17.3: Special Theory of Relativity

Einstein Postulates

In November 1905, at the age of 26, Einstein published a seminal paper entitled ”On the electrodynamics of moving bodies”. He considered the relation between space and time in inertial frames of reference that are in relative motion. In this paper he made the following postulates.

1. The laws of nature are the same in all inertial frames of reference.
2. The velocity of light in vacuum is the same in all inertial frames of reference.

Note that Einstein’s first postulate, coupled with Maxwell’s equations, leads to the statement that the velocity of light in vacuum is a universal constant. Thus the second postulate is unnecessary since it is an obvious consequence of the first postulate plus Maxwell’s equations which are basic laws of physics. This second postulate explained the null result of the Michelson-Morley experiment. However, it was not this experimental result that led Einstein to the theory of special relativity; he deduced the Special Theory of Relativity from consideration of Maxwell’s equations of electromagnetism. Although Einstein’s postulates appear reasonable, they lead to the following surprising implications.

Lorentz transformation

Galilean invariance leads to violation of the Einstein postulate that the velocity of light is a universal constant in all frames of reference. It is necessary to assume a new transformation law that renders physical laws relativistically invariant. Maxwell’s equations are relativistically invariant, which led to some electromagnetic phenomena that could not be explained using Galilean invariance. In 1904 Lorentz proposed a new transformation to replace the Galilean transformation in order to explain such electromagnetic phenomena. Einstein’s genius was that he derived the transformation, that had been proposed by Lorentz, directly from the postulates of the Special Theory of Relativity. The Lorentz transformation satisfies Einstein’s theory of relativity, and has been confirmed to be correct by many experiments.

For the geometry shown in Figure $17.2.1$, the **Lorentz transformations** are:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned} \tag{17.3.1}$$

where the Lorentz γ factor

$$\gamma \equiv \frac{1}{\sqrt{1-(\frac{v}{c})^2}} \tag{17.3.2}$$

The inverse transformations are

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \end{aligned} \tag{17.3.3}$$

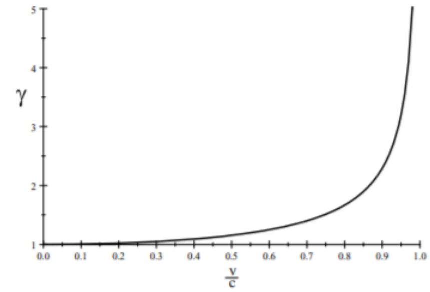

Figure 17.3.1: The dependence of the Lorentz γ factor on $\frac{v}{c}$.

The Lorentz γ factor, defined above, is the key feature differentiating the Lorentz transformations from the Galilean transformation. Note that $\gamma \geq 1$; also $\gamma \to 1.0$ as $v \to 0$ and increases to infinity as $\frac{v}{c} \to 1$ as illustrated in Figure $17.3.1$. A useful fact that will be used later is that for $\frac{v}{c} << 1$;

$$\gamma \to 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 \qquad \text{(Limit for } v << c)$$

Note that for $v << c$ then $\gamma = 1$ and the Lorentz transformation is identical to the Galilean transformation.

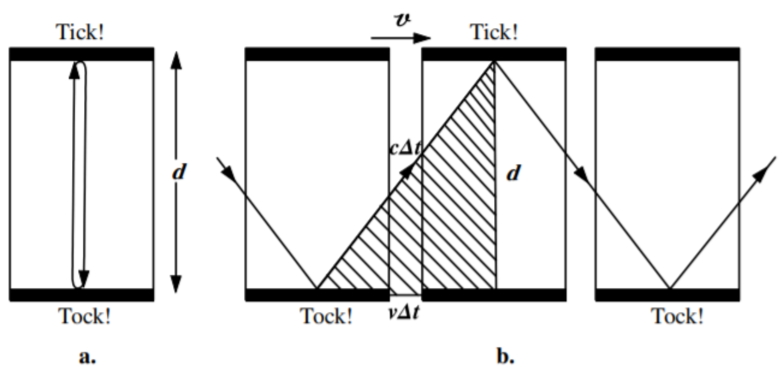

Figure 17.3.2: The observer and mirror are at rest in the left-hand frame (a). The light beam takes a time $\Delta t = \frac{d}{c}$ to travel to the mirror. In the right-hand frame (b) the source and mirror are travelling at a velocity v relative to the observer. The light travels further in the right-hand frame of reference (b) than is the stationary frame (a). Since Einstein states that the velocity of light is the same in both frames of reference then the time interval must by larger in frame (b) since the light travels further than in (a).

Time Dilation

Consider that a clock is *fixed at x'_o in a moving frame* and measures the time interval between two events in the moving frame, i.e. $\Delta t'_p = t'_1 - t'_2$. According to the Lorentz transformation, the times in the fixed frame are given by:

$$t_1 = \gamma\left(t'_1 + \frac{vx'_0}{c^2}\right) \tag{17.3.4}$$

$$t_2 = \gamma\left(t'_2 + \frac{vx'_0}{c^2}\right)$$

Thus the time interval is given by:

$$t_2 - t_1 = \gamma(t'_2 - t'_1) \tag{17.3.5}$$

The time between events in the rest frame of the clock, $\Delta\tau \equiv \Delta t'_p$ is called the *proper time* which always is the shortest time measured for a given event and is represented by the symbol τ. That is

$$\Delta t = \gamma \Delta t'_p = \gamma \Delta\tau \tag{17.3.6}$$

Note that the time interval for any other frame of reference, moving with respect to the clock frame, will show larger time intervals because $\gamma \geq 1.0$ which implies that the fixed frame perceives that the moving clock is slow by the factor γ.

The plausibility of this *time dilation* can be understood by looking at the simple geometry of the space ship example shown in Figure $17.3.2$ Pretend that the clock in the proper frame of the space ship is based on the time for the light to travel to and from the mirror in the space ship. In this proper frame the light has the shortest distance to travel, and the proper transit time is

$$\Delta\tau = \frac{2d}{c} \tag{17.3.7}$$

In the fixed frame, b, the component of velocity in the direction of the mirror is $\sqrt{c^2 - v^2}$ using the Pythagorus theorem, assuming that the light cannot travel faster the c. Thus the transit time towards and back from the mirror must be

$$\Delta t = \frac{2d}{c\sqrt{1-(\frac{v}{c})^2}} = \gamma\Delta\tau \tag{17.3.8}$$

which is the predicted time dilation.

There are many experimental verifications of time dilation in physics. For example, a stationary muon has a mean lifetime of $\tau_p = 2\ \ \mu s$, whereas the lifetime of a fast moving muon, produced in the upper atmosphere by high-energy cosmic rays, was observed in 1941 to be longer and given by $\gamma\tau_p$ as described in example $17.3.1$ In 1972 Hafely and Keating used four accurate cesium atomic clocks to confirm time dilation. Two clocks were flown on regularly scheduled airlines travelling around the World, one westward and the other eastward. The other two clocks were used for reference. The westward moving clock was slow by $(273 \pm 7)\ ns$ compared to the predicted value of $(275 \pm 10)\ ns$. The Global Positioning System of 24 geosynchronous satellites is used for locating positions to within a few meters. It has an accuracy of a few nanoseconds which requires allowance for time dilation and is a daily tribute to the correctness of Einstein's Theory of Relativity.

Length Contraction

The Lorentz transformation leads to a contraction of the apparent length of an object in a moving frame as seen from a fixed frame. The length of a ruler in its own frame of reference is called the *proper length*. Consider an accurately measured rod of known proper length $L_p = x'_2 - x'_1$ that is, at rest in the moving primed frame. The locations of both ends of this rod are measured at a *given time in the stationary frame*, $t_1 = t_2$, by taking a photograph of the moving rod. The corresponding locations in the moving frame are:

$$\begin{aligned} x'_2 &= \gamma(x_2 - vt_2) \\ x'_1 &= \gamma(x_1 - vt_1) \end{aligned} \tag{17.3.9}$$

Since $t_2 = t_1$, the measured lengths in the two frames are related by:

$$x'_2 - x'_1 = \gamma(x_2 - x_1) \tag{17.3.10}$$

That is, the lengths are related by:

$$L = \frac{1}{\gamma}L_p \tag{17.3.11}$$

Note that the moving rod appears shorter in the direction of motion. As $v \rightarrow c$ the apparent length shrinks to zero in the direction of motion while the dimensions perpendicular to the direction of motion are unchanged. This is called the *Lorentz contraction*. If you could ride your bicycle at close to the speed of light, you would observe that stationary cars, buildings, people, all would appear to be squeezed thin along the direction that you are travelling. Also objects that are further away down any side street would be distorted in the direction of travel. A photograph taken by a stationary observer would show the moving bicycle to be Lorentz contracted along the direction of travel and the stationary objects would be normal.

Simultaneity

The Lorentz transformations imply a new philosophy of space and time. A surprising consequence is that the concept of simultaneity is frame dependent in contrast to the prediction of Newtonian mechanics.

Consider that two events occur in frame S at (x_1, t_1) and (x_2, t_2). In frame S' these two events occur at (x'_1, t'_1) and (x'_2, t'_2). From the Lorentz transformation the time difference is

$$t'_2 - t'_1 = \gamma \left[(t_2 - t_1) - \frac{v(x_2 - x_1)}{c^2} \right] \tag{17.3.12}$$

If an event is simultaneous in frame S, that is $(t_2 - t_1) = 0$ then

$$t'_2 - t'_1 = \gamma \left[\frac{v(x_1 - x_2)}{c^2} \right] \tag{17.3.13}$$

Thus the event is not simultaneous in frame S' if $(x_2 - x_1) = L_p \neq 0$. That is, an event that is simultaneous in one frame is not simultaneous in the other frame if the events are spatially separated. The equivalent statement is that for two clocks, spatially separated by a distance L_p, which are synchronized in their rest frame, then in a moving frame they are not simultaneous.

Einstein discussed the problem of lightning striking both ends of a railway carriage that is moving at a velocity v. Assume that the lightning strikes both the front and rear of the carriage simultaneously, according to a stationary observer. A woman riding in the center of the train will se the lightning flash arrive from the front of the carriage before the wavefront from the rear of the carriage arrives since the carriage is moving towards the approaching wavefront and away from the wavefront from the rear of the train. If the length of the carriage is L, then the time difference between the light flash from front and rear of the carriage will be $\Delta t = \gamma L_p \frac{v}{c^2}$. As a consequence she observes that the two signals are not simultaneous. Thus a photograph of a rapidly moving body will appear to have a shorter distance. The relativistic snake discussed in chapter 17, exercise 1 is a similar example of the role of simultaneity in relativistic mechanics.

Example 17.3.1: Muon lifetime

Many people had trouble comprehending time dilation and Lorentz contraction predicted by the Special Theory of Relativity. The predictions appear to be crazy, but there are many examples where time dilation and Lorentz contraction are observed experimentally such as the decay in flight of the muon. At rest, the muon decays with a mean lifetime of 2 μs. Muons are created high in the atmosphere due to cosmic ray bombardment. A typical muon travels at $v = 0.998c$ which corresponds to $\gamma = 15$. Time dilation implies that the lifetime of the moving muon in the earth's frame of reference is 30 μs. The speed of the muon is essentially c in both frames of reference, and it would travel $600\,m$ in 2 μs and $9000\,m$ in 30 μs. In fact, it is observed that the muon does travel, on average, $9000\,m$ in the earth frame of reference before decaying. Is this inconsistent with the view of someone travelling with the muon? In the muon's moving frame, the lifetime is only 2 μs, but the Lorentz contraction of distance means that $9000\,m$ in the earth frame appears to be only $600\,m$ in the muon moving frame; a distance it travels is 2 μs. Thus in both frames of reference we have consistent explanations, that is, the muon travels the height of the mountain in one lifetime.

Example 17.3.2: Relativistic Doppler Effect

The relativistic Doppler effect is encountered frequently in physics and astronomy. Consider monochromatic electromagnetic radiation from a source, such as a star, that is moving towards the detector at a velocity v. During the time Δt in the frame of the receiver, the source emits n cycles of the sinusoidal waveform. Thus the length of this waveform, as seen by the receiver, is $n\lambda$ which equals

$$n\lambda = (c - v)\Delta t$$

The frequency as measured by the receiver is

$$\nu = \frac{c}{\lambda} = \frac{cn}{(c - v)\Delta t}$$

According to the source, it emits n waves of frequency ν_0 during the proper time interval $\Delta t'$, that is

$$n = \nu_0 \Delta t'$$

This proper time interval $\Delta t'$, in the source frame, corresponds to a time interval Δt in the receiver frame where

$$\Delta t = \gamma \Delta t'$$

Thus the frequency measured by the receiver is

$$\nu = \frac{1}{(1-\frac{v}{c})}\frac{\nu_0}{\gamma} = \frac{\sqrt{1-(\frac{v}{c})^2}}{(1-\frac{v}{c})}\nu_0 = \sqrt{\frac{1+\beta}{1-\beta}}\nu_0$$

where $\beta \equiv \frac{v}{c}$. This formula for source and receiver approaching each other also gives the correct answer for source and receiver receding if the sign of β is changed.

This relativistic Doppler Effect accounts for the red shift observed for light emitted by receding stars and galaxies, as well as many examples in atomic and nuclear physics involving moving sources of electromagnetic radiation.

Example 17.3.3: Twin paradox

A problem that troubled physicists for many years is called the twin paradox. Consider two identical twins, Jack and Jill. Assume that Jill travels in a space ship at a speed of $\gamma = 4$ for 20 years, as measured by Jack's clock, and then returns taking another 20 years, according to Jack. Thus, Jack has aged 40 years by the time his twin sister returns home. However, Jill's clock measures $20/4 = 5$ years for each half of the trip so that she thinks she travelled for 10 years total time according to her clock. Thus she has aged only 10 years on the trip, that is, now she is 30 years younger that her twin brother. Note that, according to Jill, the distance she travelled out and back was $1/4$ the distance according to Jack, so she perceives no inconsistency in her clock, and the speed of the space ship. This was called a paradox because some people claimed that Jill will perceive that the earth and Jack moved away at the same relative speed in the opposite direction and thus according to Jill, Jack should be 30 years younger, not her. Moreover, some claimed that this problem is symmetric and therefore both twins must still be the same age since there is no way of telling who was moving away from whom. This argument is incorrect because Jill was able to sense that she accelerated to $\gamma = 4$ which destroys the symmetry argument. The effect is observed with accelerated beams of unstable nuclei such as the muon and was confirmed by the results of the experiment where cesium atomic clocks were flown around the Earth. Thus the Twin paradox is not a paradox; the fact is that Jill will be younger than her twin brother.