14.3: Normal Modes

The **normal modes** of the two-coupled oscillator system are obtained by a transformation to a pair of **normal coordinates** (η_1, η_2) that are independent and correspond to the two normal modes. The pair of normal coordinates for this case are

$$\begin{aligned}\eta_1 &\equiv x_1 - x_2 \\ \eta_2 &\equiv x_1 + x_2\end{aligned} \tag{14.3.1}$$

that is

$$\begin{aligned}x_1 &= \frac{1}{2}(\eta_2 + \eta_1) \\ x_2 &= \frac{1}{2}(\eta_2 - \eta_1)\end{aligned} \tag{14.3.2}$$

Substitute these into the equations of motion $(14.2.1)$, gives

$$\begin{aligned}m(\ddot{\eta}_1 + \ddot{\eta}_2) + (\kappa + 2\kappa')\eta_1 + \kappa'\eta_2 = 0 \\ m(\ddot{\eta}_1 - \ddot{\eta}_2) + (\kappa + 2\kappa')\eta_1 - \kappa'\eta_2 = 0\end{aligned} \tag{14.3.3}$$

Adding and subtracting these two equations gives

$$\begin{aligned}m\ddot{\eta}_1 + (\kappa + 2\kappa')\eta_1 = 0 \\ m\ddot{\eta}_2 + \kappa\eta_2 = 0\end{aligned} \tag{14.3.4}$$

Note that the two coordinates η_1 and η_2 are uncoupled and therefore are independent. The solutions of these equations are

$$\eta_1(t) = C_1^+ e^{i\omega_1 t} + C_1^- e^{-i\omega_1 t} \tag{14.3.5}$$

$$\eta_2(t) = C_2^+ e^{i\omega_2 t} + C_2^- e^{-i\omega_2 t} \tag{14.3.6}$$

where η_1 corresponds to angular frequencies ω_1, and η_2 corresponds to ω_2. The two coordinates η_1 and η_2 are called the **normal coordinates and the two solutions are the normal modes with corresponding angular frequencies,** ω_1 and ω_2.

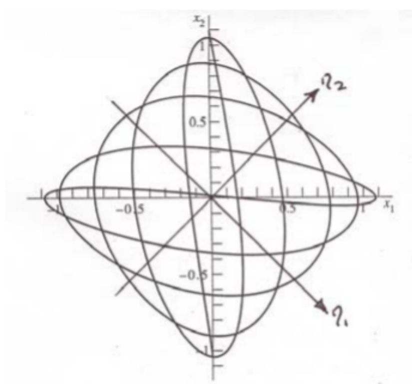

Figure 14.3.1: Motion of two coupled harmonic oscillators in the (x_1, x_2) spatial configuration space and in terms of the normal modes (η_1, η_2). Initial conditions are $x_2 = D, x_1 = \dot{x}_1 = \dot{x}_2 = 0$.

The (η_1, η_2) axes of the two normal modes correspond to a rotation of 45° in configuration space, Figure $14.3.1$. The initial conditions chosen correspond to $\eta_1 = -\eta_2$ and thus both modes are excited with equal intensity. Note that there are 5 lobes along the η_2 axis versus 4 lobes along the η_1 axis reflecting the ratio of the eigenfrequencies ω_1 and ω_2. Also note that the diamond shape of the motion in the (x_1, x_2) configuration space illustrates that the extrema amplitudes for x_2 are a maximum when x_1 is zero, and vise versa. This is equivalent to the statement that the energies in the two modes are coupled with the energy for the first oscillator being a maximum when the energy is a minimum for the second oscillator, and vise versa. By contrast, in the (η_1, η_2) configuration space, the motion is bounded by a rectangle parallel to the (η_1, η_2) axes reflecting the fact that the extrema amplitudes, and corresponding energies, for the η_1 normal mode are constant and independent of the motion for the η_2 normal mode, and vise versa. The decoupling of the two normal modes is best illustrated by considering the case when only one of these two normal modes is excited. For the initial conditions $x_1(0) = -x_2(0)$, and $\dot{x}_1(0) = -\dot{x}_2(0)$, then $\eta_2(t) = 0$. That is, only the $\eta_1(t)$ normal mode is excited with frequency ω_1 which corresponds to motion confined to the η_1 axis of Figure $14.3.1$.

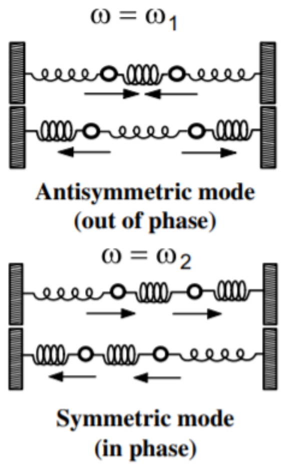

Figure 14.3.2: Normal modes for two coupled oscillators.

As shown in Figure $14.3.2$, $\eta_1(t)$ is the *antisymmetric mode* in which the two masses oscillate out of phase such as to keep the center of mass of the two masses stationary. For the initial conditions $x_1(0) = x_2(0)$, and $\dot{x}_1(0) = \dot{x}_2(0)$, then $\eta_1(t) = 0$, that is, only the $\eta_2(t)$ normal mode is excited. The $\eta_2(t)$ normal mode is the *symmetric mode* where the two masses oscillate in phase with frequency ω_2; it corresponds to motion along the η_2 axis. For the symmetric phase, both masses move together leading to a constant extension of the coupling spring. As a result the frequency ω_2 of the symmetric mode $\eta_2(t)$ is lower than the frequency ω_1 of the asymmetric mode $\eta_1(t)$. That is, the asymmetric mode is stiffer since all three springs provide active restoring forces, compared to the symmetric mode where the coupling spring is uncompressed. In general, for attractive forces the lowest frequency always occurs for the mode with the highest symmetry