4.S: Nonlinear Systems and Chaos (Summary)

The study of the dynamics of non-linear systems remains a vibrant and rapidly evolving field in classical mechanics as well as many other branches of science. This chapter has discussed examples of non-linear systems in classical mechanics. It was shown that the superposition principle is broken even for weak nonlinearity. It was shown that increased nonlinearity leads to bifurcation, point attractors, limit-cycle attractors, and sensitivity to initial conditions.

Limit-cycle attractors

The Poincaré-Bendixson theorem for limit cycle attractors states that the paths, both in state-space and phase-space, can have three possible paths:

1. closed paths, like the elliptical paths for the undamped harmonic oscillator,
2. terminate at an equilibrium point as $t \rightarrow \infty$, like the point attractor for a damped harmonic oscillator,
3. tend to a limit cycle as $t \rightarrow \infty$.

The limit cycle is unusual in that the periodic motion tends asymptotically to the limit-cycle attractor independent of whether the initial values are inside or outside the limit cycle. The balance of dissipative forces and driving forces often leads to limit-cycle attractors, especially in biological applications. Identification of limit-cycle attractors, as well as the trajectories of the motion towards these limit-cycle attractors, is more complicated than for point attractors.

The van der Pol oscillator is a common example of a limit-cycle system that has an equation of motion of the form

$$\frac{d^2x}{dt^2} + \mu\left(x^2 - 1\right)\frac{dx}{dt} + \omega_0^2 x = 0 \tag{4.S.1}$$

The van der Pol oscillator has a limit-cycle attractor that includes non-linear damping and exhibits periodic solutions that asymptotically approach one attractor solution independent of the initial conditions. There are many examples in nature that exhibit similar behavior.

Harmonically-driven, linearly-damped, plane pendulum

The non-linearity of the well-known driven linearly-damped plane pendulum was used as an example of the behavior of non-linear systems in nature. It was shown that non-linearity leads to discontinuous period bifurcation, extreme sensitivity to initial conditions, rolling motion and chaos.

Differentiation between ordered and chaotic motion

Lyapunov exponents, bifurcation diagrams, and Poincaré sections were used to identify the transition from order to chaos. Chapter 16.8 discusses the non-linear Navier-Stokes equations of viscous-fluid flow which leads to complicated transitions between laminar and turbulent flow. Fluid flow exhibits remarkable complexity that nicely illustrates the dominant role that non-linearity can have on the solutions of practical non-linear systems in classical mechanics.

Wave propagation for non-linear systems

Non-linear equations can lead to unexpected behavior for wave packet propagation such as fast or slow light as well as soliton solutions. Moreover, it is notable that some non-linear systems can lead to analytic solutions.

The complicated phenomena exhibited by the above non-linear systems is not restricted to classical mechanics, rather it is a manifestation of the mathematical behavior of the solutions of the differential equations involved. That is, this behavior is a general manifestation of the behavior of solutions for second-order differential equations. Exploration of this complex motion has only become feasible with the advent of powerful computer facilities during the past three decades. The breadth of phenomena exhibited by these examples is manifest in other nonlinear systems, ranging from many-body motion, weather patterns, growth of biological species, epidemics, motion of electrons in atoms, etc. Other examples of non-linear equations of motion not discussed here, are the three-body problem, which is mentioned in chapter 11, and turbulence in fluid flow which is discussed in chapter 16.

It is stressed that the behavior discussed in this chapter is very different from the random walk problem which is a stochastic process where each step is purely random and not deterministic. This chapter has assumed that the motion is fully deterministic and rigorously follows the laws of classical mechanics. Even though the motion is fully deterministic, and follows the laws of classical mechanics, the motion is extremely sensitive to the initial conditions and the non-linearities can lead to chaos. Computer modelling is the only viable approach for predicting the behavior of such non-linear systems. The complexity of solving non-linear equations is the reason that this book will continue to consider only linear systems. Fortunately, in nature, non-linear systems can be approximately linear when the small-amplitude assumption is applicable.