7.8: Generalized energy theorem

The Hamilton function, $(7.7.6)$ plus equation $(7.7.4)$ lead to the *generalized energy theorem*

$$\frac{dH(\mathbf{q},\mathbf{p},t)}{dt}=\frac{dh(\mathbf{q},\dot{\mathbf{q}},t)}{dt}=\sum_j \dot{q}_j\left[Q_j^{EXC}+\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)\right]-\frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial t} \tag{7.8.1}$$

Note that for the special case where all the external forces $\left[Q_j^{EXC}+\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)\right]=0$, then

$$\frac{dH}{dt}=-\frac{\partial L}{\partial t} \tag{7.8.2}$$

Thus the Hamiltonian is time independent if both $\left[Q_j^{EXC}+\sum_{k=1}^{m}\lambda_k\frac{\partial g_k}{\partial q_j}(\mathbf{q},t)\right]=0$ and the Lagrangian are time-independent. For an isolated closed system having no external forces acting, then the Lagrangian is time independent because the velocities are constant, and there is no external potential energy. That is, the Lagrangian is time-independent, and

$$\frac{d}{dt}\left[\sum_j\left(\dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right)-L\right]=\frac{dH}{dt}=-\frac{\partial L}{\partial t}=0 \tag{7.8.3}$$

As a consequence, the Hamiltonian $H(\mathbf{q},\mathbf{p},t)$, and generalized energy $h(\mathbf{q},\dot{\mathbf{q}},t)$, both are constants of motion if the Lagrangian is a constant of motion, and if the external non-potential forces are zero. This is an example of Noether's theorem, where the symmetry of time independence leads to conservation of the conjugate variable, which is the Hamiltonian or Generalized energy.