# 7.2: Generalized Momentum

Consider a holonomic system of $N$ masses under the influence of conservative forces that depend on position $q_j$ but not velocity $\dot{q}_j$, that is, the potential is velocity independent. Then for the $x$ coordinate of particle $i$ for $N$ particles

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_i} &= \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial U}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} \\ &= \frac{\partial}{\partial \dot{x}_i} \sum_{i=1}^{N} \frac{1}{2} m_i \left( \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right) \\ &= m_i \dot{x}_i = p_{i,x} \end{aligned} \tag{7.2.1}$$

Thus for a holonomic, conservative, velocity-independent potential we have

$$\frac{\partial L}{\partial \dot{x}_i} = p_{i,x} \tag{7.2.2}$$

which is the $x$ component of the linear momentum for the $i^{th}$ particle.

This result suggests an obvious extension of the concept of momentum to generalized coordinates. The **generalized momentum** associated with the coordinate $q_j$ is defined to be

$$\frac{\partial L}{\partial \dot{q}_j} \equiv p_j \tag{7.2.3}$$

Note that $p_j$ also is called the **conjugate momentum** or **canonical momentum** to $q_j$ where $q_j, p_j$ are conjugate, or canonical, variables. Remember that the linear momentum $p_j$ is the first-order time integral given by equation $(2.4.1)$. If $q_j$ is not a spatial coordinate, then $p_j$ is the generalized momentum, not the kinematic linear momentum. For example, if $q_j$ is an angle, then $p_j$ will be angular momentum. That is, the generalized momentum may differ from the usual linear or angular momentum since the definition 7.2.3 is more general than the usual $p_x = m\dot{x}$ definition of linear momentum in classical mechanics. This is illustrated by the case of a moving charged particles $m_j, e_j$ in an electromagnetic field. Chapter 6 showed that electromagnetic forces on a charge $e_j$ can be described in terms of a scalar potential $U_j$ where

$$U_j = e_j(\Phi - \mathbf{A} \cdot \mathbf{v}_j) \tag{7.2.4}$$

Thus the Lagrangian for the electromagnetic force can be written as

$$L = \sum_{j=1}^{N} \left[ \frac{1}{2} m_j \mathbf{v}_j \cdot \mathbf{v}_j - e_j(\Phi - \mathbf{A} \cdot \mathbf{v}_j) \right] \tag{7.2.5}$$

The generalized momentum to the coordinate $x_j$ for charge $e_j$, and mass $m_j$, is given by the above Lagrangian

$$p_{j,x} = \frac{\partial L}{\partial \dot{x}_j} = m_j \dot{x}_j + e_j A_x \tag{7.2.6}$$

Note that this includes both the mechanical linear momentum plus the correct electromagnetic momentum. The fact that the electromagnetic field carries momentum should not be a surprise since electromagnetic waves also carry energy as is illustrated by the transmission of radiant energy from the sun.

### Example 7.2.1: Feynman's angular-momentum paradox

Feynman posed the following paradox [Fey84]. A circular insulating disk, mounted on frictionless bearings, has a circular ring of total charge $q$ uniformly distributed around the perimeter of the circular disk at the radius $R$. A superconducting long solenoid of radius $s$, where $s < R$, is fixed to the disk and is mounted coaxial with the bearings. The moment of inertia of the system about the rotation axis is $I$. Initially the disk plus superconducting solenoid are stationary with a steady current producing a uniform magnetic field $B_0$ inside the solenoid. Assume that a rise in temperature of the solenoid destroys the superconductivity leading to a rapid dissipation of the electric current and resultant magnetic field. Assume that the system is free to rotate, no other forces or torques are acting on the system, and that the charge carriers in the solenoid have zero mass and thus do not contribute to the angular momentum. Does the system rotate when the current in the solenoid stops?

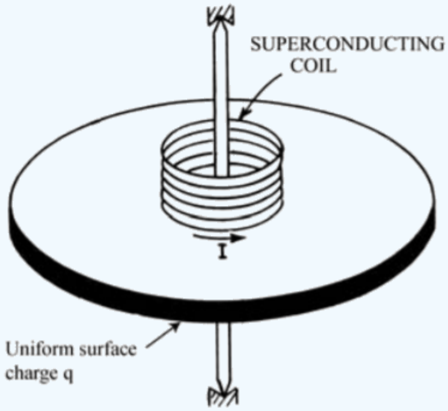


Figure 7.2.1

Initially the system is stationary with zero mechanical angular momentum. Faraday's Law states that, when the magnetic field dissipates from $B_0$ to zero, there will be a torque $\mathbf{N}$ acting on the circumferential charge $q$ at radius $R$ due to the change in magnetic flux $\Phi$.

$$\mathbf{N}(t) = -qR\frac{d\Phi}{dt}$$

Since $\frac{d\Phi}{dt} < 0$ , this torque leads to an angular impulse which will equal the final mechanical angular momentum.

$$\mathbf{L}^{MECH}_{final} = \mathbf{T} = \int_t \mathbf{N}(t)dt = qR\mathbf{\Phi}$$

The initial angular momentum in the electromagnetic field can be derived using Equation 7.2.6, plus Stoke's theorem (Appendix $19.8.3$) . Equation $2.12.56$gives that the final angular momentum equals the angular impulse

$$\mathbf{L}^{EM}_{initial} = R\int_t \oint r\dot{p}_\phi dldt = R\oint rp_\phi dl = qR\oint A_\phi dl = qR\int \mathbf{B}\cdot\mathbf{dS} = qR\mathbf{\Phi}$$

where $\Phi = \oint A_\phi dl = \int \mathbf{B}\cdot\mathbf{dS}$  is the initial total magnetic flux through the solenoid. Thus the total initial angular momentum is given by

$$\mathbf{L}^{TOTAL}_{initial} = 0 + \mathbf{L}^{EM}_{initial} = qR\mathbf{\Phi}$$

Since the final electromagnetic field is zero the final total angular momentum is given by

$$\mathbf{L}^{TOTAL}_{final} = \mathbf{L}^{MECH}_{final} + 0 = qR\mathbf{\Phi}$$

Note that the total angular momentum is conserved. That is, initially all the angular momentum is stored in the electromagnetic field, whereas the final angular momentum is all mechanical. This explains the paradox that the mechanical angular momentum is not conserved, only the total angular momentum of the system is conserved, that is, the sum of the mechanical and electromagnetic angular momenta.