13.12: Kinetic Energy of Rotating Rigid Body

An important observable is the kinetic energy of rotation of a rigid body. Consider a rigid body composed of N particles of mass m_α where $\alpha = 1, 2, 3, \ldots N$. If the body rotates with an instantaneous angular velocity $\boldsymbol{\omega}$ about some fixed point, with respect to the body coordinate system, and this point has an instantaneous translational velocity $\mathbf{V}$ with respect to the fixed (inertial) coordinate system, see Figure $13.3.1$, then the instantaneous velocity $\mathbf{v}_\alpha$ of the α^{th} particle in the fixed frame of reference is given by

$$\mathbf{v}_\alpha = \mathbf{V} + \mathbf{v}''_\alpha + \boldsymbol{\omega} \times \mathbf{r}'_\alpha \tag{13.12.1}$$

However, for a rigid body, the velocity of a body-fixed point with respect to the body is zero, that is $\mathbf{v}''_\alpha = 0$, thus

$$\mathbf{v}_\alpha = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}'_\alpha \tag{13.12.2}$$

The total kinetic energy is given by

$$\begin{aligned} T = & \sum_\alpha^N \frac{1}{2} m_\alpha \mathbf{v}_\alpha \cdot \mathbf{v}_\alpha = \sum_\alpha^N \frac{1}{2} m_\alpha (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}'_\alpha) \cdot (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}'_\alpha) \\ = & \frac{1}{2} \sum_\alpha^N m_\alpha V^2 + \sum_i^N m_\alpha \mathbf{V} \cdot \boldsymbol{\omega} \times \mathbf{r}'_\alpha + \frac{1}{2} \sum_\alpha^N m_\alpha (\boldsymbol{\omega} \times \mathbf{r}'_\alpha) \cdot (\boldsymbol{\omega} \times \mathbf{r}'_\alpha) \end{aligned} \tag{13.12.3}$$

This is a general expression for the kinetic energy that is valid for any choice of the origin from which the body-fixed vectors $\mathbf{r}'_\alpha$ are measured. However, if the origin is chosen to be the center of mass, then, and only then, the middle term cancels. That is, since $\mathbf{V} \cdot \boldsymbol{\omega}$ is independent of the specific particle, then

$$\sum_\alpha^N m_\alpha \mathbf{V} \cdot \boldsymbol{\omega} \times \mathbf{r}'_\alpha = \mathbf{V} \cdot \boldsymbol{\omega} \times \left(\sum_\alpha^N m_\alpha \mathbf{r}'_\alpha \right) \tag{13.12.4}$$

But the definition of the center of mass is

$$\sum_\alpha m_\alpha \mathbf{r}' = M\mathbf{R} \tag{13.12.5}$$

and $\mathbf{R} = 0$ in the body-fixed frame if the selected point in the body is the center of mass. Thus, *when using the center of mass frame*, the middle term of Equation 13.12.3 is zero. Therefore, for the center of mass frame, the kinetic energy separates into two terms in the body-fixed frame

$$T = T_{trans} + T_{rot} \tag{13.12.6}$$

where

$$T_{trans} = \frac{1}{2} \sum_\alpha^N m_\alpha V^2 \tag{13.12.7}$$

$$T_{rot} = \frac{1}{2} \sum_\alpha^N m_i (\boldsymbol{\omega} \times \mathbf{r}'_\alpha) \cdot (\boldsymbol{\omega} \times \mathbf{r}'_\alpha)$$

The vector identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = A^2 B^2 - (\mathbf{A} \cdot \mathbf{B})^2 \tag{13.12.8}$$

can be used to simplify T_{rot}

$$T_{rot} = \frac{1}{2} \sum_\alpha^N m_\alpha \left[\omega^2 r'^2_\alpha - (\boldsymbol{\omega} \cdot \mathbf{r}'_\alpha)^2 \right] \tag{13.12.9}$$

The rotational kinetic energy T_{rot} can be expressed in terms of components of $\boldsymbol{\omega}$ and $\mathbf{r}'_\alpha$ in the body-fixed frame. Also the following formulae are greatly simplified if $\mathbf{r}'_\alpha = (x_\alpha, y_\alpha, z_\alpha)$ in the rotating body-fixed frame is written in the form

$\mathbf{r}'_{\alpha} = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$ where the axes are defined by the numbers $1, 2, 3$ rather than x, y, z. In this notation the rotational kinetic energy is written as

$$T_{rot} = \frac{1}{2}\sum_{\alpha}^{N} m_{\alpha}\left[\left(\sum_{i}\omega_i^2\right)\left(\sum_{k}x_{\alpha,k}^2\right) - \left(\sum_{i}\omega_i x_{\alpha,i}\right)\left(\sum_{j}\omega_j x_{\alpha,j}\right)\right] \tag{13.12.10}$$

Assume the Kronecker delta relation

$$\omega_i = \sum_{j}^{3}\omega_j\delta_{ij} \tag{13.12.11}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Then the kinetic energy can be written more compactly

$$\begin{aligned} T_{rot} &= \frac{1}{2}\sum_{\alpha}^{N} m_{\alpha}\left[\left(\sum_{i}\omega_i^2\right)\left(\sum_{k}x_{\alpha,k}^2\right) - \left(\sum_{i}\omega_i x_{\alpha,i}\right)\left(\sum_{j}\omega_j x_{\alpha,j}\right)\right] \\ &= \frac{1}{2}\sum_{\alpha}^{N}\sum_{i,j}^{3} m_{\alpha}\left[\left(\omega_i\omega_j\delta_{ij}\right)\left(\sum_{k}^{3}x_{\alpha,k}^2\right) - \left(\omega_i x_{\alpha,i}\right)\left(\omega_j x_{\alpha,j}\right)\right] \\ &= \frac{1}{2}\sum_{i,j}^{3}\omega_i\omega_j\left[\sum_{\alpha}^{N} m_{\alpha}\left[\delta_{ij}\left(\sum_{k}^{3}x_{\alpha,k}^2\right) - x_{\alpha,i}x_{\alpha,j}\right]\right] \end{aligned} \tag{13.12.12}$$

The term in the outer square brackets is the inertia tensor defined in equation $(13.4.1)$ for a discrete body. The inertia tensor components for a continuous body are given by equation $(13.4.2)$.

Thus the rotational component of the kinetic energy can be written in terms of the inertia tensor as

$$T_{rot} = \frac{1}{2}\sum_{i,j}^{3} I_{ij}\omega_i\omega_j \tag{13.12.13}$$

Note that when the inertia tensor is diagonal, then the evaluation of the kinetic energy simplifies to

$$T_{rot} = \frac{1}{2}\sum_{i}^{3} I_{ii}\omega_i^2 \tag{13.12.14}$$

which is the familiar relation in terms of the scalar moment of inertia I discussed in elementary mechanics.

Equation 13.12.13also can be factored in terms of the angular momentum $\mathbf{L}$.

$$T_{rot} = \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j = \frac{1}{2}\sum_{i}\omega_i\sum_{j} I_{ij}\omega_j = \frac{1}{2}\sum_{i}\omega_i L_i \tag{13.12.15}$$

As mentioned earlier, tensor algebra is an elegant and compact way of expressing such matrix operations. Thus it is possible to express the rotational kinetic energy as

$$T_{rot} = \frac{1}{2}(\omega_1\ \omega_2\ \omega_3)\cdot\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}\cdot\begin{pmatrix}\omega_1 \\ \omega_2 \\ \omega_3\end{pmatrix} \tag{13.12.16}$$

$$T_{rot} \equiv \mathbf{T} = \frac{1}{2}\boldsymbol{\omega}\cdot\{\mathbf{I}\}\cdot\boldsymbol{\omega} \tag{13.12.17}$$

where the rotational energy $\mathbf{T}$ is a scalar. Using equation $(13.11.1)$ the rotational component of the kinetic energy also can be written as

$$T_{rot} \equiv \mathbf{T} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} \tag{13.12.18}$$

which is the same as given by 13.12.15. It is interesting to realize that even though $\mathbf{L} = \{\mathbf{I}\} \cdot \boldsymbol{\omega}$ is the inner product of a tensor and a vector, it is a vector as illustrated by the fact that the inner product $T_{rot} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2}\boldsymbol{\omega} \cdot (\{\mathbf{I}\} \cdot \boldsymbol{\omega})$ is a scalar. Note that the translational kinetic energy T_{trans} must be added to the rotational kinetic energy T_{rot} to get the total kinetic energy as given by Equation 13.12.6.