3.11: Wave Propagation

Wave motion typically involves a packet of waves encompassing a finite number of wave cycles. Information in a wave only can be transmitted by starting, stopping, or modulating the amplitude of a wave train, which is equivalent to forming a wave packet. For example, a musician will play a note for a finite time, and this wave train propagates out as a wave packet of finite length. You have no information as to the frequency and amplitude of the sound prior to the wave packet reaching you, or after the wave packet has passed you. The velocity of the wavelets contained within the wave packet is called the **phase velocity**. For a dispersive system the phase velocity of the wavelets contained within the wave packet is frequency dependent and the shape of the wave packet travels at the **group velocity** which usually differs from the phase velocity. If the shape of the wave packet is time dependent, then neither the phase velocity, which is the velocity of the wavelets, nor the group velocity, which is the velocity of an instantaneous point fixed to the shape of the wave packet envelope, represent the actual velocity of the overall wavepacket.

A third wavepacket velocity, the **signal velocity**, is defined to be the velocity of the leading edge of the energy distribution, and corresponding information content, of the wave packet. For most linear systems the shape of the wave packet is not time dependent and then the group and signal velocities are identical. However, the group and signal velocities can be very different for non-linear systems as discussed in chapter 4.7. Note that even when the phase velocity of the waves within the wave packet travels faster than the group velocity of the shape, or the signal velocity of the energy content of the envelope of the wave packet, the information contained in a wave packet is only manifest when the wave packet envelope reaches the detector and this energy and information travel at the signal velocity.

The modern ideas of wave propagation, including Hamilton's concept of group velocity, were developed by Lord Rayleigh when applied to the theory of sound[Ray1887]. The concept of phase, group, and signal velocities played a major role in discussion of electromagnetic waves as well as de Broglie's development of the concept of wave-particle duality and the development of wave mechanics by Schrödinger.

Phase, group, and signal velocities of wave packets

The concepts of wave packets, as well as their phase, group, and signal velocities, are of considerable importance for propagation of information and other manifestations of wave motion in science and engineering which warrants further discussion at this juncture.

Consider a particular k, ω, component of a one-dimensional wave,

$$q(x,t) = Ee^{i(kx \pm \omega t)} \tag{3.11.1}$$

The argument of the exponential is called the **phase** ϕ of the wave where

$$\phi \equiv kx - \omega t \tag{3.11.2}$$

If we move along the x axis at a velocity such that the phase is constant then we perceive a stationary wave. The velocity of this wave is called the **phase velocity**. To ensure constant phase we require that ϕ is constant or, assuming real k and ω

$$\omega dt = kdx \tag{3.11.3}$$

Therefore the **phase velocity** is defined to be

$$v_{phase} = \frac{\omega}{k} \tag{3.11.4}$$

The velocity we have used so far is just the phase velocity of the individual wavelets at the carrier frequency. If k or ω are complex then one must take the real parts to ensure that the velocity is real.

If the phase velocity of a wave is dependent on the wavelength, that is, $v_{phase}(k)$, then the system is said to be dispersive in that the wave is dispersed according the wavelength. The simplest illustration of dispersion is the refraction of light in glass prism which leads to dispersion of the light into the spectrum of wavelengths. Dispersion leads to development of wave packets that travel at group and signal velocities that usually differ from the phase velocity. To illustrate this consider two equal amplitude travelling waves having slightly different wave number k and angular frequency ω. Superposition of these waves gives

$$\begin{aligned} q(x,t) &= A(e^{i[kx-\omega t]} + e^{i[(k+\Delta k)x-(\omega+\Delta\omega)t]}) \\ &= Ae^{i[(k+\frac{\Delta k}{2})x-(\omega+\frac{\Delta w}{2})t]} \cdot \{e^{-i[\frac{\Delta k}{2}x-\frac{\Delta\omega}{2}t]} + e^{i[\frac{\Delta k}{2}x-\frac{\Delta\omega}{2}t]}\} \\ &= 2Ae^{i[(k+\frac{\Delta k}{2})x-(\omega+\frac{\Delta\omega}{2})t]}\cos[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t] \end{aligned} \tag{3.11.5}$$

This corresponds to a wave with the average carrier frequency modulated by the cosine term which has a wavenumber of $\frac{\Delta k}{2}$ and angular frequency $\frac{\Delta\omega}{2}$, that is, this is the usual example of beats. The cosine term modulates the average wave producing wave packets as shown in figure (3.9.1). The velocity of these wave packets is called the **group velocity** given by requiring that the phase of the modulating term is constant, that is

$$\frac{\Delta k}{2} dx = \frac{\Delta \omega}{2} dt \tag{3.11.6}$$

Thus the **group velocity** is given by

$$v_{group} = \frac{dx}{dt} = \frac{\Delta \omega}{\Delta k} \tag{3.11.7}$$

If dispersion is present then the group velocity $v_{group} = \frac{\Delta\omega}{\Delta k}$ does not equal the phase velocity $v_{phase} = \frac{\omega}{k}$.

Expanding the above example to superposition of n waves gives

$$q(x,t) = \sum_{r=1}^{n} A_r e^{i(k_r x \pm \omega_r t)} \tag{3.11.8}$$

In the event that $n \to \infty$ and the frequencies are continuously distributed, then the summation is replaced by an integral

$$q(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx \pm \omega t)} dk \tag{3.11.9}$$

where the factor $A(k)$ represents the distribution amplitudes of the component waves, that is the spectral decomposition of the wave. This is the usual Fourier decomposition of the spatial distribution of the wave.

Consider an extension of the linear superposition of two waves to a well defined wave packet where the amplitude is nonzero only for a small range of wavenumbers $k_0 \pm \Delta k$.

$$q(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} A(k) e^{i(kx-\omega t)} dk \tag{3.11.10}$$

This functional shape is called a wave packet which only has meaning if $\Delta k << k_0$. The angular frequency can be expressed by making a Taylor expansion around k_0

$$\omega(k) = \omega(k_0) + \left(\frac{d\omega}{dk}\right)_{k_0} (k - k_0) + \dots \tag{3.11.11}$$

For a linear system the phase then reduces to

$$kx - \omega t = (k_0 x - \omega_0 t) + (k - k_0)x - \left(\frac{d\omega}{dk}\right)_{k_0} (k - k_0)t \tag{3.11.12}$$

The summation of terms in the exponent given by 3.11.13leads to the amplitude 3.11.11having the form of a product where the integral becomes

$$q(x,t) = e^{i(k_0 x - \omega_0 t)} \int_{k_0-\Delta k}^{k_0+\Delta k} A(k) e^{i(k-k_0)[x - (\frac{d\omega}{dk})_{k_0} t]} dk \tag{3.11.13}$$

The integral term modulates the $e^{i(k_0 x - \omega_0 t)}$ first term.

The group velocity is defined to be that for which the phase of the exponential term in the integral is constant. Thus

$$v_{group} = \left(\frac{d\omega}{dk}\right)_{k_0} \tag{3.11.14}$$

Since $\omega = k v_{phase}$ then

$$v_{group} = v_{phase} + k \frac{\partial v_{phase}}{\partial k} \tag{3.11.15}$$

For non-dispersive systems the phase velocity is independent of the wave number k or angular frequency ω and thus $v_{group} = v_{phase}$. The case discussed earlier, equation (3.9.3), for beating of two waves gives the same relation in the limit that $\Delta\omega$ and Δk are infinitessimal.

The group velocity of a wave packet is of physical significance for dispersive media where $v_{group} = (\frac{d\omega}{dk})_{k_0} \neq \frac{\omega}{k} = v_{phase}$. Every wave train has a finite extent and thus we usually observe the motion of a group of waves rather than the wavelets moving within the wave packet. In general, for non-linear dispersive systems the derivative $\frac{\partial v_{phase}}{\partial k}$ can be either positive or negative and thus in principle the group velocity can either be greater than, or less than, the phase velocity. Moreover, if the group velocity is frequency dependent, that is, when group velocity dispersion occurs, then the overall shape of the wave packet is time dependent and thus the speed of a specific relative location defined by the shape of the envelope of the wave packet does not represent the signal velocity of the wave packet. Brillouin showed that the distribution of the energy, and corresponding information content, in any wave packet travels at the signal velocity which can be different from the group velocity if the shape of the envelope of the wave packet is time dependent. For electromagnetic waves one has the possibility that the group velocity $v_{group} > v_{phase} = c$. In 1914 Brillouin[Bri14][Bri60] showed that the signal velocity of electromagnetic waves, defined by the leading edge of the time-dependent envelope of the wave packet, never exceeds c even though the group velocity corresponding to the velocity of the instantaneous shape of the wave packet may exceed c. Thus, there is no violation of Einstein's fundamental principle of relativity that the velocity of an electromagnetic wave cannot exceed c.

Example 3.11.1: Water waves breaking on a beach

The concepts of phase and group velocity are illustrated by the example of water waves moving at velocity v incident upon a straight beach at an angle α to the shoreline. Consider that the wavepacket comprises many wavelengths of wavelength λ. During the time it takes the wave to travel a distance λ, the point where the crest of one wave breaks on the beach travels a distance $\frac{\lambda}{\cos\alpha}$ along beach. Thus the phase velocity of the crest of the one wavelet in the wave packet is

$$v_{phase} = \frac{v}{\cos\alpha}$$

The velocity of the wave packet along the beach equals

$$v_{group} = v\cos\alpha$$

Note that for the wave moving parallel to the beach $\alpha = 0$ and $v_{phase} = v_{group} = v$. However, for $\alpha = \frac{\pi}{2} v_{phase} \to \infty$ and $v_{group} \to 0$. In general for waves breaking on the beach

$$v_{phase} v_{group} = v^2$$

The same behavior is exhibited by surface waves bouncing off the sides of the Erie canal, sound waves in a trombone, and electromagnetic waves transmitted down a rectangular wave guide. In the latter case the phase velocity exceeds the velocity of light c in apparent violation of Einstein's theory of relativity. However, the information travels at the signal velocity which is less than c.

Example 3.11.2: Surface waves for deep water

In the "Theory of Sound" Rayleigh discusses the example of surface waves for water where he derives a dispersion relation for the phase velocity v_{phase} and wavenumber k which are related to the density ρ, depth l, gravity g, and surface tension T, by

$$\omega^2 = gk + \frac{Tk^3}{\rho}\tanh(kl)$$

For deep water where the wavelength is short compared with the depth, that is $kl >> 1$, then $\tanh(kl) \to 1$ and the dispersion relation is given approximately by

$$\omega^2 = gk + \frac{Tk^3}{\rho}$$

For long surface waves for deep water, that is, small k, then the gravitational first term in the dispersion relation dominates and the group velocity is given by

$$v_{group} = \left(\frac{d\omega}{dk}\right) = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\frac{\omega}{k} = \frac{v_{phase}}{2}$$

That is, the group velocity is half of the phase velocity. Here the wavelets are building at the back of the wave packet, progress through the wave packet and dissipate at the front. This can be demonstrated by dropping a pebble into a calm lake. It will be seen that the surface disturbance comprises a wave packet moving outwards at the group velocity with the individual waves within the wave packet expanding at twice the group velocity of the wavepacket, that is, they appear at the inner radius of the wave packet and disappear at the outer radius of the wave packet.

For small wavelength ripples, where k is large, then the surface tension term dominates and the dispersion relation is approximately given by

$$\omega^2 \simeq \frac{Tk^3}{\rho}$$

leading to a group velocity of

$$v_{group} = \left(\frac{d\omega}{dk}\right) = \frac{3}{2} v_{phase}$$

Here the group velocity exceeds the phase velocity and wavelets are building at the front of the wave packet and dissipate at the back. Note that for this linear system the Brillion signal velocity equals the group velocity for both gravity and surface tension waves for deep water.

Example 3.11.3: Electromagnetic waves in ionosphere

The response to radio waves of the free electron plasma in the ionosphere provides an excellent example that involves cut-off frequency, complex wavenumber k, as well as the phase, group, and signal velocities.

Maxwell's equations give the most general wave equation for electromagnetic waves to be

$$\nabla^2 \mathbf{E} - \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu \frac{\partial \mathbf{j}_{free}}{\partial t} + \nabla \cdot \left(\frac{\rho_{free}}{\varepsilon}\right)$$

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = -\nabla \times \mathbf{j}_{free}$$

where ρ_{free} and $\mathbf{j}_{free}$ are the unbound charge and current densities. The effect of the bound charges and currents are absorbed into ε and μ. Ohm's Law can be written in terms of the electrical conductivity σ which is a constant

$$\mathbf{j} = \sigma \mathbf{E}$$

Assuming Ohm's Law plus assuming $\rho_{free} = 0$, in the plasma gives the relations

$$\nabla^2 \mathbf{E} - \varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0$$

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t} = 0$$

The third term in both of these wave equations is a damping term that leads to a damped solution of an electromagnetic wave in a good conductor.

The solution of these damped wave equations can be solved by considering an incident wave

$$\mathbf{E} = E_o \hat{\mathbf{x}} e^{i(\omega t - kz)}$$

Substituting for $\mathbf{E}$ in the first damped wave equation gives

$$-k^2 + \omega^2 \varepsilon\mu - i\omega\sigma\mu = 0$$

That is

$$k^2 = \omega^2 \varepsilon\mu \left[1 - \frac{i\sigma}{\omega\varepsilon}\right]$$

In general k is complex, that is, it has real k_R and imaginary k_1 parts that lead to a solution of the form

$$\mathbf{E} = E_o e^{-k_I z} e^{i(\omega t - k_R z)}$$

The first exponential term is an exponential damping term while the second exponential term is the oscillating term.

Consider that the plasma involves the motion of a bound damped electron, of charge q of mass m, bound in a one dimensional atom or lattice subject to an oscillatory electric field of frequency ω. Assume that the electromagnetic wave is travelling in the $\hat{z}$ direction with the transverse electric field in the $\hat{x}$ direction. The equation of motion of an electron can be written as

$$\ddot{\mathbf{x}} + \Gamma \dot{\mathbf{x}} + \omega_0^2 x = \hat{\mathbf{x}} q E_0 e^{i(\omega t - kz)}$$

where Γ is the damping factor. The instantaneous displacement of the oscillating charge equals

$$\mathbf{x} = \frac{q}{m} \frac{1}{(\omega_0^2 - \omega^2) + i\Gamma\omega} \hat{\mathbf{x}} E_0 e^{i(\omega t - kz)}$$

and the velocity is

$$\dot{\mathbf{x}} = \frac{q}{m}\frac{i\omega}{(\omega_0^2-\omega^2)+i\Gamma\omega}\hat{\mathbf{x}}E_0e^{i(\omega t-kz)}$$

Thus the instantaneous current density is given by

$$\mathbf{j} = Nq\dot{\mathbf{x}} = \frac{Nq^2}{m}\frac{i\omega}{(\omega_0^2-\omega^2)+i\Gamma\omega}\hat{\mathbf{x}}E_0e^{i(\omega t-kz)}$$

therefore the electrical conductivity is given by

$$\sigma = \frac{Nq^2}{m}\frac{i\omega}{(\omega_0^2-\omega^2)+i\Gamma\omega}$$

Let us consider only unbound charges in the plasma, that is let $\omega_0 = 0$. Then the conductivity is given by

$$\sigma = \frac{Nq^2}{m}\frac{i\omega}{i\Gamma\omega-\omega^2}$$

For a low density ionized plasma $\omega >> \Gamma$ thus the conductivity is given approximately by

$$\sigma \approx -i\frac{Nq^2}{m\omega}$$

Since σ is pure imaginary, then $\mathbf{j}$ and $\mathbf{E}$ have a phase difference of $\frac{\pi}{2}$ which implies that the average of the Joule heating over a complete period is $\langle\mathbf{j}\cdot\mathbf{E}\rangle = 0$. Thus there is no energy loss due to Joule heating implying that the electromagnetic energy is conserved.

Substitution of σ into the relation for k^2

$$k^2 = \omega^2\varepsilon\mu\left[1-\frac{i\sigma}{\omega\varepsilon}\right] = \omega^2\varepsilon\mu\left[1-\frac{Nq^2}{\varepsilon m\omega^2}\right]$$

Define the Plasma oscillation frequency ω_P to be

$$\omega_P \equiv \sqrt{\frac{Nq^2}{\varepsilon m}}$$

then k^2 can be written as

$$k^2 = \omega^2\varepsilon\mu\left[1-\left(\frac{\omega_P}{\omega}\right)^2\right] \qquad (\alpha)$$

For a low density plasma the dielectric constant $\kappa_E \simeq 1$ and the relative permeability $\kappa_B \simeq 1$ and thus $\varepsilon = \kappa_E\varepsilon_0 \simeq \varepsilon_0$ and $\mu = \kappa_B\mu_0 \simeq \mu_0$. The velocity of light in vacuum $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$. Thus for low density equation α can be written as

$$\omega^2 = \omega_p^2 + c^2k^2 \qquad (\beta)$$

Differentiation of equation β with respect to k gives $2\omega\frac{d\omega}{dk} = 2c^2k$. That is, $v_{phase}v_{group} = c^2$ and the phase velocity is

$$v_{phase} = \sqrt{c^2+\frac{\omega_p^2}{k^2}}$$

There are three cases to consider.

1) $\omega > \omega_P$: For this case $\left[1-(\frac{\omega_P}{\omega})^2\right] > 1$ and thus k is a pure real number. Therefore the electromagnetic wave is transmitted with a phase velocity that exceeds c while the group velocity is less than c.

2) $\omega < \omega_P$: For this case $\left[1-(\frac{\omega_P}{\omega})^2\right] < 1$ and thus k is a pure imaginary number. Therefore the electromagnetic wave is not transmitted and in the ionosphere it is attenuated rapidly as $e^{-(\frac{\omega_P}{c})z}$. However, since there are no Joule heating losses then the electromagnetic wave must be complete reflected. Thus the Plasma oscillation frequency serves as a cut-off frequency. For this example the signal and group velocities are identical.

For the ionosphere $N = 10^{-11}$ electrons/m^3, which corresponds to a Plasma oscillation frequency of $v = \omega_P/2\pi = 3\ MHz$. Thus electromagnetic waves in the AM waveband (< 1.6 MHz) are totally reflected by the ionosphere and bounce repeatedly around the Earth, whereas for VHF frequencies above 3 MHz, the waves are transmitted and refracted passing through the atmosphere. Thus light is transmitted by the ionosphere. By contrast, for a good conductor like silver, the Plasma oscillation frequency is around 10^{16} Hz

which is in the far ultraviolet part of the spectrum. Thus, all lower frequencies, such as light, are totally reflected by such a good conductor, whereas X-rays have frequencies above the Plasma oscillation frequency and are transmitted.

Fourier transform of wave packets

The relation between the time distribution and the corresponding frequency distribution, or equivalently, the spatial distribution and the corresponding wave-number distribution, are of considerable importance in discussion of wave packets and signal processing. It directly relates to the uncertainty principle that is a characteristic of all forms of wave motion. The relation between the time and corresponding frequency distribution is given via the Fourier transform discussed in appendix 19.9. The following are two examples of the Fourier transforms of typical but rather different wavepacket shapes that are encountered frequently in science and engineering.

Example $3.11.4$: Fourier transform of a Gaussian wave packet

Assuming that the amplitude of the wave is a Gaussian wave packet shown in the adjacent figure where

$$G(\omega) = ce^{-\frac{(\omega-\omega_0)^2}{2\sigma_\omega^2}}$$

This leads to the Fourier transform

$$f(t) = c\sqrt{2\pi}\sigma_\omega e^{-\frac{\sigma_\omega^2 t^2}{2}}\cos(\omega_0 t)$$

Note that the wavepacket has a standard deviation for the amplitude of the wavepacket of $\sigma_t = \frac{1}{\sigma_\omega}$, that is $\sigma_t \cdot \sigma_\omega = 1$. The Gaussian wavepacket results in the minimum product of the standard deviations of the frequency and time representations for a wavepacket. This has profound importance for all wave phenomena, and especially to quantum mechanics. Because matter exhibits wave-like behavior, the above property of wave packet leads to Heisenberg’s Uncertainty Principle. For signal processing, it shows that if you truncate a wavepacket you will broaden the frequency distribution.

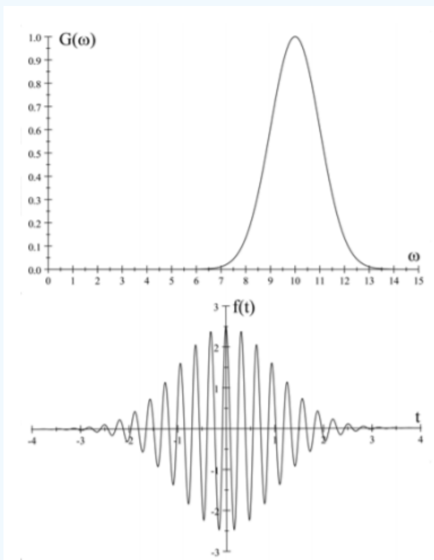

Figure $3.11.1$: Fourier transform of a Gaussian frequency distribution.

Example $3.11.5$: Fourier transform of a rectangular wave packet

Assume unity amplitude of the frequency distribution between $\omega_0 - \Delta\omega \leq \omega \leq \omega_0 + \Delta\omega$, that is, a single isolated square pulse of width τ that is described by the rectangular function $\prod$ defined as

$$\prod(\omega) = \begin{cases} 1 & |\omega-\omega_0| < \Delta\omega \\ 0 & |\omega-\omega_0| > \Delta\omega \end{cases}$$

Then the Fourier transform us given by

$$f(t) = \left[\frac{\sin \Delta\omega t}{\Delta\omega t}\right]\cos\omega_0 t$$

That is, the transform of a rectangular wavepacket gives a cosine wave modulated by an unnormalized sinc function which is a nice example of a simple wave packet. That is, on the right hand side we have a wavepacket $\Delta t = \pm \frac{2\pi}{\Delta\omega}$ wide. Note that the product of the two measures of the widths $\Delta\omega \cdot \Delta t = \pm\pi$. Example $I.2$ considers a rectangular pulse of unity amplitude between $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ which resulted in a Fourier transform $G(\omega) = \tau\left(\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right)$. That is, for a pulse of width $\Delta t = \pm\frac{\tau}{2}$ the frequency envelope has the first zero at $\Delta\omega = \pm\frac{\pi}{\tau}$. Note that this is the complementary system to the one considered here which has $\Delta\omega \cdot \Delta t = \pm\pi$ illustrating the symmetry of the Fourier transform and its inverse.

Wave-packet Uncertainty Principle

The Uncertainty Principle states that for all types of wave motion there is a minimum product of the uncertainty in the width of a wave packet and the distribution width of the frequency decomposition of the wave packet. This was illustrated by the Fourier transforms of wave packets discussed above where it was shown the product of the widths is minimized for a Gaussian-shaped wave packet. The Uncertainty Principle implies that to make a precise measurement of the frequency of a sinusoidal wave requires that the wave packet be infinitely long. If the length of the wave packet is reduced then the frequency distribution broadens. Then the crucial aspect needed for this discussion, is that, for the *amplitudes* of any wavepacket, the *standard deviations* $\sigma(t) = \sqrt{\langle t^2\rangle - \langle t\rangle^2}$ characterizing the width of the spectral distribution in the angular frequency domain, $\sigma_A(\omega)$, and the width in time $\sigma_A(t)$ are related:

$$\sigma_A(t) \cdot \sigma_A(\omega) \geqslant 1 \qquad \text{(Relation between amplitude uncertainties.)}$$

This product of the *standard deviations equals unity only for the special case of Gaussian-shaped spectral distributions, and is greater than unity for all other shaped spectral distributions.*

The *intensity* of the wave is the square of the amplitude leading to standard deviation widths for a Gaussian distribution where $\sigma_I(t)^2 = \frac{1}{2}\sigma_A(t)^2$, that is, $\sigma_I(t) = \frac{\sigma_A(t)}{\sqrt{2}}$. Thus the standard deviations for the spectral distribution and width of the intensity of the wavepacket are related by:

$$\sigma_I(t) \cdot \sigma_I(\omega) \geqslant \frac{1}{2} \qquad \text{(Uncertainty principle for frequency-time intensities)}$$

This states that the uncertainties with which you can simultaneously measure the time and frequency for the intensity of a given wavepacket are related. If you try to measure the frequency within a short time interval $\sigma_I(t)$ then the uncertainty in the frequency measurement $\sigma_I(\omega) \geqslant \frac{1}{2\sigma_I(t)}$. Accurate measurement of the frequency requires measurement times that encompass many cycles of oscillation, that is, a long wavepacket.

Exactly the same relations exist between the spectral distribution as a function of wavenumber k_x and the spatial dependence of a wave x which are conjugate representations. Thus the spectral distribution plotted versus k_x is directly related to the amplitude as a function of position x; the spectral distribution versus k_y is related to the amplitude as a function of y; and the k_z spectral distribution is related to the spatial dependence on z. Following the same arguments discussed above, the standard deviation, $\sigma_I(k_x)$ characterizing the width of the *spectral intensity* distribution of k_x, and the standard deviation $\sigma_I(x)$, characterizing the spatial width of the wave packet intensity as a function of x, are related by the Uncertainty Principle for position-wavenumber. Thus in summary the uncertainty principle for the intensity of wave motion is,

$$\sigma_I(t) \cdot \sigma_I(\omega) \geqslant \frac{1}{2} \tag{3.11.16}$$
$$\sigma_I(x) \cdot \sigma_I(k_x) \geqslant \frac{1}{2} \quad \sigma_I(y) \cdot \sigma_I(k_y) \geqslant \frac{1}{2} \quad \sigma_I(z) \cdot \sigma_I(k_z) \geqslant \frac{1}{2}$$

This *applies to all forms of wave motion*, be they, sound waves, water waves, electromagnetic waves, or matter waves.

As discussed in chapter 18, the transition to quantum mechanics involves relating the matter-wave properties to the energy and momentum of the corresponding particle. That is, in the case of matter waves, multiplying both sides of Equation 3.11.16by $\hbar$ and using the de Broglie relations gives that the particle energy is related to the angular frequency by $E = \hbar\omega$ and the particle momentum is related to the wavenumber, that is $\mathbf{p} = \hbar\mathbf{k}$. These lead to the **Heisenberg Uncertainty Principle**:

$$\sigma_I(t) \cdot \sigma_I(E) \geqslant \frac{\hbar}{2} \tag{3.11.17}$$
$$\sigma_I(x) \cdot \sigma_I(p_x) \geqslant \frac{\hbar}{2} \quad \sigma_I(y) \cdot \sigma_I(p_y) \geqslant \frac{\hbar}{2} \quad \sigma_I(z) \cdot \sigma_I(p_z) \geqslant \frac{\hbar}{2}$$

This uncertainty principle applies equally to the wavefunction of the electron in the hydrogen atom, proton in a nucleus, as well as to a wavepacket describing a particle wave moving along some trajectory. Thus, this implies that, for a particle of given momentum, the wavefunction is spread out spatially. Planck's constant $\hbar = 1.05410^{-34} J \cdot s = 6.58210^{-16} eV \cdot s$ is extremely small compared with energies and times encountered in normal life, and thus the effects due to the Uncertainty Principle are not manifest for macroscopic dimensions.

Confinement of a particle, of mass m, within $\pm\sigma(x)$ of a fixed location implies that there is a corresponding uncertainty in the momentum

$$\sigma(p_x) \geq \frac{\hbar}{2\sigma(x)} \tag{3.11.18}$$

Now the variance in momentum $\mathbf{p}$ is given by the difference in the average of the square $\langle(\mathbf{p}\cdot\mathbf{p})^2\rangle$, and the square of the average of $\langle\mathbf{p}\rangle^2$. That is

$$\sigma(\mathbf{p})^2 = \langle(\mathbf{p}\cdot\mathbf{p})^2\rangle - \langle\mathbf{p}\rangle^2 \tag{3.11.19}$$

Assuming a fixed average location implies that $\langle\mathbf{p}\rangle = 0$, then

$$\langle(\mathbf{p}\cdot\mathbf{p})^2\rangle = \sigma(p)^2 \geq \left(\frac{\hbar}{2\sigma(r)}\right)^2 \tag{3.11.20}$$

Since the kinetic energy is given by:

$$\text{Kinetic energy } = \frac{p^2}{2m} \geq \frac{\hbar^2}{8m\sigma(r)^2} \tag{Zero-point energy}$$

This zero-point energy is the minimum kinetic energy that a particle of mass m can have if confined within a distance $\pm\sigma(r)$. This zero-point energy is a consequence of wave-particle duality and the uncertainty between the size and wavenumber for any wave packet. It is a quantal effect in that the classical limit has $\hbar \rightarrow 0$ for which the zero-point energy $\rightarrow 0$.

Inserting numbers for the zero-point energy gives that an electron confined to the radius of the atom, that is $\sigma(x) = 10^{-10} m$, has a zero-point kinetic energy of $\sim 1\, eV$. Confining this electron to $3 \times 10^{-15} m$, the size of a nucleus, gives a zero-point energy of $10^9\ eV (1\ GeV)$. Confining a proton to the size of the nucleus gives a zero-point energy of 0.5 MeV. These values are typical of the level spacing observed in atomic and nuclear physics. If $\hbar$ was a large number, then a billiard ball confined to a billiard table would be a blur as it oscillated with the minimum zero-point kinetic energy. The smaller the spatial region that the ball was confined, the larger would be its zero-point energy and momentum causing it to rattle back and forth between the boundaries of the confined region. Life would be dramatically different if $\hbar$ was a large number.

In summary, Heisenberg's Uncertainty Principle is a well-known and crucially important aspect of quantum physics. What is less well known, is that the Uncertainty Principle exists for all forms of wave motion, that is, it is not restricted to matter waves. The following three examples illustrate application of the Uncertainty Principle to acoustics, the nuclear Mössbauer effect, and quantum mechanics.

Example 3.11.6: Acoustic Wave Packet

A violinist plays the note middle C (261.625 Hz) with constant intensity for precisely 2 seconds. Using the fact that the velocity of sound in air is 343.2 m/s calculate the following:

1. The wavelength of the sound wave in air: λ = 343.2/261.625 = 1.312 m.
2. The length of the wavepacket in air: Wavepacket length = 343.2 × 2 = 686.4 m
3. The fractional frequency width of the note: Since the wave packet has a square pulse shape of length $\tau = 2s$, then the Fourier transform is a sinc function having the first zeros when sin $\frac{\omega\tau}{2} = 0$, that is, $\Delta\nu = \frac{1}{\tau}$.

Therefore the fractional width is $\frac{\Delta\nu}{\nu} = \frac{1}{\nu\tau}$ = 0.0019. Note that to achieve a purity of $\frac{\Delta\nu}{\nu} = 10^{-6}$ the violinist would have to play the note for 1.06 hours.

Example 3.11.7: Gravitational Red Shift

The Mössbauer effect in nuclear physics provides a wave packet that has an exceptionally small fractional width in frequency. For example, the ^{57}Fe nucleus emits a 14.4 keV deexcitation-energy photon which corresponds to $\omega \approx 2 \times 10^{25}\ rad/s$ that has a decay time of $\tau \approx 10^{-7}\ s$. Thus the fractional width is $\frac{\Delta\omega}{\omega} \approx 3 \times 10^{-18}$. In 1959 Pound and Rebka used this to test Einstein's general theory of relativity by measurement of the gravitational red shift between the attic and basement of the 22.5 m high physics building at Harvard. The magnitude of the predicted relativistic red shift is $\frac{\Delta E}{E} = 2.5 \times 10^{-15}$ which is what was observed with a fractional precision of about 1%.

Example 3.11.8: Quantum Baseball

George Gamow, in his book "Mr. Tompkins in Wonderland", describes the strange world that would exist if $\hbar$ was a large number. As an example, consider you play baseball in a universe where $\hbar$ is a large number. The pitcher throws a 150 g ball 20 m to the batter at a speed of 40 m/s. For a strike to be thrown, the ball's position must be pitched within the 30 cm radius of the strike zone, that is, it is required that $\Delta x \leq 0.3\, m$. The uncertainty relation tells us that the transverse velocity of the ball cannot be less than $\Delta v = \frac{\hbar}{2m\Delta x}$. The time of flight of the ball from the mound to batter is $t = 0.5\ s$. Because of the transverse velocity uncertainty, Δv, the ball will deviate $t\Delta v$ transversely from the strike zone. This also must not exceed the size of the strike zone, that is;

$$t\Delta v = \frac{\hbar t}{2m\Delta x} \leq 0.3m \qquad \text{(Due to transverse velocity uncertainty)}$$

Combining both of these requirements gives

$$\hbar \leq \frac{2m\Delta x^2}{t} = 5.4\ 10^{-2} J \cdot s.$$

This is 32 orders of magnitude larger than $\hbar$ so quantal effects are negligible. However, if $\hbar$ exceeded the above value, then the pitcher would have difficulty throwing a reliable strike.