6.7: Barrier Penetration and Tunneling

Estimate the tunneling probability for an 10 MeV proton incident on a potential barrier of height 20 MeV and width 5 fm.

Consider the square barrier shown above. In the regions x < 0 and x > L the wavefunction has the oscillatory behavior we've seen before, and can be modeled by linear combinations of sines and cosines. These regions are referred to as allowed regions because the kinetic energy of the particle (KE = E – U) is a real, positive value.

Now consider the region 0 < x < L. In this region, the wavefunction decreases exponentially, and takes the form

$$\Psi(x) = Ae^{-\alpha X} \tag{6.7.1}$$

This is referred to as a forbidden region since the kinetic energy is negative, which is forbidden in classical physics. However, the probability of finding the particle in this region is not zero but rather is given by:

$$P(x) = A^2 e^{-2aX} \tag{6.7.2}$$

Thus, the particle can penetrate into the forbidden region. If the particle penetrates through the entire forbidden region, it can "appear" in the allowed region x > L. This is referred to as quantum tunneling and illustrates one of the most fundamental distinctions between the classical and quantum worlds.

A typical measure of the extent of an exponential function is the distance over which it drops to 1/e of its original value. This occurs when $x = \frac{1}{2a}$. This distance, called the penetration depth, δ, is given by

$$\delta = \frac{1}{2\alpha} \tag{6.7.3}$$

$$\delta = \frac{\hbar x}{\sqrt{8mc^2(U-E)}} \tag{6.7.4}$$

where

- U is the depth of the potential and
- E is the energy state of the wavefunction.

The penetration depth defines the approximate distance that a wavefunction extends into a forbidden region of a potential. Using this definition, the tunneling probability (T), the probability that a particle can tunnel through a classically impermeable barrier, is given by

$$T \approx e^{-x/\delta} \tag{6.7.5}$$

For this example, the probability that the proton can pass through the barrier is

$$\delta = \frac{\hbar c}{\sqrt{8mc^2(U-E)}} \tag{6.7.6}$$

$$\delta = \frac{197.3 \text{ MeVfm}}{\sqrt{8(938 \text{ MeV}}} (20 \text{ MeV -10 MeV}) \tag{6.7.7}$$

$$\delta = 0.720 \text{ fm} \tag{6.7.8}$$

Thus, there is about a one-in-a-thousand chance that the proton will tunnel through the barrier.

Tunneling In and Out

In a crude approximation of a collision between a proton and a heavy nucleus, imagine an 10 MeV proton incident on a symmetric potential well of barrier height 20 MeV, barrier width 5 fm, well depth -50 MeV, and well width 15 fm. Estimate the probability that the proton tunnels into the well. If the proton successfully tunnels into the well, estimate the lifetime of the resulting state.

In this approximation of nuclear fusion, an incoming proton can tunnel into a pre-existing nuclear well. Once in the well, the proton will remain for a certain amount of time until it tunnels back out of the well. Although the potential outside of the well is due to electric repulsion, which has the 1/r dependence shown below,

we will approximate it by a rectangular barrier:

The tunneling probability into the well was calculated above and found to be

$$T \approx 0.97x10^{-3} \tag{6.7.9}$$

All that remains is to determine how long this proton will remain in the well until tunneling back out.
First, notice that the probability of tunneling out of the well is exactly equal to the probability of tunneling in, since all of the parameters of the barrier are exactly the same.

Remember, T is now the probability of escape per collision with a well wall, so the inverse of T must be the number of collisions needed, on average, to escape. If we can determine the number of seconds between collisions, the product of this number and the inverse of T should be the lifetime ( ) of the state:

The time per collision is just the time needed for the proton to traverse the well. This is simply the width of the well (L) divided by the speed of the proton:

$$\tau = \left(\frac{L}{v}\right)\left(\frac{1}{T}\right) \tag{6.7.10}$$

The speed of the proton can be determined by relativity,

$$KE = (\gamma - 1)mc^2 \tag{6.7.11}$$

$$60 \text{ MeV} = (\gamma - 1)(938.3 \text{ MeV} \tag{6.7.12}$$

$$\gamma = 1.064 \tag{6.7.13}$$

$$v = 0.34c \tag{6.7.14}$$

$$v = 1.0x10^8 \text{ m/s} \tag{6.7.15}$$

Therefore the lifetime of the state is:

$$\tau = \left(\frac{15x10^{-15} \text{ m}}{1.0x10^8 \text{ m/s}}\right)\left(\frac{1}{0.97x10^{-3}}\right) \tag{6.7.16}$$

$$\tau = 1.5x10^{-19} \text{ s} \tag{6.7.17}$$