13.4: Inertia Tensor

The square bracket term in $(13.3.9)$ is called the **moment of inertia tensor**, $\mathbf{I}$, which is usually referred to as the **inertia tensor**

$$I_{ij} \equiv \sum_{\alpha}^{N} m_{\alpha} \left[\delta_{ij} \left(\sum_{k}^{3} x_{\alpha,k}^{2} \right) - x_{\alpha,i} x_{\alpha,j} \right] \tag{13.4.1}$$

In most cases it is more useful to express the components of the inertia tensor in an integral form over the mass distribution rather than a summation for N discrete bodies. That is,

$$I_{ij} = \int \rho(\mathbf{r}') \left(\delta_{ij} \left(\sum_{k}^{3} x_{k}^{2} \right) - x_{i} x_{j} \right) dV \tag{13.4.2}$$

The inertia tensor is easier to understand when written in cartesian coordinates $\mathbf{r}'_{\alpha} = (x_{\alpha}, y_{\alpha}, z_{\alpha})$ rather than in the form $\mathbf{r}'_{\alpha} = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$. Then, the diagonal **moments of inertia** of the inertia tensor are

$$\begin{aligned} I_{xx} &\equiv \sum_{\alpha}^{N} m_{\alpha} \left[x_{\alpha}^{2} + y_{\alpha}^{2} + z_{\alpha}^{2} - x_{\alpha}^{2} \right] = \sum_{\alpha}^{N} m_{\alpha} \left[y_{\alpha}^{2} + z_{\alpha}^{2} \right] \\ I_{yy} &\equiv \sum_{\alpha}^{N} m_{\alpha} \left[x_{\alpha}^{2} + y_{\alpha}^{2} + z_{\alpha}^{2} - y_{\alpha}^{2} \right] = \sum_{\alpha}^{N} m_{\alpha} \left[x_{\alpha}^{2} + z_{\alpha}^{2} \right] \\ I_{zz} &\equiv \sum_{\alpha}^{N} m_{\alpha} \left[x_{\alpha}^{2} + y_{\alpha}^{2} + z_{\alpha}^{2} - z_{\alpha}^{2} \right] = \sum_{\alpha}^{N} m_{\alpha} \left[x_{\alpha}^{2} + y_{\alpha}^{2} \right] \end{aligned} \tag{13.4.3}$$

while the off-diagonal **products of inertia** are

$$\begin{aligned} I_{yx} = I_{xy} &\equiv -\sum_{\alpha}^{N} m_{\alpha} [x_{\alpha} y_{\alpha}] \\ I_{zx} = I_{xz} &\equiv -\sum_{\alpha}^{N} m_{\alpha} [x_{\alpha} z_{\alpha}] \\ I_{zy} = I_{yz} &\equiv -\sum_{\alpha}^{N} m_{\alpha} [y_{\alpha} z_{\alpha}] \end{aligned} \tag{13.4.4}$$

Note that the products of inertia are symmetric in that

$$I_{ij} = I_{ji} \tag{13.4.5}$$

The above notation for the inertia tensor allows the angular momentum 13.4.1 to be written as

$$L_{i} = \sum_{j}^{3} I_{ij} \omega_{j} \tag{13.4.6}$$

Expanded in cartesian coordinates

$$\begin{aligned} L_{x} &= I_{xx} \omega_{x} + I_{xy} \omega_{y} + I_{xz} \omega_{z} \\ L_{y} &= I_{yx} \omega_{x} + I_{yy} \omega_{y} + I_{yz} \omega_{z} \\ L_{z} &= I_{zx} \omega_{x} + I_{zy} \omega_{y} + I_{zz} \omega_{z} \end{aligned} \tag{13.4.7}$$

Note that every fixed point in a body has a specific inertia tensor. The components of the inertia tensor at a specified point depend on the orientation of the coordinate frame whose origin is located at the specified fixed point. For example, the inertia tensor for a cube is very different when the fixed point is at the center of mass compared with when the fixed point is at a corner of the cube.