2.11: Virial Theorem

The virial theorem is an important theorem for a system of moving particles both in classical physics and quantum physics. The Virial Theorem is useful when considering a collection of many particles and has a special importance to central-force motion. For a general system of mass points with position vectors $\mathbf{r}_i$ and applied forces $\mathbf{F}_i$, consider the scalar product G

$$G \equiv \sum_i \mathbf{p}_i \cdot \mathbf{r}_i \tag{2.11.1}$$

where i sums over all particles. The time derivative of G is

$$\frac{dG}{dt} = \sum_i \mathbf{p}_i \cdot \dot{\mathbf{r}}_i + \sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i \tag{2.11.2}$$

However,

$$\sum_i \mathbf{p}_i \cdot \dot{\mathbf{r}}_i = \sum_i m\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \sum_i mv^2 = 2T \tag{2.11.3}$$

Also, since $\dot{\mathbf{p}}_i = \mathbf{F}_i$

$$\sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \tag{2.11.4}$$

Thus

$$\frac{dG}{dt} = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \tag{2.11.5}$$

The time average over a period τ is

$$\frac{1}{T}\int_0^\tau \frac{dG}{dt}dt = \frac{G(\tau) - G(0)}{\tau} = \langle 2T\rangle + \left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle \tag{2.11.6}$$

where the $\langle\rangle$ brackets refer to the time average. Note that if the motion is periodic and the chosen time τ equals a multiple of the period, then $\frac{G(\tau)-G(0)}{\tau} = 0$. Even if the motion is not periodic, if the constraints and velocities of all the particles remain finite, then there is an upper bound to G.This implies that choosing $\tau \to \infty$ means that $\frac{G(\tau)-G(0)}{\tau} \to 0$. In both cases the left-hand side of the equation tends to zero giving the *virial theorem*

$$\langle T\rangle = -\frac{1}{2}\left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle \tag{2.11.7}$$

The right-hand side of this equation is called the *virial of the system*. For a single particle subject to a conservative central force $\mathbf{F} = -\nabla U$ the Virial theorem equals

$$\langle T\rangle = \frac{1}{2}\langle \nabla U \cdot \mathbf{r}\rangle = \frac{1}{2}\left\langle r\frac{\partial U}{\partial r}\right\rangle \tag{2.11.8}$$

If the potential is of the form $U = kr^{n+1}$ that is, $F = -k(n+1)r^n$, then $r\frac{\partial U}{\partial r} = (n+1)U$. Thus for a single particle in a central potential $U = kr^{n+1}$ the Virial theorem reduces to

$$\langle T\rangle = \frac{n+1}{2}\langle U\rangle \tag{2.11.9}$$

The following two special cases are of considerable importance in physics.

Hooke's Law: Note that for a linear restoring force $n = 1$ then

$$\langle T\rangle = +\langle U\rangle \tag{n=1}$$

You may be familiar with this fact for simple harmonic motion where the average kinetic and potential energies are the same and both equal half of the total energy.

Inverse-square law: The other interesting case is for the inverse square law $n=-2$ where

$$\langle T\rangle = -\frac{1}{2}\langle U\rangle \tag{n = -2}$$

The Virial theorem is useful for solving problems in that knowing the exponent n of the field makes it possible to write down directly the average total energy in the field. For example, for

$$\langle E\rangle = \langle T\rangle + \langle U\rangle \tag{2.11.10}$$

$$= -\frac{1}{2}\langle U\rangle + \langle U\rangle = \frac{1}{2}\langle U\rangle \tag{2.11.11}$$

This occurs for the Bohr model of the hydrogen atom where the kinetic energy of the bound electron is half of the potential energy. The same result occurs for planetary motion in the solar system.

Example 2.11.1: The ideal gas law

The Virial theorem deals with average properties and has applications to statistical mechanics. Consider an ideal gas. According to the equipartition theorem the average kinetic energy per atom in an ideal gas is $\frac{3}{2}kT$ where T is the absolute temperature and k is the Boltzmann constant. Thus the average total kinetic energy for N atoms is $\langle KE\rangle = \frac{3}{2}NkT$. The right-hand side of the Virial theorem contains the force $\mathbf{F}_i$. For an ideal gas it is assumed that there are no interaction forces between atoms, that is the only force is the force of constraint of the walls of the pressure vessel. The pressure P is force per unit area and thus the instantaneous force on an area of wall dA is $d\mathbf{F}_i = -\hat{\mathbf{n}}PdA$ where $\hat{n}$ designates the unit vector normal to the surface. Thus the right-hand side of the Virial theorem is

$$-\frac{1}{2}\left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle = \frac{P}{2}\int \hat{\mathbf{n}}\cdot\mathbf{r}_i dA$$

Use of the divergence theorem thus gives that $\int \hat{\mathbf{n}}\cdot\mathbf{r}_i dA = \int \nabla\cdot\mathbf{r}dV = 3\int dV = 3V$. Thus the Virial theorem leads to the ideal gas law, that is

$$NkT = PV$$

Example 2.11.2: The mass of galaxies

The Virial theorem can be used to make a crude estimate of the mass of a cluster of galaxies. Assuming a spherically-symmetric cluster of N galaxies, each of mass m then the total mass of the cluster is $M = Nm$. A crude estimate of the cluster potential energy is

$$\langle U\rangle \approx \frac{GM^2}{R} \tag{α}$$

where R is the radius of a cluster. The average kinetic energy per galaxy is $\frac{1}{2}m\langle v\rangle^2$ where $\langle v\rangle^2$ is the average square of the galaxy velocities with respect to the center of mass of the cluster. Thus the total kinetic energy of the cluster is

$$\langle KE\rangle \approx \frac{Nm\langle v\rangle^2}{2} = \frac{M\langle v\rangle^2}{2} \tag{β}$$

The Virial theorem tells us that a central force having a radial dependence of the form $F \propto r^n$ gives $\langle KE\rangle = \frac{n+1}{2}\langle U\rangle$. For the inverse-square gravitational force then

$$\langle KE\rangle = -\frac{1}{2}\langle U\rangle. \tag{γ}$$

Thus equations α, β, and γ give an estimate of the total mass of the cluster to be

$$M \approx \frac{R\langle v\rangle^2}{G}$$

This estimate is larger than the value estimated from the luminosity of the cluster implying a large amount of "dark matter" must exist in galaxies which remains an open question in physics.