6.4: Lagrange equations from Hamilton's Principle

Lagrange equations from Hamilton's Action Principle

Hamilton published two papers in 1834 and 1835, announcing a fundamental new dynamical principle that underlies both Lagrangian and Hamiltonian mechanics. Hamilton was seeking a theory of optics when he developed Hamilton's Action Principle, plus the field of Hamiltonian mechanics, both of which play a crucial role in classical mechanics and modern physics. Hamilton's Action Principle states *"dynamical systems follow paths that minimize the time integral of the Lagrangian"*. That is, the *action functional* S

$$S=\int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt \tag{6.4.1}$$

has a minimum value for the correct path of motion. **Hamilton's Action Principle** can be written in terms of a virtual infinitessimal displacement δ, as

$$\delta S=\delta \int_{t_1}^{t_2} L dt=0 \tag{6.4.2}$$

Variational calculus therefore implies that a system of s independent generalized coordinates must satisfy the basic Lagrange-Euler equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=0 \tag{6.4.3}$$

Note that for $Q_j^{EX}=0$, this is the same as equation $(6.3.28)$ which was derived using d'Alembert's Principle.

This discussion has shown that Euler's variational differential equation underlies both the differential variational d'Alembert Principle, and the more fundamental integral Hamilton's Action Principle. As discussed in chapter 9.2, Hamilton's Principle of Stationary Action adds a fundamental new dimension to classical mechanics which leads to derivation of both Lagrangian and Hamiltonian mechanics. That is, both Hamilton's Action Principle, and d'Alembert's Principle, can be used to derive Lagrangian mechanics leading to the most general Lagrange equations that are applicable to both holonomic and non-holonomic constraints, as well as conservative and non-conservative systems. In addition, Chapter 6.2 presented a plausibility argument showing that Lagrangian mechanics can be justified based on Newtonian mechanics. Hamilton's Action Principle, and d'Alembert's Principle, can be expressed in terms of generalized coordinates which is much broader in scope than the equations of motion implied using Newtonian mechanics.