6.12: Impulsive Forces

Colliding bodies often involve large impulsive forces that act for a short time. As discussed in chapter $2.12.8$, the treatment of impulsive forces or torques is greatly simplified if they act for a sufficiently short time that the displacement during the impact can be ignored, even though the instantaneous change in velocities may be large. The simplicity is achieved by taking the time integral of the Euler-Lagrange equations over the duration τ of the impulse and assuming $\tau \rightarrow 0$.

The impact of the impulse on a system can be handled two ways. The first approach is to use the Euler-Lagrange equation during the impulse to determine the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{EXC} \tag{6.12.1}$$

where the impulsive force is introduced using the generalized force Q_j^{EXC}. Knowing the initial conditions at time t, the conditions at the time $t+\tau$ are given by integration of Equation 6.12.1 over the duration τ of the impulse which gives

$$\int_t^{t+\tau} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) d\tau - \int_t^{t+\tau} \frac{\partial L}{\partial q_j} d\tau = \int_t^{t+\tau} Q_j^{EXC} d\tau \tag{6.12.2}$$

This integration determines the conditions at time $t+\tau$ which then are used as the initial conditions for the motion when the impulsive force Q_j^{EXC} is zero.

The second approach is to realize that Equation 6.12.2 can be rewritten in the form

$$\lim_{\tau\to 0} \int_t^{t+\tau} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) dt = \lim_{\tau\to 0} \frac{\partial L}{\partial \dot{q}_j}\Bigg|_t^{t+\tau} = \Delta p_j = \lim_{\tau\to 0} \int_t^{t+\tau} \left(\left(\frac{\partial L}{\partial q_j}\right) + Q_j^{EXC}\right) d\tau \tag{6.12.3}$$

Note that in the limit that $\tau \rightarrow 0$ then the integral of the generalized momentum $p_j = \frac{\partial L}{\partial \dot{q}_j}$ simplifies to give the change in generalized momentum Δp_j. In addition, assuming that the non-impulsive forces $\left(\frac{\partial L}{\partial q_j}\right)$ are finite and independent of the instantaneous impulsive force during the infinitessimal duration τ, then the contribution of the non-impulsive forces $\int_t^{t+\tau}\left(\frac{\partial L}{\partial q_j}\right) d\tau$ during the impulse can be neglected relative to the large impulsive force term; $\lim_{\tau\to 0}\int_t^{t+\tau} Q_j^{EXC} d\tau$. Thus it can be assumed that

$$\Delta p_j = \lim_{\tau\to 0} \int_t^{t+\tau} Q_j^{EXC} d\tau = \tilde{Q}_j \tag{6.12.4}$$

where $\tilde{Q}_j$ is the generalized impulse associated with coordinate $j = 1, 2, 3, \ldots, n$. This generalized impulse can be derived from the time integral of the impulsive forces $\mathbf{P}_i$ given by equation $(2.12.49)$ using the time integral of Equation 6.12.3, that is

$$\Delta p_j = \tilde{Q}_j = \lim_{\tau\to 0} \int_t^{t+\tau} Q_j^{EXC} d\tau \equiv \lim_{\tau\to 0} \int_t^{t+\tau} \sum_i \mathbf{P}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} d\tau = \sum_i \tilde{\mathbf{P}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{6.12.5}$$

Note that the generalized impulse $\tilde{Q}_j$ can be a translational impulse $\tilde{\mathbf{P}}_j$ with corresponding translational variable q_j, or an angular impulsive torque $\tilde{\tau}_j$ with corresponding angular variable ϕ_j.

Impulsive force problems usually are solved in two stages. Either equations 6.12.2 or 6.12.5 are used to determine the conditions of the system immediately following the impulse. If $\tau \rightarrow 0$ then impulse changes the generalized velocities $\dot{q}_j$ but not the generalized coordinates q_j. The subsequent motion then is determined using the Lagrangian equations of motion with the impulsive generalized force being zero, and assuming that the initial condition corresponds to the result of the impulse calculation.