6.E: Lagrangian Dynamics (Exercises)

1. A disk of mass M and radius R rolls without slipping down a plane inclined from the horizontal by an angle α. The disk has a short weightless axle of negligible radius. From this axis is suspended a simple pendulum of length $l < R$ and whose bob has a mass m. Assume that the motion of the pendulum takes place in the plane of the disk.
   a. What generalized coordinates would be appropriate for this situation?
   b. Are there any equations of constraint? If so, what are they?
   c. Find Lagrange's equations for this system.
2. A Lagrangian for a particular system can be written as

$$L = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{K}{2}(ax^2 + 2bxy + cy^2)$$

   where a, b, and c are arbitrary constants, but subject to the condition that $b^2 - 4ac \neq 0$.
   a. What are the equations of motion?
   b. Examine the case $a = 0 = c$. What physical system does this represent?
   c. Examine the case $b = 0$ and $a = -c$. What physical system does this represent?
   d. Based on your answers to (b) and (c), determine the physical system represented by the Lagrangian given above.
3. Consider a particle of mass m moving in a plane and subject to an inverse square attractive force.
   a. Obtain the equations of motion.
   b. Is the angular momentum about the origin conserved?
   c. Obtain expressions for the generalized forces. Recall that the generalized forces are defined by

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j}.$$

4. Consider a Lagrangian function of the form $L(q_i, \dot{q}_i, \ddot{q}_i, t)$. Here the Lagrangian contains a time derivative of the generalized coordinates that is higher than the first. When working with such Lagrangians, the term "generalized mechanics" is used.
   a. Consider a system with one degree of freedom. By applying the methods of the calculus of variations, and assuming that Hamilton's principle holds with respect to variations which keep both q and $\dot{q}$ fixed at the end points, show that the corresponding Lagrange equation is

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} = 0.$$

   Such equations of motion have interesting applications in chaos theory.
   b. Apply this result to the Lagrangian

$$L = -\frac{m}{2} q\ddot{q} - \frac{k}{2} q^2.$$

   Do you recognize the equations of motion?
5. A bead of mass m slides under gravity along a smooth wire bent in the shape of a parabola $x^2 = az$ in the vertical (x, z) plane.
   a. What kind (holonomic, nonholonomic, scleronomic, rheonomic) of constraint acts on m?
   b. Set up Lagrange's equation of motion for x with the constraint embedded.
   c. Set up Lagrange's equations of motion for both x and z with the constraint adjoined and a Lagrangian multiplier λ introduced.
   d. Show that the same equation of motion for x results from either of the methods used in part (b) or part (c).
   e. Express λ in terms of x and $\dot{x}$.
   f. What are the x and z components of the force of constraint in terms of x and $\dot{x}$?
6. Consider the two Lagrangians

$$L(q, \dot{q}; t) \quad \text{and} \quad L'(q, \dot{q}; t) = L(q, \dot{q}; t) + \frac{dF(q, t)}{dt}$$

where $F(q, t)$ is an arbitrary function of the generalized coordinates $q(t)$. Show that these two Lagrangians yield the same Euler-Lagrange equations. As a consequence two Lagrangians that differ only by an exact time derivative are said to be equivalent.

7. Consider the double pendulum comprising masses m_1 and m_2 connected by inextensible strings as shown in the figure. Assume that the motion of the pendulum takes place in a vertical plane.
   a. Are there any equations of constraint? If so, what are they?
   b. Find Lagrange's equations for this system.

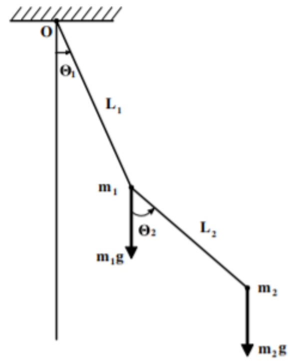

Figure $6.E.1$

8. Consider the system shown in the figure which consists of a mass m suspended via a constrained massless link of length L where the point A is acted upon by a spring of spring constant k. The spring is unstretched when the massless link is horizontal. Assume that the holonomic constraints at A and B are frictionless.
   a. Derive the equations of motion for the system using the method of Lagrange multipliers.

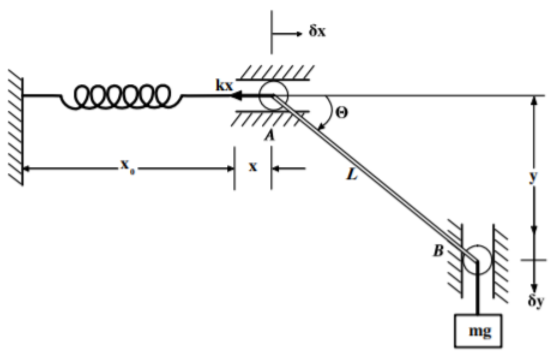

Figure $6.E.2$

9. Consider a pendulum, with mass m, connected to a (horizontally) moveable support of mass M.
   a. Determine the Lagrangian of the system.
   b. Determine the equations of motion for $\theta \ll 1$.
   c. Find an equation of motion in θ alone. What is the frequency of oscillation?
   d. What is the frequency of oscillation for $M \gg m$? Does this make sense?
10. A sphere of radius ρ is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of radius R. Determine the Lagrangian function, the equation of constraint, and the Lagrange equations of motion. Find the frequency of small oscillations.
11. A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; A and $\alpha (> 0)$ are constants. Choose generalized coordinates with the potential energy zero at the origin.
   a. Find the Lagrangian equations of motion.
   b. Is the angular momentum about the origin conserved?
   c. Is the total energy conserved?

12. Two blocks, each of mass M, are connected by an extensionless, uniform string of length l. One block is placed on a frictionless horizontal surface, and the other block hangs over the side, the string passing over a frictionless pulley. Describe the motion of the system:
    a. when the mass of the string is negligible
    b. when the string has mass m.
13. Two masses m_1 and m_2 $(m_1 \neq m_2)$ are connected by a rigid rod of length d and of negligible mass. An extensionless string of length l_1 is attached to m_1 and connected to a fixed point of the support P. Similarly a string of length l_2 $(l_1 \neq l_2)$ connects m_2 and P. Obtain the equation of motion describing the motion in the plane of m_1, m_2, and P, and find the frequency of small oscillation around the equilibrium position.
14. A thin uniform rigid rod of length $2L$ and mass M is suspended by a massless string of length l. Initially the system is hanging vertically downwards in the gravitational field g. Use as generalized coordinates the angles given in the diagram.
    a. Derive the Lagrangian for the system.
    b. Use the Lagrangian to derive the equations of motion
    c. A horizontal impulsive force F_x in the x direction strikes the bottom end of the rod for an infinitessimal time τ. Derive the initial conditions for the system immediately after the impulse has occurred.
    d. Draw a diagram showing the geometry of the pendulum shortly after the impulse when the displacement angles are significant.

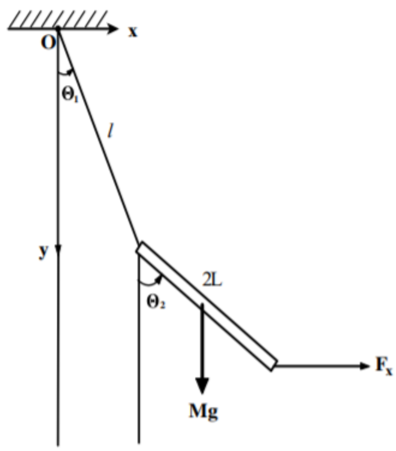

Figure $6.E.3$