5.S: Calculus of Variations (Summary)

Euler's differential equation

The calculus of variations has been introduced and Euler's differential equation was derived. The calculus of variations reduces to varying the functions $y_i(x)$, where $i = 1, 2, 3, \ldots n$, such that the integral

$$F = \int_{x_1}^{x_2} f\left[y_i(x), y_i'(x); x\right] dx \tag{5.S.1}$$

is an extremum, that is, it is a maximum or minimum. Here x is the independent variable, $y_i(x)$ are the dependent variables plus their first derivatives $y_i' \equiv \frac{dy_i}{dx}$. The quantity $f\left[y(x), y'(x); x\right]$ has some given dependence on y_i, y_i' and x. The calculus of variations involves varying the functions $y_i(x)$ until a stationary value of F is found which is presumed to be an extremum. It was shown that *if the $y_i(x)$ are independent,* then the extremum value of F leads to n independent Euler equations

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0 \tag{5.S.2}$$

where $i = 1, 2, 3..n$. This can be used to determine the functional form $y_i(x)$ that ensures that the integral $F = \int_{x_1}^{x_2} f\left[y(x), y'(x); x\right] dx$ is a stationary value, that is, presumably a maximum or minimum value.

Note that Euler's equation involves partial derivatives for the dependent variables y_i, y_i', and the total derivative for the independent variable x.

Euler's integral equation

It was shown that if the function $\int_{x_1}^{x_2} f\left[y_i(x), y_i'(x); x\right]$ does not depend on the independent variable, then Euler's differential equation can be written in an integral form. This integral form of Euler's equation is especially useful *when $\frac{\partial f}{\partial x} = 0$, that is, when f does not depend explicitly on x*, then the first integral of the Euler equation is a constant

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} \tag{5.S.3}$$

Constrained variational systems

Most applications involve constraints on the motion. The equations of constraint can be classified according to whether the constraints are holonomic or non-holonomic, the time dependence of the constraints, and whether the constraint forces are conservative.

Generalized coordinates in variational calculus

Independent generalized coordinates can be chosen that are perpendicular to the rigid constraint forces and therefore the constraint does not contribute to the functional being minimized. That is, the constraints are embedded into the generalized coordinates and thus the constraints can be ignored when deriving the variational solution.

Minimal set of generalized coordinates

If the constraints are holonomic then the m holonomic equations of constraint can be used to transform the n coupled generalized coordinates to $s = n - m$ independent generalized variables q_i, q_i'. The generalized coordinate method then uses Euler's equations to determine these $s = n - m$ independent generalized coordinates.

$$\frac{\partial f}{\partial q_i} - \frac{d}{dx}\frac{\partial f}{\partial q_i'} = 0 \tag{5.S.4}$$

Lagrange multipliers for holonomic constraints

The Lagrange multipliers approach for n variables, plus m holonomic equations of constraint, determines all $N + m$ unknowns for the system. The holonomic forces of constraint acting on the N variables, are related to the Lagrange multiplier terms $\lambda_k(x)\frac{\partial g_k}{\partial y_i}$) that are introduced into the Euler equations.

That is,

$$\frac{\partial f}{\partial y_i} - \frac{df}{dx}\frac{\partial f}{\partial y_i'} + \sum_k^m \lambda_k(x)\frac{\partial g_k}{\partial y_i} \tag{5.S.5}$$

where the holonomic equations of constraint are given by

$$g_k(y_i; x) = 0 \tag{5.S.6}$$

The advantage of using the Lagrange multiplier approach is that the variational procedure simultaneously determines both the equations of motion for the N variables plus the m constraint forces acting on the system.