3.2: Schwarzchild Metric

In General Relativity, the flatspace Minkowski metric cannot be used to describe spacetime. In fact, the metric depends (in a very complicated way) on the exact distribution of mass and energy in its vicinity. For a perfectly spherical distribution of mass and energy, the metric is

$$(ds)^2 = \left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 - \frac{(dr)^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2(d\phi)^2 \tag{3.2.1}$$

This metric is referred to as the Schwarzchild metric, and describes the shape of space near a spherical mass such as (approximately) the earth or the sun, as well as the space surrounding a black hole. There are a number of subtle points you need to understand to use this metric.

1. This is the metric for a slice of spacetime that contains the mass center. Since the mass is spherical, all slices through the mass center are identical. The metric is expressed in polar coordinates, (r, ϕ), with the mass center at the origin.
2. Notice that the tangential component of the metric is unchanged from the Minkowski metric, meaning there is no deformation in that direction. However, both the temporal and radial portions are deformed by multiplicative constants, so radial lengths and time intervals are different in different locations of space.
3. Notice that as M goes to zero, or r gets very large, the metric approaches the Minkowski (flatspace) metric.
4. t is the *flatspace time*, the time measured on clocks very far from the central mass, where spacetime is assumed to be flat.
5. r is the *reduced circumference*. There are several ways to measure your distance from an object, such as physically traveling to the object or bouncing a signal off the object. If you tried to measure your distance from a black hole in either of these manners, you would have a very tough time, because either you (or your signal) would never return. Therefore we need a different method of determining radial distance. To do this, we will imagine wrapping a tape measure around the black hole, measuring its circumference, and then dividing the circumference by 2π. The resulting number is termed the reduced circumference, and, in flat space, would actually equal the value of our radial distance. (It won't equal the "real" radial distance from the black hole because the "real" radial distance is undefined (and undefinable!).)
6. Notice that the metric diverges (becomes infinite) at $r_{horizon} = 2GM/c^2$. Thus a single radial step at this location is infinitely long (and it appears that a single clock tick has no duration)! This "radius" (actually reduced circumference but we'll be sloppy and call it radius from now on) is termed the *Schwarzchild radius* and forms the *event horizon* of the black hole. At, or within, this radius, events are "beyond the horizon", meaning they are unseen and unseeable from radii greater than the Schwarzchild radius. Basically, once you pass over the horizon, you are no longer in contact with the rest of the universe. Ever.

Using the Schwarzchild Metric: Time

How close to a black hole of 5 solar masses can you approach before your spaceship's clock differs from time measured in flat spacetime by no more than 1%?

Regardless of where you are in space, if you make your measurements over a small enough region of spacetime that region of spacetime will appear locally flat, just like a straight tangent line can be drawn at any point on a smooth curve. Therefore, the ship's measurements are made using a standard Minkowski metric while the faraway observer must use the Schwarzchild metric since the spaceship's clock is far from her location.

$$(ds)^2_{ship} = (ds)^2_{far\,away} \tag{3.2.2}$$

$$\underbrace{(c\,dt)^2 - \left((dx)^2 + (dy)^2 + (dz)^3\right)}_{\text{Minkowski metric}} = \underbrace{\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 - \frac{(dr)^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2(d\phi)^2}_{\text{Schwarzchild metric}} \tag{3.2.3}$$

We'll assume your spaceship is at rest, in both frames of reference, so $dx = dy = dz = dr = 0$ and $d\phi = 0$.

$$(c\,dt_{ship})^2 = \left(1 - \frac{2GM}{c^2 r}\right)(c\,dt_{far\,away})^2 \tag{3.2.4}$$

$$dt_{ship} = \sqrt{1 - \frac{2GM}{c^2 r}}\, dt_{far\ away} \tag{3.2.5}$$

Equation 3.2.5 plays a similar role in general relativity that the time dilation relationship plays in special relativity. It relates the time interval measured by a “special” observer (one at rest in curved space) to another observer’s time measurements. Mathematically, it even has a similar structure, with the term “$2GM/r$” playing the role of “v^2” in the time dilation formula.

Continuing with the question:

$$\frac{dt_{ship}}{dt_{far\ away}} = \sqrt{1 - \frac{2GM}{c^2 r}} \tag{3.2.6}$$

$$(0.99)^2 = 1 - \frac{(2)(6.67 \times 10^{-11})(10 \times 10^{30})}{(3.0 \times 10^8)^2 r} \tag{3.2.7}$$

$$0.9801 = 1 - \frac{14,800}{r} \tag{3.2.8}$$

$$r = 744,000\, m = 744\, km \tag{3.2.9}$$

Since the event horizon is at

$$r_{horizon} = \frac{2GM}{c^2} = 15\, km \tag{3.2.10}$$

you are about 50 event horizons away from the black hole.

Using the Schwarzchild Metric: Length

Two astronauts are creating a (metric) football field near a 10 solar mass black hole. The reduced circumference between the two astronauts is 100 m, and the astronauts lie along the same radial line. What is the radial separation between the astronauts as measured by the inner astronaut, if the inner astronaut is at twice the event horizon?

Again, we’ll assume the spacetime immediately surrounding the inner astronaut is locally flat, allowing the astronaut to use the Minkowski metric. Since the separation between the astronauts is expressed in terms of reduced circumference, this can be incorporated into the Schwarzchild metric. Thus,

$$(ds)^2_{inner\ astronaut} = (ds)^2_{outer\ astronaut} \tag{3.2.11}$$

$$\underbrace{(c\, dt)^2 - \left((dx)^2 + (dy)^2 + (dz)^3\right)}_{\text{Minkowski metric}} = \underbrace{\left(1 - \frac{2GM}{c^2 r}\right)(c\, dt)^2 - \frac{(dr)^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2 (d\phi)^2}_{\text{Schwarzchild metric}} \tag{3.2.12}$$

To measure the distance between two points, the location of the two points must be determined at the same time, so $dt = 0$ in both reference systems. Additionally, since the points lie along the same radial line, $d\phi = 0$. Calling this line the x-axis allows us to set $dy = dz = 0$. Thus,

$$-(dx)^2 = -\frac{(dr)^2}{\left(1 - \frac{2GM}{c^2 r}\right)} \tag{3.2.13}$$

$$dx = \frac{dr}{\sqrt{1 - \frac{2GM}{c^2 r}}} \tag{3.2.14}$$

Equation 3.2.14 plays a similar role in general relativity that the length contraction relationship plays in special relativity. It relates the spatial interval measured by a “special” observer (one at rest in curved space) to another observer’s spatial measurements. Mathematically, it even has a similar structure, with the term “$2GM/r$” playing the role of “v^2” in the length contraction formula.

Substituting $r = 2r_{horizon} = 4GM/c^2$ and $dr = 100\, m$ into Equation 3.2.14 yields

$$dx_{astronout} = \frac{100\,m}{\sqrt{1-\frac{1}{2}}} \tag{3.2.15}$$

$$= 141.42\,m \tag{3.2.16}$$

Thus, even though the two astronauts only differ by 100 m in reduced circumference, the distance between the astronauts, measured by the inner astronaut, is 141.42 m. Thus, spacetime is stretched by a factor of 41% compared to flat spacetime. This provides a measure of how "warped" spacetime is at this location.

Are you upset with my sloppy use of calculus in the previous example? You should be. The metrics relate differential changes in time and space (dt and dr) and I just plugged in $100\,m$ for dr. Is 100 m infinitesimally small? It depends …

More carefully, I should integrate the expression for $dx_{astronaut}$ between the two limits, from $2r_{horizon}$ to $2r_{horizon} + 100\,m$.

$$x = \int_{2r_{horizon}}^{2r_{horizon}+100\,m} \frac{dr}{\sqrt{1-\frac{2GM}{c^2 r}}} \tag{3.2.17}$$

This integral is ugly for two reasons: the variable is in the denominator of a fraction that's in the denominator of the expression, and the integral has a bunch of constants. It's easy to get rid of the constants by using the definition of the event horizon,

$$x = \int_{2r_h}^{2r_h+100\,m} \frac{dr}{\sqrt{1-\frac{r_h}{r}}} \tag{3.2.18}$$

where

$$r_h = \frac{2GM}{c^2} \tag{3.2.19}$$

To solve the more difficult problem, multiply the numerator and denominator by a skillfully chosen factor:

$$x_{astronaut} = \int_{2r_h}^{2r_k+100m} \frac{\left(\frac{r}{r_h}\right)^{1/2} dr}{\left(\frac{r}{r_h}\right)^{1/2} (1-\frac{r_h}{r})^{1/2}} \tag{3.2.20}$$

$$= \int_{2r_h}^{2r_h+100m} \frac{\sqrt{\frac{r}{r_h}}\,dr}{\sqrt{\frac{r}{r_h}-1}} \tag{3.2.21}$$

Next, notice that the combination of terms that appears in the integral is dimensionless, meaning it has no units. It is always a very good idea to try to simplify complicated integrals in terms of dimensionless factors.

Perform a u-substitution where u is equal to this dimensionless factor and simplify:

with

$$u = \frac{R}{r_h}$$

$$du = \frac{1}{r_h} dr$$

with the limits of integration

$$\text{lower limit}: u = \frac{(2r_h)}{r_h} = 2$$

$$\text{upper limit}: u = \frac{(2r_h + 100m)}{r_h} = 2 + \frac{100c^2}{2GM} = 2.03385$$

So Equation 3.2.21 then becomes

$$x_{astronaut} = r_h \int_2^{2.03385} \sqrt{\frac{u}{u-1}} du \tag{3.2.22}$$

$$= \frac{2GM}{c^2} \left[\sqrt{u(u-1)} + \ln(\sqrt{u} + \sqrt{u-1}) \right]_2^{2.03385} \tag{3.2.23}$$

Thus, the actual distance between the astronauts, measured by the inner astronaut, is 140.83 m. For this problem, 100 m is “small enough” to be considered infinitesimally small, since the correct answer differs from the approximate answer by less than 1%. The correct answer is less than the approximate answer because the correct answer takes into account that space is less stretched as you move out toward the second astronaut, while the approximate answer approximates the stretching of space as being constant between the astronauts.

Contributors and Attributions

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