3.6: Falling into a Black Hole - Hard Version (Project)

The text describes how the Schwarzchild metric can be used to directly compare time and length intervals measured by observers at rest at different locations in spacetime. However, measurements made by, and of, moving observers are slightly more difficult to compare. In order to determine how the observations of a person falling into a black hole compare to observers at rest, additional mathematical machinery is necessary.

I. Energy

The concept of energy conservation is central to all of physics. For a system acted on by no outside forces (and remember, gravity is not a force), energy must remain constant. Therefore, an observer in freefall into a black hole has constant energy. Once we learn how to quantify energy in general relativity, we will be ready to “jump in”.

A. A Plausible Relationship for Energy

Far from the black hole, spacetime is flat and the formula for energy must agree with the special relativity formula for energy,

$$E=\gamma mc^2 \tag{3.6.1}$$

The question is, how does this relationship vary as you move closer to the black hole?

An examination of the Schwarzchild metric,

$$(ds)^2=\left(1-\frac{2GM}{c^2r}\right)(c\,dt)^2-\frac{(dr)^2}{\left(1-\frac{2GM}{c^2r}\right)}-r^2(d\phi)^2 \tag{3.6.2}$$

should indicate the presence of a very common mathematical factor that describes the change in space and time as you approach the hole. This factor, designated by the Greek letter capital gamma (Γ), approaches one at large distance so wouldn’t alter the energy in flatspace:

$$\Gamma=\left(1-\frac{2GM}{c^2r}\right) \tag{3.6.3}$$

Hopefully, it’s plausible that the correct mathematical relationship for energy could be:

$$E=\Gamma\gamma mc^2 \tag{3.6.4}$$

In fact, this is the correct relationship, with one subtle twist. From the time dilation formula in special relativity,

$$\Delta t=\gamma(\Delta t_0) \tag{3.6.5}$$

γ can be defined as

$$\gamma=\frac{\Delta t}{\Delta t_o}=\frac{dt}{dt_o} \tag{3.6.6}$$

or, in words, γ is the rate at which time varies with proper time. In flat spacetime, this can also be expressed as

$$\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.6.7}$$

The subtle twist is that the first definition of γ (Equation 3.6.6) is always true, while the second definition of γ (Equation 3.6.7) is *only* true in flat spacetime. Thus, the proper expression for energy is

$$E=\Gamma\frac{dt}{dt_o}mc^2 \tag{3.6.8}$$

II. Falling into a Black Hole (from Rest at Infinity)

Imagine being at rest a large distance from a black hole. On a dare, you step out of your spaceship and take a ride on curved spacetime toward the black hole.

A. Determining Your Initial Energy

Since you are far from the black hole, you are in flat spacetime ($\Gamma = 1$ since $r = \infty$). Since you are at rest in flat spacetime

$$\frac{dt}{dt_o} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 1 \tag{3.6.9}$$

Therefore, your initial energy is solely due to your rest mass (Equation 3.6.8)

$$E_{initial} = \Gamma \frac{dt}{dt_o} mc^2 \tag{3.6.10}$$

$$= (1)(1)mc^2 \tag{3.6.11}$$

$$= mc^2 \tag{3.6.12}$$

B. Using Energy Conservation

By energy conservation,

$$E_{at\ any\ time} = E_{initial} \tag{3.6.13}$$

$$\Gamma \frac{dt}{dt_o} mc^2 = mc^2 \tag{3.6.14}$$

$$\Gamma dt = dt_o \tag{3.6.15}$$

$$\Gamma^2 dt_2 = dt_o^2 \tag{3.6.16}$$

dt_o, the proper time, is the time measured by you, the falling observer. This can be determined by relating measurements made by you to measurements made by the friends you left behind.

$$(ds)^2_{you} = (ds)^2_{far\ away\ freiends} \tag{3.6.17}$$

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In your frame, you are at rest, so $dt = dt_o$ and $dx = dy = dz = 0$. In your friends' frame, you fall radially inward so $d\varphi = 0$.

Plugging this into the energy conservation equation (Equation 3.6.16) and collecting like terms yields:

Ok, so what's the point? The point is that $\frac{dr}{dt}$ is your speed as measured by your friends! dr is your change in reduced circumference (which to your friends in flat space is equal to distance traveled) and dt is the elapsed time as measured on faraway clocks (your friends' clocks).

Since you are falling inward (to smaller r) we should take the negative root of the square root, and we can also simplify the expression $1 - \Gamma$.

You should find this relationship *stunning*. Why?

At $r = \infty$, when you step out of the ship, your speed (as measured by your friends) is zero. Not stunning.

As you slide into the hole, r decreases and your speed increases. Again, not stunning.

But then something truly weird happens. Examine the relationship as you approach the event horizon of the black hole, $r = 2GM/c^2$. Your speed, as measured by your friends, decreases to zero! Your friends see you slow down and stop exactly at the event horizon. You never pass the event horizon in your friends' frame of reference. In fact, observers outside a black hole never see anything ever enter a black hole! All the stuff that falls into the hole just "accumulates" at the event horizon. Stunning!

C. Another Frame of Reference

Just because your friends never see you pass the event horizon does not necessarily mean you never pass the event horizon. Remember, this is called relativity for a reason.

Consider a pair of stationary observers located at reduced circumference r. What do they measure as your speed? One observer measures the time interval between your feet passing by and then your head passing by (assuming you jump in feet first). These two measurements occur at the same location, so $dr = 0$, and thus

$$dt_{\textit{in the hole}} = \sqrt{1 - \frac{2GM}{c^2 r}}\, dt \tag{3.6.18}$$

$$= \Gamma^{1/2} dt \tag{3.6.19}$$

The other observer determines the location of your feet and head at the same instant of time, so $dt = 0$, and thus

Combining these observations results in

This is also stunning. Why?

Stationary observers at $r = \infty$ measure your speed to be zero. Not stunning.

Stationary observers "deeper" in the hole (smaller r) measure your speed as faster and faster. Again, not stunning.

But then (again) something truly weird happens. Examine the relationship for stationary observers very close to the event horizon of the black hole, $r = 2GM/c^2$. Your speed, as measured by these observers, increases to c! Observers standing arbitrarily close to the event horizon see you zoom by at arbitrarily close to the speed of light. (No one sees you move exactly at the speed of light, because no stationary observer can exist exactly at the event horizon.) These observers see you zoom past the event horizon (to your ultimate demise) while your friends see you step on the brakes and come to a complete stop. Stunning!

D. Another Frame of Reference? Yours.

You should probably be concerned with yet another frame of reference, the frame of reference of the jumper. How fast does the jumper see the "stationary" observers discussed in the last section zooming by?

That's easy. If the "stationary" observers see the jumper moving at

$$v_{\textit{jumper from the obsever's frame}} = +\sqrt{\frac{2GM}{c^2 r}}\, c \tag{3.6.20}$$

then the jumper sees the "stationary" observers moving at

$$v_{\textit{observers from the jumper's frame}} = +\sqrt{\frac{2GM}{c^2 r}}\, c \tag{3.6.21}$$

However, a more interesting question is how long, in the jumpers frame, does it take the jumper to go from the event horizon to the singularity (and certain death) at the center of the black hole? The event horizon and the singularity are separated by a known change in reduced circumference (dr from $r = 2GM/c^2$ to $r = 0$). The time measured by the jumper is a proper time, dt_o. Thus, construct the ratio

$$\frac{dr}{dt_o} = \frac{dr}{dt}\frac{dt}{dt_o} \tag{3.6.22}$$

$\frac{dr}{dt}$ is calculated above, and $\frac{dt}{dt_o}$ is determined by the energy conservation equation. Thus,

Solving for dt_o and integrating from the event horizon to the singularity yields

Consider a typical 10 solar mass black hole. This would yield a time interval between crossing the event horizon and being shredded into nothingness of $6.6 \times 10^{-5}\ s$, hardly enough time to even enjoy the weird spacetime geometry inside the event horizon (or to really experience the agony of being shredded).

On the other hand, a typical galactic-center-sized black hole (2 million solar masses) yields a time interval of 13.1 s, enough to possibly both admire the geometry and feel the agony.

III. Thrown into a Black Hole (from Infinity)

Imagine being at rest a large distance from a black hole. On a dare, you allow your friends to launch you directly toward the black hole, with initial velocity v.

A. Determining Your Initial Energy

1. You are initially far from the black hole, in flat spacetime, moving at velocity v. Write an expression for your initial energy.

B. Using Energy Conservation

2. By energy conservation, set your expression for initial energy equal to the general formula for energy.

3. Replace dt0 with the relationship for proper time measured by the falling observer.

4. Collect like terms and solve for , your speed as measured by your friends

5. Based on your result above, how fast do your friends see you moving as you approach the event horizon? Did being launched at high speed toward the black hole have any effect on your speed at the event horizon?

6. (Prepare to be stunned.) Instead of launching you at the black hole, your friends decide to shine a laser at the black hole. How fast do your friends see the light moving as it approaches the event horizon? Can your friends ever see anything pass the event horizon?

C. Another Frame of Reference

Just because your friends never see you (or a laser beam) pass the event horizon doesn't necessarily mean you (or a laser beam) never pass the event horizon.

Recall that your speed, as measured by a stationary observer "in the hole", is

7. Determine your speed as measured by stationary observers in the hole.

8. What does a stationary observer very close to the event horizon measure for your speed? Did being launched at high speed toward the black hole have any effect on your speed at the event horizon as measured by a stationary observer?

9. (Prepare to not be stunned.) What do stationary observers (anywhere in the hole) measure for the speed of the laser launched by your friends?

IV. Dropped into a Black Hole (from Nearby)

Imagine being at rest at reduced circumference r0 near a black hole. On a dare, you step out of your spaceship and take a ride on curved spacetime toward the black hole.

A. Determining Your Initial Energy

10. You are initially at rest at reduced circumference r0. Write an expression for your initial energy. (Note that the relationship between faraway time and time measured by someone at rest in a black hole is .) Hint: Let to keep the expression manageable.

B. Using Energy Conservation

11. By energy conservation, set your expression for initial energy equal to the general formula for energy.

12. Replace dt0 with the relationship for proper time measured by the falling observer.

13. Collect like terms and solve for , your speed as measured by your friends

14. Based on your result above, how fast do your friends see you moving as you approach the event horizon? Does it matter where you were when you stepped out of the spaceship?

C. Another Frame of Reference

15. Determine your speed as measured by stationary observers in the hole.

16. What does a stationary observer very close to the event horizon measure for your speed? Does it matter if you stepped out of a spaceship 1 mm from the event horizon or 1 million miles from the event horizon?

D. Another Frame of Reference? Yours.

You will shortly prove that although stationary observers will always see you cross the horizon at exactly the same speed (~c) that doesn't mean you will always take the same amount of time to reach the singularity. (Think about this. You are the same "distance" from the singularity, traveling at the same speed, but it won't always take you the same amount of time to get there. Stunning!)

17. Construct the ratio

Hint: You calculated above, and is determined from the energy conservation equation.

18. Solve your expression for dt0.

19. Set-up the integral from the event horizon to the singularity. Simplify the integrand until it is in the form of .

20. Substitute the expressions for Γ0 and Γ. You should be able to reduce the integrand to a function of only r and r0. Eliminate fractions in the denominator, and using the dimensionless substitution u = r/r0, put the integral into the form of .

21. Note that is equal to . This latter integral is listed in your integral table. Assume you “dropped” into the black hole from a point just beyond the event horizon. This will maximize the time between crossing the horizon and reaching the singularity. Using this assumption, evaluate and simplify your result.

22. Consider a typical 10 solar mass stellar black hole. If you dropped into this hole from just across the horizon, how much time would you have to enjoy the ride?

23. Consider a typical 2 million solar mass galactic black hole. If you dropped into this hole from just across the horizon, how much time would you have to enjoy the ride?
