6.10: Velocity-dependent Lorentz force

The Lorentz force in electromagnetism is unusual in that it is a velocity-dependent force, as well as being a conservative force that can be treated using the concept of potential. That is, the Lorentz force is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{6.10.1}$$

It is interesting to use Maxwell's equations and Lagrangian mechanics to show that the Lorentz force can be represented by a conservative potential in Lagrangian mechanics.

Maxwell's equations can be written as

$$\begin{aligned} \nabla \cdot \mathbf{E} = & \quad \frac{\rho}{\varepsilon_0} \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = & \quad 0 \\ \nabla \cdot \mathbf{B} = & \quad 0 \\ \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = & \quad \mathbf{J} \end{aligned} \tag{6.10.2}$$

Since $\nabla \cdot \mathbf{B} = 0$ then it follows from Appendix 19.8 that $\mathbf{B}$ can be represented by the curl of a vector potential, $\mathbf{A},$ that is

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6.10.3}$$

Substituting this into $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$ gives that

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \nabla \times \mathbf{A}}{\partial t} = & \quad 0 \\ \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = & \quad 0 \end{aligned} \tag{6.10.4}$$

Since this curl is zero it can be represented by the gradient of a scalar potential U

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla U \tag{6.10.5}$$

The following shows that this relation corresponds to taking the gradient of a potential U for the charge q where the potential U is given by the relation

$$U = q(\Phi - \mathbf{A} \cdot \mathbf{v}) \tag{6.10.6}$$

where Φ is the scalar electrostatic potential. This scalar potential U can be employed in the Lagrange equations using the Lagrangian

$$L = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} - q(\Phi - \mathbf{A} \cdot \mathbf{v}) \tag{6.10.7}$$

The Lorentz force can be derived from this Lagrangian by considering the Lagrange equation for the cartesian coordinate x

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{6.10.8}$$

Using the above Lagrangian 6.10.7 gives

$$m\ddot{x} + q \left[\frac{dA_x}{dt} + \frac{\partial \Phi}{\partial x} - \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{v} \right] = 0 \tag{6.10.9}$$

But

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_x}{\partial y} \dot{y} + \frac{\partial A_x}{\partial z} \dot{z} \tag{6.10.10}$$

and

$$\frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{v} = \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z} \tag{6.10.11}$$

Inserting equations 6.10.10 and 6.10.11 into 6.10.9 gives

$$F_x = m\ddot{x} = q\left[\left(-\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\dot{y} - \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\dot{z}\right] = q\left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right]_x \tag{6.10.12}$$

Corresponding expressions can be obtained for F_y and F_z. Thus the total force is the well-known Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{6.10.13}$$

This has demonstrated that the electromagnetic scalar potential

$$U = q(\Phi - \mathbf{A} \cdot \mathbf{v}) \tag{6.10.14}$$

satisfies Maxwell's equations, gives the Lorentz force, and it can be absorbed into the Lagrangian. Note that the velocity-dependent Lorentz force is conservative since $\mathbf{E}$ is conservative, and because $(\mathbf{v} \times \mathbf{B} \times \mathbf{v})dt = 0$, therefore the magnetic force does no work since it is perpendicular to the trajectory. The velocity-dependent conservative Lorentz force is an important and ubiquitous force that features prominently in many branches of science. It will be discussed further for the case of relativistic motion in chapter 16.6.