12.4: Reference Frame Undergoing Rotation Plus Translation

Consider the case where the system is accelerating in translation as well as rotating, that is, the primed frame is the non-rotating translating frame. The position vector $\mathbf{r}_{fix}$ is taken with respect to the inertial fixed unprimed frame which can be written in terms of the fixed unit basis vectors $(\hat{\mathbf{i}}_{fix}, \hat{\mathbf{j}}_{fix}, \widehat{\mathbf{k}}_{fix})$. This $\mathbf{r}_{fix}$ vector can be written as the vector sum of the translational motion $\mathbf{R}_{fix}$ of the origin of the rotating system with respect to the fixed frame, plus the position $\mathbf{r}'_{mov}$ with respect to this translating primed frame basis

$$\mathbf{r}_{fix} = \mathbf{R}_{fix} + \mathbf{r}'_{mov} \tag{12.4.1}$$

The time differential is

$$\left(\frac{d\mathbf{r}}{dt}\right)_{fix} = \left(\frac{d\mathbf{R}}{dt}\right)_{fix} + \left(\frac{d\mathbf{r}'_{mov}}{dt}\right) \tag{12.4.2}$$

The vector $d\mathbf{r}'$ is the position with respect to the translating frame of reference which can be expressed in terms of the unit vectors $\left(\widehat{\mathbf{i}'}_{mov}, \widehat{\mathbf{j}'}_{mov}, \widehat{\mathbf{k}'}_{mov}\right)$.

Equation 12.4.2 takes into account the translational motion of the moving primed frame basis. Now, assuming that the double primed frame rotates about the origin of the moving primed frame, then the net displacement with respect to the original inertial frame basis can be combined with equation $(12.3.3)$ leading to the relation

$$\left(\frac{d\mathbf{r}}{dt}\right)_{fix} = \left(\frac{d\mathbf{R}}{dt}\right)_{fix} + \left(\frac{d\mathbf{r}''}{dt}\right)_{rot} + \omega \times \mathbf{r}'_{mov} \tag{12.4.3}$$

Here the double-primed frame is both rotating and translating. Vectors in this frame are expressed in terms of the unit basis vectors $\left(\widehat{\mathbf{i}''}_{rot}, \widehat{\mathbf{j}''}_{rot}, \widehat{\mathbf{k}''}_{rot}\right)$.

Expressed as velocities, Equation 12.4.3 can be written as

$$\mathbf{v}_{fix} = \mathbf{V}_{fix} + \mathbf{v}''_{rot} + \omega \times \mathbf{r}'_{mov} \tag{12.4.4}$$

where:

- $\mathbf{v}_{fix}$ is the velocity measured with respect to the inertial (unprimed) frame basis.
- $\mathbf{V}_{fix}$ is the velocity of the origin of the non-inertial translating (primed) frame basis with respect to the origin of the inertial (unprimed) frame basis.
- $\mathbf{v}''_{rot}$ is the velocity of the particle with respect to the non-inertial rotating (double-primed) frame basis the origin of which is both translating and rotating.
- $\omega \times \mathbf{r}'_{mov}$ is the motion of the rotating (double-primed) frame with respect to the linearly-translating (primed) frame basis. Thus this relation takes into account both the translational velocity plus rotation of the reference coordinate frame basis vectors.