2.E: Review of Newtonian Mechanics (Exercises)

1. Two particles are projected from the same point with velocities v_1 and v_2, at elevations α_1 and α_2, respectively $(\alpha_1 > \alpha_2)$. Show that if they are to collide in mid-air the interval between the firings must be

$$\frac{2v_1v_2\sin(\alpha_1-\alpha_2)}{g(v_1\cos\alpha_1+v_2\cos\alpha_2)}.$$

2. The teeter totter comprises two identical weights which hang on drooping arms attached to a peg as shown. The arrangement is unexpectedly stable and can be spun and rocked with little danger of toppling over.

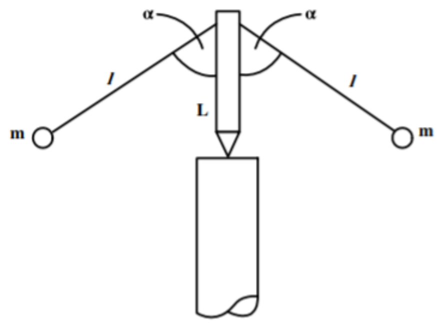

Figure $2.E.1$

a. Find an expression for the potential energy of the teeter toy as a function of θ when the teeter toy is cocked at an angle θ about the pivot point. For simplicity, consider only rocking motion in the vertical plane.
b. Determine the equilibrium values(s) of θ.
c. Determine whether the equilibrium is stable, unstable, or neutral for the value(s) of θ found in part (b).
d. How could you determine the answers to parts (b) and (c) from a graph of the potential energy versus θ?
e. Expand the expression for the potential energy about $\theta=0$ and determine the frequency of small oscillations.

3. A particle of mass m is constrained to move on the frictionless inner surface of a cone of half-angle α.

a. Find the restrictions on the initial conditions such that the particle moves in a circular orbit about the vertical axis.
b. Determine whether this kind of orbit is stable. A particle of mass m is constrained to move on the frictionless inner surface of a cone of half-angle α, as shown in the figure.

4. Consider a thin rod of length L and mass M.

a. Draw gravitational field lines and equipotential lines for the rod. What can you say about the equipotential surfaces of the rod?
b. Calculate the gravitational potential at a point P that is a distance r from one end of the rod and in a direction perpendicular to the rod.
c. Calculate the gravitational field at P by direct integration.
d. Could you have used Gauss's law to find the gravitational field at P? Why or why not?

5. Consider a single particle of mass m.

a. Determine the position r and velocity v of a particle in spherical coordinates.
b. Determine the total mechanical energy of the particle in potential V.
c. Assume the force is conservative. Show that $F=-\nabla V$. Show that it agrees with Stoke's theorem.
d. Show that the angular momentum $L=r\times p$ of the particle is conserved. Hint: $\frac{d}{dt}(A\times B)=A\times\frac{d\mathbf{B}}{dt}+\frac{d\mathbf{A}}{dt}\times B$.

6. Consider a fluid with density ρ and velocity v in some volume V. The mass current $J=\rho v$ determines the amount of mass exiting the surface per unit time by the integral $\int_S J\cdot dA$.

a. Using the divergence theorem, prove the continuity equation, $\nabla\cdot J+\frac{\partial\rho}{\partial t}=0$

7. A rocket of initial mass M burns fuel at constant rate k (kilograms per second), producing a constant force f. The total mass of available fuel is m_o. Assume the rocket starts from rest and moves in a fixed direction with no external forces acting on it.

a. Determine the equation of motion of the rocket.
b. Determine the final velocity of the rocket.
c. Determine the displacement of the rocket in time.

8. Consider a solid hemisphere of radius a. Compute the coordinates of the center of mass relative to the center of the spherical surface used to define the hemisphere.

9. A 2000 kg Ford was travelling south on Mt. Hope Avenue when it collided with your 1000 kg sports car travelling west on Elmwood Avenue. The two badly-damaged cars became entangled in the collision and leave a skid mark that is 20 meters long in a direction 14° to the west of the original direction of travel of the Excursion. The wealthy Excursion driver hires a high-powered lawyer who accuses you of speeding through the intersection. Use your P235 knowledge, plus the police officer's report of the recoil direction, the skid length, and knowledge that the coefficient of sliding friction between the tires and road is $\mu = 0.6$, to deduce the original velocities of both cars. Were either of the cars exceeding the 30 mph speed limit?

10. A particle of mass m moving in one dimension has potential energy $U(x) = U_0[2(\frac{x}{a})^2 - (\frac{x}{a})^4]$, where U_0 and a are positive constants.

a. Find the force $F(x)$ that acts on the particle.
b. Sketch $U(x)$. Find the positions of stable and unstable equilibrium.
c. What is the angular frequency ω of oscillations about the point of stable equilibrium?
d. What is the minimum speed the particle must have at the origin to escape to infinity?
e. At $t = 0$ the particle is at the origin and its velocity is positive and equal to the escape velocity. Find $x(t)$ and sketch the result.

11.

a. Consider a single-stage rocket travelling in a straight line subject to an external force F^{ext} acting along the same line where v_{ex} is the exhaust velocity of the ejected fuel relative to the rocket. Show that the equation of motion is

$$m\dot{v} = -\dot{m}v_{ex} + F^{ext}$$

b. Specialize to the case of a rocket taking off vertically from rest in a uniform gravitational field g. Assume that the rocket ejects mass at a constant rate of $\dot{m} = -k$ where k is a positive constant. Solve the equation of motion to derive the dependence of velocity on time.
c. The first couple of minutes of the launch of the Space Shuttle can be described roughly by; initial mass $= 2 \times 10^6$ kg, mass after 2 minutes = 1×10^6 kg, exhaust speed $v_{ex} = 3000$ m/s and initial velocity is zero. Estimate the velocity of the Space Shuttle after two minutes of flight.
d. Describe what would happen to a rocket where $\dot{m}v_{ex} < mg$.

12. A time independent field F is conservative if $\nabla \times F = 0$. Use this fact to test if the following fields are conservative, and derive the corresponding potential U.

a. $F_x = ayz + bx + c, F_y = axz + bz, F_z = axy + by$
b. $F_x = -ze^{-x}, F_y = \ln z, F_z = e^{-x} + \frac{y}{z}$

13. Consider a solid cylinder of mass m and radius r sliding without rolling down the smooth inclined face of a wedge of mass M that is free to slide without friction on a horizontal plane floor. Use the coordinates shown in the figure.

a. How far has the wedge moved by the time the cylinder has descended from rest a vertical distance h?
b. Now suppose that the cylinder is free to roll down the wedge without slipping. How far does the wedge move in this case if the cylinder rolls down a vertical distance h?
c. In which case does the cylinder reach the bottom faster? How does this depend on the radius of the cylinder?

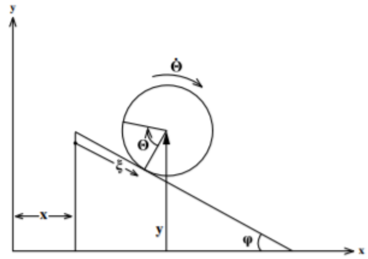

Figure $2.E.2$

14. If the gravitational field vector is independent of the radial distance within a sphere, find the function describing the mass density $\rho(r)$ of the sphere.