# 13.16: Rotational Invariants

The scalar properties of a rotating body, such as mass $M$, Lagrangian $L$, and Hamiltonian $H$, are rotationally invariant, that is, they are the same in any body-fixed or laboratory-fixed coordinate frame. This fact also applies to scalar products of all vector observables such as angular momentum. For example the scalar product

$$\mathbf{L}\cdot\mathbf{L}=l^2$$

where $l$ is the root mean square value of the angular momentum. An example of a scalar invariant is the scalar product of the angular velocity

$$\boldsymbol{\omega}\cdot\boldsymbol{\omega}=\omega^2$$

where $\omega^2$ is the mean square angular velocity. The scalar product $\omega\cdot\omega=|\omega|^2$ can be calculated using the Euler-angle velocities for the body-fixed frame, equations $(13.14.1-13.14.3)$ to be

$$\boldsymbol{\omega}\cdot\boldsymbol{\omega}=|\omega|^2=\omega_1^2+\omega_2^2+\omega_3^2=\dot{\phi}^2+\dot{\theta}^2+\dot{\psi}^2+2\dot{\phi}\dot{\psi}\cos\theta$$

Similarly, the scalar product can be calculated using the Euler angle velocities for the space-fixed frame using equations $(13.14.4-13.14.6)$

$$\boldsymbol{\omega}\cdot\boldsymbol{\omega}=|\omega|^2=\omega_x^2+\omega_y^2+\omega_z^2=\dot{\phi}^2+\dot{\theta}^2+\dot{\psi}^2+2\dot{\phi}\dot{\psi}\cos\theta$$

This shows the obvious result that the scalar product $\omega\cdot\omega=|\omega|^2$ is invariant to rotations of the coordinate frame, that is, it is identical when evaluated in either the space-fixed, or body-fixed frames.

Note that for $\theta=0$, the $\hat{3}$ and $\hat{z}$ axes are parallel, and perpendicular to the $\hat{\theta}$ axis, then

$$|\omega|^2=\left(\dot{\phi}+\dot{\psi}\right)^2+\dot{\theta}^2$$

For the case when $\theta=180^\circ$, the $\hat{3}$ and $\hat{z}$ axes are antiparallel, and perpendicular to the $\hat{\theta}$ axis, then

$$|\omega|^2=\left(\dot{\phi}-\dot{\psi}\right)^2+\dot{\theta}^2$$

For the case when $\theta=90^\circ$, the $\hat{3}$, $\hat{z}$, and $\hat{\theta}$ axes are mutually perpendicular, that is, orthogonal, and then

$$|\omega|^2=\dot{\phi}^2+\dot{\psi}^2+\dot{\theta}^2$$

The time-averaged shape of a rapidly-rotating body, as seen in the fixed inertial frame, is very different from the actual shape of the body, and this difference depends on the rotational frequency. For example, a pencil rotating rapidly about an axis perpendicular to the body-fixed symmetry axis has an average shape that is a flat disk in the laboratory frame which bears little resemblance to a pencil. The actual shape of the pencil could be determined by taking high-speed photographs which display the instantaneous body-fixed shape of the object at given times. Unfortunately for fast rotation, such as rotation of a molecule or a nucleus, it is not possible to take photographs with sufficient speed and spatial resolution to observe the instantaneous shape of the rotating body. What is measured is the average shape of the body as seen in the fixed laboratory frame. In principle the shape observed in the fixed inertial frame can be related to the shape in the body-fixed frame, but this requires knowing the body-fixed shape which in general is not known. For example, a deformed nucleus may be both vibrating and rotating about some triaxially deformed average shape which is a function of the rotational frequency. This is not apparent from the shapes measured in the fixed frame for each of the excited states.

The fact that scalar products are rotationally invariant, provides a powerful means of transforming products of observables in the body-fixed frame, to those in the laboratory frame. In 1971 Cline developed a powerful model-independent method that utilizes rotationally-invariant products of the electromagnetic quadrupole operator $E2$ to relate the electromagnetic $E2$ properties for the observed levels of a rotating nucleus measured in the laboratory frame, to the electromagnetic $E2$ properties of the deformed rotating nucleus measured in the body-fixed frame.[Cli71, Cli72, Cli86] The method uses the fact that scalar products of the electromagnetic multipole operators are rotationally invariant. This allows transforming scalar products of a complete set of measured electromagnetic matrix elements, measured in the laboratory frame, into the electromagnetic properties in the body-fixed frame of the rotating nucleus. These rotational invariants provide a model-independent determination of the magnitude, triaxiality, and vibrational amplitudes of the average shapes in the body-fixed frame for individual observed nuclear states that may be undergoing both rotation and vibration. When the bombarding energy is below the Coulomb barrier, the scattering of a projectile nucleus by a target nucleus is due purely to the electromagnetic interaction since the distance of closest approach exceeds the range of the nuclear force. For such pure Coulomb collisions, the electromagnetic excitation of collective nuclei populates many excited states with cross sections that are a direct measure of the $E2$ matrix elements. These measured matrix elements are precisely those required to evaluate, in the laboratory frame, the $E2$ rotational invariants from which it is possible to deduce the intrinsic quadrupole shapes of the rotating-vibrating nuclear states in the body-fixed frame[Cli86].