6.8: The Time-Dependent Schrödinger Equation

In this chapter, we investigated solutions of the time-independent Schrödinger equation. These solutions are offered referred to as stationary states because their spatial shape does not change with time, leading to probabilities that are constant in time. However, they are not "stationary" in the sense of having no time dependence. Imagine a guitar string vibrating in its fundamental mode. The "shape" of the string is constant; it just vibrates back and forth through space. To explore the rate at which the quantum wavefunction "vibrates" we need to solve the time-dependent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x^2}\Psi(x,t)+U(x)\Psi(x,t)=i\hbar\frac{\delta}{\delta t}\Psi(x,t) \tag{6.8.1}$$

Note that the wavefunction is now a function of both space and time, and the derivatives in the equation are partial derivatives. Also note that the potential energy function, U, is constant with respect to time.

Let's try to solve this partial differential equation by separation of variables. To do this, we'll assume the solution takes the form:

$$\Psi(x,t)=\psi(x)T(t) \tag{6.8.2}$$

Substituting this into the differential equation yields:

$$-\frac{\hbar^2}{2m}T(t)\frac{d^2}{dx^2}\psi(x)+U(x)\psi(x)T(t)=i\hbar\psi(x)\frac{d}{dt}T(t) \tag{6.8.3}$$

Dividing both sides by the wavefunction gives:

$$-\frac{\hbar^2}{2m}\frac{\frac{d^2\psi(x)}{dx^2}}{\psi(x)}+U(x)=i\hbar\frac{\frac{dT(t)}{dt}}{T(t)} \tag{6.8.4}$$

Since the left-hand side is only a function of x and the right-hand side is only a function of t, they can only be equal if both sides equal a constant value. If we call that constant E,

$$-\frac{\hbar^2}{2m}\frac{\frac{d^2\psi(x)}{dx^2}}{\psi(x)}+U(x)=E=i\hbar\frac{\frac{dT(t)}{dt}}{T(t)} \tag{6.8.5}$$

the left-side becomes the time-independent Schrödinger equation and the right-hand side becomes:

$$i\hbar\frac{dT(t)}{dt}=ET(t) \tag{6.8.6}$$

$$\frac{dT(t)}{dt}=-i\omega T(t)\text{ with }\omega=\frac{E}{\hbar} \tag{6.8.7}$$

This equation has the solution

$$T(t)=Ae^{-i\omega t} \tag{6.8.8}$$

Although you may not be familiar with imaginary arguments in the exponential function, mathematicians will tell you it is a really cool way to compactly write the sine and cosine functions:

$$T(t)=Ae^{-i\omega t}=A\left[\cos(\omega t)-i\sin(\omega t)\right] \tag{6.8.9}$$

So what does this mean?

Sines and cosines are familiar functions used to describe oscillations, so this means that the wavefunction simply oscillates at an angular frequency (□) proportional to the energy. Unlike a guitar string, however, this oscillation is not through physical space, but rather through some much more abstract space in which the square of the amplitude of oscillation is related to probability. Moreover, this space cannot be adequately represented without using imaginary numbers.

You may be concerned that the imaginary nature of the wavefunction will somehow creep into the probability of measuring some physical quantity. You shouldn't be. Although we've stated numerous times that probabilities depend on the square of the wavefunction, actually they depend on the product of the wavefunction and its complex conjugate . Thus, the effect of the temporal part of the wavefunction on all probabilities is given by:

$$\begin{aligned} \mathrm{P}rob(t) &= T\backslash *(t)T(t) \\ \mathrm{P}rob(t) &= (Ae^{+\omega t})(Ae^{-i\omega t}) \\ \mathrm{P}rob(t) &= A^2 e^0 \\ \mathrm{P}rob(t) &= A^2 \end{aligned} \tag{6.8.10}$$

In fact, if we set A = 1, the temporal part of the wavefunction will have no effect on the probabilities calculated earlier in this chapter. Thus, as long as the potential energy function is constant in time, Schrödinger's equation is separable and all of our work studying the time-independent equation is valid, as long as we remember that these solutions are actually oscillating in time according to the description given above.