1.5: Variational methods in physics

Pierre de Fermat (1601-1665) revived the principle of least time, which states that *light travels between two given points along the path of shortest time* and was used to derive Snell's law in 1657. This enunciation of variational principles in physics played a key role in the historical development of the variational principle of least action that underlies the analytical formulations of classical mechanics.

Gottfried Leibniz (1646-1716) made significant contributions to the development of variational principles in classical mechanics. In contrast to Newton's laws of motion, which are based on the concept of momentum, Leibniz devised a new theory of dynamics based on kinetic and potential energy that anticipates the analytical variational approach of Lagrange and Hamilton. Leibniz argued for a quantity called the "vis viva", which is Latin for living force, that equals twice the kinetic energy. Leibniz argued that the change in kinetic energy is equal to the work done. In 1687 Leibniz proposed that the optimum path is based on minimizing the time integral of the vis viva, which is equivalent to the action integral. Leibniz used both philosophical and causal arguments in his work which were acceptable during the Age of Enlightenment. Unfortunately for Leibniz, his analytical approach based on energies, which are scalars, appeared contradictory to Newton's intuitive vectorial treatment of force and momentum. There was considerable prejudice and philosophical opposition to the variational approach which assumes that nature is thrifty in all of its actions. The variational approach was considered to be speculative and "metaphysical" in contrast to the causal arguments supporting Newtonian mechanics. This opposition delayed full appreciation of the variational approach until the start of the 20^{th} century.

Johann Bernoulli (1667-1748) was a Swiss mathematician who was a student of Leibniz's calculus, and sided with Leibniz in the Newton-Leibniz dispute over the credit for developing calculus. Also Bernoulli sided with the Descartes' vortex theory of gravitation which delayed acceptance of Newton's theory of gravitation in Europe. Bernoulli pioneered development of the calculus of variations by solving the problems of the catenary, the brachistochrone, and Fermat's principle. Johann Bernoulli's son Daniel played a significant role in the development of the well-known Bernoulli Principle in hydrodynamics.

Pierre Louis Maupertuis (1698-1759) was a student of Johann Bernoulli and conceived the universal hypothesis that in nature there is a certain quantity called action which is minimized. Although this bold assumption correctly anticipates the development of the variational approach to classical mechanics, he obtained his hypothesis by an entirely incorrect method. He was a dilettante whose mathematical prowess was behind the high standards of that time, and he could not establish satisfactorily the quantity to be minimized. His teleological[1] argument was influenced by Fermat's principle and the corpuscle theory of light that implied a close connection between optics and mechanics.

Leonhard Euler (1707-1783) was the preeminent Swiss mathematician of the 18^{th} century and was a student of Johann Bernoulli. Euler developed, with full mathematical rigor, the calculus of variations following in the footsteps of Johann Bernoulli. Euler used variational calculus to solve minimum/maximum isoperimetric problems that had attracted and challenged the early developers of calculus, Newton, Leibniz, and Bernoulli. Euler also was the first to solve the rigid-body rotation problem using the three components of the angular velocity as kinematical variables. Euler became blind in both eyes by 1766 but that did not hinder his prolific output in mathematics due to his remarkable memory and mental capabilities. Euler's contributions to mathematics are remarkable in quality and quantity; for example during 1775 he published one mathematical paper per week in spite of being blind. Euler implicitly implied the principle of least action using vis visa which is not the exact form explicitly developed by Lagrange.

Jean le Rond d'Alembert (1717-1785) was a French mathematician and physicist who had the clever idea of extending use of the principle of virtual work from statics to dynamics. D'Alembert's Principle rewrites the principle of virtual work in the form

$$\sum_{i=1}^{N}(\mathbf{F}_i - \dot{\mathbf{p}}_i)\delta\mathbf{r}_i = 0$$

where the inertial reaction force $\dot{\mathbf{p}}$ is subtracted from the corresponding force $\mathbf{F}$. This extension of the principle of virtual work applies equally to both statics and dynamics leading to a single variational principle.

Joseph Louis Lagrange (1736-1813) was an Italian mathematician and a student of Leonhard Euler. In 1788 Lagrange published his monumental treatise on analytical mechanics entitled Mécanique Analytique which introduces his Lagrangian mechanics analytical technique which is based on d'Alembert's Principle of Virtual Work. Lagrangian mechanics is a remarkably powerful technique that is equivalent to minimizing the action integral S defined as

$$S=\int_{t_1}^{t_2} L dt$$

The Lagrangian L frequently is defined to be the difference between the kinetic energy T and potential energy V. His theory only required the analytical form of these scalar quantities. In the preface of his book he refers modestly to his extraordinary achievements with the statement “The reader will find no figures in the work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure.” Lagrange also introduced the concept of undetermined multipliers to handle auxiliary conditions which plays a vital part of theoretical mechanics. William Hamilton, an outstanding figure in the analytical formulation of classical mechanics, called Lagrange the “Shakespeare of mathematics,” on account of the extraordinary beauty, elegance, and depth of the Lagrangian methods. Lagrange also pioneered numerous significant contributions to mathematics. For example, Euler, Lagrange, and d’Alembert developed much of the mathematics of partial differential equations. Lagrange survived the French Revolution, and, in spite of being a foreigner, Napoleon named Lagrange to the Legion of Honour and made him a Count of the Empire in 1808. Lagrange was honoured by being buried in the Pantheon.

Carl Friedrich Gauss (1777-1855) was a German child prodigy who made many significant contributions to mathematics, astronomy and physics. He did not work directly on the variational approach, but Gauss’s law, the divergence theorem, and the Gaussian statistical distribution are important examples of concepts that he developed and which feature prominently in classical mechanics as well as other branches of physics, and mathematics.

Simeon Poisson (1781-1840), was a brilliant mathematician who was a student of Lagrange. He developed the Poisson statistical distribution as well as the Poisson equation that features prominently in electromagnetic and other field theories. His major contribution to classical mechanics is development, in 1809, of the Poisson bracket formalism which featured prominently in development of Hamiltonian mechanics and quantum mechanics.

The zenith in development of the variational approach to classical mechanics occurred during the 19^{th} century primarily due to the work of Hamilton and Jacobi.

William Hamilton (1805-1865) was a brilliant Irish physicist, astronomer and mathematician who was appointed professor of astronomy at Dublin when he was barely 22 years old. He developed the Hamiltonian mechanics formalism of classical mechanics which now plays a pivotal role in modern classical and quantum mechanics. He opened an entirely new world beyond the developments of Lagrange. Whereas the Lagrange equations of motion are complicated second-order differential equations, Hamilton succeeded in transforming them into a set of first-order differential equations with twice as many variables that consider momenta and their conjugate positions as independent variables. The differential equations of Hamilton are linear, have separated derivatives, and represent the simplest and most desirable form possible for differential equations to be used in a variational approach. Hence the name “canonical variables” given by Jacobi. Hamilton exploited the d’Alembert principle to give the first exact formulation of the principle of least action which underlies the variational principles used in analytical mechanics. The form derived by Euler and Lagrange employed the principle in a way that applies only for conservative (scleronomic) cases. A significant discovery of Hamilton is his realization that classical mechanics and geometrical optics can be handled from one unified viewpoint. In both cases he uses a “characteristic” function that has the property that, by mere differentiation, the path of the body, or light ray, can be determined by the same partial differential equations. This solution is equivalent to the solution of the equations of motion.

Carl Gustave Jacob Jacobi (1804-1851), a Prussian mathematician and contemporary of Hamilton, made significant developments in Hamiltonian mechanics. He immediately recognized the extraordinary importance of the Hamiltonian formulation of mechanics. Jacobi developed canonical transformation theory and showed that the function, used by Hamilton, is only one special case of functions that generate suitable canonical transformations. He proved that any complete solution of the partial differential equation, without the specific boundary conditions applied by Hamilton, is sufficient for the complete integration of the equations of motion. This greatly extends the usefulness of Hamilton’s partial differential equations. In 1843 Jacobi developed both the Poisson brackets, and the Hamilton-Jacobi, formulations of Hamiltonian mechanics. The latter gives a single, first-order partial differential equation for the action function in terms of the n generalized coordinates which greatly simplifies solution of the equations of motion. He also derived a principle of least action for time-independent cases that had been studied by Euler and Lagrange. Jacobi developed a superior approach to the variational integral that, by eliminating time from the integral, determined the path without saying anything about how the motion occurs in time.

James Clerk Maxwell (1831-1879) was a Scottish theoretical physicist and mathematician. His most prominent achievement was formulating a classical electromagnetic theory that united previously unrelated observations, plus equations of electricity, magnetism and optics, into one consistent theory. Maxwell's equations demonstrated that electricity, magnetism and light are all manifestations of the same phenomenon, namely the electromagnetic field. Consequently, all other classic laws and equations of electromagnetism were simplified cases of Maxwell's equations. Maxwell's achievements concerning electromagnetism have been called the "second great unification in physics". Maxwell demonstrated that electric and magnetic fields travel through space in the form of waves, and at a constant speed of light. In 1864 Maxwell wrote "A Dynamical Theory of the Electromagnetic Field" which proposed that light was in fact undulations in the same medium that is the cause of electric and magnetic phenomena. His work in producing a unified model of electromagnetism is one of the greatest advances in physics. Maxwell, in collaboration with **Ludwig Boltzmann (1844-1906)**, also helped develop the Maxwell—Boltzmann distribution, which is a statistical means of describing aspects of the kinetic theory of gases. These two discoveries helped usher in the era of modern physics, laying the foundation for such fields as special relativity and quantum mechanics. Boltzmann founded the field of statistical mechanics and was an early staunch advocate of the existence of atoms and molecules.

Henri Poincaré (1854-1912) was a French theoretical physicist and mathematician. He was the first to present the Lorentz transformations in their modern symmetric form and discovered the remaining relativistic velocity transformations. Although there is similarity to Einstein's Special Theory of Relativity, Poincaré and Lorentz still believed in the concept of the ether and did not fully comprehend the revolutionary philosophical change implied by Einstein. Poincaré worked on the solution of the three-body problem in planetary motion and was the first to discover a chaotic deterministic system which laid the foundations of modern chaos theory. It rejected the long-held deterministic view that if the position and velocities of all the particles are known at one time, then it is possible to predict the future for all time.

The last two decades of the 19^{th} century saw the culmination of classical physics and several important discoveries that led to a revolution in science that toppled classical physics from its throne. The end of the 19^{th} century was a time during which tremendous technological progress occurred; flight, the automobile, and turbine-powered ships were developed, Niagara Falls was harnessed for power, etc. During this period, **Heinrich Hertz (1857-1894)** produced electromagnetic waves confirming their derivation using Maxwell's equations. Simultaneously he discovered the photoelectric effect which was crucial evidence in support of quantum physics. Technical developments, such as photography, the induction spark coil, and the vacuum pump played a significant role in scientific discoveries made during the 1890's. At the end of the 19^{th} century, scientists thought that the basic laws were understood and worried that future physics would be in the fifth decimal place; some scientists worried that little was left for them to discover. However, there remained a few, presumed minor, unexplained discrepancies plus new discoveries that led to the revolution in science that occurred at the beginning of the 20^{th} century.

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[1]Teleology is any philosophical account that holds that final causes exist in nature, analogous to purposes found in human actions, nature inherently tends toward definite ends.
