1.4: Velocity Addition

Since the Lorentz transformation allows you to relate the position and time of an event in one coordinate system to the position and time in any other coordinate system, it also allows you to relate quantities that depend on position and time, like velocity and acceleration. Therefore, using Lorentz we can derive equations that allow use to transform velocities measured by one observer to velocities measured by other observers.

Refer back to the Lorentz transformation derivation. This time, imagine that the event of interest is a particle, launched from the origin when the two origins overlapped at time zero, striking some detector. Thus,

$$x = v_x t \tag{1.4.1}$$

$$x' = v_x' t' \tag{1.4.2}$$

Substituting these relationships into the Lorentz transformation yields

$$x' = \gamma(x - ut) \tag{1.4.3}$$

$$v_x' t' = \gamma(v_x - ut) \tag{1.4.4}$$

$$v_x' t' = \gamma(v_x - u)t \tag{1.4.5}$$

and

$$t' = \gamma\left(t - \frac{ux}{c^2}\right) \tag{1.4.6}$$

$$= \gamma\left(t - \frac{uv_x t}{c^2}\right) \tag{1.4.7}$$

$$= \gamma\left(1 - \frac{uv_x}{c^2}\right) t \tag{1.4.8}$$

dividing the Equation ??? by Equation ??? yields

$$\frac{v_x' t'}{t'} = \frac{\cancel{\gamma}(v_x - u)\cancel{t}}{\cancel{\gamma}\left(1 - \frac{uv_x}{c^2}\right)\cancel{t}} \tag{1.4.9}$$

$$= \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \tag{1.4.10}$$

This directly relates the x-speed of an object in one reference system (v_x) to the speed of the same object measured in a different system (v_x').

Although y- and z-positions are not effected by the Lorentz transformation, y- and z-velocities are different in different systems. Starting with,

$$y' = y \tag{1.4.11}$$

$$v_y t = v_y t \tag{1.4.12}$$

divide by the time transformation derived above

$$\frac{v_y' t'}{t'} = \frac{v_y \cancel{t}}{\gamma\left(1 - \frac{uv_x}{c^2}\right)\cancel{t}} \tag{1.4.13}$$

$$v_y' = \frac{v_y}{\gamma\left(1 - \frac{uv_x}{c^2}\right)} \tag{1.4.14}$$

The same type of transformation holds for z-velocity.

$$v'_z = \frac{v_z}{\gamma\left(1 - \frac{uv_x}{c^2}\right)} \tag{1.4.15}$$

Using Velocity Addition

A spaceship travels at 0.8c with respect to the solar system. An unmanned probe is ejected at 0.6c at an angle of 30° from the direction of travel of the ship (both the speed and angle of the probe are measured with respect to the ship). What are the speed and angle of launch of the probe as measured in the solar system frame?

Just as distance and time depend on the relative motion between observers, so does velocity. The following relationships allow you to compare velocity measurements between two observers in relative motion.

$$v'_x = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \tag{1.4.16}$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{uv_x}{c^2}\right)} \tag{1.4.17}$$

were v'_x and v'_y are the velocities of an object measured in a frame (the “primed” frame) moving at speed u relative to the “unprimed” frame (where observers measure v_x and v_y).

These equations are written in a form that easily allows the determination of v'_x and v'_y if v_x and v_y are known. If the situation requires the inverse of this task, the equations can be easily inverted (by changing the sign of u and flipping the primed and unprimed notation) to yield

$$v_x = \frac{v'_x + u}{1 + \frac{uv'_x}{c^2}} \tag{1.4.18}$$

$$v_y = \frac{v'_y}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \tag{1.4.19}$$

Using this form of the equations, with the spaceship the primed frame and $u = 0.8c$ ($\gamma = 1.67$), yields

$$\begin{aligned} v_x &= \frac{v'_x + u}{1 + \frac{uv'_x}{c^2}} \\ &= \frac{0.6c(\cos(30^\circ)) + 0.8c}{1 + \frac{(0.8c)(0.6c(\cos(30^\circ))}{c^2}} \\ &= 0.932c \\ v_y &= \frac{v'_y}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \\ &= \frac{0.6c(\sin(30^\circ))}{(1.67)\left(1 + \frac{(0.8c)(0.6c)(\cos(30^\circ))}{c^2}\right)} \\ &= 0.127c \end{aligned}$$

The solar system observers detect the probe’s motion as

$$\begin{aligned}
v &= \sqrt{v_x^2 + v_y^2} \\
&= \sqrt{(0.932c)^2 + (0.127c)^2} \\
&= 0.941c \\
\theta &= \tan^{-1}\left(\frac{v_y}{v_x}\right) \\
&= \tan^{-1}\left(\frac{0.127c}{0.932c}\right) \\
&= 7.76^\circ
\end{aligned}$$