13.3: Rigid-body Rotation about a Body-Fixed Point

With respect to some point O fixed in the body coordinate system, the angular momentum of the body α is given by

$$\mathbf{L}=\sum_i^n \mathbf{L}_i=\sum_i^n \mathbf{r}_i \times \mathbf{p}_i \tag{13.3.1}$$

There are two especially convenient choices for the fixed point O. If no point in the body is fixed with respect to an inertial coordinate system, then it is best to choose O as the center of mass. If one point of the body is fixed with respect to a fixed inertial coordinate system, such as a point on the ground where a child's spinning top touches, then it is best to choose this stationary point as the body-fixed point O.

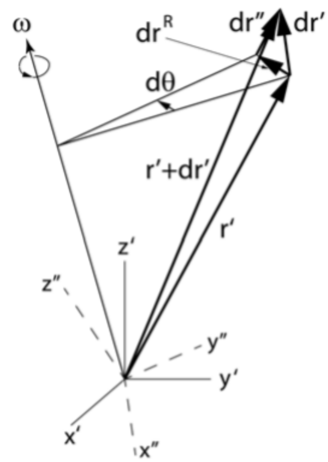

Figure 13.3.1: Infinitessimal displacement dr' in the primed frame, broken into a part dr^R due to rotation of the primed frame plus a part dr'' due to displacement with respect to this rotating frame.

Consider a rigid body composed of N particles of mass m_α where $\alpha=1,2,3,\ldots N$. As discussed in chapter 12.4, if the body rotates with an instantaneous angular velocity $\boldsymbol{\omega}$ about some fixed point, with respect to the body-fixed coordinate system, and this point has an instantaneous translational velocity $\mathbf{V}$ with respect to the fixed (inertial) coordinate system, see Figure $13.3.1$, then the instantaneous velocity $\mathbf{v}_\alpha$ of the α^{th} particle in the fixed frame of reference is given by

$$\mathbf{v}_\alpha=\mathbf{V}+\mathbf{v}_\alpha''+\boldsymbol{\omega}\times\mathbf{r}_\alpha' \tag{13.3.2}$$

However, for a rigid body, the velocity of a body-fixed point with respect to the body is zero, that is $\mathbf{v}_\alpha''=0$, thus

$$\mathbf{v}_\alpha=\mathbf{V}+\boldsymbol{\omega}\times\mathbf{r}_\alpha' \tag{13.3.3}$$

Consider the translational velocity of the body-fixed point O to be zero, i.e. $\mathbf{V}=0$ and let $\mathbf{R}=0$, then $\mathbf{r}_\alpha=\mathbf{r}_\alpha'$. These assumptions allow the linear momentum of the particle α to be written as

$$\mathbf{p}_\alpha=m_\alpha\mathbf{v}_\alpha=m_\alpha\boldsymbol{\omega}\times\mathbf{r}_\alpha \tag{13.3.4}$$

Therefore

$$\mathbf{L}=\sum_\alpha^N \mathbf{r}_\alpha\times\mathbf{p}_\alpha=\sum_\alpha^N m_\alpha\mathbf{r}_\alpha\times(\boldsymbol{\omega}\times\mathbf{r}\alpha) \tag{13.3.5}$$

Using the vector identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{A})=A^2\mathbf{B}-\mathbf{A}(\mathbf{A}\cdot\mathbf{B})$$

leads to

$$\mathbf{L}=\sum_\alpha^N m_\alpha[r_\alpha^2\boldsymbol{\omega}-\mathbf{r}_\alpha(\mathbf{r}_\alpha\cdot\boldsymbol{\omega})] \tag{13.3.6}$$

The angular momentum can be expressed in terms of components of $\boldsymbol{\omega}$ and $\mathbf{r}'_\alpha$ relative to the body-fixed frame. The following formulae can be written more compactly if $\mathbf{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$, in the rotating body-fixed frame, is written in the form $\mathbf{r}_\alpha = (x_{\alpha,1}, y_{\alpha,2}, z_{\alpha,3})$ where the axes are defined by the numbers $1, 2, 3$ rather than x, y, z. In this notation, the angular momentum is written in component form as

$$L_i = \sum_\alpha^N m_\alpha \left[\omega_i \sum_k x^2_{\alpha,k} - x_{\alpha,i} \left(\sum_j x_{\alpha,j}\omega_j \right) \right] \tag{13.3.7}$$

Assume the Kronecker delta relation

$$\omega_i = \sum_j^3 \omega_j \delta_{ij} \tag{13.3.8}$$

where

$$\begin{aligned} \delta_{ij} = & \quad 1 \qquad i = j \\ \delta_{ij} = & \quad 0 \qquad i \neq j \end{aligned}$$

Substitute 13.3.8 in 13.3.7 gives

$$\begin{aligned} L_i &= \sum_\alpha^N m_\alpha \sum_j \left[\omega_j \delta_{ij} \sum_k x^2_{\alpha,k} - \omega_j x_{\alpha,i} x_{\alpha,j} \right] \\ &= \sum_j^3 \omega_j \left[\sum_\alpha^N m_\alpha \left(\delta_{ij} \sum_k x^2_{\alpha,k} - x_{\alpha,i} x_{\alpha,j} \right) \right] \end{aligned} \tag{13.3.9}$$