# 13.S: Rigid-body Rotation (Summary)

This chapter has introduced the important, topic of rigid-body rotation which has many applications in physics, engineering, sports, etc.

## Inertia tensor

The concept of the inertia tensor was introduced where the 9 components of the inertia tensor are given by

$$I_{ij} = \int \rho(\mathbf{r}') \left( \delta_{ij} \left( \sum_k^3 x_k^2 \right) - x_i x_j \right) dV \tag{13.S.1}$$

Steiner's parallel-axis theorem

$$J_{11} \equiv I_{11} + M((a_1^2 + a_2^2 + a_3^3)\delta_{11} - a_1^2) = I_{11} + M(a_2^2 + a_3^2) \tag{13.S.2}$$

relates the inertia tensor about the center-of-mass to that about parallel axis system not through the center of mass.

Diagonalization of the inertia tensor about any point was used to find the corresponding Principal axes of the rigid body.

## Angular momentum

The angular momentum $\mathbf{L}$ for rigid-body rotation is expressed in terms of the inertia tensor and angular frequency $\omega$ by

$$\mathbf{L} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \{\mathbf{I}\} \cdot \boldsymbol{\omega} \tag{13.S.3}$$

## Rotational kinetic energy

The rotational kinetic energy is

$$T_{rot} = \frac{1}{2}(\omega_1\ \omega_2\ \omega_3) \cdot \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{13.S.4}$$

$$T_{rot} \equiv \mathbf{T} = \frac{1}{2}\boldsymbol{\omega} \cdot \{\mathbf{I}\} \cdot \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} \tag{13.S.5}$$

## Euler angles

The Euler angles relate the space-fixed and body-fixed principal axes. The angular velocity $\boldsymbol{\omega}$ expressed in terms of the Euler angles has components for the angular velocity in the *body-fixed axis system* $(1, 2, 3)$

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \tag{13.S.6}$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \tag{13.S.7}$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi} \cos\theta + \dot{\psi} \tag{13.S.8}$$

Similarly, the components of the angular velocity for the *space-fixed axis system* $(x, y, z)$ are

$$\omega_x = \dot{\theta} \cos\phi + \dot{\psi} \sin\theta \sin\phi \tag{13.S.9}$$

$$\omega_y = \dot{\theta} \sin\phi - \dot{\psi} \sin\theta \cos\phi \tag{13.S.10}$$

$$\omega_z = \dot{\phi} + \dot{\psi} \cos\theta \tag{13.S.11}$$

## Rotational invariants

The powerful concept of the rotational invariance of scalar properties was introduced. Important examples of rotational invariants are the Hamiltonian, Lagrangian, and Routhian.

## Euler equations of motion for rigid-body motion

The dynamics of rigid-body rotational motion was explored and the Euler equations of motion were derived using both Newtonian and Lagrangian mechanics.

$$\begin{aligned} N_1^{ext} &= I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 \\ N_2^{ext} &= I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \\ N_3^{ext} &= I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 \end{aligned} \tag{13.S.12}$$

## Lagrange equations of motion for rigid-body motion

The Euler equations of motion for rigid-body motion, given in Equation 13.S.12, were derived using the Lagrange-Euler equations.

## Torque-free motion of rigid bodies

The Euler equations and Lagrangian mechanics were used to study torque-free rotation of both symmetric and asymmetric bodies including discussion of the stability of torquefree rotation.

## Rotating symmetric body subject to a torque

The complicated motion exhibited by a symmetric top, that is spinning about one fixed point and subject to a torque, was introduced and solved using Lagrangian mechanics.

## The rolling wheel

The non-holonomic motion of rolling wheels was introduced, as well as the importance of static and dynamic balancing of rotating machinery..

## Rotation of deformable bodies

The complicated non-holonomic motion involving rotation of deformable bodies was introduced.