# 4.1: Introduction to Nonlinear Systems and Chaos

In nature only a subset of systems have equations of motion that are linear. Contrary to the impression given by the analytic solutions presented in undergraduate physics courses, most dynamical systems in nature exhibit non-linear behavior that leads to complicated motion. The solutions of non-linear equations usually do not have analytic solutions, superposition does not apply, and they predict phenomena such as attractors, discontinuous period bifurcation, extreme sensitivity to initial conditions, rolling motion, and chaos. During the past four decades, exciting discoveries have been made in classical mechanics that are associated with the recognition that nonlinear systems can exhibit chaos. Chaotic phenomena have been observed in most fields of science and engineering such as, weather patterns, fluid flow, motion of planets in the solar system, epidemics, changing populations of animals, birds and insects, and the motion of electrons in atoms. The complicated dynamical behavior predicted by non-linear differential equations is not limited to classical mechanics, rather it is a manifestation of the mathematical properties of the solutions of the differential equations involved, and thus is generally applicable to solutions of first or second-order non-linear differential equations. It is important to understand that the systems discussed in this chapter follow a fully deterministic evolution predicted by the laws of classical mechanics, the evolution for which is based on the prior history. This behavior is completely different from a random walk where each step is based on a random process. The complicated motion of deterministic non-linear systems stems in part from sensitivity to the initial conditions. There are many examples of turbulent and laminar flow.

The French mathematician Poincaré is credited with being the first to recognize the existence of chaos during his investigation of the gravitational three-body problem in celestial mechanics. At the end of the nineteenth century Poincaré noticed that such systems exhibit high sensitivity to initial conditions characteristic of chaotic motion, and the existence of nonlinearity which is required to produce chaos. Poincaré’s work received little notice, in part it was overshadowed by the parallel development of the Theory of Relativity and quantum mechanics at the start of the $20^{th}$ century. In addition, solving nonlinear equations of motion is difficult, which discouraged work on nonlinear mechanics and chaotic motion. The field blossomed during the $1960's$ when computers became sufficiently powerful to solve the nonlinear equations required to calculate the long-time histories necessary to document the evolution of chaotic behavior.

Laplace, and many other scientists, believed in the deterministic view of nature which assumes that if the position and velocities of all particles are known, then one can unambiguously predict the future motion using Newtonian mechanics. Researchers in many fields of science now realize that this “clockwork universe" is invalid. That is, knowing the laws of nature can be insufficient to predict the evolution of nonlinear systems in that the time evolution can be extremely sensitive to the initial conditions even though they follow a completely deterministic development. There are two major classifications of nonlinear systems that lead to chaos in nature. The first classification encompasses nondissipative Hamiltonian systems such as Poincaré’s three-body celestial mechanics system. The other main classification involves driven, damped, non-linear oscillatory systems.

Nonlinearity and chaos is a broad and active field and thus this chapter will focus only on a few examples that illustrate the general features of non-linear systems. Weak non-linearity is used to illustrate bifurcation and asymptotic attractor solutions for which the system evolves independent of the initial conditions. The common sinusoidally-driven linearly-damped plane pendulum illustrates several features characteristic of the evolution of a non-linear system from order to chaos. The impact of non-linearity on wavepacket propagation velocities and the existence of soliton solutions is discussed. The example of the three-body problem is discussed in chapter $11$. The transition from laminar flow to turbulent flow is illustrated by fluid mechanics discussed in chapter $16.8$. Analytic solutions of nonlinear systems usually are not available and thus one must resort to computer simulations. As a consequence the present discussion focusses on the main features of the solutions for these systems and ignores how the equations of motion are solved.