9.1: Introduction to Hamilton's Action Principle

Hamilton's principle of stationary action was introduced in two papers published by Hamilton in 1834 and 1835. Hamilton's Action Principle provides the foundation for building Lagrangian mechanics that had been pioneered 46 years earlier. Hamilton's Principle now underlies theoretical physics and many other disciplines in mathematics and economics. In 1834 Hamilton was seeking a theory of optics when he developed both his principle of stationary action, plus the field of Hamiltonian mechanics.

Hamilton's Action Principle is based on defining the **action functional**[1] S for n generalized coordinates which are expressed by the vector $\mathbf{q},$ and their corresponding velocity vector $\dot{\mathbf{q}}$.

$$S=\int_{t_i}^{t_f} L(\mathbf{q},\dot{\mathbf{q}},t)dt \tag{9.1.1}$$

The scalar action S, is a functional of the Lagrangian $L(\mathbf{q},\dot{\mathbf{q}},t)$, integrated between an initial time t_i and final time t_f. In principle, higher order time derivatives of the generalized coordinates could be included, but most systems in classical mechanics are described adequately by including only the generalized coordinates, plus their velocities. The definition of the action functional allows for more general Lagrangians than the standard Lagrangian $L(\mathbf{q},\dot{\mathbf{q}},t)=T(\dot{\mathbf{q}},t)-U(\mathbf{q},t)$ that has been used throughout chapters $5-8$. Hamilton stated that the actual trajectory of a mechanical system is that given by requiring that the action functional is stationary with respect to change of the variables. The action functional is stationary when the variational principle can be written in terms of a virtual infinitessimal displacement, δ, to be

$$\delta S=\delta\int_{t_i}^{t_f} L(\mathbf{q},\dot{\mathbf{q}},t)dt=0 \tag{9.1.2}$$

Typically the stationary point corresponds to a minimum of the action functional. Applying variational calculus to the action functional leads to the same Lagrange equations of motion for systems as the equations derived using d'Alembert's Principle, if the additional generalized force terms, $\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)+Q_j^{EXC}$, are omitted in the corresponding equations of motion.

These are used to derive the equations of motion, which then are solved for an assumed set of initial conditions. Prior to Hamilton's Action Principle, Lagrange developed Lagrangian mechanics based on d'Alembert's Principle in contrast to Newtonian equations of motion which are defined in terms of Newton's Laws of Motion.

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[1]The term "action functional" was named "Hamilton's Principal Function" in older texts. The name usually is abbreviated to "action" in modern mechanics.
