11.5: Differential Orbit Equation

The differential orbit equation relates the shape of the orbital motion, in plane polar coordinates, to the radial dependence of the two-body central force. A **Binet coordinate transformation**, which depends on the functional form of $\mathbf{F}(\mathbf{r})$, can simplify the differential orbit equation. For the inverse-square law force, the best Binet transformed variable is u which is defined to be

$$u \equiv \frac{1}{r} \tag{11.5.1}$$

Inserting the transformed variable u into equation $(11.4.2)$ gives

$$\dot{\psi} = \frac{lu^2}{\mu} \tag{11.5.2}$$

From the definition of the new variable

$$\frac{dr}{dt} = -u^{-2}\frac{du}{dt} = -u^{-2}\frac{du}{d\psi}\dot{\psi} = -\frac{l}{\mu}\frac{du}{d\psi} \tag{11.5.3}$$

Differentiating again gives

$$\frac{d^2r}{dt^2} = -\frac{l}{\mu}\frac{d}{dt}\left(\frac{du}{d\psi}\right) = -\left(\frac{lu}{\mu}\right)^2\frac{d^2u}{d\psi^2} \tag{11.5.4}$$

Substituting these into Lagrange's radial equation of motion gives

$$\frac{d^2u}{d\psi^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(\frac{1}{u}) \tag{11.5.5}$$

Binet's *differential orbit equation* directly relates ψ and r which determines the overall shape of the orbit trajectory. This shape is crucial for understanding the orbital motion of two bodies interacting via a two-body central force. Note that for the special case of an inverse square-law force, that is where $F(\frac{1}{u}) = ku^2$, then the right-hand side of Equation 11.5.5 equals a constant $-\frac{\mu k}{l^2}$ since the orbital angular momentum is a conserved quantity.

Example 11.5.1: Central force leading to a circular orbit $r = 2R\cos\theta$

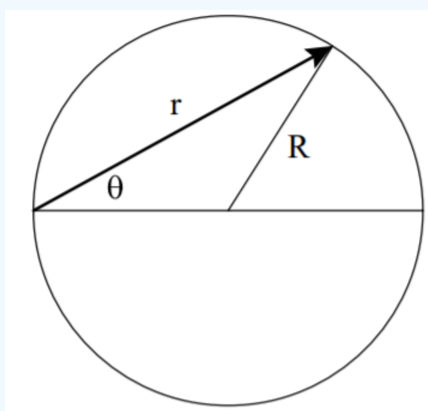

Figure 11.5.1: Circular trajectory passing through the origin of the central force.

Binet's differential orbit equation can be used to derive the central potential that leads to the assumed circular trajectory of $r = 2R\cos\theta$ where R is the radius of the circular orbit. Note that this circular orbit passes through the origin of the central force when $r = 2R\cos\theta = 0$

Inserting this trajectory into Binet's differential orbit Equation 11.5.5 gives

$$\frac{1}{2R}\frac{d^2(\cos\theta)^{-1}}{d\theta^2} + \frac{1}{2R}(\cos\theta)^{-1} = -\frac{\mu}{l^2}4R^2(\cos\theta)^2 F(\frac{1}{u}) \tag{α}$$

Note that the differential is given by

$$\frac{d^2(\cos\theta)^{-1}}{d\theta^2} = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos^3\theta}\right) = \frac{2\sin^2\theta}{\cos^3\theta} + \frac{1}{\cos\theta}$$

Inserting this differential into equation α gives

$$\frac{2\sin^2\theta}{\cos^3\theta} + \frac{1}{\cos\theta} + \frac{1}{\cos\theta} = \frac{2}{\cos^3\theta} = -\frac{\mu}{l^2}8R^3(\cos\theta)^2F(\frac{1}{u})$$

Thus the radial dependence of the required central force is

$$F = -\frac{l^2}{8R^3\mu}\frac{2}{\cos^5\theta} = -\frac{8R^2l^2}{\mu}\frac{1}{r^5} = -\frac{k}{r^5}$$

This corresponds to an attractive central force that depends to the fifth power on the inverse radius r. Note that this example is unrealistic since the assumed orbit implies that the potential and kinetic energies are infinite when $r \to 0$ at $\theta \to \frac{\pi}{2}$.