11.4: Equations of Motion

The equations of motion for two bodies interacting via a conservative two-body central force can be determined using the center of mass Lagrangian, L_{cm}, given by equation $(11.3.3)$. For the radial coordinate, the operator equation $\Lambda_r L_{cm} = 0$ for Lagrangian mechanics leads to

$$\frac{d}{dt}(\mu\dot{r}) - \mu r\dot{\psi}^2 + \frac{\partial U}{\partial r} = 0 \tag{11.4.1}$$

But

$$\dot{\psi} = \frac{l}{\mu r^2} \tag{11.4.2}$$

therefore the radial equation of motion is

$$\mu\ddot{r} = -\frac{\partial U}{\partial r} + \frac{l^2}{\mu r^3} \tag{11.4.3}$$

Similarly, for the angular coordinate, the operator equation $\Lambda_\psi L_{cm} = 0$ leads to equation $(11.3.5)$. That is, the angular equation of motion for the magnitude of p_ψ is

$$p_\psi = \frac{\partial L}{\partial\dot{\psi}} = \mu r^2\dot{\psi} = l \tag{11.4.4}$$

Lagrange's equations have given two equations of motion, one dependent on radius r and the other on the polar angle ψ. Note that the radial acceleration is just a statement of Newton's Laws of motion for the radial force F_r in the center-of-mass system of

$$F_r = -\frac{\partial U}{\partial r} + \frac{l^2}{\mu r^3} \tag{11.4.5}$$

This can be written in terms of an effective potential

$$U_{eff}(r) \equiv U(r) + \frac{l^2}{2\mu r^2} \tag{11.4.6}$$

which leads to an equation of motion

$$F_r = \mu\ddot{r} = -\frac{\partial U_{eff}(r)}{\partial r} \tag{11.4.7}$$

Since $\frac{l^2}{\mu r^3} = \mu r\dot{\psi}^2$, the second term in Equation 11.4.6 is the usual centrifugal force that originates because the variable r is in a non-inertial, rotating frame of reference. Note that the angular equation of motion is independent of the radial dependence of the conservative two-body central force.

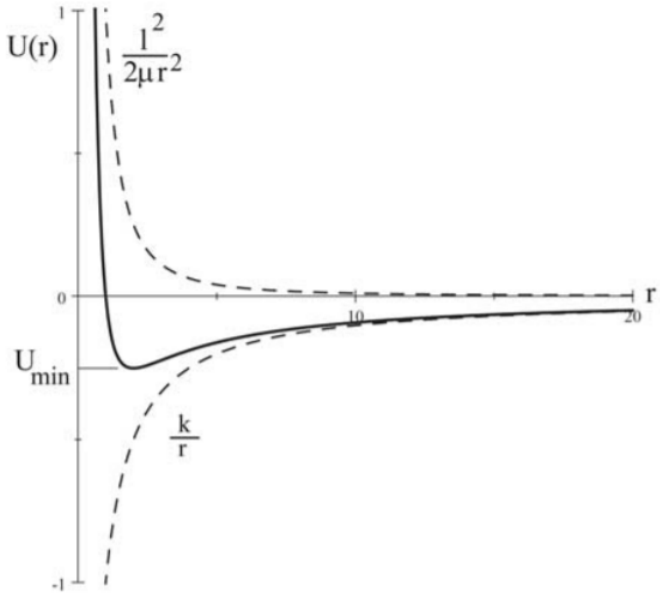

Figure 11.4.1: The attractive inverse-square law potential ($\frac{k}{r}$), the centrifugal potential ($\frac{l^2}{2\mu r^2}$), and the combined effective bound potential.

Figure $11.4.1$ shows, by dashed lines, the radial dependence of the potential corresponding to the attractive inverse square law force, that is $U = -\frac{k}{r}$, and the potential corresponding to the centrifugal term $\frac{l^2}{2\mu r^2}$ corresponding to a repulsive centrifugal force. The sum of these two potentials $U_{eff}(r)$, shown by the solid line, has a minimum $U_{\min}$ value at a certain radius similar to that manifest by the diatomic molecule discussed in example $(2.12.1)$.

It is remarkable that the six-dimensional equations of motion, for two bodies interacting via a two-body central force, has been reduced to trivial center-of-mass translational motion, plus a *one-dimensional one-body problem* given by $11.4.7$ in terms of the relative separation r and an effective potential $U_{eff}(r)$.