\partial\phi} = 0$, then the angular momentum about the z axis, p_ϕ, is a constant of motion. Similarly, if z is cyclic, then p_z is a constant of motion.

Spherical coordinates, r, θ, ϕ

Appendix table $19.3.4$ shows that the spherical coordinates are related to the cartesian coordinates by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{8.4.14}$$

The Lagrangian is

$$L = T_i - U = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r\theta\phi) \tag{8.4.15}$$

The conjugate momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \tag{8.4.16}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{8.4.17}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi} \tag{8.4.18}$$

Assuming a conservative force then H is conserved. Since the transformation from cartesian to generalized spherical coordinates is time independent, then $H = E$. Thus using 8.4.16-8.4.18 the Hamiltonian is given in spherical coordinates by

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_i p_i \dot{q}_i - L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{8.4.19}$$

$$= \left(p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) - \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + U(r, \theta, \phi) \tag{8.4.20}$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r, \theta, \phi) \tag{8.4.21}$$

Then the canonical equations of motion in spherical coordinates are

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{1}{mr^3}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - \frac{\partial U}{\partial r} \tag{8.4.22}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{mr^2}\left(\frac{p_\phi^2\cos\theta}{\sin^3\theta}\right) - \frac{\partial U}{\partial \theta} \tag{8.4.23}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial U}{\partial \phi} \tag{8.4.24}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \tag{8.4.25}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \tag{8.4.26}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta} \tag{8.4.27}$$

Note that if the coordinate ϕ is cyclic, that is $\frac{\partial U}{\partial \phi} = 0$ then the angular momentum p_ϕ is conserved. Also if the θ coordinate is cyclic, and $p_\phi = 0$, that is, there is no change in the angular momentum perpendicular to the z axis, then p_θ is conserved.

An especially important spherically-symmetric Hamiltonian is that for a central field. Central fields, such as the gravitational or Coulomb fields of a uniform spherical mass, or charge, distributions, are spherically symmetric and then both θ and ϕ are cyclic. Thus the projection of the angular momentum p_ϕ about the z axis is conserved for these spherically symmetric potentials. In addition, since both p_θ and p_ϕ, are conserved, then the total angular momentum also must be conserved as is predicted by Noether's theorem.