3.7: Wave equation

Wave motion is a ubiquitous feature in nature. Mechanical wave motion is manifest by transverse waves on fluid surfaces, longitudinal and transverse seismic waves travelling through the Earth, and vibrations of mechanical structures such as suspended cables. Acoustical wave motion occurs on the stretched strings of the violin, as well as the cavities of wind instruments. Electromagnetic wave motion includes wavelengths ranging from $10^5\ m$ radiowaves, to $10^{-13}\ m$ γ-rays. Matter waves are a prominent feature of quantum physics. All these manifestations of waves exhibit the same general features of wave motion.

Wave motion occurs for deformable bodies where elastic forces acting between the nearest-neighbor atoms of the body exert time-dependent forces on one another. Chapter 14 will introduce the collective modes of motion, called the normal modes, of coupled, many-body, linear oscillators which act as independent modes of motion. However, it is useful to introduce wavemotion at this juncture because the equations of wave motion are simple, and wave motion features prominently in several chapters of this book.

Consider a travelling wave in one dimension for a linear system. If the wave is moving, then the wave function $\Psi\ (x,t)$ describing the shape of the wave, is a function of both x and t. The instantaneous amplitude of the wave $\Psi\ (x,t)$ could correspond to the transverse displacement of a wave on a string, the longitudinal amplitude of a wave on a spring, the pressure of a longitudinal sound wave, the transverse electric or magnetic fields in an electromagnetic wave, a matter wave, etc. If the wave train maintains its shape as it moves, then one can describe the wave train by the function $f(\phi)$ where the coordinate ϕ is measured relative to the shape of the wave, that is, it could correspond to the phase of a crest of the wave. Consider that $f(\phi = 0)$ corresponds to a constant phase, e.g. the peak of the travelling pulse, then assuming that the wave travels at a phase velocity v in the x direction and the peak is at $x = 0$ for $t = 0$, then it is at $x = vt$ at time t. That is, a point with phase ϕ fixed with respect to the waveform shape of the wave profile $f(\phi)$ moves in the $+x$ direction for $\phi = x - vt$ and in $-x$ direction for $\phi = x + vt$.

General wave motion can be described by solutions of a wave equation. The wave equation can be written in terms of the spatial and temporal derivatives of the wave function $\Psi(xt)$. Consider the first partial derivatives of $\Psi(xt)$ = $f(x \mp vt) = f(\phi)$.

$$\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{d\Psi}{d\phi} \tag{3.7.1}$$

and

$$\frac{\partial \Psi}{\partial t} = \frac{d\Psi}{d\phi}\frac{\partial \phi}{\partial t} = \mp v\frac{d\Psi}{d\phi} \tag{3.7.2}$$

Factoring out $\frac{d\Psi}{d\phi}$ for the first derivatives gives

$$\frac{\partial \Psi}{\partial t} = \mp v\frac{\partial \Psi}{\partial x} \tag{3.7.3}$$

The sign in this equation depends on the sign of the wave velocity making it not a generally useful formula.

Consider the second derivatives

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{d^2\Psi}{d\phi^2}\frac{\partial \phi}{\partial x} = \frac{d^2\Psi}{d\phi^2} \tag{3.7.4}$$

and

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{d^2\Psi}{d\phi^2}\frac{\partial \phi}{\partial t} = +v^2\frac{d^2\Psi}{d\phi^2} \tag{3.7.5}$$

Factoring out $\frac{d^2\Psi}{d\phi^2}$ gives

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} \tag{3.7.6}$$

This *wave equation in one dimension for a linear system* is independent of the sign of the velocity. There are an infinite number of possible shapes of waves both travelling and standing in one dimension, all of these must satisfy this one-dimensional wave equation. The converse is that any function that satisfies this one dimensional wave equation must be a wave in this one dimension.

The *Wave Equation in three dimensions* is

$$\nabla^2\Psi \equiv \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} \tag{3.7.7}$$

There are an infinite number of possible solutions Ψ to this wave equation, any one of which corresponds to a wave motion with velocity v.

The Wave Equation is applicable to all manifestations of wave motion, both transverse and longitudinal, for linear systems. That is, it applies to waves on a string, water waves, seismic waves, sound waves, electromagnetic waves, matter waves, etc. If it can be shown that a wave equation can be derived for any system, discrete or continuous, then this is equivalent to proving the existence of waves of any waveform, frequency, or wavelength travelling with the phase velocity given by the wave equation.[Cra65]