11.12: Two-body Scattering

Two moving bodies, that are interacting via a central force, scatter when the force is repulsive, or when an attractive system is unbound. Two-body scattering of bodies is encountered extensively in the fields of astronomy, atomic, nuclear, and particle physics. The probability of such scattering is most conveniently expressed in terms of scattering cross sections defined below.

Total two-body scattering cross section

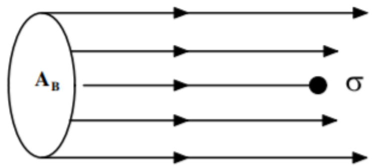

Figure 11.12.1: Scattering probability for an incident beam of cross sectional area A by a target body of cross sectional area σ.

The concept of scattering cross section for two-body scattering is most easily described for the total two-body cross section. The probability P that a beam of n_B incident point particles/second, distributed over a cross sectional area A_B, will hit a single solid object, having a cross sectional area σ, is given by the ratio of the areas as illustrated in Figure $11.12.1$ That is,

$$P = \frac{\sigma}{A_B} \tag{11.12.1}$$

where it is assumed that $A_B >> \sigma$. For a spherical target body of radius r, the cross section $\sigma = \pi r^2$. The scattering probability P is proportional to the cross section σ which is the cross section of the target body perpendicular to the beam; thus σ has the units of area.

Since the incident beam of n_B incident point particles/second, has a cross sectional area A_B, then it will have an areal density I given by

$$I = \frac{n_B}{A_B} \text{ beam particles}/m^2/s \tag{11.12.2}$$

The number of beam particles scattered per second N_S by this single target scatterer equals

$$N_S = P n_B = \frac{\sigma}{A_B} I A_B = \sigma I \tag{11.12.3}$$

Thus the cross section for scattering by this single target body is

$$\sigma = \frac{N_S}{I} = \frac{\text{Scattered particles}/s}{\text{incident beam}/\text{m}^2/s} \tag{11.12.4}$$

Realistically one will have many target scatterers in the target and the total scattering probability increases proportionally to the number of target scatterers. That is, for a target comprising an areal density of η_T target bodies per unit area of the incident beam, then the number scattered will increase proportional to the target areal density η_T. That is, there will be $\eta_T A_B$ scattering bodies that interact with the beam assuming that the target has a larger area than the beam. Thus the total number scattered per second N_S by a target that comprises multiple scatterers is

$$N_S = \sigma \frac{n_B}{A_B} \eta_T A_B = \sigma n_B \eta_T \tag{11.12.5}$$

Note that this is independent of the cross sectional area of the beam assuming that the target area is larger than that of the beam. That is, the number scattered per second is proportional to the cross section σ times the product of the number of incident particles per second, n_B, and the areal density of target scatterers, η_T. Typical cross sections encountered in astrophysics are $\sigma \approx 10^{14} m^2$, in atomic physics: $\sigma \approx 10^{-20} m^2$, and in nuclear physics; $\sigma \approx 10^{-28} m^2 = barns$.[3]

N. B., the above proof assumed that the target size is larger than the cross sectional area of the incident beam. If the size of the target is smaller than the beam, then n_B is replaced by the areal density/s of the beam η_B and η_T is replaced by the number of target particles n_T and the cross-sectional size of the target cancels.

Differential two-body scattering cross section

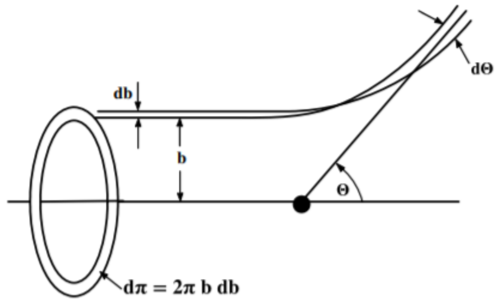

Figure 11.12.2: The equivalent one-body problem for scattering of a reduced mass μ by a force centre in the centre of mass system.

The differential two-body scattering cross section gives much more detailed information of the scattering force than does the total cross section because of the correlation between the impact parameter and the scattering angle. That is, a measurement of the number of beam particles scattered into a given solid angle as a function of scattering angles θ, ϕ probes the radial form of the scattering force.

The differential cross section for scattering of an incident beam by a single target body into a solid angle $d\Omega$ at scattering angles θ, ϕ is defined to be

$$\frac{d\sigma}{d\Omega}(\theta\phi) \equiv \frac{1}{I}\frac{dN_S(\theta,\phi)}{d\Omega} \tag{11.12.6}$$

where the right-hand side is the ratio of the number scattered per target nucleus into solid angle $d\Omega(\theta,\phi)$, to the incident beam intensity $I\ particles/m^2/s$.

Similar reasoning used to derive Equation 11.12.4 leads to the number of beam particles scattered into a solid angle $d\Omega$ for n_B beam particles incident upon a target with areal density η_T is

$$\frac{dN_S(\theta,\phi)}{d\Omega} = n_B\eta_T\frac{d\sigma}{d\Omega}(\theta\phi) \tag{11.12.7}$$

Consider the equivalent one-body system for scattering of one body by a scattering force center in the center of mass. As shown in figures $(11.8.2)$ and $11.12.2$, the perpendicular distance between the center of force of the two body system and trajectory of the incoming body at infinite distance is called the *impact parameter* b. For a central force the scattering system has cylindrical symmetry, therefore the solid angle $d\Omega(\theta\phi) = \sin\theta d\theta d\phi$ can be integrated over the azimuthal angle ϕ to give $d\Omega(\theta) = 2\pi\sin\theta d\theta$.

For the inverse-square, two-body, central force there is a one-to-one correspondence between impact parameter b and scattering angle θ for a given bombarding energy. In this case, assuming conservation of flux means that the incident beam particles passing through the impact-parameter annulus between b and $b+db$ must equal the the number passing between the corresponding angles θ and $\theta + d\theta$. That is, for an incident beam flux of $I\ particles/m^2/s$ the number of particles per second passing through the annulus is

$$I2\pi b\,|db| = 2\pi\frac{d\sigma}{d\Omega}I\sin\theta\,|d\theta| \tag{11.12.8}$$

The modulus is used to ensure that the number of particles is always positive. Thus

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{11.12.9}$$

Impact parameter dependence on scattering angle

If the function $b = f(\theta, E_{cm})$ is known, then it is possible to evaluate $\left|\frac{db}{d\theta}\right|$ which can be used in Equation 11.12.8 to calculate the differential cross section. A simple and important case to consider is two-body elastic scattering for the inverse-square law force such as the Coulomb or gravitational forces. To avoid confusion in the following discussion, the center-of-mass scattering angle will be called θ, while the angle used to define the hyperbolic orbits in the discussion of trajectories for the inverse square law, will be called ψ.

In chapter 11.8 the equivalent one-body representation gave that the radial distance for a trajectory for the inverse square law is given by

$$\frac{1}{r} = -\frac{\mu k}{l^2}[1+\epsilon\cos\psi] \tag{11.12.10}$$

Note that closest approach occurs when $\psi = 0$ while for $r \to \infty$ the bracket must equal zero, that is

$$\cos\psi_{\infty} = \pm\left|\frac{1}{\epsilon}\right| \tag{11.12.11}$$

The polar angle ψ is measured with respect to the symmetry axis of the two-body system which is along the line of distance of closest approach as shown in Figure $(11.8.2)$. The geometry and symmetry show that the scattering angle θ is related to the trajectory angle ψ_{∞} by

$$\theta = \pi - 2\psi_{\infty} \tag{11.12.12}$$

Equation $(11.7.1)$ gives that

$$\psi_{\infty} = \int_{r_{\min}}^{\infty} \frac{\pm l dr}{r^2\sqrt{2\mu\left(E_{cm} - U - \frac{l^2}{2\mu r^2}\right)}} \tag{11.12.13}$$

Since

$$l^2 = b^2p^2 = b^2 2\mu E_{cm} \tag{11.12.14}$$

then the scattering angle can be written as.

$$\psi_{\infty} = \frac{\pi-\theta}{2} = \int_{r_{\min}}^{\infty} \frac{bdr}{r^2\sqrt{\left(1 - \frac{U}{E_{cm}} - \frac{b^2}{r^2}\right)}} \tag{11.12.15}$$

Let $u = \frac{1}{r}$, then

$$\psi_{\infty} = \frac{\pi-\theta}{2} = \int_{r_{\min}}^{\infty} \frac{bdu}{\sqrt{\left(1 - \frac{U}{E_{cm}} - b^2u^2\right)}} \tag{11.12.16}$$

For the repulsive inverse square law

$$U = -\frac{k}{r} = -ku \tag{11.12.17}$$

where k is taken to be positive for a repulsive force. Thus the scattering angle relation becomes

$$\psi_{\infty} = \frac{\pi-\theta}{2} = \int_{r_{\min}}^{\infty} \frac{bdu}{\sqrt{\left(1 + \frac{ku}{E_{cm}} - b^2u^2\right)}} \tag{11.12.18}$$

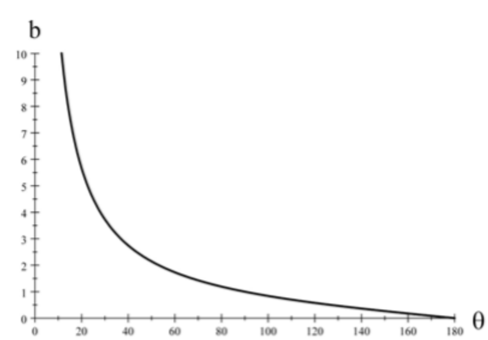

Figure 11.12.3: Impact parameter dependence on scattering angle for Rutherford scattering.

The solution of this equation is given by equation $(11.8.12)$to be

$$u=\frac{1}{r}=-\frac{\mu k}{l^{2}}[1+\epsilon \cos \psi] \tag{11.12.19}$$

where the eccentricity

$$\epsilon=\sqrt{1+\frac{2 E_{cm} l^{2}}{\mu k^{2}}} \tag{11.12.20}$$

For $r \rightarrow \infty, \ u=0$ then, as shown previously,

$$\left|\frac{1}{\epsilon}\right|=\cos \psi_{\infty}=\cos \frac{\pi-\theta}{2}=\sin \frac{\theta}{2} \tag{11.12.21}$$

Therefore

$$\frac{2 E_{cm} b}{k}=\sqrt{\epsilon^{2}-1}=\cot \frac{\theta}{2} \tag{11.12.22}$$

that is, the impact parameter b is given by the relation

$$b=\frac{k}{2 E_{cm}} \cot \frac{\theta}{2} \tag{11.12.23}$$

Thus, for an inverse-square law force, the two-body scattering has a one-to-one correspondence between impact parameter b and scattering angle θ as shown schematically in Figure $11.12.3$

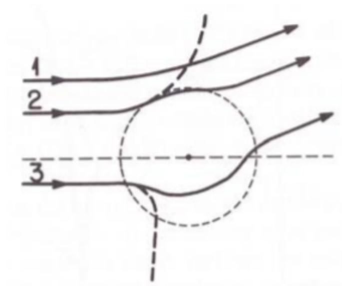

Figure 11.12.4: Classical trajectories for scattering to a given angle by the repulsive Coulomb field plus the attractive nuclear field for three different impact parameters. Path 1 is pure Coulomb. Paths 2 and 3 include Coulomb plus nuclear interactions. The dashed parts of trajectories 2 and 3 correspond to only the Coulomb force acting, i.e. zero nuclear force

If k is negative, which corresponds to an attractive inverse square law, then one gets the same relation between impact parameter and scattering angle except that the sign of the impact parameter b is opposite. This means that the hyperbolic trajectory has an interior rather than exterior focus. That is, the trajectory partially orbits around the center of force rather than being repelled away.

$$r_{\min }=\frac{k}{2 E_{cm}}\left(1+\frac{1}{\sin \frac{\theta}{2}}\right) \tag{11.12.24}$$

Note that for $\theta=180^{o}$ then

$$E_{cm}=\frac{k}{r_{\min }}=U\left(r_{\min)}\right. \tag{11.12.25}$$

which is what you would expect from equating the incident kinetic energy to the potential energy at the distance of closest approach.

For scattering of two nuclei by the repulsive Coulomb force, if the impact parameter becomes small enough, the attractive nuclear force also acts leading to impact-parameter dependent effective potentials illustrated in Figure $11.12.4$ Trajectory 1 does not overlap the nuclear force and thus is pure Coulomb. Trajectory 2 interacts at the periphery of the nuclear potential and the trajectory deviates from pure Coulomb shown dashed. Trajectory 3 passes through the interior of the nuclear potential. These three trajectories all can lead to the same scattering angle and thus there no longer is a one-to-one correspondence between scattering angle and impact parameter.

Rutherford scattering

Two models of the nucleus evolved in the 1900's, the Rutherford model assumed electrons orbiting around a small nucleus like planets around the sun, while J.J. Thomson's "plum-pudding" model assumed the electrons were embedded in a uniform sphere of positive charge the size of the atom. When Rutherford derived his classical formula in 1911 he realized that it can be used to determine the size of the nucleus since the electric field obeys the inverse square law only when outside of the charged spherical nucleus. Inside a uniform sphere of charge the electric field is $\mathbf{E} \propto \mathbf{r}$ and thus the scattering cross section will not obey the Rutherford relation for distances of closest approach that are less than the radius of the sphere of negative charge. Observation of the angle beyond which the Rutherford formula breaks down immediately determines the radius of the nucleus.

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{k}{2E_{cm}}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} \tag{11.12.26}$$

This cross section assumes elastic scattering by a repulsive two-body inverse-square central force. For scattering of nuclei in the Coulomb potential, the constant k is given to be

$$k = \frac{Z_p Z_T e^2}{4\pi\varepsilon_o} \tag{11.12.27}$$

The cross section, scattering angle and E_{cm} of Equation 11.12.26 are evaluated in the center-of-mass coordinate system, whereas usually two-body elastic scattering data involve scattering of the projectiles by a stationary target as discussed in chapter 11.13.

Gieger and Marsden performed scattering of 7.7 MeV α particles from a thin gold foil and proved that the differential scattering cross section obeyed the Rutherford formula back to angles corresponding to a distance of closest approach of $10^{-14}m$ which is much smaller that the $10^{-10}m$ size of the atom. This validated the Rutherford model of the atom and immediately led to the Bohr model of the atom which played such a crucial role in the development of quantum mechanics. Bohr showed that the agreement with the Rutherford formula implies the Coulomb field obeys the inverse square law to small distances. This work was performed at Manchester University, England between 1908 and 1913. It is fortunate that the classical result is identical to the quantal cross section for scattering, otherwise the development of modern physics could have been delayed for many years.

Scattering of very heavy ions, such as ^{208}Pb, can electromagnetically excite target nuclei. For the Coulomb force the impact parameter b and the distance of closest approach, $r_{\min}$ are directly related to the scattering angle θ by Equation 11.12.23. Thus observing the angle of the scattered projectile unambiguously determines the hyperbolic trajectory and thus the electromagnetic impulse given to the colliding nuclei. This process, called Coulomb excitation, uses the measured angular distribution of the scattered ions for inelastic excitation of the nuclei to precisely and unambiguously determine the Coulomb excitation cross section as a function of impact parameter. This unambiguously determines the shape of the nuclear charge distribution.

Example 11.12.1: Two-body scattering by an inverse cubic force

Assume two-body scattering by a potential $U = \frac{k}{r^2}$ where $k > 0$. This corresponds to a repulsive two-body force $\mathbf{F} = \frac{2k}{r^3}\hat{\mathbf{r}}$. Insert this force into Binet's differential orbit, equation $(11.5.5)$, gives

$$\frac{d^2u}{d\phi^2} + u\left(1 + \frac{2k\mu}{l^2}\right) = 0$$

The solution is of the form $u = A\sin(\omega\psi + \beta)$ where A and β are constants of integration, $l = \mu r^2\dot{\psi}$, and

$$\omega^2 = \left(1 + \frac{2k\mu}{l^2}\right)$$

Initially $r = \infty$, $u = 0$, and therefore $\beta = 0$. Also at $r = \infty$, $E = \frac{1}{2}\mu\dot{r}_\infty^2$, that is $|\dot{r}_\infty| = \sqrt{\frac{2E}{\mu}}$. Then

$$\dot{r} = \frac{dr}{d\psi}\dot{\psi} = \frac{dr}{d\psi}\frac{l}{\mu r^2} = -\frac{l}{\mu}\frac{du}{d\psi} = -A\frac{l}{\mu}\omega\cos(\omega\psi)$$

The initial energy gives that $A = \frac{1}{l\omega}\sqrt{2\mu E}$. Hence the orbit equation is

$$u = \frac{1}{r} = \frac{\sqrt{2\mu E}}{l\omega}\sin(\omega\psi)$$

The above trajectory has a distance of closest approach, $r_{\min}$, when $\psi_{\min}=\frac{\pi}{2\omega}$. Moreover, due to the symmetry of the orbit, the scattering angle θ is given by

$$\theta=\pi-2\psi_0=\pi\left(1-\frac{1}{\omega}\right)$$

Since $l^2=\mu^2b^2\dot{r}_\infty^2=2b^2\mu E$ then

$$1-\frac{\theta}{\pi}=\left(1+\frac{2k\mu}{l^2}\right)^{-\frac{1}{2}}=\left(1+\frac{k}{b^2E}\right)^{-\frac{1}{2}}$$

This gives that the impact parameter b is related to scattering angle by

$$b^2=\frac{k}{E}\frac{(\pi-\theta)^2}{(2\pi-\theta)\theta}$$

This impact parameter relation can be used in Equation 11.12.8to give the differential cross section

$$\frac{d\sigma}{d\Omega}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|=\frac{k}{Esin\theta}\frac{\pi^2(\pi-\theta)}{(2\pi-\theta)^2\theta^2}$$

These orbits are called Cotes spirals.

[3]The term "barn" was chosen because nuclear physicists joked that the cross sections for neutron scattering by nuclei were as large as a barn door.