6.2: Newtonian plausibility argument for Lagrangian mechanics

Insight into the physics underlying Lagrange mechanics is given by showing the direct relationship between Newtonian and Lagrangian mechanics. The variational approaches to classical mechanics exploit the first-order spatial integral of the force, equation $(2.4.8)$, which equals the work done between the initial and final conditions. The work done is a simple scalar quantity that depends on the initial and final location for conservative forces. Newton's equation of motion is

$$\mathbf{F}=\frac{d\mathbf{p}}{dt} \tag{6.2.1}$$

The kinetic energy is given by

$$T=\frac{1}{2}mv^2=\frac{\mathbf{p}\cdot\mathbf{p}}{2m}=\frac{p_x^2}{2m}+\frac{p_y^2}{2m}+\frac{p_z^2}{2m}$$

It can be seen that

$$\frac{\partial T}{\partial \dot{x}}=p_x \tag{6.2.2}$$

and

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}}=\frac{dp_x}{dt}=F_x \tag{6.2.3}$$

Consider that the force, acting on a mass m, is arbitrarily separated into two components, one part that is conservative, and thus can be written as the gradient of a scalar potential U, plus the excluded part of the force, F^{EX}. The excluded part of the force F^{EX} could include non-conservative frictional forces as well as forces of constraint which may be conservative or non-conservative. This separation allows the force to be written as

$$\mathbf{F}=-\nabla U+\mathbf{F}^{EX} \tag{6.2.4}$$

Along each of the x_i axes,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i}=-\frac{\partial U}{\partial x_i}+F_{x_i}^{EX} \tag{6.2.5}$$

Equation 6.2.5 can be extended by transforming the cartesian coordinate x_i to the generalized coordinates q_i.

Define the standard Lagrangian to be the difference between the kinetic energy and the potential energy, which can be written in terms of the generalized coordinates q_i as

$$L(q_i,\dot{q}_i)\equiv T(\dot{q}_i)-U(q_i) \tag{6.2.6}$$

Assume that the potential is only a function of the generalized coordinates q_i, that is $\frac{\partial U}{\partial \dot{q}_i}=0$, then

$$\frac{\partial L}{\partial \dot{q}_i}=\frac{\partial T}{\partial \dot{q}_i}+\frac{\partial U}{\partial \dot{q}_i}=\frac{\partial T}{\partial \dot{q}_i} \tag{6.2.7}$$

Using the above equations allows Newton's equation of motion 6.2.5 to be expressed as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=F_{q_i}^{EX} \tag{6.2.8}$$

The excluded force $F_{q_i}^{EX}$ can be partitioned into a holonomic constraint force $F_{q_i}^{HC}$, plus any remaining excluded forces F^{EXC}, as given by

$$F_{q_i}^{EX}=F_{q_i}^{HC}+F^{EXC} \tag{6.2.9}$$

A comparison of equations 6.2.8 and $(6.1.4)$ shows that the holonomic constraint forces $F_{q_i}^{HC}$, that are contained in the excluded force F^{EX}, can be identified with the Lagrange multiplier term in equation $(6.1.4)$.

$$F_{q_i}^{HC}\equiv\sum_k^m \lambda_k(t)\frac{\partial g_k}{\partial q_i} \tag{6.2.10}$$

That is the Lagrange multiplier terms can be used to account for holonomic constraint forces $F_{q_i}^{HC}$. Thus Equation 6.2.8 can be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=\sum_k^m \lambda_k(t)\frac{\partial g_k}{\partial q_i}+F_{q_i}^{EXC} \tag{6.2.11}$$

where the Lagrange multiplier term accounts for holonomic constraint forces, and $F_{q_i}^{EXC}$ includes all the remaining forces that are not accounted for by the scalar potential U, or the Lagrange multiplier terms $F_{q_i}^{HC}$.

For holonomic, conservative forces it is possible to absorb all the forces into the potential U plus the Lagrange multiplier term, that is $F_{q_i}^{EXC}=0$. Moreover, the use of a minimal set of generalized coordinates allows the holonomic constraint forces to be ignored by explicitly reducing the number of coordinates from n dependent coordinates to $s=n-m$ independent generalized coordinates. That is, the correlations due to the constraint forces are embedded into the generalized coordinates. Then Equation 6.2.12 reduces to the basic Euler differential equations.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=0 \tag{6.2.12}$$

Note that Equation 6.2.12 is identical to Euler's equation (5.8.1), if the independent variable x is replaced by time t. Thus Newton's equation of motion are equivalent to minimizing the action integral $S=\int_{t_1}^{t_2} Ldt$, that is

$$\delta S=\delta\int_{t_1}^{t_2} L(q_i,\dot{q}_i;t)dt=0 \tag{6.2.13}$$

which is Hamilton's Principle. Hamilton's Principle underlies many aspects of physics as discussed in chapter 9, and is used as the starting point for developing classical mechanics. Hamilton' Principle was postulated 46 years after Lagrange introduced Lagrangian mechanics.

The above plausibility argument, which is based on Newtonian mechanics, illustrates the close connection between the vectorial Newtonian mechanics and the algebraic Lagrangian mechanics approaches to classical mechanics.