9.S: Hamilton's Action Principle (Summary)

The Hamilton's 1834 publication, introducing both Hamilton's Principle of Stationary Action and Hamiltonian mechanics, marked the crowning achievements for the development of variational principles in classical mechanics. A fundamental advantage of Hamiltonian mechanics is that it uses the conjugate coordinates $\mathbf{q}$, $\mathbf{p}$, plus time t, which is a considerable advantage in most branches of physics and engineering. Compared to Lagrangian mechanics, Hamiltonian mechanics has a significantly broader arsenal of powerful techniques that can be exploited to obtain an analytical solution of the integrals of the motion for complicated systems, as described in chapter 15. In addition, Hamiltonian dynamics provides a means of determining the unknown variables for which the solution assumes a soluble form, and is ideal for study of the fundamental underlying physics in applications to fields such as quantum or statistical physics. As a consequence, Hamiltonian mechanics has become the preeminent variational approach used in modern physics.

This chapter has introduced and discussed Hamilton's Principle of Stationary Action, which underlies the elegant and remarkably powerful Lagrangian and Hamiltonian representations of algebraic mechanics. The basic concepts employed in algebraic mechanics are summarized below.

Hamilton's Action Principle

As discussed in chapter 9.2, Hamiltonian mechanics is built upon Hamilton's action functional

$$S(\mathbf{q},\mathbf{p},t) = \int_{t_i}^{t_f} L(\mathbf{q},\dot{\mathbf{q}},t)dt \tag{9.S.1}$$

Hamilton's Principle of least action states that

$$\delta S(\mathbf{q},\mathbf{p},t) = \delta \int_{t_i}^{t_f} L(\mathbf{q},\dot{\mathbf{q}},t)dt = 0 \tag{9.S.2}$$

Generalized momentum $\boldsymbol{p}$

In chapter 7.2, the generalized (canonical) momentum was defined in terms of the Lagrangian L to be

$$p_i \equiv \frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial \dot{q}_i} \tag{9.S.3}$$

Chapter $9.2.2$ defined the generalized momentum in terms of the action functional S to be

$$p_j \equiv \frac{\partial S(\mathbf{q},\mathbf{p},t)}{\partial \dot{q}_j} \tag{9.S.4}$$

Generalized energy $\boldsymbol{h(\mathbf{q},\dot{q},t)}$

Jacobi's Generalized Energy $h(\mathbf{q},\dot{q},t)$ was defined in Equation 9.S.5 as

$$h(\mathbf{q},\dot{q},t) \equiv \sum_j \left(\dot{q}_j \frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial \dot{q}_j} \right) - L(\mathbf{q},\dot{\mathbf{q}},t) \tag{9.S.5}$$

Hamiltonian function $\boldsymbol{H(\mathbf{q},\mathbf{p},t)}$

The Hamiltonian $H(\mathbf{q},\mathbf{p},t)$ was defined in terms of the generalized energy $h(\mathbf{q},\dot{q},t)$ plus the generalized momentum. That is

$$H(\mathbf{q},\mathbf{p},t) \equiv h(\mathbf{q},\dot{q},t) = \sum_j p_j \dot{q}_j - L(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{p}\cdot\dot{\mathbf{q}} - L(\mathbf{q},\dot{\mathbf{q}},t) \tag{9.S.6}$$

where $\mathbf{p}$, $\mathbf{q}$ correspond to n-dimensional vectors, e.g. $\mathbf{q} \equiv (q_1, q_2, \ldots, q_n)$ and the scalar product $\mathbf{p}\cdot\dot{\mathbf{q}} = \sum_i p_i \dot{q}_i$. Chapter 8.2 used a Legendre transformation to derive this relation between the Hamiltonian and Lagrangian functions. Note that whereas the Lagrangian $L(\mathbf{q},\dot{\mathbf{q}},t)$ is expressed in terms of the coordinates $\mathbf{q}$, plus conjugate velocities $\dot{\mathbf{q}}$, the Hamiltonian $H(\mathbf{q},\mathbf{p},t)$ is expressed in terms of the coordinates $\mathbf{q}$ plus their conjugate momenta $\mathbf{p}$. For scleronomic systems, using the standard Lagrangian, in equations $(7.9.4)$ and $(7.6.14)$, shows that the Hamiltonian simplifies to be equal to the total mechanical energy, that is, $H = T + U$.

Generalized energy theorem

The equations of motion lead to the generalized energy theorem which states that the time dependence of the Hamiltonian is related to the time dependence of the Lagrangian.

$$\frac{dH(\mathbf{q},\mathbf{p},t)}{dt} = \sum_j \dot{q}_j \left[Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q},t) \right] - \frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial t} \tag{9.S.7}$$

Note that if all the generalized non-potential forces and Lagrange multiplier terms are zero, and if the Lagrangian is not an explicit function of time, then the Hamiltonian is a constant of motion.

Lagrange equations of motion

Equation 9.S.8 gives that the N Lagrange equations of motion are

$$\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right\} = \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q},t) + Q_j^{EXC} \tag{9.S.8}$$

where $j = 1, 2, 3, \ldots . N$.

Hamilton’s equations of motion

Chapter 8.3 showed that a Legendre transform, plus the Lagrange-Euler equations, $(8.3.11, 8.3.12, 8.3.13)$lead to Hamilton’s equations of motion. Hamilton derived these equations of motion directly from the action functional, as shown in chapter 9.2.

$$\dot{q}_j = \frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial p_j} \tag{9.S.9}$$

$$\dot{p}_j = \qquad -\frac{\partial H}{\partial q_j}(\mathbf{q},\mathbf{p},t) + \left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q},t) + Q_j^{EXC} \right] \tag{9.S.10}$$

$$\frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial t} = -\frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial t} \tag{9.S.11}$$

Note the symmetry of Hamilton’s two canonical equations. The canonical variables p_k, q_k are treated as independent canonical variables. Lagrange was the first to derive the canonical equations but he did not recognize them as a basic set of equations of motion. Hamilton derived the canonical equations of motion from his fundamental variational principle and made them the basis for a far-reaching theory of dynamics. Hamilton’s equations give $2s$ first-order differential equations for p_k, q_k for each of the s degrees of freedom. Lagrange’s equations give s second-order differential equations for the variables q_k, $\dot{q}_k$.

Hamilton-Jacobi equation

Hamilton used Hamilton’s Principle plus Equation 9.S.12 to derive the Hamilton-Jacobi equation.

$$\frac{\partial S}{\partial t} + H(\mathbf{q},\mathbf{p},t) = 0 \tag{9.S.12}$$

The solution of Hamilton’s equations is trivial if the Hamiltonian is a constant of motion, or when a set of generalized coordinate can be identified for which all the coordinates q_i are constant, or are cyclic (also called ignorable coordinates). Jacobi developed the mathematical framework of canonical transformation required to exploit the Hamilton-Jacobi equation.

Hamilton’s Principle applied using initial boundary conditions

The definition of Hamilton’s Principle assumes integration between the initial time t_i and final time t_f. A recent development has extended applications of Hamilton’s Principle to apply to systems that are defined in terms of only the initial boundary conditions. This method doubles the number of degrees of freedom and uses a coupling Lagrangian $K(\mathbf{q}_2, \dot{\mathbf{q}}_2, \mathbf{q}_1, \dot{\mathbf{q}}_1, t)$ between the corresponding $\mathbf{q}_1$ and $\mathbf{q}_2$ doubled degrees of freedom

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_-^I} - \frac{\partial L}{\partial q_-^I} = \left[\frac{\partial K}{\partial q_-^I} - \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_-^I} \right]_{PL} \equiv Q^I(\mathbf{q}_1, \dot{\mathbf{q}}_1, t) \tag{9.S.13}$$

and where Q^I is a generalized nonconservative force derived from K.

Standard Lagrangians

Derivation of Lagrangian mechanics, using d'Alembert's principle of virtual work, assumed that the Lagrangian is defined by Equation 9.S.14

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\dot{\mathbf{q}}, t) - U(\mathbf{q}, t) \tag{9.S.14}$$

This was used in equation $(9.2.1)$ to derive the action in terms of the fundamental Lagrangian defined by Equation 9.S.14. The assumption that the action S is the fundamental property inverts this procedure and now equation $(9.2.1)$ is used to derived the Lagrangian. That is, the assumption that Hamilton's Principle is the foundation of algebraic mechanics defines the Lagrangian in terms of the fundamental action S.

Non-standard Lagrangians

The flexibility and power of Lagrangian mechanics can be extended to a broader range of dynamical systems by employing an extended definition of the Lagrangian that assumes that the action is the fundamental property, and then the Lagrangian is defined in terms of Hamilton's variational action principle using Equation 9.S.2. It was illustrated that the inverse variational calculus formalism can be used to identify non-standard Lagrangians that generate the required equations of motion. These nonstandard Lagrangians can be very different from the standard Lagrangian and do not separate into kinetic and potential energy components. These alternative Lagrangians can be used to handle dissipative systems which are beyond the range of validity when using standard Lagrangians. That is, it was shown that several very different Lagrangians and Hamiltonians can be equivalent for generating useful equations of motion of a system. Currently the use of non-standard Lagrangians is a narrow, but active, frontier of classical mechanics with important applications to relativistic mechanics.

Gauge invariance of the standard Lagrangian

It was shown that there is a continuum of equivalent standard Lagrangians that lead to the same set of equations of motion for a system. This feature is related to gauge invariance in mechanics. The following transformations change the standard Lagrangian, but leave the equations of motion unchanged.

1. The Lagrangian is indefinite with respect to addition of a constant to the scalar potential which cancels out when the derivatives in the Euler-Lagrange differential equations are applied.
2. Similarly the Lagrangian is indefinite with respect to addition of a constant kinetic energy.
3. The Lagrangian is indefinite with respect to addition of a total time derivative of the form $L + \frac{d}{dt}[\Lambda(q_i, t)]$ for any differentiable function $\Lambda(q_i t)$ of the generalized coordinates, plus time, that has continuous second derivatives.

Application of Hamilton's Action Principle to mechanics

The derivation of the equations of motion for any system can be separated into a hierarchical set of three stages in both sophistication and understanding. Variational principles are employed during the primary "action" stage and secondary "Hamilton/Lagrangian" stage to derive the required equations of motion, which then are solved during the third "equations-of-motion stage". Hamilton's Action Principle, is a scalar function that is the basis for deriving the Lagrangian and Hamiltonian functions. The primary "action stage" uses Hamilton's Action functional, $S = \int_{t_i}^{t_f} L(\mathbf{q}, \dot{\mathbf{q}}, t)dt$ to derive the Lagrangian and Hamiltonian functionals that are based on Hamilton's action functional and provide the most fundamental and sophisticated level of understanding. The second "Hamiltonian/Lagrangian stage" involves using the Lagrangian and Hamiltonian functionals to derive the equations of motion. The third "equations-of-motion stage" uses the derived equations of motion to solve for the motion subject to a given set of initial boundary conditions. The Newtonian mechanics approach bypasses the primary "action" stage, as well as the secondary "Hamiltonian/Lagrangian" stage. That is, Newtonian mechanics starts at the third "equations-of-motion" stage, which does not allow exploiting the considerable advantages provided by use of action, the Lagrangian, and the Hamiltonian. Newtonian mechanics requires that all the active forces be included when deriving the equations of motion, which involves dealing with vector quantities. This is in contrast to the action, Lagrangian, and Hamiltonian which are scalar functionals. Both the primary "action" stage, and the secondary "Lagrangian/Hamiltonian" stage, exploit the powerful arsenal of mathematical techniques that have been developed for exploiting variational principles.