3.E: Linear Oscillators (Exercises)

1. Consider a simple harmonic oscillator consisting of a mass m attached to a spring of spring constant k. For this oscillator $x(t) = A\sin(\omega_0 t - \delta)$.

a. Find an expression for $\dot{x}(t)$.
b. Eliminate t between $x(t)$ and $\dot{x}(t)$ to arrive at one equation similar to that for an ellipse.
c. Rewrite the equation in part (b) in terms of $x, \dot{x}, k, m,$ and the total energy E.
d. Give a rough sketch of the phase space diagram ($\dot{x}$ versus x) for this oscillator. Also, on the same set of axes, sketch the phase space diagram for a similar oscillator with a total energy that is larger than the first oscillator.
e. What direction are the paths that you have sketched? Explain your answer.
f. Would different trajectories for the same oscillator ever cross paths? Why or why not?

2. Consider a damped, driven oscillator consisting of a mass m attached to a spring of spring constant k.

a. What is the equation of motion for this system?
b. Solve the equation in part (a). The solution consists of two parts, the complementary solution and the particular solution. When might it be possible to safely neglect one part of the solution?
c. What is the difference between amplitude resonance and kinetic energy resonance?
d. How might phase space diagrams look for this type of oscillator? What variables would affect the diagram?

3. A particle of mass m is subject to the following force

$$\mathbf{F} = A(x^3 - 4x^2 + 3x)\hat{\mathbf{x}}$$

where A is a constant.

a. Determine the points when the particle is in equilibrium.
b. Which of these points is stable and which are unstable?
c. Is the motion bounded or unbounded?

4. A very long cylindrical shell has a mass density that depends upon the radial distance such that $\rho(r) = \frac{k}{r}$, where k is a constant. The inner radius of the shell is a and the outer radius is b.

a. Determine the direction and the magnitude of the gravitational field for all regions of space.
b. If the gravitational potential is zero at the origin, what is the difference between the gravitational potential at $r = b$ and $r = a$?

5. A mass m is constrained to move along one dimension. Two identical springs are attached to the mass, one on each side, and each spring is in turn attached to a wall. Both springs have the same spring constant k.

a. Determine the frequency of the oscillation, assuming no damping.
b. Now consider damping. It is observed that after n oscillations, the amplitude of the oscillation has dropped to one-half of its initial value. Find an expression for the damping constant.
c. How long does it take for the amplitude to decrease to one-quarter of its initial value?

6. Discuss the motion of a continuous string when plucked at one third of the length of the string. That is, the initial condition is $\ddot{q}(x, 0) = 0$, and $q(x, 0) = \left\{ \begin{array}{ll} \frac{3A}{L}x, & 0 \leq x \leq \frac{L}{3} \\ \frac{3A}{2L}(L - x), & \frac{L}{3} \leq x \leq L \end{array} \right\}$

7. When a particular driving force is applied to a stretched string it is observed that the string vibration in purely of the n^{th} harmonic. Find the driving force.

8. Consider the two-mass system pivoted at its vertex where $M \neq m$. It undergoes oscillations of the angle θ with respect to the vertical in the plane of the triangle.

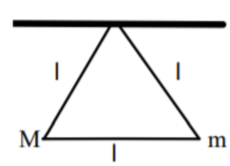

Figure $3.E.1$

a. Determine the angular frequency of small oscillations.
b. Use your result from part (a) to show $\omega 2 \approx \frac{g}{l}$ for $M \gg m$.
c. Show that your result from part (a) agrees with $\omega^2 = \frac{U''(\theta_e)}{I}$ where θ_e is the equilibrium angle and I is the moment of inertia.
d. Assume the system has energy E. Setup an integral that determines the period of oscillation.

9. An unusual pendulum is made by fixing a string to a horizontal cylinder of radius R, wrapping the string several times around the cylinder, and then tying a mass m to the loose end. In equilibrium the mass hangs a distance l_0 vertically below the edge of the cylinder. Find the potential energy if the pendulum has swung to an angle ϕ from the vertical. Show that for small angles, it can be written in the Hooke's Law form $U = \frac{1}{2}k\phi^2$. Comment of the value of k.

10. Consider the two-dimensional anisotropic oscillator with motion with $\omega_x = p\omega$ and $\omega_y = q\omega$.

a. Prove that if the ratio of the frequencies is rational (that is, $\frac{\omega_x}{\omega_y} = \frac{p}{q}$ where p and q are integers) then the motion is periodic. What is the period?
b. Prove that if the same ratio is irrational, the motion never repeats itself.

11. A simple pendulum consists of a mass m suspended from a fixed point by a weight-less, extensionless rod of length l.

a. Obtain the equation of motion, and in the approximation $\sin\theta \approx \theta$, show that the natural frequency is $\omega_0 = \sqrt{\frac{g}{l}}$, where g is the gravitational field strength.
b. Discuss the motion in the event that the motion takes place in a viscous medium with retarding force $2m\sqrt{gl}\dot{\theta}$.

12. Derive the expression for the State Space paths of the plane pendulum if the total energy is $E > 2\ mgl$. Note that this is just the case of a particle moving in a periodic potential $U(\theta) = mgl(1-\cos\theta)$. Sketch the State Space diagram for both $E > 2\ mgl$ and $E < 2\ mgl$.

13. Consider the motion of a driven linearly-damped harmonic oscillator after the transient solution has died out, and suppose that it is being driven close to resonance, $\omega = \omega_o$.

a. Show that the oscillator's total energy is $E = \frac{1}{2}m\omega^2 A^2$.
b. Show that the energy ΔE_{dis} dissipated during one cycle by the damping force $\Gamma\dot{x}$ is $\pi\Gamma m\omega A^2$

14. Two masses m_1 and m_2 slide freely on a horizontal frictionless rail and are connected by a spring whose force constant is k. Find the frequency of oscillatory motion for this system.

15. A particle of mass m moves under the influence of a resistive force proportional to velocity and a potential U, that is l.

$$F(x,\dot{x}) = -b\dot{x} - \frac{\partial U}{\partial x}$$

where $b > 0$ and $U(x) = (x^2 - a^2)^2$

a. Find the points of stable and unstable equilibrium.
b. Find the solution of the equations of motion for small oscillations around the stable equilibrium points
c. Show that as $t \to \infty$ the particle approaches one of the stable equilibrium points for most choices of initial conditions. What are the exceptions? (Hint: You can prove this without finding the solutions explicitly.)