17.6: Lorentz-Invariant Formulation of Lagrangian Mechanics

Parametric Formulation

The Lagrangian and Hamiltonian formalisms in classical mechanics are based on the Newtonian concept of absolute time t which serves as the system evolution parameter in Hamilton's Principle. This approach violates the Special Theory of Relativity. The extended Lagrangian and Hamiltonian formalism is a parametric approach, pioneered by Lanczos[La49], that introduces a system evolution parameter s that serves as the independent variable in the action integral, and all the space-time variables $q_i(s), t(s)$ are dependent on the evolution parameter s. This extended Lagrangian and Hamiltonian formalism renders it to a form that is compatible with the Special Theory of Relativity. The importance of the Lorentz-invariant extended formulation of Lagrangian and Hamiltonian mechanics has been recognized for decades.[La49, Go50, Sy60] Recently there has been a resurgence of interest in the extended Lagrangian and Hamiltonian formalism stimulated by the papers of Struckmeier[Str05, Str08] and this formalism has featured prominently in recent textbooks by Johns[Jo05] and Greiner[Gr10]. This parametric approach develops manifestly-covariant Lagrangian and Hamiltonian formalisms that treat equally all $2n+1$ space-time canonical variables. It provides a plausible manifestly-covariant Lagrangian for the one-body system, but serious problems exist extending this to the N-body system when $N>1$. Generalizing the Lagrangian and Hamiltonian formalisms into the domain of the Special Theory of Relativity is of fundamental importance to physics, while the parametric approach gives insight into the philosophy underlying use of variational methods in classical mechanics.[1]

In conventional Lagrangian mechanics, the equations of motion for the n generalized coordinates are derived by minimizing the action integral, that is, Hamilton's Principle.

$$\delta S(\mathbf{q}, \dot{\mathbf{q}}, t)=\delta \int_a^b L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) d t=0 \tag{17.6.1}$$

where $L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)$ denotes the conventional Lagrangian. This approach implicitly assumes the Newtonian concept of absolute time t which is chosen to be the independent variable that characterizes the evolution parameter of the system. The actual path $[\mathbf{q}(t), \dot{\mathbf{q}}(t)]$ the system follows is defined by the extremum of the action integral $S(\mathbf{q}, \dot{\mathbf{q}}, t)$ which leads to the corresponding Euler-Lagrange equations. This assumption is contrary to the Theory of Relativity which requires that the space and time variables be treated equally, that is, the Lagrangian formalism must be covariant.

Extended Lagrangian

Lanczos[La49] proposed making the Lagrangian covariant by introducing a general evolution parameter s, and treating the time as a dependent variable $t(s)$ on an equal footing with the configuration space variables $q^i(s)$. That is, the time becomes a dependent variable $q_0(s)=ct(s)$ similar to the spatial variables $q_\mu(s)$ where $1 \leq \mu \leq n$. The dynamical system then is described as motion confined to a hypersurface within an extended space where the value of the extended Hamiltonian and the evolution parameter s constitute an additional pair of canonically conjugate variables in the extended space. That is, the canonical momentum p_0, corresponding to $q_0=ct$, is $p_0=\frac{E}{c}$ similar to the momentum-energy four vector, equation $(17.5.21)$.

An *extended Lagrangian* $\mathbb{L}(\mathbf{q}(s), \frac{d\mathbf{q}(s)}{ds}, t(s), \frac{dt(s)}{ds})$ can be defined which can be written compactly as $\mathbb{L}(q^\mu(s), \frac{dq^\mu(s)}{ds})$ where the index $0 \leq \mu \leq n$ denotes the entire range of space-time variables.

This extended Lagrangian can be used in an extended action functional $\mathbb{S}(\mathbf{q}, \frac{d\mathbf{q}}{ds}, t, \frac{dt}{ds})$ to give an extended version of Hamilton's Principle[2]

$$\delta \mathbb{S}(\mathbf{q}, \frac{d\mathbf{q}}{ds}, t, \frac{dt}{ds})=\delta \int_a^b \mathbb{L}(q^\mu(s), \frac{dq^\mu(s)}{ds}) ds=0 \tag{17.6.2}$$

The conventional action S, and extended action $\mathbb{S}$, address alternate characterizations of the same underlying physical system, and thus the action principle implies that $\delta S=\delta \mathbb{S}=0$ must hold simultaneously. That is,

$$\delta \int_a^b L(\mathbf{q}, \frac{d\mathbf{q}}{dt}, t) \frac{dt}{ds} ds=\delta \int_a^b \mathbb{L}(\mathbf{q}, \frac{d\mathbf{q}}{ds}, t, \frac{dt}{ds}) ds \tag{17.6.3}$$

As discussed in chapter 9.3, there is a continuous spectrum of equivalent gauge-invariant Lagrangians for which the Euler-Lagrange equations lead to identical equations of motion. Equation 17.6.3 is satisfied if the conventional and extended Lagrangians are related by

$$\mathbb{L}(\mathbf{q}, \frac{d\mathbf{q}}{ds}, t, \frac{dt}{ds}) = L(\mathbf{q}, \frac{d\mathbf{q}}{dt}, t)\frac{dt}{ds} + \frac{d\Lambda(\mathbf{q}, t)}{ds} \tag{17.6.4}$$

where $\Lambda(\mathbf{q}, t)$ is a continuous function of $\mathbf{q}$ and t that has continuous second derivatives. It is acceptable to assume that $\frac{d\Lambda(\mathbf{q},t)}{ds} = 0$, then the extended and conventional Lagrangians have a unique relation requiring no simultaneous transformation of the dynamical variables. That is, assume

$$\mathbb{L}(\mathbf{q}, \frac{d\mathbf{q}}{ds}, t, \frac{dt}{ds}) = L(\mathbf{q}, \frac{d\mathbf{q}}{dt}, t)\frac{dt}{ds} \tag{17.6.5}$$

Note that the time derivative of $\mathbf{q}$ can be expressed in terms of the s derivatives by

$$\frac{d\mathbf{q}}{dt} = \frac{d\mathbf{q}/ds}{dt/ds} \tag{17.6.6}$$

Thus, for a conventional Lagrangian with n variables, the corresponding extended Lagrangian is a function of $n+1$ variables while the conventional and extended Lagrangians are related using equations 17.6.5, and 17.6.6.

The derivatives of the relation between the extended and conventional Lagrangians lead to

$$\frac{\partial \mathbb{L}}{\partial q^\mu} = \frac{\partial L}{\partial q^\mu}\frac{dt}{ds} \tag{17.6.7}$$

$$\frac{\partial \mathbb{L}}{\partial t} = \frac{\partial L}{\partial t}\frac{dt}{ds} \tag{17.6.8}$$

$$\frac{\partial \mathbb{L}}{\partial\left(\frac{dq^\mu}{ds}\right)} = \frac{\partial L}{\partial\left(\frac{dq^\mu}{dt}\right)} \tag{17.6.9}$$

$$\frac{\partial \mathbb{L}}{\partial\left(\frac{dt}{ds}\right)} = L - \sum_{\mu=1}^{n}\frac{\partial L}{\partial\left(\frac{dq^\mu}{dt}\right)}\frac{dq^\mu}{dt} \tag{17.6.10}$$

where $1 \leq \mu \leq n$ since the $\mu = 0$ time derivatives are written explicitly in equations 17.6.8, 17.6.10.

Equations 17.6.9— 17.6.10, summed over the extended range $0 \leq \mu \leq n$ of time and spatial dynamical variables, imply

$$\sum_{\mu=0}^{n}\frac{\partial \mathbb{L}}{\partial\left(\frac{dq^\mu}{ds}\right)}\left(\frac{dq^\mu}{ds}\right) = L\frac{dt}{ds} - \sum_{\mu=1}^{n}\frac{\partial L}{\partial\left(\frac{dq^\mu}{dt}\right)}\frac{dq^\mu}{dt}\frac{dt}{ds} + \sum_{i=1}^{n}\frac{\partial L}{\partial\left(\frac{dq^\mu}{dt}\right)}\frac{dq^\mu}{ds} = \mathbb{L} \tag{17.6.11}$$

Equation 17.6.11 can be written in the form

$$\mathbb{L} - \sum_{\mu=0}^{n}\frac{\partial \mathbb{L}}{\partial\left(\frac{dq^\mu}{ds}\right)}\frac{dq^\mu}{ds} = \begin{cases} \stackrel{\neq}{=} 0 \text{ if } \mathbb{L} \text{ is not homogeneous in } \frac{dq^\mu}{ds} \\ \equiv 0 \text{ if } \mathbb{L} \text{ is homogeneous in } \frac{dq^\mu}{ds} \end{cases} \tag{17.6.12}$$

If the extended Lagrangian $\mathbb{L}(\mathbf{q}, \frac{d\mathbf{q}}{ds}, t, \frac{dt}{ds})$ is homogeneous to first order in the $n+1$ variables $\frac{dq^\mu}{ds}$, then Euler's theorem on homogeneous functions trivially implies the relation given in Equation 17.6.12. Struckmeier[Str08] identified a subtle but important point that if $\mathbb{L}$ is not homogeneous in $\frac{dq^\mu}{ds}$, then Equation 17.6.12 is not an identity but is an implicit equation that is always satisfied as the system evolves according to the solution of the extended Euler-Lagrange equations. Then Equation 17.6.5 is satisfied without it being a homogeneous form in the $n+1$ velocities $\frac{dq^\mu}{ds}$. This introduces a new class of non-homogeneous Lagrangians. The relativistic free particle, discussed in example $17.6.1$, is a case of a non-homogeneous extended Lagrangian.

Extended generalized momenta

The generalized momentum is defined by

$$p_{\mu}=\frac{\partial L}{\partial\left(\frac{\partial q^{\mu}}{\partial t}\right)} \tag{17.6.13}$$

Assume that the definitions of the extended Lagrangian $\mathbb{L}$, and the extended Hamiltonian $\mathbb{H}$, are related by a Legendre transformation, and are based on variational principles, analogous to the relation that exists between the conventional Lagrangian L and Hamiltonian H. The Legendre transformation requires defining the extended generalized (canonical) momentum-energy four vector $\mathbb{P}(s)=(\frac{\mathbb{E}(s)}{c},\mathbf{p}(s))$. The momentum components of the momentum-energy four vector $\mathbb{P}(s)=(\frac{\mathbb{E}(s)}{c},\mathbf{p}(s))$ are given by the $1\leq\mu\leq n$ components using Equation 17.6.9.

$$p_{\mu}(s)=\frac{\partial\mathbb{L}}{\partial\left(\frac{dq^{\mu}}{ds}\right)}=\frac{\partial L}{\partial\left(\frac{dq^{\mu}}{dt}\right)} \tag{17.6.14}$$

The $\mu=0$ component of the momentum-energy four vector can be derived by recognizing that the right-hand side of Equation 17.6.10is equal to $-H(p_{\mu},q^{\mu},t)$. That is, the corresponding generalized momentum p_0, that is conjugate to $q_0=ct$, is given by

$$p_{0}=\frac{\partial\mathbb{L}}{\partial\left(\frac{dq^{0}}{ds}\right)}=\frac{1}{c}\left(\frac{\partial\mathbb{L}}{\partial\left(\frac{dt}{ds}\right)}\right)=\frac{1}{c}\left(L-\sum_{\mu=1}^{n}\frac{\partial L}{\partial\left(\frac{dq^{\mu}}{dt}\right)}\frac{dq^{\mu}}{dt}\right)=-\frac{H(p_{\mu},q^{\mu},t)}{c} \tag{17.6.15}$$

Extended Lagrange equations of motion

By direct analogy with the non-relativistic action integral 17.6.1, the extremum for the relativistic action integral $S(\mathbf{q},\frac{d\mathbf{q}}{ds},t,\frac{dt}{ds})$ is obtained using the Euler-Lagrange equations derived from Equation 17.6.2 where the independent variable is s. This implies that for $0\leq\mu\leq n$

$$\frac{d}{ds}\left(\frac{\partial\mathbb{L}}{\partial\left(\frac{dq^{\mu}}{ds}\right)}\right)-\frac{\partial\mathbb{L}}{\partial q^{\mu}}=\mathbb{Q}_{\mu}^{EX}=\sum_{k=1}^{m}\frac{dt}{ds}\lambda_{k}\frac{\partial g_{k}}{\partial q^{\mu}}+Q_{\mu}^{EXC}\frac{dt}{ds} \tag{17.6.16}$$

where the extended generalized force $\mathbb{Q}_{\mu}^{EX}$ shown on the right-hand side of Equation 17.6.16, accounts for all forces not included in the potential energy term in the Lagrangian. The extended generalized force $\mathbb{Q}_{\mu}^{EX}$ can be factored into two terms as discussed in chapter 6, equation (6.5.12). The Lagrange multiplier term includes $1\leq k\leq m$ holonomic constraint forces where the m holonomic constraints, which do no work, are expressed in terms of the m algebraic equations of holonomic constraint g_k. The Q_{μ}^{EXC} term includes the remaining constraint forces and generalized forces that are not included in the Lagrange multiplier term or the potential energy term of the Lagrangian.

For the case where $\mu=0$, since $q_0=ct$, then Equation 17.6.16reduces to

$$\frac{d}{ds}\left(\frac{\partial\mathbb{L}}{\partial\left(\frac{dt}{ds}\right)}\right)-\frac{\partial\mathbb{L}}{\partial t}=\sum_{k=1}^{m}\frac{dt}{ds}\lambda_{k}\frac{\partial g_{k}}{\partial t}-\sum_{\nu=1}^{n}Q_{\nu}^{EXC}\frac{dq^{\nu}}{ds} \tag{17.6.17}$$

These Euler-Lagrange equations of motion 17.6.16, 17.6.17 determine the $1\leq\mu\leq n$ generalized coordinates $q^{\mu}(s)$, plus $q^0=ct(s)$ in terms of the independent variable s.

If the holonomic equations of constraint are time independent, that is $\frac{\partial g_k}{\partial t}=0$ and if $\mathbb{Q}_0^{EXC}=0$, then the $\mu=0$ term of the Euler-Lagrange equations simplifies to

$$\frac{d}{ds}\left(\frac{\partial\mathbb{L}}{\partial\left(\frac{dt}{ds}\right)}\right)-\frac{\partial\mathbb{L}}{\partial t}=0 \tag{17.6.18}$$

One interpretation is to select L to be primary. Then $\mathbb{L}$ is derived from L using Equation 17.6.5 and $\mathbb{L}$ must satisfy the identity given by Equation 17.6.12 while the Euler-Lagrange equations containing $\frac{dt}{ds}$ yield an identity which implies that L does not provide an equation of motion in terms of $t(s)$. Conversely, if $\mathbb{L}$ is chosen to be primary, then $\mathbb{L}$ is no longer a homogeneous function and Equation 17.6.12 serves as a constraint on the motion that can be used to deduce L, while $\frac{dt}{ds}$ yields a non-trivial equation of motion in terms of $t(s)$. In both cases the occurrence of a constraint surface results from the fact that the extended space has $2n+2$ variables to describe $2n+1$ degrees of freedom, that is, one more degree of freedom than required for the actual system.

Example 17.6.1: Lagrangian for a relativistic free particle

The standard Lagrangian $L=T-U$ is not Lorentz invariant. The extended Lagrangian $L(\mathbf{q}, \frac{d\mathbf{q}}{ds}, t, \frac{dt}{ds})$ introduces the independent variable s which treats both the space variables $q(s)$ and time variable $q_0 = ct(s)$ equally. This can be achieved by defining the non-standard Lagrangian

$$\mathbb{L}\left(\mathbf{q}, \frac{d\mathbf{q}}{ds}, t, \frac{dt}{ds}\right) = \frac{1}{2}mc^2\left[\frac{1}{c^2}\left(\frac{d\mathbf{q}}{ds}\right)^2 - \left(\frac{dt}{ds}\right)^2 - 1\right] \tag{α}$$

The constant third term in the bracket is included to ensure that the extended Lagrangian converges to the standard Lagrangian in the limit $\frac{dt}{ds} \rightarrow 1$.

Note that the extended Lagrangian α is not homogeneous to first order in the velocities $\frac{d\mathbf{q}}{ds}$ as is required. Equation 17.6.12 must be used to ensure that Equation α is homogeneous. That is, it must satisfy the constraint relation

$$\left(\frac{dt}{ds}\right)^2 - \frac{1}{c^2}\left(\frac{d\mathbf{q}}{ds}\right)^2 - 1 = 0 \tag{β}$$

Inserting β into the extended Lagrangian α yields that the square bracket in Equation α must equal 2. Thus

$$|\mathbb{L}| = \frac{1}{2}mc^2[-2] = -mc^2 \tag{γ}$$

The constraint Equation β implies that

$$\frac{ds}{dt} = \sqrt{1 - \frac{1}{c^2}\left(\frac{d\mathbf{q}}{dt}\right)^2} = \frac{1}{\gamma} \tag{δ}$$

Using Equation δ gives that the relativistic Lagrangian is

$$L = \frac{\mathbb{L}}{\gamma} = -\frac{mc^2}{\gamma} = -mc^2\sqrt{1-\beta^2} \tag{ϵ}$$

Equation ϵ is the conventional relativistic Lagrangian derived by assuming that the system evolution parameter s is transformed to be along the world line ds, where the invariant length ds replaces the proper time interval

$$ds = cd\tau = \frac{cdt}{\gamma} \tag{ε}$$

The definition of the generalized (canonical) momentum

$$p_i = \frac{\partial L}{d\dot{q}_i} = \gamma m\dot{q}_i \tag{ς}$$

leads to the relativistic expression for momentum given in equation $(17.4.6)$.

The relativistic Lagrangian is an important example of a non-standard Lagrangian. Equation α does not equal the difference between the kinetic and potential energies, that is, the relativistic expression for kinetic energy is given by $(17.4.13)$to be

$$T = (\gamma - 1)mc^2 \tag{η}$$

The non-standard relativistic Lagrangian ϵ can be used with the Euler-Lagrange equations to derive the second-order equations of motion for both relativistic and non-relativistic problems within the Special Theory of Relativity.

Example 17.6.2: Relativistic particle in an external elctromagnetic field

A charged particle moving at relativistic speed in an external electromagnetic field provides an example of the use of the relativistic Lagrangian.

In the discussion of classical mechanics it was shown that the velocity-dependent Lorentz force can be absorbed into the scalar electric potential Φ plus the vector magnetic potential $\mathbf{A}$. That is, the potential energy is given by equation $(17.3.4)$ to be $U = q(\Phi - \mathbf{A}\cdot\mathbf{v})$. Including this in the Lagrangian, 17.6.17, gives

$$L = -\frac{mc^2}{\gamma} - U = -mc^2\sqrt{1-\beta^2} - q\Phi + q\mathbf{A}\cdot\mathbf{v}$$

The three spatial partial derivatives can be written in vector notation as

$$\frac{\partial L}{\partial \mathbf{r}} = -q\boldsymbol{\nabla}\Phi + \frac{q}{c}\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}) \tag{a}$$

and the generalized momentum is given by

$$\mathbf{p} = \frac{\partial L}{d\mathbf{v}} = \gamma m\mathbf{v} + q\mathbf{A}$$

which is identical to the non-relativistic answer given by equation 7.6. That is, it includes the momentum of the electromagnetic field plus the classical linear momentum of the moving particle.

The total time derivative of the generalized momentum is

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left(\frac{\partial L}{d\mathbf{v}}\right) = \frac{d}{dt}(\gamma m\mathbf{v}) + q\frac{d\mathbf{A}}{dt} \tag{b}$$

where the last term is given by the chain rule

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A} \tag{c}$$

Using equations a, b, c in the Euler-Lagrange equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{d\mathbf{v}}\right) = \frac{\partial L}{\partial \mathbf{r}}$$

$$\frac{d}{dt}(\gamma m\mathbf{v}) + q\frac{d\mathbf{A}}{dt} = -q\boldsymbol{\nabla}\Phi + q\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A})$$

Collecting terms and using the well-known vector-product identity, plus the definition $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, gives

$$\begin{aligned} \frac{d}{dt}(\gamma m\mathbf{v}) &= -\left[q\boldsymbol{\nabla}\Phi - q\frac{\partial \mathbf{A}}{\partial t}\right] + q[\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}] \\ &= -q\left[\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}\right] + q[\mathbf{v}\times\boldsymbol{\nabla}\times\mathbf{A}] \\ \mathbf{F} &= q[\mathbf{E} + \mathbf{v}\times\mathbf{B}] \end{aligned}$$

If we adopt the definition that the relativistic canonical momentum is $p = \gamma mv$ then the left hand side is the relativistic force while the right-hand side is the well-known Lorentz force of electromagnetism. Thus the extended Lagrangian formulation correctly reproduces the well-known Lorentz force for a charged particle moving in an electromagnetic field.

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[1] Chapters 17.6 and 17.7 reproduce the Struckmeier presentation.[Str08]

[2] These formula involve total and partial derivatives with respect to both time, t and parameter s. For clarity, the derivatives are written out in full because Lanczos[La49] and Johns[Jo05] use the opposite convention for the dot and prime superscripts as abbreviations for the differentials with respect to t and s. The blackboard bold format is used to designate the extended versions of the action $\mathbb{S}$, Lagrangian $\mathbb{L}$ and Hamiltonian $\mathbb{H}$.