9.4: Application of Hamilton's Action Principle to Mechanics

Knowledge of the equations of motion is required to predict the response of a system to any set of initial conditions. Hamilton's action principle, that is built into Lagrangian and Hamiltonian mechanics, coupled with the availability of a wide arsenal of variational principles and techniques, provides a remarkably powerful and broad approach to deriving the equations of motions required to determine the system response.

As mentioned in the Prologue, derivation of the equations of motion for any system, based on Hamilton's Action Principle, separates naturally into a hierarchical set of three stages that differ in both sophistication and understanding, as described below.

1. **Action stage:** The primary "action stage" employs Hamilton's Action functional, $S=\int_{t_i}^{t_f} L(\mathbf{q},\dot{\mathbf{q}},t)dt$ to derive the Lagrangian and Hamiltonian functionals. This action stage provides the most fundamental and sophisticated level of understanding. It involves specifying all the active degrees of freedom, as well as the interactions involved. Symmetries incorporated at this primary action stage can simplify subsequent use of the Hamiltonian and Lagrangian functionals.
2. **Hamiltonian/Lagrangian stage:** The "Hamiltonian/Lagrangian stage" uses the Lagrangian or Hamiltonian functionals, that were derived at the action stage, in order to derive the equations of motion for the system of interest. Symmetries, not already incorporated at the primary action stage, may be included at this secondary stage.
3. **Equations of motion stage:** The "equations-of-motion stage" uses the derived equations of motion to solve for the motion of the system subject to a given set of initial boundary conditions. Nonconservative forces, such as dissipative forces, that were not included at the primary and secondary stages, may be added at the equations of motion stage.

Lagrange omitted the action stage when he used d'Alembert's Principle to derive Lagrangian mechanics. The Newtonian mechanics approach omits both the primary "action" stage, as well as the secondary "Hamiltonian/Lagrangian" stage, since Newton's Laws of Motion directly specify the "equations-of-motion stage". Thus these do not exploit the considerable advantages provided by the use of the action, the Lagrangian, and the Hamiltonian. Newtonian mechanics requires that all the active forces be included when deriving the equations of motion, which involves dealing with vector quantities. In Newtonian mechanics, symmetries must be incorporated directly at the equations of motion stage, which is more difficult than when done at the primary "action" stage, or the secondary "Lagrangian/Hamiltonian" stage. The "action" and "Hamiltonian/Lagrangian" stages allow for use of the powerful arsenal of mathematical techniques that have been developed for applying variational principles.

There are considerable advantages to deriving the equations of motion based on Hamilton's Principle, rather than derive them using Newtonian mechanics. It is significantly easier to use variational principles to handle the scalar functionals, action, Lagrangian, and Hamiltonian, rather than starting at the equationsof-motion stage. For example, utilizing all three stages of algebraic mechanics facilitates accommodating extra degrees of freedom, symmetries, and interactions. The symmetries identified by Noether's theorem are more easily recognized during the primary "action" and secondary "Hamiltonian/Lagrangian" stages rather than at the subsequent "equations of motion" stage. Approximations made at the "action" stage are easier to implement than at the "equations-of-motion" stage. Constrained motion is much more easily handled at the primary "action", or secondary "Hamilton/Lagrangian" stages, than at the equations-of-motion stage. An important advantage of using Hamilton's Action Principle, is that there is a close relationship between action in classical and quantal mechanics, as discussed in chapters 15 and 18. Algebraic principles, that underly analytical mechanics, naturally encompass applications to many branches of modern physics, such as relativistic mechanics, fluid motion, and field theory.

In summary, the use of the single fundamental invariant quantity, action, as described above, provides a powerful and elegant framework, that was developed first for classical mechanics, but now is exploited in a wide range of science, engineering, and economics. An important feature of using the algebraic approach to classical mechanics is the tremendous arsenal of powerful mathematical techniques that have been developed for use of variational calculus applied to Lagrangian and Hamiltonian mechanics. Some of these variational techniques were presented in chapters 6, 7, 8, and 9, while others will be introduced in chapter 15.