11.3: Angular Momentum

Angular momentum $\mathbf{L}$

The notation used for the angular momentum vector is $\mathbf{L}$ where the magnitude is designated by $|\mathbf{L}| = l$. Be careful not to confuse the angular momentum vector $\mathbf{L}$ with the Lagrangian L_{cm}. Note that the angular momentum for two-body rotation about the center of mass with angular velocity ω is identical when evaluated in either the laboratory or equivalent two-body representation. That is, using equations $(11.2.6)$ and $(11.2.7)$

$$\mathbf{L} = \mathbf{m}_1 r_1'^2 \omega + \mathbf{m}_2 r_2'^2 \omega = \mu r^2 \omega \tag{11.3.1}$$

The center-of-mass Lagrangian leads to the following two general properties regarding the angular momentum vector $\mathbf{L}$.

1) The motion lies entirely in a plane perpendicular to the fixed direction of the total angular momentum vector. This is because

$$\mathbf{L} \cdot \mathbf{r} = \mathbf{r} \times \mathbf{p} \cdot \mathbf{r} = 0 \tag{11.3.2}$$

that is, *the radius vector is in the plane perpendicular to the total angular momentum vector*. Thus, it is possible to express the Lagrangian in polar coordinates, (r, ψ) rather than spherical coordinates. In polar coordinates the center-of-mass Lagrangian becomes

$$L_{cm} = \frac{1}{2}\mu \left(\dot{r}^2 + r^2 \dot{\psi}^2\right) - U(r) \tag{11.3.3}$$

2) If the potential is spherically symmetric, then the polar angle ψ is cyclic and therefore Noether's theorem gives that the angular momentum $\mathbf{p}_\psi \equiv \mathbf{L} = \mathbf{r} \times \mathbf{p}$ is a constant of motion. That is, since $\frac{\partial L_{cm}}{\partial \psi} = 0$, then the Lagrange equations imply that

$$\dot{\mathbf{p}}_\psi = \frac{d}{dt}\frac{\partial L_{cm}}{\partial \dot{\psi}} = 0 \tag{11.3.4}$$

where the vectors $\dot{\mathbf{p}}_\psi$ and $\dot{\psi}$ imply that Equation 11.3.4 refers to three independent equations corresponding to the three components of these vectors. *Thus the angular momentum* $\mathbf{p}_\psi$*, conjugate to* ψ*, is a constant of motion.* The generalized momentum $\mathbf{p}_\psi$ is a first integral of the motion which equals

$$\mathbf{p}_\psi = \frac{\partial L_{cm}}{\partial \dot{\psi}} = \mu r^2 \dot{\psi} = \hat{\mathbf{p}}_\psi l \tag{11.3.5}$$

where the magnitude of the angular momentum l, and the direction $\hat{\mathbf{p}}_\psi$, both are constants of motion.

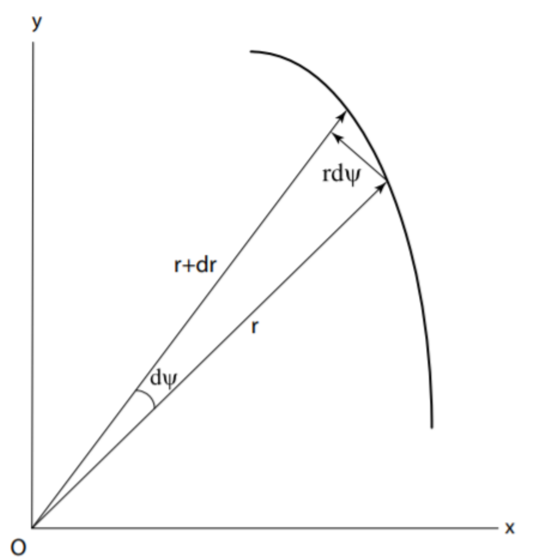

Figure 11.3.1: Area swept out by the radius vector in the time dt.

A simple geometric interpretation of Equation 11.3.5 is illustrated in Figure 11.3.1. The radius vector sweeps out an area $d\mathbf{A}$ in time dt where

$$d\mathbf{A} = \frac{1}{2}\mathbf{r} \times \mathbf{v}dt \tag{11.3.6}$$

and the vector $\mathbf{A}$ is perpendicular to the $x - y$ plane. The rate of change of area is

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2}\mathbf{r} \times \mathbf{v} \tag{11.3.7}$$

But the angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mu\mathbf{r} \times \mathbf{v} = 2\mu\frac{d\mathbf{A}}{dt} \tag{11.3.8}$$

Thus the conservation of angular momentum implies that the areal velocity $\frac{dA}{dt}$ also is a constant of motion. This fact is called Kepler's second law of planetary motion which he deduced in 1609 based on Tycho Brahe's 55 years of observational records of the motion of Mars. Kepler's second law implies that a planet moves fastest when closest to the sun and slowest when farthest from the sun. Note that Kepler's second law is a statement of *the conservation of angular momentum which is independent of the radial form of the central potential.*