7.9: Generalized energy and total energy

The generalized kinetic energy, equation $(7.6.4)$, can be used to write the generalized Lagrangian as

$$L(\mathbf{q},\dot{\mathbf{q}},t)=T_2(\mathbf{q},\dot{\mathbf{q}},t)+T_1(\mathbf{q},\dot{\mathbf{q}},t)+T_0(\mathbf{q},t)-U(\mathbf{q},t) \tag{7.9.1}$$

If the potential energy U does not depend explicitly on velocities $\dot{q}_i$ or time, then

$$p_j=\frac{\partial L}{\partial \dot{q}_j}=\frac{\partial (T-U)}{\partial \dot{q}_j}=\frac{\partial T}{\partial \dot{q}_j} \tag{7.9.2}$$

Equation 7.9.2 can be used to write the **Hamiltonian**, equation $(7.7.6)$, as

$$H(\mathbf{q},\mathbf{p},t)=\sum_i\left(\dot{q}_j\frac{\partial T_2}{\partial \dot{q}_j}\right)+\sum_i\left(\dot{q}_j\frac{\partial T_1}{\partial \dot{q}_j}\right)+\sum_i\left(\dot{q}_j\frac{\partial T_0}{\partial \dot{q}_j}\right)-L(\mathbf{q},\dot{\mathbf{q}},t) \tag{7.9.3}$$

Using equations $(7.6.12)$, $(7.6.13)$, $(7.6.14)$ gives that the total generalized Hamiltonian $H(\mathbf{q},\mathbf{p},t)$ equals

$$H(\mathbf{q},\mathbf{p},t)=2T_2+T_1-(T_2+T_1+T_0-U)=T_2-T_0+U \tag{7.9.4}$$

But the sum of the kinetic and potential energies equals the total energy. Thus Equation 7.9.4 can be rewritten in the form

$$H(\mathbf{q},\mathbf{p},t)=(T+U)-(T_1+2T_0)=E-(T_1+2T_0) \tag{7.9.5}$$

Note that Jacobi's generalized energy and the Hamiltonian do not equal the total energy E. However, in the special case where the transformation is scleronomic, then $T_1=T_0=0$, and if the potential energy U does not depend explicitly of $\dot{q}_i$, then the generalized energy (Hamiltonian) equals the total energy, that is, $H=E$. Recognition of the relation between the Hamiltonian and the total energy facilitates determining the equations of motion.