# 14.12: Collective Synchronization of Coupled Oscillators

Collective synchronization of coupled oscillators is a multifaceted phenomenon where large ensembles of coupled oscillators, with comparable natural frequencies, self synchronize leading to coherent collective modes of motion. Biological examples include congregations of synchronously flashing fireflies, crickets that chirp in unison, an audience clapping at the end of a performance, networks of pacemaker cells in the heart, insulin-secreting cells in the pancreas, as well as neural networks in the brain and spinal cord that control rhythmic behaviors such as breathing, walking, and eating. Example **14.13** illustrates an application to nuclei.

An ensemble of coupled oscillators will have a frequency distribution with a finite width. It is interesting to elucidate how an ensemble of coupled oscillators, that have a finite width frequency distribution, can self synchronize their motion to a unique common frequency, and how that synchronization is maintained over long time periods. The answers to these issues provide insight into the dynamics of coupled oscillators.

The discussion of coupled oscillators has implicitly assumed $n$ identical undamped linear oscillators that have identical, infinitely-sharp, natural frequencies $\omega_i$. In nature typical coupled oscillators can have a finitewidth frequency distribution $g(\omega)$ about some average value, due to the natural variability of the oscillator parameters for biological systems, the manufacturing tolerances for mechanical oscillators, or the natural Lorentzian frequency distribution associated with the uncertainty principle that occurs even for atomic clocks where the oscillator frequencies are defined directly by the physical constants. Assume that the ensemble of coupled oscillators has a frequency distribution $g(\omega)$ about some average value.

Undamped linear oscillators have elliptical closed-path trajectories in phase space whereas dissipation leads to a spiral attractor unless the system is driven such as to preserve the total energy. As described in chapter $4.4$ many systems in nature, especially biological systems, have closed limit cycles in phase space where the energy lost to dissipation is replenished by a driving mechanism. The simplest systems for understanding collective synchronization of coupled oscillators are those that involve closed limit cycles in phase space.

N. Wiener first recognized the ubiquity of collective synchronization in the natural world, but his mathematical approach, based on Fourier integrals, was not suited to this problem. A more fruitful approach was pioneered in 1975 by an undergraduate student A.T. Winfree[Win67] who recognized that the long-time behavior of a large ensemble of limit-cycle oscillators can be characterized in the simplest terms by considering only the phase of closed phase-space trajectories. He assumed that the instantaneous state of an ensemble of oscillators can be represented by points distributed around the circular phase-space diagram shown in Figure $14.12.1$. For uncoupled oscillators these points will be distributed randomly around the circle, whereas coupling of the oscillators will result in a spatial correlation of the points. That is, the dynamics of the phases can be visualized as a swarm of points running around the unit circle in the complex plane of the phase space diagram. The complex order parameter of this swarm can be defined to be the magnitude and phase of the centroid of this swarm

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \tag{14.12.1}$$

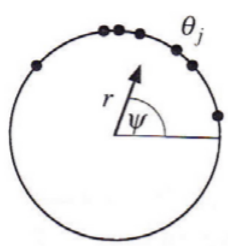


Figure 14.12.1: Order parameter for weakly-coupled oscillators.

The centroid of the ensemble of points on the phase diagram has a magnitude $r$, designating the offset of the centroid from the center of the circular phase diagram, and $\psi$ which is the phase of this centroid. A uniform distribution of points around the unit circle will lead to a centroid $r = 0$. Correlated motion leads to a bunching of the points around some phase value leading to a non-zero centroid $r$ and angle $\psi$. If the swarm acts like a fully-coupled single oscillator then $r \approx 1$ with an appropriate phase $\psi$.

The **Kuramoto model**[Kur75, Str00] incorporates Winfree's intuition by mapping the limit cycles onto a simple circular phase diagram and incorporating the long-term dynamics of coupled oscillators in terms of the relative phases for a mean-field system. That is, the angular velocity of the phase $\dot{\phi}_i$ for the $i^{th}$ oscillator is

$$\dot{\phi}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\phi_j - \phi_i) \tag{14.12.2}$$

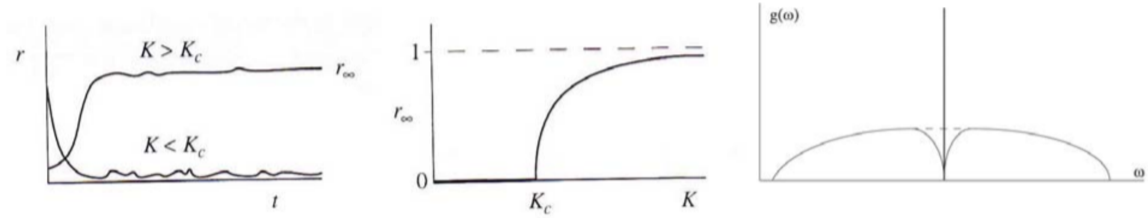


Figure 14.12.2: Kuramoto model of collective synchronization of coupled oscillators. The left and center plots show the time and coupling strength dependence of the order parameter $r$. The right plot shows the frequency dependence including coupling (solid line) and without coupling (dashed line).

where $i = 1, 2, , , N$. Kuramoto recognized that mean-field coupling was the most tractable system to solve, that is, a system where the coupling is applicable equally to all the oscillators. Moreover, he assumed an equally-weighted, pure sinusoidal coupling for the coupling term $\Gamma_{ij}(\theta_j - \theta_i)$ between the coupled oscillators. That is, he assumed

$$\Gamma_{ij}(\phi_j - \phi_i) = \frac{K}{N}\sin(\phi_j - \phi_i) \tag{14.12.3}$$

where $K \geq 0$ is the coupling strength, and the factor $\frac{1}{N}$ ensures that the model is well behaved as $N \to \infty$. Kuramoto assumed that the frequency distribution $g(\omega)$ was unimodular and symmetric about the mean frequency $\Omega$, that is $g(\Omega + \omega) = g(\Omega - \omega)$ .

This problem can be simplified by exploiting the rotational symmetry and transforming to a frame of reference that is rotating at an angular frequency $\Omega$. That is, use the transformation $\theta_i = \phi_i - \Omega t$ where $\theta_i$ is measured in the rotating frame. This makes $g(\omega)$ unimodular with a symmetric frequency distribution about $\omega = 0$. The phase velocity in this rotating frame is

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \frac{K}{N}\sin(\theta_j - \theta_i) \tag{14.12.4}$$

Kuramoto observed that the phase-space distribution can be expressed in terms of the order parameters $r, \psi$ in that Equation 14.12.1can be multiplied on both sides by $e^{-i\theta_i}$ to give

$$re^{i(\psi - \theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j - \theta_i)} \tag{14.12.5}$$

Equating the imaginary parts yields

$$r\sin(\psi - \theta_i) = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) \tag{14.12.6}$$

This allows Equation 14.12.4to be written as

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i) \tag{14.12.7}$$

for $i = 1, 2, , N$. Equation 14.12.7reflects the mean-field aspect of the model in that *each oscillator $\theta_i$ is attracted to the phase of the mean field $\psi$ rather than to the phase of another individual oscillator.*

Simulations showed that the evolution of the order parameter with coupling strength $K$ is as illustrated in Figure $14.12.2$ This simulation shows (1) for all $K$, when below a certain threshold $K_c$, the order parameter decays to an incoherent jitter as expected for random scatter of $N$ points. (2) When $K > K_c$ this incoherent state becomes unstable and the order parameter $r$ grows exponentially reflecting the nucleation of small clusters of oscillators that are mutually synchronized. (3) The population of individual oscillators splits into two groups. The oscillators near the center of the distribution lock together in phase at the mean angular frequency $\Omega$ and co-rotate with average phase $\psi(t)$, whereas those frequencies lying further from the center continue to rotate independently at their natural frequencies and drift relative to the coherent cluster frequency $\Omega$. As a consequence this mixed state is only partially synchronized as illustrated on the right side of Figure $14.12.2$ The synchronized fraction has a $\delta$-function behavior for the frequency distribution which grows in intensity with further increase in $K$. The unsynchronized component has nearly the original frequency distribution $g(\omega)$ except that it is depleted in the region of the locked frequency due to strength absorbed by the $\delta$-function component.

Kuramoto's toy model nicely illustrates the essential features of the evolution of collective synchronization with coupling strength. It has been applied to the study neuronal synchronization in the brain[Cum07]. The model illustrates that the collective synchronization of coupled oscillators leads to a component that has a single frequency for correlated motion which can be much narrower than the inherent frequency distribution of the ensemble of coupled oscillators.

### Example 14.12.1: Collective motion in nuclei

The nucleus is an unusual quantal system that involves the coupled motion of the many nucleons. It exhibits features characteristic of the many-body classical coupled oscillator with coupling between all the valence nucleons. Nuclear structure can be described by a shell model of individual nucleons bound in weakly interacting orbits in a central average mean field that is produced by the summed attraction of all the nucleons in the nucleus. However, nuclei also exhibit features characteristic of collective rotation and vibration of a quantal fluid. For example, beautiful rotational bands up to spin over $60\hbar$ are observed in heavy nuclei. These rotational bands are similar to those observed in the rotational structure of diatomic molecules. Actinide nuclei also can fission into two large fragments which is another manifestation of collective motion.

The essential general feature of weakly-coupled identical oscillators is illustrated by the solutions of the three linearly-coupled identical oscillators where the most symmetric state is displaced in frequency from the remaining states. For $n$ identical oscillators, one state is displaced significantly in energy from the remaining $n-1$ degenerate states. This most symmetric state is pushed downwards in energy if the residual coupling force is attractive, and it is pushed upwards if the coupling force is repulsive. This symmetric state corresponds to the coherent oscillation of all the coupled oscillators, and carries all of the strength for the corresponding dominant multipole for the coupling force. In the nucleus this state corresponds to coherent shape oscillations of many nucleons.

The weak residual electric quadrupole and octupole nucleon-nucleon correlations in the nucleon-nucleon interactions generate collective quadrupole and octupole motion in nuclei. The collective synchronization of such coherent quadrupole and octupole excitation leads to collective bands of states, that correspond to synchronized in-phase motion of the protons and neutrons in the valence oscillator shell. These modes correspond to rotations and vibrations about the center of mass. The attractive residual nucleon-nucleon interaction couples the many individual particle excitations in a given shell producing one coherent state that is pushed downwards in energy far from the remaining $n-1$ degenerate states. This coherent state involves correlated motion of the nucleons that corresponds to a macroscopic oscillation of a charged fluid. For nonclosed shell nuclei like $^{238}U$, the dominant quadrupole multipole in the residual nucleon-nucleon interaction leads to the ground state being a coherent state corresponding to $\approx 16$ protons plus $\approx 20$ neutrons oscillating in phase. The collective motion of the charged protons leads to electromagnetic $E2$ radiation with a transition decay amplitude being about 16 times larger than for a single proton. This corresponds to radiative decay probability being enhanced by a factor of $\approx 256$ relative to radiation by a single proton. This collective state corresponds to a macroscopic quadrupole deformation at low excitation energies that exhibits both collective rotational and vibrational degrees of freedom. This coherent state is analogous to the correlated flow of individual water molecules in a tidal wave. The weaker octupole term in the residual interaction leads to an octupole [pear-shaped] coupled oscillator coherent state lying slightly above the quadrupole coherent state. In contrast to the rotational motion of strongly-deformed quadrupole-deformed nuclei, the octupole deformation exhibits more vibrational-like properties than rotational motion of a charged tidal wave. Hamiltonian mechanics, based on the Routhian $R_{noncyclic}$, is used to make theoretical model calculations of the nuclear structure of $^{238}U$ in the rotating body-fixed frame for comparison with the experimental data.