6.S: Lagrangian Dynamics (Summary)

Newtonian plausibility argument for Lagrangian mechanics

A justification for introducing the calculus of variations to classical mechanics becomes apparent when the concept of the Lagrangian $L \equiv T - U$ is used in the functional and time t is the independent variable. It was shown that Newton's equation of motion can be rewritten as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_{q_i}^{EX} \tag{6.S.1}$$

where $F_{y_i}^{EX}$ are the excluded forces of constraint plus any other conservative or non-conservative forces not included in the potential U. This corresponds to the Euler-Lagrange equation for determining the minimum of the time integral of the Lagrangian.

Equation 6.S.1 can be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_k^m \lambda_k(t)\frac{\partial g_k}{\partial q_i} + F_{q_i}^{EXC} \tag{6.S.2}$$

where the Lagrange multiplier term accounts for holonomic constraint forces, and $F_{q_i}^{EXC}$ includes all additional forces not accounted for by the scalar potential U, or the Lagrange multiplier terms $F_{q_i}^{HC}$. The constraint forces can be included explicitly as generalized forces in the excluded term $F_{q_i}^{EXC}$ of Equation 6.S.2.

d'Alembert's Principle

It was shown that d'Alembert's Principle

$$\sum_i^N (\mathbf{F}_i^A - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0 \tag{6.S.3}$$

cleverly transforms the principle of virtual work from the realm of statics to dynamics. Application of virtual work to statics primarily leads to algebraic equations between the forces, whereas d'Alembert's principle applied to dynamics leads to differential equations.

Lagrange equations from d'Alembert's Principle

After transforming to generalized coordinates, d'Alembert's Principle leads to

$$\sum_j^N \left[\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right\} - Q_j\right]\delta q_j = 0 \tag{6.S.4}$$

If all the n coordinates q_j are independent, then Equation 6.S.4 implies that the term in the square brackets is zero for each individual value of j. That is, this implies the basic Euler-Lagrange equations of motion.

The handling of both conservative and non-conservative generalized forces Q_j is best achieved by assuming that the generalized force $Q_j = \sum_i^n \mathbf{F}_i^A \cdot \frac{\partial \overline{\mathbf{r}}_i}{\partial q_j}$ can be partitioned into a conservative velocity-independent term, that can be expressed in terms of the gradient of a scalar potential, $-\nabla U_i$, plus an excluded generalized force Q_j^{EX} which contains the non-conservative, velocity-dependent, and all the constraint forces not explicitly included in the potential U_j. That is,

$$Q_j = -\nabla U_j + Q_j^{EX} \tag{6.S.5}$$

Inserting 6.S.5 into 6.S.4, and *assuming that the potential U is velocity independent*, allows 6.S.4 to be rewritten as

$$\sum_j \left[\left\{\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j}\right\} - Q_j^{EX}\right]\delta q_j = 0 \tag{6.S.6}$$

Expressed in terms of the standard Lagrangian $L = T - U$ this gives

$$\sum_j^N \left[\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right\} - Q_j^{EX}\right]\delta q_j = 0 \tag{6.S.7}$$

Note that Equation 6.S.7 contains the basic Euler-Lagrange Equation 6.S.4 for the special case when $U = 0$. In addition, note that *if all the generalized coordinates are independent,* then the square bracket terms are zero for each value of j, which leads to the n *general Euler-Lagrange equations of motion*

$$\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right\} = Q_j^{EX} \tag{6.S.8}$$

where $n \geq j \geq 1$. Newtonian mechanics has trouble handling constraint forces because they lead to coupling of the degrees of freedom. Lagrangian mechanics is more powerful since it provides the following three ways to handle such correlated motion.

1) Minimal set of generalized coordinates

If the n coordinates q_j are independent, then the square bracket equals zero for each value of j in Equation 6.S.7, which corresponds to Euler's equation for each of the n independent coordinates. If the n generalized coordinates are coupled by m constraints, then the coordinates can be transformed to a minimal set of $s = n - m$ independent coordinates which then can be solved by applying Equation 6.S.8 to the minimal set of s independent coordinates.

2) Lagrange multipliers approach

The Lagrangian method concentrates solely on active forces, completely ignoring all other internal forces. In Lagrangian mechanics the generalized forces, corresponding to each generalized coordinate, can be partitioned three ways

$$Q_j = -\nabla U + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t) + Q_j^{EXC}$$

where the velocity-independent conservative forces can be absorbed into a scalar potential U, the holonomic constraint forces can be handled using the Lagrange multiplier term $\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t)$, and the remaining part of the active forces can be absorbed into the generalized force Q_j^{EXC}. The scalar potential energy U is handled by absorbing it into the standard Lagrangian $L = T - U$. If the constraint forces are holonomic then these forces are easily and elegantly handled by use of Lagrange multipliers. All remaining forces, including dissipative forces, can be handled by including them explicitly in the the generalized force Q_j^{EXC}.

Combining the above two equations gives

$$\sum_{j}^{N} \left[\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right\} - Q_j^{EXC} - \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t)\right] \delta q_j = 0 \tag{6.S.9}$$

Use of the Lagrange multipliers to handle the m constraint forces ensures that all n infinitessimals δq_j are independent implying that the expression in the square bracket must be zero for each of the n values of j. This leads to n Lagrange equations plus m constraint relations

$$\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right\} = Q_j^{EXC} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t) \tag{6.S.10}$$

where $j = 1, 2, 3, \ldots n$.

3) Generalized forces approach

The two right-hand terms in 6.S.10 can be understood to be those forces acting on the system that are not absorbed into the scalar potential U component of the Lagrangian L. The Lagrange multiplier terms $\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(\mathbf{q}, t)$ account for the holonomic forces of constraint that are not included in the conservative potential or in the generalized forces Q_j^{EXC}. The generalized force

$$Q_j^{EXC} = \sum_{i}^{n} \mathbf{F}_i^A \cdot \frac{\partial \mathbf{r}_i}{\partial p_j} \tag{6.S.11}$$

is the sum of the components in the q_j direction for all external forces that have not been taken into account by the scalar potential or the Lagrange multipliers. Thus the non-conservative generalized force Q_j^{EXC} contains non-holonomic constraint forces, including dissipative forces such as drag or friction, that are not included in U, or used in the Lagrange multiplier terms to account for the holonomic constraint forces.

Applying the Euler-Lagrange equations in mechanics:

The optimal way to exploit Lagrangian mechanics is as follows:

1. Select a set of independent generalized coordinates.
2. Partition the active forces into three groups:
   1. Conservative one-body forces
   2. Holonomic constraint forces
   3. Generalized forces
3. Minimize the number of generalized coordinates.
4. Derive the Lagrangian
5. Derive the equations of motion

Velocity-dependent Lorentz force:

Usually velocity-dependent forces are non-holonomic. However, electromagnetism is a special case where the velocity-dependent Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ can be obtained from a velocity-dependent potential function $U(q, \dot{q}, t)$. It was shown that the velocity-dependent potential

$$U = q\Phi - q\mathbf{v} \cdot \mathbf{A} \tag{6.S.12}$$

leads to the Lorentz force where Φ is the scalar electric potential and $\mathbf{A}$ the vector potential.

Time-dependent forces:

It was shown that time-dependent forces can lead to complicated motion having both stable regions and unstable regions of motion that can exhibit chaos.

Impulsive forces:

A generalized impulse $\tilde{Q}_j$ can be derived for an instantaneous impulsive force from the time integral of the impulsive forces $\mathbf{P}_i$ given by equation $(3.12.49)$using the time integral of equation $(7.2.13)$, that is

$$\Delta p_j = \tilde{Q}_j = \lim_{\tau \to 0} \int_t^{t+\tau} Q_j^{EXC} d\tau \equiv \lim_{\tau \to 0} \int_t^{t+\tau} \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} d\tau = \sum_i \tilde{\mathbf{P}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{6.S.13}$$

Note that the generalized impulse $\tilde{Q}_j$ can be a translational impulse $\tilde{\mathbf{P}}_j$ with corresponding translational variable q_j or an angular impulsive torque $\tilde{\mathbf{T}}_j$ with corresponding angular variable ϕ_j.

Comparison of Newtonian and Lagrangian mechanics:

In contrast to Newtonian mechanics, which is based on knowing all the vector forces acting on a system, Lagrangian mechanics can derive the equations of motion using generalized coordinates without requiring knowledge of the constraint forces acting on the system. Lagrangian mechanics provides a remarkably powerful, and incredibly consistent, approach to solving for the equations of motion in classical mechanics which is especially powerful for handling systems that are subject to holonomic constraints.
