15.6: Canonical Perturbation Theory

Most examples in classical mechanics discussed so far have been capable of exact solutions. In real life, the majority of problems cannot be solved exactly. For example, in celestial mechanics the two-body Kepler problem can be solved exactly, but solution of the three-body problem is intractable. Typical systems in celestial mechanics are never as simple as the two-body Kepler system because of the influence of additional bodies. Fortunately in most cases the influence of additional bodies is sufficiently small to allow use of perturbation theory. That is, the restricted three-body approximation can be employed for which the system is reduced to considering it as an exactly solvable two-body problem, subject to a small perturbation to this solvable two-body system. Note that even though the change in the Hamiltonian due to the perturbing term may be small, the impact on the motion can be especially large near a resonance.

Consider the Hamiltonian, subject to a time-dependent perturbation, is written as

$$H(q,p,t) = H_0(q,p,t) + \Delta H(q,p,t)$$

where $H_0(q,p,t)$ designates the unperturbed Hamiltonian and $\Delta H(q,p,t)$ designates the perturbing term. For the unperturbed system the Hamilton-Jacobi equation is given by

$$\mathcal{H}(Q_i,P_i,t) = H_0(q_1,\dots q_n; \frac{\partial S}{\partial q_1}\dots, \frac{\partial S}{\partial q_n}; t) + \frac{\partial S}{\partial t} = 0 \tag{15.6.1}$$

where $S(q_i,P_i,t)$ is the generating function for the canonical transformation $(q,p)\rightarrow(Q,P)$. The perturbed $S(q_i,P_i,t)$ remains a canonical transformation, but the transformed Hamiltonian $\mathcal{H}(Q_i,P_i,t)\neq 0$. That is,

$$\mathcal{H}(Q_i,P_i,t) = H_0 + \Delta H(q,p,t) + \frac{\partial S}{\partial t} = \Delta H(q,p,t) \tag{15.6.2}$$

The equations of motion satisfied by the transformed variables now are

$$\begin{aligned} \dot{Q}_i &= \frac{\partial \Delta H}{\partial P_i} \\ \dot{P}_i &= \frac{\partial \Delta H}{\partial Q_i} \end{aligned} \tag{15.6.3}$$

These equations remain as difficult to solve as the full Hamiltonian. However, the perturbation technique assumes that ΔH is small, and that one can neglect the change of (Q_i,P_i) over the perturbing interval. Therefore, to a first approximation, the unperturbed values of $\frac{\partial \Delta H}{\partial P_i}$ and $\frac{\partial \Delta H}{\partial Q_i}$ can be used in equations 15.6.3. A detailed explanation of canonical perturbation theory is presented in chapter 12 of Goldstein[Go50].

Example 15.6.1: Harmonic oscillator perturbation

(a) Consider first the Hamilton-Jacobi equation for the generating function $S(q,\alpha,t)$ for the case of a single free particle subject to the Hamiltonian $H=\frac{1}{2}p^2$. Find the canonical transformation $q=q(\beta,\alpha)$ and $p=p(\beta,\alpha)$ where β and α are the transformed coordinate and momentum respectively.

The Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(q,p,t) = 0$$

Using $p=\frac{\partial S}{\partial q}$ in the Hamiltonian $H=\frac{1}{2}p^2$ gives

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 = 0$$

Since H does not depend on q,t explicitly, then the two terms on the left hand side of the equation can be set equal to $-\gamma,\gamma$ respectively, where γ is at most a function of p. Then the generating function is

$$S = \sqrt{2\gamma}q - \gamma t$$

Set $\alpha = \sqrt{2\gamma}$ then the generating function can be written as

$$S = \alpha q - \frac{1}{2}\alpha^2 t$$

The constant α can be identified with the new momentum P. Then the transformation equations become

$$p = \frac{\partial S}{\partial q} = \alpha \quad Q = \frac{\partial S}{\partial P} = \frac{\partial S}{\partial \alpha} = q - \alpha t = \beta$$

That is

$$q = \beta + \alpha t$$

which corresponds to motion with a uniform velocity α in the q, p system.

(b) Consider that the Hamiltonian is perturbed by addition of potential $U = \frac{q^2}{2}$ which corresponds to the harmonic oscillator. Then

$$H = \frac{1}{2}p^2 + \frac{q^2}{2}$$

Consider the transformed Hamiltonian

$$\mathcal{H} = H + \frac{\partial S}{\partial t} = \frac{1}{2}p^2 + \frac{q^2}{2} - \frac{\alpha^2}{2} = \frac{q^2}{2} = \frac{1}{2}(\beta + \alpha t)^2$$

Hamilton's equations of motion

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} \quad \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}$$

give that

$$\dot{\beta} = (\beta + \alpha t)t$$

$$\dot{\alpha} = -(\beta + \alpha t)$$

These two equations can be solved to give

$$\ddot{\alpha} + \alpha = 0$$

which is the equation of a harmonic oscillator showing that α is harmonic of the form $\alpha = \alpha_0 \sin(t + \delta)$ where α_0, δ are constants of motion. Thus

$$\beta = -\dot{\alpha} - t = -\alpha_0[\cos(t + \delta) + t\sin(t + \delta)]$$

The transformation equations then give

$$p = \alpha = \alpha_0 \sin(t + \delta)$$

$$q = \beta + \alpha t = -\dot{\alpha} = -\alpha_0 \cos(t + \delta)$$

Hence the solution for the perturbed system is harmonic, which is to be expected since the potential has a quadratic dependence of position.

Example 15.6.2: Lindblad resonance in planetary and galactic motion

Use of canonical perturbation theory in celestial mechanics has been exploited by Professor Alice Quillen and her group. They combine use of action-angle variables and Hamilton-Jacobi theory to investigate the role of Lindblad resonance to planetary motion, and also for stellar motion in galaxies. A Lindblad resonance is an orbital resonance in which the orbital period of a celestial body is a simple multiple of some forcing frequency. Even for very weak perturbing forces, such resonance behavior can lead to orbit capture and chaotic motion.

For planetary motion the planet masses are about $1/1000$that of the central star, so the perturbations to Kepler orbits are small. However, Lindblad resonance for planetary motion led to Saturn's rings which result from perturbations produced by the moons of Saturn that skulpt and clear dust rings. Stellar orbits in disk galaxies are perturbed a few percent by non axially-symmetric galactic features such as spiral arms or bars. Lindblad resonances perturb stellar motion and drive spiral density waves at distances from the center of a galactic disk where the natural frequency of the radial component of a star's orbital velocity is close to the frequency of the fluctuations in the gravitational field due to passage through spiral arms or bars. If a stars orbital speed around a galactic center is greater than that of the part of a spiral arm through which it is traversing, then an inner Lindblad resonance occurs which speeds up the star's orbital speed moving the orbit outwards. If the orbital speed is less than that of a spiral arm, an inner Lindblad resonance occurs causing inward movement of the orbit.