# 11.10: Closed-orbit Stability

Bertrand's theorem states that the linear oscillator and the inverse-square law are the only two-body, central forces for which all bound orbits are single-valued, and stable closed orbits. The stability of closed orbits can be illustrated by studying their response to perturbations. For simplicity, the following discussion of stability will focus on circular orbits, but the general principles are the same for elliptical orbits.

A circular orbit occurs whenever the attractive force just balances the effective "centrifugal force" in the rotating frame. This can occur for any radial functional form for the central force. The effective potential, equation $(11.4.6)$ will have a stationary point when

$$\left(\frac{\partial U_{eff}}{\partial r}\right)_{r=r_0}=0 \tag{11.10.1}$$

that is, when

$$\left(\frac{\partial U}{\partial r}\right)_{r=r_0}-\frac{l^2}{\mu r_0^3}=0 \tag{11.10.2}$$

This is equivalent to the statement that the net force is zero. Since the central attractive force is given by

$$F(r)=-\frac{\partial U_{eff}}{\partial r} \tag{11.10.3}$$

then the stationary point occurs when

$$F(r_0)=-\frac{l^2}{\mu r_0^3}=-\mu r_0\dot{\psi}^2 \tag{11.10.4}$$

This is the so-called centrifugal force in the rotating frame. The Hamiltonian, equation $(11.6.5)$, gives that

$$\dot{r}=\pm\sqrt{\frac{2}{\mu}\left(E_{cm}-U-\frac{l^2}{2\mu r^2}\right)} \tag{11.10.5}$$

For a circular orbit $\dot{r}=0$ that is

$$E_{cm}=U-\frac{l^2}{2\mu r^2} \tag{11.10.6}$$

A stable circular orbit is possible if both equations 11.10.2and 11.10.6are satisfied. Such a circular orbit will be a **stable orbit** at the minimum when

$$\left(\frac{d^2U_{eff}}{dr^2}\right)_{r=r_0}>0 \tag{11.10.7}$$

Examples of stable and unstable orbits are shown in Figure $11.10.1$

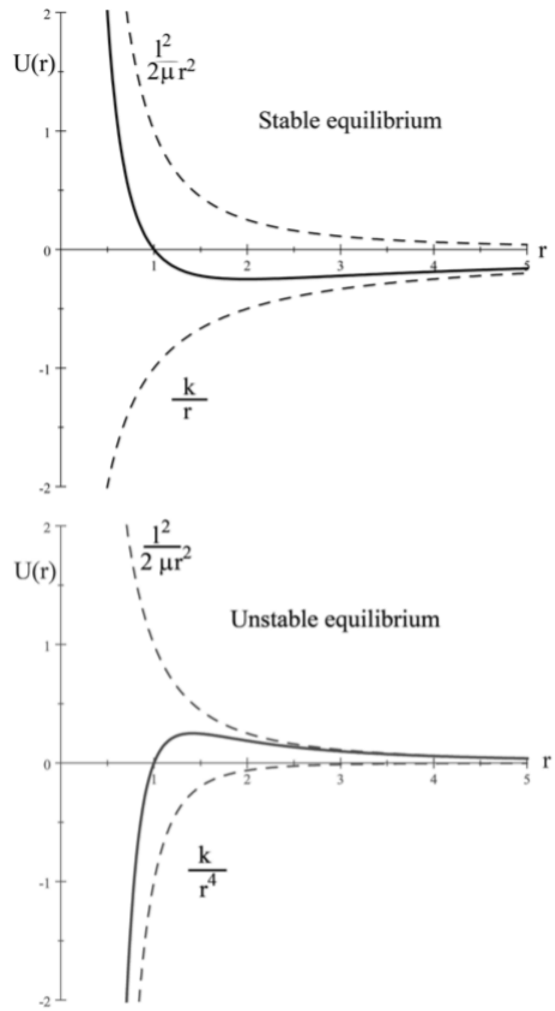


Figure 11.10.1: Stable and unstable effective central potentials. The repulsive centrifugal and the attractive potentials $(k<0)$ are shown dashed. The solid curve is the effective potential.

Stability of a circular orbit requires that

$$\left(\frac{\partial^2 U}{\partial r^2}\right)_{r=r_0}+\frac{3l^2}{\mu r_0^4}>0 \tag{11.10.8}$$

which can be written in terms of the central force for a **stable orbit** as

$$-\left(\frac{\partial F}{\partial r}\right)_{r_0}+\frac{3F\left(r_0\right)}{r_0}>0 \tag{11.10.9}$$

If the attractive central force can be expressed as a power law

$$F(r)=-kr^n \tag{11.10.10}$$

then stability requires

$$kr_0^{n-1}(3+n)>0 \tag{11.10.11}$$

or

$$n>-3 \tag{11.10.12}$$

Stable equivalent orbits will undergo oscillations about the stable orbit if perturbed. To first order, the restoring force on a bound reduced mass $\mu$ is given by

$$F_{restore} = -\left(\frac{d^2U_{eff}}{dr^2}\right)_{r=r_0}(r-r_0) = \mu\ddot{r} \tag{11.10.13}$$

To the extent that this linear restoring force dominates over higher-order terms, then a perturbation of the stable orbit will undergo simple harmonic oscillations about the stable orbit with angular frequency

$$\omega = \sqrt{\frac{\left(\frac{d^2U_{eff}}{dr^2}\right)_{r=r_0}}{\mu}} \tag{11.10.14}$$

The above discussion shows that a small amplitude radial oscillation about the stable orbit with amplitude $\xi$ will be of the form

$$\xi = A\sin(2\pi\omega t + \delta) \tag{11.10.15}$$

The orbit will be closed if the product of the oscillation frequency $\omega$, and the orbit period $\tau$ is an integer value.

The fact that planetary orbits in the gravitational field are observed to be closed is strong evidence that the gravitational force field must obey the inverse square law. Actually there are small precessions of planetary orbits due to perturbations of the gravitational field by bodies other than the sun, and due to relativistic effects. Also the gravitational field near the earth departs slightly from the inverse square law because the earth is not a perfect sphere, and the field does not have perfect spherical symmetry. The study of the precession of satellites around the earth has been used to determine the oblate quadrupole and slight octupole (pear shape) distortion of the shape of the earth.

The most famous test of the inverse square law for gravitation is the precession of the perihelion of Mercury. If the attractive force experienced by Mercury is of the form

$$\mathbf{F}(r) = -G\frac{m_s m_m}{r^{2+\alpha}}\hat{\mathbf{r}}$$

where $|\alpha|$ is small, then it can be shown that, for approximate circular orbitals, the perihelion will advance by a small angle $\pi\alpha$ per orbit period. That is, the precession is zero if $\alpha = 0$, corresponding to an inverse square law dependence which agrees with Bertrand's theorem. The position of the perihelion of Mercury has been measured with great accuracy showing that, after correcting for all known perturbations, the perihelion advances by $43(\pm 5)$ seconds of arc per century, that is $5\times 10^{-7}$ radians per revolution. This corresponds to $\alpha = 1.6\times 10^{-7}$ which is small but still significant. This precession remained a puzzle for many years until $1915$ when Einstein predicted that one consequence of his general theory of relativity is that the planetary orbit of Mercury should precess at $43$ seconds of arc per century, which is in remarkable agreement with observations.

**Example 11.10.1: Linear two-body restoring force**

The effective potential for a linear two-body restoring force $F = -kr$ is

$$U_{eff} = \frac{1}{2}kr^2 + \frac{l^2}{2\mu r^2}$$

At the minimum

$$\left(\frac{\partial U_{eff}}{\partial r}\right)_{r=r_0} = kr - \frac{l^2}{\mu r^3} = 0$$

Thus

$$r_0 = \left(\frac{l^2}{\mu k}\right)^{\frac{1}{4}}$$

and

$$\left(\frac{d^2U_{eff}}{dr^2}\right)_{r=r_0} = \frac{3l^2}{\mu r_0^4} + k = 4k > 0$$

which is a stable orbit. Small perturbations of such a stable circular orbit will have an angular frequency

$$\omega=\sqrt{\frac{\left(\frac{d^2U_{eff}}{dr^2}\right)_{r=r_0}}{\mu}}=2\sqrt{\frac{k}{\mu}}$$

Note that this is twice the frequency for the planar harmonic oscillator with the same restoring coefficient. This is due to the central repulsion, the effective potential well for this rotating oscillator example has about half the width for the corresponding planar harmonic oscillator. Note that the kinetic energy for the rotational motion, which is $\frac{l^2}{2\mu r^2}$, equals the potential energy $\frac{1}{2}kr^2$ at the minimum as predicted by the Virial Theorem for a linear two-body restoring force.

### Example 11.10.2: Inverse square law attractive force

The effective potential for an inverse square law restoring force $F=-\frac{k}{r^2}\hat{r}$, where $k$ is assumed to be positive,

$$U_{eff}=-\frac{k}{r}+\frac{l^2}{2\mu r^2}$$

At the minimum

$$\left(\frac{\partial U_{eff}}{\partial r}\right)_{r=r_0}=\frac{k}{r^2}-\frac{l^2}{\mu r^3}=0$$

Thus

$$r_0=\frac{l^2}{\mu k}$$

and

$$\left(\frac{d^2U_{eff}}{dr^2}\right)_{r=r_0}=\frac{3l^2}{\mu r_0^4}-\frac{2k}{r_0^3}=\frac{k}{r_0^3}>0$$

which is a stable orbit. Small perturbations about such a stable circular orbit will have an angular frequency

$$\omega=\sqrt{\frac{\left(\frac{d^2U_{eff}}{dr^2}\right)_{r=r_0}}{\mu}}=\frac{\mu k^2}{l^3}$$

The kinetic energy for oscillations about this stable circular orbit, which is $\frac{l^2}{2\mu r^2}$, equals half the magnitude of the potential energy $-\frac{k}{r}$ at the minimum as predicted by the Virial Theorem.

### Example 11.10.3: Attractive inverse cubic central force

The inverse cubic force is an interesting example to investigate the stability of the orbit equations. One solution of the inverse cubic central force, for a reduced mass $\mu$, is a spiral orbit

$$r=r_0e^{\alpha\psi}$$

That this is true can be shown by inserting this orbit into the differential orbit equation.

Using a Binet transformation of the variable $r$ to $u$ gives

$$u=\frac{1}{r}=\frac{1}{r_0}e^{-\alpha\psi}$$

$$\frac{du}{d\psi}=-\frac{\alpha}{r_0}e^{-\alpha\psi}$$

$$\frac{d^2u}{d\psi^2} = \frac{\alpha^2}{r_0} e^{-\alpha\psi}$$

Substituting these into the differential equation of the orbit

$$\frac{d^2u}{d\psi^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(\frac{1}{u})$$

gives

$$\frac{\alpha^2}{r_0} e^{-\alpha\psi} + \frac{1}{r_0} e^{-\alpha\psi} = -\frac{\mu}{l^2} r_0^2 e^{2\alpha\psi} F\left(\frac{1}{u}\right)$$

That is

$$F\left(\frac{1}{u}\right) = -\frac{\left(\alpha^2+1\right) l^2}{\mu} r_0^{-3} e^{-3\alpha\psi} = -\frac{\left(\alpha^2+1\right) l^2}{\mu r^3}$$

which is a central attractive inverse cubic force.

The time dependence of the spiral orbit can be derived since the angular momentum gives

$$\dot{\psi} = \frac{l}{\mu r^2} = \frac{l}{\mu r_0^2 e^{2\alpha\psi}}$$

This can be written as

$$e^{2\alpha\psi} d\psi = \frac{l}{\mu r_0^2} dt$$

Integrating gives

$$\frac{e^{2\alpha\psi}}{2\alpha} = \frac{lt}{\mu r_0^2} + \beta$$

where $\beta$ is a constant. But the orbit gives

$$r^2 = r_0^2 e^{2\alpha\psi} = \frac{2\alpha lt}{\mu} + 2\alpha\beta$$

Thus the radius increases or decreases as the square root of the time. That is, an attractive cubic central force does not have a stable orbit which is what is expected since there is no minimum in the effective potential energy. Note that it is obvious that there will be no minimum or maximum for the summation of effective potential energy since, if the force is $F = -\frac{k}{r^3}$, then the effective potential energy is

$$U_{eff} = -\frac{k}{2r^2} + \frac{l^2}{2\mu r^2} = \left(\frac{l^2}{\mu} - k\right)\frac{1}{2r^2}$$

which has no stable minimum or maximum.

### Example 11.10.4: Spiralling mass attached by a string to a hanging mass

An example of an application of orbit stability is the case shown in the adjacent figure. A particle of mass $m$ moves on a horizontal frictionless table. This mass is attached by a light string of fixed length $b$ and rotates about a hole in the table. The string is attached to a second equal mass $m$ that is hanging vertically downwards with no angular motion.

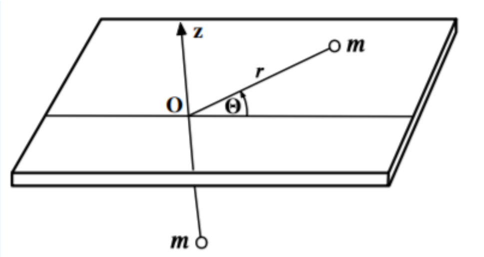


Figure 11.10.2: Rotating mass $m$ on a frictionless horizontal table connected to a suspended mass $m$.

The equations are most conveniently expressed in cylindrical coordinates $(r, \theta, z)$ with the origin at the hole in the table, and $z$ vertically upward. The fixed length of the string requires $z = r - b$ . The potential energy is

$$U = mgz = mg(r - b)$$

The system is central and conservative, thus the Hamiltonian can be written as

$$H = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{m}{2}\dot{r}^2 + mg(r - b) = E$$

The Lagrangian is independent of $\theta$, that is, $\theta$ is cyclic, thus the angular momentum $mr^2\dot{\theta} = l$ is a constant of motion. Substituting this into the Hamiltonian equation gives

$$m\dot{r}^2 + \frac{l^2}{2mr^2} + mg(r - b) = E$$

The effective potential is

$$U_{eff} = \frac{l^2}{2mr^2} + mg(r - b)$$

which is shown in the adjacent figure. The stationary value occurs when

$$\left(\frac{\partial U_{eff}}{\partial r}\right)_{r_0} = -\frac{l^2}{mr_0^3} + mg = 0$$

That is, when the angular momentum is related to the radius by

$$l^2 = m^2 g r_0^3$$

Note that $r_0 = 0$ if $l = 0$.

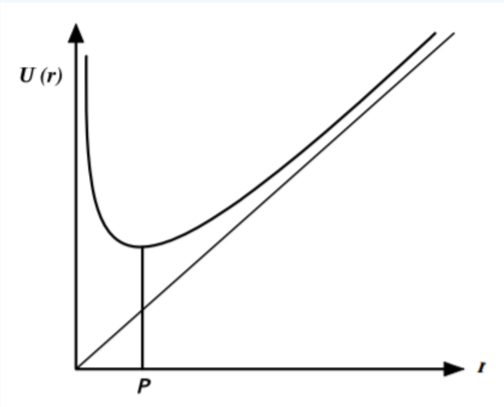


Figure 11.10.3: Effective potential for two connected masses.

The stability of the solution is given by the second derivative

$$\left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)_{r_0} = \frac{3l^2}{mr_0^4} = \frac{3mg}{r_0} > 0$$

Therefore the stationary point is stable.

Note that the equation of motion for the minimum can be expressed in terms of the restoring force on the two masses

$$2m\ddot{r} = -\left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)_{r_0} (r - r_0)$$

Thus the system undergoes harmonic oscillation with frequency

$$\omega = \sqrt{\frac{\frac{3mg}{r_0}}{2m}} = \sqrt{\frac{3g}{2r_0}}$$

The solution of this system is stable and undergoes simple harmonic motion.