12.13: Foucault pendulum

A classic example of motion in non-inertial frames is the rotation of the **Foucault pendulum** on the surface of the earth. The Foucault pendulum is a spherical pendulum with a long suspension that oscillates in the $x-y$ plane with sufficiently small amplitude that the vertical velocity $\dot{z}$ is negligible.

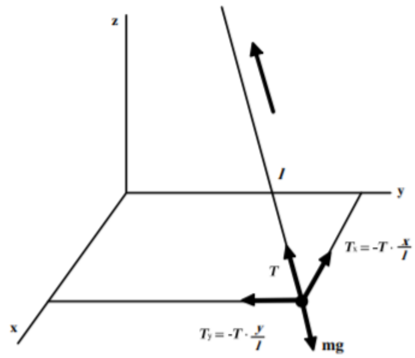

Figure 12.13.1: Foucault pendulum.

Assume that the pendulum is a simple pendulum of length l and mass m as shown in Figure $12.13.1$. The equation of motion is given by

$$\ddot{\mathbf{r}} = \mathbf{g} + \frac{\mathbf{T}}{m} - 2\boldsymbol{\Omega} \times \dot{\mathbf{r}} \tag{12.13.1}$$

where $\frac{T}{m}$ is the acceleration produced by the tension in the pendulum suspension and the rotation vector of the earth is designated by $\boldsymbol{\Omega}$ to avoid confusion with the oscillation frequency of the pendulum ω. The effective gravitational acceleration $\mathbf{g}$ is given by

$$\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\Omega} \times [\boldsymbol{\Omega} \times (\mathbf{r} + \mathbf{R})] \tag{12.13.2}$$

that is, the true gravitational field $\mathbf{g}_0$ corrected for the centrifugal force.

Assume the small angle approximation for the pendulum deflection angle β, then $T_z = T\cos\beta \simeq T$ and $T_z = mg$, thus $T \simeq mg$. Then has shown in Figure $12.13.1$, the horizontal components of the restoring force are

$$T_z = -mg\frac{x}{l} \tag{12.13.3}$$

$$T_y = -mg\frac{y}{l} \tag{12.13.4}$$

Since $\mathbf{g}$ is vertical, and neglecting terms involving $\dot{z}$, then evaluating the cross product in Equation 12.13.2simplifies to

$$\ddot{x} = -g\frac{x}{l} + 2\dot{y}\Omega\cos\theta \tag{12.13.5}$$

$$\ddot{y} = -g\frac{y}{l} + 2\dot{x}\Omega\cos\theta \tag{12.13.6}$$

where θ is the colatitude which is related to the latitude λ by

$$\cos\theta = \sin\lambda \tag{12.13.7}$$

The natural angular frequency of the simple pendulum is

$$\omega_0 = \sqrt{\frac{g}{l}} \tag{12.13.8}$$

while the z component of the earth’s angular velocity is

$$\Omega_z = \Omega\ cos\theta \tag{12.13.9}$$

Thus equations 12.13.5and 12.13.6can be written as

$$\begin{aligned}\ddot{x}-2\Omega_z\dot{y}+\omega_0^2x&=0\\ \ddot{y}-2\Omega_z\dot{x}+\omega_0^2y&=0\end{aligned}\tag{12.13.10}$$

These are two coupled equations that can be solved by making a coordinate transformation.

Define a new coordinate that is a complex number

$$\eta=x+iy\tag{12.13.11}$$

Multiply the second of the coupled equations 12.13.10by i and add to the first equation gives

$$(\ddot{x}+i\ddot{y})+2i\Omega_z(\dot{x}+i\dot{y})+\omega_0^2(x+iy)=0$$

which can be written as a differential equation for η

$$\ddot{\eta}+2i\Omega_z\dot{\eta}+\omega_0^2\eta=0\tag{12.13.12}$$

Note that the complex number η contains the same information regarding the position in the $x-y$ plane as equations 12.13.10. The plot of η in the complex plane, the Argand diagram, is a birds-eye view of the position coordinates (x,y) of the pendulum. This second-order homogeneous differential equation has two independent solutions that can be derived by guessing a solution of the form

$$\eta(t)=A_e^{-i\alpha t}\tag{12.13.13}$$

Substituting Equation 12.13.13into 12.13.12gives that

$$\alpha^2-2\Omega_z\alpha-\omega_o^2=0$$

That is

$$\alpha=\Omega_z\pm\sqrt{\Omega_z^2+\omega_0^2}\tag{12.13.14}$$

If the angular velocity of the pendulum $\omega_0\gg\Omega$, then

$$\alpha\simeq\Omega_Z\pm\omega_0\tag{12.13.15}$$

Thus the solution is of the form

$$\eta(t)=e^{-i\Omega_zt}\left(A_+e^{i\omega_0t}+A_-e^{i\omega_0t}\right)\tag{12.13.16}$$

This can be written as

$$\eta(t)=Ae^{-i\Omega_zt}\cos(\omega_0t+\delta)\tag{12.13.17}$$

where the phase δ and amplitude A depend on the initial conditions. Thus the plane of oscillation of the pendulum is defined by the ratio of the x and y coordinates, that is the phase angle $i\Omega_zt$. This phase angle rotates with angular velocity Ω_z where

$$\Omega_z=\Omega\cos\theta=\Omega\sin\lambda\tag{12.13.18}$$

At the north pole the earth rotates under the pendulum with angular velocity Ω and the axis of the pendulum is fixed in an inertial frame of reference. At lower latitudes, the pendulum precesses at the lower angular frequency $\Omega_z=\Omega\sin\lambda$ that goes to zero at the equator. For example, in Rochester, NY, $\lambda=43^\circ N$, and therefore a Foucault pendulum precesses at $\Omega_Z=0.682\Omega$. That is, the pendulum precesses 245.5°/day.