# 5.2: The Double Slit with Matter

## Exercise 5.2.1: Neutron beams

A beam of very cold neutrons with kinetic energy $5.0 \times 10^{-6}\ eV$ is directed toward a double slit foil with slit separation 1 mm. What is the angular separation between adjacent interference maxima?

**Solution**

In addition to Bragg diffraction, the wave-like nature of matter can be demonstrated in the same experimental manner as the wave-like nature of light was first demonstrated, by passing the matter wave through a pair of adjacent slits. You should remember the result for the location of interference maxima in a double slit experiment, but nonetheless I'll remind you:

$$d \sin \theta = n\lambda \tag{5.2.1}$$

where

- $d$ is the distance between adjacent slits,
- $\theta$ is the angle at which constructive interference occurs,
- and $\lambda$ is the wavelength of the disturbance.

The kinetic energy of the neutrons is so small we can use classical physics to determine the momentum. Remembering the classical relationship between kinetic energy and momentum

$$KE = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m} = \frac{(pc)^2}{2mc^2} \tag{5.2.2}$$

leads to

$$\begin{aligned} pc &= \sqrt{2(KE)mc^2} \\ pc &= \sqrt{2(5 \times 10^{-6}(939.6 \times 10^6)} \\ pc &= 96.9 \text{ eV} \end{aligned} \tag{5.2.3}$$

and, by DeBroglie's relation, a wavelength of

$$\begin{aligned} \lambda &= \frac{hc}{pc} \\ \lambda &= \frac{1240 \text{ eVnm}}{96.9 \text{ eV}} \\ \lambda &= 12.8 \text{ nm} \end{aligned} \tag{5.2.4}$$

Inserting this result into the double slit relation results in

$$\begin{aligned} & d \sin \theta = n\lambda \\ & (1000 \text{ nm}) \sin \theta = (1)(12.8 \text{ nm}) \\ & \theta = 0.73^0 \end{aligned} \tag{5.2.5}$$

Thus, adjacent maxima are separated by 0.73 degrees.

## Thermal Wavelength

*How "cold" is a beam of very cold neutrons with kinetic energy 5.0 x $10^{-6}$ eV?*

You may have been confused when I referred to the neutron beam in the previous example as being "very cold". However, physicists routinely talk about temperature, mass and energy using the same language. An ideal (non-interacting) gas of particles at an equilibrium temperature will have a range of kinetic energies. You may recall from your study of the ideal gas[1] that:

$$KE_{mean} = \frac{3}{2}kT \tag{5.2.6}$$

where

- $KE_{mean}$ is the mean kinetic energy of a particle in the sample,
- k is *Boltzmann's constant,*
- and T is the temperature of the sample, in Kelvin.

Technically, we shouldn't talk about the temperature of a mono-energetic beam, since by definition a temperature implies a range of energies. However, let's be sloppy and assume the energy of the beam corresponds to the mean kinetic energy of a (hypothetical) sample. Then:

$$\begin{aligned} KE_{mean} &= \frac{3}{2}kT \\ T &= \frac{2KE_{mean}}{3k} \\ T &= \frac{2(5\times10^{-6}\text{ eV}}{3(8.617\times10^{-5}\text{ eV / K}} \\ T &= 0.039\text{ K} \end{aligned} \tag{5.2.7}$$

So the neutron beam really is pretty cold!

Note that if we wanted to find the DeBroglie wavelength corresponding to this mean kinetic energy, we would find (assuming non-relativistic speeds)

$$\begin{aligned} KE &= \frac{(pc)^2}{2mc^2} \\ pc &= \sqrt{2(KE)mc^2} \\ pc &= \sqrt{2\left(\frac{3}{2}KT\right)mc^2} \\ pc &= \sqrt{3mc^2kT} \end{aligned} \tag{5.2.8}$$

and thus

$$\lambda = \frac{hc}{pc} \tag{5.2.9}$$

$$\lambda = \frac{hc}{\sqrt{3mc^2kT}} \tag{5.2.10}$$

This is the DeBroglie wavelength corresponding to the mean kinetic energy of a gas at temperature, T. However, a more useful value would be the *mean wavelength* of all of the particles in the gas. The mean wavelength is not equal to the wavelength of the mean energy. Calculating this mean wavelength, termed the *thermal DeBroglie wavelength* is a bit beyond our skills at this point, but it is the same as the result above but with a different numerical factor in the denominator:

$$\lambda_{thermal} = \frac{hc}{\sqrt{2\pi nc^2kT}} \tag{5.2.11}$$

For an ideal gas sample at a known temperature, we can quickly determine the average wavelength of the particles comprising the sample.

One important use for this relationship is to determine when the gas sample is no longer ideal. If the mean wavelength becomes comparable to the separation between the particles in the gas, this means that the waves begin to overlap and the particles begin to interact. When these waves begin to overlap, it becomes impossible, even in principle, to think of each of the particles as a separate entity. When this occurs, some really cool stuff starts to happen (like superfluidity, superconductivity, Bose-Einstein condensation, etc.).

## A Plausibility Argument for the Heisenberg Uncertainty Principle

Imagine a wave passing through a small slit in an opaque barrier. As the wave passes through the slit, it will form the diffraction pattern shown below.

Remember that the location of the first minima of the pattern is given by

$$a \sin \theta = \lambda \tag{5.2.12}$$

From the geometry of the situation,

$$\tan \theta = \frac{y}{D} \tag{5.2.13}$$

If the detecting screen is far from the opening,

$$\sin \theta \approx \tan \theta \tag{5.2.14}$$

so

$$a \sin \theta = \lambda \tag{5.2.15}$$

$$a \tan \theta = \lambda \tag{5.2.16}$$

$$a \left( \frac{y}{D} \right) = \lambda \tag{5.2.17}$$

$$y = \frac{\lambda D}{a} \tag{5.2.18}$$

Now, consider the “wave” to be a “particle”. The time to traverse the distance from slit to screen is given by

$$t = \frac{D}{v_x} \tag{5.2.19}$$

while during this time interval the particle also travels a distance in the y-direction given by:

$$y = v_y t \tag{5.2.20}$$

Combining these relations yields

$$y = v_y \left( \frac{D}{v_x} \right) \tag{5.2.21}$$

Combining this “particle” expression with the “wave” expression above gives:

$$\frac{\lambda D}{a} = v_y \left( fracDv_x \right) \tag{5.2.22}$$

$$\lambda = \frac{v_y a}{v_x} \tag{5.2.23}$$

Substituting the DeBroglie relation results in,

$$\frac{h}{m v_x} = \frac{v_y a}{v_x} \tag{5.2.24}$$

$$h = m v_y a \tag{5.2.25}$$

Notice that the term $mv_y$ is the *uncertainty in the y-momentum* ($\sigma_{p_y}$) of the particle, since the particle is just as likely to move in the +y or the –y-direction with this momentum. Also, a is twice the *uncertainty in the y-position* ($\sigma_y$) of the particle, since the particle has a range of possible positions of +a/2 to –a/2.

Therefore, our expression can be written as

$$(2\sigma_y)(\sigma_{p_y}) = h \tag{5.2.26}$$

$$(\sigma_y)(\sigma_{p_y}) = \frac{h}{2} \tag{5.2.27}$$

Thus, the uncertainty in the y-position of the particle is inversely proportional to the uncertainty in the y-momentum. Neither of these quantities can be determined precisely, because the act of restricting one of these parameters automatically has a compensating effect on the other parameter, i.e., making the hole smaller spreads out the pattern, and the only way to make the pattern smaller is to increase the size of the hole!

A more careful analysis (for circular openings rather than slits) shows that the *minimum* uncertainty in the product of position and momentum can be reduced by a factor of 2p, resulting in:

$$(\sigma_y)(\sigma_{p_y}) \geq \frac{1}{2\pi}\frac{h}{2} \tag{5.2.28}$$

$$(\sigma_y)(\sigma_{p_y}) \geq \frac{\hbar}{2} \tag{5.2.29}$$

where the symbol ħ is defined to be Planck's constant divided by $2\pi$.

## Contributors and Attributions

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[1] The factor 3/2 is true only for particles. If the gas is comprised of diatomic molecules the factor is 5/2 at low temperatures and 7/2 at high temperatures. What counts as a low or high temperature depends on the molecule. More complex molecules have even more complex factors.