3.4: Global Positioning System (Project)

There is no better illustration of the unpredictable payback of fundamental science than the story of Albert Einstein and the Global Positioning System (GPS). The next time your plane approaches an airport in bad weather and you just happen to be wondering "what good is basic science," think about Einstein and the GPS tracker in the cockpit, guiding you to a safe landing. - Clifford Will

Do you think that general relativity concerns only events far from common experience? Think again! Your life may be saved by a hand held receiver that "listens" to overhead satellites, the system telling you where you are at any place on Earth. In this project you will show that this system would be useless without corrections provided by general relativity.

The Global Positioning System (GPS) includes 24 satellites, in circular orbits around Earth with orbital period of 12 hours, distributed in six orbital planes equally spaced in angle. Each satellite carries an operating atomic clock (along with several backup clocks) and emits timed signals that include a code telling its location. By analyzing signals from at least four of these satellites, a receiver on the surface of Earth with a built in microprocessor can display the location of the receiver (latitude, longitude, and altitude). GPS satellites are gradually revolutionizing driving, flying, hiking, exploring, rescuing, and map making.

The timing accuracy required by the GPS system is so great that general relativistic effects are central to its performance. First, clocks run at different rates when they are at different distances from a center of gravitational attraction. Second, both satellite motion and Earth rotation must be taken into account; neither the moving satellite nor Earth's surface corresponds to a stationary frame of reference. In this project you will investigate these effects.

I. Getting Started

Earth rotates and is not perfectly spherical, so, strictly speaking, the Schwarzschild metric does not describe spacetime above Earth's surface. But Earth rotates slowly and the Schwarzschild metric is a good approximation for purposes of analyzing the Global Positioning System.

We will apply the Schwarzschild metric to both the orbiting satellite clock and a clock fixed at Earth’s equator and rotating as Earth turns. Both the Earth clock and the satellite clock travel at constant radius around Earth's center, so $dr = 0$ for each clock. Also, each clock measures the proper time at its location. This yields

$$(cdt0)^2 = (1 - \frac{2GM}{c^2 r})(cdt)^2 - r^2 (d\phi)^2 \tag{3.4.1}$$

for each clock.

Divide the Schwarzschild metric through by the square of the flatspace time $(cdt)^2$ to obtain, for either clock,

$$\left(\frac{dt_0}{dt}\right)^2 = (1 - \frac{2GM}{c^2 r}) - r^2 \left(\frac{d\phi}{cdt}\right)^2 \tag{3.4.2}$$

Note that $d\phi/dt$ is the angular velocity of either clock, so r $d\phi/dt$ is the tangential velocity along the circular path of each clock as measured using flatspace time. Therefore,

$$\left(\frac{dt_0}{td}\right)^2 = (1 - \frac{2GM}{c^2 r}) - \frac{v^2}{c^2} \tag{3.4.3}$$

This equation relates time measured on Earth to flatspace time, or time measured on the satellite to flatspace time. Since we can’t directly measure flatspace time, we can eliminate this term by applying the relationship to both the satellite clock and the Earth clock, and dividing the two expressions.

$$\left(\frac{dt_{satellite}}{dt_{earth}}\right)^2 = \frac{1 - \frac{2GM}{c^2 r_{satellite}} - \frac{vsatellite^2}{c^2}}{1 - \frac{2GM}{c^2 r_{earth}} - \frac{v_{earth}^2}{c^2}} \tag{3.4.4}$$

$$\frac{dt_{satellite}}{dt_{earth}} = \frac{\left(1 - \frac{2GM}{c^2 r_{satellite}} - \frac{vsatellite^2}{c^2}\right)^{1/2}}{\left(1 - \frac{2GM}{c^2 r_{earth}} - \frac{v_{earth}^2}{c^2}\right)^{1/2}} \tag{3.4.5}$$

II. Stationary Clocks

1. We will start our analysis by ignoring the motions of both clocks. Rewrite the above expression assuming the clocks are stationary.

2. The radius of the Earth is $6.37x10^6$ m and the radius of a 12 hour circular orbit is about $26.6x10^6$ m (You will calculate this value later in the activity). Calculate the value of $2GM/c^2r$ for both the Earth clock and the satellite clock. Are these large or small values?

3. You should find that the value of $2GM/c^2r$ is extremely small. Therefore, you can use the binomial expansion to simplify the expression for the ratio of satellite time to earth time. (First move the term in the denominator into the numerator using a negative exponent, then apply the binomial expansion to both terms, and then multiply out the result. Do this symbolically. Note that the product of two extremely small numbers is really small and can be ignored.)

4. Evaluate your expression, leaving your result in the form “1 + δ” where δ is the small factor by which the two clocks are not synchronized.

The number represented by δ is an estimate of the fractional difference in rates between stationary clocks at the position of the satellite and at Earth's surface. Is this difference negligible or important to the operation of the GPS system? In 1 nanosecond, light signals (and radio waves) propagate approximately 30 centimeters, or about one foot. So for each nanosecond of discrepancy per day, the GPS will give incorrect position readings of about one foot.

5. To three significant figures, find the discrepancy between earth time and satellite time after one day.

You should find that the satellite clock will "run fast" by more than 40,000 nanoseconds per day compared with the clock on Earth’s surface due to position effects alone. After one day, the GPS would be off by over 40,000 feet (7.5 miles)! Clearly general relativity is needed for correct operation of the Global Positioning Satellite System!

III. Moving Clocks

1. In addition to effects of position, we must include effects due to the motion of the satellite and Earth observer. Will the effects of velocity make the time discrepancy larger or smaller than that calculated above? Explain.

We need to calculate the speed of both the satellite clock and the earth clock to complete our analysis.

2. Determine the speed of a clock at rest on the surface of the earth, relative to a hypothetical observer at rest relative to the center of the earth.

Finding the speed of the satellite is slightly more complicated. The satellite is in a circular orbit and must obey Newton’s Law of Gravitation (there is a slight error made in using Newton’s theory, but that error is much smaller than the discrepancies we are examining).

$$F = ma \tag{3.4.6}$$

$$\frac{GM_{earth} m_{satellite}}{r_{satellite}^2} = m_{satellite} \frac{v_{satellite}^2}{r_{satellite}} \tag{3.4.7}$$

$$\frac{GM_{earth}}{r_{satellite}} = v_{satellite}^2 \tag{3.4.8}$$

and the satellite’s orbital period, T, must obey

$$T_{satellite} = \frac{2\pi r_{satellite}}{v_{satellite}} \tag{3.4.9}$$

3. Using these two equations, find an expression for v that does not include the radius of the satellite's orbit.

4. Evaluate your expression for the velocity of the satellite in a 12-hr orbit.

5. Calculate the radius of this orbit to justify the value used earlier.

Returning to our original relationship between earth and satellite time,

$$\frac{dt_{satellite}}{dt_{earth}} = \frac{\left(1 - \frac{2GM}{c^2 r_{satellite}} - \frac{vsatellite^2}{c^2}\right)^{1/2}}{\left(1 - \frac{2GM}{c^2 r_{earth}} - \frac{v_{earth}^2}{c^2}\right)^{1/2}} \tag{3.4.10}$$

we are now prepared to calculate this discrepancy including the relative velocities of the two clocks.

6. Use the binomial expansion to simplify the expression for the ratio of satellite time to earth time. (Again, first move the term in the denominator into the numerator using a negative exponent, then apply the binomial expansion to both terms, and then multiply out the result. Do this symbolically. Note that the product of two extremely small numbers is really small and can be ignored.)

7. Evaluate your expression, leaving your result in the form "$1+\delta$" where δ is the small factor by which the two clocks are not synchronized.

8. To three significant figures, find the discrepancy between earth time and satellite time after one day.

You should find that the satellite clock will "run fast" by slightly less than 40,000 nanoseconds per day compared with the clock on Earth's surface. Even after including the time dilation of the satellite clock due to its greater velocity, general relativity is still needed for correct operation of the Global Positioning Satellite System.
