16.2: The Continuous Uniform Linear Chain

The Lagrangian for the discrete lattice chain, for longitudinal modes, is given by equation $(14.10.3)$ to be

$$L=\frac{1}{2}\sum_{j=1}^{n+1}\left(m\dot{q}_j^2-\kappa(q_{j-1}-q_j)^2\right) \tag{16.2.1}$$

where the n masses are attached in series to $n+1$ identical springs of length d and spring constant κ. Assume that the spring has a uniform cross-section area A and length d. Then each spring volume element $\Delta\tau=Ad$ has a mass m, that is, the volume mass density $\rho=\frac{m}{\Delta\tau}$ or $m=\rho\Delta\tau$. Chapter 16.5 will show that the spring constant $\kappa=\frac{EA}{d}$ where E is Young's modulus, A is the cross sectional area of the chain element, and d is the length of the element. Then the spring constant can be written as $\kappa=\frac{E\Delta\tau}{d^2}$. Therefore Equation 16.2.1 can be expressed as a sum over volume elements $\Delta\tau=Ad$

$$L=\frac{1}{2}\sum_{j=1}^{n+1}\left(\rho\dot{q}_j^2-E\left(\frac{q_{j-1}-q_j}{d}\right)^2\right)\Delta\tau \tag{16.2.2}$$

In the limit that $n\rightarrow\infty$ and the spacing $d=dx\rightarrow 0$, then the summation in Equation 16.2.2 can be written as a volume integral where $x=jd$ is the distance along the linear chain and the volume element $\boldsymbol{\Delta\tau}\rightarrow$ o. Then the Lagrangian can be written as the integral over the volume element $d\tau$ rather than a summation over $\Delta\tau$. That is,

$$L=\frac{1}{2}\int\left(\rho\dot{q}^2-E\left(\frac{dq(x,t)}{dx}\right)^2\right)d\tau \tag{16.2.3}$$

The discrete-chain coordinate $q(t)$ is assumed to be a continuous function $q(x,t)$ for the uniform chain. Thus the integral form of the Lagrangian can be expressed as

$$L=\frac{1}{2}\int\left(\rho\dot{q}^2-E\left(\frac{dq(x,t)}{dx}\right)^2\right)d\tau=\int\mathfrak{L}d\tau \tag{16.2.4}$$

where the function $\mathfrak{L}$ is called the **Lagrangian density** defined by

$$\mathfrak{L}\equiv\frac{1}{2}\left(\rho\dot{q}^2-E\left(\frac{dq(x,t)}{dx}\right)^2\right) \tag{16.2.5}$$

The variable x in the Lagrangian density is not a generalized coordinate; it only serves the role of a continuous index played previously by the index j. For the discrete case, each value of j defined a different generalized coordinate q_i. Now for each value of x there is a continuous function $q(x,t)$ which is a function of both position and time.

Lagrange's equations of motion applied to the continuous Lagrangian in Equation 16.2.4 gives

$$\rho\frac{d^2q}{dt^2}-E\frac{d^2q}{dx^2}=0 \tag{16.2.6}$$

This is the familiar wave equation in one dimension for a longitudinal wave on the continuous chain with a phase velocity

$$v_{phase}=\sqrt{\frac{E}{\rho}} \tag{16.2.7}$$

The continuous linear chain also can exhibit transverse modes which have a Lagrangian density were the Young's modulus E is replaced by the tension τ in the chain, and ρ is replaced by the linear mass density μ of the chain, leading to a phase velocity for a transverse wave $v_{phase}=\sqrt{\frac{\tau}{\mu}}$.