7.7: Generalized Energy and the Hamiltonian Function

Consider the time derivative of the Lagrangian, plus the fact that time is the independent variable in the Lagrangian. Then the total time derivative is

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t} \tag{7.7.1}$$

The Lagrange equations for a conservative force are given by equation $(6.5.12)$ to be

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t) \tag{7.7.2}$$

The holonomic constraints can be accounted for using the Lagrange multiplier terms while the generalized force Q_j^{EXC} includes non-holonomic forces or other forces not included in the potential energy term of the Lagrangian, or holonomic forces not accounted for by the Lagrange multiplier terms.

Substituting Equation 7.7.2 into Equation 7.7.1 gives

$$\begin{aligned} \frac{dL}{dt} = & \sum_j \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \sum_j \dot{q}_j \left[Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t)\right] + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t} \\ = & \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - \sum_j \dot{q}_j \left[Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t)\right] + \frac{\partial L}{\partial t} \end{aligned} \tag{7.7.3}$$

This can be written in the form

$$\frac{d}{dt}\left[\sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - L\right] = \sum_j \dot{q}_j \left[Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t)\right] - \frac{\partial L}{\partial t} \tag{7.7.4}$$

Define Jacobi's **Generalized Energy**[1] $h(\mathbf{q}, \dot{\mathbf{q}}, t)$ by

$$h(\mathbf{q}, \dot{\mathbf{q}}, t) \equiv \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{7.7.5}$$

Jacobi's generalized momentum, equation $7.2.3$, can be used to express the generalized energy $h(q, \dot{q}, t)$ in terms of the canonical coordinates $\dot{q}_i$ and p_i, plus time t. Define the **Hamiltonian function** to equal the generalized energy expressed in terms of the conjugate variables (q_j, p_j), that is,

$$H(\mathbf{q}, \mathbf{p}, t) \equiv h(\mathbf{q}, \dot{\mathbf{q}}, t) \equiv \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - L(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_j (\dot{q}_j p_j) - L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{7.7.6}$$

This Hamiltonian $H(\mathbf{q}, \mathbf{p}, t)$ underlies Hamiltonian mechanics which plays a profoundly important role in most branches of physics as illustrated in chapters 8, 15 and 18.

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[1]Most textbooks call the function $h(\mathbf{q}, \dot{\mathbf{q}}, t)$ *Jacobi's energy integral*. This book adopts the more descriptive name *Generalized energy* in analogy with use of generalized coordinates $\mathbf{q}$ and generalized momentum $\mathbf{p}$.