2.4: First-order Integrals in Newtonian mechanics

A fundamental goal of mechanics is to determine the equations of motion for an $n-$body system, where the force $\mathbf{F}_i$ acts on the individual mass m_i where $1 \leq i \leq n$. Newton's second-order equation of motion, equation $(2.2.6)$ must be solved to calculate the instantaneous spatial locations, velocities, and accelerations for each mass m_i of an n-body system. Both $\mathbf{F}_i$ and $\ddot{\mathbf{r}}_i$ are vectors each having three orthogonal components. The solution of equation $(2.2.6)$ involves integrating second-order equations of motion subject to a set of initial conditions. Although this task appears simple in principle, it can be exceedingly complicated for many-body systems. Fortunately, solution of the motion often can be simplified by exploiting three first-order integrals of Newton's equations of motion, that relate directly to conservation of either the linear momentum, angular momentum, or energy of the system. In addition, for the special case of these three first-order integrals, the internal motion of any many-body system can be factored out by a simple transformations into the center of mass of the system. As a consequence, the following three first-order integrals are exploited extensively in classical mechanics.

Linear Momentum

Newton's Laws can be written as the differential and integral forms of the first-order time integral which equals the change in linear momentum.

$$\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt} \qquad\qquad \int_1^2 \mathbf{F}_i dt = \int_1^2 \frac{d\mathbf{p}_i}{dt} dt = (\mathbf{p}_2 - \mathbf{p}_1)_i \tag{2.4.1}$$

This allows Newton's law of motion to be expressed directly in terms of the linear momentum $\mathbf{p}_i = m_i \dot{\mathbf{r}}_i$ of each of the $1 < i < n$ bodies in the system. This first-order time integral features prominently in classical mechanics since it connects to the important concept of linear momentum $\mathbf{p}$. This first-order time integral gives that the total linear momentum is a constant of motion when the sum of the external forces is zero.

Angular Momentum

The angular momentum $\mathbf{L}_i$ of a particle i with linear momentum $\mathbf{p}_i$ with respect to an origin from which the position vector $\mathbf{r}_i$ is measured, is defined by

$$\mathbf{L}_i \equiv \mathbf{r}_i \times \mathbf{p}_i \tag{2.4.2}$$

The torque, or moment of the force $\mathbf{N}_i$ with respect to the same origin is defined to be

$$\mathbf{N}_i \equiv \mathbf{r}_i \times \mathbf{F}_i \tag{2.4.3}$$

where $\mathbf{r}_i$ is the position vector from the origin to the point where the force $\mathbf{F}_i$ is applied. Note that the torque $\mathbf{N}_i$ can be written as

$$\mathbf{N}_i \equiv \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \tag{2.4.4}$$

Consider the time differential of the angular momentum, $\frac{d\mathbf{L}_i}{dt}$

$$\frac{d\mathbf{L}_i}{dt} = \frac{d}{dt}(\mathbf{r}_i \times \mathbf{p}_i) = \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \tag{2.4.5}$$

However,

$$\frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i = m\frac{d\mathbf{r}_i}{dt} \times \frac{d\mathbf{r}_i}{dt} = 0 \tag{2.4.6}$$

Equations 2.4.4 – 2.4.6 can be used to write the first-order time integral for angular momentum in either differential or integral form as

$$\frac{d\mathbf{L}_i}{dt} = \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} = \mathbf{N}_i \qquad\qquad \int_1^2 \mathbf{N}_i dt = \int_1^2 \frac{d\mathbf{L}_i}{dt} dt = (\mathbf{L}_2 - \mathbf{L}_1)_i \tag{2.4.7}$$

Newton's Law relates torque and angular momentum about the same axis. When the torque about any axis is zero then angular momentum about that axis is a constant of motion. If the total torque is zero then the total angular momentum, as well as the components about three orthogonal axes, all are constants.

Kinetic Energy

The third first-order integral, that can be used for solving the equations of motion, is the first-order spatial integral $\int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$. Note that this spatial integral is a scalar in contrast to the first-order time integrals for linear and angular momenta which are vectors. The work done on a mass m_i by a force $\mathbf{F}_i$ in transforming from condition 1 to 2 is defined to be

$$[W_{12}]_i \equiv \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i \tag{2.4.8}$$

If $\mathbf{F}_i$ is the net resultant force acting on a particle i then the integrand can be written as

$$F_i \cdot d\mathbf{r}_i = \frac{d\mathbf{p}_i}{dt} \cdot d\mathbf{r}_i = m_i \frac{d\mathbf{v}_i}{dt} \cdot \frac{d\mathbf{r}_i}{dt} dt = m_i \frac{d\mathbf{v}_i}{dt} \cdot \mathbf{v}_i dt = \frac{m_i}{2} \frac{d}{dt}(\mathbf{v}_i \cdot \mathbf{v}_i) dt = d\left(\frac{1}{2} m_i v_i^2\right) = d[T]_i \tag{2.4.9}$$

where the kinetic energy of a particle i is defined as

$$[T]_i \equiv \frac{1}{2} m_i v_i^2 \tag{2.4.10}$$

Thus the work done on the particle i, that is, $[W_{12}]_i$ equals the change in kinetic energy of the particle if there is no change in other contributions to the total energy such as potential energy, heat dissipation, etc. That is

$$[W_{12}]_i = \left[\frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2\right]_i = [T_2 - T_1]_i \tag{2.4.11}$$

Thus the differential, and corresponding first integral, forms of the kinetic energy can be written as

$$\mathbf{F}_i = \frac{d\mathbf{T}_i}{dt} \qquad \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = (\mathbf{T}_2 - \mathbf{T}_1)_i \tag{2.4.12}$$

If the work done on the particle is positive, then the final kinetic energy $T_2 > T_1$ Especially noteworthy is that the kinetic energy $[T]_i$ is a scalar quantity which makes it simple to use. This first-order spatial integral is the foundation of the analytic formulation of mechanics that underlies Lagrangian and Hamiltonian mechanics.