5.1: Introduction to the Calculus of Variations

During the 18^{th} century, Bernoulli, who was a student of Leibniz, developed the field of variational calculus which underlies the integral variational approach to mechanics. He solved the brachistochrone problem which involves finding the path for which the transit time between two points is the shortest. The integral variational approach also underlies Fermat's principle in optics, which can be used to derive that the angle of reflection equals the angle of incidence, as well as derive Snell's law. Other applications of the calculus of variations include solving the catenary problem, finding the maximum and minimum distances between two points on a surface, polygon shapes having the maximum ratio of enclosed area to perimeter, or maximizing profit in economics. Bernoulli, developed the principle of virtual work used to describe equilibrium in static systems, and d'Alembert extended the principle of virtual work to dynamical systems. Euler, the preeminent Swiss mathematician of the 18^{th} century and a student of Bernoulli, developed the calculus of variations with full mathematical rigor. The culmination of the development of the Lagrangian variational approach to classical mechanics was done by Lagrange (1736-1813), who was a student of Euler.

The Euler-Lagrangian approach to classical mechanics stems from a deep philosophical belief that the laws of nature are based on the principle of economy. That is, the physical universe follows paths through space and time that are based on extrema principles. The standard **Lagrangian** L is defined as the difference between the kinetic and potential energy, that is

$$L = T - U \tag{5.1.1}$$

Chapters 6 through 9 will show that the laws of classical mechanics can be expressed in terms of **Hamilton's variational principle** which states that the motion of the system between the initial time t_1 and final time t_2 follows a path that minimizes the scalar **action integral** S defined as the time integral of the Lagrangian.

$$S = \int_{t_1}^{t_2} L dt \tag{5.1.2}$$

The calculus of variations provides the mathematics required to determine the path that minimizes the action integral. This variational approach is both elegant and beautiful, and has withstood the rigors of experimental confirmation. In fact, not only is it an exceedingly powerful alternative approach to the intuitive Newtonian approach in classical mechanics, but Hamilton's variational principle now is recognized to be more fundamental than Newton's Laws of Motion. The Lagrangian and Hamiltonian variational approaches to mechanics are the only approaches that can handle the Theory of Relativity, statistical mechanics, and the dichotomy of philosophical approaches to quantum physics.