2.1: Relativistic Momentum, Force and Energy

Once Einstein revolutionized our understanding of space and time, physicists were faced with a monumental task. All of physics, before Einstein, was based on the idea of absolute space and time. Once these concepts were found to be erroneous, all of classical physics had to be re-examined in this light. In this section, we will "re-examine" our understanding of momentum, force, and energy.

Relativistic Momentum

In classical physics, momentum is defined as

$$\vec{p} = m\vec{v} \tag{2.1.1}$$

However, using this definition of momentum results in a quantity that is not conserved in all frames of reference during collisions. However, if momentum is re-defined as

$$\vec{p} = \gamma m\vec{v} \tag{2.1.2}$$

it *is* conserved during particle collisions. Therefore, experimentally, relativistic momentum is defined by Equation 2.1.2.

Relativistic Force

Once nature tells us the proper formula to use for calculating momentum, mathematics tells us how to measure force and energy. Force is defined as the time derivative of momentum

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{2.1.3}$$

(In classical physics, where $\vec{p} = m\vec{v}$, Equation 2.1.3 reduces to $\vec{F} = m\vec{a}$.)

> Equation 2.1.3 is the scalar form of this relationship and is only true for motion in one-dimension. The full vector treatment of force is more complicated than its worth.

Substituting correct relationship for momentum (Equation 2.1.2) into Equation 2.1.3 yields

$$\begin{aligned}
\vec{F} &= \frac{d(\gamma m\vec{v})}{dt} \\
&= \left(\frac{d\gamma}{dt}\right) m\vec{v} + \gamma m \left(\frac{d\vec{v}}{dt}\right) \\
&= m\vec{v}\frac{d}{dt}\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right] + \gamma m\vec{a}
\end{aligned}$$

Use the chain rule to evaluate the derivative of γ

$$\begin{aligned}
\vec{F} &= m\vec{v}\left[-\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right)\left(\frac{dv}{dt}\right)\right] + \gamma m\vec{a} \\
&= m\vec{v}\left[\frac{v}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\vec{a}\right] + \gamma m\vec{a}
\end{aligned}$$

Factor out the common factor γma,

$$\vec{F} = \gamma m\vec{a}\left[\frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} + 1\right] \tag{2.1.4}$$

Find a common denominator and simplify,

$$\vec{F}=\gamma m\vec{a}\left[\frac{\cancel{\frac{v^2}{c^2}}+1-\cancel{\frac{v^2}{c^2}}}{1-\frac{v^2}{c^2}}\right] \tag{2.1.5}$$

$$=\gamma m\vec{a}\left[\frac{1}{1-\frac{v^2}{c^2}}\right] \tag{2.1.6}$$

$$=\gamma^3 m\vec{a} \tag{2.1.7}$$

Equation 2.1.7 is the relativistically correct form of Newton's Second Law, for motion constrained in one dimension. (Physicists seldom use a force approach when analyzing motion since a momentum and energy approach is almost always more useful.)

Relativistic Energy

The kinetic energy of an object is defined to be the work done on the object in accelerating it from rest to speed v.

$$KE=\int_0^v F\,dx \tag{2.1.8}$$

Using our result for relativistic force (Equation 2.1.7) yields

$$KE=\int_0^v \gamma^3 ma\,dx \tag{2.1.9}$$

The variable of integration in Equation 2.1.9 is x, yet the integrand is expressed in terms of a and v (v is hidden inside γ). To solve this problem,

$$\begin{aligned}KE&=\int_0^v \gamma^3 m\left(\frac{dv}{dt}\right)dx\\&=\int_0^v \gamma^3 m\left(\frac{dx}{dt}\right)dv\\&=\int_0^v \gamma^3 mv\,dv\\&=\int_0^v \frac{mv\,dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\end{aligned}$$

This integral can be done by a simple u-substitution,

$$u=1-\frac{v^2}{c^2} \tag{2.1.10}$$

$$du=-\frac{2v}{c^2}dv \tag{2.1.11}$$

$$
\begin{aligned}
KE &= -\frac{mc^2}{2}\int_o^v \frac{du}{u^{3/2}} \\
&= -\frac{mc^2}{2}\left[-2u^{-1/2}\right] \\
&= mc^2\left[\frac{1}{1-\frac{v^2}{c^2}}\right]_o^v \\
&= mc^2\left[\frac{1}{(1-\frac{v^2}{c^2})^1/2}-1\right] \\
&= \gamma mc^2 - mc^2
\end{aligned}
$$

Rearranging this yields,

$$\gamma mc^2 = KE + mc^2 \tag{2.1.12}$$

Einstein identified the term γmc^2 as the *total* energy of the particle. Thus, the total energy is the sum of the kinetic energy and a completely new form of energy, the **rest energy**. Particles have rest energy just by virtue of having mass. In fact, mass is simply a form of energy.

$$E_{total} = KE + E_{Rest} \tag{2.1.13}$$

✔ Example 2.1.1: Using Momentum and Energy

An electron is accelerated through a potential difference of 80 kilovolts. Find the kinetic energy, total energy, momentum and velocity of the electron.

Solution

The following collection of equations express the relationships between momentum, energy, and velocity in special relativity. (Momentum is often easier expressed as "*pc*" rather than "*p*" as you will see once you begin working problems.)

$$p = \gamma mv \tag{2.1.14}$$

$$pc = \gamma mc^2\left(\frac{v}{c}\right) \tag{2.1.15}$$

$$
\begin{aligned}
E_{total} &= \gamma mc^2 \\
&= KE + mc^2
\end{aligned}
$$

$$KE = (\gamma - 1)mc^2 \tag{2.1.16}$$

$$E_{total}^2 = (pc)^2 + (mc^2)^2 \tag{2.1.17}$$

Equation 2.1.17 is particularly useful in that it allows a direct relationship between energy and momentum without the need to calculate the velocity. The proof of this relationship is left as an exercise.

From electrodynamics, the kinetic energy of a charge accelerated through a potential difference V is simply the product of the charge (q) and the potential difference (V),

$$
\begin{aligned}
KE &= qV \\
&= e(80 \times 10^3 V) = 80\,keV
\end{aligned}
$$

Rather than substituting the numerical value of the charge on an electron

$$q = -e = -1.6 \times 10^{-19}\,C \tag{2.1.18}$$

into this expression (and obtaining the kinetic energy in joules), we will leave “e” in the equation and use “eV” as a unit of energy (i.e., $1.0\,eV = 1.6 \times 10^{-19}\,J$.

Thus, the kinetic energy of the electron is 80 keV.

The total energy of the electron is then

$$\begin{aligned} E_{total} &= KE + mc^2 \\ &= 80\,keV + 511\,keV \\ &= 591\,keV \end{aligned}$$

The momentum is

$$E_{total}^2 = (pc)^2 + (mc^2)^2 \tag{2.1.19}$$

$$\begin{aligned} pc &= \sqrt{E_{total}^2 - (mc^2)^2} \\ &= \sqrt{591^2 - (511)^2} \\ &= 297\,keV \end{aligned}$$

(Again, momentum is often easier expressed as “pc” rather than “p”)

The speed of the electron is

$$pc = \gamma mc^2 \left(\frac{v}{c}\right) \tag{2.1.20}$$

$$296\,keV = 591\,keV \left(\frac{v}{c}\right) \tag{2.1.21}$$

$$v = 0.503c \tag{2.1.22}$$