17.4: Relativistic Kinematics

Velocity Transformations

Consider the two parallel coordinate frames with the primed frame moving at a velocity v along the x_1' axis as shown in Figure $17.2.1$. Velocities of an object measured in both frames are defined to be

$$\begin{aligned} u_i &= \frac{dx_i}{dt} \\ u_i' &= \frac{dx_i'}{dt'} \end{aligned} \tag{17.4.1}$$

Using the Lorentz transformations $(17.3.1)$, $(17.3.3)$ between the two frames moving with relative velocity v along the x_1 axis, gives that the velocity along the x_1' axis is

$$u_1' = \frac{dx_1'}{dt'} = \frac{dx_1 - vdt}{dt - \frac{v}{c^2}dx_1} = \frac{u_1 - v}{1 - \frac{u_1 v}{c^2}} \tag{17.4.2}$$

Similarly we get the velocities along the perpendicular x_2' and x_3' axes to be

$$\begin{aligned} u_2' &= \frac{dx_2'}{dt'} = \frac{u_2}{1 - \frac{u_1 v}{c^2}} \\ u_3' &= \frac{dx_3'}{dt'} = \frac{u_3}{1 - \frac{u_1 v}{c^2}} \end{aligned} \tag{17.4.3}$$

When $\frac{u_1 v}{c^2} \to 0$ these velocity transformations become the usual Galilean relations for velocity addition. Do not confuse $\mathbf{u}$ and $\mathbf{u}'$ with $\mathbf{v}$; that is, $\mathbf{u}$ and $\mathbf{u}'$ are the velocities of some object measured in the unprimed and primed frames of reference respectively, whereas $\mathbf{v}$ is the relative velocity of the origin of one frame with respect to the origin of the other frame.

Momentum

Using the classical definition of momentum, that is $\mathbf{p} = m\mathbf{u}$, the linear momentum is not conserved using the above relativistic velocity transformations if the mass m is a scalar quantity. This problem originates from the fact that both $\mathbf{x}$ and t have non-trivial transformations and thus $\mathbf{u} = \frac{d\mathbf{x}}{dt}$ is frame dependent.

Linear momentum conservation can be retained by redefining momentum in a form that is identical in all frames of reference, that is by referring to the *proper time* τ as measured in the rest frame of the moving object. Therefore we define relativistic linear momentum as

$$\mathbf{p} \equiv m\frac{d\mathbf{x}}{d\tau} = m\frac{d\mathbf{x}}{dt}\frac{dt}{d\tau} \tag{17.4.4}$$

But we know the time dilation relation

$$dt = \frac{d\tau}{\sqrt{(1 - \frac{u^2}{c^2})}} = \gamma_u d\tau \tag{17.4.5}$$

Note that the γ_u in this relation refers to the velocity u between the moving object and the frame; *this is quite different* from the $\gamma = \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}}$ which refers to the transformation between the two frames of reference. Thus the new relativistic definition of momentum is

$$\mathbf{p} \equiv m\frac{d\mathbf{x}}{d\tau} \tag{17.4.6}$$

$$= m\gamma_u \frac{d\mathbf{x}}{dt} \tag{17.4.7}$$

$$= \gamma_u m\mathbf{u} \tag{17.4.8}$$

The relativistic definition of linear momentum is the same as the classical definition with the rest mass m replaced by the relativistic mass γm.[1]

Center of momentum coordinate system

The classical relations for handling the kinematics of colliding objects, carry over to special relativity when the relativistic definition of linear momentum, Equation 17.4.8, is assumed. That is, one can continue to apply conservation of linear momentum. However, there is one important conceptual difference for relativistic dynamics in that the center of mass no longer is a meaningful concept due to the interrelation of mass and energy. However, this problem is eliminated by considering the center of momentum coordinate system which, as in the non-relativistic case, is the frame where the total linear momentum of the system is zero. Using the concept of center of momentum incorporates the formalism of classical non-relativistic kinematics.

Force

Newton's second law $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ is covariant under a Galilean transformation. In special relativity this definition also applies using the relativistic definition of momentum $\mathbf{p}$. The fact that the **relativistic momentum $\mathbf{p}$** is conserved in the force-free situation, leads naturally to using the definition of force to be

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{17.4.9}$$

Then the relativistic momentum is conserved if $\mathbf{F} = 0$.

Energy

The classical definition of work done is defined by

$$W_{12} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = T_2 - T_1 \tag{17.4.10}$$

Assume $T_1 = 0$, let $d\mathbf{r} = \mathbf{u}dt$ and insert the relativistic force relation in Equation 17.4.10, gives

$$W = T = \int_0^t \frac{d}{dt}(\gamma_u m\mathbf{u})\cdot\mathbf{u}dt = m\int_0^u ud(\gamma_u u) \tag{17.4.11}$$

Integrate by parts, followed by algebraic manipulation, gives

$$T = \gamma_u mu^2 - m\int_0^u \frac{udu}{\sqrt{1-\frac{u^2}{c^2}}} \tag{17.4.12}$$

$$= \gamma_u mu^2 + mc^2\sqrt{1-\frac{u^2}{c^2}} - mc^2 \tag{17.4.13}$$

$$= \frac{mu^2}{\sqrt{1-\frac{u^2}{c^2}}} + \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}\left(1-\frac{u^2}{c^2}\right) - mc^2 \tag{17.4.14}$$

$$= mc^2(\gamma_u - 1) \tag{17.4.15}$$

Define the **rest energy** E_0

$$E_0 \equiv mc^2 \tag{17.4.16}$$

and **total relativistic energy** E

$$E \equiv \gamma_u mc^2 \tag{17.4.17}$$

then Equation 17.4.15can be written as

$$E = T + E_0 \tag{17.4.18}$$

$$= \gamma_u mc^2 \tag{17.4.19}$$

This is the famous Einstein relativistic energy that relates the equivalence of mass and energy. The total relativistic energy E is a conserved quantity in nature. It is an extension of the conservation of energy and manifestations of the equivalence of energy and mass occur extensively in the real world.

In nuclear physics we often convert mass to energy and back again to mass. For example, gamma rays with energies greater than $1.022\,MeV$, which are pure electromagnetic energy, can be converted to an electron plus positron both of which have rest mass. The positron can then annihilate a different electron in another atom resulting in emission of two $511\,keV$ gamma rays in back to back directions to conserve linear momentum. A dramatic example of Einstein's equation is a nuclear reactor. One gram of material, the mass of a paper clip, provides $E = 9 \times 10^{13}$ joules. This is the daily output of a 1 $GWatt$ nuclear power station or the explosive power of the Nagasaki or Hiroshima bombs.

As the velocity of a particle v approaches c, then γ and the relativistic mass γm both approach infinity. This means that the force needed to accelerate the mass also approaches infinity, and thus no particle can exceed the velocity of light. The energy continues to increase not by increasing the velocity but by increase of the relativistic mass. Although the relativistic relation for kinetic energy is quite different from the Newtonian relation, the Newtonian form is obtained for the case of $u \ll c$ in that

$$T = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2 \tag{17.4.20}$$

$$= mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - mc^2 \tag{17.4.21}$$

$$= \frac{1}{2}mu^2 \tag{17.4.22}$$

An especially useful relativistic relation that can be derived from the above is

$$E^2 = p^2c^2 + E_0^2 \tag{17.4.23}$$

This is useful because it provides a simple relation between total energy of a particle and its relativistic linear momentum plus rest energy.

Example 17.4.1: Rocket Propulsion

Consider a rocket, having initial mass M, is accelerated in a straight line in free space by exhausting propellant at a constant speed v_p relative to the rocket. Let u be the speed of the rocket relative to it's initial rest frame S, when its rest mass has decreased to m. At this instant the rocket is at rest in the inertial frame S'. At a proper time $\tau + d\tau$ the rest mass is $m - dm$ and it has acquired a velocity increment du relative to S' and propellant of rest mass dm_p has been expelled with velocity v_p relative to S'. At proper time τ in S' the rest mass is mc^2. At the time $\tau + d\tau$, energy conservation requires that

$$\gamma_{u'}(m - dm)c^2 + \gamma_{v_p} m_p c^2 = mc^2$$

At the same instant, conservation of linear momentum requires

$$\gamma_{u'}(m - dm)du' - \gamma_p v_p dm_p = 0$$

To first order these two equations simplify to

$$dm_p = \sqrt{1 - \left(\frac{v_p}{c}\right)^2}\, dm$$

$$mdu' = dm_p \gamma_{v_p} v_p$$

Therefore

$$mdu' = v_p dm \tag{a}$$

The velocity increment du' in frame S' can be transformed back to frame S using equation $(17.3.3)$, that is

$$d + du = \frac{u + du'}{1 + \frac{udu'}{c^2}} \approx u + \left(1 - \left(\frac{u}{c}\right)^2\right) du' \tag{b}$$

Equations a and b yield a differential equation for $u(m)$ of

$$\frac{du}{1 - (\frac{u}{c})^2} = v_p \frac{dm}{m}$$

Integrate the left-hand side between 0 and u and the right-hand side between M and m gives

$$\frac{1}{2} c \ln\left(\frac{1 + \frac{u}{c}}{1 - \frac{u}{c}}\right) = -v_p \ln\left(\frac{m}{M}\right)$$

This reduces to

$$\frac{u}{c} = \frac{1 - (\frac{m}{M})^{2v_p/c}}{1 + (\frac{m}{M})^{2v_p/c}}$$

When $\frac{u}{c} \rightarrow 0$ this equation reduces to the non-relativistic answer given in equation $(2.12.34)$.

[1]Note that, until recently, the rest mass was denoted by m_0 and the relativistic mass was referred to as m. Modern texts denote the rest mass by m and the relativistic mass by γm. This book follows the modern nomenclature for rest mass to avoid confusion.