5.6: A Problem with DeBroglie’s Hypothesis?

Aside from the “minor” issue of trying to understand what it means for a particle to have a frequency and a wavelength, DeBroglie’s hypothesis also leads to a more technical issue regarding how the frequency and wavelength of the particle are related to its velocity. Hopefully you remember that the standard relationship between a wave’s velocity, frequency and wavelength is

$$v = f\lambda \tag{5.6.1}$$

Substituting DeBroglie’s relationships yields:

$$\begin{aligned} v &= \left(\frac{E}{h}\right)\left(\frac{h}{p}\right) \\ &= \frac{E}{p} \\ &= \frac{\sqrt{(pc)^2+(mc^2)^2}}{p} \\ &= \sqrt{c^2+\frac{(mc^2)^2}{p^2}} \end{aligned}$$

This should bother you. Why? Because the second term under the radical is obviously positive and if added to \(\ce{c^2\) seems to require that the velocity of the “matter wave” is greater than the speed of light! To resolve this apparent contradiction, we have to be much more careful in how we conceptualize the wave-representation of a particle.

The key is to distinguish between the phase velocity and group velocity as defined above. The wave packet, and hence the group velocity, is what represents the particle. The constituent waves that add together to form the packet are not, individually, physically meaningful. The simple relation between frequency and wavelength mentioned above is actually only correct for these individual constituent waves. Hence, it holds for the phase velocity only:

$$v_{phase} = f\lambda \tag{5.6.2}$$

What we need to worry about is not the value of these phase velocities, but the value of the group velocity. If the group velocity is greater than c, we’ve got some serious problems.

So how do we define the group velocity? Well, let’s first review how we define the phase velocity. The phase velocity is the speed at which a particular point on the wave moves through space. This particular point has a constant phase, so let’s set the phase in the standard definition of a wave,

$$\Psi(x,t) = A\sin(kx-\omega t) \tag{5.6.3}$$

equal to a constant. For simplicity we’ll call that constant zero.

$$(kx-\omega t) = 0 \tag{5.6.4}$$

$$kx = \omega t \tag{5.6.5}$$

$$\frac{x}{t} = \frac{\omega}{k} \tag{5.6.6}$$

$$v_{phase} = \frac{\omega}{k} \tag{5.6.7}$$

If you recall the definitions of angular velocity ($omega$) and wave number (k) from your study of waves, this reduces to:

$$\begin{aligned} v_{phase} &= \frac{2\pi f}{\left(\frac{2\pi}{\lambda}\right)} \\ &= f\lambda \end{aligned}$$

Ok, so that was easy. Now what do we change to get an expression for group velocity?

In a wave packet, each constituent wave has a constant phase velocity but the packet as a whole experiences dispersion and changes its shape with time. Because of this continually changing shape, let's try to define the group velocity not as the ratio of the angular frequency to the wave number but rather as the *derivative* of the angular frequency with respect to wave number:

$$v_{phase} = \frac{\omega}{k}$$

and

$$v_{group} = \frac{\partial \omega}{\partial k}$$

Before we tackle this derivative, let's express DeBroglie's hypothesis in terms of w and k:

$$E = hf = h\left(\frac{\omega}{2\pi}\right) = \hbar\omega \tag{5.6.8}$$

$$p = \frac{h}{\lambda} = \frac{h}{\left(\frac{2\pi}{k}\right)} = \hbar k \tag{5.6.9}$$

Thus,

$$\begin{aligned} v_{group} &= \frac{\partial \omega}{\partial k} \\ &= \frac{\partial\left(\frac{E}{\hbar}\right)}{\partial\left(\frac{p}{\hbar}\right)} \\ &= \frac{\partial E}{\partial p} \end{aligned}$$

This means that the velocity of the wave packet is the rate at which the particle's energy changes with respect to its momentum.

Using the relativistic relationship between energy and momentum yields

$$\begin{aligned} v_{group} &= \frac{\partial}{\partial p}\left(\sqrt{pc^2 + mc^4}\right) \\ &= \frac{1}{2}(pc^2 + mc^4)^{-\frac{1}{2}}(2pc^2) \\ &= \frac{pc^2}{\sqrt{pc^2 + mc^4}} \\ &= \frac{(\gamma mv)c^2}{E} \\ &= \frac{(\gamma mv)c^2}{\gamma mc^2} \\ &= v \end{aligned}$$

Thus the group velocity of the wave packet is exactly the same as the velocity of the particle the packet is intended to represent. Even though each of the constituent waves travels with a phase velocity greater than c, they combine to form a wave packet that moves slower than light. Isn't mathematics amazing?
