4.5: Gradient

The gradient operator is an important and useful tool in electromagnetic theory. Here's the main idea:

> The *gradient* of a scalar field is a vector that points in the direction in which the field is most rapidly increasing, with the scalar part equal to the rate of change.

A particularly important application of the gradient is that it relates the electric field intensity $\mathbf{E}(\mathbf{r})$ to the electric potential field $V(\mathbf{r})$. This is apparent from a review of Section 2.2; see in particular, the battery-charged capacitor example. In that example, it is demonstrated that:

- The *direction* of $\mathbf{E}(\mathbf{r})$ is the direction in which $V(\mathbf{r})$ decreases most quickly, and
- The *scalar part* of $\mathbf{E}(\mathbf{r})$ is the rate of change of $V(\mathbf{r})$ in that direction. Note that this is also implied by the units, since $V(\mathbf{r})$ has units of V whereas $\mathbf{E}(\mathbf{r})$ has units of V/m.

The gradient is the mathematical operation that relates the vector field $\mathbf{E}(\mathbf{r})$ to the scalar field $V(\mathbf{r})$ and is indicated by the symbol "∇" as follows:

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

or, with the understanding that we are interested in the gradient as a function of position $\mathbf{r}$, simply

$$\mathbf{E} = -\nabla V$$

At this point we should note that the gradient is a very general concept, and that we have merely identified one application of the gradient above. In electromagnetics there are many situations in which we seek the gradient ∇f of some scalar field $f(\mathbf{r})$. Furthermore, we find that other differential operators that are important in electromagnetics can be interpreted in terms of the gradient operator ∇. These include *divergence* (Section 4.6), *curl* (Section 4.8), and the *Laplacian* (Section 4.10).

In the Cartesian system:

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z} \tag{4.5.1}$$

Example 4.5.1: Gradient of a ramp function.

Find the gradient of $f = ax$ (a "ramp" having slope a along the x direction).

Solution

Here, $\partial f/\partial x = a$ and $\partial f/\partial y = \partial f/\partial z = 0$. Therefore $\nabla f = \hat{\mathbf{x}}a$. Note that ∇f points in the direction in which f most rapidly increases, and has magnitude equal to the slope of f in that direction.

The gradient operator in the cylindrical and spherical systems is given in Appendix B2.