3.S: Linear Oscillators (Summary)

Linear systems have the feature that the solutions obey the *Principle of Superposition*, that is, the amplitudes add linearly for the superposition of different oscillatory modes. Applicability of the Principle of Superposition to a system provides a tremendous advantage for handling and solving the equations of motion of oscillatory systems.

Geometric representations of the motion of dynamical systems provide sensitive probes of periodic motion. Configuration space $(\mathbf{q}, \mathbf{q}, t)$, state space $(\mathbf{q}, \dot{\mathbf{q}}, t)$ and phase space $(\mathbf{q}, \mathbf{p}, t)$, are powerful geometric representations that are used extensively for recognizing periodic motion where $\mathbf{q}$, $\dot{\mathbf{q}}$, and $\mathbf{p}$ are vectors in n-dimensional space.

Linearly-damped free linear oscillator

The free linearly-damped linear oscillator is characterized by the equation

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = 0 \tag{3.S.1}$$

The solutions of the linearly-damped free linear oscillator are of the form

$$z = e^{-\left(\frac{\Gamma}{2}\right)t}\left[z_1 e^{i\omega_1 t} + z_2 e^{-i\omega_1 t}\right] \qquad \omega_1 \equiv \sqrt{\omega_o^2 - \left(\frac{\Gamma}{2}\right)^2} \tag{3.S.2}$$

The solutions fall into three categories

Table $3.S.1$

$x(t) = Ae^{-\left(\frac{\Gamma}{2}\right)t}\cos(\omega_1 t - \beta)$	underdamped	$\omega_1 = \sqrt{\omega_o^2 - \left(\frac{\Gamma}{2}\right)^2} > 0$
$x(t) = [A_1 e^{-\omega_+ t} + A_2 e^{-\omega_- t}]$	overdamped	$\omega_\pm = -\left[-\frac{\Gamma}{2} \pm \sqrt{\left(\frac{\Gamma}{2}\right)^2 - \omega_o^2}\right]$
$x(t) = (A + Bt)e^{-\left(\frac{\Gamma}{2}\right)t}$	critically damped	$\omega_1 = \sqrt{\omega_o^2 - \left(\frac{\Gamma}{2}\right)^2} = 0$

The energy dissipation for the linearly-damped free linear oscillator time averaged over one period is given by

$$\langle E \rangle = E_0 e^{-\Gamma t} \tag{3.S.3}$$

The quality factor Q characterizing the damping of the free oscillator is define to be

$$Q = \frac{E}{\Delta E} = \frac{\omega_1}{\Gamma} \tag{3.S.4}$$

where ΔE is the energy dissipated per radian.

Sinusoidally-driven, linearly-damped, linear oscillator

The linearly-damped linear oscillator, driven by a harmonic driving force, is of considerable importance to all branches of physics, and engineering. The equation of motion can be written as

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \tag{3.S.5}$$

where $F(t)$ is the driving force. The complete solution of this second-order differential equation comprises two components, the complementary solution (*transient response*), and the particular solution (*steady-state response*). That is,

$$x(t)_{Total} = x(t)_T + x(t)_S \tag{3.S.6}$$

For the underdamped case, the transient solution is the complementary solution

$$x(t)_T = \frac{F_0}{m} e^{-\frac{\Gamma}{2}t}\cos(\omega_1 t - \delta) \tag{3.S.7}$$

and the steady-state solution is given by the particular solution

$$x(t)_S = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2-\omega^2)^2+(\Gamma\omega)^2}}\cos(\omega t-\delta) \tag{3.S.8}$$

Resonance

A detailed discussion of resonance and energy absorption for the driven linearly-damped linear oscillator was given. For resonance the maximum amplitudes occur at frequencies

Table $3.S.2$

Resonant system	Resonant frequency
undamped free linear oscillator	$\omega_0=\sqrt{\frac{k}{m}}$
linearly-damped free linear oscillator	$\omega_1=\sqrt{\omega_0^2-\left(\frac{\Gamma}{2}\right)^2}$
driven linearly-damped linear oscillator	$\omega_R=\sqrt{\omega_0^2-2\left(\frac{\Gamma}{2}\right)^2}$

The energy absorption for the steady-state solution for resonance is given by

$$x(t)_S = A_{el}\cos\omega t + A_{abs}\sin\omega t \tag{3.S.9}$$

where the **elastic amplitude**

$$A_{el} = \frac{\frac{F_0}{m}}{(\omega_0^2-\omega^2)^2+(\Gamma\omega)^2}(\omega_0^2-\omega^2) \tag{3.S.10}$$

while the **absorptive amplitude**

$$A_{abs} = \frac{\frac{F_0}{m}}{(\omega_0^2-\omega^2)^2+(\Gamma\omega)^2}\Gamma\omega \tag{3.S.11}$$

The time average power input is given by only the absorptive term

$$\langle P\rangle = \frac{1}{2}F_0\omega A_{abs} = \frac{F_0^2}{2m}\frac{\Gamma\omega^2}{(\omega_0^2-\omega^2)^2+(\Gamma\omega)^2} \tag{3.S.12}$$

This power curve has the classic Lorentzian shape.

Wave propagation

The wave equation was introduced and both travelling and standing wave solutions of the wave equation were discussed. Harmonic wave-form analysis, and the complementary time-sampled wave form analysis techniques, were introduced in this chapter and in appendix 19.9. The relative merits of Fourier analysis and the digital Green's function waveform analysis were illustrated for signal processing.

The concepts of phase velocity, group velocity, and signal velocity were introduced. The phase velocity is given by

$$v_{phase} = \frac{\omega}{k} \tag{3.S.13}$$

and group velocity

$$v_{group} = \left(\frac{d\omega}{dk}\right)_{k_0} = v_{phase} + k\frac{\partial v_{phase}}{\partial k} \tag{3.S.14}$$

If the group velocity is frequency dependent then the information content of a wave packet travels at the signal velocity which can differ from the group velocity.

The Wave-packet Uncertainty Principle implies that making a precise measurement of the frequency of a sinusoidal wave requires that the wave packet be infinitely long. The *standard deviation* $\sigma(t)=\sqrt{\langle t^2\rangle-\langle t\rangle^2}$ characterizing the width of the *amplitude* of the wavepacket spectral distribution in the angular frequency domain, $\sigma_A(\omega)$, and the corresponding width in time $\sigma_A(t)$, are related by :

$$\sigma_A(t) \cdot \sigma_A(\omega) \geqslant 1 \qquad \text{(Relation between amplitude uncertainties.)}$$

The standard deviations for the spectral distribution and width of the *intensity* of the wave packet are related by:

$$\sigma_I(t) \cdot \sigma_I(\omega) \geqslant \frac{1}{2} \tag{3.S.15}$$

$$\sigma_I(x) \cdot \sigma_I(k_x) \geqslant \frac{1}{2} \quad \sigma_I(y) \cdot \sigma_I(k_y) \geqslant \frac{1}{2} \quad \sigma_I(z) \cdot \sigma_I(k_z) \geqslant \frac{1}{2}$$

This applies to all forms of wave motion, including sound waves, water waves, electromagnetic waves, or matter waves.
