# 19.3: Appendix - Vector algebra

## Linear operations

The important force fields in classical mechanics, namely, gravitation, electric, and magnetic, are vector fields that have a position-dependent magnitude and direction. Thus, it is useful to summarize the algebra of vector fields.

A vector $\mathbf{a}$ has both a magnitude $|a|$ and a direction defined by the *unit vector* $\hat{\mathbf{e}}_a$, that is, the vector can be written as a bold character $\mathbf{a}$ where

$$\mathbf{a} = a \cdot \hat{\mathbf{e}}_a \tag{19.3.1}$$

where by convention the implied modulus sign is omitted. The hat symbol on the vector $\hat{\mathbf{e}}_a$ designates that this is a unit vector with modulus $|\hat{\mathbf{e}}_a| = 1$.

Vector force fields are assumed to be linear, and consequently they obey the principle of superposition, are commutative, associative, and distributive as illustrated below for three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ plus a scalar multiplier $\gamma$.

$$\mathbf{a} \pm \mathbf{b} = \pm \mathbf{b} + \mathbf{a} \tag{19.3.2}$$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \tag{19.3.3}$$

$$\gamma(\mathbf{a} + \mathbf{b}) = \gamma\mathbf{a} + \gamma\mathbf{b} \tag{19.3.4}$$

The manipulation of vectors is greatly facilitated by use of components along an orthogonal coordinate system defined by three orthogonal unit vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$. For example the cartesian coordinate system is defined by three unit vectors which, by convention, are called $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$.

## Scalar product

Multiplication of two vectors can produce a $9-$component tensor that can be represented by a $3 \times 3$ matrix as discussed in appendix $19.5$. There are two special cases for vector multiplication that are important for vector algebra; the first is the scalar product, and the second is the vector product.

The *scalar product* of two vectors is defined to be

$$\mathbf{a} \cdot \mathbf{b} = |a||b| \cos\theta \tag{19.3.5}$$

where $\theta$ is the angle between the two vectors. It is a scalar and thus is independent of the orientation of the coordinate axis system. Note that the scalar product commutes, is distributive, and associative with a scalar multiplier, that is

$$\begin{gathered}
\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \\
\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \\
(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{b} \cdot \mathbf{a})
\end{gathered} \tag{19.3.6}$$

Note that $\mathbf{a} \cdot \mathbf{a} = |a|^2$ and if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular then $\cos\theta = 0$ and thus $\mathbf{a} \cdot \mathbf{b} = 0$

If the three unit vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ form an orthonormal basis, that is, they are orthogonal unit vectors, then from equations 19.3.5 and 19.3.6

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k = \delta_{ik} \tag{19.3.7}$$

If $\hat{\mathbf{a}}$ is the unit vector for the vector $\mathbf{a}$ then the scalar product of a vector $\mathbf{a}$ with one of these unit vectors $\hat{\mathbf{e}}_n$ gives the cosine of the angle between the vector $\mathbf{a}$ and $\hat{\mathbf{e}}_n$, that is

$$\begin{gathered}
\mathbf{a} \cdot \hat{\mathbf{e}}_1 = |a|(\hat{\mathbf{a}} \cdot \hat{\mathbf{e}}_1) = |a| \cos\alpha \\
\mathbf{a} \cdot \hat{\mathbf{e}}_2 = |a|(\hat{\mathbf{a}} \cdot \hat{\mathbf{e}}_2) = |a| \cos\beta \\
\mathbf{a} \cdot \hat{\mathbf{e}}_3 = |a|(\hat{\mathbf{a}} \cdot \hat{\mathbf{e}}_3) = |a| \cos\gamma
\end{gathered} \tag{19.3.8}$$

where the cosines are called the direction cosines since they define the direction of the vector a with respect to each orthogonal basis unit vector. Moreover, $\mathbf{a} \cdot \hat{\mathbf{e}}_1 = |a|\hat{\mathbf{a}} \cdot \hat{\mathbf{e}}_1 = |a| \cos\alpha$ is the component of $\mathbf{a}$ along the $\hat{\mathbf{e}}_1$ axis. Thus the three components of the vector $\mathbf{a}$ is fully defined by the magnitude $|a|$ and the direction cosines, corresponding to the angles $\alpha, \beta, \gamma$. That is,

$$\begin{aligned} a_1 &= |a|(\hat{\mathbf{a}} \cdot \hat{\mathbf{e}}_1) = |a|\cos\alpha \\ a_2 &= |a|(\hat{\mathbf{a}} \cdot \hat{\mathbf{e}}_2) = |a|\cos\beta \\ a_3 &= |a|(\hat{\mathbf{a}} \cdot \hat{\mathbf{e}}_3) = |a|\cos\gamma \end{aligned} \tag{19.3.9}$$

If the three unit vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ form an orthonormal basis then the vector is fully defined by

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3 \tag{19.3.10}$$

Consider two vectors

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3$$

$$\mathbf{b} = b_1\hat{\mathbf{e}}_1 + b_2\hat{\mathbf{e}}_2 + b_3\hat{\mathbf{e}}_3$$

Then using 19.3.7

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |a||b|\cos\theta \tag{19.3.11}$$

where $\theta$ is the angle between the two vectors. In particular, since the direction cosine $\cos\alpha_a = \frac{a_1}{|a|}$ , then Equation 19.3.11gives

$$\cos\theta = \cos\alpha_a \cos\alpha_b + \cos\beta_a \cos\beta_b + \cos\gamma_a \cos\gamma_b \tag{19.3.12}$$

Note that when $\theta = 0$ then 19.3.12gives

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{19.3.13}$$

## Vector product

The vector product of two vectors is defined to be

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = |a||b|\sin\theta\hat{\mathbf{n}} \tag{19.3.14}$$

where $\theta$ is the angle between the vectors and $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane defined by $\mathbf{a}$ and $\mathbf{b}$ such that the unit vectors $\left(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{n}}\right)$ obey a right-handed screw rule. The vector product acts like a pseudovector which comprises a normal vector multiplied by a sign factor that depends on the handedness of the system as described in appendix 19.4.3.

The components of $\mathbf{c}$ are defined by the relation

$$c_i \equiv \sum_{jk} \varepsilon_{ijk} a_j b_k \tag{19.3.15}$$

where the (Levi-Civita) permutation symbol $\varepsilon_{ijk}$ has the following properties

$$\begin{aligned} \varepsilon_{ijk} &= 0 && \text{if an index is equal to any another index} \\ \varepsilon_{ijk} &= +1 && \text{if } i, j, k, \text{ form an even permutation of } 1,2,3 \\ \varepsilon_{ijk} &= -1 && \text{if } i, j, k, \text{ form an odd permutation of } 1,2,3 \end{aligned} \tag{19.3.16}$$

For example, if the three unit vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ form an orthonormal basis, then $\hat{\mathbf{e}}_i \equiv \sum_{jk} \varepsilon_{ijk} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k$ , i.e.

$$\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \qquad \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \qquad \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 \tag{19.3.17}$$

$$\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3 \qquad \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_2 = -\hat{\mathbf{e}}_1 \qquad \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_3 = -\hat{\mathbf{e}}_2 \tag{19.3.18}$$

$$\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_1 = \mathbf{0} \qquad \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_2 = \mathbf{0} \qquad \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_0 = \mathbf{0} \tag{19.3.19}$$

The vector product anticommutes in that

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \tag{19.3.20}$$

However, it is distributive and associative with a scalar multiplier

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \tag{19.3.21}$$

$$(\lambda\mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) \tag{19.3.22}$$

Note that when $\sin\theta = 0$ then $\mathbf{a} \times \mathbf{b} = 0$ and in particular, $\mathbf{a} \times \mathbf{a} = 0$ .

Consider two vectors

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3$$

$$\mathbf{b} = b_1\hat{\mathbf{e}}_1 + b_2\hat{\mathbf{e}}_2 + b_3\hat{\mathbf{e}}_3$$

Then using equations 19.3.14and 19.3.17− 19.3.19

$$\mathbf{a}\times\mathbf{b} = |a||b|\sin\theta = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\mathbf{e}}_1(a_2b_3 - a_3b_2) + \hat{\mathbf{e}}_2(a_3b_1 - a_1b_3) + \hat{\mathbf{e}}_3(a_1b_2 - a_2b_1)$$

where $\theta$ is the angle between the two vectors and the determinant is evaluated for the top row. Examples of vector products are torque $\mathbf{N} = \mathbf{r}\times\mathbf{F}$ , angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ , and the magnetic force $\mathbf{F}_B = q\mathbf{v}\times\mathbf{B}$ .

## Triple products

The following scalar and vector triple products can be formed from the product of three vectors and are used frequently.

### Scalar triple products

There are several permutations of scalar triple products of three vectors $[\mathbf{a},\mathbf{b},\mathbf{c}]$ that are identical.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = -\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}) \tag{19.3.23}$$

That is, the scalar product is invariant to cyclic permutations of the three vectors but changes sign for interchange of two vectors. The scalar product is unchanged by swapping the scalar $(dot)$ and vector $(cross)$ .

Because of the symmetry the scalar triple product can be denoted as $[\mathbf{a},\mathbf{b},\mathbf{c}]$ and

$$\begin{aligned} [\mathbf{a},\mathbf{b},\mathbf{c}] > 0 \qquad & \text{if } [\mathbf{a},\mathbf{b},\mathbf{c}] \text{ is right-handed} \\ [\mathbf{a},\mathbf{b},\mathbf{c}] = 0 \qquad & \text{if } [\mathbf{a},\mathbf{b},\mathbf{c}] \text{ is coplanar} \\ [\mathbf{a},\mathbf{b},\mathbf{c}] < 0 \qquad & \text{if } [\mathbf{a},\mathbf{b},\mathbf{c}] \text{ is left-handed} \end{aligned} \tag{19.3.24}$$

The scalar triple product can be written in terms of the components using a determinant

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{19.3.25}$$

### Vector triple product

The vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is a vector. Since $(\mathbf{b}\times\mathbf{c})$ is perpendicular to the plane of $\mathbf{b},\mathbf{c}$, then $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ must lie in the plane containing $\mathbf{b},\mathbf{c}$. Therefore the triple product can be expanded in terms of $\mathbf{b},\mathbf{c}$, as given by the following identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{19.3.26}$$

## Problems

1. Partition the following exercises among your collaborators. Once you have completed your problem, check with a classmate before writing it on the board. After you have verified that you have found the correct solution, write your answer in the space provided on the board, taking care to include the steps that you used to arrive at your solution. The following information is needed.

| $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 9\mathbf{k}$ | $\mathbf{b} = -2\mathbf{i} + 3\mathbf{k}$ | $\mathbf{c} = -2\mathbf{i} + \mathbf{j} - 6\mathbf{k}$ | $\mathbf{d} = \mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$ |
|---|---|---|---|
| $\mathbf{E} = \begin{pmatrix} 2 & 7 & -4 \\ 3 & 1 & -2 \\ -2 & 0 & 5 \end{pmatrix}$ | $\mathbf{F} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ | $\mathbf{G} = \begin{pmatrix} 2 & -4 \\ 7 & 1 \\ -1 & 1 \end{pmatrix}$ | $\mathbf{H} = \begin{pmatrix} -8 & -1 & -3 \\ -4 & 2 & -2 \\ -1 & 0 & 0 \end{pmatrix}$ |

Calculate each of the following

| | |
|---|---|
| 1. $\lvert\mathbf{a} - (\mathbf{b} + \mathbf{3c})\rvert$ | 7. $(\mathbf{EH})^T$ |
| 2. Component of $\mathbf{c}$ along $\mathbf{a}$ | 8. $\lvert\mathbf{HE}\rvert$ |

| | |
|---|---|
| 3. Angle between $\mathbf{c}$ and $\mathbf{d}$ | 9. $\mathbf{EHG}$ |
| 4. $(\mathbf{b} \times \mathbf{d}) \cdot \mathbf{a}$ | 10. $\mathbf{EG} - \mathbf{HG}$ |
| 5. $(\mathbf{b} \times \mathbf{d}) \times \mathbf{a}$ | 11. $\mathbf{EH} - \mathbf{H}^T\mathbf{E}^T$ |
| 6. $\mathbf{b} \times (\mathbf{d} \times \mathbf{a})$ | 12. $\mathbf{F}^{-1}$ |

2. For what values of $a$ are the vectors $\mathbf{A} = 2a\hat{i} - 2\hat{j} + a\hat{k}$ and $\mathbf{B} = a\hat{i} + 2a\hat{j} + 2\hat{k}$ perpendicular?

3. Show that the triple scalar product $(A \times B) \cdot C$ can be written as

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Show also that the product is unaffected by interchange of the scalar and vector product operations or by change in the order of $A, B, C$ as long as they are in cyclic order, that is

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$$

Therefore we may use the notation $ABC$ to denote the triple scalar product. Finally give a geometric interpretation of $ABC$ by computing the volume of the parallelepiped defined by the three vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$.