15.S: Advanced Hamiltonian mechanics (Summary)

This chapter has gone beyond what is normally covered in an undergraduate course in classical mechanics, in order to illustrate the power of the remarkable arsenal of methods available for solution of the equations of motion using Hamiltonian mechanics. This has included the Poisson bracket representation of Hamiltonian formulation of mechanics, canonical transformations, Hamilton-Jacobi theory, action-angle variables, and canonical perturbation theory. The purpose was to illustrate the power of variational principles in Hamiltonian mechanics and how they relate to fields such as quantum mechanics and astronomy. The following are the key points made in this chapter.

Poisson brackets:

The elegant and powerful Poisson bracket formalism of Hamiltonian mechanics was introduced. The Poisson bracket of any two continuous functions of generalized coordinates $F(p, q)$ and $G(p, q)$, is defined to be

$$\{F, G\}_{pq} \equiv \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) \tag{15.S.1}$$

The fundamental Poisson brackets equal

$$\{q_k, q_l\} = 0 \tag{15.S.2}$$

$$\{p_k, p_l\} = 0 \tag{15.S.3}$$

$$\{q_k, p_l\} = -\{p_l, q_k\} = \delta_{kl} \tag{15.S.4}$$

The Poisson bracket is invariant to a canonical transformation from (q, p) to (Q, P). That is

$$\{F, G\}_{qp} = \sum_k \left(\frac{\partial F}{\partial Q_k} \frac{\partial G}{\partial P_k} - \frac{\partial F}{\partial P_k} \frac{\partial G}{\partial Q_k} \right) = \{F, G\}_{QP} \tag{15.S.5}$$

There is a one-to-one correspondence between the commutator and Poisson Bracket of two independent functions,

$$(F_1 G_1 - G_1 F_1) = \lambda \{F_1, G_1\} \tag{15.S.6}$$

where λ is an independent constant. In particular $F_1 G_1$ commute of the Poisson Bracket $\{F_1, G_1\} = 0$.

Poisson Bracket representation of Hamiltonian mechanics:

It has been shown that the Poisson bracket formalism contains the Hamiltonian equations of motion and is invariant to canonical transformations. Also this formalism extends Hamilton's canonical equations to non-commuting canonical variables. Hamilton's equations of motion can be expressed directly in terms of the Poisson brackets

$$\dot{q}_k = \{q_k, H\} = \frac{\partial H}{\partial p_k} \tag{15.S.7}$$

$$\dot{p}_k = \{p_k, H\} = -\frac{\partial H}{\partial q_k} \tag{15.S.8}$$

An important result is that the total time derivative of any operator is given by

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \{G, H\} \tag{15.S.9}$$

Poisson brackets provide a powerful means of determining which observables are time independent and whether different observables can be measured simultaneously with unlimited precision. It was shown that the Poisson bracket is invariant to canonical transformations, which is a valuable feature for Hamiltonian mechanics. Poisson brackets were used to prove Liouville's theorem which plays an important role in the use of Hamiltonian phase space in statistical mechanics. The Poisson bracket is equally applicable to continuous solutions in classical mechanics as well as discrete solutions in quantized systems.

Canonical transformations:

A transformation between a canonical set of variables (q, p) with Hamiltonian $H(q, p, t)$ to another set of canonical variable (Q, P) with Hamiltonian $\mathcal{H}(Q, P, t)$ can be achieved using a generating functions F such that

$$\mathcal{H}(Q, P, t) = H(q, p, t) + \frac{\partial F}{\partial t} \tag{15.S.10}$$

Possible generating functions are summarized in the following table.

Table $15.S.1$

Generating function	Generating function derivatives	Trivial special case
$F = F_1(\mathbf{q}, \mathbf{Q}, t)$	$p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i \quad Q_i = p_i \quad P_i = -q_i$
$F = F_2(\mathbf{q}, \mathbf{P}, t) - \mathbf{Q} \cdot \mathbf{P}$	$p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i \quad Q_i = q_i \quad P_i = p_i$
$F = F_3(\mathbf{p}, \mathbf{Q}, t) + \mathbf{q} \cdot \mathbf{p}$	$q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i \quad Q_i = -q_i \quad P_i = -p_i$
$F = F_4(\mathbf{p}, \mathbf{P}, t) + \mathbf{q} \cdot \mathbf{p} - \mathbf{Q} \cdot \mathbf{P}$	$q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_1 = p_i P_i \quad Q_i = p_i \quad P_i = -q_i$

If the canonical transformation makes $\mathcal{H}(Q, P, t) = 0$ then the conjugate variables (Q, P) are constants of motion. Similarly if $\mathcal{H}(Q, P, t)$ is a cyclic function then the corresponding P are constants of motion.

Hamilton-Jacobi theory:

Hamilton-Jacobi theory determines the generating function required to perform canonical transformations that leads to a powerful method for obtaining the equations of motion for a system. The Hamilton-Jacobi theory uses the action function $S \equiv F_2$ as a generating function, and the canonical momentum is given by

$$p_i = \frac{\partial S}{\partial q_i} \tag{15.S.11}$$

This can be used to replace p_i in the Hamiltonian H leading to the **Hamilton-Jacobi equation**

$$H(q; \frac{\partial S}{\partial q}; t) + \frac{\partial S}{\partial t} = 0 \tag{15.S.12}$$

Solutions of the Hamilton-Jacobi equation were obtained by separation of variables. The close optical-mechanical analogy of the Hamilton-Jacobi theory is an important advantage of this formalism that led to it playing a pivotal role in the development of wave mechanics by Schrödinger.

Action-angle variables:

The action-angle variables exploits a canonical transformation from $(q, p) \rightarrow (\phi, I)$ where

$$I_i \equiv \frac{1}{2\pi} J_i = \frac{1}{2\pi} \oint p_i dq_i \tag{15.S.13}$$

For periodic motion the phase-space trajectory is closed with area given by J and this area is conserved for the above canonical transformation. For a conserved Hamiltonian the action variable I is independent of the angle variable ϕ. The time dependence of the angle variable ϕ directly determines the frequency of the periodic motion without recourse to calculation of the detailed trajectory of the periodic motion.

Canonical perturbation theory:

Canonical perturbation theory is a valuable method of handling multibody interactions. The adiabatic invariance of the action-angle variables provides a powerful approach for exploiting canonical perturbation theory.

Comparison of Lagrangian and Hamiltonian formulations:

The remarkable power, and intellectual beauty, provided by use of variational principles to exploit the underlying principles of natural economy in nature, has had a long and rich history. It has led to profound developments in many branches of theoretical physics. However, it is noted that although the above algebraic formulations of classical mechanics have been used for over two centuries, the important limitations of these algebraic formulations to non-linear systems remain a challenge that still is being addressed.

It has been shown that the Lagrangian and Hamiltonian formulations represent the vector force fields, and the corresponding equations of motion, in terms of the Lagrangian function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, or the action functional $S(\mathbf{q}, \mathbf{p}, t)$, which are scalars under rotation. The Lagrangian function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is related to the action functional $S(\mathbf{q}, \mathbf{p}, t)$ by

$$S(\mathbf{q}, \mathbf{p}, t) = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt \tag{15.S.14}$$

These functions are analogous to electric potential, in that the observables are derived by taking derivatives of the Lagrangian function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ or the action functional $S(\mathbf{q}, \mathbf{p}, t)$. The Lagrangian formulation is more convenient for deriving the equations of motion for simple mechanical systems. The Hamiltonian formulation has a greater arsenal of techniques for solving complicated problems plus it uses the canonical variables (q_i, p_i) which are the variables of choice for applications to quantum mechanics and statistical mechanics.