13.24: The Rolling Wheel

As discussed in chapter 5.7, the rolling wheel is a non-holonomic system that is simple in principle, but in practice the solution can be complicated, as illustrated by the Tippe Top. Chapter 13.23 discussed the motion of a symmetric top rotating about a fixed point on the symmetry axis when subject to a torque. The rolling wheel involves rotation of a symmetric rigid body that is subject to torques. However, the point of contact of the wheel with a static plane is on the periphery of the wheel, and friction at the point of contact is assumed to ensure zero slip. Note that friction is necessary to ensure that the rotating object rolls without slipping, but the frictional force does no work for pure rolling of an undeformable rigid wheel.

The coordinate system employed is shown in Figure $13.24.1$. For simplicity it is better to use a moving coordinate frame $(\mathbf{1},\mathbf{2},\mathbf{3})$ that is fixed to the orientation of the wheel with the origin at the center of mass of the wheel, but this moving reference frame *does not* include the angular velocity $\dot{\psi}$ of the disk about the $\mathbf{3}$ axis. That is, the moving $(\mathbf{1},\mathbf{2},\mathbf{3})$ frame has angular velocities

$$\begin{aligned}\omega_1 &= \dot{\theta} \\ \omega_2 &= \dot{\phi}\sin\theta \\ \omega_3 &= \dot{\phi}\cos\theta\end{aligned} \tag{13.24.1}$$

The frame fixed in the rotating wheel must include the additional angular velocity of the disk $\dot{\psi}$ about the $\hat{\mathbf{e}}_3$ axis, that is

$$\begin{aligned}\Omega_1 &= \omega_1 = \dot{\theta} \\ \Omega_2 &= \omega_2 = \dot{\phi}\sin\theta \\ \Omega_3 &= \omega_3 + \dot{\psi} = \dot{\phi}\cos\theta + \dot{\psi}\end{aligned} \tag{13.24.2}$$

where Ω designates the angular velocity of the rotating disk, while $\boldsymbol{\omega}$ designates the rotation of the moving frame $(\mathbf{1},\mathbf{2},\mathbf{3})$.

The principle moments of inertia of a thin circular disk are related by the perpendicular axis theorem (chapter 13.9)

$$I_1 + I_2 = I_3$$

Since $I_1 = I_2$ for a uniform disk, therefore $I_3 = 2I_1$.

Equation $(12.3.10)$ can be used to relate the vector forces $\mathbf{F}$ in the space-fixed frame to the rate of change of momenta in the moving frame $(\mathbf{1},\mathbf{2},\mathbf{3})$.

$$\mathbf{F} = \dot{\mathbf{p}}_{space} = \dot{\mathbf{p}}_{moving} + \boldsymbol{\omega}\times\mathbf{p} \tag{13.24.3}$$

This leads to the following relations for the three components in the moving frame

$$\begin{aligned}F_1 &= \dot{p}_1 + \omega_2 p_3 - \omega_3 p_2 \\ F_2 - Mg\sin\theta &= \dot{p}_2 + \omega_3 p_1 - \omega_1 p_3 \\ F_3 - Mg\cos\theta &= \dot{p}_3 + \omega_1 p_2 - \omega_2 p_1\end{aligned} \tag{13.24.4}$$

where F_1, F_2, F_3 are the reactive forces acting shown in Figure $13.24.1$

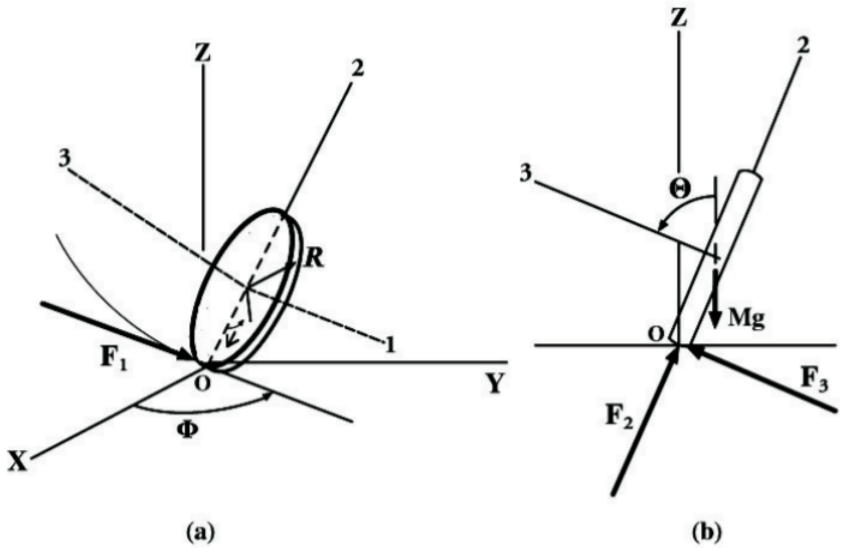

Figure 13.24.1: Uniform disk rolling on a horizontal plane as viewed in the (a) fixed frame, and (b) rolling disk frame. The space-fixed axis system is $(\mathbf{x},\mathbf{y},\mathbf{z})$, while the moving reference frame $(\mathbf{1},\mathbf{2},\mathbf{3})$ is centered at the center of mass of the disk with the $\mathbf{1},\mathbf{2}$ axes in the plane of the disk. The disk is rotating with a uniform angular velocity $\dot{\psi}$ about the $\mathbf{3}$ axis and rolling in the direction that is at an angle ϕ relative to the x axis.

Similarly, the torques $\mathbf{N}$ in the space-fixed frame can be related to the rate of change of angular momentum by

$$\mathbf{N} = \dot{\mathbf{L}}_{space} = \dot{\mathbf{L}}_{moving} + \boldsymbol{\omega} \times \mathbf{L} \tag{13.24.5}$$

where $L_i = \mathbf{I}_i\Omega_i$. This leads to the following relations for the three torque equations in the moving frame

$$\begin{aligned} N_1 = -F_3R = I_1\dot{\Omega}_1 + I_3\Omega_3\omega_2 - I_2\Omega 2\omega_3 \\ N_2 = 0 = I_1\dot{\Omega}_2 + I_1\Omega_1\omega_3 - I_3\Omega_3\omega_1 \\ N_3 = F_1R = I_3\dot{\Omega}_3 + I_2\Omega_2\omega_1 - I_1\Omega_1\omega_2 \end{aligned} \tag{13.24.6}$$

The rolling constraints are

$$\begin{aligned} p_1 + MR\Omega_3 = 0 \\ p_2 = 0 \\ p_3 - MR\Omega_1 = 0 \end{aligned} \tag{13.24.7}$$

where $p_i = Mv_i$. Combining equations 13.24.4, 13.24.6, 13.24.7 gives

$$\begin{aligned} (I_1 + MR^2)\dot{\Omega}_1 + (I_3 + MR^2)\omega_2\Omega_3 - I_2\omega_3\Omega_2 = -MgR\cos\theta \\ I_1\dot{\Omega}_2 + I_1\omega_3\Omega_1 - I_3\omega_1\Omega_3 = 0 \\ (I_3 + MR^2)\dot{\Omega}_3 + I_2\omega_1\Omega_2 - (I_1 + MR^2)\omega_2\Omega_1 = 0 \end{aligned} \tag{13.24.8}$$

These are the torque equations about the point of contact O.

Introduction of equations 13.24.1 and 13.24.2 into Equation 13.24.8 expresses the equations of motion in terms of the Euler angles to be

$$\begin{aligned} (I_1 + MR^2)\ddot{\theta} + (I_3 + MR^2)\dot{\phi}\sin\theta\left(\dot{\phi}\cos\theta + \dot{\psi}\right) - I_1\dot{\phi}^2\sin\theta\cos\theta = -MgR\cos\theta \\ I_1\ddot{\phi}\sin\theta + 2I_1\dot{\phi}\dot{\theta}\cos\theta - I_3\dot{\theta}\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = 0 \\ (I_3 + MR^2)\left(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta + \ddot{\psi}\right) - MR^2\dot{\theta}\dot{\phi}\sin\theta = 0 \end{aligned} \tag{13.24.9}$$

Equations 13.24.9 are non-linear, and a closed-form solution is possible only for limited cases such as when $\theta = 90^\circ$.

Note that the above equations of motion also can be derived using Lagrangian mechanics knowing that

$$L = \frac{1}{2}M(v_1^2 + v_2^2 + v_3^2) + \frac{1}{2}I_1(\Omega_1^2 + \Omega_2^2) + \frac{1}{2}I_3\Omega_3^2 - MgR\cos\theta$$

The differential equations of constraint can be derived from equations 13.24.7 to be

$$dx - R\cos\phi d\psi = 0$$

$$dy - R\sin\phi d\psi = 0 \tag{13.24.10}$$

Use of generalized forces plus the Lagrange-Euler equations $(6.3.28)$ can be used to derive the equations of motion and solve for the components of the constraint force F_1, F_2, and F_3.

Example 13.24.1: Tipping stability of a rolling wheel

A circular wheel rolling in a vertical plane at high angular velocity initially rolls in a straight line and remains vertical. However, below a certain angular velocity, gyroscopic forces become weaker and the wheel will tip sideways and veer rapidly from the initial direction. It is interesting to estimate the minimum angular velocity of the disk such that it does not start to tip over sideways.

Note that equations 13.24.9 are satisfied for $\theta = \frac{\pi}{2}$, $\phi = 0$ and $\dot{\psi} = \Omega_3 =$ constant. Assume a small disturbance causes the tilt angle to be $\theta = \frac{\pi}{2} + \alpha$ where α is small and that ϕ is non-zero but small, that is $\dot{\theta} = \dot{\alpha}$ and $\dot{\phi}$ are small. Keeping only terms to first order in the third of equations 13.24.9, and integrating gives

$$\dot{\phi}\cos\theta + \dot{\psi} = \Omega_3 \tag{a}$$

The first two of equations 13.24.8 become

$$(I_1 + MR^2)\ddot{\alpha} + (I_3 + MR^2)\dot{\phi}\Omega_3 - MgR\alpha = 0 \tag{b}$$

$$I_1\ddot{\phi} - I_3\Omega_3\dot{\alpha} = 0 \tag{c}$$

Integrating Equation c gives

$$\dot{\phi} = \frac{I_3\Omega_3}{I_1}\alpha \tag{d}$$

Inserting d into b gives

$$(I_1 + MR^2)\ddot{\alpha} + \left[(I_3 + MR^2)\frac{I_3\Omega_3^2}{I_1} - MgR\right]\alpha = 0 \tag{e}$$

Equation e has a stable oscillatory solution when the square bracket in positive, that is,

$$\Omega_3^2 > \frac{I_1 MgR}{I_3(I_3 + MR^2)} \tag{f}$$

which gives the minimum angular velocity required for stable rolling motion. For angular velocity less than the minimum, the square bracket in Equation e is negative leading to an exponentially decaying and divergent solution. For a uniform disk the perpendicular axis theorem gives $I_3 = 2I_1 = \frac{1}{2}MR^2$ for which Equation f gives

$$\Omega_3^2 \frac{g}{3R} \tag{g}$$

Therefore the critical linear velocity of the wheel is

$$v = R\Omega_3 > \sqrt{\frac{gR}{3}} \tag{h}$$

The bicycle wheel provides a common example of the tipping of a rolling wheel. For the typical $0.35\ m$ radius of a bicycle wheel, this gives a critical velocity of $v > 1.07\ m/s = 2.4\ mph$.[4]

[4]The stability of the bicycle is sensitive to the castor and other aspects of the steering geometry of the front wheel, in addition to the gyroscopic effects. Excellent articles on this sub ject have been written by D.E.H. Jones *Physics Today* **23**(4) (1970) 34, and also by J. Lowell & H.D. McKell, *American Journal of Physics* **50** (1982) 1106.