6.1: Schrödinger's Equation

In 1925 Erwin Schrödinger proposed a differential equation that, when solved, produced a complete mathematical description of the wavefunction, $\psi(x)$, of a "particle" moving in a region of space with potential energy function $U(x)$. The one-dimensional, time-independent, non-relativistic form of this equation is:

$$-\frac{\hbar^2}{2m}\frac{d}{dx^2}\psi(x)+U(x)\psi x=E\psi(x) \tag{6.1.1}$$

where E is the total energy of the system. The wavefunction can be thought of as the amplitude of the "wave" representing the "particle". Physically, it turns out that the square of the wavefunction is equal to the probability of finding the particle at a particular region of space.

Although this equation cannot be "derived" from any other physics principle, it can be shown to at least be consistent with the conservation of energy. Assuming the wavefunction takes the form of a sum of sine waves of the form:

$$\psi(x)=A\sin(kx) \tag{6.1.2}$$

with the wavelength (λ) defined as

$$k=\frac{2\pi}{\lambda} \tag{6.1.3}$$

Substituting Equation 6.1.2 into Equation 6.1.1 yields

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d}{dx^2}A\sin(kx)+U(x)A\sin(kx) &= EA\sin(kx) \\ +\frac{\hbar^2}{2m}k^2A\sin(kx)+U(x)A\sin(kx) &= EA\sin(kx) \\ \frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2+U(x) &= E \\ \frac{h^2}{2m\lambda^2}+U(x) &= E \end{aligned} \tag{6.1.4}$$

Substituting DeBroglie's relation into Equation 6.1.4 yields

$$\begin{aligned} \frac{h^2}{2m\left(\frac{h}{p}\right)^2}+U(x) &= E \\ \frac{p^2}{2m}+U(x) &= E \\ \frac{(mv)^2}{2m}+U(x) &= E \\ \frac{1}{2}mv^2+U(x) &= E \\ KE+U(x) &= E \end{aligned} \tag{6.1.5}$$

For different potential energy functions, we will solve Schrödinger's equation for the allowed values of total energy, E, and the exact mathematical form of the function, $\psi(x)$, describing the object moving in the region of potential energy. Schrödinger was awarded the Nobel Prize in 1933 primarily for the development of this equation.