4.10: The Laplacian Operator

The Laplacian $\nabla^2 f$ of a field $f(\mathbf{r})$ is the divergence of the gradient of that field:

$$\nabla^2 f \triangleq \nabla \cdot (\nabla f) \tag{4.10.1}$$

Note that the Laplacian is essentially a definition of the second derivative with respect to the three spatial dimensions. For example, in Cartesian coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{4.10.2}$$

as can be readily verified by applying the definitions of gradient and divergence in Cartesian coordinates to Equation 4.10.1.

The Laplacian relates the electric potential (i.e., V, units of V) to electric charge density (i.e., ρ_v, units of C/m^3). This relationship is known as *Poisson's Equation*:

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

where ϵ is the permittivity of the medium. The fact that V is related to ρ_v in this way should not be surprising, since electric field intensity ($\mathbf{E}$, units of V/m) is proportional to the derivative of V with respect to distance (via the gradient) and ρ_v is proportional to the derivative of $\mathbf{E}$ with respect to distance (via the divergence).

The Laplacian operator can also be applied to *vector* fields; for example, Equation 4.10.2 is valid even if the scalar field "f" is replaced with a vector field. In the Cartesian coordinate system, the Laplacian of the vector field $\mathbf{A} = \hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z$ is

$$\nabla^2 \mathbf{A} = \hat{\mathbf{x}} \nabla^2 A_x + \hat{\mathbf{y}} \nabla^2 A_y + \hat{\mathbf{z}} \nabla^2 A_z$$

An important application of the Laplacian operator of vector fields is the *wave equation*; e.g., the wave equation for $\mathbf{E}$ in a lossless and source-free region is

$$\nabla^2 \mathbf{E} + \beta^2 \mathbf{E} = 0$$

where β is the **phase propagation constant**.

It is sometimes useful to know that the Laplacian of a vector field can be expressed in terms of the gradient, divergence, and curl as follows:

$$\nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

The Laplacian operator in the cylindrical and spherical coordinate systems is given in Appendix B2.