5.8: Generalized coordinates in Variational Calculus

Newtonian mechanics is based on a vectorial treatment of mechanics which can be difficult to apply when solving complicated problems in mechanics. Constraint forces acting on a system usually are unknown. In Newtonian mechanics constrained forces must be included explicitly so that they can be determined simultaneously with the solution of the dynamical equations of motion. The major advantage of the variational approaches is that solution of the dynamical equations of motion can be simplified by expressing the motion in terms of n independent **generalized coordinates.** These generalized coordinates can be any set of **independent variables**, q_i, where $1 \leq i \leq n$, plus the corresponding velocities $\dot{q}_i$ for Lagrangian mechanics, or the corresponding canonical variables, q_i, p_i for Hamiltonian mechanics. These generalized coordinates for the n variables are used to specify the scalar functional dependence on these generalized coordinates. The variational approach employs this scalar functional to determine the trajectory. The generalized coordinates used for the variational approach do not need to be orthogonal, they only need to be independent since they are used only to completely specify the *magnitude* of the scalar functional. This greatly expands the arsenal of possible generalized coordinates beyond what is available using Newtonian mechanics. For example, generalized coordinates can be the dimensionless amplitudes for the n normal modes of coupled oscillator systems, or action-angle variables. In addition, generalized coordinates having different dimensions can be used for each of the n variables. *Each generalized coordinate, q_i specifies an independent mode of the system, not a specific particle*. For example, each normal mode of coupled oscillators can involve correlated motion of several coupled particles. The major advantage of using generalized coordinates is that they can be chosen to be perpendicular to a corresponding constraint force, and therefore that specific constraint force does no work for motion along that generalized coordinate. Moreover, the constrained motion does no work in the direction of the constraint force for rigid constraints. Thus generalized coordinates allow specific constraint forces to be ignored in evaluation of the minimized functional. This freedom and flexibility of choice of generalized coordinates allows the correlated motion produced by the constraint forces to be embedded directly into the choice of the independent generalized coordinates, and the actual constraint forces can be ignored. Embedding of the constraint induced correlations into the generalized coordinates, effectively "sweeps the constraint forces under the rug" which greatly simplifies the equations of motion for any system that involve constraint forces. Selection of the appropriate generalized coordinates can be obvious, and often it is performed subconsciously by the user.

Three variational approaches are used that employ generalized coordinates to derive the equations of motion of a system that has n generalized coordinates subject to m constraints.

1) Minimal set of generalized coordinates: When the m equations of constraint are holonomic, then the m algebraic constraint relations can be used to transform the coordinates into $s = n - m$ independent *generalized coordinates* q_i. This approach reduces the number of unknowns, n, by the number of constraints m, to give a minimal set of $s = n - m$ independent generalized dynamical variables. The forces of constraint are not explicitly discussed, or determined, when this generalized coordinate approach is employed. This approach greatly simplifies solution of dynamical problems by avoiding the need for explicit treatment of the constraint forces. This approach is straight forward for holonomic constraints, since the n *spatial coordinates* $y_1(x), \ldots y_N(x)$, are coupled by m algebraic equations which can be used to make the transformation to generalized coordinates. Thus the n *coupled spatial coordinates* are transformed to $s = n - m$ *independent generalized dynamical coordinates* $q_1(x), \ldots . q_s(x)$, and their generalized first derivatives $\dot{q}_1(x), \ldots . \dot{q}_s(x)$. These *generalized coordinates are independent,* and thus it is possible to use Euler's equation for each independent parameter q_i

$$\frac{\partial f}{\partial q_i} - \frac{d}{dx}\frac{\partial f}{\partial q_i'} = 0 \tag{5.8.1}$$

where $i = 1, 2, 3 \ldots s$. There are $s = n - m$ such Euler equations. The freedom to choose generalized coordinates underlies the tremendous advantage of applying the variational approach.

2) Lagrange multipliers: The n Lagrange equations, plus the m equations of constraint, can be used to explicitly determine the n generalized coordinates plus the m constraint forces. That is, $n + m$ unknowns are determined. This approach is discussed in chapter 5.9.

3) Generalized forces: This approach introduces the constraint forces explicitly. This approach, applied to Lagrangian mechanics, is discussed in chapter 6.6.3.

The above three approaches exploit generalized coordinates to handle constraint forces as described in chapter 6.