16.3: The Lagrangian density formulation for continuous systems

One spatial dimension

In general the Lagrangian density can be a function of q, ∇q, $\frac{dq}{dt}$, x, y, z, and t. It is of interest that Hamilton's principle leads to a set of partial differential equations of motion, based on the Lagrangian density, that are analogous to the Lagrange equations of motion for discrete systems. When deriving the Lagrangian equations of motion in terms of the Lagrangian density using Hamilton's principle, the notation is simplified if the system is limited to one spatial coordinate x. In addition, it is convenient to use the compact notation where the spatial derivative is written $q' \equiv \frac{dq}{dx}$ and the time derivative is $\dot{q} \equiv \frac{dq}{dt}$, and the one-dimensional Lagrangian density is assumed to be a function $\mathfrak{L}(q, q', \dot{q}, x, t)$. The appearance of the derivative $q' \equiv \frac{dq}{dx}$ as an argument of the Lagrange density is a consequence of the continuous dependence of q on x. In principle, higher-order derivatives could occur but they do not arise in most problems of physical interest.

Assuming that the one spatial dimension is x, then Hamilton's principle of least action can be expressed in terms of the Lagrangian density as

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = \delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathfrak{L}(q, q', \dot{q}, x, t) dx dt \tag{16.3.1}$$

Following the same approach used in chapter 5.2, it is assumed that the stationary path for the action integral is described by the function $q(x,t)$. Define a neighboring function using a parametric representation $q(x,t;\epsilon)$ such that when $\epsilon = 0$, the extremum function $q = q(x,t)$ yields the stationary action integral S.

Assume that an infinitessimal fraction ϵ of a neighboring function $\eta(x,t)$ is added to the extremum path $q(x,t)$. That is, assume

$$q(x,t;\epsilon) = q(x,t) + \epsilon\eta(x,t) \tag{16.3.2}$$

$$q'(x,t;\epsilon) \equiv \frac{dq(x,t;\epsilon)}{dx} = \frac{dq(x,t)}{dx} + \epsilon\frac{d\eta(x,t)}{dx} = q'(x,t) + \epsilon\eta'(x,t) \tag{16.3.3}$$

$$\dot{q}(x,t;\epsilon) \equiv \frac{dq(x,t;\epsilon)}{dt} = \frac{dq(x,t)}{dt} + \epsilon\frac{d\eta(x,t)}{dt} = \dot{q}(x,t) + \epsilon\dot{\eta}(x,t) \tag{16.3.4}$$

where it is assumed that both the extremum function $q(x,t)$ and the auxiliary function $\eta(x,t)$ are well behaved functions of x and t, with continuous first derivatives, and that $\eta(x,t) = 0$ at (x_1, t_1) and (x_2, t_2) because, for all possible paths, the function $q(x,t;\epsilon)$ must be identical with $q(x,t)$ at the end points of the path, i.e. $\eta(x_1,t_1) = \eta(x_2,t_2) = 0$.

A parametric family of curves $S(\epsilon)$, as a function of the admixture coefficient ϵ, is described by the function

$$S(\epsilon) = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathfrak{L}(q(x,t;\epsilon), q'(x,t;\epsilon), \dot{q}(x,t;\epsilon), x, t) dx dt \tag{16.3.5}$$

Then Hamilton's principle requires that the action integral be a stationary function value for $\epsilon = 0$, that is, $S(\epsilon)$ is independent of ϵ which is satisfied if

$$\frac{\partial S(\epsilon)}{\partial \epsilon} = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left(\frac{\partial \mathfrak{L}}{\partial q}\frac{\partial q}{\partial \epsilon} + \frac{\partial \mathfrak{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \epsilon} + \frac{\partial \mathfrak{L}}{\partial q'}\frac{\partial q'}{\partial \epsilon} \right) dx dt = 0 \tag{16.3.6}$$

Equations 16.3.2, 16.3.3,and 16.3.4give the partial differentials

$$\frac{\partial q}{\partial \epsilon} = \eta(x,t) \tag{16.3.7}$$

$$\frac{\partial q'}{\partial \epsilon} = \eta'(x,t) \tag{16.3.8}$$

$$\frac{\partial \dot{q}}{\partial \epsilon} = \dot{\eta}(x,t) \tag{16.3.9}$$

Integration by parts in both the x and t terms in Equation 16.3.6, plus using the fact that $\eta(x_1,t_1) = \eta(x_2,t_2) = 0$ at both end points, yields

$$\int_{t_1}^{t_2} \frac{\partial \mathfrak{L}}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial \epsilon} dt = -\int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}}\right) \frac{\partial q}{\partial \epsilon} dt \tag{16.3.10}$$

$$\int_{x_1}^{x_2} \frac{\partial \mathfrak{L}}{\partial q'} \frac{\partial q'}{\partial \epsilon} dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{\partial \mathfrak{L}}{\partial q'}\right) \frac{\partial q}{\partial \epsilon} dx \tag{16.3.11}$$

Therefore Hamilton's principle, Equation 16.3.6 becomes

$$\frac{\partial S(\epsilon)}{\partial \epsilon} = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left[\frac{\partial \mathfrak{L}}{\partial q} - \frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathfrak{L}}{\partial q'}\right)\right] \eta(x,t) dx dt = 0 \tag{16.3.12}$$

Since the auxiliary function $\eta(x,t)$ is arbitrary, then the integrand term in the square brackets of Equation 16.3.12 must equal zero. That is,

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathfrak{L}}{\partial q'}\right) - \frac{\partial \mathfrak{L}}{\partial q} = 0 \tag{16.3.13}$$

Equation 16.3.13 gives the equations of motion in terms of the Lagrangian density that has been derived based on Hamilton's principle.

Three spatial dimensions

Equation $(16.2.4)$ expresses the Lagrangian as an integral of the Lagrangian density over a single continuous index $q(x,t)$ where the Lagrangian density is a function $\mathfrak{L}(q, \frac{dq}{dt}, \frac{dq}{dx}, x, t)$. The derivation of the Lagrangian equations of motion in terms of the Lagrangian density for three spatial dimensions involves the straightforward addition of the y, and z coordinates. That is, in three dimensions the vector displacement is expressed by the vector $\mathbf{q}(x,y,z,t)$ and the Lagrangian density is related to the Lagrangian by integration over three dimensions. That is, they are related by the equation

$$L = \int \mathfrak{L}(\mathbf{q}, \frac{d\mathbf{q}}{dt}, \boldsymbol{\nabla}\cdot\mathbf{q}, x, y, z, t) d\tau \tag{16.3.14}$$

where, in cartesian coordinates, the volume element $d\tau = dxdydz$. The Lagrangian density is a function $\mathfrak{L}(\mathbf{q}, \frac{d\mathbf{q}}{dt}, \boldsymbol{\nabla}\cdot\mathbf{q}, x, y, z, t)$ where the one field quantity $q(x,t)$ has been extended to a spatial vector $\mathbf{q}(x,y,z,t)$ and the spatial derivatives q' have been transformed into $\boldsymbol{\nabla}\cdot\mathbf{q}$. Applying the method used for the one-dimensional spatial system, to the three-dimensional system, leads to the following set of equations of motion

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}}{\frac{\partial \mathbf{q}}{\partial t}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathfrak{L}}{\frac{\partial \mathbf{q}}{\partial x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathfrak{L}}{\frac{\partial \mathbf{q}}{\partial y}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathfrak{L}}{\frac{\partial \mathbf{q}}{\partial z}}\right) - \frac{\partial \mathfrak{L}}{\partial \mathbf{q}} = 0 \tag{16.3.15}$$

where the x, y, z spatial derivatives have been written explicitly for clarity.

Note that the equations of motion, Equation 16.3.15, treat the spatial and time coordinates symmetrically. This symmetry between space and time is unchanged by multiplying the spatial and time coordinate by arbitrary numerical factors. This suggests the possibility of introducing a four-dimensional coordinate system

$$\phi_u \equiv \{x, y, z, \alpha t\}$$

where the parameter α is freely chosen. Using this 4-dimensional formalism allows Equation 16.3.15 to be written more compactly as

$$\sum_{\mu}^{4} \frac{\partial}{\partial \phi_u}\left(\frac{\partial \mathfrak{L}}{\frac{\partial \mathbf{q}}{\partial \phi_u}}\right) - \frac{\partial \mathfrak{L}}{\partial \mathbf{q}} = 0 \tag{16.3.16}$$

As discussed in chapter 17, relativistic mechanics treats time and space symmetrically, that is, a four-dimensional vector $\mathbf{q}(x,y,z,t)$ can be used that treats time and the three spatial dimensions symmetrically and equally. This four-dimensional space-time formulation allows the first four terms in Equation 16.3.15 to be condensed into a single term which illustrates the symmetry underlying Equation 16.3.16. If the Lagrangian density is Lorentz invariant, and if $\alpha = ic$, then Equation 16.3.16 is covariant. Thus the Lagrangian density formulation is ideally suited to the development of relativistically covariant descriptions of fields.