Monroe Community College

Spiral Modern Physics

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This text was compiled on 04/15/2025

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CHAPTER OVERVIEW

1: The Special Theory of Relativity - Kinematics

Contributors and Attributions

- Paul D’Alessandris (Monroe Community College)

1.1: Time Dilation

Einstein developed much of his understanding of relativity through the use of gedanken, or thought, experiments. In a gedanken experiment, Einstein would imagine an experiment that could not be performed due to technological limitations, and “perform” the experiment in his head. By analyzing the results of these experiments, he was lead to a deeper understanding of his theory.

In developing his understanding of the relativity of time, Einstein imagined what he considered the simplest possible clock, consisting of a mirror on the floor and a mirror on the ceiling of a train car, and a light pulse bouncing back and forth between the mirrors. Each reflection of the light pulse serves as a “tick” of the clock.

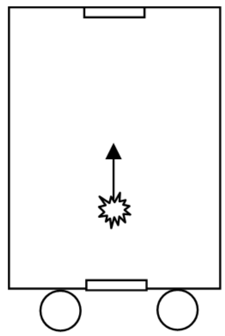

***Figure 1.1.1**: Einstein's simple "clock".*

Now imagine two observers and the train car traveling to the right at speed v. A passenger on the train will see the light pulse travel a vertical path from floor to ceiling. An observer at rest on the earth, however, will see the train displaced to the right during the time it takes the pulse to travel from floor to ceiling, and hence see the light pulse follow an angled path.

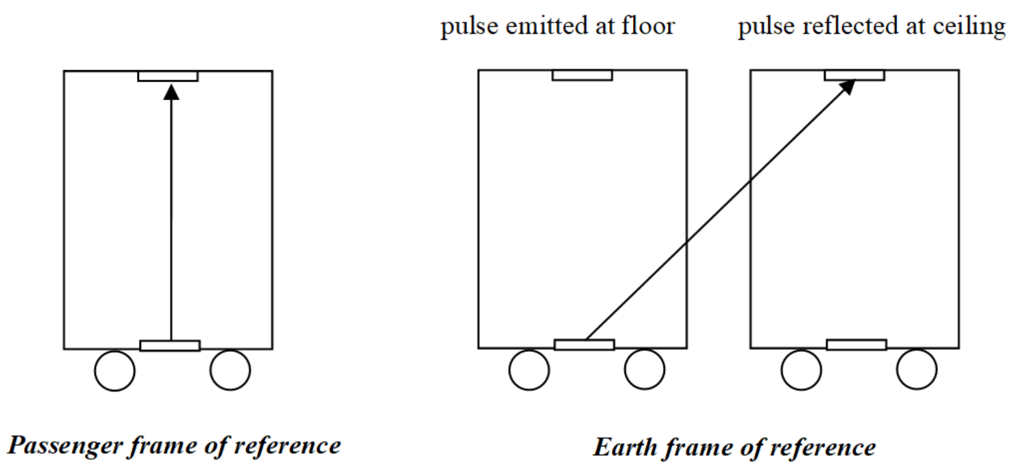

***Figure 1.1.2**: Einstein's simple "clock" in motion.*

Let Δt represent the travel time from floor to ceiling measured on the earth, and Δt_o represent the travel time from floor to ceiling measured on the train. Based on these definitions, and the speed of the train (v), the distances in the diagram can be determined.

> You do remember from your study of electromagnetism that the speed of electromagnetic waves is the same in all frames of reference, don’t you?

By Pythagoras’ Theorem,

$$(c\Delta T)^2 = (v\Delta t)^2 + (c\Delta t_o)^2 \tag{1.1.1}$$

$$(c\Delta T)^2 - (v\Delta t)^2 = (c\Delta t_o)^2 \tag{1.1.2}$$

$$(c^2 - v^2)\Delta t^2 = c^2\Delta t_o^2 \tag{1.1.3}$$

$$\left(1-\frac{v^2}{c^2}\right)\Delta t^2=\Delta t_o^2 \tag{1.1.4}$$

$$\Delta t=\frac{\Delta_o}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.1.5}$$

Therefore, since the denominator is less than 1 for any speed $v<c$, the time interval between the "ticks" of the clock measured on the earth is greater than the time interval measured on the train. To earth-bound observers, the clock is ticking slower than normal. Einstein proposed that this is not a property of the clock, but rather a reflection of the fact that time itself passes slower for an object is in motion.

Using Time Dilation

The half-life of a muon at rest is $2.2\,\mu s$. One can store muons for a much longer time (as measured in the laboratory) by accelerating them to a speed very close to that of light and then keeping them circulating at that speed in an evacuated ring. Assume that you want to design a ring that can keep muons moving so fast that they have a laboratory half-life of $20 \mu s$. How fast will the muons have to be moving?

The time between two events, for example the "birth" and "death" of a muon, depends on who makes the measurements. Within any particular reference system, the familiar results of classical physics are valid. However, in comparing results between observers in different reference systems, a method of relating one observer's measurements to another is needed.

Equation 1.1.5 for time dilation is typically written in the form

$$\Delta t=\gamma(\Delta t_o) \tag{1.1.6}$$

where

- Δt_o is the proper time, the time between two events in the frame of reference in which both events occur at the same spatial point,
- Δt is the time between the same two events in a different frame, moving at relative speed v,
- and γ is the Lorentz factor, given by

$$\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.1.7}$$

In the muon's frame of reference, the muon's are at rest and hence have a half-life of $2.2\,\mu s$. Moreover, this is a proper time since in this frame of reference both the birth and death of the muons occur at the same point. (Since the muons are not moving, everything that happens to them happens at the same point.)

$$\Delta t=\gamma(\Delta t_o) \tag{1.1.8}$$

$$20\,\mu s=\gamma(2.2\,\mu s) \tag{1.1.9}$$

$$\gamma=\frac{20\,\mu s}{2.2\,\mu s} \tag{1.1.10}$$

$$=9.09 \tag{1.1.11}$$

Using Equation 1.1.7

$$9.09=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.1.12}$$

$$1-\frac{v^2}{c^2}=\left(\frac{1}{9.09}\right)^2 \tag{1.1.13}$$

$$\frac{v^2}{c^2}=1-\left(\frac{1}{9.09}\right)^2 \tag{1.1.14}$$

$$\frac{v^2}{c^2} = 0.9879 \tag{1.1.15}$$

$$v = 0.994c \tag{1.1.16}$$

To get this level of time dilation, the muons have be me moving at 99.4% the speed of light.

Time Dilation at Everyday Speeds

The cruising speed of a jet airplane is approximately 250 m/s relative to the earth. For each hour of earth-time that passes, how much less time passes for the passengers on the plane?

The time interval measured by the passengers is the proper time, Δt_o, so we start with Equation 1.1.5

$$\Delta t = \frac{\Delta t_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Delta t_o = \Delta t\sqrt{1 - \frac{v^2}{c^2}} \tag{1.1.17}$$

$$\Delta t_o = (1\,hr)\sqrt{1 - \frac{250^2}{(3 \times 10^8)^2}} \tag{1.1.18}$$

$$\Delta t_o = (1\,hr)\sqrt{1 - 6.94 \times 10^{-13}} \tag{1.1.19}$$

Unfortunately, at this point my calculator tells me that $\Delta t_o = 1\,hr$, which is obviously not true. I need a better way to deal with the extremely small numbers that sometimes show up in relativity. One way to do this is with the *binomial approximation*.

Expressions of the form:

$$(1 + x)^n \tag{1.1.20}$$

where x is much less than 1 can be approximated by:

$$(1 + x)^n \approx 1 + nx \tag{1.1.21}$$

if $x \ll 1$

Using this approximation allows Equation 1.1.19to be simplified:

$$\Delta t_o = (1\,hr)\sqrt{1 - 6.94 \times 10^{-13}} \tag{1.1.22}$$

$$\Delta t_o = (1\,hr)\left(1 - 6.94 \times 10^{13}\right)^{1/2} \tag{1.1.23}$$

$$\Delta t_o \approx (1\,hr)\left(1 - \left(\frac{1}{2}\right) 6.94 \times 10^{-13}\right) \tag{1.1.24}$$

$$\Delta t_o \approx 1\,hr - 3.47 \times 10^{-13} hr \tag{1.1.25}$$

$$\Delta t_o \approx 1\,hr - 1.25 \times 10^{-9}\,s \tag{1.1.26}$$

Thus, for every hour that passes on earth, 1.25 nanoseconds less time passes on the airplane! This may seem like an incredibly small amount of time (and it is) but this effect has been measured in numerous experiments.

1.2: Length Contraction

If the time interval between two events depends on the relative motion of the observer, Einstein realized that the spatial separation between the events must also be observer-dependent. Consider a hypothetical spaceship journey from earth to a distant star. Assume the star is a distance L_o from the earth, as measured by stationary earth-bound observers. Therefore, the elapsed time for the spaceship to reach the star, as measured on earth, is

$$\Delta t = \frac{L_o}{v} \tag{1.2.1}$$

where v is the speed of the spaceship measured on earth.

By time dilation, however, the elapsed time for the spaceship to reach the star, as measured on the spaceship (a proper time), is

$$\Delta t_o = \frac{\Delta t}{\gamma} \tag{1.2.2}$$

$$= \frac{L_o}{v\gamma} \tag{1.2.3}$$

The distance the spaceship travels (L), as measured on the spaceship, is simply the product of its speed and the elapsed time measured on the ship

$$L = v\Delta t_o \tag{1.2.4}$$

$$= v\left(\frac{L_o}{v\gamma}\right) \tag{1.2.5}$$

$$= \frac{L_o}{\gamma} \tag{1.2.6}$$

Since gamma is greater than one, the distance between the earth and the star as measured on the ship is less than the distance as measured on the earth. To moving observers, the distance to the star shrinks.

Using Length Contraction

The star Vega is approximately 25 light-years from Earth (as measured by observers on Earth).

a. *How fast must a spaceship travel in order to reach Vega in 30 years, as measured on Earth?*
b. *How fast must a spaceship travel in order to reach Vega in 30 years, as measured on the spaceship?*

The distance between two events, for example leaving Earth and arriving at Vega, depends on who makes the measurements. Within any particular reference system, the familiar results of classical physics are valid. However, in comparing results between observers in different reference systems, a method of relating one observer's measurements to another is needed.

The formula for length contraction is

where

L0 the proper length, the distance between two events in the frame of reference in which both events are at rest,

and L is the distance between the same two events in a different frame, moving at relative speed v.

For part a, relativity theory is unnecessary. Both the distance and time measurements are made from the same frame of reference. Therefore, results from classical physics are valid.

(Note the use of the speed of light as a unit. Rather than substituting 3.0 x 108 m/s for c, simply leave c as a unit of velocity.)

For part b, the distance is measured in the Earth's frame while the time is in the frame of the spaceship. To solve part b, you must either convert the distance into the spaceship frame or the time into the Earth frame. You can convert the distance into the spaceship frame by realizing that the distance to Vega as measured on Earth is a proper length. Therefore,

1.3: Lorentz Transformation

The time dilation and length contraction relationships actually have very limited applicability. To use the time dilation relationship, one of the two observers must measure a proper time, where the two events occur at exactly the same spatial point. To use the length contraction relationship, one of the two observers must measure a proper length, where the two events must be at rest with respect to the observer. What if you want to compare measurements concerning more general events? To do this requires the Lorentz Transformation, which allows you to transform the spacetime coordinates of an event in one inertial reference system to any other inertial reference system.

To derive the *Lorentz transformation*, imagine two inertial reference systems, labeled O and O'. Let the origins of O and O' overlap at time zero, and allow O' to move with speed u relative to O. (Therefore, at a later time t, the origins are separated by a distance ut). Call the direction of motion the x-direction.

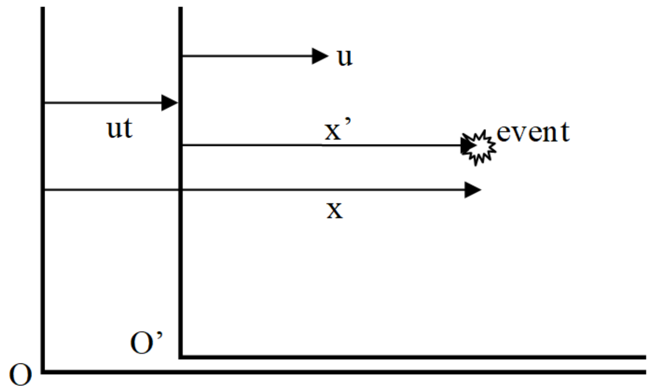

Figure* 1.3.1*:

Now imagine an event that occurs somewhere in spacetime. This event is located at position x relative to the O system, and position x' relative to the O' system. How are these two locations related?

You may be tempted to state that

$$x = x' + ut \tag{1.3.1}$$

however, this can't be correct because x and x' are measured in different reference systems. However, imagine that the event is the tip of a meterstick, fixed in O', striking some object. Since x' is now a proper length in O', it will appear contacted in O by the gamma factor. Therefore, the correct relationship between x and x' is

$$x = \frac{x'}{\gamma} + ut \tag{1.3.2}$$

rearranging yields

$$x' = \gamma(x - ut) \tag{1.3.3}$$

Since there is no relative motion in the y and z directions, these positions are the same in both coordinate systems

$$y' = y \tag{1.3.4}$$

$$z' = z \tag{1.3.5}$$

This completes the spatial part of the Lorentz transformation, but what about the temporal part? To determine how t and t' are related, now imagine that the event under investigation is the result of a light pulse, emitted from the origin when the two origins overlapped at time zero, striking some detector. Since the speed of light is the same in both systems, the distance measured in each system must be equal to the product of c and the elapsed time

$$x' = \gamma(x - ut) \tag{1.3.6}$$

$$ct' = \gamma\left(ct - u\frac{x}{c}\right) \tag{1.3.7}$$

$$t' = \gamma\left(t - u\frac{x}{c^2}\right) \tag{1.3.8}$$

Using the Lorentz Transformation

Inside of a spaceship zooming past earth at $0.5c$, I fire a laser (in the same direction as the ship's motion) and let it strike a mirror 10 m in front of the laser.

a. *What is the elapsed time measured on the earth between turning on the laser and the light striking the mirror?*
b. *How far has the light traveled before hitting the mirror, as measured on earth?*

Since neither the earth's observers nor the observers on the ship measure a proper time or a proper length between the two events (turning on the laser and the laser striking the mirror), a more general method of relating different observers' measurements is needed. This general method of relating measurements is the Lorentz Transformation. The Lorentz Transformation relates the coordinates of a spacetime event, (x, y, z, t), measured in one frame to the coordinates of the same event in a frame moving with relative velocity u, (x', y', z', t') as follows:

$$x' = \gamma(x - ut)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{ux}{c^2}\right)$$

These equations are written in a form that easily allows the determination of the primed coordinates from the unprimed. If the situation requires the inverse of this task, the equations can be easily inverted (by changing the sign of u and flipping the primed and unprimed notation) to yield

$$x = \gamma(x' - ut') \tag{1.3.9}$$

$$y = y' \tag{1.3.10}$$

$$z = z' \tag{1.3.11}$$

$$t = \gamma\left(t' - \frac{ux'}{c^2}\right) \tag{1.3.12}$$

Let the two coordinate systems overlap at the first event (the laser is fired). Thus, the position and time of the laser's firing is zero in both coordinate systems. We now must find the position and time of the second event (the laser strikes the mirror). This is relatively easy to determine in the frame of the spaceship (the primed frame):

$$x' = 10\,m \tag{1.3.13}$$

$$t' = \frac{10\,m}{c} \approx 3.33 \times 10^{-8}\,s \tag{1.3.14}$$

Since we know the spacetime location of the event in the primed frame, the Lorentz Transformation allows us to *transform* this information into the earth frame. With $u = 0.5c$ ($\gamma = 1.155$),

$$t = \gamma\left(t' + \frac{ux'}{c^2}\right) \tag{1.3.15}$$

$$= 1.155\left(\left(\frac{10}{c}\right) + \frac{(0.5c)(10)}{c^2}\right) \tag{1.3.16}$$

$$= \frac{17.3\,m}{c} = 5.7 \times 10^{-8}\,s \tag{1.3.17}$$

and for the x-direction

$$x = \gamma(x' + ut') \tag{1.3.18}$$

$$= 1.155\left(10 + (0.5c)\left(\frac{10}{c}\right)\right) \tag{1.3.19}$$

$$= 17.3\,m \tag{1.3.20}$$

The light travels $17.3\,m$ and takes $5.78 \times 10^{-8}\ s$ to strike the mirror in the earth's frame.

Contributors

- Paul D'Alessandris (Monroe Community College)

1.4: Velocity Addition

Since the Lorentz transformation allows you to relate the position and time of an event in one coordinate system to the position and time in any other coordinate system, it also allows you to relate quantities that depend on position and time, like velocity and acceleration. Therefore, using Lorentz we can derive equations that allow use to transform velocities measured by one observer to velocities measured by other observers.

Refer back to the Lorentz transformation derivation. This time, imagine that the event of interest is a particle, launched from the origin when the two origins overlapped at time zero, striking some detector. Thus,

$$x = v_x t \tag{1.4.1}$$

$$x' = v'_x t' \tag{1.4.2}$$

Substituting these relationships into the Lorentz transformation yields

$$x' = \gamma(x - ut) \tag{1.4.3}$$

$$v'_x t' = \gamma(v_x - ut) \tag{1.4.4}$$

$$v'_x t' = \gamma(v_x - u)t \tag{1.4.5}$$

and

$$t' = \gamma\left(t - \frac{ux}{c^2}\right) \tag{1.4.6}$$

$$= \gamma\left(t - \frac{uv_x t}{c^2}\right) \tag{1.4.7}$$

$$= \gamma\left(1 - \frac{uv_x}{c^2}\right) t \tag{1.4.8}$$

dividing the Equation ??? by Equation ??? yields

$$\frac{v'_x t'}{t'} = \frac{\cancel{\gamma}(v_x - u)\cancel{t}}{\cancel{\gamma}\left(1 - \frac{uv_x}{c^2}\right)\cancel{t}} \tag{1.4.9}$$

$$= \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \tag{1.4.10}$$

This directly relates the x-speed of an object in one reference system (v_x) to the speed of the same object measured in a different system (v'_x).

Although y- and z-positions are not effected by the Lorentz transformation, y- and z-velocities are different in different systems. Starting with,

$$y' = y \tag{1.4.11}$$

$$v_y t = v_y t \tag{1.4.12}$$

divide by the time transformation derived above

$$\frac{v'_y t'}{t'} = \frac{v_y \cancel{t}}{\gamma\left(1 - \frac{uv_x}{c^2}\right)\cancel{t}} \tag{1.4.13}$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{uv_x}{c^2}\right)} \tag{1.4.14}$$

The same type of transformation holds for z-velocity.

$$v'_z = \frac{v_z}{\gamma\left(1 - \frac{uv_x}{c^2}\right)} \tag{1.4.15}$$

Using Velocity Addition

A spaceship travels at 0.8c with respect to the solar system. An unmanned probe is ejected at 0.6c at an angle of 30° from the direction of travel of the ship (both the speed and angle of the probe are measured with respect to the ship). What are the speed and angle of launch of the probe as measured in the solar system frame?

Just as distance and time depend on the relative motion between observers, so does velocity. The following relationships allow you to compare velocity measurements between two observers in relative motion.

$$v'_x = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \tag{1.4.16}$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{uv_x}{c^2}\right)} \tag{1.4.17}$$

were v'_x and v'_y are the velocities of an object measured in a frame (the "primed" frame) moving at speed u relative to the "unprimed" frame (where observers measure v_x and v_y).

These equations are written in a form that easily allows the determination of v'_x and v'_y if v_x and v_y are known. If the situation requires the inverse of this task, the equations can be easily inverted (by changing the sign of u and flipping the primed and unprimed notation) to yield

$$v_x = \frac{v'_x + u}{1 + \frac{uv'_x}{c^2}} \tag{1.4.18}$$

$$v_y = \frac{v'_y}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \tag{1.4.19}$$

Using this form of the equations, with the spaceship the primed frame and $u = 0.8c$ ($\gamma = 1.67$), yields

$$\begin{aligned} v_x &= \frac{v'_x + u}{1 + \frac{uv'_x}{c^2}} \\ &= \frac{0.6c(\cos(30^\circ)) + 0.8c}{1 + \frac{(0.8c)(0.6c(\cos(30^\circ))}{c^2}} \\ &= 0.932c \\ v_y &= \frac{v'_y}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \\ &= \frac{0.6c(\sin(30^\circ))}{(1.67)\left(1 + \frac{(0.8c)(0.6c)(\cos(30^\circ))}{c^2}\right)} \\ &= 0.127c \end{aligned}$$

The solar system observers detect the probe's motion as

$$\begin{aligned} v &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{(0.932c)^2 + (0.127c)^2} \\ &= 0.941c \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{v_y}{v_x}\right) \\ &= \tan^{-1}\left(\frac{0.127c}{0.932c}\right) \\ &= 7.76^\circ \end{aligned}$$

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1.6: Interstellar Travel – Kinematic Issues (Project)

Imagine you would like to visit Vega, a nearby star approximately 25 light-years from earth, and then return to earth to tell your friends. At first glance, it would appear that the Special Theory of Relativity requires this to be a very long trip. Since no material object can travel faster than light, it would seem to require more than 50 years to make the roundtrip. Then you remember time dilation! If you traveled fast enough, you could make the roundtrip in much less than 50 years (of course, your poor friends would still have aged 50+ years). However, as you will see in this and a following investigation, other, potentially overwhelming, problems may make your trip to Vega incredibly unlikely.

I. Kinematic Considerations

A. Vega in one year, in spaceship frame
Assume you would like to be able to reach Vega in one year, as measured on your clock.

1. Ignoring accelerations at the beginning and end of your journey, how fast must you travel (in units of c) to get to Vega (25 c yr from earth) in 1.0 yr?

B. Time needed to accelerate, in spaceship frame
In part A, we neglected the time it would take to reach cruising speed. This could be a major concern. The human body cannot withstand extended periods of acceleration greater than 9.8 m/s2, or 1 g. Imagine the sensation of riding a rollercoaster around a sharp bend. This is approximately an acceleration of 2g. Could you withstand that feeling for the days, weeks or months required to reach very high speed? An acceleration of 2g is like a 150 lb person walking around all day and night carrying another 150 lb person on their back. This cannot be tolerated for long, so we will restrict our acceleration to 1 g.

1. With an acceleration of 1 g, as measured on the spaceship, how long will it take (in days measured on the ship) to reach the cruising speed calculated in A?

You should have found that it would take approximately a year just to reach cruising speed. So much for the idea of reaching Vega in a year! You have only just reached cruising speed! And don't forget you would need an additional year just to slow down and land or even turn around.

C. Distance needed to accelerate, in spaceship frame
Even though it takes nearly a year to accelerate to cruising speed, you would be covering some of the 25 c yr to Vega.

1. Calculate the distance traveled, in light-years in the frame of the spaceship, while the spaceship is accelerating.

The answer above would be the odometer reading on the spaceship when it has reached cruising speed. However, it's not clear what the total distance to Vega is in this frame, because the distance is continually being contracted by different amounts as the spaceship accelerates.

Therefore, a more useful number would be the distance traveled by the spaceship in the frame of Earth. This number could then be compared to the 25 c yr to Vega.

D. Distance needed to accelerate, in Earth's frame
This calculation is much trickier than the preceding ones. If the ship traveled at constant speed, the distance traveled in the earth's frame is proportional to the distance traveled in the ship's frame:
dearth = dship

However, changes as the ship accelerates. To handle a changing , we will have to perform an integral over the ship's journey.

First, the distance traveled by the ship, in the ship's frame, can be written:

Note that if you evaluated this integral, you would get the familiar result for distance traveled from rest. The key observation is that the distance traveled measured on earth is still simply the product of and the distance traveled by the ship, just remembering that since is not constant, it is inside the integral.

Insert the term "a (1/a)" into the integral.

Note that "a" can be written "dv/dt".

Since the acceleration of the ship is assumed constant, "(1/a)" can come out of the integral, and note that the dt's "cancel".

1. Evaluate the integral, from rest to final speed vc. Simplify the result but do not yet plug in numerical values.

2. Using the cruising speed determined in A, calculate the distance in light-years the spaceship travels while accelerating, in the earth's frame.

You should have found that the ship has traveled less than one light-year during acceleration, and would, of course, need to begin to slow down one light-year, as measured on earth, before reaching Vega.

E. Total travel time, in spaceship frame
The total time to travel to Vega includes the time needed to accelerate, the time needed to decelerate, and the time needed to cruise at constant speed between these two portions of the journey.

1. Calculate the total travel time, in years in the spaceship frame, to Vega.

2. To summarize this process in the spaceship frame, complete the following motion table:

Ship's Frame
Blast-off!

t1 = 0 yr
r1* = 0 c-yr
v1* = 0 c
a12 = Reaches cruising speed

t2 =
r2 =
v2 =
a23 = Begins to slow

t3 =
r3 =
v3 =
a34 = Reaches Vega

t4 =
r4 =
v4 =

* Of course, the position (r) and velocity (v) of the ship in the ship's frame are always zero. Use these rows to tabulate the odometer reading of the ship and the speed with which Vega approaches the ship.

F. Total travel time, in Earth's frame
Just as we needed to do an integral to find the acceleration distance in the earth's frame (since the distance traveled by the ship is contracted by a varying amount), we need to do an integral to determine the acceleration time in the earth's frame since time is dilated by a varying amount.

If the ship traveled at constant speed, the time measured in the earth's frame is proportional to the time measured in the ship's frame:
tearth = tship

However, changes as the ship accelerates. To handle a changing , we have to perform an integral over the ship's journey. Thus,

Since the ship starts at rest, v = at, and

1. Evaluate the integral from rest to the time at which cruising speed is reached in the ship frame, tc.

2. To summarize this process in the Earth frame, complete the following motion table:

Earth's Frame
Blast-off!

t1 = 0 yr
r1 = 0 c-yr

v1 = 0 c
a12 = XXX Reaches cruising speed

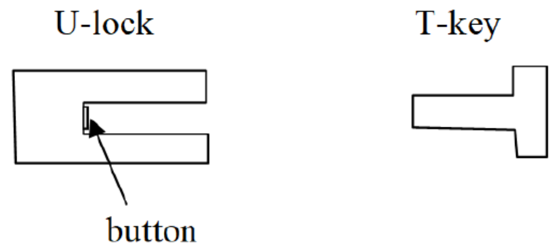
t2 =
r2 =
v2 =
a23 = Begins to slow

t3 =
r3 =
v3 =
a34 = XXX Reaches Vega

t4 =
r4 =
v4 =

1.7: The Lock and Key Paradox (Project)

Imagine a U-shaped lock that opens only when a T-shaped key strikes a button in the deepest part of the lock, as illustrated below:



To prevent the lock from being easily opened, the depth of the lock (L_o, when at rest) is greater than the length of the key (K0, when at rest). Naïve locksmiths unfamiliar with special relativity would therefore believe the lock is unopenable. However, educated locksmiths argue that if the lock is in motion, relative to the key, the depth of the lock will shrink allowing the key to open the lock. Unfortunately, other educated locksmiths argue that in the same situation, in the frame of reference of the lock, the key would be in motion, causing it to shrink, thereby not opening the lock.

This situation is more complicated than the rather mundane pole-vaulter-and-the-barn paradox (or equivalently the farmer-and-the-tractor paradox). In those situations, the “solution” to the paradox involves understanding that the order of two events is not the same in both frames. In this situation, we don’t really care when the key strikes the button (if it does), but whether it objectively happens or not. To resolve this paradox, we will need to directly address the limits of information travel time.

I. The Lock Frame

Let the relative speed between the lock and key be v, corresponding to a Lorentz factor . Answer the following questions in the frame of reference of the lock.

1. Write an expression for the length of the key, K.

Let’s assume that when the rear of the key strikes the front edge of the lock, the rear of the key instantaneously stops. Although any real material would deform and come to rest over a finite time, no fundamental law of physics limits the lower value of this time. However, a fundamental law of physics does imply that the front edge of the key will continue to move forward until the information that the rear edge has stopped is transmitted to it. The front edge cannot stop simultaneously with the rear edge. The information that the rear edge has stopped can be transmitted no faster than the speed of light!

2. Write an expression for T, the time it takes for a message to be sent from the rear edge of the key to its front edge. Assume the message is sent at maximum speed, c. (Hint: Note that the tip of the key is moving away from this propagating signal.)

3. Write an expression for the distance the tip travels (K) after the rear has been stopped.

4. Write an expression for the length of the key when the front tip finally stops, K*. If K* is greater than or equal to the rest length of the lock, L0, the button will be pressed.

5. Show that K* is greater than K0. This means that the length of the key is larger than its proper length when it finally stops moving. Thus, during the process of stopping, the key will overshoot its proper length before settling back to its proper length!

II. The Key Frame

Answer the following questions in the frame of reference of the key.

1. Write an expression for the depth of the lock, L.

Again assume that when the front of the lock strikes the rear of the key, the front of the lock instantaneously stops. However, the rear of the lock (and hence the button) will continue to move forward until the information that the front edge has stopped is transmitted to it.

2. Write an expression for T, the time it takes for a message to be sent from the front edge of the lock to the location of the button. Assume the message is sent at maximum speed, c. (Hint: Note that the button is moving toward this propagating signal.)

3. Write an expression for the distance the button travels (L) after the rear has been stopped.

4. Write an expression for the depth of the lock when the button finally stops, L*. If L* is less than or equal to the rest length of the key, K0, the button will be pressed.

5. Show that L* is smaller than L. This means that the depth of the lock is smaller than its Lorentz contracted length when it finally stops moving. Thus, during the process of stopping, the lock will overcontract before re-expanding back to its proper length!

III. Consistent Results?

1. Show that the necessary condition to trigger the button in the lock frame (K* = L0, from I. 4.) is exactly the same as the necessary condition to trigger the button in the key frame (L* = K0, from II. 4.)

IV. A Numerical Example

Let L0 = 0.10 m, K0 = 0.09 m, and v = 0.5 c.

1. Ignoring the overcontraction of the lock, show that in the key frame the button is hit.

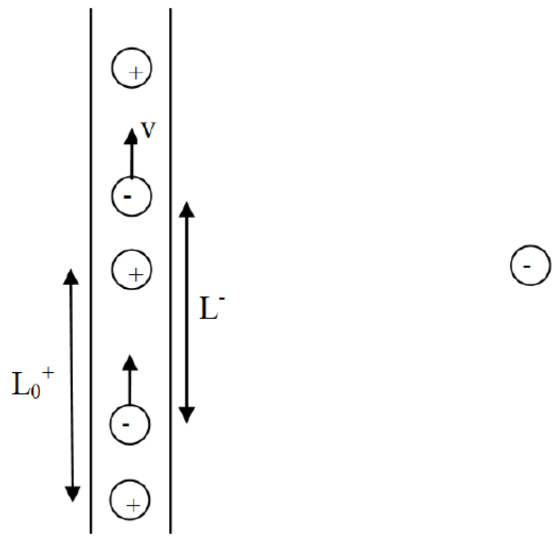
2. Ignoring the overexpansion of the key, show that in the lock frame the button would not be hit.

3. Including the overexpansion of the key, how long is the key when it finally stops? Is this sufficient to contact the button?

1.8: Length Contraction and the Magnetic Force (Project)

As I'm sure you recall, the magnetic force between a charged particle and a current-carrying wire depends on whether the particle is moving. If the particle is stationary, no magnetic force is present. However, if the particle moves, a magnetic force acts on the particle. Of course, whether something is moving or not depends on the frame of reference of the observer, i.e., motion is a purely relative concept. Does that mean that the magnetic force, and possibly the magnetic field, have no absolute, frame-independent existence and are also purely relative concepts? If so, perhaps the effects of magnetism can be explained in terms of some more fundamental concepts.

Imagine a current-carrying wire, illustrated below in the laboratory frame of reference. The average separation between the stationary positive nuclei is L_o^+. Note that this is a proper length for the separation between positive charges. The negative charges are moving at speed v and have an average separation L^- in the laboratory frame. A stationary negative charge is a distance, r, to the right of the wire.



I. Charge Density in the Laboratory Frame

The total electric charge density on the wire λ, is the sum of the positive and negative charge densities. The positive charge density is simply the ratio of the positive charge, +e, over the separation between positive charges, L_o^+.

1. Write an expression for the total electric charge density on the wire, λ.

2. Since current-carrying wires are electrically neutral, this density must equal zero. Based on this observation, what is the relationship between L_o^+ and L^-?

Obviously, there is no electric or magnetic force on the particle for this scenario. Boring. So let's imagine the particle moving with the same speed and direction as the current-carrying electrons.

3. What is the direction of the magnetic field from the current-carrying wire at the location of the moving particle?

4. What is the direction of the magnetic force on the moving particle?

(You should get the same direction for the magnetic force on the moving particle if you consider the wire and particle as two parallel currents flowing in the same direction.)

5. Using the result for the magnetic field from a long wire carrying current i, write an expression for the magnitude of the magnetic force acting on the moving particle.

II. Charge Density in the Moving Particle Frame

Let's see what's happening in the frame of reference of the moving particle. Note that in this frame of reference, the positive nuclei are moving "backward" with speed v and the electrons in the wire are at rest.

A. Conceptual Comparison

1. As you move from the laboratory frame into the moving particle frame, what happens to the separation between the electrons in the wire (i.e., does it stay the same, become larger, or become smaller)?

2. As you move from the laboratory frame into the moving particle frame, what happens to the separation between the positive nuclei in the wire (i.e., does it stay the same, become larger, or become smaller)?

3. Therefore, as you move from the laboratory frame into the moving particle frame, what happens to the electric charge density on the wire (i.e., does it stay neutral, become positive, or become negative)?

4. Based on the electric charge density on the wire, what is the direction of the electric force on the moving particle?

It appears that due to relativistic length contraction the neutral (in the laboratory frame) wire is positively charged in the moving particle frame and attracts the moving particle electrically. Could it be that what we have called the magnetic force is really simply the electric force from a length contracted charge distribution? To prove this conjecture we are going to have to get quantitative.

B. Quantitative Comparison

1. Write an expression for the total electric charge density on the wire in the moving frame λ'. Clearly designate whether a separation is a proper length (L_o) or not (L).

2. Using the length contraction relationship, replace the separation between electrons in this frame with the separation between electrons in the laboratory frame (L_-).

3. Using your result from I.2., replace L^- in your expression.

4. Using the length contraction relationship, replace the proper separation between nuclei (L_o^+) with the separation in the moving particle frame.

5. Factor out the common term in your expression and simplify. Your final result should be in terms of e, L^+, v and c only, and should be clearly a positive value.

6. Using Gauss' Law (or looking in your old notes), determine the electric field a distance r from a wire with linear charge density λ'.

7. Based on this electric field, write an expression for the electric force acting on the moving charge.

If we can show that this expression is exactly the same as the “magnetic” force acting on the moving charge, we will have shown that what we have traditionally called magnetic force is really simply the electric force from a length-contracted charge distribution.

8. Current is defined as the product of a charge density and a velocity. Write an expression for the current in the moving frame, i’. (Note, only the positive nuclei are moving in this frame so only their charge density contributes to the current.)

9. Using this result, replace the charge density (□’) in your electric force expression with current (i').

10. Using

$$c=\frac{1}{\sqrt{\epsilon_o\mu_o}} \tag{1.8.1}$$

show that your expression for electric force is identical to the expression for magnetic force in I.5.

C. Summary

So, what does this all mean? Basically, certain objects that are neutral in one frame of reference are actually electrically charged in other frames of reference due to length contraction. Moving charges (and hence electric currents) see these “charged-because-of-length-contraction” objects and feel electric forces of either attraction or repulsion. However, in our frame the objects are neutral so we “invent” a new force, magnetism, to make sense of the behavior of these moving charges. Now that we understand length contraction, there really is no need to think of the magnetic force as being different from the electric force. (It’s often more convenient to talk about magnetism, but it’s never really necessary.) Although it may be hard to believe, the pictures attached to your refrigerator are held in place because of relativistic length contraction!

What’s really strange, however, is how small of a length contraction is responsible for this effect. Most people have the idea that current-carrying electrons are “zipping” through a wire at high speeds. However, the average speed of an electron in a current-carrying wire (the drift speed) is typically only a couple millimeters per second! Electrons moving at this tiny speed experience ridiculously small length contractions, on the order of one part in 10^{22}! This tiny amount of “shrinkage” gives rise to the measurable forces of magnetism because of the incredible strength of the electric force. Basically, the electric force is so strong that even this little bit of shrinking gives rise to a tiny amount of net charge that results in a force large enough to notice with the naked eye. Or to hold a photo on a refrigerator.

CHAPTER OVERVIEW

2: The Special Theory of Relativity - Dynamics

Contributors and Attributions

- Paul D’Alessandris (Monroe Community College)

2.1: Relativistic Momentum, Force and Energy

Once Einstein revolutionized our understanding of space and time, physicists were faced with a monumental task. All of physics, before Einstein, was based on the idea of absolute space and time. Once these concepts were found to be erroneous, all of classical physics had to be re-examined in this light. In this section, we will "re-examine" our understanding of momentum, force, and energy.

Relativistic Momentum

In classical physics, momentum is defined as

$$\vec{p} = m\vec{v} \tag{2.1.1}$$

However, using this definition of momentum results in a quantity that is not conserved in all frames of reference during collisions. However, if momentum is re-defined as

$$\vec{p} = \gamma m\vec{v} \tag{2.1.2}$$

it *is* conserved during particle collisions. Therefore, experimentally, relativistic momentum is defined by Equation 2.1.2.

Relativistic Force

Once nature tells us the proper formula to use for calculating momentum, mathematics tells us how to measure force and energy. Force is defined as the time derivative of momentum

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{2.1.3}$$

(In classical physics, where $\vec{p} = m\vec{v}$, Equation 2.1.3 reduces to $\vec{F} = m\vec{a}$.)

> Equation 2.1.3 is the scalar form of this relationship and is only true for motion in one-dimension. The full vector treatment of force is more complicated than its worth.

Substituting correct relationship for momentum (Equation 2.1.2) into Equation 2.1.3 yields

$$\begin{aligned}\vec{F} &= \frac{d(\gamma m\vec{v})}{dt} \\ &= \left(\frac{d\gamma}{dt}\right) m\vec{v} + \gamma m \left(\frac{d\vec{v}}{dt}\right) \\ &= m\vec{v}\frac{d}{dt}\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right] + \gamma m\vec{a}\end{aligned}$$

Use the chain rule to evaluate the derivative of γ

$$\begin{aligned}\vec{F} &= m\vec{v}\left[-\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right)\left(\frac{dv}{dt}\right)\right] + \gamma m\vec{a} \\ &= m\vec{v}\left[\frac{v}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\vec{a}\right] + \gamma m\vec{a}\end{aligned}$$

Factor out the common factor γma,

$$\vec{F} = \gamma m\vec{a}\left[\frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} + 1\right] \tag{2.1.4}$$

Find a common denominator and simplify,

$$\vec{F} = \gamma m\vec{a}\left[\frac{\cancel{\frac{v^2}{c^2}} + 1 - \cancel{\frac{v^2}{c^2}}}{1-\frac{v^2}{c^2}}\right] \tag{2.1.5}$$

$$= \gamma m\vec{a}\left[\frac{1}{1-\frac{v^2}{c^2}}\right] \tag{2.1.6}$$

$$= \gamma^3 m\vec{a} \tag{2.1.7}$$

Equation 2.1.7 is the relativistically correct form of Newton's Second Law, for motion constrained in one dimension. (Physicists seldom use a force approach when analyzing motion since a momentum and energy approach is almost always more useful.)

Relativistic Energy

The kinetic energy of an object is defined to be the work done on the object in accelerating it from rest to speed v.

$$KE = \int_0^v F\,dx \tag{2.1.8}$$

Using our result for relativistic force (Equation 2.1.7) yields

$$KE = \int_0^v \gamma^3 ma\,dx \tag{2.1.9}$$

The variable of integration in Equation 2.1.9 is x, yet the integrand is expressed in terms of a and v (v is hidden inside γ). To solve this problem,

$$\begin{aligned} KE &= \int_0^v \gamma^3 m\left(\frac{dv}{dt}\right)\,dx \\ &= \int_0^v \gamma^3 m\left(\frac{dx}{dt}\right)\,dv \\ &= \int_0^v \gamma^3 mv\,dv \\ &= \int_0^v \frac{mv\,dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \end{aligned}$$

This integral can be done by a simple u-substitution,

$$u = 1-\frac{v^2}{c^2} \tag{2.1.10}$$

$$du = -\frac{2v}{c^2}dv \tag{2.1.11}$$

$$
\begin{aligned}
KE &= -\frac{mc^2}{2}\int_o^v \frac{du}{u^{3/2}} \\
&= -\frac{mc^2}{2}\left[-2u^{-1/2}\right] \\
&= mc^2\left[\frac{1}{1-\frac{v^2}{c^2}}\right]_o^v \\
&= mc^2\left[\frac{1}{(1-\frac{v^2}{c^2})^1/2}-1\right] \\
&= \gamma mc^2 - mc^2
\end{aligned}
$$

Rearranging this yields,

$$\gamma mc^2 = KE + mc^2 \tag{2.1.12}$$

Einstein identified the term γmc^2 as the *total* energy of the particle. Thus, the total energy is the sum of the kinetic energy and a completely new form of energy, the **rest energy**. Particles have rest energy just by virtue of having mass. In fact, mass is simply a form of energy.

$$E_{total} = KE + E_{Rest} \tag{2.1.13}$$

✔ Example 2.1.1: Using Momentum and Energy

An electron is accelerated through a potential difference of 80 kilovolts. Find the kinetic energy, total energy, momentum and velocity of the electron.

Solution

The following collection of equations express the relationships between momentum, energy, and velocity in special relativity. (Momentum is often easier expressed as "pc" rather than "p" as you will see once you begin working problems.)

$$p = \gamma mv \tag{2.1.14}$$

$$pc = \gamma mc^2\left(\frac{v}{c}\right) \tag{2.1.15}$$

$$
\begin{aligned}
E_{total} &= \gamma mc^2 \\
&= KE + mc^2
\end{aligned}
$$

$$KE = (\gamma - 1)mc^2 \tag{2.1.16}$$

$$E_{total}^2 = (pc)^2 + (mc^2)^2 \tag{2.1.17}$$

Equation 2.1.17 is particularly useful in that it allows a direct relationship between energy and momentum without the need to calculate the velocity. The proof of this relationship is left as an exercise.

From electrodynamics, the kinetic energy of a charge accelerated through a potential difference V is simply the product of the charge (q) and the potential difference (V),

$$
\begin{aligned}
KE &= qV \\
&= e(80 \times 10^3 V) = 80\, keV
\end{aligned}
$$

Rather than substituting the numerical value of the charge on an electron

$$q = -e = -1.6 \times 10^{-19}\, C \tag{2.1.18}$$

into this expression (and obtaining the kinetic energy in joules), we will leave "e" in the equation and use "eV" as a unit of energy (i.e., $1.0\, eV = 1.6 \times 10^{-19} J$.

Thus, the kinetic energy of the electron is 80 keV.

The total energy of the electron is then

$$\begin{aligned} E_{total} &= KE + mc^2 \\ &= 80\, keV + 511\, keV \\ &= 591\, keV \end{aligned}$$

The momentum is

$$E_{total}^2 = (pc)^2 + (mc^2)^2 \tag{2.1.19}$$

$$\begin{aligned} pc &= \sqrt{E_{total}^2 - (mc^2)^2} \\ &= \sqrt{591^2 - (511)^2} \\ &= 297\, keV \end{aligned}$$

(Again, momentum is often easier expressed as "pc" rather than "p")

The speed of the electron is

$$pc = \gamma mc^2 \left(\frac{v}{c}\right) \tag{2.1.20}$$

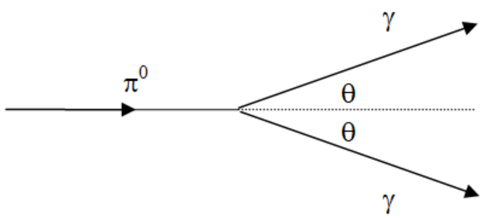
$$296\, keV = 591\, keV \left(\frac{v}{c}\right) \tag{2.1.21}$$

$$v = 0.503c \tag{2.1.22}$$

2.2: Collisions and Decays

A neutral pion (rest energy 135 MeV) moving at 0.7c decays into a pair of photons. The photons each travel at the same angle from the initial pion velocity. Find this angle and the energy of each photon.

Any process that occurs in nature must obey energy and momentum conservation. To analyze this particle decay, apply both conservation laws to the process.



First, find the Lorentz factor for the pion.

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{2.2.1}$$

$$= \frac{1}{\sqrt{1-\frac{(0.7c)^2}{c^2}}} \tag{2.2.2}$$

$$= 1.4 \tag{2.2.3}$$

Applying energy conservation yields:

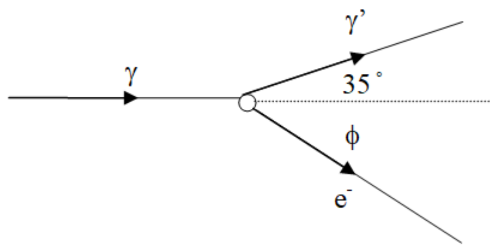
The two photons must have the same energy since they travel in the same direction relative to the initial pion velocity. This is the only way that momentum in this perpendicular direction can be conserved.

Applying momentum conservation (actually conservation of “pc”) along the initial direction of travel and using the relationship yields:

The photons each travel at 45.60 from the direction of the pions initial path.

Collisions and Decays II

A photon of energy 500 keV scatters from an electron at rest. The photon is redirected to an angle of 35° from its initial direction of travel. Find the energy of the scattered photon and the angle and energy of the scattered electron.



To analyze, apply energy conservation:

$$E_{photon} + E_{electron} = E'_{photon} + E'_{electron} \tag{2.2.4}$$

$$500 + 511 = E'_{photon} + E'_{electron} \tag{2.2.5}$$

$$1011 = E'_{photon} + E'_{electron} \tag{2.2.6}$$

note that the electron initially has only rest energy.

Apply x-momentum conservation (and use $pc=\sqrt{E_{total}^2-(mc^2)^2}$):

Apply y-momentum conservation:

This yields three equations with the requested three unknowns (E'_{photon}, $E'_{electron}$, and ϕ).

If you enjoy algebra, solve this system of equations by hand. If you have better things to do with your life, use a solver to find:

$$E'_{photon} = 425\ keV \tag{2.2.7}$$

$$E'_{photon} = 586\, keV \tag{2.2.8}$$

$$\phi = 58.1^\circ \tag{2.2.9}$$

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2.4: Interstellar Travel – Energy Issues (Project)

In a previous investigation, you discovered some of the fundamental difficulties in traveling to even a relatively nearby star. In this investigation, you will be introduced to a much more important factor, the incredible amounts of energy needed to accelerate a macroscopic object to relativistic speeds.

I. Energy Considerations

Consider a spaceship fully fueled of mass M. Some portion of the mass of the spaceship is in the form of its fuel, such that after "burning" its fuel the ship will have a smaller mass. This mass is the "useful" mass of the spacecraft, the payload, and we will designate it fM (a fraction, f, of the total loaded mass). For example, if the ship is 5% fuel, $f = 0.95$.

1. What is an expression for the amount of fuel on the fully loaded spacecraft?

A. Fusion powered spaceship

The most efficient energy source available using today's technology is fusion energy, and given that it is the energy source used by stars, it's probably a good bet that it may be the most efficient large scale energy source available in the universe.

Imagine a spacecraft powered by fusion energy. The most efficient fusion reaction effectively combines four hydrogen nuclei into a single helium nucleus. Since a helium nucleus is less massive than four hydrogen nuclei, the energy released by this reaction can be calculated using Einstein's relation:

$$E = mc^2. \tag{2.4.1}$$

1. Determine the energy released by this reaction by converting the mass difference between the helium nucleus and the four hydrogen nuclei into a measure of energy.

2. The efficiency of the reaction, ϵ, is defined to be the ratio of energy released to energy present in the initial hydrogen nuclei. Calculate the efficiency of this fusion reaction.

3. Using your expression for the total amount of fuel on the spacecraft, and the efficiency of the fusion reaction, write a symbolic expression (use ϵ, not its numerical value) for the total energy released by fusing all of the fuel.

The energy released via fusion will go toward providing kinetic energy to the spacecraft as well as kinetic energy to the exhaust helium. Calculating the portion of this energy that goes into moving the spacecraft is a complicated problem, so we'll simply approximate that **all** of the energy goes into kinetic energy of the ship! (This is a huge over-estimation, but we will find that even with this approximation, a fusion powered spaceship is not very feasible.)

4. Equate the kinetic energy of the payload (fM) with the total energy released by fusing all of the fuel. From this equation, solve for f, the fraction of the initial mass of the ship that is payload.

5. In the last investigation, you found that at a cruising speed of $v = 0.9992c$, a spacecraft could reach Vega in a little less than 3 years. Using the result above, what fraction of the mass of that spaceship can be payload, and how much of the ship must be made up of fuel?

You should have found that 99.97% of the mass of your spaceship must be fuel! A more useful way to express the amount of fuel needed for acceleration is to calculate the fuel to payload ratio. This is the number of kg of fuel needed to accelerate a single kg of payload.

6. Calculate the fuel to payload ratio for your spaceship.

For every 1 kg of payload you want to take to Vega, you would have to supply the ship with over 3000 kg of fuel. Actually, the situation is far worse. This is simply the fuel required to accelerate the payload up to the cruising speed. For a complete round trip, one would have to carry enough fuel to boost the payload to the cruising speed, decelerate it to rest at the destination, boost it to cruising speed again for the return trip home, and decelerate it upon reaching earth.

7. How much fuel is required for a complete round trip? Express your answer as a multiple of the payload mass, i.e., the number of kg of fuel per kg of payload. Explain why the answer is not simply four times the fuel required to boost the payload alone to the cruising speed.

This number borders on the ridiculous. You would need over $10^{14}\ kg$ of fuel to take a single one kg mass on a round-trip to Vega.

8. If you aren't as anxious to get to Vega quickly, you could travel at a cruising speed of $0.5\,c$. Calculate the amount of fuel, as a multiple of payload mass, needed for this roundtrip.

9. This should be substantially less fuel, but how long, ignoring acceleration and deceleration, would a roundtrip journey to Vega (25 c yr away) take at the relatively slow speed of 0.5 c?

B. Collecting fuel along the way

One possible way around the problem discovered above is to collect the hydrogen needed for fuel along the way to Vega. Interstellar space contains approximately one hydrogen atom per cubic meter. With a large enough collecting "mouth" perhaps this would solve the fuel problem.

In addition to the problem of the size of the "mouth" needed to collect enough hydrogen, a more fundamental problem remains. The hydrogen is approximately at rest with respect to Vega. However, in the frame of the spaceship, it is moving toward the ship at the same speed that the ship is moving toward Vega. Thus, the hydrogen atoms have a large kinetic energy as they are scooped up by the ship. The collision between the atoms and the ship will effectively slow the ship. When the kinetic energy of the atoms (which effectively slows the ship) is approximately equal to the energy that can be extracted from the atoms to accelerate the ship, this technique of collecting fuel along the way becomes relatively useless.

1. Setting the kinetic energy of the collected atoms equal to the energy that can be extracted from the atoms, calculate the speed above which collecting fuel as you go becomes a losing proposition.

2. Cruising at the speed calculated above, ignoring acceleration and deceleration, how long would it take you to reach Vega?

C. Matter-Antimatter Powered Spaceship

There is one final possibility. There exists a reaction that has an efficiency of 1.0, where all of the initial mass energy present is converted into kinetic energy. This is the annihilation of matter with antimatter. If a proton is combined with an antiproton, the total mass of both protons is converted into the kinetic energy of the resulting photons. If this reaction does not provide a reasonable fuel source for our trip to Vega, then a trip to Vega is simply not feasible.

There are many very serious problems with using matter-antimatter annihilation as a fuel source. The first is that there is effectively no naturally occurring antimatter in the universe! All of the antimatter has to be "built". We currently are technologically able to manufacture small amounts of antimatter, so we'll assume that in the future this may not be an insurmountable problem. Of course, the energy that is released when antimatter is annihilated has to be "put into" the antimatter in the first place, but this manufacturing can be done on a home planet and need not weigh down our spaceship. Storing large amounts of antimatter and combining it in a controlled manner on the spaceship is a different story, but let's pretend we've mastered that as well.

1. Imagine a rocket engine that combines matter and antimatter in a controlled way, and focuses the resulting photons into a tight beam traveling away from the stern of the spaceship. For this engine, determine f, the payload fraction of the initial mass of the ship, to achieve a cruising speed v. (Hint: The ship can essentially be considered to be a particle of mass M at rest that decays into a big flash of light and a smaller particle (the payload) of known mass fM traveling at a known speed v. Conserve both energy and momentum.)

2. Using the result above, what fraction of the mass of the spaceship can be payload, and how much of the ship must be made up of fuel?

3. Calculate the fuel to payload ratio for a matter-antimatter spaceship. How much fuel, per kilogram of payload, is needed for a roundtrip journey?

This is less ridiculous, but still requires over a million kg of fuel per kg of payload.

4. If you aren't as anxious to get to Vega quickly, you could travel at a cruising speed of 0.8 c. Calculate the amount of fuel, as a multiple of payload mass, needed for this roundtrip.

5. How long would such a roundtrip journey take at the relatively slow speed of 0.8 c.

D. One last concern

1. If relativistic travel were ever possible, we would be wise to avoid bumping into any interstellar dust. Calculate the kinetic energy of a speck of dust (10-6 g) traveling at 0.8 c. This energy is comparable to how many 1000 kg cars traveling at 65 mph?

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CHAPTER OVERVIEW

3: Spacetime and General Relativity

Contributors and Attributions

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3.1: Minkowski Metric

Two observers both measure the time separation and spatial separation of two explosions that occur in interstellar space. One observer finds the explosions to be separated by $22\ s$ and $5.5\times10^{9}\ m$.

a. *Is it possible that the other observer detects the two explosions as simultaneous? If so, how far apart are the events in this second frame?*
b. *Is it possible that the other observer detects the two explosions as occurring at the same point? If so, how far apart in time are the two explosions in this second frame?*

In classical three-dimensional space, the distance between two events can be determined by Pythagoras' Theorem,

$$(dr)^2=(dx)^2+(dy)^2+(dz)^3 \tag{3.1.1}$$

This distance is the same for any two observers, even if their individual measurements of $x-$, $y-$, and z-separation are different. This relationship forms a method of calculating distance in three-dimensional space and is referred to as the metric of three-dimensional *flat space*.

In Special Relativity, Pythagoras' Theorem is not a valid way to calculate the distance between two events. Hermann Minkowski discovered that if the temporal (dt) and spatial $(dx,\ dy,\ dz)$ separation between two events are combined in the following way,

$$(ds)^2=(c\,dt)^2-\left((dx)^2+(dy)^2+(dz)^3\right) \tag{3.1.2}$$

the resulting quantity, the spacetime interval, is the same for all observers. This result is the metric of the four-dimensional flat spacetime that obeys Special Relativity. This metric is referred to as the *Minkowski metric*.

Since this combination of spatial and temporal separations is the same for all observers, we can use it to answer the above question. Label the two observers #1 and #2, and, if the events are simultaneous for observer #2, dt2=0.

Since the distance between events in frame #2 cannot be the square-root of a negative number, it is impossible for any other observer to see these two events as simultaneous.
For part b, if the two events are to be located at the same point,

Therefore, it is possible that another observer can see the two explosions as occurring at exactly the same point in space, separated by 12.2 s of time.

Solving Relativity Problems Geometrically

Particle accelerators routinely accelerate particles to close to the speed of light. Imagine an electron traveling at 0.9995c in a linear accelerator of length 3.2 km. How long does it take the electron to travel the length of the accelerator, as measured by the electron?

First, find the time in the laboratory frame using basic kinematics:

$$t=\frac{\Delta x}{v}=\frac{3200\,m}{0.9995c}=1.07\times10^{-5}\,s \tag{3.1.3}$$

This problem could now be easily solved using the time dilation relationship from special relativity. However, it can also be solved by a purely geometrical approach using the Minkowski metric.

With observer #1 the laboratory and #2 the electron, and the two events the electron beginning its journey and ending its journey (which both occur at the same point in the electron's frame),

The Minkowski metric automatically incorporates all of the relationships we discussed while studying special relativity. Those relationships are properties of spacetime, not really relationships between objects occupying spacetime, and are thus built into the basic metric of spacetime. This view of examining the metric of spacetime to determine what happens to objects in spacetime forms the conceptual framework of general relativity.

Minkowski Metric in Polar Coordinates

We are free to express the Minkowski metric in whatever coordinate system is most useful for the problem under investigation. For example, the metric expressed in polar coordinates is:

$$(ds)^2 = (c\,dt)^2 - (dr)^2 - r^2(d\phi)^2 \tag{3.1.4}$$

Notice (in Figure $3.1.1$) that each small “step” in the radial direction, dr, is exactly the same length. The metric reflects this fact because there is no multiplicative factor in front of the dr term.

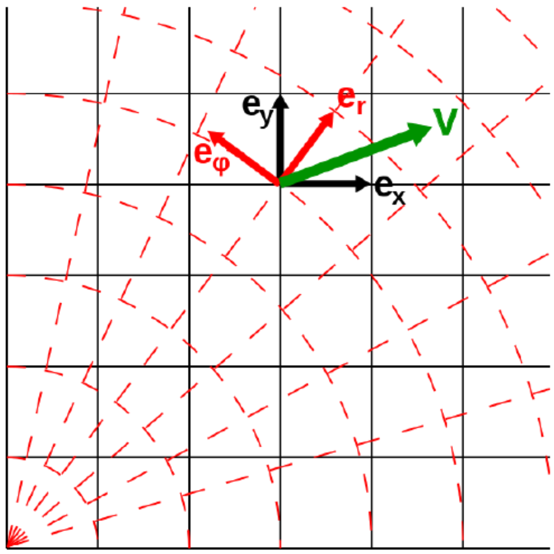

***Figure $3.1.1$**: Transformation to polar coordinates, showing unit basis vectors. (CC BY-SA; Maksim).*

However, the length of a “step” in the tangential direction, $d\phi$, depends on how far you are from the origin. The farther you are from the origin, the longer a single step in $d\phi$ is. This is reflected by the multiplicative factor of ‘r’ in the tangential part of the metric. (As r gets big, a single step in $d\phi$ gets big.)

What would it mean if there were similar multiplicative factors in front of the temporal and radial parts of the metric? A factor in front of the dt part of the metric would mean that steps in dt (i.e., clicks of a clock) would be of different duration in different places. A factor in front of the radial part would mean that radial steps were of different lengths in different places. It would be like taking the coordinate grid shown above and stretching it by different amounts in different places, resulting in a coordinate system (and underlying space that the coordinate system is trying to represent) that is no longer flat, but rather warped or curved (Figure $3.1.2$).

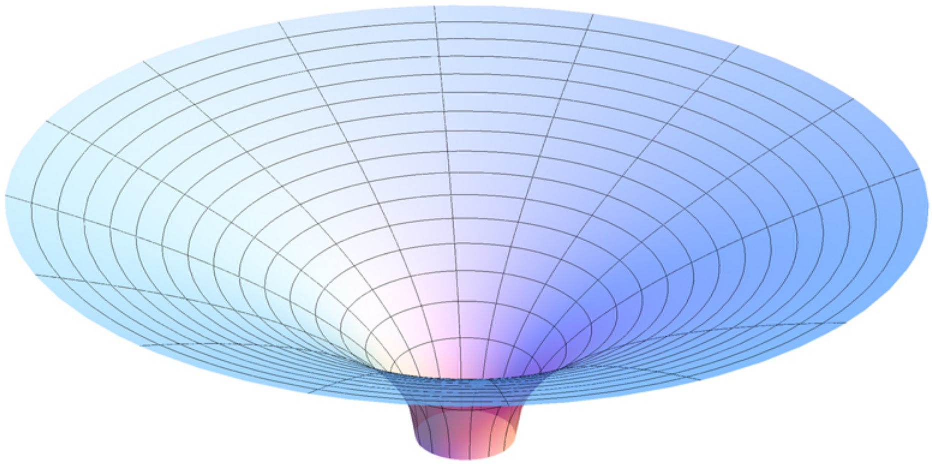

***Figure 3.1.2**: (CC BY-SA; Kes47)*

For example, if the radial part of the metric was multiplied by a factor that got large as r got small, this would result in a coordinate system that looked something like the one in Figure $3.1.2$. Notice that the radial steps get larger and larger as the radial distance gets smaller and smaller. Also notice that there is really no way to adequately draw this coordinate system on a flat surface; the coordinate system is intrinsically curved.

3.2: Schwarzchild Metric

In General Relativity, the flatspace Minkowski metric cannot be used to describe spacetime. In fact, the metric depends (in a very complicated way) on the exact distribution of mass and energy in its vicinity. For a perfectly spherical distribution of mass and energy, the metric is

$$(ds)^2 = \left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 - \frac{(dr)^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2(d\phi)^2 \tag{3.2.1}$$

This metric is referred to as the Schwarzchild metric, and describes the shape of space near a spherical mass such as (approximately) the earth or the sun, as well as the space surrounding a black hole. There are a number of subtle points you need to understand to use this metric.

1. This is the metric for a slice of spacetime that contains the mass center. Since the mass is spherical, all slices through the mass center are identical. The metric is expressed in polar coordinates, (r, ϕ), with the mass center at the origin.
2. Notice that the tangential component of the metric is unchanged from the Minkowski metric, meaning there is no deformation in that direction. However, both the temporal and radial portions are deformed by multiplicative constants, so radial lengths and time intervals are different in different locations of space.
3. Notice that as M goes to zero, or r gets very large, the metric approaches the Minkowski (flatspace) metric.
4. t is the *flatspace time*, the time measured on clocks very far from the central mass, where spacetime is assumed to be flat.
5. r is the *reduced circumference*. There are several ways to measure your distance from an object, such as physically traveling to the object or bouncing a signal off the object. If you tried to measure your distance from a black hole in either of these manners, you would have a very tough time, because either you (or your signal) would never return. Therefore we need a different method of determining radial distance. To do this, we will imagine wrapping a tape measure around the black hole, measuring its circumference, and then dividing the circumference by 2π. The resulting number is termed the reduced circumference, and, in flat space, would actually equal the value of our radial distance. (It won't equal the "real" radial distance from the black hole because the "real" radial distance is undefined (and undefinable!).)
6. Notice that the metric diverges (becomes infinite) at $r_{horizon} = 2GM/c^2$. Thus a single radial step at this location is infinitely long (and it appears that a single clock tick has no duration)! This "radius" (actually reduced circumference but we'll be sloppy and call it radius from now on) is termed the *Schwarzchild radius* and forms the *event horizon* of the black hole. At, or within, this radius, events are "beyond the horizon", meaning they are unseen and unseeable from radii greater than the Schwarzchild radius. Basically, once you pass over the horizon, you are no longer in contact with the rest of the universe. Ever.

Using the Schwarzchild Metric: Time

How close to a black hole of 5 solar masses can you approach before your spaceship's clock differs from time measured in flat spacetime by no more than 1%?

Regardless of where you are in space, if you make your measurements over a small enough region of spacetime that region of spacetime will appear locally flat, just like a straight tangent line can be drawn at any point on a smooth curve. Therefore, the ship's measurements are made using a standard Minkowski metric while the faraway observer must use the Schwarzchild metric since the spaceship's clock is far from her location.

$$(ds)^2_{ship} = (ds)^2_{far\ away} \tag{3.2.2}$$

$$\underbrace{(c\,dt)^2 - \left((dx)^2 + (dy)^2 + (dz)^3\right)}_{\text{Minkowski metric}} = \underbrace{\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 - \frac{(dr)^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2(d\phi)^2}_{\text{Schwarzchild metric}} \tag{3.2.3}$$

We'll assume your spaceship is at rest, in both frames of reference, so $dx = dy = dz = dr = 0$ and $d\phi = 0$.

$$(c\,dt_{ship})^2 = \left(1 - \frac{2GM}{c^2 r}\right)(c\,dt_{far\ away})^2 \tag{3.2.4}$$

$$dt_{ship} = \sqrt{1 - \frac{2GM}{c^2 r}}\, dt_{far\ away} \tag{3.2.5}$$

Equation 3.2.5 plays a similar role in general relativity that the time dilation relationship plays in special relativity. It relates the time interval measured by a "special" observer (one at rest in curved space) to another observer's time measurements. Mathematically, it even has a similar structure, with the term "$2GM/r$" playing the role of "v^2" in the time dilation formula.

Continuing with the question:

$$\frac{dt_{ship}}{dt_{far\ away}} = \sqrt{1 - \frac{2GM}{c^2 r}} \tag{3.2.6}$$

$$(0.99)^2 = 1 - \frac{(2)(6.67 \times 10^{-11})(10 \times 10^{30})}{(3.0 \times 10^8)^2 r} \tag{3.2.7}$$

$$0.9801 = 1 - \frac{14,800}{r} \tag{3.2.8}$$

$$r = 744,000\, m = 744\, km \tag{3.2.9}$$

Since the event horizon is at

$$r_{horizon} = \frac{2GM}{c^2} = 15\, km \tag{3.2.10}$$

you are about 50 event horizons away from the black hole.

Using the Schwarzchild Metric: Length

Two astronauts are creating a (metric) football field near a 10 solar mass black hole. The reduced circumference between the two astronauts is 100 m, and the astronauts lie along the same radial line. What is the radial separation between the astronauts as measured by the inner astronaut, if the inner astronaut is at twice the event horizon?

Again, we'll assume the spacetime immediately surrounding the inner astronaut is locally flat, allowing the astronaut to use the Minkowski metric. Since the separation between the astronauts is expressed in terms of reduced circumference, this can be incorporated into the Schwarzchild metric. Thus,

$$(ds)^2_{inner\ astronaut} = (ds)^2_{outer\ astronaut} \tag{3.2.11}$$

$$\underbrace{(c\,dt)^2 - \left((dx)^2 + (dy)^2 + (dz)^3\right)}_{\text{Minkowski metric}} = \underbrace{\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 - \frac{(dr)^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2 (d\phi)^2}_{\text{Schwarzchild metric}} \tag{3.2.12}$$

To measure the distance between two points, the location of the two points must be determined at the same time, so $dt = 0$ in both reference systems. Additionally, since the points lie along the same radial line, $d\phi = 0$. Calling this line the x-axis allows us to set $dy = dz = 0$. Thus,

$$-(dx)^2 = -\frac{(dr)^2}{\left(1 - \frac{2GM}{c^2 r}\right)} \tag{3.2.13}$$

$$dx = \frac{dr}{\sqrt{1 - \frac{2GM}{c^2 r}}} \tag{3.2.14}$$

Equation 3.2.14 plays a similar role in general relativity that the length contraction relationship plays in special relativity. It relates the spatial interval measured by a "special" observer (one at rest in curved space) to another observer's spatial measurements. Mathematically, it even has a similar structure, with the term "$2GM/r$" playing the role of "v^2" in the length contraction formula.

Substituting $r = 2r_{horizon} = 4GM/c^2$ and $dr = 100\, m$ into Equation 3.2.14 yields

$$dx_{astronout} = \frac{100\,m}{\sqrt{1-\frac{1}{2}}} \tag{3.2.15}$$

$$= 141.42\,m \tag{3.2.16}$$

Thus, even though the two astronauts only differ by 100 m in reduced circumference, the distance between the astronauts, measured by the inner astronaut, is 141.42 m. Thus, spacetime is stretched by a factor of 41% compared to flat spacetime. This provides a measure of how “warped” spacetime is at this location.

Are you upset with my sloppy use of calculus in the previous example? You should be. The metrics relate differential changes in time and space (dt and dr) and I just plugged in $100\,m$ for dr. Is 100 m infinitesimally small? It depends …

More carefully, I should integrate the expression for $dx_{astronaut}$ between the two limits, from $2r_{horizon}$ to $2r_{horizon} + 100\,m$.

$$x = \int_{2r_{horizon}}^{2r_{horizon}+100\ m} \frac{dr}{\sqrt{1-\frac{2GM}{c^2 r}}} \tag{3.2.17}$$

This integral is ugly for two reasons: the variable is in the denominator of a fraction that’s in the denominator of the expression, and the integral has a bunch of constants. It’s easy to get rid of the constants by using the definition of the event horizon,

$$x = \int_{2r_h}^{2r_h+100\ m} \frac{dr}{\sqrt{1-\frac{r_h}{r}}} \tag{3.2.18}$$

where

$$r_h = \frac{2GM}{c^2} \tag{3.2.19}$$

To solve the more difficult problem, multiply the numerator and denominator by a skillfully chosen factor:

$$x_{astronaut} = \int_{2r_h}^{2r_k+100m} \frac{\left(\frac{r}{r_h}\right)^{1/2} dr}{\left(\frac{r}{r_h}\right)^{1/2} (1-\frac{r_h}{r})^{1/2}} \tag{3.2.20}$$

$$= \int_{2r_h}^{2r_h+100m} \frac{\sqrt{\frac{r}{r_h}}dr}{\sqrt{\frac{r}{r_h}-1}} \tag{3.2.21}$$

Next, notice that the combination of terms that appears in the integral is dimensionless, meaning it has no units. It is always a very good idea to try to simplify complicated integrals in terms of dimensionless factors.

Perform a u-substitution where u is equal to this dimensionless factor and simplify:

with

$$u = \frac{R}{r_h}$$

$$du = \frac{1}{r_h}dr$$

with the limits of integration

$$\text{lower limit}: u = \frac{(2r_h)}{r_h} = 2$$

$$\text{upper limit}: u = \frac{(2r_h + 100m)}{r_h} = 2 + \frac{100c^2}{2GM} = 2.03385$$

So Equation 3.2.21 then becomes

$$x_{astronaut} = r_h \int_2^{2.03385} \sqrt{\frac{u}{u-1}} du \tag{3.2.22}$$

$$= \frac{2GM}{c^2} \left[\sqrt{u(u-1)} + \ln(\sqrt{u} + \sqrt{u-1}) \right]_2^{2.03385} \tag{3.2.23}$$

Thus, the actual distance between the astronauts, measured by the inner astronaut, is 140.83 m. For this problem, 100 m is “small enough” to be considered infinitesimally small, since the correct answer differs from the approximate answer by less than 1%. The correct answer is less than the approximate answer because the correct answer takes into account that space is less stretched as you move out toward the second astronaut, while the approximate answer approximates the stretching of space as being constant between the astronauts.

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3.4: Global Positioning System (Project)

There is no better illustration of the unpredictable payback of fundamental science than the story of Albert Einstein and the Global Positioning System (GPS). The next time your plane approaches an airport in bad weather and you just happen to be wondering "what good is basic science," think about Einstein and the GPS tracker in the cockpit, guiding you to a safe landing. - Clifford Will

Do you think that general relativity concerns only events far from common experience? Think again! Your life may be saved by a hand held receiver that "listens" to overhead satellites, the system telling you where you are at any place on Earth. In this project you will show that this system would be useless without corrections provided by general relativity.

The Global Positioning System (GPS) includes 24 satellites, in circular orbits around Earth with orbital period of 12 hours, distributed in six orbital planes equally spaced in angle. Each satellite carries an operating atomic clock (along with several backup clocks) and emits timed signals that include a code telling its location. By analyzing signals from at least four of these satellites, a receiver on the surface of Earth with a built in microprocessor can display the location of the receiver (latitude, longitude, and altitude). GPS satellites are gradually revolutionizing driving, flying, hiking, exploring, rescuing, and map making.

The timing accuracy required by the GPS system is so great that general relativistic effects are central to its performance. First, clocks run at different rates when they are at different distances from a center of gravitational attraction. Second, both satellite motion and Earth rotation must be taken into account; neither the moving satellite nor Earth's surface corresponds to a stationary frame of reference. In this project you will investigate these effects.

I. Getting Started

Earth rotates and is not perfectly spherical, so, strictly speaking, the Schwarzschild metric does not describe spacetime above Earth's surface. But Earth rotates slowly and the Schwarzschild metric is a good approximation for purposes of analyzing the Global Positioning System.

We will apply the Schwarzschild metric to both the orbiting satellite clock and a clock fixed at Earth's equator and rotating as Earth turns. Both the Earth clock and the satellite clock travel at constant radius around Earth's center, so $dr = 0$ for each clock. Also, each clock measures the proper time at its location. This yields

$$(cdt0)^2 = (1 - \frac{2GM}{c^2 r})(cdt)^2 - r^2 (d\phi)^2 \tag{3.4.1}$$

for each clock.

Divide the Schwarzschild metric through by the square of the flatspace time $(cdt)^2$ to obtain, for either clock,

$$\left(\frac{dt_0}{dt}\right)^2 = (1 - \frac{2GM}{c^2 r}) - r^2 \left(\frac{d\phi}{cdt}\right)^2 \tag{3.4.2}$$

Note that $d\phi/dt$ is the angular velocity of either clock, so r $d\phi/dt$ is the tangential velocity along the circular path of each clock as measured using flatspace time. Therefore,

$$\left(\frac{dt_0}{td}\right)^2 = (1 - \frac{2GM}{c^2 r}) - \frac{v^2}{c^2} \tag{3.4.3}$$

This equation relates time measured on Earth to flatspace time, or time measured on the satellite to flatspace time. Since we can't directly measure flatspace time, we can eliminate this term by applying the relationship to both the satellite clock and the Earth clock, and dividing the two expressions.

$$\left(\frac{dt_{satellite}}{dt_{earth}}\right)^2 = \frac{1 - \frac{2GM}{c^2 r_{satellite}} - \frac{vsatellite^2}{c^2}}{1 - \frac{2GM}{c^2 r_{earth}} - \frac{v_{earth}^2}{c^2}} \tag{3.4.4}$$

$$\frac{dt_{satellite}}{dt_{earth}} = \frac{\left(1 - \frac{2GM}{c^2 r_{satellite}} - \frac{vsatellite^2}{c^2}\right)^{1/2}}{\left(1 - \frac{2GM}{c^2 r_{earth}} - \frac{v_{earth}^2}{c^2}\right)^{1/2}} \tag{3.4.5}$$

II. Stationary Clocks

1. We will start our analysis by ignoring the motions of both clocks. Rewrite the above expression assuming the clocks are stationary.

2. The radius of the Earth is $6.37x10^6$ m and the radius of a 12 hour circular orbit is about $26.6x10^6$ m (You will calculate this value later in the activity). Calculate the value of $2GM/c^2r$ for both the Earth clock and the satellite clock. Are these large or small values?

3. You should find that the value of $2GM/c^2r$ is extremely small. Therefore, you can use the binomial expansion to simplify the expression for the ratio of satellite time to earth time. (First move the term in the denominator into the numerator using a negative exponent, then apply the binomial expansion to both terms, and then multiply out the result. Do this symbolically. Note that the product of two extremely small numbers is really small and can be ignored.)

4. Evaluate your expression, leaving your result in the form "1 + δ" where δ is the small factor by which the two clocks are not synchronized.

The number represented by δ is an estimate of the fractional difference in rates between stationary clocks at the position of the satellite and at Earth's surface. Is this difference negligible or important to the operation of the GPS system? In 1 nanosecond, light signals (and radio waves) propagate approximately 30 centimeters, or about one foot. So for each nanosecond of discrepancy per day, the GPS will give incorrect position readings of about one foot.

5. To three significant figures, find the discrepancy between earth time and satellite time after one day.

You should find that the satellite clock will "run fast" by more than 40,000 nanoseconds per day compared with the clock on Earth's surface due to position effects alone. After one day, the GPS would be off by over 40,000 feet (7.5 miles)! Clearly general relativity is needed for correct operation of the Global Positioning Satellite System!

III. Moving Clocks

1. In addition to effects of position, we must include effects due to the motion of the satellite and Earth observer. Will the effects of velocity make the time discrepancy larger or smaller than that calculated above? Explain.

We need to calculate the speed of both the satellite clock and the earth clock to complete our analysis.

2. Determine the speed of a clock at rest on the surface of the earth, relative to a hypothetical observer at rest relative to the center of the earth.

Finding the speed of the satellite is slightly more complicated. The satellite is in a circular orbit and must obey Newton's Law of Gravitation (there is a slight error made in using Newton's theory, but that error is much smaller than the discrepancies we are examining).

$$F = ma \tag{3.4.6}$$

$$\frac{GM_{earth} m_{satellite}}{r_{satellite}^2} = m_{satellite} \frac{v_{satellite}^2}{r_{satellite}} \tag{3.4.7}$$

$$\frac{GM_{earth}}{r_{satellite}} = v_{satellite}^2 \tag{3.4.8}$$

and the satellite's orbital period, T, must obey

$$T_{satellite} = \frac{2\pi r_{satellite}}{v_{satellite}} \tag{3.4.9}$$

3. Using these two equations, find an expression for v that does not include the radius of the satellite's orbit.

4. Evaluate your expression for the velocity of the satellite in a 12-hr orbit.

5. Calculate the radius of this orbit to justify the value used earlier.

Returning to our original relationship between earth and satellite time,

$$\frac{dt_{satellite}}{dt_{earth}} = \frac{\left(1 - \frac{2GM}{c^2 r_{satellite}} - \frac{vsatellite^2}{c^2}\right)^{1/2}}{\left(1 - \frac{2GM}{c^2 r_{earth}} - \frac{v_{earth}^2}{c^2}\right)^{1/2}} \tag{3.4.10}$$

we are now prepared to calculate this discrepancy including the relative velocities of the two clocks.

6. Use the binomial expansion to simplify the expression for the ratio of satellite time to earth time. (Again, first move the term in the denominator into the numerator using a negative exponent, then apply the binomial expansion to both terms, and then multiply out the result. Do this symbolically. Note that the product of two extremely small numbers is really small and can be ignored.)

7. Evaluate your expression, leaving your result in the form "$1+\delta$" where δ is the small factor by which the two clocks are not synchronized.

8. To three significant figures, find the discrepancy between earth time and satellite time after one day.

You should find that the satellite clock will "run fast" by slightly less than 40,000 nanoseconds per day compared with the clock on Earth's surface. Even after including the time dilation of the satellite clock due to its greater velocity, general relativity is still needed for correct operation of the Global Positioning Satellite System.

3.5: Falling into a Black Hole - Easy Version (Project)

Consider falling feet first into a black hole of mass M. Although you may think that this would be a rather unpleasant experience, the truth may surprise you. Sure, you will be increasingly stretched until you are torn apart near the end of your journey, but for most of your journey you feel quite pleasant. You are freely floating through space, enjoying the ride. It's only when you are so close to the hole that your feet feel a greater "force" than your head that you begin to be stretched into spaghetti and the pain increases. We can approximate the duration of this painful period (t_{ouch}) to see if it's over quickly or is a slow, agonizing demise. We will use Newtonian physics to calculate t_{ouch}. (You get a very similar answer when doing this calculation correctly with General Relativity.)

In Newton's Theory of Gravitation,

$$\frac{GM_{blackhole}m_{you}}{r^2} = m_{you}g \tag{3.5.1}$$

$$\frac{GM_{blackhole}}{r^2} = g \tag{3.5.2}$$

We will assume that pain begins when the *difference* between the acceleration of your head and your feet is approximately the value of g at the earth's surface (i.e., your head is "pulled" upward at $4.9\, m/s^2$ and your feet are "pulled" downward at $4.9\, m/s^2$). Thus, pain begins when $\Delta g = -9.8\, m/s^2$ and $\Delta r \approx 2m$, the distance between your head and feet.

1. Take the derivative, dg/dr, of the expression for gravitational acceleration in Newtonian physics.

2. Substitute the values of Δg and Δr in for dg and dr and solve for r_{pain}, the radius at which you begin to feel "uncomfortable", as a function of M, the mass of the black hole.

Black holes come in various sizes. A typical stellar black hole (a star that has collapsed to form a black hole) may have a mass of 5 solar masses, while a galactic black hole (formed from millions of stars collapsing together) may have a mass a million solar masses.

3. Find the radius of pain for a 5 solar mass black hole.

4. Is r_{pain} inside or outside the event horizon of the hole? (If outside the horizon, others can witness you screaming in agony.)

5. Find the radius of pain for a 10^6 solar mass black hole.

6. Is r_{pain} inside or outside the event horizon of the hole? (If inside the horizon, when you begin to scream in agony no one can ever hear you!)

How long you feel uncomfortable (and whether you even have time to scream) depends on how long it takes you to reach $r = 0$ (where you will surely be dead). To find this time, we need your speed when you reach r_{pain}.

7. Assume you fell into the black hole from rest from a great distance. Using energy conservation in Newtonian physics, find an expression for your speed when you reach r_{pain}.

8. Calculate your speed at r_{pain} for the 5 solar mass hole. If this speed is slow compared to c, the Newtonian approximation is valid. Is the approximation valid?

9. Calculate your speed at r_{pain} for the 10^6 solar mass hole. Is the Newtonian approximation valid for a galactic black hole?

You should find that the approximation is not valid for a very large black hole. Therefore, we will not be able to calculate touch for this hole. However, trust me, it won't hurt for long.

10. Even though you will continue to accelerate as you fall further into the 5 solar mass hole, find the time it would take to reach the center traveling at the speed calculated above. This is the maximum amount of time for which you would suffer. Can you take the pain for this long?

3.6: Falling into a Black Hole - Hard Version (Project)

The text describes how the Schwarzchild metric can be used to directly compare time and length intervals measured by observers at rest at different locations in spacetime. However, measurements made by, and of, moving observers are slightly more difficult to compare. In order to determine how the observations of a person falling into a black hole compare to observers at rest, additional mathematical machinery is necessary.

I. Energy

The concept of energy conservation is central to all of physics. For a system acted on by no outside forces (and remember, gravity is not a force), energy must remain constant. Therefore, an observer in freefall into a black hole has constant energy. Once we learn how to quantify energy in general relativity, we will be ready to “jump in”.

A. A Plausible Relationship for Energy

Far from the black hole, spacetime is flat and the formula for energy must agree with the special relativity formula for energy,

$$E=\gamma mc^2 \tag{3.6.1}$$

The question is, how does this relationship vary as you move closer to the black hole?

An examination of the Schwarzchild metric,

$$(ds)^2=\left(1-\frac{2GM}{c^2r}\right)(c\,dt)^2-\frac{(dr)^2}{\left(1-\frac{2GM}{c^2r}\right)}-r^2(d\phi)^2 \tag{3.6.2}$$

should indicate the presence of a very common mathematical factor that describes the change in space and time as you approach the hole. This factor, designated by the Greek letter capital gamma (Γ), approaches one at large distance so wouldn’t alter the energy in flatspace:

$$\Gamma=\left(1-\frac{2GM}{c^2r}\right) \tag{3.6.3}$$

Hopefully, it’s plausible that the correct mathematical relationship for energy could be:

$$E=\Gamma\gamma mc^2 \tag{3.6.4}$$

In fact, this is the correct relationship, with one subtle twist. From the time dilation formula in special relativity,

$$\Delta t=\gamma(\Delta t_0) \tag{3.6.5}$$

γ can be defined as

$$\gamma=\frac{\Delta t}{\Delta t_o}=\frac{dt}{dt_o} \tag{3.6.6}$$

or, in words, γ is the rate at which time varies with proper time. In flat spacetime, this can also be expressed as

$$\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.6.7}$$

The subtle twist is that the first definition of γ (Equation 3.6.6) is always true, while the second definition of γ (Equation 3.6.7) is *only* true in flat spacetime. Thus, the proper expression for energy is

$$E=\Gamma\frac{dt}{dt_o}mc^2 \tag{3.6.8}$$

II. Falling into a Black Hole (from Rest at Infinity)

Imagine being at rest a large distance from a black hole. On a dare, you step out of your spaceship and take a ride on curved spacetime toward the black hole.

A. Determining Your Initial Energy

Since you are far from the black hole, you are in flat spacetime ($\Gamma = 1$ since $r = \infty$). Since you are at rest in flat spacetime

$$\frac{dt}{dt_o} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 1 \tag{3.6.9}$$

Therefore, your initial energy is solely due to your rest mass (Equation 3.6.8)

$$E_{initial} = \Gamma \frac{dt}{dt_o} mc^2 \tag{3.6.10}$$

$$= (1)(1)mc^2 \tag{3.6.11}$$

$$= mc^2 \tag{3.6.12}$$

B. Using Energy Conservation

By energy conservation,

$$E_{at\ any\ time} = E_{initial} \tag{3.6.13}$$

$$\Gamma \frac{dt}{dt_o} mc^2 = mc^2 \tag{3.6.14}$$

$$\Gamma dt = dt_o \tag{3.6.15}$$

$$\Gamma^2 dt_2 = dt_o^2 \tag{3.6.16}$$

dt_o, the proper time, is the time measured by you, the falling observer. This can be determined by relating measurements made by you to measurements made by the friends you left behind.

$$(ds)^2_{you} = (ds)^2_{far\ away\ freiends} \tag{3.6.17}$$

more

In your frame, you are at rest, so $dt = dt_o$ and $dx = dy = dz = 0$. In your friends' frame, you fall radially inward so $d\varphi = 0$.

Plugging this into the energy conservation equation (Equation 3.6.16) and collecting like terms yields:

Ok, so what's the point? The point is that $\frac{dr}{dt}$ is your speed as measured by your friends! dr is your change in reduced circumference (which to your friends in flat space is equal to distance traveled) and dt is the elapsed time as measured on faraway clocks (your friends' clocks).

Since you are falling inward (to smaller r) we should take the negative root of the square root, and we can also simplify the expression $1-\Gamma$.

You should find this relationship *stunning*. Why?

At $r = \infty$, when you step out of the ship, your speed (as measured by your friends) is zero. Not stunning.

As you slide into the hole, r decreases and your speed increases. Again, not stunning.

But then something truly weird happens. Examine the relationship as you approach the event horizon of the black hole, $r = 2GM/c^2$. Your speed, as measured by your friends, decreases to zero! Your friends see you slow down and stop exactly at the event horizon. You never pass the event horizon in your friends' frame of reference. In fact, observers outside a black hole never see anything ever enter a black hole! All the stuff that falls into the hole just "accumulates" at the event horizon. Stunning!

C. Another Frame of Reference

Just because your friends never see you pass the event horizon does not necessarily mean you never pass the event horizon. Remember, this is called relativity for a reason.

Consider a pair of stationary observers located at reduced circumference r. What do they measure as your speed? One observer measures the time interval between your feet passing by and then your head passing by (assuming you jump in feet first). These two measurements occur at the same location, so $dr = 0$, and thus

$$dt_{\text{in the hole}} = \sqrt{1 - \frac{2GM}{c^2 r}}\, dt \tag{3.6.18}$$

$$= \Gamma^{1/2} dt \tag{3.6.19}$$

The other observer determines the location of your feet and head at the same instant of time, so $dt = 0$, and thus

Combining these observations results in

This is also stunning. Why?

Stationary observers at $r = \infty$ measure your speed to be zero. Not stunning.

Stationary observers "deeper" in the hole (smaller r) measure your speed as faster and faster. Again, not stunning.

But then (again) something truly weird happens. Examine the relationship for stationary observers very close to the event horizon of the black hole, $r = 2GM/c^2$. Your speed, as measured by these observers, increases to c! Observers standing arbitrarily close to the event horizon see you zoom by at arbitrarily close to the speed of light. (No one sees you move exactly at the speed of light, because no stationary observer can exist exactly at the event horizon.) These observers see you zoom past the event horizon (to your ultimate demise) while your friends see you step on the brakes and come to a complete stop. Stunning!

D. Another Frame of Reference? Yours.

You should probably be concerned with yet another frame of reference, the frame of reference of the jumper. How fast does the jumper see the "stationary" observers discussed in the last section zooming by?

That's easy. If the "stationary" observers see the jumper moving at

$$v_{\text{jumper from the obsever's frame}} = +\sqrt{\frac{2GM}{c^2 r}}\, c \tag{3.6.20}$$

then the jumper sees the "stationary" observers moving at

$$v_{\text{observers from the jumper's frame}} = +\sqrt{\frac{2GM}{c^2 r}}\, c \tag{3.6.21}$$

However, a more interesting question is how long, in the jumpers frame, does it take the jumper to go from the event horizon to the singularity (and certain death) at the center of the black hole? The event horizon and the singularity are separated by a known change in reduced circumference (dr from $r = 2GM/c^2$ to $r = 0$). The time measured by the jumper is a proper time, dt_o. Thus, construct the ratio

$$\frac{dr}{dt_o} = \frac{dr}{dt}\frac{dt}{dt_o} \tag{3.6.22}$$

$\frac{dr}{dt}$ is calculated above, and $\frac{dt}{dt_o}$ is determined by the energy conservation equation. Thus,

Solving for dt_o and integrating from the event horizon to the singularity yields

Consider a typical 10 solar mass black hole. This would yield a time interval between crossing the event horizon and being shredded into nothingness of $6.6 \times 10^{-5}\ s$, hardly enough time to even enjoy the weird spacetime geometry inside the event horizon (or to really experience the agony of being shredded).

On the other hand, a typical galactic-center-sized black hole (2 million solar masses) yields a time interval of 13.1 s, enough to possibly both admire the geometry and feel the agony.

III. Thrown into a Black Hole (from Infinity)

Imagine being at rest a large distance from a black hole. On a dare, you allow your friends to launch you directly toward the black hole, with initial velocity v.

A. Determining Your Initial Energy

1. You are initially far from the black hole, in flat spacetime, moving at velocity v. Write an expression for your initial energy.

B. Using Energy Conservation

2. By energy conservation, set your expression for initial energy equal to the general formula for energy.

3. Replace dt0 with the relationship for proper time measured by the falling observer.

4. Collect like terms and solve for , your speed as measured by your friends

5. Based on your result above, how fast do your friends see you moving as you approach the event horizon? Did being launched at high speed toward the black hole have any effect on your speed at the event horizon?

6. (Prepare to be stunned.) Instead of launching you at the black hole, your friends decide to shine a laser at the black hole. How fast do your friends see the light moving as it approaches the event horizon? Can your friends ever see anything pass the event horizon?

C. Another Frame of Reference

Just because your friends never see you (or a laser beam) pass the event horizon doesn't necessarily mean you (or a laser beam) never pass the event horizon.

Recall that your speed, as measured by a stationary observer "in the hole", is

7. Determine your speed as measured by stationary observers in the hole.

8. What does a stationary observer very close to the event horizon measure for your speed? Did being launched at high speed toward the black hole have any effect on your speed at the event horizon as measured by a stationary observer?

9. (Prepare to not be stunned.) What do stationary observers (anywhere in the hole) measure for the speed of the laser launched by your friends?

IV. Dropped into a Black Hole (from Nearby)

Imagine being at rest at reduced circumference r0 near a black hole. On a dare, you step out of your spaceship and take a ride on curved spacetime toward the black hole.

A. Determining Your Initial Energy

10. You are initially at rest at reduced circumference r0. Write an expression for your initial energy. (Note that the relationship between faraway time and time measured by someone at rest in a black hole is .) Hint: Let to keep the expression manageable.

B. Using Energy Conservation

11. By energy conservation, set your expression for initial energy equal to the general formula for energy.

12. Replace dt0 with the relationship for proper time measured by the falling observer.

13. Collect like terms and solve for , your speed as measured by your friends

14. Based on your result above, how fast do your friends see you moving as you approach the event horizon? Does it matter where you were when you stepped out of the spaceship?

C. Another Frame of Reference

15. Determine your speed as measured by stationary observers in the hole.

16. What does a stationary observer very close to the event horizon measure for your speed? Does it matter if you stepped out of a spaceship 1 mm from the event horizon or 1 million miles from the event horizon?

D. Another Frame of Reference? Yours.

You will shortly prove that although stationary observers will always see you cross the horizon at exactly the same speed (~c) that doesn't mean you will always take the same amount of time to reach the singularity. (Think about this. You are the same "distance" from the singularity, traveling at the same speed, but it won't always take you the same amount of time to get there. Stunning!)

17. Construct the ratio

Hint: You calculated above, and is determined from the energy conservation equation.

18. Solve your expression for dt0.

19. Set-up the integral from the event horizon to the singularity. Simplify the integrand until it is in the form of .

20. Substitute the expressions for Γ0 and Γ. You should be able to reduce the integrand to a function of only r and r0. Eliminate fractions in the denominator, and using the dimensionless substitution u = r/r0, put the integral into the form of .

21. Note that is equal to . This latter integral is listed in your integral table. Assume you “dropped” into the black hole from a point just beyond the event horizon. This will maximize the time between crossing the horizon and reaching the singularity. Using this assumption, evaluate and simplify your result.

22. Consider a typical 10 solar mass stellar black hole. If you dropped into this hole from just across the horizon, how much time would you have to enjoy the ride?

23. Consider a typical 2 million solar mass galactic black hole. If you dropped into this hole from just across the horizon, how much time would you have to enjoy the ride?

CHAPTER OVERVIEW

4: The Photon

Topic hierarchy

Contributors and Attributions

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4.1: Light as a Stream of Particles

Although the first suggestion that light acts as a particle rather than a wave can be dated to Planck's explanation of blackbody radiation, the explanation of the photoelectric effect by Einstein is both simple and convincing. In the photoelectric effect, a beam of light is directed onto a metal plate. It had been noted that the energy deposited by the light on the plate is sufficient (under certain circumstances) to free electrons from the plate.

The energy of the freed electrons (measured by the voltage needed to stop the flow of electrons) and the number of freed electrons (measured as a current) could then be explored as a function of the intensity and frequency of the incident light. These early experiments revealed several surprises:

- The energy of the freed electrons was independent of light intensity. Thus, even when the energy per second striking the plate increased, the electrons did not respond by leaving the plate with more energy, nor when an extremely weak light was used were the electrons emitted with less energy. If light is a wave, a more intense wave should deposit more energy to each electron.
- Below a certain frequency (the threshold frequency) no electrons were emitted, regardless of light intensity. Thus, an extremely bright red light, for example, would free no electrons while an extremely faint blue light would. In fact, as the frequency increased, the electron energy increased proportionally. If light is a wave, all frequencies should emit electrons since at all frequencies enough light would ultimately be deposited on the electron.
- The electrons were emitted the instant (within 10-9 s) the light struck the metal. If the energy in the light was distributed over some spatial volume (as it is in a wave) a small time lag should occur before the electrons are emitted, since a small amount of time is necessary for the electron to "collect" enough energy to leave the metal.

Einstein realized that all of these "surprises" were not surprising at all if you considered light to be a stream of particles, termed *photons*. In Einstein's model of light, light is a stream of photons where the energy of each individual photon is directly proportional to its frequency

$$E_{photon} = hf \tag{4.1.1}$$

where f is the frequency and h is Planck's constant, $6.626 \times 10^{-34}\ J\,s$, introduced several years earlier. This model resolves all of the issues raised by the photoelectric effect experiments:

- A more intense light source contains more photons, but each individual photon has exactly the same energy. Since the electrons are freed by absorbing individual photons, every electron is freed with exactly the same energy. Increasing intensity increases the *number* of photons and hence the *number* of free electrons, but not their individual energy.
- Below a certain frequency the individual photons in the light beam do not have enough energy to overcome the bonds holding the electrons in the metal. Regardless of the number of photons, if each individual photon is too "weak" to free an electron, no electrons will ever be freed.
- The energy in the light beam is not spread over a finite spatial volume; it is concentrated into individual, infinitesimal bundles (the photons). as soon as the light strikes the metal, photons strike electrons, and electrons are freed.

For his explanation of the photoelectric effect in terms of photons, Einstein was awarded the Nobel Prize in 1921.

The Photoelectric Effect

A metal is illuminated with 400 nm light and the stopping potential is measured to be 0.87 V.

a. *What is the workfunction for the metal?*
b. *For what wavelength light is the stopping potential 1.0 V?*

Applying energy conservation to the photoelectric effect results in a relationship for the kinetic energy of the ejected electrons. The incoming energy is in the form of photon energy. Some of this energy goes toward freeing the electrons from the metal surface (this "binding" energy of the electrons to the surface is called the workfunction, ϕ) while the remainder (if any) appears as kinetic energy of the ejected electrons. Therefore,

$$E_{photon} = \psi + KE_{electrons} \tag{4.1.2}$$

or

$$KE_{electrons} = E_{photon} - \psi \tag{4.1.3}$$

In Einstein’s model of the photon the energy of a photon is given by

where f is the frequency and the wavelength of the light, and h is Planck’s constant, 6.626 x 10-34 Js. A more useful factor, with more “friendly” units, is
hc = 1240 eV nm.
Combining this with the result from energy conservation yields

Additionally, the electrons can be “stopped” by the application of an appropriately biased potential difference. The electron current will stop when the maximum kinetic energy of the electrons is matched by the electrostatic energy of the potential difference. This potential difference is termed the stopping potential, and is given by

Therefore, in part a, if the stopping potential is 0.87 V, then the maximum kinetic energy of the emitted electrons must be 0.87 eV. So,

In part b, if the stopping potential is measured to be 1.0 V, then the wavelength of the incident light must be

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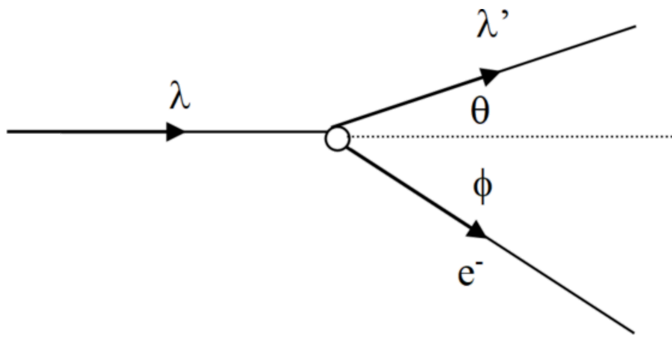
4.2: Compton Scattering

Compton scattering refers to the scattering of light off of free electrons. Experimentally, it's impossible to create a target of completely free electrons. However, if the incident photons have energy much greater than the typical binding energies of electrons to atoms, the electrons will be "knocked off" of the atoms by the photons and act as free particles. Therefore, Compton scattering typically refers to scattering of high energy photons off of atomic targets.

If light was purely a wave phenomenon, an incoming wave with a specific frequency would cause the electron to oscillate with the same frequency. The oscillating electron would then emit electromagnetic waves of this frequency. Thus, the scattered light and incoming light would have, to within a slight variation due to the Doppler effect for light, the same frequency. This is not what is seen experimentally.

Instead, let's imagine light to be a stream of photons and analyze the collision of a photon and an electron by energy and momentum conservation. Consider an incident photon of wavelength λ striking a stationary electron. The photon scatters to angle θ' (and new wavelength λ') and the electron to angle ϕ.



To analyze, apply energy conservation:

$$E+mc^2=E'+E_e \tag{4.2.1}$$

x-momentum conservation:

$$pc+0=p'c\cos\theta+p_ec\cos\phi \tag{4.2.2}$$

and y-momentum conservation:

$$0+0=p'c\sin\theta+p_ec\sin\phi \tag{4.2.3}$$

For photons, $E=pc$, so the momentum equations can be written as:

$$p_x: E=E'\cos\theta+p_e\cos\phi \tag{4.2.4}$$

$$p_x: 0=E'\cos\theta-p_e\sin\phi \tag{4.2.5}$$

Experimentally, it's easier to detect the scattered photon than the scattered electron, so we'll eliminate the electron parameters and derive an interrelationship between the various photon parameters. To eliminate ϕ, solve x-momentum and y-momentum for $\cos\phi$ and $\sin\phi$, and then square and add them together:

Solve the energy conservation equation for E_e:

$$E_e=E+mc^2-E' \tag{4.2.6}$$

$$E_e^2=(E+mc^2-E')^2 \tag{4.2.7}$$

Plug the two previous results into Equation 4.2.7 to eliminate the electron variables:

$$E_e^2=(p_ec)^2+(mc^2)^2 \tag{4.2.8}$$

$$(E+mc^2-E')^2=E^2-2EE'\cos\theta E'^2+(mc^2)^2 \tag{4.2.9}$$

Six terms cancel and the equation greatly simplifies

$$2Emc^2 - 2E'mc^2 - 2EE' = -2EE' \cos\theta \tag{4.2.10}$$

Rearranging yields

This result directly relates the incoming wavelength to the scattered wavelength and the scattering angle. All of these parameters are easily measured experimentally. For his theoretical explanation and experimental verification of high energy photon scattering, the American Arthur Compton was awarded the Nobel Prize in 1927.

Using the Compton Scattering Relationship

An 800 keV photon collides with an electron at rest. After the collision, the photon is detected with 650 keV of energy. Find the kinetic energy and angle of the scattered electron.

The fundamental relationship for Compton scattering is

$$\lambda' - \lambda = \frac{hc}{mc^2}(1 - \cos\theta) \tag{4.2.11}$$

where

- λ' is the scattered photon wavelength,
- λ is the incident photon wavelength,
- and θ is the angle of the scattered photon.

To find the kinetic energy of the scattered electron does not require using the Compton formula. If the photon initially has $800\,\text{keV}$, and after scattering has 650keV, then $150,\text{keV}$ must have been transferred to the electron. Thus, $KE_{electron} = 150\text{keV}$.

Finding the angle of the scattered electron does involve the Compton relation. First, convert the photon energies into wavelengths:

$$E_{photon} = \frac{hc}{\lambda} \tag{4.2.12}$$

$$\lambda = \frac{hc}{E} \tag{4.2.13}$$

$$= \frac{1240 \text{ eVnm}}{800 \text{ keV}} = 1.55 \times 10^{-3} \text{ nm} \tag{4.2.14}$$

$$= \frac{1240 \text{ eVnm}}{650 \text{ keV}} = 1.91 \times 10^{-3} \text{ nm} \tag{4.2.15}$$

then use the relationship

$$\lambda' - \lambda = \frac{hc}{mc^2}(1 - \cos\theta) \tag{4.2.16}$$

$$1.91 \times 10^{-3} - 1.55 \times 10^{-3} = \lambda = \frac{1240 \text{ eVnm}}{511 \text{ keV}}(1 - \cos\theta) \tag{4.2.17}$$

$$0.1484 = (1 - \cos\theta) \tag{4.2.18}$$

$$\theta = 31.6^\circ \tag{4.2.19}$$

However, this is the scattering angle of the photon, not the electron!

To find the electron's scattering angle, apply momentum conservation in the direction perpendicular to the initial photon direction.

$$0 = p_{\text{scattered photon}} c(\sin\theta) - p_{electron}\, c(\sin\phi) \tag{4.2.20}$$

$$p_{\text{scattered photon}} c(\sin\theta) = p_{electron}\, c(\sin\phi) \tag{4.2.21}$$

$$E_{\text{scattered photon}}c(\sin\theta) = \sqrt{E_{electron}^2 - (mc^2)^2}\,(\sin\theta) \tag{4.2.22}$$

$$650\sin(31.6) = \sqrt{(511+150)^2 - (511)^2}(\sin\phi) \tag{4.2.23}$$

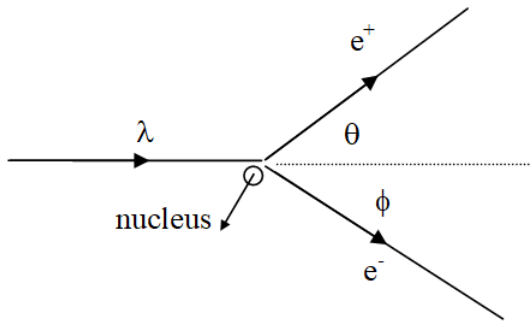
$$\sin\phi = 0.813 \tag{4.2.24}$$

$$\phi = 54.4 \tag{4.2.25}$$

4.3: Pair Production

In addition to the photoelectric effect (photon absorption) and Compton scattering (photon scattering), there is a third process by which photons can lose energy in their interaction with matter. In this process, termed *pair production*, a photon can simply vanish and in its place a matter-antimatter pair of particles can appear. This phenomenon is a wonderful illustration of the fact that *mass is not conserved*, since the mass of the electron and positron can be created from the energy of the massless photon. Of course, the photon must have sufficient energy to create the rest masses of the two new particles.

Typically, this process occurs in the vicinity of a nucleus and an electron-positron pair is formed. The effect is sketched below:



First, let’s try to imagine a simpler version of this phenomenon, with no nucleus present and the electron and positron both traveling in the same direction as the initial photon. Energy conservation would lead to:

$$E_{photon} = E_{electron} + E_{positron} \tag{4.3.1}$$

and x-momentum conservation:

$$pc_{photon} = pc_{electron} + pc_{positron} \tag{4.3.2}$$

Using

$$E^2_{total} = (pc)^2 + (mc^2)^2 \tag{4.3.3}$$

energy conservation can be written as:

$$\sqrt{pc^2_{photon} + (0)^2} = \sqrt{pc^2_{electron} + 0.511^2} + \sqrt{pc^2_{photon} + 0.511^2} \tag{4.3.4}$$

$$pc_{photon} = \sqrt{pc^2_{electron} + 0.511^2} + \sqrt{pc^2_{photon} + 0.511^2} \tag{4.3.5}$$

Setting this equation equal to the momentum conservation equation leads to:

$$\sqrt{pc^2_{electron} + 0.511^2} + \sqrt{pc^2_{positron} + 0.511^2} = pc_{electron} + pc_{electron} \tag{4.3.6}$$

Hopefully it’s clear that this equation is garbage! The left-side of the equation is larger than the right-side of the equation for any real values of momentum. This means that the “simplified” version of pair production discussed above is impossible. The only way pair production can occur is if a third body (the nucleus) is present to participate in the sharing of energy and momentum.

However, if we try to set-up and solve the conservation laws for the real pair production process we will get bogged down in a rather large amount of algebra. Instead, let’s make an approximation that since the nucleus is much, much more massive than the electron and positron it can “absorb” the proper amount of momentum to guarantee momentum conservation without “absorbing” very much of the kinetic energy. In a sense, we will ignore the nucleus during the solution of the problem, and then once we have a solution we will check to see if ignoring the nucleus was a reasonable choice.

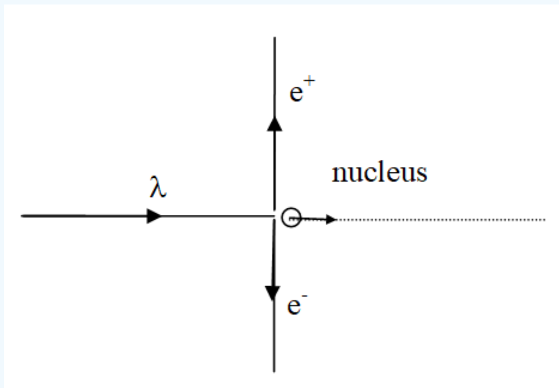
As an example, consider the problem below.

Example 4.3.1

A 3.0 MeV photon interacts with a lead nucleus and creates an electron-positron pair. The electron and positron travel perpendicular to the initial direction of travel of the photon.

a. Find the kinetic energies of the electron and positron assuming the nucleus is at rest after the collision.
b. Find the kinetic energy of the lead nucleus needed to ensure momentum conservation. Would this amount of kinetic energy greatly affect the result in part a?

Solution



Energy conservation (ignoring the energy of the nucleus) leads to:

$$E_{photon} = E_{electron} + E_{positron} \tag{4.3.7}$$

$$3 = E_{electron} + E_{positron} \tag{4.3.8}$$

and y-momentum conservation (ignoring the nucleus again):

$$0 = -pc_{electron} + pc_{positron} \tag{4.3.9}$$

$$pc_{electron} = pc_{positron} \tag{4.3.10}$$

Since the pair have equal momenta, they must have equal energy, so:

$$3 = 2E_{either} \tag{4.3.11}$$

$$E_{either} - 1.5\text{ MeV} \tag{4.3.12}$$

$$KE_{either} = (1.5 - 0.511)\text{ MeV} \tag{4.3.13}$$

$$KE_{either}\ 0.989\text{ MeV} \tag{4.3.14}$$

To ensure that x-momentum is conserved,

$$pc_{photon} = pc_{nucleus} \tag{4.3.15}$$

$$3\text{ MeV} = pc_{nucleus} \tag{4.3.16}$$

Using $E^2_{total} = (pc)^2 + (mc^2)^2$ and noting that the atomic mass of a lead atom is $207.2\text{ u} = 193007\text{ MeV}$ yields,

$$E_{nucleus} = \sqrt{3^2 + 193007^2} \tag{4.3.17}$$

$$E_{nucleus} = 193007\text{ MeV}\sqrt{1 + \frac{9}{193007^2}} \tag{4.3.18}$$

then simplify using the binomial expansion,

$$E_{nucleus} = 193007\text{ MeV}\left(1 + \frac{1}{2}\frac{9}{193007^2}\right) \tag{4.3.19}$$

$$E_{nucleus} = 193007\text{ MeV}(1 + 1.2 \times 10^{-10}) \tag{4.3.20}$$

$$E_{nucleus} = 193007\text{ MeV} + 2.33 \times 10^{-5}\text{ MeV} \tag{4.3.21}$$

$$KE_{nucleus} = 2.33 \times 10^{-5}\text{ MeV} \tag{4.3.22}$$

$$KE_{nucleus} = 23.3\text{ eV} \tag{4.3.23}$$

Thus, the nucleus can preserve momentum conservation while only “stealing” a ridiculously small portion of the total energy available. Thus, we can ignore the presence of the nucleus while dividing up the energy of the incoming photon between the electron and positron.

4.4: Photons and Matter

Consider a beam of photons incident on a chunk of matter. The photons can be absorbed by the electrons, scatter from the electrons, and create pairs of particles by interacting with the nucleus. Two questions remains, however. Which of these processes will occur for any particular photon and how far will the photon penetrate the matter before one of these processes occurs. These questions require an understanding of the *probability* of each of these processes occurring. The probability of a particular process occurring is represented by the *cross section* for that process, denoted σ.

Basically, the idea is to imagine the atom as a dart board. Each process (photoelectric effect, scattering, pair production) is represented by an area of the dart board. If the photon strikes that area, that process occurs.

Thus, the probability of a process occurring per atom is given by the ratio of the cross section for that process per atom (the area of that portion of the dart board) divided by the total area of the target:

$$\frac{\text{Probability}}{\text{atom}} = \frac{\sigma}{A} \tag{4.4.1}$$

$$\text{Probability} = \frac{\sigma}{A} N_{atoms} \tag{4.4.2}$$

where N_atoms is the total number of atoms in the target.

To convert this idea into a more useable form, imagine a beam of parallel photons incident on a thick slab of material (containing many microscopic targets).

The decrease in the number of photons in the beam due to interactions in a thin slab of material is given by the product of the probability of interaction and the number of photons present:

$$\Delta N_{photons} = -\left(\frac{\sigma}{A} N_{atoms}\right) N_{photons} \tag{4.4.3}$$

The number of atoms in the slim slab of material is given by the product of the number of atoms per unit volume (n) and the volume of the slab ($A\Delta x$). Thus,

$$\Delta N_{photons} = -\left(\frac{\sigma}{A} n A \Delta x\right) N_{photons} \tag{4.4.4}$$

$$\Delta N_{photons} = -\sigma n (\Delta x) N_photons \tag{4.4.5}$$

Since the mass per unit volume (ρ) is a more commonly known value than the number per unit volume (n), let's replace n with ρ and redefine the cross section to have units of area per unit mass (typically cm^2/g).

$$\Delta N_{photons} = -\sigma \rho (\Delta x) N_{photons} \tag{4.4.6}$$

$$\frac{\Delta N_{photons}}{\Delta x} = -\sigma \rho N_{photons} \tag{4.4.7}$$

In the limit of very thin slabs, this becomes a differential equation with a known solution:

$$\frac{dN_{photons}}{dx} = -\sigma \rho N_{photons} \tag{4.4.8}$$

$$N_{photons}(x) - N_0 e^{-\sigma \rho X} \tag{4.4.9}$$

Thus, the number of photons in the beam decreases exponentially, with dependence on both the cross section for the interaction of interest and the density of the target material.

Photons and Lead

The wonderful website:
physics.nist.gov/PhysRefData/...Text/XCOM.html
tabulates the cross sections for all of the photon interactions we have discussed for almost any element or compound you can imagine. These cross sections are crucial information for a wide variety of important activities, from calculating dosage for radiation therapy for cancer to determining the necessary shielding for nuclear reactors.

Below is a graphical representation of the cross sections for photons with energy between 1.0 keV and 1000 MeV interacting with lead. Note that at different energies different processes dominate. At lower energy, the dominant process is photoelectric absorption. Around 1.0 MeV, however, incoherent scattering begins to dominate. (Incoherent scattering is Compton scattering, in which the wavelength of the photon changes during the scattering event. Coherent scattering is when the photon scatters without changing its wavelength. This occurs, for example, when the photon scatters off of the nucleus. Note that if you substitute the nuclear mass into the Compton formula, you would find that the wavelength of the photon does not change.) Finally, above about 10 MeV pair production involving the nucleus dominates. Pair production can also occur with the electron "absorbing" the necessary momentum but this is much less likely.

Consider the problem below:

1.0 MeV photons are incident on a thick lead (ρ = 11.34 g/cm^3) slab.

a. If the slab is 1.0 cm thick, what percentage of the photons will undergo the photoelectric effect?

b. What thickness of lead is needed to stop 95% of the photons?
c. What thickness of lead is needed to stop or scatter 95% of the photons?

Since the first question concerns just the photoelectric effect, we need the photoelectric effect cross section at 1.0 MeV. From the website, σ photoelectric = $1.81x10^{-2}$ cm2/g. Thus,

$$N_{photons}(x) = N_0 e^{\sigma\rho x} \tag{4.4.10}$$

$$N_{photons} = N_0 e^{-(.0181)(11.34)(1)} \tag{4.4.11}$$

$$N_{photons} = N_0(0.814) \tag{4.4.12}$$

If 81.4 percent of the photons have not undergone the photoelectric effect, then 18.6% of them have.

Normally, photons can be "stopped" by either absorption (photoelectric effect) or pair production. Since 1.0 MeV is too low an energy for pair production, the only stopping mechanism is the photoelectric effect. Thus,

$$N_{photons}(x) = N_0 e^{\sigma\rho x} \tag{4.4.13}$$

$$\frac{N_{photons}}{N_0} = e^{-(.0181)(11.34)x} \tag{4.4.14}$$

$$0.05 = e^{-0.205x} \tag{4.4.15}$$

$$\ln(0.05) = -0.205x \tag{4.4.16}$$

$$x = 14.6 \text{ cm} \tag{4.4.17}$$

The cross section for stopping and scattering is the total cross section at 1.0 MeV, (\sigma\)total = 7.1 x 10-2 cm2/g. Thus,

$$N_{photons}(x) = N_0 e^{\sigma\rho x} \tag{4.4.18}$$

$$\frac{N_{photons}}{N_0} = e^{-(.071)(11.34)x} \tag{4.4.19}$$

$$0.05 = e^{-0.805x} \tag{4.4.20}$$

$$\ln(0.05) = -0.805x \tag{4.4.21}$$

$$x = 3.72 \text{ cm} \tag{4.4.22}$$

The decrease in beam intensity due to both absorption and scattering is referred to as the attenuation of the beam.

4.5: Activities

Beams of different frequency electromagnetic radiation are described below.

A gamma ray
B green light
C x-ray
D yellow light
E AM radio wave
F FM radio wave

a. Rank these beams on the basis of their frequency.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

b. If each beam has the same total energy, rank these beams on the number of photons in each beam.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

c. If each beam has the same number of photons, rank these beams on their total energy.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

The six metals listed below have the work functions indicated.

Metal Work Function (eV)
A aluminum 4.1
B beryllium 5.0
C cesium 2.1
D magnesium 3.7
E platinum 6.4
F potassium 2.3

a. Rank these metals on the basis of their threshold frequency.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

b. Rank these metals on the basis of the maximum wavelength of light needed to free electrons from their surface.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

c. Each metal is illuminated with 400 nm (3.10 eV) light. Rank the metals on the basis of the maximum kinetic energy of the emitted electrons. (If no electrons are emitted from a metal, the maximum kinetic energy is zero, so rank that metal as smallest.)

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

Six different light sources of different intensity (I) and frequency (f) are directed onto identical sodium surfaces. Sodium has a threshold frequency of 0.55 x 1015 Hz.

I (W/m2) f (x 1015 Hz)
A 1.0 1.0
B 2.0 0.5
C 1.0 2.0
D 0.5 2.0
E 4.0 0.5
F 0.5 1.0

a. Which of the sodium surfaces emit electrons?

b. Of the surfaces that emit electrons, rank the scenarios on the basis of the maximum kinetic energy of the emitted electrons.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

c. Of the surfaces that emit electrons, rank the scenarios on the basis of the stopping potential needed to “stop” the emitted electrons.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

The line below represents the maximum kinetic energy of photoelectrons emitted from sodium metal as a function of the frequency of the incident light.

a. If the intensity of the light striking the sodium metal was doubled, and the experiment re-performed, sketch and label as (a) the line that would represent the results of the experiment.

b. If a metal having half the work function of sodium was used in the experiment, and the experiment re-performed, sketch and label as (b) the line that would represent the results of the experiment.

c. What feature of the graph (i.e., slope, y-intercept, x-intercept, etc.) represents the work function?

d. What feature of the graph (i.e., slope, y-intercept, x-intercept, etc.) represents Planck’s constant?

A bar chart for the energy of the each incident photon, the kinetic energy of each emitted electron, and the work function of the metal surface is shown for a specific run of a photoelectric effect experiment.

For each change in the experiment listed below, construct a bar chart that illustrates the changes in each of the variables.

1. Use a brighter light source

2.Use a higher frequency light source

3. Use a metal with a lower threshold frequency

4. Use a metal with a work function greater than the photon energy

For each of the following experimental observations, select the correct choice from below:

A Can best be explained by thinking of light as a wave
B Can best be explained by thinking of light as a stream of particles
C Can be explained by either conception of light
D Cannot be explained by either conception of light

________ 1. In the photoelectric effect experiment, the current is directly proportional to the intensity of the incident light.

________ 2. In the photoelectric effect experiment, a threshold frequency exists below which no electrons are emitted.

________ 3. In the double slit experiment, light spreads out as it exits each hole of the apparatus.

________ 4. In the Compton scattering experiment, the wavelength of scattered light is longer than that of the incident light by an amount dependent on scattering angle.

________ 5. In the photoelectric effect experiment, the stopping voltage is independent of the intensity of the incident light.

________ 6. In the double slit experiment, a pattern of many alternating bright and dark spots appears on the screen opposite the slits.

________ 7. In the Compton scattering experiment, light is scattered from electrons to all angles.

________ 8. In the photoelectric effect experiment, electrons are emitted immediately after turning on the light, even at very low intensity.

________ 9. In the photoelectric effect experiment with constant light intensity, the electrons emitted from different metals require different stopping voltages.

________ 10. Light can pass through certain objects (like glass) but not through other objects (like concrete).

The image below shows the typical set-up for a Compton scattering experiment and the number of photons detected at different wavelengths with the photon detector at 900 from the initial beam direction. The incident photon wavelength is .

a. The peak present at ' corresponds to the result expected from the Compton scattering formula. Why is there also a peak present at the initial photon wavelength? (Hint: These photons are also scattered from the target.)

b. The peaks present in actual data have a width, i.e., not all of the photons scattered from electrons are scattered to exactly '. Why is there a spread in wavelength of the scattered photons?

c. Which of the two graphs below corresponds to the data collected with the detector at 450? Explain your reasoning.

The image below shows the photon cross sections for lead.

a. Using a pair of vertical lines, divide the graph into the three regions where the photoelectric effect, Compton scattering, and pair production are the dominant processes. Label these regions.

b. Using a dashed vertical line, indicate the energy at which photons would penetrate deepest into a lead target.

c. At the energy at which pair production becomes the dominant process, approximately how much more likely is pair production than the photoelectric effect?

d. Clearly explain why the cross section for the photoelectric effect has sharp "knife edge" jumps. Which lead electrons are beginning to absorb photons at the rightmost of these jumps?

e. Clearly explain why coherent scattering is much more likely than incoherent scattering at low energies. (Hint: Consider the photon wavelength and the size of a lead atom.)

f. Clearly explain why the pair production cross sections abruptly disappear at around 1 MeV.

a. How many photons per second are emitted by a 10 kW FM transmitter broadcasting at 89.7 MHz?
b. How many photons per second are emitted by a 5.0 mW, 634 nm laser?
c. Approximately how far apart are the individual photons in the laser beam described in (b), assuming the beam has negligible cross-sectional area?

Mathematical Analysis

When the human eye is fully dark-adapted, it requires approximately 1000 photons per second striking the retina for an object to be visible. These photons enter the eye through the approximately 3 mm diameter pupil.
a. Approximately how far apart are the individual photons striking our eye in the situation above?
b. The sun radiates at about 3.9 x 1026 W, with peak emission around 500 nm. Approximately how many photons per second are emitted by the sun?

c. Assuming these photons are radiated equally in all directions and very few are absorbed by intervening materials, approximately how far away could a star like the sun be and still be visible to humans?

Mathematical Analysis

Lead is illuminated with UV light of wavelength 250 nm. This results in a stopping potential of 0.82 V.

a. What is the work function for lead?

b. What is the stopping potential when lead is illuminated with 200 nm light?

c. What is the threshold frequency for lead?

Mathematical Analysis

When cesium is illuminated with 500 nm light the stopping potential is 0.57 V.

a. What is the work function for cesium?

b. For what wavelength light is the stopping potential 1.0 V?

c. What is the threshold wavelength for cesium?

Mathematical Analysis

A 190 mW laser operating at 650 nm is directed onto a photosensitive metal with work function 0.7 eV.

a. How many photons per second are emitted by the laser?

b. What is the maximum kinetic energy of the ejected electrons?

c. If 15% of the laser light is absorbed (with 85% reflected), what is the electron current in this experiment?

Mathematical Analysis

Light of wavelength 500 nm is incident on a photosensitive metal. The stopping potential is found to be 0.45 V.

a. Find the maximum kinetic energy of the ejected electrons.

b. Find the work function of the metal.

c. Find the cutoff wavelength of the metal.

Mathematical Analysis

The photoelectric effect involves the absorption of a photon by an electron bound in a metal. Show that it is impossible for a free electron at rest to absorb a photon. (Hint: Show that combining the conservation of \energy and the conservation of momentum for a free electron absorbing a photon results in a contradiction.)

Mathematical Analysis

A 0.03 nm photon collides with an electron at rest. After the collision, the photon is detected at 400 relative to its initial direction of travel.

a. Find the energy of the scattered photon.

b. Find the kinetic energy of the scattered electron.

c. Find the angle of the scattered electron.

Mathematical Analysis

An 800 keV photon collides with an electron at rest. After the collision, the photon is detected with 650 keV of energy.

a. Find the angle of the scattered photon.

b. Find the kinetic energy of the scattered electron.

c. Find the angle of the scattered electron.

Mathematical Analysis

A 0.01 nm photon collides with an electron at rest. After the collision, the photon is detected with 0.1 MeV of energy.

a. Find the angle of the scattered electron.

b. Find the velocity of the scattered electron.

Mathematical Analysis

In a Compton scattering experiment, it is noted that the maximum energy transferred to an electron is 45 keV. What was the initial photon energy used in the experiment?

Mathematical Analysis

In a symmetric collision, the scattering angle of the photon and electron are equal. If the incoming photon has 1.0 MeV of energy, and the scattering is symmetric, find the scattering angle.

Mathematical Analysis

A 2.0 MeV photon interacts with a carbon nucleus and creates an electron-positron pair. The electron and positron travel perpendicular to the initial direction of travel of the photon.

a. Find the kinetic energy of the electron and positron assuming the nucleus is at rest after the interaction.

b. Find the kinetic energy of the carbon nucleus needed to ensure momentum conservation. Would this amount of kinetic energy greatly affect the result in part a?

Mathematical Analysis

A 1.5 MeV photon interacts with a carbon nucleus and creates an electron-positron pair. The electron travels parallel and the positron antiparallel, at equal speeds, to the initial direction of travel of the photon.

a. Find the kinetic energy of the electron and positron assuming the nucleus is at rest after the interaction.

b. Find the kinetic energy of the carbon nucleus needed to ensure momentum conservation. Would this amount of kinetic energy greatly affect the result in part a?

Mathematical Analysis

A photon interacts with a lead nucleus and creates a proton-antiproton pair. The proton travels parallel and the antiproton antiparallel, both at 0.4c, to the initial direction of travel of the photon.

a. Find the kinetic energy of the proton and antiproton assuming the nucleus is at rest after the interaction.

b. Find the kinetic energy of the lead nucleus needed to ensure momentum conservation. Would this amount of kinetic energy greatly affect the result in part a?

Mathematical Analysis

A 1.8 MeV photon interacts with a lead nucleus and creates an electron-positron pair. The electron and positron travel at 350 from the initial direction of travel of the photon.

a. Find the kinetic energy of the electron and positron assuming the nucleus is at rest after the interaction.

b. Find the kinetic energy of the lead nucleus needed to ensure momentum conservation. Would this amount of kinetic energy greatly affect the result in part a?

Mathematical Analysis

A 1.8 MeV photon interacts with a hydrogen nucleus and creates an electron-positron pair. The electron and positron travel at 350 from the initial direction of travel of the photon.

a. Find the kinetic energy of the electron and positron assuming the nucleus is at rest after the interaction.

b. Find the kinetic energy of the hydrogen nucleus needed to ensure momentum conservation. Would this amount of kinetic energy greatly affect the result in part a?

Mathematical Analysis

Photons are incident on a 1.0 cm thick aluminum target.
a. If the incident energy is 1.0 keV, what percentage of the photons will undergo the photoelectric effect?
b. If the incident energy is 1.0 MeV, what percentage of the photons will undergo the photoelectric effect?
c. If the incident energy is 1.0 MeV, what percentage of the photons will be stopped or scattered?

Mathematical Analysis

Photons are incident on a very thick aluminum target.
a. At what energy will the beam penetrate deepest into the aluminum?
b. At the above energy, what thickness is needed to decrease the beam intensity by 50%?
c. At the above energy, what is the dominant photon process? What percentage of photons undergo this process?

Mathematical Analysis

Photons are incident on a very thick “depleted” uranium target.
a. At what energy will the beam penetrate deepest into the uranium?
b. At the above energy, what thickness is needed to decrease the beam intensity by 50%?
c. At the above energy, what is the dominant photon process? What percentage of photons undergo this process?

Mathematical Analysis

Equal intensity photon beams are incident on slabs of iron and lead.
a. At 1.0 MeV, find the thickness of lead needed to provide the same attenuation as 10 cm of iron.
b. At 100 MeV, find the thickness of lead needed to provide the same attenuation as 10 cm of iron.

Mathematical Analysis

a. For a 1.0 keV photon beam, find the thickness of air needed to decrease the beam intensity by 99%?
b. For a 100 MeV photon beam, find the thickness of air needed to decrease the beam intensity by 99%?

Mathematical Analysis

a. For a 1.0 keV photon beam, find the thickness of water needed to decrease the beam intensity by 99%?
b. For a 100 MeV photon beam, find the thickness of water needed to decrease the beam intensity by 99%?

Mathematical Analysis

Diagnostic x-rays have energy of around 100 keV. Imagine getting an x-ray for a broken arm. Model the non-bony part of the arm as a 5 cm thick bag of water.
a. What percentage of x-rays are absorbed by the non-bony part of the arm?
b. What percentage of x-rays are scattered by the non-bony part of the arm?
c. With all of these scattered x-rays flying around, you may want to wear a 1.0 cm thick lead apron to protect yourself. What percentage of x-rays (incident or scattered) are absorbed by the lead apron?

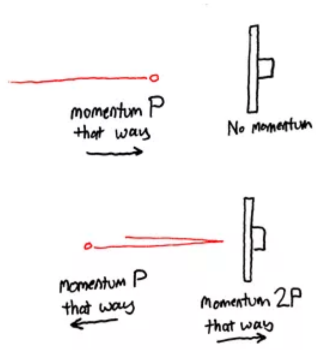
Mathematical Analysis

1.0 MeV photons are incident on a lead target. What is the probability per atom that a photon will be absorbed? (Hint: Consider a 1.0 cm thick target and use the density and molar mass of lead.)

4.6: The Laser Elevator (Project)

Text and drawings from: http://blag.xkcd.com/2008/02/15/the-laser-elevator/
Solar sails suck.

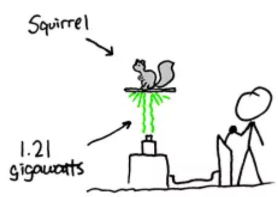
In a 2002 paper, *Laser Elevator: Momentum Transfer Using an Optical Resonator*, Thomas R. Meyer et. al. talk about a neat way to get a lot more speed out of light reflection than with a regular solar sail. The basic physics are pretty simple, and it's a fun subject to think about. When a photon hits a solar sail, it gives the sail momentum. If the photon has momentum P and bounces off a stationary sail, it looks like this:



Think of where the energy is in this system. Before it hits, the photon has energy E. After it bounces, the photon still has roughly energy E. But the sail's moving, so where did it get its kinetic energy? (Remember, energy — unlike momentum — has no direction.)

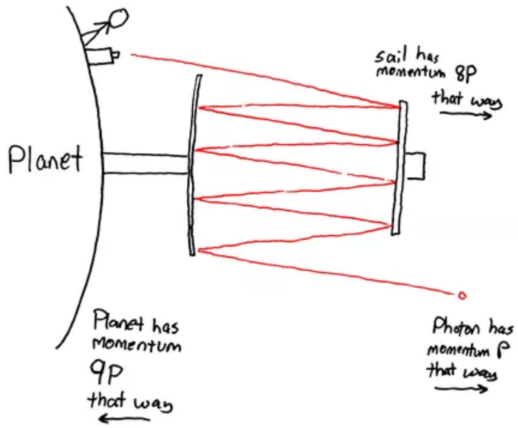
The answer lies in the word "roughly". The photon loses a tiny fraction of its energy to Doppler shifting when it's reflected, but only a tiny fraction. It is this tiny fraction that goes into pushing the sail. This is a phenomenally small amount of energy — far less than a percent of what the photon has. That is, not much of the photon's energy is being used for motion here.

This is why solar sails are so slow. It's not that light doesn't have that much energy, it's that it has so little momentum. If you set a squirrel on a solar sail and shone a laser on the underside, do you know how much power would be required to lift the squirrel? About 1.21 gigawatts.



This is awful. If we were lifting the squirrel with a motor, railgun, or electric catapult, with 1.21 gigawatts we could send it screaming upward at ridiculous speeds.

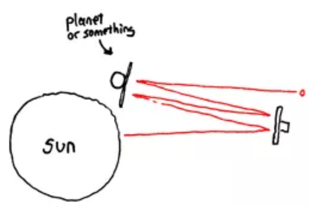
This is where Meyer and friends come in. They've point out a novel way to extract momentum from the photon: bounce it back and forth between the sail and a large mirror (on a planet or moon, perhaps).



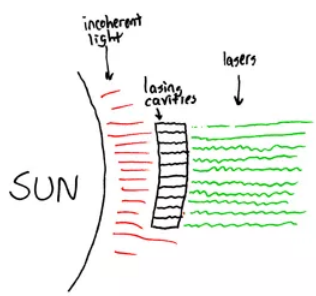
With each bounce, the photon loses a little more energy and adds another 2P to the sail's momentum. The photon can keep this up for thousands of bounces — in their paper, Meyer et. al. found that with reasonable assumptions about available materials and a lot of precision, you could extract 1,000 times the momentum from a photon before diffraction and Dopper shifts killed you. This means you only need 1/1,000th the energy to levitate the squirrel — a mere megawatt.

This isn't too practical for interstellar travel. It requires something to push off from, and probably couldn't get you up to the necessary speeds. It may, they suggest, be useful for getting stuff to Pluto and back, since (somewhat like a space elevator) it lets you generate the power any old way you want (a ground nuclear station, solar, etc). But more importantly, it's kind of neat — it helped me realize some things about photon momentum that I hadn't quite gotten before. It's like Feynman says, physics is like sex — it may give practical results, but that's not why we do it.

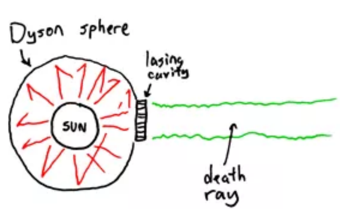
Now we'll let things get sillier. I spent a while trying to brainstorm how to use this with a solar sail (that is, using the sun). I imagined mirrors catching the sun's light and letting it resonate with a sail.



But you really need lasers for this — regular light spreads out too fast. Maybe a set of lasing cavities orbiting the sun



supplemented by a **Dyson sphere** …



And since by this point we'll probably have found aliens …

Why settle for interstellar communication when you can have interstellar war? And we could modulate the beam to carry a message — in this case, “F&%# YOU GUYS!

1. Analyze the collision (i.e., conserve energy and momentum) between a single photon of energy E and a solar sail of mass M. Call the energy of the photon after reflection E'. Derive a relationship for the energy lost by the photon $(E - E')$ in terms of E and M (and constants). Feel free to use the approximation that $(E + E') \approx 2E$.

2. Show that for a 1.0 MeV photon and a 1000 kg solar sail, the statement “This is a phenomenally small amount of energy — far less than a percent of what the photon has.” is a gross underestimation. The photon actually only loses about 10-32% of its energy!

3. Find the mass of the squirrel and platform that can be levitated with a 1.21 GW laser. You may need to recall that force is equal to the derivative of momentum:

$$F = \frac{dp}{dt}. \tag{4.6.1}$$

CHAPTER OVERVIEW

5: Matter Waves

Inspired by the dual nature of light, in 1923 Louis DeBroglie postulated, in his PhD thesis, that material particles also have both a particle-like and a wave-like nature. He conjectured that the frequency and wavelength of a “particle” are related to its energy and momentum in the same way as the frequency and wavelength of light are related to its energy and momentum, namely

$$E-hf \tag{5.1}$$

$$p=\frac{h}{\lambda} \tag{5.2}$$

After the experimental verification of this prediction, DeBroglie was awarded the Nobel Prize in 1929.

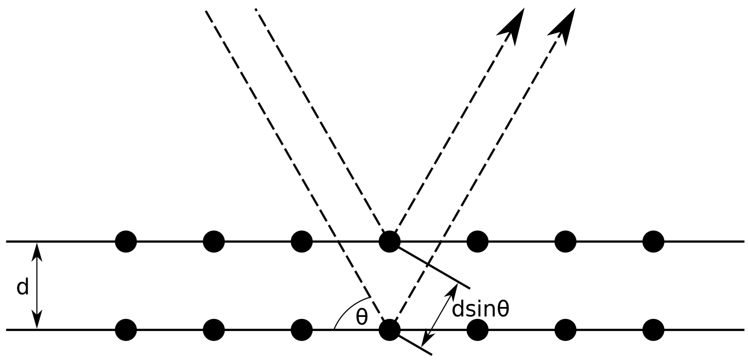
Contributors and Attributions

- Paul D’Alessandris (Monroe Community College)

5.1: Bragg Diffraction

You should immediately ask, "How was the wave-like nature of matter experimentally verified?" If matter has a wave-like nature, it should exhibit interference in a manner completely analogous to the interference of light. Thus, when passing through a regular array of slits, or reflecting from a regular array of atoms, an interference pattern should form. In 1927 Clinton Davisson and David Germer tested this hypothesis by directing a beam of electrons at a crystal of nickel.

Incoming waves reflecting from the first crystal plane will interfere with waves reflecting from the second (and subsequent) crystal planes forming an interference pattern. This interference, termed *Bragg diffraction*, had been initially investigated using x-rays.



For constructive interference, the path length difference between the two reflected beams must differ by an integer multiple of a complete wavelength. From the diagram above, the wave reflecting from the second crystal plane travels an additional distance of \(2d \sin q\(. Thus, the relation for constructive interference is:

$$2d\sin\theta - n\lambda \tag{5.1.1}$$

where

- d is the distance between adjacent crystal planes, termed the *lattice spacing*,
- θ is the angle, measured from the crystal face, at which constructive interference occurs,
- and l is the wavelength of the disturbance.

A beam of electrons is accelerated through a potential difference of 54 V and is incident on a nickel crystal. The primary interference maximum is detected at 13.7^o from the crystal face. What is the lattice spacing of the crystal?

If a beam of electrons is accelerated through a potential difference of 54 V, it gains a kinetic energy of 54 eV. This results in a momentum of

$$l\,E_{total}^2 = (pc)^2 + (mc^2)^2 \tag{5.1.2}$$

$$pc = \sqrt{E_{total}^2 - (mc^2)^2} \tag{5.1.3}$$

$$pc = \sqrt{54 + 511000)^2 - (511000)^2} \tag{5.1.4}$$

$$pc - 7.43 \text{ keV} \tag{5.1.5}$$

and, by DeBroglie's relation, a wavelength of

$$\lambda = \frac{hc}{pc} \tag{5.1.6}$$

$$\lambda = \frac{1240 \text{ eVnm}}{7430 \text{ eV}} \tag{5.1.7}$$

$$\lambda = 0.167 \text{ nm} \tag{5.1.8}$$

Inserting this result into the Bragg relation results in

$$2d\sin\theta = n\lambda \tag{5.1.9}$$

$$2d \sin 13.7 = (1)(0.167 \text{ nm}) \tag{5.1.10}$$

$$d = 0.352 \backslash\text{texT} nm \tag{5.1.11}$$

This value agrees with the known lattice spacing of nickel.

The presence of distinct interference maxima validates the idea that matter has a wave-like nature, and the agreement in lattice spacing illustrates that DeBroglie's relationship between the momentum and wavelength of matter is correct. For their experimental validation of DeBroglie's relation, Davisson (but not poor Mr. Germer) was awarded the Nobel Prize in 1937.

5.2: The Double Slit with Matter

Exercise 5.2.1: Neutron beams

A beam of very cold neutrons with kinetic energy $5.0 \times 10^{-6}\ eV$ is directed toward a double slit foil with slit separation 1 mm. What is the angular separation between adjacent interference maxima?

Solution

In addition to Bragg diffraction, the wave-like nature of matter can be demonstrated in the same experimental manner as the wave-like nature of light was first demonstrated, by passing the matter wave through a pair of adjacent slits. You should remember the result for the location of interference maxima in a double slit experiment, but nonetheless I'll remind you:

$$d \sin\theta = n\lambda \tag{5.2.1}$$

where

- d is the distance between adjacent slits,
- θ is the angle at which constructive interference occurs,
- and λ is the wavelength of the disturbance.

The kinetic energy of the neutrons is so small we can use classical physics to determine the momentum. Remembering the classical relationship between kinetic energy and momentum

$$KE = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m} = \frac{(pc)^2}{2mc^2} \tag{5.2.2}$$

leads to

$$\begin{aligned} pc &= \sqrt{2(KE)mc^2} \\ pc &= \sqrt{2(5 \times 10^{-6}(939.6 \times 10^6)} \\ pc &= 96.9 \text{ eV} \end{aligned} \tag{5.2.3}$$

and, by DeBroglie's relation, a wavelength of

$$\begin{aligned} \lambda &= \frac{hc}{pc} \\ \lambda &= \frac{1240 \text{ eVnm}}{96.9 \text{ eV}} \\ \lambda &= 12.8 \text{ nm} \end{aligned} \tag{5.2.4}$$

Inserting this result into the double slit relation results in

$$\begin{aligned} & d \sin\theta = n\lambda \\ & (1000 \text{ nm}) \sin\theta = (1)(12.8 \text{ nm}) \\ & \theta = 0.73^0 \end{aligned} \tag{5.2.5}$$

Thus, adjacent maxima are separated by 0.73 degrees.

Thermal Wavelength

How "cold" is a beam of very cold neutrons with kinetic energy 5.0 x 10^{-6} eV?

You may have been confused when I referred to the neutron beam in the previous example as being "very cold". However, physicists routinely talk about temperature, mass and energy using the same language. An ideal (non-interacting) gas of particles at an equilibrium temperature will have a range of kinetic energies. You may recall from your study of the ideal gas[1] that:

$$KE_{mean} = \frac{3}{2}kT \tag{5.2.6}$$

where

- KE_{mean} is the mean kinetic energy of a particle in the sample,
- k is *Boltzmann's constant,*
- and T is the temperature of the sample, in Kelvin.

Technically, we shouldn't talk about the temperature of a mono-energetic beam, since by definition a temperature implies a range of energies. However, let's be sloppy and assume the energy of the beam corresponds to the mean kinetic energy of a (hypothetical) sample. Then:

$$\begin{aligned} KE_{mean} &= \frac{3}{2}kT \\ T &= \frac{2KE_{mean}}{3k} \\ T &= \frac{2(5\times 10^{-6}\text{ eV}}{3(8.617\times 10^{-5}\text{ eV / K}} \\ T &= 0.039\text{ K} \end{aligned} \tag{5.2.7}$$

So the neutron beam really is pretty cold!

Note that if we wanted to find the DeBroglie wavelength corresponding to this mean kinetic energy, we would find (assuming non-relativistic speeds)

$$\begin{aligned} KE &= \frac{(pc)^2}{2mc^2} \\ pc &= \sqrt{2(KE)mc^2} \\ pc &= \sqrt{2\left(\frac{3}{2}KT\right)mc^2} \\ pc &= \sqrt{3mc^2kT} \end{aligned} \tag{5.2.8}$$

and thus

$$\lambda = \frac{hc}{pc} \tag{5.2.9}$$

$$\lambda = \frac{hc}{\sqrt{3mc^2kT}} \tag{5.2.10}$$

This is the DeBroglie wavelength corresponding to the mean kinetic energy of a gas at temperature, T. However, a more useful value would be the *mean wavelength* of all of the particles in the gas. The mean wavelength is not equal to the wavelength of the mean energy. Calculating this mean wavelength, termed the *thermal DeBroglie wavelength* is a bit beyond our skills at this point, but it is the same as the result above but with a different numerical factor in the denominator:

$$\lambda_{thermal} = \frac{hc}{\sqrt{2\pi nc^2kT}} \tag{5.2.11}$$

For an ideal gas sample at a known temperature, we can quickly determine the average wavelength of the particles comprising the sample.

One important use for this relationship is to determine when the gas sample is no longer ideal. If the mean wavelength becomes comparable to the separation between the particles in the gas, this means that the waves begin to overlap and the particles begin to interact. When these waves begin to overlap, it becomes impossible, even in principle, to think of each of the particles as a separate entity. When this occurs, some really cool stuff starts to happen (like superfluidity, superconductivity, Bose-Einstein condensation, etc.).

A Plausibility Argument for the Heisenberg Uncertainty Principle

Imagine a wave passing through a small slit in an opaque barrier. As the wave passes through the slit, it will form the diffraction pattern shown below.

untitled

Remember that the location of the first minima of the pattern is given by

$$a \sin\theta = \lambda \tag{5.2.12}$$

From the geometry of the situation,

$$\tan\theta = \frac{y}{D} \tag{5.2.13}$$

If the detecting screen is far from the opening,

$$\sin\theta \approx \tan\theta \tag{5.2.14}$$

so

$$a \sin\theta = \lambda \tag{5.2.15}$$

$$a \tan\theta = \lambda \tag{5.2.16}$$

$$a\left(\frac{y}{D}\right) = \lambda \tag{5.2.17}$$

$$y = \frac{\lambda D}{a} \tag{5.2.18}$$

Now, consider the “wave” to be a “particle”. The time to traverse the distance from slit to screen is given by

$$t = \frac{D}{v_x} \tag{5.2.19}$$

while during this time interval the particle also travels a distance in the y-direction given by:

$$y = v_y t \tag{5.2.20}$$

Combining these relations yields

$$y = v_y \left(\frac{D}{v_x}\right) \tag{5.2.21}$$

Combining this “particle” expression with the “wave” expression above gives:

$$\frac{\lambda D}{a} = v_y \left(fracDv_x\right) \tag{5.2.22}$$

$$\lambda = \frac{v_y a}{v_x} \tag{5.2.23}$$

Substituting the DeBroglie relation results in,

$$\frac{h}{m v_x} = \frac{v_y a}{v_x} \tag{5.2.24}$$

$$h = m v_y a \tag{5.2.25}$$

Notice that the term mv_y is the *uncertainty in the y-momentum* (σ_{p_y}) of the particle, since the particle is just as likely to move in the +y or the –y-direction with this momentum. Also, a is twice the *uncertainty in the y-position* (σ_y) of the particle, since the particle has a range of possible positions of +a/2 to –a/2.

Therefore, our expression can be written as

$$(2\sigma_y)(\sigma_{p_y}) = h \tag{5.2.26}$$

$$(\sigma_y)(\sigma_{p_y}) = \frac{h}{2} \tag{5.2.27}$$

Thus, the uncertainty in the y-position of the particle is inversely proportional to the uncertainty in the y-momentum. Neither of these quantities can be determined precisely, because the act of restricting one of these parameters automatically has a compensating effect on the other parameter, i.e., making the hole smaller spreads out the pattern, and the only way to make the pattern smaller is to increase the size of the hole!

A more careful analysis (for circular openings rather than slits) shows that the *minimum* uncertainty in the product of position and momentum can be reduced by a factor of 2p, resulting in:

$$(\sigma_y)(\sigma_{p_y}) \geq \frac{1}{2\pi}\frac{h}{2} \tag{5.2.28}$$

$$(\sigma_y)(\sigma_{p_y}) \geq \frac{\hbar}{2} \tag{5.2.29}$$

where the symbol ħ is defined to be Planck's constant divided by 2π.

Contributors and Attributions

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[1] The factor 3/2 is true only for particles. If the gas is comprised of diatomic molecules the factor is 5/2 at low temperatures and 7/2 at high temperatures. What counts as a low or high temperature depends on the molecule. More complex molecules have even more complex factors.

5.3: The Spatial Form of the Uncertainty Principle

The electrons in atoms are confined to a region of space approximately 10^{-10} m across. What is the minimum uncertainty in the velocity of atomic electrons?

The wave-like nature of matter forces certain restrictions on the precision with which a particle can be located in space and time. These restrictions are known as the Heisenberg Uncertainty Principle. (The word "uncertainty" is a poor choice. It is not that we are uncertain of the speed and location of the particle at a specific time; rather it is that the "particle" does not *have* a definite speed and location! The wave-like nature of the "particle" forces it to be spread out in space and time, analogous to the spreading of classical waves in space.)

The spatial form of Heisenberg's Uncertainty Principle is

$$(\sigma_X)(\sigma_{p_x}) \geq \frac{\hbar}{2} \tag{5.3.1}$$

where

- σ_x is the uncertainty, or variation, in the particle's position,
- \sigma\)$_{px}$ is the uncertainty, or variation, in the particle's momentum in the same direction,
- and ħ is Planck's constant divided by 2p.

With the center of the atom designated as the origin, the position of the electron can be represented as

\[x = (0\pm0.5)\times10^{-10) \text{ m}\]

thus

$$\sigma_x = 0.05 \text{ nm} \tag{5.3.2}$$

Using the uncertainty principle results in

$$(\sigma_X)(\sigma_{p_x}) \approx \frac{\hbar}{2} \tag{5.3.3}$$

\sigma_{p_x} \approx \frac{\hbar}{2(\sigma_x}}

m\sigma_{v_x} \approx \frac{\hbar}{2(\sigma_x}}

\sigma_{v_x} \approx \frac{\hbar}{2m(\sigma_x}}

\sigma_{v_x} \approx \frac{\hbar c}{2mc^2(\sigma_x}}c

Just as "hc" will pop up in numerous equations throughout this course, the constant "ħc" is also quite common and has an equally friendly value, 197.4 eV nm. Thus,

$$\sigma_{v_x} \approx \frac{197.4 \text{ eVnm}}{2(511000 \text{ eV})(0.05 \text{ nm}} c \tag{5.3.4}$$

$$\sigma_{v_x} \approx 3.86 \times 10^{-3} c \tag{5.3.5}$$

$$\sigma_{v_x} \approx 1.2 \times 10^6 \text{ m/s} \tag{5.3.6}$$

Thus the velocity of an atomic electron has an inherent, irreducible uncertainty of about a million meters per second! If anyone tells you they know how fast an atomic electron is moving to a greater precision than a million meters per second, you know what to tell them…

5.4: The Temporal Form of the Uncertainty Principle

Empty space can never be completely empty. Particles can spontaneously "pop" into existence and then disappear. Imagine a proton and antiproton spontaneously created from the vacuum with kinetic energy 1.0 MeV each. For how long can these particles exist and how far could they travel in this time?

An analogous argument to the one that led to the spatial form of the uncertainty principle can be made that leads to the temporal form of Heisenberg's Uncertainty Principle:

$$(\sigma_E)(\sigma_t) \geq \frac{\hbar}{2} \tag{5.4.1}$$

where

- s$_E$ is the uncertainty, or variation, in the particle's total energy,
- and s$_t$ is the time interval over which the energy was measured.

This form of the uncertainty principle implies that the precise value of the energy of a particle or system can never be known, since the time interval over which the value is measured inversely affects the precision of the measurement. This even applies to a region of space in which the energy is, nominally, zero.

In this example, the energy of a certain region of empty space, naively thought to be equal to zero, spontaneously fluctuates by an amount equal to the total energy of the two created particles. This variation can only last for

$$(\sigma_E)(\sigma_t) \geq \frac{\hbar}{2} \tag{5.4.2}$$

$$\sigma_T \approx \frac{\hbar}{2\sigma_E} \tag{5.4.3}$$

$$\sigma_t \approx \frac{0.658 \times 10^{-15} \text{ eVs}}{2[2(938 \text{ MeV} + 1.0 \text{ MeV})]} \tag{5.4.4}$$

$$\sigma_t \approx 1.8 \times 10^{-25} s \tag{5.4.5}$$

The particle's speed is given by

$$KE = (\gamma - 1)mc^2 \tag{5.4.6}$$

$$1.0 = (\gamma - 1)(938) \tag{5.4.7}$$

$$\gamma = 1.001066 \tag{5.4.8}$$

$$v = 0.046c \tag{5.4.9}$$

In this incredibly short amount of time the particles will be able to travel

$$d = vt \tag{5.4.10}$$

$$d = (0.046c)(1.8 \times 10^{-25}) \tag{5.4.11}$$

$$d = 2.4 \times 10^{-18} m \tag{5.4.12}$$

This distance is approximately *one-thousandth* the width of a single proton. Although this is an incredibly short distance, our modern understanding of the nature of forces and the evolution and fate of the universe depend on the affects of these *virtual particles*.

5.5: The Meaning of the Uncertainty Principle

There is much confusion regarding the meaning of the Uncertainty Principle. In fact, the uncertainty principle is really just a statement about waves and how simple waves can be combined to form *wave packets*. A wave packet is a localized wave disturbance.

Unlike simple sine and cosine representations of waves, such as:

$$\Psi(x, t)=A \sin (k x-\omega t) \tag{5.5.1}$$

which extend at equal amplitude to $\pm\infty$, a wave packet has amplitude that is larger in one region of space than another. Mathematically, wave packets are formed by adding together appropriately chosen simple sine and cosine waves.

For example,

note that the wave packet (C) has larger amplitude in some regions of space than in other regions. We can say that the wave packet C is *localized* in space. If we want to more narrowly localize C in space we will need to add together a larger range of different wavelength waves. Thus, the spatial localization of C is inversely related to the range of wavelengths used to construct C. Since a larger range of wavelengths corresponds to a larger range of momenta (by DeBroglie's hypothesis), spatial localization comes at the price of an increased range of momenta. This is all the uncertainty principle says! The width of spatial localization is inversely proportional to the range in momenta. This is true of ***all*** waves and is not special, in any way, to matter waves.

Additionally, since the wave packet is mathematically comprised of many waves with different momenta, these constituent waves all move through space at different rates. The speed of each constituent wave is referred to as the *phase velocity*. The speed of the wave packet is the *group velocity*. Since the different constituent waves have different phase velocities, this leads, over time, to a change in the overall shape and extent of the wave packet. This change in shape of the wave packet over time is termed *dispersion* and is illustrated at left (time increases as you scan from top to bottom).

This spreading of the wave packet is natural and is completely analogous to, for example, the spreading of water waves on a pond. In the case of a matter wave, however, this spreading is interpreted as the increasing uncertainty as to the location of the "particle" that the wave represents. The wave (actually the square of the wave) represents the probability of finding the particle at a certain location in space when a measurement is performed.

Before the measurement is made, however, the "particle" must be thought of as existing at all of the locations where the probability is non-zero.

5.6: A Problem with DeBroglie's Hypothesis?

Aside from the "minor" issue of trying to understand what it means for a particle to have a frequency and a wavelength, DeBroglie's hypothesis also leads to a more technical issue regarding how the frequency and wavelength of the particle are related to its velocity. Hopefully you remember that the standard relationship between a wave's velocity, frequency and wavelength is

$$v = f\lambda \tag{5.6.1}$$

Substituting DeBroglie's relationships yields:

$$\begin{aligned} v &= \left(\frac{E}{h}\right)\left(\frac{h}{p}\right) \\ &= \frac{E}{p} \\ &= \frac{\sqrt{(pc)^2 + (mc^2)^2}}{p} \\ &= \sqrt{c^2 + \frac{(mc^2)^2}{p^2}} \end{aligned}$$

This should bother you. Why? Because the second term under the radical is obviously positive and if added to \(\ce{c^2\) seems to require that the velocity of the "matter wave" is greater than the speed of light! To resolve this apparent contradiction, we have to be much more careful in how we conceptualize the wave-representation of a particle.

The key is to distinguish between the phase velocity and group velocity as defined above. The wave packet, and hence the group velocity, is what represents the particle. The constituent waves that add together to form the packet are not, individually, physically meaningful. The simple relation between frequency and wavelength mentioned above is actually only correct for these individual constituent waves. Hence, it holds for the phase velocity only:

$$v_{phase} = f\lambda \tag{5.6.2}$$

What we need to worry about is not the value of these phase velocities, but the value of the group velocity. If the group velocity is greater than c, we've got some serious problems.

So how do we define the group velocity? Well, let's first review how we define the phase velocity. The phase velocity is the speed at which a particular point on the wave moves through space. This particular point has a constant phase, so let's set the phase in the standard definition of a wave,

$$\Psi(x,t) = A\sin(kx - \omega t) \tag{5.6.3}$$

equal to a constant. For simplicity we'll call that constant zero.

$$(kx - \omega t) = 0 \tag{5.6.4}$$

$$kx = \omega t \tag{5.6.5}$$

$$\frac{x}{t} = \frac{\omega}{k} \tag{5.6.6}$$

$$v_{phase} = \frac{\omega}{k} \tag{5.6.7}$$

If you recall the definitions of angular velocity ($omega$) and wave number (k) from your study of waves, this reduces to:

$$\begin{aligned} v_{phase} &= \frac{2\pi f}{\left(\frac{2\pi}{\lambda}\right)} \\ &= f\lambda \end{aligned}$$

Ok, so that was easy. Now what do we change to get an expression for group velocity?

In a wave packet, each constituent wave has a constant phase velocity but the packet as a whole experiences dispersion and changes its shape with time. Because of this continually changing shape, let's try to define the group velocity not as the ratio of the angular frequency to the wave number but rather as the *derivative* of the angular frequency with respect to wave number:

$$v_{phase} = \frac{\omega}{k}$$

and

$$v_{group} = \frac{\partial \omega}{\partial k}$$

Before we tackle this derivative, let's express DeBroglie's hypothesis in terms of w and k:

$$E = hf = h\left(\frac{\omega}{2\pi}\right) = \hbar\omega \tag{5.6.8}$$

$$p = \frac{h}{\lambda} = \frac{h}{\left(\frac{2\pi}{k}\right)} = \hbar k \tag{5.6.9}$$

Thus,

$$\begin{aligned} v_{group} &= \frac{\partial \omega}{\partial k} \\ &= \frac{\partial\left(\frac{E}{\hbar}\right)}{\partial\left(\frac{p}{\hbar}\right)} \\ &= \frac{\partial E}{\partial p} \end{aligned}$$

This means that the velocity of the wave packet is the rate at which the particle's energy changes with respect to its momentum.

Using the relativistic relationship between energy and momentum yields

$$\begin{aligned} v_{group} &= \frac{\partial}{\partial p}\left(\sqrt{pc^2 + mc^4}\right) \\ &= \frac{1}{2}(pc^2 + mc^4)^{-\frac{1}{2}}(2pc^2) \\ &= \frac{pc^2}{\sqrt{pc^2 + mc^4}} \\ &= \frac{(\gamma mv)c^2}{E} \\ &= \frac{(\gamma mv)c^2}{\gamma mc^2} \\ &= v \end{aligned}$$

Thus the group velocity of the wave packet is exactly the same as the velocity of the particle the packet is intended to represent. Even though each of the constituent waves travels with a phase velocity greater than c, they combine to form a wave packet that moves slower than light. Isn't mathematics amazing?

5.7: Virtual Pair Production (Project)

The Heisenberg Uncertainty Principle allows for short-term violations of the law of energy conservation. The amount of energy "borrowed" from empty space (ΔE) and the elapsed time (Δt) before the energy must be "repaid" are related by

$$(\Delta E)(\Delta t) \approx \frac{\hbar}{2} \tag{5.7.1}$$

These fluctuations in energy often take the form of matter/antimatter pairs of particles spontaneously "popping" into existence from otherwise empty space. These pairs of particles, called virtual particles, travel finite distances through space during their brief lifetimes and have repercussions that may influence the ultimate fate of the universe. In this activity, you will develop a spreadsheet that calculates the maximum distance these particles can travel as a function of their rest mass and kinetic energy.

I. Constructing the Spreadsheet

Construct a spreadsheet that has the following general form. (The template VirtualParticles is available in the PHY 262 course folder.)

Virtual Particle Production
your name:
your name:

rest energy = MeV

$$\text{Kinetic Energy (MeV)} \quad \Delta t(s)\text{gamma Velocity }(c) \quad \text{Distance (m)} \tag{5.7.2}$$

The spreadsheet should allow you to enter the rest energy of one of the pair of particles produced (its antimatter partner has the same rest energy). Then, for kinetic energies ranging from 0 to the rest energy of the particle, in 20 even increments, and from the rest energy to 20 times the rest energy, in 20 even increments, the spreadsheet should complete the above table of values.

The spreadsheet should then construct a graph of distance traveled vs. kinetic energy.

II. Using the Spreadsheet

A. Testing your Spreadsheet

An electron and positron, each with kinetic energy 1.0 MeV, spontaneously appear from the vacuum. Calculate below the distance each particle can travel before disappearing.

If your spreadsheet does not agree with your calculation above, correct your spreadsheet (or your calculation) before continuing!

B. Electron/positron pair production

1. What is the maximum distance a virtually produced electron or positron can travel? How does this distance compare to the size of an atom? The size of a nucleus?

2. At what kinetic energy (in MeV) does the pair travel a maximum distance?

3. What is this kinetic energy as a fraction of the rest energy of the particle?

C. Proton/antiproton pair production

1. What is the maximum distance a virtually produced proton or antiproton can travel? How does this distance compare to the size of an atom? The size of a nucleus?

2. At what kinetic energy (in MeV) does the pair travel a maximum distance?

3. What is this kinetic energy as a fraction of the rest energy of the particle?

You should notice that the graph of distance traveled vs. kinetic energy has the same functional shape regardless of rest energy, with the maximum distance traveled always occurring for the same kinetic energy value (expressed as a fraction of rest energy).

D. Neutrino/antineutrino pair production

The least massive (non-zero mass) particle known to exist is the electron neutrino, with rest energy of approximately 0.05 eV. Print the graph of distance vs. kinetic energy, and your evaluated spreadsheet, for this pair production. Attach your printouts to the end of this activity.

CHAPTER OVERVIEW

6: The Schrödinger Equation

Contributors and Attributions

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6.1: Schrödinger’s Equation

In 1925 Erwin Schrödinger proposed a differential equation that, when solved, produced a complete mathematical description of the wavefunction, $\psi(x)$, of a “particle” moving in a region of space with potential energy function $U(x)$. The one-dimensional, time-independent, non-relativistic form of this equation is:

$$-\frac{\hbar^2}{2m}\frac{d}{dx^2}\psi(x)+U(x)\psi x=E\psi(x) \tag{6.1.1}$$

where E is the total energy of the system. The wavefunction can be thought of as the amplitude of the “wave” representing the “particle”. Physically, it turns out that the square of the wavefunction is equal to the probability of finding the particle at a particular region of space.

Although this equation cannot be “derived” from any other physics principle, it can be shown to at least be consistent with the conservation of energy. Assuming the wavefunction takes the form of a sum of sine waves of the form:

$$\psi(x)=A\sin(kx) \tag{6.1.2}$$

with the wavelength (λ) defined as

$$k=\frac{2\pi}{\lambda} \tag{6.1.3}$$

Substituting Equation 6.1.2 into Equation 6.1.1 yields

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d}{dx^2}A\sin(kx)+U(x)A\sin(kx) &= EA\sin(kx) \\ +\frac{\hbar^2}{2m}k^2A\sin(kx)+U(x)A\sin(kx) &= EA\sin(kx) \\ \frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2+U(x) &= E \\ \frac{h^2}{2m\lambda^2}+U(x) &= E \end{aligned} \tag{6.1.4}$$

Substituting DeBroglie’s relation into Equation 6.1.4 yields

$$\begin{aligned} \frac{h^2}{2m\left(\frac{h}{p}\right)^2}+U(x) &= E \\ \frac{p^2}{2m}+U(x) &= E \\ \frac{(mv)^2}{2m}+U(x) &= E \\ \frac{1}{2}mv^2+U(x) &= E \\ KE+U(x) &= E \end{aligned} \tag{6.1.5}$$

For different potential energy functions, we will solve Schrödinger’s equation for the allowed values of total energy, E, and the exact mathematical form of the function, $\psi(x)$, describing the object moving in the region of potential energy. Schrödinger was awarded the Nobel Prize in 1933 primarily for the development of this equation.

6.2: Solving the 1D Infinite Square Well

Imagine a (non-relativistic) particle trapped in a one-dimensional well of length L. Inside the well there is no potential energy, and the particle is trapped inside the well by "walls" of infinite potential energy.

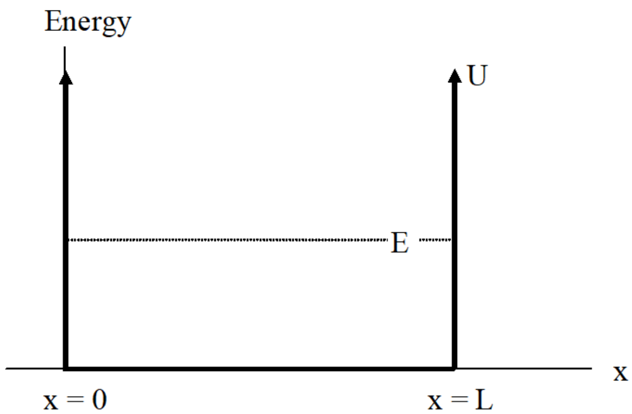

Figure $6.2.1$

Since this potential is a piece-wise function, Schrödinger's equation must be solved in the three regions separately. In the region $x > L$ (and $x < 0$), the equation is:

$$-\frac{\hbar^2}{2m}\frac{d}{dx^2}\psi(x) + U(x)\psi x = E\psi(x) \tag{6.2.1}$$

$$-\frac{h^2}{2m}\frac{d^2}{dx^2}\psi(x) + (\infty)\psi(x) = E\psi(x) \tag{6.2.2}$$

$$(\infty)\psi(x) = E\psi(x) \tag{6.2.3}$$

This has solutions of $E = \infty$, which is impossible (no particle can have infinite energy) or $\psi = 0$. Since $\psi = 0$, the particle can never be found outside of the well.

In the region $0 < x < L$, the equation is:

$$\begin{aligned} &-\tfrac{h^2}{2m}\tfrac{d^2}{dx^2}\psi(x) + U(x)\psi(x) = E\psi(x) \\ &-\tfrac{h^2}{2m}\tfrac{d^2}{dx^2}\psi(x) + (0)\psi(x) = E\psi(x) \\ &-\tfrac{h^2}{2m}\tfrac{d^2}{dx^2}\psi(x) = E\psi(x) \\ &\tfrac{d^2}{dx^2}\psi(x) = -\tfrac{2mE}{h^2}\psi(x) \end{aligned} \tag{6.2.4}$$

The general solution to this equation is

$$\psi(x) = A\sin(kx) + B\cos(kx) \text{ with } \quad k = \sqrt{\tfrac{2mE}{h^2}}$$

The solutions in the different regions of space must be continuous across the boundaries at $x = 0$ and $x = L$ (the derivatives of the solutions must also be continuous if the potential energy is continuous at these points).

At $x = 0$, equate the two solutions:

$$\begin{aligned} &\psi\left(x = 0^-\right) = \psi\left(x = 0^+\right) \\ &0 = A\sin(k0) + B\cos(k0) \\ &0 = \mathrm{B} \end{aligned} \tag{6.2.5}$$

therefore, the cosine portion of the solution has amplitude zero.

At $x = L$, equate the two solutions:

$$\begin{aligned}
&\psi\left(x=L^{-}\right)=\psi\left(x=L^{+}\right) \\
&A \sin (k L)=0 \\
&k L=n \pi \\
&k=\frac{n \pi}{L}
\end{aligned} \tag{6.2.6}$$

This leads to allowed wavefunctions:

$$\psi(x)=A \sin \left(\frac{n \pi x}{L}\right) \tag{6.2.7}$$

and allowed energies:

$$\begin{aligned}
&k=\sqrt{\frac{2 m E}{h^{2}}} \\
&\left(\frac{n \pi}{L}\right)^{2}=\frac{2 m E}{h^{2}} \\
&\left(\frac{n \pi}{L}\right)^{2}=\left(\frac{2 \pi}{h}\right)^{2} 2 m E \\
&E=\frac{n^{2} h^{2}}{8 m L^{2}} \\
&E=n^{2} \frac{(h c)^{2}}{8 m c^{2} L^{2}}
\end{aligned} \tag{6.2.8}$$

These solutions are represented below. On the left is the standard textbook representation, where the wavefunction is superimposed on the graph of energy. Remember, however, that what is "waving" above and below the energy level isn't a measure of energy (i.e., the energy is a constant value at each level), but rather related to the probability of finding the particle at each location in space. The representation on the right tries to represent the regions where it is more likely to find the particle (the darker regions) and the regions where it is less likely to find the particle.

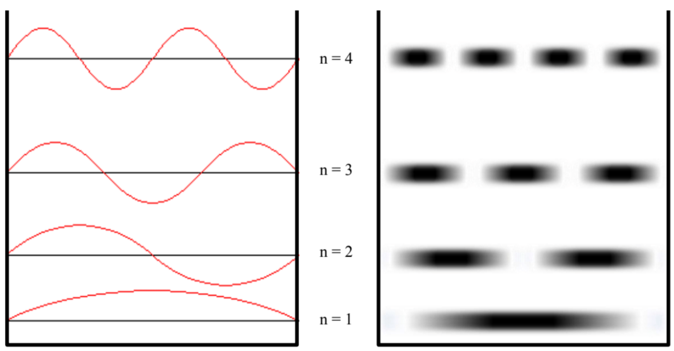

Figure 6.2.1: To the left the wavefunction, to the right a representation of the probability of finding the particle at a specific position for the various quantum states.

Additionally, since the probability of finding the particle somewhere in the well is equal to 1, and the probability of finding the particle is equal to the wavefunction squared,

using an integral table yields the result

This result has a number of extremely important features.

- The particle can only have certain, discrete values for energy. In classical physics, a particle trapped in a region of space can have any, continuous value for energy. The restriction of a bound particle to specific, quantized values of allowed energy is a hallmark of quantum mechanics.

- The particle cannot be at rest. Notice that the lowest possible energy for the particle is in the $n=1$ state, which has non-zero energy. This is termed the zero-point energy, and can be understood as a consequence of the Heisenberg uncertainty principle. This is in complete contrast to classical physics.
- The particle has regions of high and low probability of being found in the well. The probability of finding the particle is equal to the square of the wavefunction, which is not a constant value. Unlike classical physics, where the particle is equally likely to be anywhere in the well, in quantum mechanics there exist positions where the particle will never be found, and regions where the probability of finding the particle is greatly enhanced. for example, in the $n=2$ state, the particle will never be found in the center of the well, while it is very likely to find the particle at the positions $x=L/4$ and $x=3L/4$.

The 1D Infinite Well

An electron is trapped in a one-dimensional infinite potential well of length $4.0\times 10^{-10}\ m$. Find the three longest wavelength photons emitted by the electron as it changes energy levels in the well.

The allowed energy states of a particle of mass m trapped in an infinite potential well of length L are

$$E=n^2\frac{(hc)^2}{8mc^2L^2} \tag{6.2.9}$$

Therefore, the electron has allowed energy levels given by

$$\begin{aligned} E &= n^2\frac{(hc)^2}{8mc^2L^2} \\ E &= n^2\frac{(1240)^2}{8(511000\mathrm{eV})(0.4\mathrm{nm})^2} \\ E &= n^2(2.35\mathrm{eV}) \end{aligned} \tag{6.2.10}$$

As the electron changes energy levels, the energy released by the electron is in the form of a photon.

$$\begin{aligned} E_{photon} &= E_{initial\ electron}-E_{final\ electron} \\ E_{photon} &= (n_i^2-n_f^2)2.35\mathrm{eV} \end{aligned} \tag{6.2.11}$$

Thus, the emitted wavelengths are

$$\begin{aligned} \lambda &= \frac{hc}{E_{photon}} \\ \lambda &= \frac{1240\mathrm{eV\ nm}}{(n_i^2-n_f^2)2.35\mathrm{eV\ nm}} \\ \lambda &= \frac{528\ \mathrm{nm}}{(n_i^2-n_f^2)} \end{aligned} \tag{6.2.12}$$

The longest wavelength photons involve the smallest value of $n_i^2-n_f^2$. These are:

n_i	n_f	λ
2	1	176nm
3	2	106 nm
4	3	75.6 nm

(6.2.13)

6.3: The Pauli Exclusion Principle

Imagine five neutrons and four protons trapped in an atomic nucleus. Model the nucleus as an infinite potential well of length 10 fm. Find the total kinetic energy of the nucleons. Ignore the mass difference between protons and neutrons.

The allowed energy levels in this infinite square well are:

$$\begin{aligned} E &= n^2 \frac{(hc)^2}{8mc^2L^2} \\ E &= n^2 \frac{(1240 \text{ MeV fm})^2}{8(938 \text{ MeV})(10 \text{ fm})^2} \\ E &= n^2(2.35 \text{ MeV}) \end{aligned} \tag{6.3.1}$$

Although all of the particles would love to occupy the lowest energy state, the Pauli Exclusion Principle states that no two identical fermions can occupy the exact same quantum state. Thus, only two neutrons (and two protons) can occupy the lowest energy state, one with spin "up" and one with spin "down". In this way, the well is filled from the bottom up as indicted below:

n	E (2.05 MeV)	# neutrons	# protons
1	1	2	2
2	4	2	2
3	9	1	

(6.3.2)

Therefore, the total kinetic energy of the nucleons is:

$$KE = [4(1) + 4(4) + 1(9)]2.05 \text{ MeV} = 59.5 \text{ MeV} \tag{6.3.3}$$

6.4: Expectation Values, Observables, and Uncertainty

An electron is trapped in a one-dimensional infinite potential well of length L. Find the expectation values of the electron's position and momentum in the ground state of this well. Show that the uncertainties in these values do not violate the uncertainty principle.

Imagine an electron was trapped in the well described above and we repeatedly measured its location in the well. Due to the wave nature of the electron, we would get different values for these positions but, after many measurements, we could average these values to determine the expectation value of the electron's position.

You may be tempted to just refer to this as the average value of the electron's position, but if you take the wave-like nature of the electron seriously, and you should, the electron does not have a position until it is measured, so it is senseless to refer to the average value of something that doesn't even exist! What you are averaging is your measurements of the electron's position, not its pre-existing position. To avoid this metaphysical conundrum, we will call the value that we most likely expect to measure the expectation value of the variable.

The expectation value of the position (given by the symbol <x>) can be determined by a simple weighted average of the product of the probability of finding the electron at a certain position and the position, or

$$<x>=\int_0^L x\,\mathrm{Prob}(x)dx \tag{6.4.1}$$

$$<x>=\int_0^L (\Psi(x))x(\Psi(x))dx \tag{6.4.2}$$

What may strike you as somewhat strange is why I placed the factor of x between the two factors of the wavefunction. Mathematically, it doesn't matter where I place the x, but it turns out that for other variables the placement of the variable of interest must be "between" the two wavefunctions. Before we explore why this is the case, let's finish the calculation.

$$\begin{aligned} <x> &= \int_0^L \left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)x\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)dx \\ <x> &= \frac{2}{L}\int_0^L x\sin^2\left(\frac{\pi x}{L}\right)dx \end{aligned} \tag{6.4.3}$$

This integral begs for a u-substitution of:

$$u=\frac{\pi x}{L} \tag{6.4.4}$$

$$\begin{aligned} <x> &= \frac{2}{L}\int_0^\pi \left(\frac{2}{L}u\right)\sin^2(u)\left(\frac{L}{\pi}du\right) \\ <x> &= \frac{2L}{\pi^2}\int_0^\pi u\sin^2(u)du \\ <x> &= \frac{2L}{\pi^2}\frac{\pi^2}{4} \\ <x> &= \frac{L}{2} \end{aligned} \tag{6.4.5}$$

I'll agree that this seems like a stupid amount of work just to determine that the expectation value of a particle's position in an infinite well is in the center of the well, but it's always nice when learning a new mathematical technique to apply it to a situation in which you know the answer.

Now let's move on to the expectation value of the electron's momentum. You should be tempted to write:

$$<p>=\int_0^L (\Psi(x))p(\Psi(x))dx \tag{6.4.6}$$

The only problem with this, of course, is that we have to express the momentum of the electron in terms of its position in order to do the integral. How can we do that? Well, the momentum of the electron is related (by DeBroglie) to its wavelength, and the wavelength is dependent on how “curvy” the wavefunction is at any point, and the “curviness” of the wavefunction is related to the spatial derivative of the wavefunction. Thus,

$$p \propto \frac{d}{dx} \tag{6.4.7}$$

Not to get overly philosophical here, but in quantum mechanics all that exists is the wavefunction. Everything that is observable in nature must somehow be extracted from the wavefunction. This means that quantities like momentum can only be determined by manipulating the wavefunction is some way, in this case by taking a spatial derivative. Thus, quantities like momentum (or kinetic energy) are represented not by the “formulas” you are familiar with from classical mechanics but by mathematical operators, basically actions that must be taken on the wavefunction in order to squeeze from it the information you are interested in. This is why the placement of the variable in the formula for expectation values is so important. The operator for momentum acts on one “copy” of the wavefunction, and then the result is multiplied by the other “copy” and then integrated over all of space. We are almost ready to do this, but first we need to complete the description of the momentum operator.

If you recall from above, I showed that Schrödinger’s equation is consistent with the idea of energy conservation:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x)+U(x)\Psi(x)=E\Psi(x) \tag{6.4.8}$$

$$\frac{p^2}{2m}+U(x)=E \tag{6.4.9}$$

Carefully comparing these two relationships leads to the conclusion that the operator representing momentum may be,

$$p=i\hbar ddx \tag{6.4.10}$$

where

$$i=\sqrt{-1} \tag{6.4.11}$$

Thus, if you want to determine the momentum of a wavefunction, you must take a spatial derivative and then multiply the result by –ih. Should you be concerned that this implies that momentum is not “real”? The short answer is no.

Let’s determine the expectation value of the momentum of the electron:

$$\begin{aligned}
<p>&=\int_0^L\left(\sqrt{\frac{2}{L}}\sin(\frac{\pi x}{L})\right)\left(-i\hbar\frac{d}{dx}\left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\right)dx\right.\\
<p>&=-i\hbar\frac{2}{L}\int_0^L\sin\left(\frac{\pi x}{L}\right)\left(\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right)\right)dx\\
<p>&=-i\hbar 2\pi\frac{2}{L}\int_0^L\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)dx\\
<p>&=-ih\int_0^\pi\sin(u)\cos(u)\left(\frac{L}{\pi}du\right)\\
<p>&=-ih\frac{L}{\pi}\int_0^\pi\sin(u)\cos(u)du\\
<p>&=-ih\frac{L}{\pi}\int_0^\pi\frac{1}{2}\sin(2u)du\\
<p>&=-ih\frac{L}{\pi}(0)\\
<p>&=0
\end{aligned} \tag{6.4.12}$$

So the expectation value of the momentum of a particle in an infinite square well is zero? Of course it is! The allowed energy levels in a well can be thought of as the standing waves that “fit” in the well. The whole idea of a standing wave is that there is no net flow of energy (or momentum) in either direction. That’s why we call it a standing wave!

Now what about the uncertainties in these values? Obviously, every time we measure the position of the electron it won’t be in the center of the well (just equally likely on the right and the left) and every time we measure the momentum of the particle it won’t be at rest (just equally likely “moving” to the right or the left). The uncertainty in these values gives you an idea of the spread in possible measurements you should expect if you made a large number of measurements. This idea of the spread in a collection of data is simply the idea of the standard deviation.

Mathematically, the standard deviation of a set of position data is determined by

$$\sigma_x = \sqrt{<x^2> - <x>^2} \tag{6.4.13}$$

i.e., the difference between the expectation value of the square of x and the expectation value of x squared. Thus, to find the uncertainty in position, we need the expectation value of x2:

$$\begin{aligned} <x^2> &= \int_0^L \left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\right) x^2 \left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\right) dx \\ <x^2> &= \frac{2}{L}\int_0^L x^2 \sin^2\left(\frac{\pi x}{L}\right) dx \\ <x^2> &= \frac{2}{L}\int_0^\pi \left(\frac{L}{\pi}u\right)^2 \sin^2(u)\left(\frac{L}{\pi}du\right) \\ <x^2> &= \frac{2L^2}{\pi^3}\int_0^\pi u^2 \sin^2(u) du \\ <x^2> &= \frac{2L^2}{\pi^3}\left(\frac{\pi^3}{6} - \frac{\pi}{4}\right) \\ <x^2> &= 0.283L^2 \end{aligned} \tag{6.4.14}$$

So the uncertainty in position is:

$$\begin{aligned} \sigma_x &= \sqrt{0.283L^2 - (0.5L)^2} \\ \sigma_x &= \sqrt{0.283L^2 - 0.25L^2} \\ \sigma_x &= 0.182L \end{aligned} \tag{6.4.15}$$

The uncertainty in momentum is:

$$\sigma_p = \sqrt{<p^2> - <p>^2} \tag{6.4.16}$$

and the expectation value of p2 is:

$$\begin{aligned} <p^2>&= \int_0^L \left(\sqrt{\frac{2}{L}}\right) \sin bigg(\frac{\pi x}{L})\bigg)\left(-\hbar^2 \frac{d^2}{dx^2}\left(\sqrt{\frac{2}{L}}\right)\sin bigg(\frac{\pi x}{L}\right)\bigg) \\ <p^2>&= -\hbar^2 \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\left(-\frac{\pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right)\right)dx \\ <p^2>&= \frac{2\pi^2\hbar^2}{L^3}\int_0^L \sin^2\left(\frac{\pi x}{L}\right)dx \\ <p^2>&= \frac{2\pi\hbar^2}{L^2}\int_0^\pi \sin^2(u)\left(\frac{L}{\pi}du\right) \\ <p^2>&=<p^2>= \frac{2\pi\hbar^2}{L^2}\int_0^\pi \sin^2(u)du \\ <p^> &= \frac{2\pi\hbar^2}{L^2}\left(\frac{\pi}{2}\right) \\ <p^2>&= \frac{\pi^2\hbar^2}{L^2} \\ <p^2>&= \frac{h^2}{4L^2} \end{aligned} \tag{6.4.17}$$

Does this result look familiar? If not, compare it to the ground state energy …

So the uncertainty in momentum is:

$$\begin{aligned} \sigma_p &= \sqrt{\left(\frac{h^2}{4L^2}\right) - (0)^2} \\ \sigma_p &= \frac{h}{2L} \end{aligned} \tag{6.4.18}$$

Note that the uncertainty in the momentum is actually equal to the absolute value of the momentum. (The electron has a wavelength of 2L, so the above expression is actually just DeBroglie's relationship for the momentum of the electron.) This can be interpreted as the electron having a momentum magnitude of h/2L but having an unknown direction for this momentum. Thus the momentum is:

$$p = 0 \pm \frac{h}{2L} \tag{6.4.19}$$

Finally, we can verify that the uncertainty in position and momentum are consistent with the uncertainty principle:

$$\begin{aligned} &\sigma_x\sigma_p \geq \hbar/2 (0.182 \text{ L})\left(\frac{h}{2L}\right) \geq \frac{h}{4\pi} \\ &0.091h \geq 0.080h \end{aligned} \tag{6.4.20}$$

Heisenberg can sleep easy since the ground state of the infinite well displays no violation of his principle!

6.5: The 2D Infinite Square Well

Twelve electrons are trapped in a two-dimensional infinite potential well of x-length 0.40 nm and y-width 0.20 nm. Find the total kinetic energy of the system.

Since the x- and y-directions in space are independent, Schrödinger's equation can be separated into an x-equation and a y-equation. The solutions to these equations are identical to the one-dimensional infinite square well. Thus, the allowed energy states of a particle of mass m trapped in a two-dimensional infinite potential well can be written as:

$$\begin{aligned} E &= n_x^2 \frac{(hc)^2}{8mc^2 L_x^2} + n_y^2 \frac{(hc)^2}{8mc^2 L_y^2} \\ E &= \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right)\frac{(hc)^2}{8mc^2} \end{aligned} \tag{6.5.1}$$

with wavefunction:

$$\Psi(x, y) = A \sin\left(\frac{n_x \pi x}{L_x}\right) \sin\left(\frac{n_y \pi y}{L_y}\right) \tag{6.5.2}$$

Therefore, the allowed energy levels are given by

$$\begin{aligned} E &= \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right)\frac{(hc)^2}{8mc^2} \\ E &= \left(\frac{n_x^2}{4^2} + \frac{2^2}{L_y^2}\right)\frac{(1240 \text{ eV nm})^2}{8(5111000 \text{ eV})^2(0.1 \text{ nm})^2} \\ E &= \left(\frac{n_x^2}{16} + \frac{n_y^2}{4}\right)37.6 \text{ eV} \end{aligned} \tag{6.5.3}$$

Rather than deal with fractions, multiply and divide by 16:

$$E = (n_x^2 + 4n_y^2)2.35 \text{ eV} \tag{6.5.4}$$

To help calculate the total kinetic energy of the system, list the first few lowest allowed energy states:

Level	n_x	n_y	E (2.35 eV)	# electrons
1	1	1	5	2
2	2	1	8	2
3	3	1	13	2
4	1	2	17	2
5	2 4	2 1	20	4
6	3	2	25	0

(6.5.5)

The states (nx, ny) = (2, 2) and (4, 1) are termed degenerate because two completely different wavefunctions have the same energy. The state (2, 2) looks like this:

while the state (4, 1) looks like this:

Since these are different wavefunctions, two electrons (spin up and spin down) can occupy each state. Thus, four electrons have the same energy. Thus, the twelve electrons will have total kinetic energy of

$$KE = [2(5) + 2(8) + 2(13) + 2(17) + 4(20)] 2.35 \text{ eV} = 390 \text{ eV} \tag{6.5.6}$$

6.6: Solving the 1D Semi-Infinite Square Well

Imagine a particle trapped in a one-dimensional well of length L. Inside the well there is no potential energy. However, the "right-hand wall" of the well (and the region beyond this wall) has a finite potential energy. This means that it is possible for the particle to escape the well if it had enough energy.

Again, since this potential is a piece-wise function, Schrödinger's equation must be solved in the three regions separately.
In the region x < 0, we have already seen that since the potential is infinite there is no chance of finding the particle in this region. Thus, Ψ = 0 in this region.
In the region 0 < x < L, the equation and solution should look familiar:

$$\begin{aligned} -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)+U(x)\Psi(x) &= E\Psi(x) \\ -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)+(0)\Psi(x) &= E\Psi(x) \\ -\frac{\hbar}{2m}\frac{d}{dx^2}\Psi(x) &= E\Psi(x) \\ \frac{d^2}{dx^2}\Psi(x) &= -\frac{2mE}{\hbar^2}\Psi(x) \end{aligned} \tag{6.6.1}$$

The general solution to this equation is

$$\Psi(x) = A\sin(kx)+B\cos(kx)\text{with}k=\sqrt{\frac{2mE}{\hbar^2}} \tag{6.6.2}$$

In order for this solution to be continuous with the solution for x < 0, the coefficient B must equal zero. Thus,

$$\Psi(x) = A\sin(kx) \tag{6.6.3}$$

In the region x > L, the equation is:

$$\begin{aligned} -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)+U(x)\Psi(x)=E\Psi(x) \\ -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)=(E-U)\Psi(x) \\ \frac{d^2}{dx^2}\Psi(x)=\frac{2m(U-E)}{\hbar^2}\Psi(x) \end{aligned} \tag{6.6.4}$$

The general solution is:

$$\Psi(x) = Ce^{aX}+De^{-aX} \text{ with } \alpha=\sqrt{\frac{2m(U-E)}{\hbar^2}} \tag{6.6.5}$$

Since this region contains the point x = +∞, C must equal zero or the wavefunction will diverge. Therefore,

$$\Psi(x) = De^{-aX} \tag{6.6.6}$$

The wave function, and the derivative of the wave function, must be continuous across the boundary at x = L. Forcing continuity leads to:

$$\begin{aligned} \Psi(x=L^-) &= \Psi(x=L^+) \\ A\sin(kL) &= De^{-aL} \end{aligned} \tag{6.6.7}$$

and forcing the continuity of the derivative leads to:

$$\begin{aligned} \Psi(x=L^-) &= \Psi(x=L^+) \\ kA\cos(kL) &= -\alpha De^{-aL} \end{aligned} \tag{6.6.8}$$

Substituting the first equation into the second equation yields:

\[\begin}align} & kA\cos(kL) = - \alpha(A \sin(kL)) \\ \nonumber & \tan(kL) = - \dfrac{k}{\alpha} \\ \nonumber & \tan \bigg(\sqrt{\dfrac{2mE}{\hbar^2}L \bigg) = - \dfrac{

This last result is a transcendental equation for the allowed energy levels. If the potential energy and width of the well are known, the allowed energy levels can be determined by using a solver or graphing the function.

The 1D Semi-Infinite Well

Determine the allowed energy levels for a proton trapped in a semi-infinite square well of width 5.0 fm and depth 60 MeV.

Applying the previous result:

\tan \bigg \sqrt{\frac{2mc^2L^2E}{(\hbar c)^2}} \bigg) = - \sqrt{\frac{E}{U-E}}

\tan \bigg \sqrt{\frac{2(938 \text{ MeV} (5.0 \text{ fm})^2 E}{(194.7 \text{ MeV fm})^2}} \bigg) = - \sqrt{\frac{E}{60-E}}

\tan \bigg \sqrt{1.204 E} \bigg) = - \sqrt{\frac{E}{60-E}}

with E in MeV.
The solutions to this equation, which represent the allowed energy levels for the proton, are 6.53, 25.75, and 55.08 MeV.

6.7: Barrier Penetration and Tunneling

Estimate the tunneling probability for an 10 MeV proton incident on a potential barrier of height 20 MeV and width 5 fm.

Consider the square barrier shown above. In the regions x < 0 and x > L the wavefunction has the oscillatory behavior we've seen before, and can be modeled by linear combinations of sines and cosines. These regions are referred to as allowed regions because the kinetic energy of the particle (KE = E – U) is a real, positive value.

Now consider the region 0 < x < L. In this region, the wavefunction decreases exponentially, and takes the form

$$\Psi(x) = Ae^{-\alpha X} \tag{6.7.1}$$

This is referred to as a forbidden region since the kinetic energy is negative, which is forbidden in classical physics. However, the probability of finding the particle in this region is not zero but rather is given by:

$$P(x) = A^2 e^{-2aX} \tag{6.7.2}$$

Thus, the particle can penetrate into the forbidden region. If the particle penetrates through the entire forbidden region, it can "appear" in the allowed region x > L. This is referred to as quantum tunneling and illustrates one of the most fundamental distinctions between the classical and quantum worlds.

A typical measure of the extent of an exponential function is the distance over which it drops to 1/e of its original value. This occurs when $x = \frac{1}{2a}$. This distance, called the penetration depth, δ, is given by

$$\delta = \frac{1}{2\alpha} \tag{6.7.3}$$

$$\delta = \frac{\hbar x}{\sqrt{8mc^2(U-E)}} \tag{6.7.4}$$

where

- U is the depth of the potential and
- E is the energy state of the wavefunction.

The penetration depth defines the approximate distance that a wavefunction extends into a forbidden region of a potential. Using this definition, the tunneling probability (T), the probability that a particle can tunnel through a classically impermeable barrier, is given by

$$T \approx e^{-x/\delta} \tag{6.7.5}$$

For this example, the probability that the proton can pass through the barrier is

$$\delta = \frac{\hbar c}{\sqrt{8mc^2(U-E)}} \tag{6.7.6}$$

$$\delta = \frac{197.3 \text{ MeVfm}}{\sqrt{8(938 \text{ MeV}}} (20 \text{ MeV -10 MeV}) \tag{6.7.7}$$

$$\delta = 0.720 \text{ fm} \tag{6.7.8}$$

Thus, there is about a one-in-a-thousand chance that the proton will tunnel through the barrier.

Tunneling In and Out

In a crude approximation of a collision between a proton and a heavy nucleus, imagine an 10 MeV proton incident on a symmetric potential well of barrier height 20 MeV, barrier width 5 fm, well depth -50 MeV, and well width 15 fm. Estimate the probability that the proton tunnels into the well. If the proton successfully tunnels into the well, estimate the lifetime of the resulting state.

In this approximation of nuclear fusion, an incoming proton can tunnel into a pre-existing nuclear well. Once in the well, the proton will remain for a certain amount of time until it tunnels back out of the well. Although the potential outside of the well is due to electric repulsion, which has the 1/r dependence shown below,

we will approximate it by a rectangular barrier:

The tunneling probability into the well was calculated above and found to be

$$T \approx 0.97x10^{-3} \tag{6.7.9}$$

All that remains is to determine how long this proton will remain in the well until tunneling back out.
First, notice that the probability of tunneling out of the well is exactly equal to the probability of tunneling in, since all of the parameters of the barrier are exactly the same.

Remember, T is now the probability of escape per collision with a well wall, so the inverse of T must be the number of collisions needed, on average, to escape. If we can determine the number of seconds between collisions, the product of this number and the inverse of T should be the lifetime ( ) of the state:

The time per collision is just the time needed for the proton to traverse the well. This is simply the width of the well (L) divided by the speed of the proton:

$$\tau = \left(\frac{L}{v}\right)\left(\frac{1}{T}\right) \tag{6.7.10}$$

The speed of the proton can be determined by relativity,

$$KE = (\gamma - 1)mc^2 \tag{6.7.11}$$

$$60 \text{ MeV} = (\gamma - 1)(938.3 \text{ MeV} \tag{6.7.12}$$

$$\gamma = 1.064 \tag{6.7.13}$$

$$v = 0.34c \tag{6.7.14}$$

$$v = 1.0x10^8 \text{ m/s} \tag{6.7.15}$$

Therefore the lifetime of the state is:

$$\tau = \left(\frac{15x10^{-15} \text{ m}}{1.0x10^8 \text{ m/s}}\right)\left(\frac{1}{0.97x10^{-3}}\right) \tag{6.7.16}$$

$$\tau = 1.5x10^{-19} \text{ s} \tag{6.7.17}$$

6.8: The Time-Dependent Schrödinger Equation

In this chapter, we investigated solutions of the time-independent Schrödinger equation. These solutions are offered referred to as stationary states because their spatial shape does not change with time, leading to probabilities that are constant in time. However, they are not "stationary" in the sense of having no time dependence. Imagine a guitar string vibrating in its fundamental mode. The "shape" of the string is constant; it just vibrates back and forth through space. To explore the rate at which the quantum wavefunction "vibrates" we need to solve the time-dependent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x^2}\Psi(x,t)+U(x)\Psi(x,t)=i\hbar\frac{\delta}{\delta t}\Psi(x,t) \tag{6.8.1}$$

Note that the wavefunction is now a function of both space and time, and the derivatives in the equation are partial derivatives. Also note that the potential energy function, U, is constant with respect to time.

Let's try to solve this partial differential equation by separation of variables. To do this, we'll assume the solution takes the form:

$$\Psi(x,t)=\psi(x)T(t) \tag{6.8.2}$$

Substituting this into the differential equation yields:

$$-\frac{\hbar^2}{2m}T(t)\frac{d^2}{dx^2}\psi(x)+U(x)\psi(x)T(t)=i\hbar\psi(x)\frac{d}{dt}T(t) \tag{6.8.3}$$

Dividing both sides by the wavefunction gives:

$$-\frac{\hbar^2}{2m}\frac{\frac{d^2\psi(x)}{dx^2}}{\psi(x)}+U(x)=i\hbar\frac{\frac{dT(t)}{dt}}{T(t)} \tag{6.8.4}$$

Since the left-hand side is only a function of x and the right-hand side is only a function of t, they can only be equal if both sides equal a constant value. If we call that constant E,

$$-\frac{\hbar^2}{2m}\frac{\frac{d^2\psi(x)}{dx^2}}{\psi(x)}+U(x)=E=i\hbar\frac{\frac{dT(t)}{dt}}{T(t)} \tag{6.8.5}$$

the left-side becomes the time-independent Schrödinger equation and the right-hand side becomes:

$$i\hbar\frac{dT(t)}{dt}=ET(t) \tag{6.8.6}$$

$$\frac{dT(t)}{dt}=-i\omega T(t)\text{ with }\omega=\frac{E}{\hbar} \tag{6.8.7}$$

This equation has the solution

$$T(t)=Ae^{-i\omega t} \tag{6.8.8}$$

Although you may not be familiar with imaginary arguments in the exponential function, mathematicians will tell you it is a really cool way to compactly write the sine and cosine functions:

$$T(t)=Ae^{-i\omega t}=A\left[\cos(\omega t)-i\sin(\omega t)\right] \tag{6.8.9}$$

So what does this mean?

Sines and cosines are familiar functions used to describe oscillations, so this means that the wavefunction simply oscillates at an angular frequency () proportional to the energy. Unlike a guitar string, however, this oscillation is not through physical space, but rather through some much more abstract space in which the square of the amplitude of oscillation is related to probability. Moreover, this space cannot be adequately represented without using imaginary numbers.

You may be concerned that the imaginary nature of the wavefunction will somehow creep into the probability of measuring some physical quantity. You shouldn't be. Although we've stated numerous times that probabilities depend on the square of the wavefunction, actually they depend on the product of the wavefunction and its complex conjugate . Thus, the effect of the temporal part of the wavefunction on all probabilities is given by:

$$\begin{aligned} \mathrm{P}rob(t) &= T\backslash*(t)T(t) \\ \mathrm{P}rob(t) &= (Ae^{+\omega t})(Ae^{-i\omega t}) \\ \mathrm{P}rob(t) &= A^2e^0 \\ \mathrm{P}rob(t) &= A^2 \end{aligned} \tag{6.8.10}$$

In fact, if we set A = 1, the temporal part of the wavefunction will have no effect on the probabilities calculated earlier in this chapter. Thus, as long as the potential energy function is constant in time, Schrödinger's equation is separable and all of our work studying the time-independent equation is valid, as long as we remember that these solutions are actually oscillating in time according to the description given above.

6.9: The Schrödinger Equation Activities

A 1.0 kg ball is thrown directly upward at 10 m/s and the zero-point of gravitational potential energy is located at the position at which the ball leaves the thrower's hand. All heights are measured relative to the zero-point. Below is a graph of gravitational potential energy (U = mgh) vs. height. Answer the following questions.

a. Draw a line on the graph representing the total energy of the ball. Using the graph, determine the maximum height reached by the ball.

b. Using the graph, determine the kinetic energy of the ball when it is at one-half of its maximum height.

c. What would happen to the kinetic energy of the ball if it was at 1 m above its maximum height. What would this imply about the velocity of the ball at this location?

d. Does the ball spend more time, less time or the same amount time in the position interval between 3 m and 4 m or in the position interval between 4 m and 5 m? Why?

e. If a determination of the ball's position is made at a random time, is it more likely, less likely or equally likely to find the ball in the position interval between 3 m and 4 m or in the position interval between 4 m and 5 m? Why?
A 2.0 kg cart oscillates on a horizontal, frictionless surface attached to a k = 50 N/m spring. The cart passes through the spring equilibrium length (s = 0 m) at a speed of 10 m/s. Below is a graph of elastic potential energy (U = ½ ks2) vs. spring deformation (s). Answer the following questions.

a. Draw a line on the graph representing the total energy of the cart. Using the graph, determine the maximum elongation of the spring.

b. Using the graph, determine the kinetic energy of the cart when the spring is at one-half of its maximum elongation.

c. What would happen to the kinetic energy of the cart if it were 1m beyond the maximum elongation of the spring? What would this imply about the velocity of the cart at this location?

d. Does the cart spend more time, less time or the same amount time in the position interval between 0 m and 1 m or in the position interval between 1 m and 2 m? Why?

e. If a determination of the cart's position is made at a random time, is it more likely, less likely or equally likely to find the cart in the position interval between 0 m and 1 m or in the position interval between 1 m and 2 m? Why?

Below is a graph of potential energy (U) vs. position for a region of space occupied by a single macroscopic particle. There are no forces acting on the particle in this region of space other than the force that gives rise to the potential energy. The total energy of the particle is also indicated on the graph. The letters A through F mark six positions within this region of space.

a. Rank the potential energy of the particle at these positions.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

b. Rank the kinetic energy of the particle at these positions.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

c. If a determination of the particle's position is made at a random time, rank the probability of finding the particle in the immediate vicinity of these positions.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your rankings:

Below is a graph of potential energy (U) vs. position for a region of space occupied by a single macroscopic particle. There are no forces acting on the particle in this region of space other than the force that gives rise to the potential energy. The total energy of the particle is also indicated on the graph. The letters A through F mark six positions within this region of space.

a. Rank the potential energy of the particle at these positions. If the particle is never at a position leave it out of the ranking.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

b. Rank the kinetic energy of the particle at these positions. If the particle is never at a position leave it out of the ranking.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

c. If a determination of the particle's position is made at a random time, rank the probability of finding the particle in the immediate vicinity of these positions.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your rankings:

For each of the potential energy functions below, carefully sketch the wavefunction corresponding to the energy level indicated.

a.

b.

c.

d.

For each of the potential energy functions below, carefully sketch the wavefunction corresponding to the energy level indicated.

a.

b.

c.

d.

For each of the potential energy functions below, carefully sketch the continuation of the wavefunction as it passes the barrier or well.

a.

b.

c.

d.

A hypothetical atom has the four electron energy levels shown below.

a. How many spectral lines can be emitted by transitions between these four energy levels?

b. The transition from level ____ to level ____ emits the longest wavelength photon.

c. The transition from level ____ to level ____ involves the absorption of the largest energy photon.

d. If the energy of the n = 3 level was somehow reduced, how many of the six spectral lines would change energy?

e. If the energy of the n = 3 level was somehow reduced, which electron transition(s) would change to larger energy?

f. In which level(s) is the electron most likely to be detected in the immediate vicinity of x = 0?

g. In which level(s) does the electron spend the most time outside of the atom?

g. In which level(s) is the electron most likely to be detected within the right half of the atom?

A proton is trapped in an infinite potential well of length 1.0 x 10-15 m. Find the three longest wavelength photons emitted by the proton as it changes energy levels in the well.

Mathematical Analysis

An electron is trapped in an infinite potential well of length 1.0 x 10-10 m. Find the three longest wavelength photons emitted by the electron as it changes energy levels in the well.

Mathematical Analysis

Imagine eight neutrons trapped in an atomic nucleus. Model the nucleus as an infinite potential well of length 5.0 x 10-15 m.
a. Find the total kinetic energy of the eight neutrons.
b. If four of the neutrons changed into protons, calculate the new total kinetic energy. Ignore the mass difference between protons and neutrons.

Mathematical Analysis

Imagine eight neutrons and five protons trapped in an atomic nucleus. Model the nucleus as an infinite potential well of length 5.0 x 10-15 m.
a. Find the total kinetic energy of the nucleons.
b. If neutrons and protons can freely change into each other, what will happen?
c. Calculate the minimum total kinetic energy of the nucleons. Ignore the mass difference between protons and neutrons.

Mathematical Analysis

A photon is trapped in an infinite potential well of length L. Find the allowed energies for the photon. (Hint: You cannot use the Schrödinger equation to solve this problem.)

Mathematical Analysis

You may occasionally feel trapped in a classroom. If so, you may find yourself unable to be completely stationary. Assuming a 10 m wide classroom, and a 65 kg student, estimate your minimum energy and velocity.

Mathematical Analysis

An electron is trapped in an infinite potential well of length 1.0 x 10-10 m.
a. According to classical physics (i.e., common sense), what is the probability that the electron will be found in the middle fifth

(from 0.4 x 10-10 m to 0.6 x 10-10 m) of the well?
b. In its ground state, what is the probability that the electron will be found in the middle fifth of the well?
c. In its n=2 state, what is the probability that the electron will be found in the middle fifth of the well?
d. Based on the shape of the wavefunction, explain why (b) is greater than (a), and (a) is greater than (c)? (If they aren't, you did the problem incorrectly!)

Mathematical Analysis

A neutron is trapped in an infinite potential well of length 4.0 x 10-15 m.
a. According to classical physics (i.e., common sense), what is the probability that the neutron will be found in the left quarter (from 0.0 x 10-15 m to 1.0 x 10-15 m) of the well?
b. In its ground state, what is the probability that the neutron will be found in the left quarter of the well?
c. In its n=2 state, what is the probability that the neutron will be found in the left quarter of the well?
d. Based on the shape of the wavefunctions, do your answers for a, b, and c have the correct relative size?

Mathematical Analysis

A particle is trapped in a one-dimensional infinite potential well of length L.
a. Find the expectation values of the particle's position and momentum in the first excited state of this well. Compare these results to the results for the ground state.
b. Find the uncertainty in the particle's position and momentum in the first excited state of this well. Compare these results to the results for the ground state.
c. Show that the uncertainties in these values do not violate the uncertainty principle.

Mathematical Analysis

A particle is trapped in a one-dimensional infinite potential well of length L.
a. Find the expectation value of the particle's position as a function of energy level, n.
b. Find the uncertainty in the particle's position as a function of energy level, n.
c. In the limit of very large n, what is the uncertainty in the particle's position? (In classical physics, this value would be 0.289L.)

Mathematical Analysis

A particle is trapped in a one-dimensional infinite potential well of length L.
a. Find the expectation value of the particle's kinetic energy in the ground state of this well. (Hint: Kinetic energy can be expressed as p2/2m. This should allow you to construct an operator for kinetic energy.)
b. Find the uncertainty in the particle's kinetic energy in the ground state of this well.
c. Carefully explain what your answer for (b) implies about the time the particle can remain in the ground state.

Mathematical Analysis

Classically, a particle trapped in a one-dimensional infinite potential well of length L would have an equal probability of being detected anywhere in the well. Thus, its "classical" wavefunction would be:

where C is a constant.
a. Find the value of C by setting the total probability of finding the particle in the well equal to 1.
b. Find the expectation value of the particle's position.
c. Find the uncertainty in the particle's position.

Mathematical Analysis

A particle trapped in a one-dimensional parabolic potential well centered on x = 0 has a ground-state wavefunction given by:

Note that this type of well extends from x = -∞ to x = +∞.
a. Find A.
b. Find the expectation values of the particle's position and momentum in the ground state of this well.
c. Find the uncertainty in the particle's position and momentum in the ground state of this well.

d. Show that the uncertainties in these values do not violate the uncertainty principle.

Mathematical Analysis

A particle of mass m is trapped in an infinite potential well of x-length L and y-width 3L. For each of the pairs of quantum numbers below, state the locations (other than the boundaries) where the probability of detecting the particle is zero.
a. (nx, ny) = (1, 1)
b. (nx, ny) = (3, 1)
c. (nx, ny) = (2, 3)

Mathematical Analysis

A particle of mass m is trapped in an infinite potential well of x-length L, y-width L, and z-height 2L. For each of the triplets of quantum numbers below, state the locations (other than the boundaries) where the probability of detecting the particle is zero.
a. (nx, ny, nz) = (1, 1, 1)
b. (nx, ny, nz) = (2, 2, 1)
c. (nx, ny, nz) = (1, 2, 2)

Mathematical Analysis

A particle of mass m is trapped in an infinite potential well of x-length L and y-width L. Determine the 5 lowest energy levels and list them below.

Mathematical Analysis

Level nx ny E ( )
1 1 1

A particle of mass m is trapped in an infinite potential well of x-length 3L and y-width 2L. Determine the 5 lowest energy levels and list them below.

Mathematical Analysis

Level nx ny E ( )
1 1 1

Eight neutrons and five protons are trapped in an infinite potential well of x-length 4.0 x 10-15 m and y-width 4.0 x 10-15 m. Ignore the mass difference between protons and neutrons.
a. Determine the 5 lowest energy levels and list them below.
b. Find the total kinetic energy of the 13 particles.
c. Would the total energy decrease if a neutron turned into a proton? If so, by how much?

Mathematical Analysis

Level nx ny E ( ) # neutrons # protons
1 1 1 2 2

Eight protons and eleven neutrons are trapped in an infinite potential well of x-length 4.0 x 10-15 m and y-width 2.0 x 10-15 m. Ignore the mass difference between protons and neutrons.

a. Determine the 5 lowest energy levels and list them below.

b. Find the total kinetic energy of the 19 particles.

c. Would the total energy decrease if a neutron turned into a proton? If so, by how much?

Mathematical Analysis

Level nx ny E ( ) # neutrons # protons
1 1 1 2 2

A particle of mass m is trapped in an infinite potential well of x-length L, y-width L, and z-height L. Determine the 5 lowest energy levels and list them below.

Mathematical Analysis

Level nx ny nz E ( )
1 1 1 1

A particle of mass m is trapped in an infinite potential well of x-length L, y-width L, and z-height 2L. Determine the 5 lowest energy levels and list them below.

Mathematical Analysis

Level nx ny nz E ( )
1 1 1 1

A particle of mass m is trapped in an infinite potential well of x-length L, y-width 2L, and z-height 2L. Determine the 5 lowest energy levels and list them below.

Mathematical Analysis

Level nx ny nz E ( )
1 1 1 1

Find the allowed energy levels for a proton trapped in a semi-infinite potential well of width 3.0 fm and depth 40 MeV. Compare these values to those obtained assuming the well is infinitely deep.

Mathematical Analysis

Find the allowed energy levels for an electron trapped in a semi-infinite potential well of width 1.0 nm and depth 5.0 eV. Compare these values to those obtained assuming the well is infinitely deep.

Mathematical Analysis

Find the allowed energy levels for a neutron trapped in a semi-infinite potential well of width 7.0 fm and depth 50 MeV. Compare these values to those obtained assuming the well is infinitely deep.

Mathematical Analysis

A proton is incident on a rectangular potential barrier of height 50 MeV and width 5 fm. What is the approximate probability that the proton will tunnel through the barrier for each of the incident kinetic energies below?
a. 20 MeV
b. 30 MeV
c. 40 MeV
d. 45 MeV

Mathematical Analysis

a. Estimate the tunneling probability for an 18 MeV proton incident on a symmetric potential well with barrier height 20 MeV, barrier width 3 fm, well depth -50 MeV, and well width 15 fm.
b. If the proton successfully tunnels into the well, estimate the lifetime of the resulting state.

Mathematical Analysis

a. Estimate the tunneling probability for a 5 MeV alpha particle incident on a symmetric potential well with barrier height 40 MeV, barrier width 8 fm, well depth -50 MeV, and well width 15 fm.
b. If the proton successfully tunnels into the well, estimate the lifetime of the resulting state.

Mathematical Analysis

a. Estimate the tunneling probability for a 1.0 MeV proton incident on a symmetric potential well with barrier height 1.5 MeV, barrier width 1 fm, well depth -10 MeV, and well width 3.0 fm.
b. If the proton successfully tunnels into the well, estimate the lifetime of the resulting state.

Mathematical Analysis

In a sparkplug, a potential difference of about 20,000 V is needed for a spark to jump the 1.5 mm gap. Modeling this as a rectangular potential barrier of height 20 keV and width 1.5 mm:
a. Estimate the probability of an electron tunneling across the sparkplug gap when the potential difference across the gap is only 10,000 V. Based on this answer, should auto mechanics worry about quantum mechanics?
b. At what potential difference would tunneling provide a one-in-a-billion chance of a premature spark?

Mathematical Analysis

In a transistor a rule of thumb is “60 mV per decade”, meaning that a voltage change of 60 mV should cause a tenfold increase (or decrease) in current. In a tunneling transistor we can image a rectangular potential barrier of width 10 nm with a height that can be adjusted by the applied voltage. What height barrier is needed so that a change of 60 mV (60meV of energy) causes a tenfold change in current?

Mathematical Analysis

In a scanning tunneling microscope (STM), a slender metal tip is positioned very close to a sample under study. Although no electric contact is made between tip and sample, electrons from the tip can tunnel across the empty space to the sample, resulting in an electric current. Since this current is exponentially dependent on the separation between the tip and the sample, incredibly precise measurements of surface features are possible.

Approximating the potential barrier between tip and sample to be a rectangular barrier with height equal to a typical metallic work function (4.0 eV), find the change in separation between tip and sample that will result in a 10% change in tunneling current. This change in separation is approximately the resolution of the STM.

Mathematical Analysis

In an ammonia molecule (NH3), the nitrogen atom is equally likely to be “above” or “below” the plane formed by the three hydrogen atoms. In fact, the nitrogen atom tunnels back and forth between these two equivalent orientations at an incredibly stable frequency. The stable frequency of ammonia inversion was used as the standard for the first generation of atomic clocks.

Approximating the potential barrier between the above and below orientations as a rectangular barrier with height U = 0.26 eV, and width L = 0.038 nm, find the frequency with which the nitrogen oscillates between these two states. The energy of the nitrogen atom in either well is E = 0.25 eV.

Mathematical Analysis

6.A: Solving the Finite Well (Project)

Imagine a particle trapped in a one-dimensional well of length 2L. Inside the well there is no potential energy while the region outside the well has a finite potential energy. This potential energy function is referred to as the finite square well.

I. General Solution

Since the potential energy is a piece-wise function, Schrödinger’s equation must be solved in the three regions separately.

A. Region I: x < -L
1. Apply and solve Schrödinger’s equation in region I. Your solution should have two arbitrary constants. Make use of the definition:

2. Clearly explain why one of your two constants must equal zero, and then record your simplified solution below.

B. Region II: -L < x < L
1. Apply and solve Schrödinger’s equation in region II. Your solution should have two arbitrary constants. Make use of the definition:

C. Region III: x > L
1. Apply and solve Schrödinger’s equation in region III. Your solution should have two arbitrary constants. Again make use of the definition:

2. Clearly explain why one of your two constants must equal zero, and then record your simplified solution below.

D. Sketching Solutions
1. Sketch the wavefunctions of the four lowest energy states on the diagram below.

Your sketched solutions should fall into one of two categories. Symmetric solutions (n = 1, 3, …) are those that are symmetric about the y-axis. Antisymmetric solutions (n = 2, 4, …) are, you guessed it, antisymmetric when reflected about the y-axis. We will solve for these two types of solutions separately.

E. Symmetric Solutions
1. Clearly explain why one of the two constants in region II must equal zero for symmetric solutions.

2. What is the relationship between the constant in region I and the constant in region III for symmetric solutions?

3. Record your simplified solutions in the three regions below.
Region I:

Region II:

Region III:

The wave function, and the derivative of the wave function, must be continuous across the boundary at x = L. This will lead to two simultaneous equations that can be solved for the energy levels of symmetric solutions.
4. Force the wavefunction to be continuous across the x = L boundary and simplify the resulting expression.

5. Force the derivative of the wavefunction to be continuous across the x = L boundary and simplify the resulting expression.

6. Solve the two equations above for an expression involving □ and k. Simplify this expression into a transcendental equation for the energy levels of symmetric solutions.

F. Antisymmetric Solutions
1. Clearly explain why one of the two constants in region II must equal zero for antisymmetric solutions.

2. What is the relationship between the constant in region I and the constant in region III for antisymmetric solutions?

3. Record your simplified solutions in the three regions below.
Region I:

Region II:

Region III:

4. Force the wavefunction to be continuous across the x = L boundary and simplify the resulting expression.

5. Force the derivative of the wavefunction to be continuous across the x = L boundary and simplify the resulting expression.

6. Solve the two equations above for an expression involving and k. Simplify this expression into a transcendental equation for the energy levels of antisymmetric solutions.

II. Specific Solution

Find all the allowed energy levels for an electron trapped in a finite square well of total width 2.00 nm and depth 1.00 eV.

6.A: Solving the Hydrogen Atom (Project)

Enough with pretending atoms are three-dimensional, infinite square wells! It's time to tackle an atom for real. (Before we get too excited, the atom under analysis is hydrogen. All other atoms are impossible to solve analytically.)

I. Schrödinger's Equation in Spherical Coordinates

The time-independent Schrödinger's equation in Cartesian coordinates is:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi(x,y,z)}{\partial x^2}+\frac{\partial^2\Psi(x,y,z)}{\partial y^2}+\frac{\partial^2\Psi(x,y,z)}{\partial z^2}\right)+U(x,y,z)\Psi(x,y,z)=E\Psi(x,y,z) \tag{6.A.1}$$

Of course, this is not really the best coordinate system to use to address the hydrogen atom. A better coordinate system is spherical coordinates. In spherical coordinates, the Schrödinger's equation for the hydrogen atom looks like this:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi(r,\theta,\phi)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi(r,\theta,\phi)}{\partial\theta}\right)+\frac{R(r)}{r^2\sin^2\theta}\frac{\partial^2\Psi(r,\theta,\phi)}{\partial\phi^2}\right]+U(r,\theta,\phi)\Psi(r,\theta,\phi) \\ =E\Psi(r,\theta,\phi) \tag{6.A.2}$$

Granted, this doesn't make the equation look very nice, but it makes the solution of the equation possible. If you want to know more about why the spatial derivative part looks like this, go talk to a math professor.

II. Solving Schrödinger's Equation for Hydrogen

For hydrogen, the potential energy function is simply:

$$U(r,\theta,\phi)=-\frac{ke^2}{r} \tag{6.A.3}$$

Since the potential energy only depends on r, perhaps we can separate the r-dependence in the equation from the angular dependence. Let's assume:

$$\Psi(r,\theta,\phi)=R(r)Y(\theta,\phi) \tag{6.A.4}$$

Substituting these in gives:

$$-\frac{\hbar^2}{2m}\left[\frac{Y(\theta,\phi)}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{R(r)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{R(r)}{r^2\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right]-\frac{ke^2}{r}R(r)Y(\theta,\phi) \\ =ER(r)Y(\theta,\phi) \tag{6.A.5}$$

Next, divide through by the wavefunction, $R(r)Y(\theta,\phi)$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{1}{r^2Y(\theta,\phi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{1}{r^2Y(\theta,\phi)\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right]-\frac{ke^2}{r}=E \tag{6.A.6}$$

Move the potential energy to the right side and multiply through by $\frac{2mr^2}{\hbar^2}$

$$-\left[\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{1}{Y(\theta,\phi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{1}{Y(\theta,\phi)\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right]=\frac{2mr^2}{\hbar^2}\left(E+\frac{ke^2}{r}\right) \tag{6.A.7}$$

Move the remaining r-dependence to the right-hand side:

$$-\frac{1}{Y(\theta,\phi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)-\frac{1}{Y(\theta,\phi)\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}=\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left(E+\frac{ke^2}{r}\right) \tag{6.A.8}$$

The left-side of the equation is a function only of θ and ϕ, and the right-side is a function only of r. For the two sides to be equal, they must both equal the same constant. Making what may seem like an odd choice for this constant yields two differential equations. The radial equation:

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\frac{2mr^2}{\hbar^2}\left(E+\frac{ke^2}{r}\right)=\lambda(\lambda+1) \tag{6.A.9}$$

and the angular equation:

$$-\frac{1}{Y(\theta,\phi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)-\frac{1}{Y(\theta,\phi)\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}=\lambda(\lambda+1) \tag{6.A.10}$$

II.A: The Angular Equation

Re-arranging the angular equation a bit leads to:

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}=-\lambda(\lambda+1)Y(\theta,\phi)\sin^2\theta \tag{6.A.11}$$

You may not recognize this equation, but Laplace solved it in 1782 and named the solutions spherical harmonics. Just as sines and cosines can be used to model vibrations in rectangular coordinates (on guitar strings, in 3D infinite wells, etc.) spherical harmonics model vibrations in spherical coordinates. In some sense, you can imagine them as the fundamentally distinct ways in which a spherical surface can vibrate.

Spherical harmonics are labeled by a pair of integers, m and l, and typically written as $Y_\lambda^m(\theta,\phi$

- m is termed the magnetic quantum number and represents the number of complete waves that wrap around the sphere in the azimuthal (ϕ) direction. Thus, if m = 0, there is no change as you move around the sphere. If m = 1, one wavelength wraps around the sphere so that half of the sphere has positive “displacement” and the other half negative. m can also be negative. In this case, the displacements are flipped, and the “wave” travels the other way around the sphere.
- l is termed the orbital quantum number and controls the polar (θ) direction. The value of (l – |m|) represents the number of nodes in the polar direction. l cannot be negative.
- In ye olde fashioned chemistry notation, l = 0, 1, 2, 3, … are referred to as s, p, d, f, …
- |m| is always less than or equal to l.

The diagrams below may help you better visualize spherical harmonics. The nodal lines are clearly marked.

1. Draw the nodal lines for the spherical harmonics listed below. Label the sectors + or -.

The first few spherical harmonics are listed below.

$$Y_0^0(\theta,\phi)=\frac{1}{2}\sqrt{\frac{1}{\pi}} \tag{6.A.12}$$

$$Y_1^{-1}(\theta,\phi)=\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi} \tag{6.A.13}$$

$$Y_1^0(\theta,\phi)=\frac{1}{2}\sqrt{frac3\pi}\cos\theta \tag{6.A.14}$$

$$Y_1^1(\theta,\phi)=\frac{-1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi} \tag{6.A.15}$$

$$Y_2^{-2}(\theta,\phi)=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{-2i\phi} \tag{6.A.16}$$

$$Y_2^{-1}(\theta,\phi)=\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{-i\phi} \tag{6.A.17}$$

$$Y_2^0(\theta,\phi)=\frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta-1) \tag{6.A.18}$$

$$Y_2^1(\theta,\phi)=\frac{-1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{i\phi} \tag{6.A.19}$$

$$Y_2^2(\theta,\phi)=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi} \tag{6.A.20}$$

Note lack of node in θ direction
Note node at $\theta=\pi/2$

2. At what values of θ does $Y_2^0(\theta, \phi)$ have nodes?

3. $Y_3^1(\theta, \phi) = \frac{-1}{8}\sqrt{\frac{21}{\pi}}\sin\theta(5\cos^2\theta - 1)e^{i\theta}$. Where are the nodal lines?

These functions are all normalized so that the integral of $Y(\theta, phi)^2$ over the surface of the sphere is 1:

$$\int_0^{2\pi}\int_0^{\pi}(Y_i^m(\theta, \phi))^2 \sin\theta d\theta d\phi = 1 \tag{6.A.21}$$

4. $Y_3^{-3} C\sin^3(\theta)e^{-3i\phi}$. Find C by normalizing this spherical harmonic.

5. Can you see a trend in the spherical harmonics when l = |m|? Determine $Y_5^5(\theta, \phi)$, including the constant.

B. The Radial Equation

Re-arranging the radial equation a bit leads to:
\[\frac{d}{dr}\bigg(r^2\frac{dR(r)}{dr}\bigg)+\frac{2mr^2}{\hbar^2}\bigg(E+\frac{ke^2}{r}\bigg)R(r)-\lambda(\lambda+1)R(r)=0\)

If we define a0, termed the Bohr radius, as

$$a_0 = \frac{\hbar^2}{mke^2} = 5.29\times 10^{-11}\text{ m} \tag{6.A.22}$$

then

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \left[\frac{2mE}{\hbar^2}r^2 + \frac{2}{a_0}r - \lambda(\lambda+1)\right]R(r) = 0 \tag{6.A.23}$$

Again, you may not recognize this equation, but the 19th century mathematician Leguerre did and the solutions involve *Laguerre polynomials*. These solutions depend on both l and a new integer, n.

The radial wavefunction depends on a pair of integers, n and l, and is typically written as $R_{n\lambda}(r)$.
• All solutions are the product of a polynomial of degree (n -1) and the term \(e^{-r/na_0\).
• n is termed the principal quantum number and determines the spatial extent of the wavefunction and its energy. Since all solutions drop off as , large n states extend farther from the nucleus.
• The energy of the electron is dependent on n, and given by the formula:

$$E = -\frac{1}{n^2}\frac{ke^2}{2a_0} = -\frac{13.6\text{ eV}}{n^2} \tag{6.A.24}$$

• n is greater than zero, and l is always less than n.

The first few radial wavefunctions are listed below. (Note Z = 1 for hydrogen)

6. Which of the radial wavefunctions listed above are non-zero at the origin, r = 0? In ye olde fashioned chemistry notation, what are these wavefunctions called?

7. Which of the radial wavefunctions listed above have nodes, i.e., radii where the probability of finding the electron is zero? (Do not include r = 0 or r = ∞ as nodes.)

8. Find the nodes for each of the radial wavefunctions listed above, in terms of a0.

9. Can you spot a relationship between the number of nodes and the values of n and l? What is it?

These functions are all normalized so that the total probability of finding the electron somewhere is 1. In spherical coordinates, this means:

$$\int_0^{\infty}(R_{n\lambda}(r))^2 r^2 dr = 1 \tag{6.A.25}$$

10. $R_{43}(r) = C\left(\frac{r}{a_0}\right)^3 e^{-r/4a_0}$. Find C by normalizing this radial wavefunction.

The graphic below shows both the radial wavefunction and the radial probability distribution . You should be able to use this to check your node calculations.

Combining the radial wavefunctions with the spherical harmonics generates the complete wavefunction for the electron. If you want to see this in three dimensions, go stand in front of the chemistry labs. If you want to see two dimensional pictures, just look below.

Remember:
• The number of polar nodes is l –|m|
• The number of radial nodes is n – (l + 1)

III. Angular Momentum

The energy levels for hydrogen are the same as Bohr derived using a much simpler model. (In fact, you reproduced this derivation in the Emission Spectrum laboratory.) However, the angular momentum properties are quite different.

The vector expression for angular momentum is:

$$L = r \times p \tag{6.A.26}$$

Thus, a particle circling around the z-axis counter-clockwise would have angular momentum in the +z-direction.

Of course, thinking of the electron as having a well-defined trajectory like this will get you into trouble with Heisenberg.
A better picture would be the one below, at left. In this state, the electron is basically circling around the z-axis, but with quite a bit of "play" in its plane of rotation. We interpret this as the electron's angular momentum precessing around the z-axis, like a wobbly top. The z-component of its angular momentum is constant, but the total angular momentum vector traces a cone around the z-axis, providing a bit of angular momentum in both the x- and y-directions. Note that Lx and Ly are not well-defined like Lz. They each average to zero but at all times there is some angular momentum in the xy-plane.

Now consider the state at right. You should be able to see that the angular momentum vector lies in the xy-plane, with Lz = 0. (By the argument above you may believe that it occasionally "wanders" off the xy-plane creating a temporary z-component, which averages to zero, but since m = 0 there is no "wave" encircling the z-axis and hence Lz is exactly zero.)

Finally, the state at left has no angular momentum at all. The probability "cloud" is completely independent of both ▯ and ▯ so there is no "motion" in any angular direction.

Rather than examining the wavefunction for every possible situation conceptually, I'm now just going to tell you the formulas for the total angular momentum and the z-component of angular momentum:

$$|L| = \sqrt{\lambda(\lambda+1)}\hbar \tag{6.A.27}$$

$$L_s = m\hbar \tag{6.A.28}$$

When no external electric or magnetic fields are present, the allowed energy levels do not depend on the angular momentum state. For each value of l, there are (2l + 1) degenerate energy states.

In the l = 2 states illustrated at right, the magnitude of the angular momentum is always

$$|L| = \sqrt{2(2+1)}\hbar = 2.44\hbar \tag{6.A.29}$$

This represents the length of the vector precessing around the z-axis. The projection of this vector on the z-axis can take on one of five different values determined by m. With no external field to break the ±z-symmetry all five of these states have the same energy.

11. For each of the five states above, determine the angular momentum in the xy-plane.

All this talk about precessing around the z-axis may have you convinced that at any particular moment we can determine the exact direction of and therefore calculate not only the amount of angular momentum in the xy-plane, but the exact components in both x and y. However, if this was true (i.e., if we knew Lx, Ly, and Lz at the same time) it would mean we know both the position and linear momentum of the electron at the same time. Again, Heisenberg wouldn't like this! We need to think of the electron "cloud" as somehow precessing around the z-axis, with no definite direction at any particular time.

12. Consider the n = 4 state. What is the total degeneracy of this state? (Do not include electron spin.) List the angular momentum and z-component of angular momentum for each sub-state.

It may confuse you for me to say that all of the states above (4s, 4p, 4d, and 4f in chemistry notation) have the same energy, but for hydrogen they do. However, you've learned in chemistry that these states have different energies and are thus filled in a specific order. That's only the case for multi-electron atoms. Basically, if you incorporate the interactions of all the electrons the degeneracy between these states splits.

For example, s-state electrons have a radial wavefunction that is non-zero at the location of the nucleus while the other states don't. Thus, when the s-state electrons are closer to the nucleus than the other states, they effectively "screen" the nuclear charge so that the other states "see" less positive charge. This makes the other states less tightly bound. If there are Z protons in the nucleus, s-state electrons interact with all Z protons while p-states only "see" approximately Z - 2 protons, d-states "see" Z – 8, etc. This effect, electron screening, and other electron-electron interactions split the degeneracy present for a single-electron atom.

IV. Emission Spectrum

IV.A: Selection Rules

In the Bohr model of hydrogen, the emission spectrum is simple. Every time the electron drops from a higher to a lower energy level a photon is emitted with energy equal to the difference between these levels. However, the real situation is more complicated. One reason is that the photon has angular momentum.

Since the photon has angular momentum, and angular momentum is conserved, the atom can only undergo photon emission if the total angular momentum of the atom changes. This requirement creates a selection rule:

$$\Delta\lambda = \pm 1 \tag{6.A.30}$$

The angular momentum carried away by the photon can either come from the z-component or the xy-plane component:

$$\Delta m = 0, \pm 1 \tag{6.A.31}$$

Thus, s-states cannot decay to s-states, p-states cannot decay to p-states, etc. Some allowed transitions are illustrated at right.

13. Consider the state (4,1,1). List all decays from this state that are "allowed" by the selection rules above.

Decays that do not obey the selection rules above are not strictly forbidden, they just occur at greatly reduced rates.

IV.B: Zeeman Effect

Another complication to the simple emission spectrum described by Bohr occurs if the atom is in the presence of an external magnetic field, which for convenience we will imagine as oriented in the z-direction.

If m is not equal to 0, we can imagine the electron as orbiting around the z-axis. This orbiting electron creates a magnetic field that interacts with the external magnetic field. If these two fields are in opposite directions the electron "wants" to flip over and align with the external field. This can be quantified as an additional source of potential energy given by:

$$U = m\mu_B B \tag{6.A.32}$$

where μ_B is the Bohr Magneton, defined as

$$\mu_B = \frac{e\nu}{2m} \tag{6.A.33}$$

$$\mu_B = 5.79 \times 10^{-5}\ \mathrm{eV/T} \tag{6.A.34}$$

Imagine the 2p state illustrated at right in the presence of a magnetic field in the +z-direction. If the electron is orbiting counterclockwise (m = 1) it creates a magnetic field oriented in the –z-direction, since it is negatively charged. Since these fields have opposite orientation, this electron state has additional potential energy. The opposite is true for the m = -1 state. The m = 0 is unaffected by the magnetic field. Thus, the threefold degeneracy in the 2p state is broken in the presence of a magnetic field.

Due to this breaking of the energy degeneracy of 2p, rather than a single emission line corresponding to the 2p to 1s transition there are three closely spaced emission lines. This splitting of spectral lines due to the presence of an external magnetic field is termed the *Zeeman effect.*

14. Consider the 2p state in the presence of a 2.0 T magnetic field. If this state decays to the 1s state, find the shift in wavelength of the m = ±1 states relevant to the m = 0 state. What is this shift as a percentage of the original wavelength?

C. Intrinsic Spin

One last complication, I promise.

The angular momentum we have been discussing up to this point is termed orbital angular momentum and is due, not surprisingly given its name, to the orbital motion of the electron. In addition, the electron has an intrinsic angular momentum, commonly referred to as spin. This angular momentum is not really due to the electron “spinning” in space but rather represents a fundamental characteristic of the electron, like its charge or its mass.

The electron’s spin vector adds to its orbital angular momentum vector to form its total angular momentum vector, . All of these vectors may precess around the z-axis.

The spin vector obeys similar mathematics to the angular momentum vector, although the spin quantum number, s, is always equal to ½.

If you are following along you should now expect me to discuss how the z-portion of the spin can interact with an external magnetic field to further split each sub-state into two different energy levels, a spin-up and a spin-down sub-state. In fact, there is a splitting involving a potential energy term given by:

$$U=\pm\mu_B B \tag{6.A.35}$$

However, this splitting is not due to an external magnetic field but rather the magnetic field generated by the electron’s own orbital motion. If Lz and Sz are in the same direction (either both “up” or both “down”) the potential energy is positive while if they are in opposite directions the potential energy is negative. This interaction between the spin and angular momentum is termed spin-orbit coupling. The difference between these levels is given by:

$$\Delta E=E_{parallel}-E_{antiparallel}=\frac{mc^2\alpha^4}{n^5} \tag{6.A.36}$$

where , the fine structure constant, is very close to 1/137.

15. Consider the (2,1,1) state with no external magnetic field. If this state decays to the 1s state, find the shift in wavelength of the LS-parallel state with the LS-antiparallel state. What is this shift as a percentage of the original wavelength?

CHAPTER OVERVIEW

7: Nuclear Physics

Topic hierarchy

Contributors and Attributions

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7.1: The Simplified Nuclear Potential Well

The nucleus is held together by the *strong nuclear force*. The strong force is a short range (~1 fm), very strong (~100 times stronger than the electromagnetic force), attractive force that acts between protons and neutrons. Rather than focus of the force, we will focus on the potential energy well associated with this force. Conveniently, this potential well is, to a reasonable approximation, a finite three-dimensional square well.

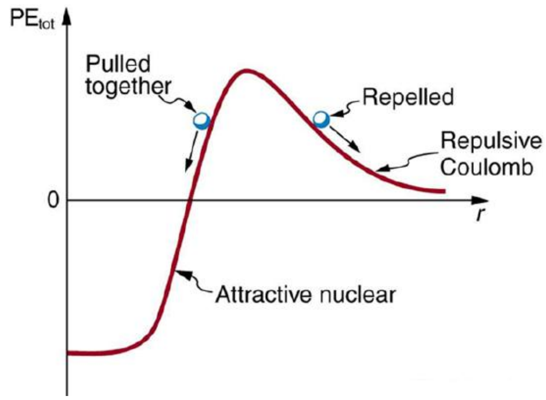
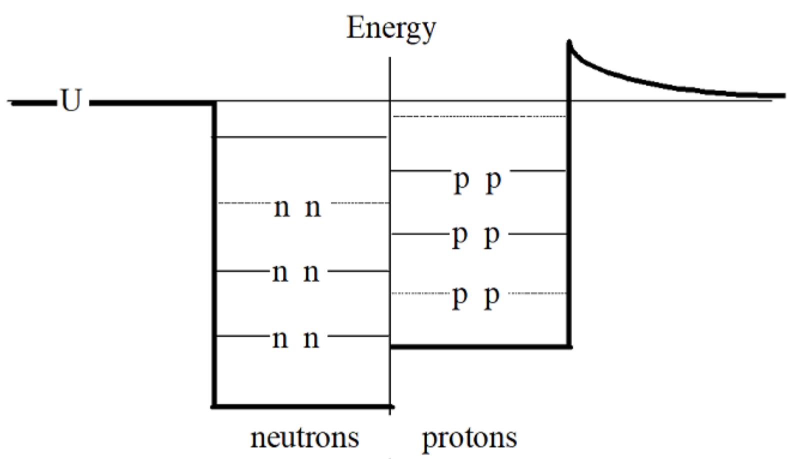
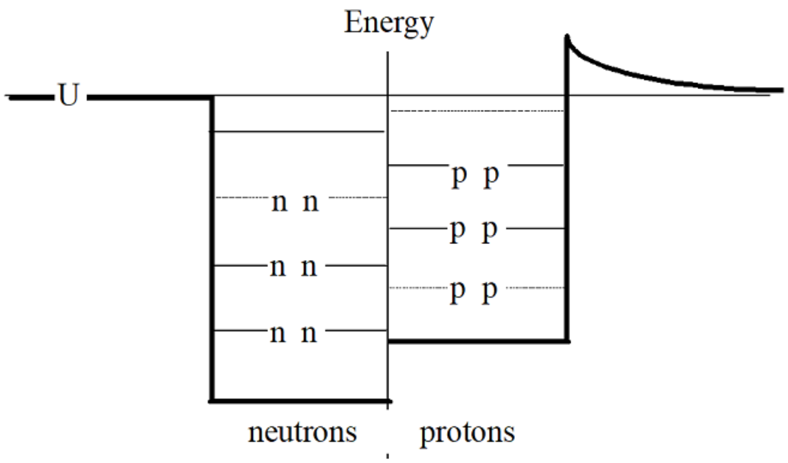


Figure 7.1.1:Potential energy between two light nuclei graphed as a function of distance between them. If the nuclei have enough kinetic energy to get over the Coulomb repulsion hump, they combine, release energy, and drop into a deep attractive well. Tunneling through the barrier is important in practice. The greater the kinetic energy and the higher the particles get up the barrier (or the lower the barrier), the more likely the tunneling. (CC BY 3.0; OpenStax).

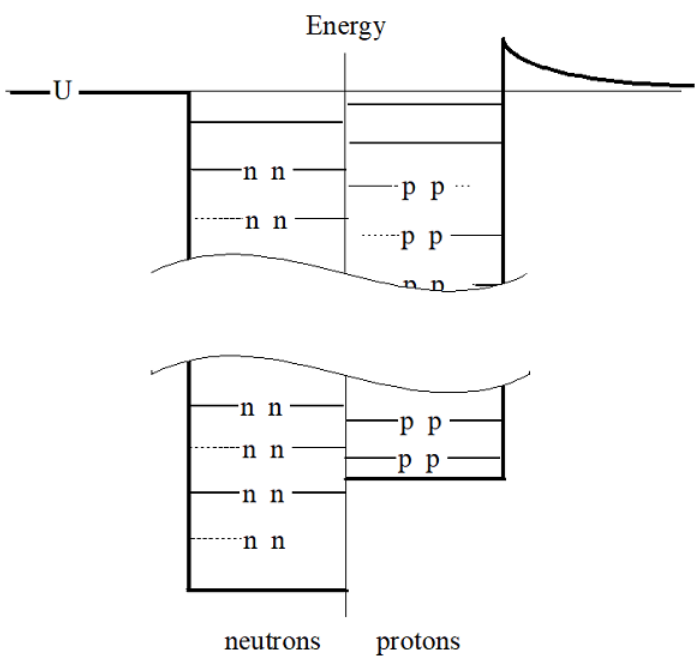
In addition to the strong force, the electromagnetic force also acts within the nucleus (as does the weak force, which we will ignore for now). Of course, the electromagnetic force acts only on the protons, not the neutrons, in the nucleus. Thus, the nuclear potential well looks slightly different for neutrons and protons, as illustrated below:



Typically, this will be drawn with half the well for neutrons and half for protons. (Of course, this is just a representation of the nuclear well, it isn't actually split like this!) The difference in energy levels between neutrons and protons grows more and more pronounced as more and more particles are added to the nucleus. For example, a nucleus with 12 particles would look like this:



Notice that the particles fill the lowest available energy levels, six in the neutron well and six in the proton well resulting in $^{12}\mathrm{C}$. For a nucleus with 56 particles, however, the well looks more like this:



In this well, there are an “extra” four neutrons because the neutron well is substantially deeper than the proton well. This nucleus would have 30 neutrons and 26 protons, making it $^{56}\mathrm{Fe}$. This difference in well depth for protons and neutrons is why light nuclei typically have equal numbers of protons and neutrons while heavier nuclei have an overabundance of neutrons. This difference in well depth will also help us later to understand a type of radioactive decay termed beta decay.

In addition to the shape, the size and depth of the nuclear well can be easily estimated. The radius of a nucleus can be determined from the relationship:

$$r = (1.2\ fm)A^{1/3} \tag{7.1.1}$$

where A is the total number of nucleons (protons and neutrons) in the nucleus.

An estimate of the depth of the well can be determined by calculating the total binding energy of the nucleus. This is the amount of energy that would be needed to remove each nucleon from the well. Surprisingly, the mass of the constituents of a nucleus is larger when the constituents are free (outside of the well) than when they are bound (inside the well). Thus, the total binding energy can be calculated by finding the mass difference between the bound-state nucleus and the total mass of its free nucleons, and converting this mass difference into an energy difference.

$$\text{Binding Energy} = (\text{mass of individual nucleons} - \text{mass of bound nucleus})c^2 \tag{7.1.2}$$

One complication with calculating binding energy via Equation 7.1.2 is that only atomic masses are tabulated, while the difference in nuclear masses determines binding energy. To convert atomic masses to nuclear masses, multiples of the electron mass must be subtracted from each term.

$$\text{Binding Energy} = (\text{mass of individual nucleons} - \text{mass of bound nucleus - mass of bound electrons})c^2 \tag{7.1.3}$$

Example 7.1.1

What is the nuclear binding energy of ${}^4_2\text{He}$?

Solution

This is a simple application of Equation 7.1.3

$$\begin{aligned} BE &= \left[(2m_{proton} - 2m_{neutron}) - (m_{He,atomic} - 2m_{electron})\right] c^2 \\ &= \left[(2(1.00727g\,u) - 2(1.008665\,u) - 4.002603\,u + 2(0.000549\,u)\right] c^2 \\ &= (0.030377\,u)c^2 \\ &= 0.030377(931.5\,MeV) \\ &= 28.3\,MeV \end{aligned}$$

Thus, 28.3 MeV would be needed to full disassemble a nucleus.

Rather than the total binding energy, the binding energy per nucleon is often calculated. This is simply the total binding energy divided by the number of nucleons in the nucleus. Thus, for helium-4, the binding energy per nucleon is:

$$\begin{aligned} BE_{per\ nucleon} &= \frac{BE}{A} \\ &= \frac{28.3\,MeV}{4} \\ &= 7.08\,MeV \end{aligned}$$

The graph of binding energy per nucleon has the interesting property that a natural maximum occurs for ${}^{62}\text{Ni}$. This means that ${}^{62}\text{Ni}$ nuclei are the most tightly bound nuclei. Additionally, nuclei with fewer nucleons can become more tightly bound (and release large amounts of energy) through the process of fusion, and nuclei with more nucleons can become more tightly bound (and release large amounts of energy) through the process of fission.

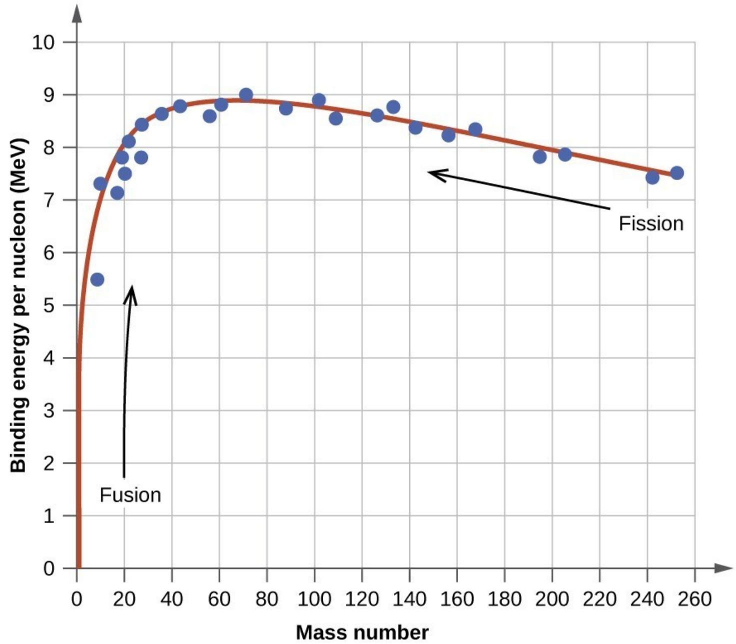


Figure 7.1.2: The binding energy per nucleon is largest for ^{62}Ni. (CC BY 3.0; OpenStax). ^{62}Ni has the maximum value of binding energy per nucleon (8.7945 MeV). It means that ^{62}Ni is the most efficiently bound nucleus meaning that it has the greatest binding energy per nucleon.

Misstatement of Stability: $^{62}\mathrm{Ni}$ vs. $^{56}\mathrm{Fe}$

It is often (and incorrectly) stated that $^{56}\mathrm{Fe}$ is the "most stable nucleus", but actually $^{56}\mathrm{Fe}$ merely has the lowest mass per nucleon (not binding energy per nucleon) of all nuclides. The lower mass per nucleon in $^{56}\mathrm{Fe}$ is enhanced by the fact that $^{56}\mathrm{Fe}$ has 26/56 = 46.43% protons, while ^{62}Ni has only 28/62 = 45.16% protons, and the relatively larger fraction of light protons in $^{56}\mathrm{Fe}$ lowers its average mass-per-nucleon ratio in a way that has no effect on its binding energy. $^{56}\mathrm{Fe}$ is actually the *third* most stable nucleus (binding energy per nucleon) behind $^{58}\mathrm{Fe}$ and $^{62}\mathrm{Ni}$.

7.2: Radioactivity Terminology

In an attempt to maximize stability, nuclei have several options. Each of these options is a distinct type of radioactive decay. Before discussing the individual types of decays, we will first describe the general terminology used to describe these transformations.

The activity, A, of a radioactive sample is defined to be the number of radioactive decays per second, typically measured in Becquerel (Bq), where 1.0 Bq = 1 decay/second, or Curie (Ci), where

$$1.0\,Ci = 3.7x10^{10}\,Bq. \tag{7.2.1}$$

The decay constant, □, is defined to be the probability per second of any particular nucleus decaying.
Therefore, the activity of a sample is the product of the decay constant and the number of nuclei (N) in the sample:

$$A = 2N \tag{7.2.2}$$

Also, the activity can be thought of as the change in the number of radioactive nuclei present, because every decay reduces the number of radioactive nuclei remaining. Therefore,

$$A = -\frac{dN}{dt} \tag{7.2.3}$$

Combining these results yields a differential equation:

$$-\frac{dN}{dt} = \lambda N \tag{7.2.4}$$

$$\frac{dN}{dt} = -\lambda N \tag{7.2.5}$$

with solution

$$N(t) = N_0 e^{-\lambda t} \tag{7.2.6}$$

The number of radioactive nuclei in a sample decreases exponentially over time.

Since the activity is proportional to the number of atoms, the activity follows the same exponential decrease,

$$A(t) = A_0 e^{-\lambda t} \tag{7.2.7}$$

The half-life is the amount of time it takes for the activity of a sample, or the number of atoms in a sample, to be reduced by one-half. This can be related to the decay constant via

$$\begin{aligned} A(t) &= A_0 e^{-\lambda t} \\ 0.5A_0 &= A_0 e^{-\lambda t_{1/2}} \\ -\lambda t_{1/2} &= \log(0.5) \\ \lambda &= \frac{-\log(0.5)}{t_{1/2}} \\ \lambda &= \frac{\log(2)}{t_{1/2}} \end{aligned} \tag{7.2.8}$$

Radioactivity

A radioactive sample has an activity of 2.3 □Ci. After 1.0 hr, the activity has decreased to 1.7 □Ci. What percentage of the sample will remain after 6.0 hr?

Since the activity decreases exponentially,

$$\begin{aligned} A(t) &= A_0 e^{-\lambda t} \\ 1.7\mu C_i &= (2.3\mu C_i)e^{-\lambda(1.0/hr)} \\ e^{-\lambda(1.0hr)} &= 0.74 \\ -\lambda(1.0hr) &= \log(0.74) \\ \lambda &= 0.302hr^{-1} \end{aligned} \tag{7.2.9}$$

The decay constant is 0.302 hr-1, meaning each atom has approximately a 30% chance of decaying each hour.

After 6 hrs,

$$\begin{aligned} A(t) &= A_0 e^{-\lambda t} \\ A(6.0hr) &= (2.3\mu C_i)e^{-(0.302hr^{-1})(6.0hr)} \\ A(6.0hr) &= 0.376\mu C_i \end{aligned} \tag{7.2.10}$$

The activity has dropped to 0.376/2.3 = 0.163 of its initial value. Since activity and number of atoms are proportional, 16% of the sample remains after 6.0 hr.

Carbon Dating

A sample of leather from a tomb is burned to obtain 0.20 g of carbon. The measured activity of the sample is 0.017 Bq. How old is the sample? In the environment, about one carbon atom in 7.7 x 1011 is 14C. The half-life of 14C is 5730 yrs.

The half-life can be related to the decay constant via

$$\begin{aligned} &= \frac{\log(2)}{t_{1/2}} \\ \lambda &= \frac{\log(2)}{5730 \text{ yr}} \\ \lambda &= 1.21x10^{-4} \text{ yr}^{-1} \end{aligned} \tag{7.2.11}$$

To find the age of the leather, we need to calculate the activity of 14C in the present environment, and assume that this was the activity of the leather when the leather sample was in equilibrium with its environment (i.e., when it was alive). To do this, find the number of carbon atoms in the sample, then find how many of those are 14C. From this number, the activity of a “new” piece of leather can be determined.

$$0.20g\left(\frac{6.02x10^{23}atomsC}{12g}\right) = 1.00x10^{23}atomsC \tag{7.2.12}$$

$$1.00x10^{23}atomsC\left(\frac{1atom^{14}C}{7.7x10^{11}atomsC}\right) = 1.30x10^{10}atoms^{14}C \tag{7.2.13}$$

Therefore, there were 1.30 x 1010 atoms of 14C present in the sample when the sample was alive. Since the decay constant gives the probability of decay as 1.21 x 10-4 yr-1, the initial activity of the sample was

$$\begin{aligned} A_0 &= (1.30x10^{10}atoms^{14}C)(1.21x10^{-4}yr^{-1}) \\ A_0 &= 1.57x10^{6}decays/yr \\ A_0 &= 0.050decays/s = 0.050Bq \end{aligned} \tag{7.2.14}$$

Therefore,

$$A(t) = A_0 e^{-\lambda t} \tag{7.2.15}$$

$$0.017Bq = (0.050Bq)e^{-(1.21x10^{-4})t} \tag{7.2.16}$$

$$e^{-(1.21x10^{-4})t} = 0.34 \tag{7.2.17}$$

$$-(1.21x10^{-4})t = \log(0.34) \tag{7.2.18}$$

$$t = 8900 \text{ yr} \tag{7.2.19}$$

The sample is 8900 years old.

7.3: Alpha and Beta Decay

$^{226}\mathrm{Ra}$ undergoes alpha decay with a half-life of 1600 years. Consider a 100 g sample of pure $^{226}\mathrm{Ra}$. What is the initial power output of this sample?

An alpha particle is a bound state of two neutrons and two protons. In many nuclei with very large numbers of nucleons (typically more than 200), alpha particle near the "top" of the potential well "see" a well with the barrier height (~20 MeV) and well depth (~ -50 MeV) shown below:

If this is the case, the alpha particle can escape the nucleus by tunneling through the barrier. This tunneling process is alpha decay. The half-life for alpha decay can be calculated from the tunneling probability.

Generically, alpha decay can be written as:

$$^{\mathrm{A}}_{\mathrm{Z}}\mathrm{X} \Rightarrow {}^{\mathrm{A}-4}_{\mathrm{Z}-2}\mathrm{X} + {}^{4}_{2}\mathrm{He} \tag{7.3.1}$$

For this example, this reduces to,

$$^{226}_{88}\mathrm{Ra} \Rightarrow {}^{222}_{86}\mathrm{Rn} + {}^{4}_{2}\mathrm{He} \tag{7.3.2}$$

The energy released by this decay can be calculated by simply finding the mass difference between the initial and final nuclear states, and converting this mass difference into an energy difference. The energy released in a nuclear transformation is typically referred to as the Q-value of the reaction.

One possible complication with calculating Q-values is that only atomic masses are tabulated, while the difference in nuclear mass determines Q. To convert atomic masses to nuclear masses, multiples of the electron mass must be subtracted from each term. Generically,

$$Q = (m_{X,atomic}c^2 - Zm_ec^2) - (m_{X,atomic}c^2 - (Z-2)m_ec^2) - (m_{He,atomic}c^2 - 2m_ec^2) \tag{7.3.3}$$

$$Q = m_{X,atomic}c^2 - Zm_ec^2 - m_{X,atomic}c^2 - (Z-2)m_ec^2 - m_{He,atomic}c^2 - 2m_ec^2 \tag{7.3.4}$$

$$Q = (m_{X,atomic} - m_{X,atomic} - m_{He,atomic}c^2 \tag{7.3.5}$$

Since the number of electrons on each side of the reaction is equal, you can use atomic masses to determine Q.

$$Q = (m_{Ra} - m_{Rn} - m_{He})c^2 \tag{7.3.6}$$

$$Q = (226.025402u - 222.017570u - 4.002603)c^2 \tag{7.3.7}$$

$$Q = 5.229 \times 10^{-3} uc^2 \tag{7.3.8}$$

$$Q = 4.87 \text{ MeV} \tag{7.3.9}$$

This is the energy released per alpha decay. If we can determine the activity of the sample (the number of decays per second), the product of activity and Q will be the power of the sample.

The number of atoms in the sample is

$$100g\left(\frac{6.02 \times 10^{23} atomsRa}{226g}\right) = 2.66 \times 10^{23} atomsRa \tag{7.3.10}$$

Since the decay constant is

$$\lambda = \frac{\log(2)}{t_{1/2}} \tag{7.3.11}$$

$$\lambda = \frac{\log(2)}{1600yr} \tag{7.3.12}$$

$$\lambda = 4.33 \times 10^{-4} yr^{-1} \tag{7.3.13}$$

$$\lambda = 1.37 \times 10^{-11} s^{-1} \tag{7.3.14}$$

the activity is

$$A = (2.66 \times 10^{23} atoms Ra)(1.37 \times 10^{-11} s^{-1} \tag{7.3.15}$$

$$A = 3.65 \times 10^{12} decays/s \tag{7.3.16}$$

and the power is

$$P = QA \tag{7.3.17}$$

$$P = (4.87 \text{ MeV}/decay)(3.65 \times 10^{12} decays/s \tag{7.3.18}$$

$$P = 1.78 \times 10^{13} \text{ MeV/s} \tag{7.3.19}$$

$$P = 2.85 J/s \tag{7.3.20}$$

$$P = 2.85 W \tag{7.3.21}$$

Beta Decay

$^{81}\mathrm{Kr}$ is unstable. How will it decay? Calculate the Q value for this decay.

In addition to alpha decay, which typically occurs only for very large nuclei, another possible nuclear transformation involves the spontaneous transformation of a proton into a neutron, or vice-versa. There are actually three different decay processes that involve this type of transformation, which is governed by the weak force. These decays are generically referred to as beta decay.

\Beta$^-$ Decay

Beta-minus decay involves the transformation of a neutron into a proton, electron, and anti-neutrino:

$$n \Rightarrow p^+ + e^- + \bar{v} \tag{7.3.22}$$

The neutron can decay by this reaction both inside the nucleus and as a free particle.

Generically, beta-minus decay can be written as

$${}^A_Z X \Rightarrow^A_{Z+1} X' + e^- + \bar{v} \tag{7.3.23}$$

resulting in a Q-value of:

$$Q = (m_{X,atomic}c^2 - Zm_e c^2) - (m_{X',atomic}c^2 - (Z+1)m_e c^2) - (m_e c^2) \tag{7.3.24}$$

$$Q = m_{X,atomic}c^2 - Zm_e c^2 - m_{X',atomic}c^2 + (Z+1)m_e c^2 - m_e c^2 \tag{7.3.25}$$

$$Q = (m_{X,atomic} - m_{X',atomic})c^2 \tag{7.3.26}$$

\Beta$^+$ Decay

Beta-plus decay involves the transformation of a proton into a neutron, positron, and neutrino:

$$p^+ \Rightarrow n + e^+ + v \tag{7.3.27}$$

This process can only occur inside the nucleus.

Generically, beta-plus decay can be written as

$${}^A_Z X \Rightarrow^A_{Z-1} X' + e^+ + v \tag{7.3.28}$$

resulting in a Q-value of:

$$Q = (m_{X,atomic}c^2 - Zm_e c^2) - (m_{X',atomic}c^2 - (Z-1)m_e c^2) - (m_e c^2) \tag{7.3.29}$$

$$Q = m_{X,atomic}c^2 - Zm_ec^2 - m_{X',atomic}c^2 + (Z-1)m_ec^2 - m_ec^2 \tag{7.3.30}$$

$$Q = (m_{X,atomic} - m_{X',atomic} - 2m_e)c^2 \tag{7.3.31}$$

Electron Capture

Electron capture involves a proton in the nucleus absorbing an inner shell electron:

$$p^+ + e^- \Rightarrow n + v \tag{7.3.32}$$

Generically, electron capture can be written as

$${}^A_ZX + e^- \Rightarrow {}^A_{Z-1}\ X' + v \tag{7.3.33}$$

resulting in a Q-value of:

$$Q = (m_{X,atomic}c^2 - Zm_ec^2) + (m_ec^2) - (m_{X',atomic}c^2 - (Z-1)m_ec^2) \tag{7.3.34}$$

$$Q = m_{X,atomic}c^2 - Zm_ec^2 + m_ec^2 - m_{X',atomic}c^2 + (Z-1)m_ec^2 \tag{7.3.35}$$

$$Q = (m_{X,atomic} - m_{X',atomic})c^2 \tag{7.3.36}$$

Applied to this example, the three processes yield the following reactions:

\[

$$\begin{array}{lll} \text{beta}^-\text{decay} & {}^{81}_{36}Kr \Rightarrow {}^{81}_{37}\ Rb + e^- + \bar{v} & Q = (m_{Kr} - m_{Rb})c^2 \\ \text{beta}^+\text{decay} & {}^{81}_{36}Kr \Rightarrow {}^{81}_{35}\ Br + e^+ + \bar{v} & Q = (m_{Kr} - m_{Br} - 2m_{e^-})c^2 \\ \text{electron capture} & {}^{81}_{36}Kr + e^- \Rightarrow {}^{81}_{35}\ Br + v & Q = (m_{Kr} - m_{Br})c^2 \end{array} \tag{7.3.37}$$

]

To determine how 81Kr will decay, calculate the Q-value for each hypothetical reaction. Only Q-values greater than zero (reactions that release energy) occur spontaneously.

\[

$$\begin{array}{lll} \text{beta}^-\text{decay} & {}^{81}_{36}Kr \Rightarrow {}^{81}_{37}\ Rb + e^- + \bar{v} & Q = (80.916593u - 80.916291u)c^2 \\ & & Q < 0 \\ \text{beta}^+\text{decay} & {}^{81}_{36}Kr \Rightarrow {}^{81}_{35}\ Br + e^+ + \bar{v} & Q = (80.916593u - 80.916291u - 2(5.4858 \times 10^{-4}u))c^2 \\ & & Q < 0 \\ \text{electron capture} & {}^{81}_{36}Kr + e^- \Rightarrow {}^{81}_{35}\ Br + v & Q = (80.916593u - 80.916291u)c^2 \\ & & Q = 0.281MeV \end{array} \tag{7.3.38}$$

]

Therefore, 81Kr will decay via electron capture, and release 0.281 MeV of energy per decay.

If more than one decay involves a positive Q, the one that releases the most energy will typically dominate. The exception to this rule involves electron capture. Both beta-plus and beta-minus, if allowed, always dominate electron capture since electron capture involves the relatively rare occurrence of a sizable overlap between electron and proton wavefunctions.

Beta decay can be understood conceptually by looking carefully at the differences in the potential wells for protons and neutrons, and the order in which the available energy levels are filled.
For example, consider 24Na, which consists of 13 neutrons and 11 protons. The filled energy levels would look like the well on the left.

The overall energy of the nucleus would be reduced (and its stability increased) if the “stray” neutron at the top of the neutron well could somehow transform itself into a proton and jump down to the lower energy state in the proton well. Well, nature allows this transformation and we call it - decay!

As another example, consider 18F, which consists of 9 neutrons and 9 protons. Its filled energy levels would look like the well on the left.

The overall energy of this nucleus would be reduced if a proton could somehow transform itself into a neutron. Well, nature allows this transformation and we call it + decay!

7.4: Fission and Fusion

Fission

Find Q for the fission reaction:

$$ {}^{1}_{0}\mathrm{n} + {}^{235}_{92}\mathrm{U} \longrightarrow {}^{141}_{56}\mathrm{Ba} + {}^{92}_{36}\mathrm{K} + 3\,{}^{1}_{0}\mathrm{n} \tag{7.4.1}$$

As we noticed in our discussion of binding energy, nuclei much more massive than iron actually have less binding energy per nucleon than iron. Thus, if some method existed by which these massive nuclei could be "broken up" into iron-sized fragments energy would be released. This method is termed nuclear fission.

Although some nuclei will undergo spontaneous fission, most fission reactions of interest involve induced fission, in which an incoming projectile, typically a neutron, collides with the target nuclei and initiates the fission process. A typical example is

$$ n + ^{235}_{92} U \Rightarrow ^{141}_{56} Ba + ^{92}_{36} Kr + 3n \tag{7.4.2}$$

The Q for this reaction is:

$$Q = (m_{U,atomic} c^2 - 92 m_e c^2) + (m_n c^2) - (m_{Bd,atomic} c^2 - (56) m_e c^2) - (m_{Kr,atomic} c^2 - 36 m_e c^2) - 3(m_n c^2) \tag{7.4.3}$$

$$Q = m_{U,atomic} c^2 - m_{Bd,atomic} c^2 - m_{Kr,atomic} c^2 - 2(m_n c^2) \tag{7.4.4}$$

$$Q = (m_{U,atomic} - m_{Bd,atomic} - m_{Kr,atomic} - 2m_n) c^2 \tag{7.4.5}$$

$$Q = (235.043922 - 140.914406 - 91.926153 - 2(1.008665)) uc^2 \tag{7.4.6}$$

$$Q = 173.3 \text{ MeV} \tag{7.4.7}$$

Thus, for every $^{235}\mathrm{U}$ nucleus that undergoes fission by this process, 173.3 MeV is released.

It should be pointed out that there is nothing special about this particular fission reaction. When struck by the incoming neutron, the uranium nucleus can fission in many, many different ways, with the vast majority of these fission processes releasing between 150 and 200 MeV.

Additionally, the fission fragments ($^{141}\mathrm{Ba}$ and $^{92}\mathrm{Kr}$) are themselves radioactive. Their subsequent decay(s) into a stable form will add to the total energy released by the fission process.

Finally, special mention should be made of the neutrons released by the fission reaction. These neutrons can be used to fission additional uranium nuclei, leading to more energy and still more neutrons. This chain reaction is crucial to the construction of a fission weapon, and crucial to control in a fission power plant.

Fusion

Find Q for the fusion reaction:

$$ {}^{2}_{1}\mathrm{H} + {}^{2}_{1}\mathrm{H} \longrightarrow {}^{3}_{2}\mathrm{He} + \gamma \tag{7.4.8}$$

Again, as we noticed in our discussion of binding energy, nuclei much less massive than iron actually have less binding energy per nucleon than iron. Thus, if some method existed by which these very low mass nuclei could be "squeezed together" into larger nuclei energy would be released. This method is termed nuclear fusion.

There are no examples of spontaneous fusion. Since all nuclei are positively charged, the electromagnetic repulsion between two nuclei makes getting them close enough together to "fall" into each other's potential wells very difficult. In fact, quantum mechanical tunneling is crucial to understanding the fusion that takes place in the sun. Without the ability of two nuclei to tunnel through the relatively large electrostatic barrier separating them, the sun would not shine!
The Q for the fusion reaction given is:

$$Q = (m_{2H,atomic} c^2 - m_e c^2) + (m_{H,atomic} c^2 - m_e c^2) - (m_{3He,atomic} c^2 - 2 m_e c^2 \tag{7.4.9}$$

$$Q = (m_{2H,atomic} + m_{H,atomic} - m_{3He,atomic}) c^2 \tag{7.4.10}$$

$$Q = (2.014102 + 1.007825 - 3.016029) uc^2 \tag{7.4.11}$$

$$Q = 5.49 \text{ MeV} \tag{7.4.12}$$

Thus, 5.49 MeV is released for every occurrence of this reaction.

This result can be compared to the fission result a number of different ways. First, although less energy is released per reaction by this fusion reaction, more energy is released per kg of reactants. For example, dividing the Q by the atomic mass of the reactants yields:

$$\frac{Q}{A_{fission}} = \frac{173.3 \text{ MeV}}{236} = 0.7 \text{ MeV} \tag{7.4.13}$$

$$\frac{Q}{A_{fission}} = \frac{5.49 \text{ MeV}}{3} = 1.8 \text{ MeV} \tag{7.4.14}$$

Additionally, not only is the input fuel for the fusion reaction incredibly common (there’s a bit more water on the earth than there is ^{235}U), the output materials have a relatively low level of radioactivity in comparison to the very nasty fragments created by fission.

There is, however, one huge downside to fusion; the energies and pressures needed to control fusion are still beyond our technological abilities! We can build fusion bombs, but not fusion power plants.

7.5: Nuclear Physics (Activities)

A radioactive sample undergoes three different types of radioactive decays and emits three different types of particles.

I. The particles are emitted into a region of space with a uniform magnetic field directed out of the page and follow the paths indicated. None of the particles bend either into or out of the screen. For each path, identify the radioactive decay (□□□□□□□□□□□□□) or state that the type of decay cannot be determined based on the information provided. Ignore the neutrinos (and antineutrinos) emitted in beta decay.

a.

b.

c.

II. The particles are directed toward a series of barriers and follow the paths indicated. For each path, identify the most likely radioactive decay (□□□□□□□□□□□□□) or state that the type of decay cannot be determined based on the information provided. Ignore the neutrinos (and antineutrinos) emitted in beta decay.

d.

e.

f.

The activity of six different radioactive samples is tracked over a 10 year period and graphed below.

a. Rank these samples on the basis of their half-life.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

b. Rank these samples on the basis of their decay constant.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

Six radioactive samples, with different half-lives and initial activities, are listed below.

Half-life (min) Activity (mCi)
A 1.0 160
B 6.0 80
C 4.0 80
D 2.0 40
E 1.0 320
F 8.0 40

a. Rank these samples on the basis of their activity after 4.0 minutes.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

b. Rank these samples on the basis of their activity after 8.0 minutes.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your ranking:

The graphic below illustrates the valley of stability, the collection of stable, metastable, and unstable nuclei in terms of atomic number and neutron number.

a. Examine the number of stable nuclei at odd values of N vs. even values of N. Does one value lead to consistently more stable nuclei? If so, explain why this occurs.

b. Examine the number of stable nuclei at odd values of Z vs. even values of Z. Does one value lead to consistently more stable nuclei? If so, explain why this occurs.

c. In addition to the simple relationships above, certain values of N have an inordinate number of stable nuclei, for example 5 or more. What are these magic numbers of neutrons?

d. Are the same numbers magic for protons? If so, explain why this happens. (Hint: Consider what you know about the allowed occupancy of different atomic energy levels.)

The graphic below illustrates the valley of stability, the collection of stable, metastable, and unstable nuclei in terms of atomic number and neutron number.

a. Examine the nucleus indicted above, with Z = 21 and N = 19. Based only on the graphic, how do you think this nucleus decays? What nucleus (or nuclei) does it decay into?

b. Examine the nucleus indicted above, with Z = 9 and N = 13. Based only on the graphic, how do you think this nucleus decays? What nucleus (or nuclei) does it decay into?

c. Examine the nucleus indicted above, with Z = 35 and N = 42. Based only on the graphic, how do you think this nucleus decays? What nucleus (or nuclei) does it decay into?

d. Examine the nucleus indicted above, with Z = 19 and N = 21. Based only on the graphic, how do you think this nucleus decays? What nucleus (or nuclei) does it decay into?

I. Why are beta particles absorbed by a sheet of metal while gamma particles require several inches of lead for absorption?

Barbie: Gamma particles move at the speed of light. The faster something moves, the more difficult it is to stop.

Kathleen: Gamma particles are smaller than betas. That’s what makes them more difficult to stop.

Kevin: Beta particles are charged, so they are easily stopped by electrical conductors.

Which, if any, of the students are correct? For each incorrect response, provide a short explanation why it is incorrect. If no one is correct, provide a correct answer below.

II. Why is it so important not to eat or drink in the vicinity of radioactive materials?

Robert: You may accidentally ingest some of the radioactive materials before they have decayed. This could be dangerous.

Paul: If your food or drink gets “hit” by the radioactive particles it could become harmful to consume.

Betty: You may accidentally ingest some of the radioactive materials after they have decayed. This could be dangerous.

Which, if any, of the students are correct? For each incorrect response, provide a short explanation why it is incorrect. If no one is correct, provide a correct answer below.

Complete the following hypothetical radioactive chain with the correct decay process on each line and the correct nucleus in each box.

Complete the following hypothetical radioactive chain with the correct decay process on each line and the correct nucleus in each box.

I. Why must 235U be separated from 238U when constructing a fission bomb?

Otto: Any 238U in the bomb material will absorb neutrons rather than be fissioned by them. Since the neutrons needed for the chain reaction are produced by fission, any 238U in the bomb would cause the chain reaction to “fizzle”.

Fritz: 238U does not have as much nuclear energy as 235U. Thus, the bomb would be much less powerful.

Lise: Any 235U in the bomb material will absorb neutrons rather than be fissioned by them. Since the neutrons needed for the chain reaction are produced by fission, any 235U in the bomb would cause the chain reaction to “fizzle”.

Which, if any, of the students are correct? For each incorrect response, provide a short explanation why it is incorrect. If no one is correct, provide a correct answer below.

II. Why can’t a fission bomb be constructed from less than the critical mass of fissionable material?

Albert: If you have less than the critical mass of fissionable material, you run the risk of the bomb spontaneously detonating (which is obviously something you’d like to avoid).

Edward: If you have less than the critical mass of fissionable material, the chain reaction that causes the bomb to denote will not occur.

Emile: If you have less than the critical mass of fissionable material, the bomb will not be powerful enough to destroy anything.

Which, if any, of the students are correct? For each incorrect response, provide a short explanation why it is incorrect. If no one is correct, provide a correct answer below.

I. What is the source of the energy released in a fission bomb?

Michala: The energy is initially stored in the chemical bonds of the uranium. As the chemical reaction unfolds, and uranium changes into other materials, this energy is released.

Fiona: A fission bomb requires a chain reaction mediated by neutrons. The kinetic energy, or energy of motion, of the neutrons provides the energy released by the bomb.

Kip: The energy released by the bomb is due to a very small decrease in the mass of the bomb materials as they undergo fission. This small decrease is mass-energy is equivalent to a huge release of other forms of energy.

Which, if any, of the students are correct? For each incorrect response, provide a short explanation why it is incorrect. If no one is correct, provide a correct answer below.

II. What is the primary difference between a fission and fusion reaction?

Ian: Fission reactions occur on earth and fusion reactions occur in the sun.

Tom: Fusion reactions release more energy than fission reactions.

Matt: Fusion reactions release no harmful by-products while fission reactions release various highly dangerous by-products.

Which, if any, of the students are correct? For each incorrect response, provide a short explanation why it is incorrect. If no one is correct, provide a correct answer below.

Find the total binding energy for 4He, 3He, and 3H.
a. Based on these results, how much energy is needed to remove a single neutron from 4He?
b. Based on these results, how much energy is needed to remove a single proton from 4He?
c. Carefully explain why the proton is easier to remove than the neutron.

Mathematical Analysis

Find the total binding energy for 56Fe, 55Fe, and 55Mn.
a. Based on these results, how much energy is needed to remove a single neutron from 56Fe?
b. Based on these results, how much energy is needed to remove a single proton from 56Fe?
c. Carefully explain why the proton is easier to remove than the neutron.

Mathematical Analysis

Find the binding energy per nucleon for 15O and 15N. Based on these results, which of these two isobars do you expect to be more plentiful in nature?

Mathematical Analysis

Find the binding energy per nucleon for 13C and 13N. Based on these results, which of these two isobars do you expect to be more plentiful in nature?

Mathematical Analysis

28Al has a half-life of 2.24 minutes.
a. What is □?
b. What percentage of a given sample will remain after 1.0 hr?

Mathematical Analysis

Tritium, 3H, has a half-life of 12.3 yrs.
a. What is □?
b. What percentage of a given sample will remain after 50 yr?

Mathematical Analysis

A radioactive sample decays at a rate of 500 per second. After 1 hr, the rate has decreased to 225 per second.
a. What is the half-life of the sample?
b. What will be the decay rate after 2 hr?
c. What percentage of the sample will remain after 10 hr?

Mathematical Analysis

A 2.00 mCi sample of 131I has a half-life of 8.04 days.
a. What will be the decay rate after 30 days?
b. What percentage of the sample will remain after 30 days?

Mathematical Analysis

Natural potassium contains 0.012 percent 40K, which has a half-life of 1.3 x 109 yrs.
a. What is the activity of 1.0 kg of potassium?
b. What was the percent 40K in natural potassium 4.4 x 109 yrs ago?

Mathematical Analysis

Humans have about 2.5 g of potassium per kilogram of body mass. The average 150 g banana contains about 0.5 g potassium. Natural potassium contains 0.012 percent 40K, which has a half-life of 1.3 x 109 yrs.
a. What is the activity of a 70 kg human due to 40K decay?
b. If he eats a banana, by what percentage does his activity increase? (This increase in activity lasts for about 3 hrs until his kidneys eliminate the excess potassium.)

Mathematical Analysis

A piece of wood from a recently cut tree shows 13 14C decays per minute. An old axe handle of the same mass shows 3 decays per minute. How old is the axe handle? The half-life of 14C is 5730 yrs.

Mathematical Analysis

A sample of charcoal from an ancient fire pit has a mass of 0.36 g and is essentially pure carbon. The measured activity of the sample is 0.01 Bq. How old is the sample? In the environment, about one carbon atom in 7.7 x 1011 is 14C. The half-life of 14C is 5730 yrs.

Mathematical Analysis

A sample of leather from a tomb is burned to obtain 0.12 g of carbon. The measured activity of the sample is 0.012 Bq. How old is the sample? In the environment, about one carbon atom in 7.7 x 1011 is 14C. The half-life of 14C is 5730 yrs.

Mathematical Analysis

In uranium-lead dating, two independent age estimates for the same substance can be made by comparing the ratio of specific isotopes of lead and uranium in a sample. 238U decays to 206Pb through a 14-step process that has an overall half-life of 4.47 x 109 yr. 235U decays to 207Pb through an 11-step process that has an overall half-life of 704 x 106 yr. Aside from supernovae explosions, these isotopes of lead are only formed by these processes. Thus, the ratio of lead to uranium in a sample can be used to determine the age of the sample. Consider a sample of zircon that contains 8 206Pb atoms for every 100 238U atoms and 62 207Pb atoms for every 100 235U atoms.

a. What is the age of the sample based on the amount of 206Pb present?

b. What is the age of the sample based on the amount of 207Pb present?

c. How would you report the age of this sample?

Mathematical Analysis

230Th undergoes alpha decay with a half-life of 75,400 years. Consider a 1 kg sample of pure 230Th.

a. What is the initial power output (in Watts) of this sample?

b. How long will it take for the activity to drop to 1 decay/s?

c. What is the kinetic energy of the emitted alpha particle?

Mathematical Analysis

239Pu undergoes alpha decay with a half-life of 24,100 years. Consider a 1 g sample of pure 239Pu.

a. What is the initial power output (in Watts) of this sample?

b. How long will it take for the activity to drop to 1 decay/s?

c. What is the kinetic energy of the emitted alpha particle?

Mathematical Analysis

211Rn undergoes alpha decay with a half-life of 14.6 h. Consider a 0.01 mol sample of pure 211Rn.

a. What is the initial activity of this sample?

b. How long will it take for the power output of this sample to drop to 1% of its initial value?

c. What is the kinetic energy of the emitted alpha particle?

Mathematical Analysis

The vital ingredient of household smoke detectors is a very small quantity (~35 kBq) of americium-241 (241Am). (This element was discovered in 1945 during the Manhattan Project. The first sample of americium was produced by bombarding plutonium with neutrons in a nuclear reactor at the University of Chicago. As if you cared.) 241Am undergoes alpha decay with a half-life of 432 years.

a. How long will it take for the activity to drop to 1 decay/s (1 Bq)?

b. What is the initial power output (in Watts) of this sample?

c. How many grams of 241Am are in a typical smoke detector?

Mathematical Analysis

24Na is unstable.

a. How can it decay? Calculate the Q value for each decay.

b. Which decay dominates?

Mathematical Analysis

28Al is unstable.

a. How can it decay? Calculate the Q value for each decay.

b. Which decay dominates?

Mathematical Analysis

18F is unstable.
a. How can it decay? Calculate the Q value for each decay.
b. Which decay dominates?

Mathematical Analysis

32P is unstable.
a. How can it decay? Calculate the Q value for each decay.
b. Which decay dominates?

Mathematical Analysis

36Cl is unstable.
a. How can it decay? Calculate the Q value for each decay.
b. Which decay dominates?

Mathematical Analysis

58Co is unstable.
a. How can it decay? Calculate the Q value for each decay.
b. Which decay dominates?

Mathematical Analysis

The atomic masses of 3918Ar, 3919K, and 3920Ca are 38.964315u, 38.963999u, and 38.970718u.
a. Which one(s) are stable?
b. How will each unstable nuclei decay?

Mathematical Analysis

The atomic masses of 8134Se, 8135Br, 8136Kr, and 8137Rb are 80.917990u, 80.916289u, 80.916589u, and 80.918990u.
a. Which one(s) are stable?
b. How will each unstable nuclei decay?

Mathematical Analysis

40K undergoes all three beta decays: 89% of the time beta-minus, 11% of the time electron capture, and very rarely beta-plus.
a. Find the Q value for each decay.
b. Find a weighted average Q value for 40K decay.

Humans have about 2.5 g of potassium per kilogram of body mass. The average 150 g banana contains about 0.5 g potassium. Natural potassium contains 0.012 percent 40K, which has a half-life of 1.3 x 109 yrs.
c. What is the average power output of an 80 kg man due to 40K decay?
d. How many bananas would it take to light a 60 W bulb? Assume all of the 40K power output is somehow channeled into the light bulb.

Mathematical Analysis

In positron-emission tomography (PET scanning), a radioactive isotope that emits positrons is introduced into the body. Upon decay, the emitted positron immediately annihilates with a nearby electron releasing a pair of oppositely directed gamma rays. Detecting these gamma rays allows a precise determination of the location of the decay. For brain scans, a common tracer is glucose incorporating a small amount (approximately 1.0 x 10-9 g) of 11C, whose half-life is 20 min. (Glucose can cross the blood-brain barrier and enter the brain.)
a. Find the Q value for 11C decay.
b. What is the initial power output due to 11C decay?
c. What is the total energy released into the brain due to 11C decay?

Mathematical Analysis

14C undergoes beta-minus decay.
a. What is the maximum kinetic energy of the emitted electron?
b. Assuming the initial and final nuclei are at rest, and the neutrino mass is zero, find the kinetic energy of the emitted electron.

Mathematical Analysis

11Be undergoes beta-minus decay.
a. What is the maximum kinetic energy of the emitted electron?
b. Assuming the initial and final nuclei are at rest, and the neutrino mass is zero, find the kinetic energy of the emitted electron.

Mathematical Analysis

15O undergoes beta-plus decay.
a. What is the maximum kinetic energy of the emitted positron?
b. Assuming the initial and final nuclei are at rest, and the neutrino mass is zero, find the kinetic energy of the emitted positron.

Mathematical Analysis

Consider the neutrino capture reaction

a. Determine the end-product of the reaction.
b. What is the Q-value for the reaction?
c. What minimum value of kinetic energy must the neutrino have for this reaction to take place?

Mathematical Analysis

Consider the neutrino capture reaction

a. Determine the end-product of the reaction.
b. What is the Q-value for the reaction?
c. What minimum value of kinetic energy must the neutrino have for this reaction to take place?

Mathematical Analysis

Consider the anti-neutrino capture reaction

a. Determine the end-product of the reaction.
b. What is the Q-value for the reaction?
c. What minimum value of kinetic energy must the anti-neutrino have for this reaction to take place?

Mathematical Analysis

Consider the fission reaction:

a. Find Q for this reaction.
b. The total energy released by the fission bomb dropped on Hiroshima was equivalent to the chemical energy in 15 kilotons of TNT, approximately 60 x 1012 J. Based on this amount of energy, how many kilograms of uranium underwent fission in the bomb? (The bomb actually contained 64 kg of uranium.)
c. The fission reactions took place over about 10-12 s. What was the power of the Hiroshima bomb? Compare this power to the power output of the sun.

Mathematical Analysis

In addition to 235U, fission can also be induced in 239Pu. However, plutonium does not exist naturally. To create plutonium, 238U is bombarded with neutrons. (99.3% of all naturally occurring uranium is 238U, with almost all of the remainder 235U.)

a. Complete the above reaction, clearly showing how 239Pu is formed.
b. Find Q for the complete reaction.

Mathematical Analysis

Consider the fission reaction:

a. Find Q for this reaction.
b. A typical power plant generates about 20 MW of power. How many kg of plutonium must undergo fission per year to supply this energy? Assume about 30% of the energy released by fission is converted to electricity.

Mathematical Analysis

The proton-proton cycle (or p-p cycle) is responsible for energy production in the sun. The cycle consists of the following steps:

The complete cycle consists of step I twice, step II twice, followed by step III. Notice that these five steps lead to the creation of a single 4He nucleus (and two positrons).
a. Find Q for this cycle. (Hint: Rather than find the Q for each step of the cycle, just find the Q between the ultimate final state and the complete initial state of the cycle.)
b. The positron created by this cycle very rapidly annihilates with an electron in the sun. Add the energy released by this annihilation to the Q found above for the “total” Q of the p-p cycle.
c. Based on the total Q above, determine the number of p-p cycles occurring per second in the sun.

Mathematical Analysis

The carbon cycle is responsible for energy production in stars much larger than the sun. The cycle consists of the following steps:

Notice that the net effect of this cycle is creation of a single 4He nucleus (and two positrons). The carbon serves solely as a catalyst for the reaction. The carbon cycle progresses at a much faster rate than the p-p cycle described in a previous problem and is the dominant energy source for large stars.
a. Find Q for this cycle. (Hint: Rather than find the Q for each step of the cycle, just find the Q between the ultimate final state and the complete initial state of the cycle.)
b. The two positrons created by this cycle very rapidly annihilate with electrons. Add the energy released by these annihilations to the Q found above for the “total” Q of the carbon cycle.
c. Based on the total Q above, determine the number of carbon cycles occurring per second in a star ten times more powerful than the sun.

Mathematical Analysis

The most promising fusion reaction for electrical power generation is the deuterium (2H)-tritium (3H), or D-T, reaction:

Tritium is a radioactive form of hydrogen that would have to be artificially manufactured, but that process is technologically simple compared to creating the conditions needed for D-T fusion.
a. Find Q for this reaction.
b. The alpha particle carries about 20% and the neutron 80% of the released energy. The energy carried by the alpha (since it is charged) can be easily captured. Capturing the neutron’s energy is much more complicated. Calculate the velocity of the neutron.
c. One scheme for capturing the neutron’s energy is to have the reactor encircled in a 1 m thick blanket of liquid lithium. The following reaction would occur:

The alpha could then be easily captured and the 3H used as fuel for additional reactions. Find the Q for this neutron capture reaction.
d. Assuming all of this can be accomplished, how many kg of hydrogen must undergo fusion per year to supply a typical 20 MW power plant? Assume about 30% of the total energy released by fusion and neutron capture is converted to electricity.

Mathematical Analysis

7.A: Alpha Decay (Project)

The radioactive process known as alpha decay involves the tunneling of an alpha particle (a bound state of two protons and two neutrons) through the Coulomb barrier to escape the nuclear potential.

In this activity, you will construct a spreadsheet that calculates the tunneling probability for the alpha particle and, using this probability, calculates the half-life of the radioactive nucleus.

I. Constructing the Spreadsheet

Construct a spreadsheet that has the following general form. (The template AlphaDecay is available in the PHY 262 course folder.)

Alpha Decay
your name:
your name:

alpha energy = MeV T =
A = f =
Z = ▯ =
half-life = s
nuclear depth = MeV
nuclear radius = fm
exit radius = fm

Position Potential integrand
(fm) (MeV)

The spreadsheet should allow you to enter the mass number (A) and atomic number (Z) of a radioactive nucleus, the kinetic energy of the emitted alpha particle, and then calculate the half-life for the transition.

One of the several approximations you will make is to imagine the alpha particle as a point particle moving in the potential well created by the rest of the nucleus. Since the alpha particle consists of 4 nucleons (two of which are protons), the effective mass number of the rest of the nucleus is (A – 4) and the effective charge of the rest of the nucleus is (Z – 2).

For heavy nuclei, the depth of the nuclear potential (UNuclear) is approximately

$$U_{Nuclear} = -50 \text{ MeV} \tag{7.A.1}$$

In addition, the radius of the nuclear potential (R) is

$$R = (1.2 \text{ fm})(A-4)^{1/3} \tag{7.A.2}$$

The electrical potential energy between the alpha particle and the rest of the nucleus (UCoulomb) is given by

$$U_{Coulomb} = qV \tag{7.A.3}$$

$$U_{Coulomb} = (2e)\left(\frac{k((Z-2)e)}{r}\right) \tag{7.A.4}$$

$$U_{Coulomb} = \frac{2(Z-2)ke^2}{r} \tag{7.A.5}$$

(Note that in “nuclear” units, ke2 = 1.44 MeV fm.)

Therefore, the exit radius (RExit) can be determined by equating the alpha energy (EAlpha) to the electrical energy

$$E_{Alpha} = U_{Coulomb} \tag{7.A.6}$$

$$E_{Alpha} = \frac{2(Z-2)ke^2}{R_{Exit}} \tag{7.A.7}$$

$$R_{Exit} = \frac{2(Z-2)ke^2}{E_{Alpha}} \tag{7.A.8}$$

Using these definitions, complete the Position and Potential columns of the spreadsheet for 500 equally spaced points between the nuclear radius and the exit radius.

A. Tunneling Approximation Scheme

The probability of tunneling through a square potential barrier of height U and width x is given by:

$$T \cong e^{-x/\delta} \tag{7.A.9}$$

where

$$\delta = \frac{\hbar c}{\sqrt{8mc^2(U-E)}} \tag{7.A.10}$$

If the barrier is not square, we can approximate the barrier by a series of extremely thin barriers which are approximately square. In the limit of infinitesimally thin barriers, this becomes

$$T \cong e^{-\int dx/\delta} \tag{7.A.11}$$

$$T \cong exp\left(-\frac{\sqrt{8mc^2}}{\hbar c}\int \sqrt{U-E}\,dx\right) \tag{7.A.12}$$

Thus, to find the probability of the alpha particle tunneling through the barrier, you should calculate, at each of the 500 equally spaced points between the nuclear radius and the exit radius, the value of the integrand,

$$\sqrt{U-E}(\Delta x) \tag{7.A.13}$$

The sum of these contributions can then be used to find T. (Note that the rest energy of the alpha particle is 3728 MeV.)

The number of collisions per second the alpha particle makes with the nuclear "wall", f, can be determined from the speed of the alpha particle inside the nucleus and the size of the nucleus. The decay rate, ▯, is the probability of decay per second, and can be determined from T and f. Finally, the half-life is directed related to ▯. The specific relationships are left for you to determine.

II. Using the Spreadsheet

1. Calculate the half-life for ${}^{222}_{92}U$ emitting an alpha particle of energy 9.50 MeV and record it below. (Don't worry if you don't get exactly the answer shown in the data table. Due to the approximations we've made your answer should be a couple hundred times larger than the experimentally measured value.) Print the first page of your spreadsheet and attach it to the end of this activity.

2. Although your value for the half-life is larger than the measured value, your spreadsheet can still be used to explore how half-life depends on alpha energy. For each of the even-A uranium isotopes listed in the data table, determine the theoretical half-life using your spreadsheet. Record these results in a suitably labeled column in the data table.

3. To compare the theoretical dependence between half-life and alpha energy to the experimentally measured dependence, create a graph of the log of the half-life vs. alpha energy. Show both the theoretical and experimental data on this graph. Print this graph and attach it to the end of the activity.

4. Does your spreadsheet accurately reflect the dependence of half-life on alpha energy? Clearly explain why or why not.

5. Clearly explain why a reduction in alpha energy by a factor of approximately two (9.50 MeV to 4.27 MeV) can result in a change in half-life by a factor of approximately 1023 (!).

7.A: Radioactive Chains (Project)

After a radioactive nucleus decays, it often leaves behind a new radioactive nucleus, termed the daughter nucleus. This daughter then decays, often leaving behind another radioactive daughter. This chain of daughter nuclei can get very complicated, as each nucleus typically has a different decay constant, may decay by a different process, and emits a radioactive particle of different energy. For example, starting from a sample of 238U, you will end up with this lovely collection of nuclei:

In this activity, you will construct a spreadsheet that calculates the relative abundance of daughter nuclei over time for a radioactive chain of three members, as well as the energy released by the sample.

I. Constructing the Spreadsheet

Construct a spreadsheet that has the following general form. (The template RadioactiveChains is available in the PHY 262 course folder.)

Radioactive Chains
name:
name:
λ (unit) Q (unit)
Nuclei #1
Nuclei #2
Nuclei #3

time step = unit?

Time (unit?) N1 N2 N3 Energy (unit?)
0 100.00 0.00 0.00 0.00

The spreadsheet should allow you to enter the decay constants for three sequential members in a radioactive chain, the Q for each decay, and a reasonable time step. The abundance of element #1 is initially set to 100, while the initial abundances of the daughters are zero. Whatever unit (s-1, day-1, etc.) you use to express the decay constant should be the same unit (s, day, etc.) used to express the time step.

The Time column should include 100 time steps of the inputted duration, and your spreadsheet should also include graphs of the relative abundance of all three nuclei vs. time (all on one graph), and the total energy released by the sample vs. time.

A. Relative Abundances

The number of radioactive particles present at one instant is related to the number present a short time earlier, adjusted by the decay constant and time step. The decay constant (λ) is, crudely speaking, the probability that a nucleus will decay in some designated time interval. For example, a decay constant of $\lambda = 0.5 textyear^{-1}$ means that there is a 50% chance of the nucleus decaying per year. Therefore, the number of nuclei actually decaying is the product of this decay constant (λ), the number of nuclei currently present (N), and the time interval (dt).

Mathematically,

$$\text{number of decays} = \lambda N dt \tag{7.A.1}$$

This means that the number of nuclei present at one instant is equal to the number present at a previous instant, minus the number decayed.

$$N(t+dt) = N(t) - \lambda N(t) dt \tag{7.A.2}$$

There is one complication, however. In addition to decaying, nuclei are created by the decay of higher mass nuclei. Therefore, the above equation is only correct for the mother, or highest mass, nucleus. For each of the daughters, you would have to add the number that are created by the decay of their mother. If the nuclei are labeled 1, 2, 3, … in order of creation, then

$$N_1(t+dt) = N_1(t) - \lambda_1 N_1(t) dt \tag{7.A.3}$$

$$N_1(t+dt) = N_2(t) - \lambda_2 N_2(t) dt + \lambda_1 N_1(t) dt \tag{7.A.4}$$

$$N_1(t+dt) = N_3(t) - \lambda_3 N_3(t)dt + \lambda_2 N_2(t)dt \tag{7.A.5}$$

Create a spreadsheet that incorporates these ideas for a chain of three hypothetical nuclei with $\lambda_1 = 0.006\text{year}^{-1}$, $\lambda_2 = 0.003\text{year}^{-1}$, and $\lambda_3 = 0.012\text{year}^{-1}$. Determine and graph the abundance of each type of nuclei as a function of time with dt = 10 years.

1. For each nuclei, record below the approximate time at which it is in greatest abundance, and the maximum number present at this time.

If nuclei #2 is not in greatest abundance after approximately 225 years, you've done something wrong.

B. Energy Released

If you were designing an enclosure for the radioactive sample described above, you would need to know how the relative abundances of the different nuclei change over time. Your spreadsheet can now perform that calculation. In addition, however, you would need to know how the amount of energy released by the sample also changes over time. In fact, contrary to what you may think, the amount of energy released by a radioactive sample often increases rather than decreases over time. This means the sample gets "hotter" before it ultimately decays toward stability. To study this process, you will need to incorporate information regarding the energy released by each of the decay processes.

For each radioactive decay reaction, a specific amount of energy is released. This energy is referred to as the Q for the reaction. Thus Q1 is released for every decay of nuclei #1, Q2 for every decay of nuclei #2, and Q3 for every decay of nuclei #3. Therefore,

Energy released = Q1 (decays of N1) + Q2 (decays of N2) + Q3 (decays of N3)

With Q1 = 0.4 MeV, Q2 = 2.6 MeV, and Q3 = 0.8 MeV, adjust your spreadsheet so that it calculates the total amount of energy released for each time step. Create a graph of energy released vs. time.

2. From your spreadsheet, determine the time at which the most energy is being released by the sample. How much energy is being released (per 100 initial nuclei of nuclei #1) at this time? How are you going to engineer a containment system that will still be operational after this amount of time? (I don't expect you to be able to answer that last one.)

II. Using the Spreadsheet

A.

$$^{210}\text{Rn} \xrightarrow{\lambda_1} {}^{210}\text{At} \xrightarrow{\lambda_2} {}^{210}\text{Po} \xrightarrow{\lambda_3} {}^{206}\text{Pb} \tag{7.A.6}$$

Analyze the above chain where $\lambda_1 = 6.93 day^{-1}$, $\lambda_2 = 2.05 day^{-1}$, and $\lambda_3 = 0.0051 day^{-1}$.

1. Determine the Q for $^{210}\text{Rn} \rightarrow {}^{210}\text{At}$.
2. Determine the Q for $^{210}\text{At} \rightarrow {}^{210}\text{Po}$.
3. Determine the Q for $^{210}\text{Po} \rightarrow {}^{206}\text{Pb}$.

Adjust your time step so that your graph shows all of the N1 and most of the N2 decayed. Print-out the first page of your spreadsheet calculations and both graphs and attach them to the end of this activity.

4. When (in days) will the amount of 210At be the greatest? What is the maximum amount present?

5. When (in days) will the amount of 210Po be the greatest? What is the maximum amount present? (You will have to use a different time step than above to answer this question.)

6. When (in days) is the maximum amount of energy released by this chain?

B. Thorium Series

Wikipedia (http://en.Wikipedia.org/wiki/Decay_chain) has a very thorough discussion of the major radioactive decay chains, including half-lives and Q values for each decay process. Create a spreadsheet model for the 3-steps leading from 228Ra to 224Ra.

Adjust your time step until you can clearly see the maximum of the 228Th abundance curve.

7. Why is it so difficult to choose the appropriate time step?

The large difference in half-life between the various daughters makes it impossible to pick a time step where, over only 100 cycles, you can get significant 228Ra decay but yet not have your calculation “blow-up” for 226Ac. There are two relatively simple solutions to this problem.

Typically, if you are interested in behavior over the span of years, half-lives on the order of days or less can be considered “instantaneous”. Thus, you can approximate 228Ra as decaying directly to 228Th with a half-life of 5.75 yrs and a Q equal to the total energy released by the two processes, 2.170 MeV.

The other option is just to increase the number of cycles of your calculation. Instead of 100 time steps, use 10,000!

Using 10,000 cycles, adjust your time step until you can clearly see the maximum of the 228Th abundance curve. Print-out the first page of your spreadsheet calculations and both graphs and attach them to the end of this activity.

8. Clearly explain why 228Ac is never present in substantial amounts.

9. When (in years) will the amount of 228Th be the greatest? What is the maximum amount present?

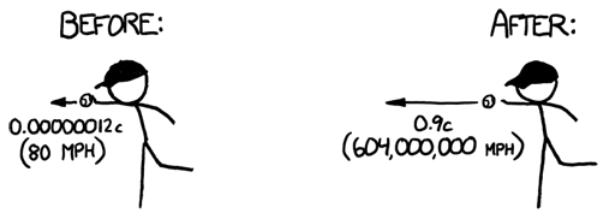
10. When (in years) is the maximum amount of energy released by this chain?

7.A: Relativistic Baseball (Project)

Text and drawings from: https://what-if.xkcd.com/1/

What would happen if you tried to hit a baseball pitched at 90% the speed of light? - Ellen McManis

Let's set aside the question of how we got the baseball moving that fast. We'll suppose it's a normal pitch, except in the instant the pitcher releases the ball, it magically accelerates to 0.9c. From that point onward, everything proceeds according to normal physics:



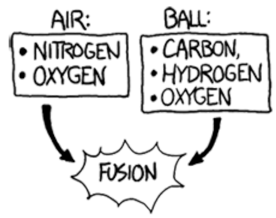
The answer turns out to be "a lot of things", and they all happen very quickly, and it doesn't end well for the batter (or the pitcher). I sat down with some physics books, a Nolan Ryan action figure, and a bunch of videotapes of nuclear tests and tried to sort it all out. What follows is my best guess at a nanosecond-by-nanosecond portrait:

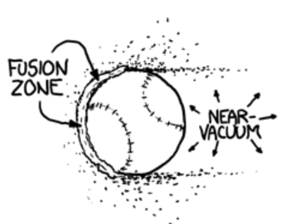
The ball is going so fast that everything else is practically stationary. Even the molecules in the air are stationary. Air molecules vibrate back and forth at a few hundred miles per hour, but the ball is moving through them at 600 million miles per hour. This means that as far as the ball is concerned, they're just hanging there, frozen.

Q1

Find the total mass of air and the total number of air molecules struck by the ball on its way to the plate. How many of these molecules are N_2 and how many are O_2? (Obviously, you are going to need to find out some basic parameters about baseballs and air.)

The ideas of aerodynamics don't apply here. Normally, air would flow around anything moving through it. But the air molecules in front of this ball don't have time to be jostled out of the way. The ball smacks into them so hard that the atoms in the air molecules actually fuse with the atoms in the ball's surface. Each collision releases a burst of gamma rays and scattered particles.

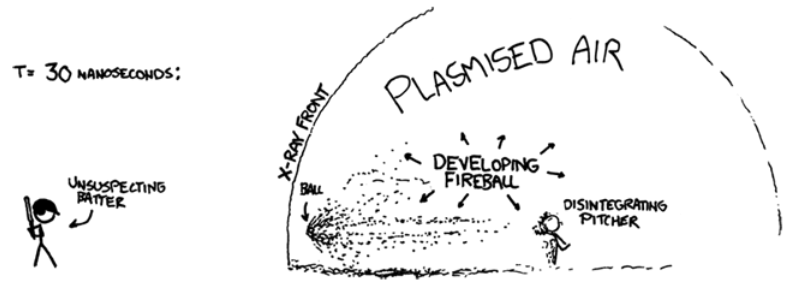




Q2

Calculate plausible reactions and Q's for the fusion of the air with the carbon of the ball.

These gamma rays and debris expand outward in a bubble centered on the pitcher's mound. They start to tear apart the molecules in the air, ripping the electrons from the nuclei and turning the air in the stadium into an expanding bubble of incandescent plasma. The wall of this bubble approaches the batter at about the speed of light—only slightly ahead of the ball itself.



The constant fusion at the front of the ball pushes back on it, slowing it down, as if the ball were a rocket flying tail-first while firing its engines. Unfortunately, the ball is going so fast that even the tremendous force from this ongoing thermonuclear explosion barely slows it down at all. It does, however, start to eat away at the surface, blasting tiny particulate fragments of the ball in all directions. These fragments are going so fast that when they hit air molecules, they trigger two or three more rounds of fusion.

Q3

Using momentum conservation, and ignoring the change in mass of the ball due to ablation, analyze the collision of the ball and the air molecules as a simple, inelastic collision (i.e., the air sticks to the ball). Find the speed of the ball as it crosses home plate. (Hint: You know the total mass of air that “sticks’ to the ball from question 1.)

After about 70 nanoseconds the ball arrives at home plate. The batter hasn't even seen the pitcher let go of the ball, since the light carrying that information arrives at about the same time the ball does.

Q4

Find the difference in arrival times between the light and the ball, ignoring the deceleration of the ball. This is the amount of time you would have to react and swing (assuming it’s a strike). (It normally takes about 30 ms to react to a visual stimulus.)

Collisions with the air have eaten the ball away almost completely, and it is now a bullet-shaped cloud of expanding plasma (mainly carbon, oxygen, hydrogen, and nitrogen) ramming into the air and triggering more fusion as it goes. The shell of x-rays hits the batter first, and a handful of nanoseconds later the debris cloud hits.

When it reaches the batter, the center of the cloud is still moving at an appreciable fraction of the speed of light. It hits the bat first, but then the batter, plate, and catcher are all scooped up and carried backward through the backstop as they disintegrate. The shell of x-rays and superheated plasma expands outward and upward, swallowing the backstop, both teams, the stands, and the surrounding neighborhood—all in the first microsecond.

Suppose you’re watching from a hilltop outside the city. The first thing you see is a blinding light, far outshining the sun. This gradually fades over the course of a few seconds, and a growing fireball rises into a mushroom cloud. Then, with a great roar, the blast wave arrives, tearing up trees and shredding houses.
Everything within roughly a mile of the park is leveled, and a firestorm engulfs the surrounding city. The baseball diamond is now a sizable crater, centered a few hundred feet behind the former location of the backstop.

Q5

Based on your Q’s from question 2, and the number of collisions from question 1, estimate the total energy released by fusion. (Assume all of the air molecules undergo fusion with carbon.) (For comparison, the total energy released by the Hiroshima bomb was about $60 \times 10^{12}\,J$, and it leveled approximately 5 square miles.)

A careful reading of official Major League Baseball Rule 6.08(b) suggests that in this situation, the batter would be considered "hit by pitch", and would be eligible to advance to first base.

CHAPTER OVERVIEW

8: Misc - Semiconductors and Cosmology

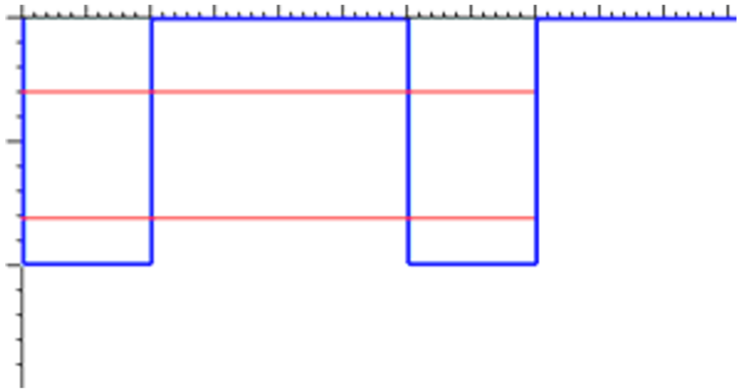
Topic hierarchy

Thumbnail: The Hubble eXtreme Deep Field (XDF) was completed in September 2012 and shows the farthest galaxies ever photographed. Except for the few stars in the foreground (which are bright and easily recognizable because only they have diffraction spikes), every speck of light in the photo is an individual galaxy, some of them as old as 13.2 billion years; the observable universe is estimated to contain more than 2 trillion galaxies. (Public domain; NASA).

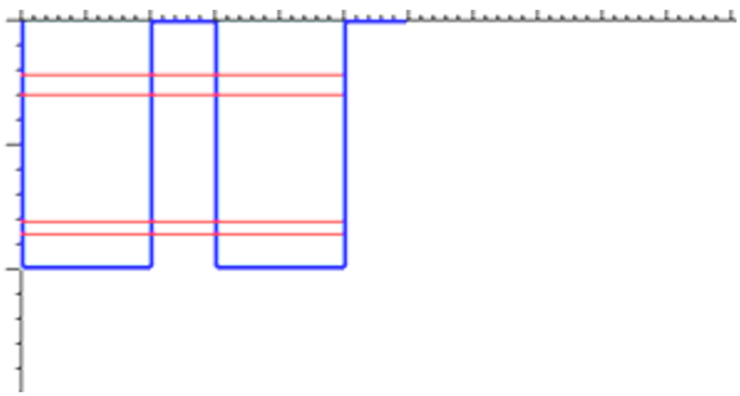
8.1: Modeling Semiconductors (Project)

I. Conductors, Semiconductors and Insulators

In a gas, where inter-atomic distances are large, the energy levels of one gas atom are not altered by the presence of the other gas atoms. Thus, you can think of each atom as a separate entity. A simplified energy level diagram for two gas atoms would look like this:

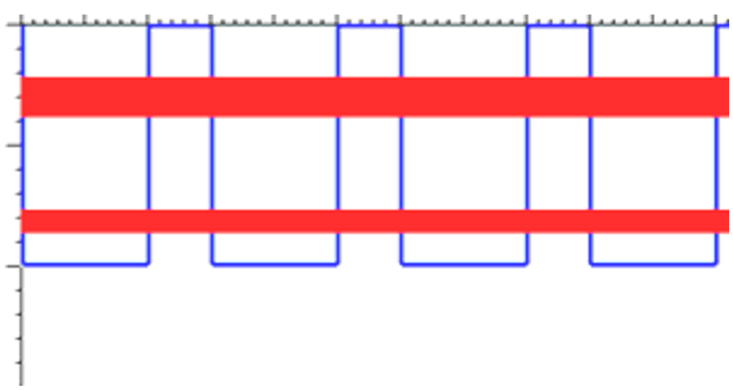


However, in a solid the atoms are close enough together that they directly influence each other's energy levels. For example, in a hypothetical two-atom solid (a pretty small solid), the energy levels look like this:



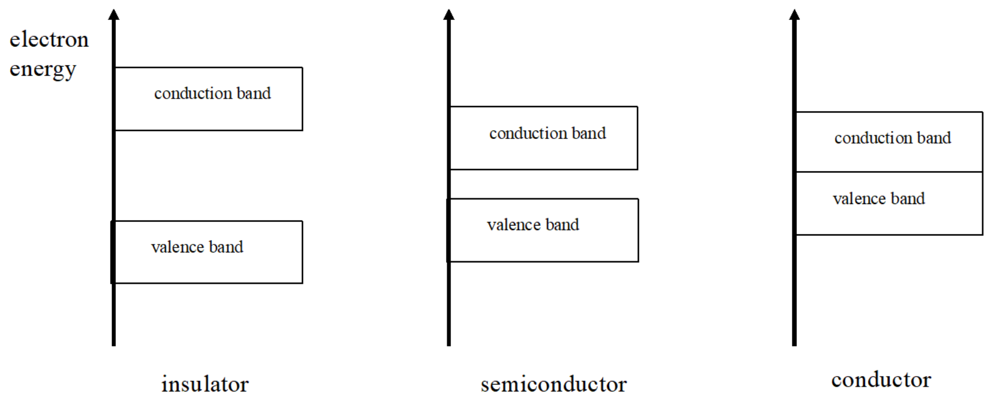
The splitting in the energy levels occurs because an "odd" wavefunction (positive in one well and negative in the other) has a different energy than an "even" wavefunction (positive in both wells).

If we were to look at realistic solids (with more than just a handful of atoms), this discreet splitting into two levels becomes a continuous blurring of energy levels into energy bands.



Thus, the allowed energy levels for electrons in solids are actually continuous bands separated by finite gaps in energy. The size of the band gap between the highest filled energy band (the valence band) and the lowest unfilled energy band (the conduction band) is of crucial importance in determining the solids properties.

For example, if this band gap is large (a few eV), electrons in the valence band will not be able to easily "jump" into the conduction band and the solid will be an electrical insulator. If the gap is basically non-existent, the electrons have free access to the conduction band and the solid is a conductor. If the gap is small (near 1 eV), the solid has a rich variety of "intermediate" properties that make into particularly intriguing. These materials are *intrinsic semiconductors*. The three possibilities are illustrated below.



II. Intrinsic Semiconductors

Imagine an electron in the valence band of a semiconductor. This electron is bound to its "parent" atom, but longs to be free to roam the entire solid. If only it could "jump" up to the conduction band! If it could, it would no longer be bound to any one atom but rather it would be free to flow wherever forces take it in the solid. In other words, it could be part of a current. But where will it get the energy needed to make this jump? From the random thermal vibrations of the solid!

In general, the relative probability of acquiring an amount of energy, E, from the thermal energy available in a material is given by the Boltzmann Factor:

$$\text{Probability} \approx e^{-E/kT} \tag{8.1.1}$$

where E is the energy acquired, T is the absolute temperature, and k is Boltzmann's constant ($8.617 \times 10^{-5}\ eVK^{-1}$).

For quantum mechanical reasons we'll gloss over, the effective energy needed to jump the gap is only about ½ of the actual band gap (and even this factor of ½ depends on the precise manufacturing of the semiconductor). Thus, the probability of an electron jumping from the valence band to the conduction band is:

$$\text{Probability} \approx e^{-E_{gap}/2kT} \tag{8.1.2}$$

where E_{gap} is the band gap energy.

For an insulator ($E_{gap} \approx 4\ eV$) at room temperature (T ~ 293 K), the probability is:

so the electron better get used to being totally bound to its parent atom.

For a semi-conductor ($E_{gap} \approx 1\ eV$) the odds are much better:

You may think that a one in a billion chance is pretty small odds, but remember that there are about 10^{22} atoms in a single gram of silicon. At these odds, quite a few electrons will win the lottery and be free. Of course, in a conductor every atom contributes a free electron so although an intrinsic semiconductor will conduct current, its conductivity is quite a bit less than a typical metal.

One interesting property of intrinsic semiconductors, in contrast to conductors, is their very strong temperature dependence. Semiconductors become exponentially *more* conductive with increasing temperature while conductors typically become linearly *less* conductive with temperature. This allows semiconductors to be very effective as thermistors. For example, over a temperature range from 0 C to 100 C, the current in a semiconductor will increase by a factor of:

This enormous change in current can be easily measured allowing a very precise temperature determination.

One final feature of intrinsic semiconductors is the concept of a hole. When an electron jumps from the valence to the conduction band (a process termed thermal generation) it leaves behind a positively charged atom. Valence electrons from other atoms can then move into the vacancy (or hole) in the first atom. This process will continue with the hole gradually moving through the solid. In fact, the motion of the hole is indistinguishable from the motion of an actual positive charge carrier. This process does not occur in conductors because in a conductor all of the atoms are missing electrons and thus all of the atoms have the same positive charge. In the graphic below, the black dots represent electrons and the white dots are holes. Notice that the electron current and the hole current are always in opposite directions.

Recombination occurs when a freely moving electron meets a freely moving hole. The energy released at recombination, when the electron "falls" into the hole, is normally absorbed by neighboring atoms, but it is possible to construct a device in which this energy is released as a photon (with energy closely matching the band gap energy). This device is a light-emitting diode, or LED. Photovoltaic cells are, in essence, this process in reverse, in which an incoming photon creates an electron and hole current.

Q1

Imagine a 5.0 g crystal of pure Ge (Egap = 0.7 eV).
a. At room temperature (293 K), determine the average number of mobile electrons.

b. At room temperature (293 K), determine the average number of mobile holes.

c. At what temperature will the number of mobile charge carriers be twice the number calculated in a and b?

d. At what temperature will a 5.0 g crystal of pure Si (Egap = 1.1 eV) have the same number of mobile charge carriers as calculated in a and b?

Q2

a. Calculate the range of energy gaps that could produce visible light on recombination.

b. Consider a mole of pure semiconducting material with the gap needed to produce violet (400 nm) light. At room temperature, now many mobile electrons and holes are available to recombine into photons? What does this imply about making LEDs out of pure (non-doped) semiconducting material?

III. Doped Semiconductors

In a conductor, the density of charge carriers is fixed by the density of the material. The amazing utility of semiconductors is because this density can be altered by carefully introducing impurities into the semiconductor. This process, termed doping, allows you to tailor many of the properties of the semiconductor.

In pure silicon, each atom makes four double-bonds with its nearest neighbors. If we replace a small number of silicon atoms with, for example, antimony atoms (or any other atom with a valence of five), one of the antimony electrons will be unbound and free to move about the solid. Thus, for each antimony atom we get one free electron. Notice that this doping does not produce a corresponding hole. This type of doping creates n(egative)-type material:

Conversely, if we replace a small number of silicon atoms with, for example, boron atoms (or any other atom with a valence of three), one of the silicon electrons will be unpaired. This creates a hole that an electron from an adjacent atom will fill. This type of doping creates p(ositive)-type material:

Material is doped by either growing a crystal from molten silicon (or germanium) containing precise amounts of doping material, or by injecting the doping material into a thin surface film of the semiconductor.

Although the contribution to the current density due to the intrinsic material (via thermal generation) is highly temperature dependent, the contribution due to doping is independent of temperature. Imagine a sample of silicon doped with phosphorus at a rate of 1 per 20,000 atoms. At 273 K, consider the ratio of free electrons from phosphorus (P) to free electrons from silicon (Si):

$$\frac{i(P)}{i(S_i)} \approx \frac{1}{20000e^{-\frac{1.1}{2(8.617x10^{-5})(273)}}} \tag{8.1.3}$$

$$\frac{i(P)}{i(S_i)} \approx 700,000 \tag{8.1.4}$$

compared to at 373 K:

$$\frac{i(P)}{i(S_i)} \approx \frac{1}{20000e^{-\frac{1.1}{2(8.617x10^{-5})(373)}}} \tag{8.1.5}$$

$$\frac{i(P)}{i(S_i)} \approx 1,400 \tag{8.1.6}$$

A temperature change of 100 K shifts this ratio by a factor of 500.

3. A crystal of Ge is doped at a rate of 1/30000 with boron. Find the ratio of doped holes (from B) to thermal holes (from Ge) at room temperature.

IV. The p-n Junction

Imagine a sample of n-type material brought into contact with p-type material. (Remember, although there are free holes in the p-material and free electrons in the n-material, both materials are initially electrically neutral.) Very rapidly after being brought into contact, electrons will drift across the boundary and quickly combine with holes (creating fixed negative ions) and holes will rapidly drift across and combine with electrons (creating fixed positive ions), as shown below:

This process will continue until the electric field formed by the creation of the negative and positive ions (pointing left within the depletion region above) grows large enough to prevent further drifting of electrons and holes. Once this equilibrium is reached, the overall neutral material will have a “built-in” electric field in the depletion region separating the two conducting regions. This electric field will also give rise to a “built-in” potential difference between the n- and p-sides of the material. What we’d like to do now is calculate the width of the depletion region (d), as well as the electric field and electric potential in this region.

To determine the electric field, let’s apply Gauss’ law to the Gaussian surface illustrated below:

+x

The only surface with flux is the right-hand side of the gaussian rectangle, with area A. The total charge within the gaussian rectangle is the product of the doping charge density (ρ), which is negative, and the volume of the rectangle within the depletion zone (Ax). Also note that since the region of interest is not vacuum, the permittivity is given by the product of the dielectric constant (κ) and ε_0.

Putting this all together yields,

$$\oint E \cdot dA = \frac{q_{enclosed}}{\varepsilon} \tag{8.1.7}$$

\[E(A) = \frac{(-\rho)(Ax)}{\kappa\varepsilon_0\}\]

$$E = -\frac{\rho x}{\kappa \varepsilon_0} \tag{8.1.8}$$

The negative sign indicates the electric field points to the left. This expression is only valid from x = 0 to x = d/2, since beyond d/2 the magnitude of the enclosed charge begins to decrease. By symmetry, the electric field in the depletion region looks like this:

Now that we know the electric field in the depletion zone we can determine an expression for the potential difference across the depletion zone.

$$\Delta V = - \in_0^d E \cdot dx \tag{8.1.9}$$

By symmetry, we can integrate from 0 to d/2 and double the result:

$$\Delta V = 2(i \int_0^{d/2} E \cdot dx) \tag{8.1.10}$$

$$\Delta V = 2(- \int_0^{d/2} -\frac{\rho x}{\kappa \varepsilon_0} dx) \tag{8.1.11}$$

$$\Delta V = 2\left(\frac{\rho\left(\frac{d}{2}\right)^2}{2\kappa\varepsilon_0} - 0\right) \tag{8.1.12}$$

$$\Delta V = \frac{\rho d^2}{4\kappa\varepsilon_0} \tag{8.1.13}$$

Thus, the potential within the depletion zone looks like this:

The fundamental remaining question is how large will the potential difference grow before the junction reaches equilibrium. To determine this, let’s examine the p-material:

• In order for a hole to roll “uphill” into the n-material it would need to have a kinetic energy at least as large as e V. The probability of this occurring is given by the Boltzmann factor with energy e V. If this did occur, the potential difference would increase. This flow of charge is referred to as the recombination current, iR, and is given by:

$$i_R \approx N_{holes} e^{-\frac{e\backslash\text{DeltaV}}{kT}} \tag{8.1.14}$$

• On the other hand, it is also possible for a thermally generated electron to form in the p-material and freely roll uphill into the n-material (remember, negative charges roll “up” electric potentials). The probability of this occurring is given by the Boltzmann factor with energy Egap/2. If this occurred, the potential difference would decrease. This flow of charge is referred to as the thermal generation current, iTh, and is given by:

$$i_{Th} \approx N_{holes} e^{-\frac{E_{gap}}{2kT}} \tag{8.1.15}$$

At equilibrium, these two currents must be equal. Of course, the same processes can occur in the n-material and, assuming symmetric doping, equilibrium will occur at the same potential difference.

Finally, we can calculate some stuff! Let’s imagine a room temperature silicon p-n junction doped (on both sides) at 1 part per 40,000 atoms. The density of silicon is 2.4 g/cm3, silicon has 28 g/mol, and the dielectric constant for silicon is 12.

First, let’s calculate the internal potential difference across the junction. Setting the two currents equal leads to:

$$i_R = i_{Th} \tag{8.1.16}$$

\[N_{holes} e^{-\frac{e\Delta V}{kT} = N_{atoms} e^{-\frac{E_{GAP}}{2kT}}\]

\[1e^{-\frac{e\Delta V}{kT} = 40000e^{\frac{E_{gap}}{2kT}\]

$$e^{\left(\frac{E_{gap}}{2} - e\Delta V\right)/kT} = 40000 \tag{8.1.17}$$

$$\left(\frac{E_{gap}}{2} - e\Delta V\right)/kT = \ln(40000) \tag{8.1.18}$$

$$e\Delta V = \frac{E_{gap}}{2} - kT\ln(40000) \tag{8.1.19}$$

$$e\Delta V = 0.29\backslash\text{tex}eV \tag{8.1.20}$$

$$\Delta V = 0.29\text{ V} \tag{8.1.21}$$

Since the doping charge density is:
\[\rho = \bigg(\frac{1.6\times 10^{-19)C}{40000\text{ atoms}}\bigg)\bigg(\frac{6.02\times 10^{-23}\text{ atoms}}{mol}\

the width of the depletion zone is:

The depletion zone is only about 24 nm wide at equilibrium.

The maximum magnitude electric field, occurring at the boundary between the p- and n-material, is

The electric field at the boundary is a whopping 24 megavolts per meter.

4. Imagine a room temperature germanium p-n junction doped (on both sides) at 1018 atoms per cm3. The density of germanium is 5.4 g/cm3, germanium has 73 g/mol, and the dielectric constant for germanium is 16. Calculate the internal potential difference, the width of the depletion zone, and the maximum electric field in the junction.

V. The Diode

Now that we understand what happens when a pn-junction is formed, our next (and final) task is to show how such a junction acts in a circuit. First, let's consider a pn-junction under reverse bias conditions, in which the positive supply voltage is attached to the n-material.

In this configuration, the bias voltage, Vb, adds to the internal potential difference of the junction effectively making the potential "hill" higher:

Now consider how this affects the two currents flowing through the junction. We will again look at the p-material but the same conclusions can be drawn from the n-material.

• It is now more difficult for a hole to roll "uphill" into the n-material since it needs more kinetic energy, at least as large as e(V+Vb). Thus the recombination current becomes:

$$i_R \approx N_{holes}e^{-\frac{e(\text{\DeltaV}+V_b)}{kT}} \tag{8.1.22}$$

• On the other hand, this has no effect on the thermal generation current, since thermally generated electrons freely roll uphill regardless of hill height. The thermal generation current is still given by:
\[i_{Th}\approx N_{atoms}e^{-\frac{E_{gap}}{2kT}\]

The total current is given by:

$$i = i_R = -i_{Th} \tag{8.1.23}$$

$$i = N_{holes}e^{-\frac{e(\Delta V+V_b}{kT}} - i_{Th} \tag{8.1.24}$$

$$i = (N_{holes}e^{-\frac{e(\Delta V}{kT}})e^{-\frac{e(\Delta V_b}{kT}} - i_{Th} \tag{8.1.25}$$

Notice that the term in parentheses is exactly equal to iTh since the junction was in equilibrium before the external voltage was applied. Thus:

$$i = i_{Th}e^{-\frac{e(\Delta V_b}{kT}} - i_{Th} \tag{8.1.26}$$

$$i = i_{Th}(e^{-\frac{e(\Delta V_b}{kT}} - 1) \tag{8.1.27}$$

Thus, the current is negative (to the left through the junction) and very small, no larger than the thermal current. In fact, the current reaches the thermal current exponentially quickly and remains relatively constant for any bias voltage less than the break-down voltage of the device.

Now let's consider a pn-junction under forward bias conditions, in which the positive supply voltage is attached to the p-material.

In this configuration, the bias voltage, Vb, subtracts from the internal potential difference of the junction effectively making the potential "hill" smaller:

Now consider how this affects the two currents flowing through the junction.

• It is now easier for a hole to roll "uphill" into the n-material since it needs less kinetic energy, now only e(V-Vb). Thus the recombination current becomes:

$$i_R \approx N_{holes}e^{-\frac{e(\Delta V+V_b}{kT}} \tag{8.1.28}$$

• Again, this has no effect on the thermal generation current, since thermally generated electrons freely roll uphill regardless of hill height.

The total current is given by:

$$i = i_R - i_{Th} \tag{8.1.29}$$

$$i = N_{holes}e^{-\frac{e(\Delta V-V_b}{kT}} - i_{Th} \tag{8.1.30}$$

$$i=(N_{holes}e^{-\frac{e(\Delta V}{kT}})e^{+\frac{e(\Delta V_b}{kT}}-i_{Th} \tag{8.1.31}$$

$$i=i_{Th}e^{+\frac{e(\Delta V_b}{kT}}-i_{Th} \tag{8.1.32}$$

$$i=i_{Th}(e^{+\frac{e(\Delta V}{kT}}-1) \tag{8.1.33}$$

Thus, the current is positive (to the right through the junction) and grows exponentially large with applied voltage.

The two results can be combined into the Shockley Diode Equation,

\[i =i_{Th}(e^{\pm\frac{eV_b}{kT}-1)\]

where the positive sign is chosen for forward and the negative sign for reverse bias.

Q5

a. Estimate the ratio of forward-bias (+0.5 V) current to reverse-bias (-0.5 V) current in a room temperature Ge pn-junction doped at 1/50000.

b. Estimate the same ratio at 50° C.

8.2: Modeling the Universe (Project)

Cosmology is the scientific study, and the associated mathematical models, of the universe as a whole. Amazingly, using only a few basic scientific principles one can construct a quantitative model of the universe that agrees surprisingly well with observational data. This model not only is in agreement with current observations, but makes clear predictions for the age of the universe and its future fate.

I. Energy Conservation and the First Friedmann Equation

Consider a large region of our universe, extending a few hundred million light years in all directions from earth. Observationally, portions of the universe this large (or larger) appear isotropic (the same in all directions) and homogeneous (the same from all locations within the region). This implies that if we were to be magically transported to a random location within this region, the universe, on large scales, would look much the same as it does from earth. This observation is consistent with the simplest predictions of relativity, which imply the lack of a "preferred" reference system in the universe. Thus, we can analyze a chunk of the universe centered on the earth (which we can easily observe) and draw conclusions that hold for the entire universe, even the portions beyond our direct observation.

Consider the large region of the universe in Figure $8.2.1$, and the one particular galaxy located a distance R from the Milky Way.

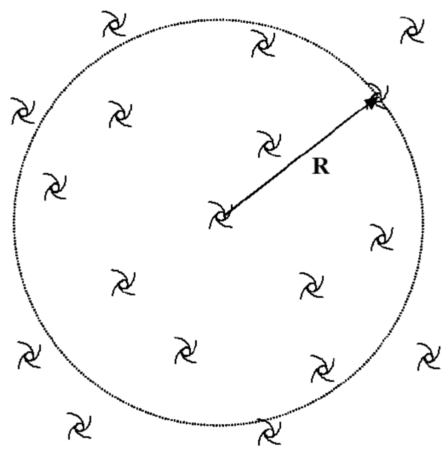


***Figure* $8.2.1$**

The total energy of this galaxy is the sum of its kinetic energy and gravitational potential energy.

$$E=\frac{1}{2}mv^2-\frac{GMm}{R} \tag{8.2.1}$$

where, by Gauss' Law, M is simply the total mass of all of the galaxies within the gaussian sphere of radius R. (The mass of the galaxies outside of the sphere do not contribute to the galaxy's potential energy.)

In addition to isotropy and homogeneity, another set of observations plays a central role in our understanding of the universe. In 1929, while measuring the velocities of distant galaxies, Edwin Hubble was stunned to find that every galaxy cluster measured was moving directly away from earth. Moreover, he noted that the velocity of each galaxy was directly proportional to its distance from earth, i.e., nearby galaxies are moving away from the earth at relatively low speeds while distant galaxies are moving away very quickly. This is, of course, the first piece of observational evidence in support of the expansion of the universe resulting from the Big Bang.

Mathematically, Hubble's measurements can be summarized as:

$$v=\frac{dR}{dt}=HR \tag{8.2.2}$$

where H is the Hubble Constant. (The Hubble constant is not actually constant, but rather is a function only of time.)

Putting this together with the energy result yields:

$$E=\frac{1}{2} m H^2 R^2-\frac{G M n}{R} \tag{8.2.3}$$

Now, what does this equation tell us about the universe? Notice that the energy of this galaxy is the difference between two terms, a positive kinetic energy term related to the Hubble constant and a negative potential energy term related to the total mass between the earth and this galaxy. Since this galaxy, or more properly galaxy cluster, is an isolated system its energy is conserved. Therefore if the total energy is positive today, it will remain positive forever. Thus, as R gets larger and larger (due to Hubble expansion) the potential energy term drops to zero leaving only the positive kinetic energy term. Thus, even as R approaches infinity, the galaxy still has positive kinetic energy. This means that the galaxy will move away from us *forever* (and the universe will expand forever because there is nothing special about this particular galaxy).

Conversely, if the energy is currently negative, it must always be negative. Since the first term in Equation 8.2.3 is proportional to R^2, the only way to keep this term from making the entire energy positive is for H to decrease to zero (i.e., the galaxy must stop moving). Once it stops moving, we've got problems. The galaxy (and hence the entire universe) will begin to collapse back toward the earth. The situation is exactly analogous to throwing an object upward from the surface of the earth. If the object's kinetic energy exceeds its gravitational potential energy it will move away forever, if it doesn't it will come falling back down. To determine the ultimate fate of the universe we need only to determine the value of the Hubble constant (which is relatively easy) and the total mass of our region of the universe (which is where the problems lie …).

Let's continue to manipulate this equation into the form typically used by cosmologists. The total mass between the earth and the galaxy in question can be written in terms of the mass density within the sphere (ρ) as:

$$M=\rho\left(\frac{4}{3} \pi R^3\right) \tag{8.2.4}$$

Thus,

$$E=\frac{1}{2} m H^2 R^2-\frac{4 \pi G m \rho R^2}{3} \tag{8.2.5}$$

$$\frac{2 E}{m}=H^2 R^2-\frac{8 \pi G \rho R^2}{3} \tag{8.2.6}$$

Since E and m (and the number 2) are constants, it would be reasonable to lump them all together into a single constant. Let's be unreasonable and define them to be the product of two different constants, k and $C2$, and throw in a negative sign for good measure:

$$-k C^2=\frac{2 E}{m} \tag{8.2.7}$$

The constant k is the *curvature* of the universe, and C^2 is chosen such that k is equal to either 1, 0, or -1.

- If $k=1$, the curvature of the universe is positive and space has a spherical shape. This requires E to be negative, and therefore the universe will collapse.
- If $k=-1$, the curvature of the universe is negative and space has a hyperbolic shape. This requires E to be positive, and therefore the universe will continually expand.
- If $k=0$, the universe has a flat, Euclidean shape. This requires the universe to have no net energy, and the expansion of the universe asymptotically approaches zero velocity as R approaches ∞.

With this definition, Equation 8.2.6 becomes

$$-k C^2=H^2 R^2-\frac{8 \pi G \rho R^2}{3} \tag{8.2.8}$$

$$H^2 R^2=\frac{8}{3} \pi G \rho R^2-k C^2 \tag{8.2.9}$$

$$\left(\frac{d R}{d t}\right)^2=\frac{8}{3} \pi G \rho R^2-k C^2 \tag{8.2.10}$$

Equation 8.2.10 is known as the *first Friedmann equation*, or the *velocity equation*, since it is a differential equation for the "velocity" of the expansion of the universe. Although we have used classical physics to derive this result, the result is relativistically correct as long as we interpret ρ as the *energy density* rather than the *mass density* of the universe. Also, R is typically referred to as the scale factor of the universe since it applies to any distance measure in the universe, not only the distance between the earth and a particular galaxy. Finally, the constant C can be shown to be everybody's favorite constant in relativity, the speed of light (c)! Therefore, the first Friedmann equation is:

$$\left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\rho R^2 - kc^2 \tag{8.2.11}$$

II. Work-Energy and the Second Friedmann Equation

In general, the only way a sample of material can change its total energy is by doing work. In general, the change in energy of a system is equal to the negative of its work done (i.e., if it does (positive) work, its change in energy is negative):

$$\Delta E = -work \tag{8.2.12}$$

$$= -\int F(dx) \tag{8.2.13}$$

The force that a system exerts on its surroundings can be represented as the product of its pressure and its area of contact

$$\Delta E = -\int pA(dx) \tag{8.2.14}$$

and the product of the area of contact and the displacement of this area (dx) is the change in volume of the system

$$\Delta E = -\int p(dV) \tag{8.2.15}$$

$$dE = -pdV \tag{8.2.16}$$

$$\frac{dE}{dt} = -p\frac{dV}{dt} \tag{8.2.17}$$

Let's now examine not a single galaxy a distance R from earth but rather the total energy within the sphere of radius R. With ρ interpreted as the energy density, the total energy within the sphere is simply

$$E = \rho\left(\frac{4}{3}\pi R^3\right) \tag{8.2.18}$$

Thus,

$$\frac{dE}{dt} = -p\frac{dV}{dt} \tag{8.2.19}$$

$$\frac{d}{dt}\left(\rho\frac{4}{3}\pi R^3\right) = -p\frac{d}{dt}\left(\frac{4}{3}\pi R^3\right) \tag{8.2.20}$$

$$\left(\frac{d\rho}{dt}\right)\frac{4}{3}\pi R^3 + \frac{4}{3}\pi\rho\left(3R^2\frac{dR}{dt}\right) = -p\frac{4}{3}\pi\left(3R^2\frac{dR}{dt}\right) \tag{8.2.21}$$

$$\left(\frac{d\rho}{dt}\right)R + \rho\left(3\frac{dR}{dt}\right) = -p\left(3\frac{dR}{dt}\right) \tag{8.2.22}$$

$$R\left(\frac{d\rho}{dt}\right) = -3(\rho + p)\left(\frac{dR}{dt}\right) \tag{8.2.23}$$

Let's call Equation 8.2.23 the *cosmological work-energy relation*. Great, it has a name, but what does it mean?

Since the universe is expanding,

$$\frac{dR}{dt} > 0 \tag{8.2.24}$$

and thus

$$\frac{d\rho}{dt} < 0. \tag{8.2.25}$$

Thus, as the universe expands, the energy density decreases (i.e., is diluted by the expansion). Just as the temperature of an ideal gas decreases as it expands, the "temperature" of the universe decreases as it expands.

There is one small twist, however. This story assumes that the energy density and pressure of the universe are positive. If, for example, the pressure is negative (whatever that means) and larger in magnitude than the energy density, you could get the strange result of a universe that expands and *increases* its energy density. More about this twist later .

So now that we have this lovely equation what should we do with it? How about if we start with a time derivative of the Friedmann Equation (Equation 8.2.11):

$$\frac{d}{dt}\left(\frac{dR}{dt}\right)^2 = \frac{d}{dt}\left[\frac{8}{3}\pi G\rho R^2 - kc^2\right] \tag{8.2.26}$$

$$2\frac{dR}{dt}\frac{d^2R}{dt^2} = \frac{8}{3}\pi GR^2\frac{d\rho}{dt} + \frac{8}{3}\pi G\rho\left(2R\frac{dR}{dt}\right) + 0 \tag{8.2.27}$$

Notice that in the middle term of Equation 8.2.27 is a factor of

$$R\frac{d\rho}{dr}. \tag{8.2.28}$$

We can substitute our result from Equation 8.2.23 in for this factor in Equation 8.2.27:

$$2\frac{dR}{dt}\frac{d^2R}{dt^2} = \frac{8}{3}\pi GR\left[-3(\rho+p)\left(\frac{dR}{dt}\right)\right] + \frac{8}{3}\pi G\rho\left(2R\frac{dR}{dt}\right) \tag{8.2.29}$$

$$2\frac{d^2R}{dt^2} = \frac{8}{3}\pi GR(-3\rho-3p) + \frac{8}{3}\pi GR(2\rho) \tag{8.2.30}$$

$$= \frac{8}{3}\pi GR(-\rho-3p) \tag{8.2.31}$$

$$\frac{d^2R}{dt^2} = -\frac{4}{3}\pi GR(\rho+3p) \tag{8.2.32}$$

Equation 8.2.32 is known as the *second Friedmann equation*, or the *acceleration equation*, since it is a differential equation for the "acceleration" of the expansion of the universe. It seems to imply that the acceleration of the universe is negative, meaning that the expansion of the universe must slow over time. The sum of the energy density and three times the pressure determines the rate of deceleration, i.e., a universe with lots of energy (and hence mass) has more gravity and slows quicker than a universe with less mass. Two interesting features need attention, however. First, it's not just matter that creates gravitational attraction since normal positive pressure also slows the expansion of the universe. (This means that a gas in a smaller container will exert more gravitational attraction than the same temperature and mass of gas in a larger container.) Second, if somehow a material had negative pressure, and there was enough of it in the universe, the expansion of the universe could speed up over time.

III. The Equation of State

An *equation of state* is a relationship between the energy density and pressure of a substance. For example, the ideal gas law,

$$pV = nRT \tag{8.2.33}$$

is an equation of state because temperature is a measure of energy density. What we are going to do next is find the equation of state for the primary constituents of the universe, photons and matter, and then use the Friedmann equations to calculate what a universe comprised of these two substances would look like.

A. Photons

Imagine a single photon, traveling in the +x-direction with momentum P, striking and elastically rebounding from a surface (Figure 8.2.2). The force exerted on the surface by this photon is given by:

$$F_{1\,photon} = \frac{\Delta P}{\Delta t} \tag{8.2.34}$$

$$= \frac{P-(-P)}{t} \tag{8.2.35}$$

$$= \frac{2P}{t} \tag{8.2.36}$$

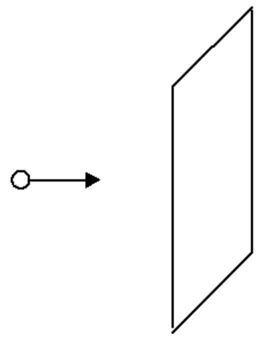


Figure **8.2.2**

The *total* force exerted on the surface by all of the photons in the sample would be the product of the above force and the total number of photons headed in the +x-direction that hit the wall at about the same time. Any photon located within a box of width *ct* and area A traveling in the +x-direction will hit the wall during this interval (Figure 8.2.3).

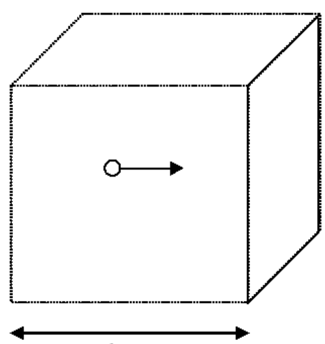


Figure **8.2.3**

The total number of photons (N) in the box is that product of the number density of photons (n) and the volume of the box:

$$N = n(ctA) \tag{8.2.37}$$

We can assume that about ⅓ of these photons are moving in the x-direction (with ⅓ in y and z) and about ½ of the x-photons are moving in the +x-direction (with ½ in $-x$). Thus:

$$F_{all\,photons} = \left(\frac{1}{3}\frac{1}{2}nctA\right)\left(\frac{2P}{t}\right) \tag{8.2.38}$$

$$= \left(\frac{1}{3}ncAP\right) \tag{8.2.39}$$

Thus, the pressure on the surface is:

$$p = \frac{F}{A} \tag{8.2.40}$$

$$= \frac{1}{3}nPc \tag{8.2.41}$$

Since the energy of a photon is given by

$$E = Pc \tag{8.2.42}$$

$$p = \frac{1}{3} nE \tag{8.2.43}$$

and energy density is the product of number density and the photon energy,

$$\rho = nE \tag{8.2.44}$$

we have the equation of state for photons

$$p_\gamma = \frac{1}{3}\rho_\gamma \tag{8.2.45}$$

B. Matter

The derivation for matter is exactly the same as the derivation for photons, except the "box" has width vt rather than ct. Thus, we are lead to this result:

$$p_M = \frac{1}{3} nPv \tag{8.2.46}$$

Notice that Pv is equal to *twice* the classical kinetic energy. Under normal conditions, the vast majority of the matter in the universe is moving at non-relativistic speeds. Therefore, the kinetic energy of matter is totally insignificant compared to its rest-mass energy. Thus, the pressure of matter is effectively zero compared to its energy density. This corresponds to an equation of state of

$$p_M = 0 \tag{8.2.47}$$

IV. Modeling a Universe of Light and Matter (and the need for more)

A. A Universe of Photons

Consider a universe consisting only of photons. First, apply the cosmological work-energy relation (Equation 8.2.23):

$$R\left(\frac{d\rho}{dt}\right) = -3(\rho - p)\left(\frac{dR}{dt}\right) \tag{8.2.48}$$

$$= -3\left(\rho + \frac{1}{3}\rho\right)\left(\frac{dR}{dt}\right) \tag{8.2.49}$$

$$= -3\left(\frac{4}{3}\rho\right)\left(\frac{dR}{dt}\right) \tag{8.2.50}$$

$$= -4\rho\left(\frac{dR}{dt}\right) \tag{8.2.51}$$

As you can hopefully see, Equation 8.2.51 is a separable differential equation:

$$R\left(\frac{d\rho}{dt}\right) = -4\rho\left(\frac{dR}{dt}\right) \tag{8.2.52}$$

$$\frac{d\rho}{\rho} = -4\frac{dR}{dt} \tag{8.2.53}$$

$$\int \frac{d\rho}{\rho} = \int -4\frac{dR}{dt} \tag{8.2.54}$$

$$\ln\rho = -4\ln R + C \tag{8.2.55}$$

$$\rho_\gamma = \frac{C}{R^4} \tag{8.2.56}$$

Equation 8.2.56 states that as the universe expands (as R increases), the energy density of the photons drops very rapidly. (If you double the radius of the universe, the energy density drops by a factor of 16.) You may have expected that the energy density would drop in inverse proportion to the volume (R^3) (i.e., if you double the radius, you get 8x the volume and 1/8 the energy density). The additional factor diluting the energy density is because the wavelengths of the photons stretch as the universe expands. Since the energy of a photon is inversely proportional to its wavelength, and its wavelength stretches in direct proportion to the scale of the universe, this results in an additional factor of R in the denominator. This “extra” dilution of the energy density of photons as the universe expands has some interesting ramifications we will explore shortly.

Now that we know the relationship between ρ and R, we can solve the Friedmann equation for a universe of photons.

$$\left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\rho R^2 - kc^2 \tag{8.2.57}$$

$$= \frac{8}{3}\pi G\left(\frac{C}{R^4}\right)R^2 - kc^2 \tag{8.2.58}$$

We can ignore the curvature term for two reasons. First, to the limits of our current ability to measure the curvature of the universe, it appears that the universe is flat, so $k=0$. Second, even if our measurements are slightly off, when the universe was much smaller than it is now (let $R \to 0$), the first term in the Friedmann equation completely dominates the second term. If we are interested in the birth of the universe (when $R \approx 0$), the curvature of the universe is irrelevant.

Thus,

$$\left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\left(\frac{C}{R^4}\right)R^2 \tag{8.2.59}$$

$$= \frac{C}{R^2} \tag{8.2.60}$$

where I’ve lumped all of the constants together into a “new” constant, but was too lazy to change its symbol. (I’m going to do this a lot so get used to it.)

Taking the square-root of both sides (and being lazy again) of Equation 8.2.60 leaves:

$$\frac{dR}{dt} = \frac{C}{R} \tag{8.2.61}$$

This is a separable differential equation:

$$R\,dR = C\,dt \tag{8.2.62}$$

$$\int R\,dR = \int C\,dt \tag{8.2.63}$$

$$\frac{1}{2}R^2 = Ct \tag{8.2.64}$$

$$R = C\sqrt{t} \tag{8.2.65}$$

This says that a universe filled with photons must begin at zero size ($R=0$ when $t=0$) with a Big Bang-type creation, and then expand as a simple square-root function of time, continuously slowing down in its expansion, but never stopping. (If k really isn’t zero, this would affect the late-time behavior of this universe, but would have no effect on the initial creation and square-root time dependence.)

Now let’s compare this universe to our own. (Of course, there are more than photons in our universe but if this model matches our universe we would say we live in a photon-dominated universe.) To do this, take a time derivative of our model universe:

$$\frac{dR}{dt} = \frac{1}{2}Ct^{-1/2} \tag{8.2.66}$$

and pull out a factor of R

$$\frac{dR}{dt} = \frac{1}{2}t^{-1}(Ct^{1/2}) \tag{8.2.67}$$

$$= \frac{1}{2t}(R) \tag{8.2.68}$$

Comparing this to the Hubble law

$$\frac{dR}{dt} = HR \tag{8.2.69}$$

where H is HUbbles constant. This results in

$$H = \frac{1}{2t} \tag{8.2.70}$$

$$t = \frac{1}{2H} \tag{8.2.71}$$

Thus, the Hubble constant is inversely related to the age of the universe. The current value of the Hubble constant is:

$$H = 70\frac{km/s}{Mpc} = 2.27 \times 10^{-18}\ s^{-1} \tag{8.2.72}$$

Thus, if our universe is photon-dominated and has this value for the Hubble constant, its age is:

$$t = \frac{1}{2(2.27 \times 10^{-18}\ s^{-1})} \tag{8.2.73}$$

$$= 2.2 \times 10^{17}\ s \tag{8.2.74}$$

$$\approx 7 \times 10^{9}\ yr \tag{8.2.75}$$

The current value of the age of the universe is about 13.7 billion years, nearly twice as old as this photon-dominated model. So, we don't (currently) live in a photon-dominated universe. Obviously, there is a bit of matter around too.

B. A Universe of Matter

Consider a universe consisting only of matter.

Q1

Apply the cosmological work-energy relation to determine the relationship between energy density and scale factor (R).

You should find that the energy density of matter decreases more slowly with expansion than the energy density of photons. This means that in a universe with both matter and photons (such as ours) matter must dominate the universe as R gets larger and larger. Regardless of how much photon energy initially exists, in the limit as R → ∞ this photon energy will disappear, leaving matter to dominate the universe.

Conversely, when the universe was very young ($R \rightarrow 0$) the photon energy density increases much faster than the matter energy density. Thus, photons dominate the early universe regardless of the amount of matter in the universe.

In our universe, the transition from a photon-dominated to a matter-dominated universe occurred when the universe was about 70,000 years old. Before this point, the photon densities were so large that no structures (atoms, molecules, etc.) could stably form in the universe and the universe expanded as modeled in the previous section, as t1/2. After this point, the photon energies were diluted by expansion and the universe began its matter-dominated epoch and expanded at a rate you will now calculate.

Q2

Solve the first Friedmann equation to determine the time dependence of the scale factor in a matter-dominated universe.

Q3

Determine the relationship between the age of the universe and the Hubble constant in a matter-dominated universe. Based on the observed Hubble constant, determine the age of our universe if it is matter-dominated.

You should find that for the observed Hubble constant a matter-dominated universe is still only around 9 billion years old. Something is seriously wrong. We know our universe consists of matter and photons, but neither of these two types of model universes can be as old as our universe. Of course, perhaps our measurements of the Hubble constant are incorrect, or perhaps our measurements of the age of the universe are incorrect, or perhaps our measurements of the curvature of space are incorrect, but all of these measurements have been verified by many independent means. So what if none of these sets of measurements are mistaken? One other option is that our universe consists of some other *type* of substance. (This is not the **dark** matter astronomers have been looking for for 50 years. This material cannot be matter of any type, luminous or not. It has to be a completely different type of substance.) Amazingly, Einstein predicted the existence of this "substance" nearly a hundred years ago. But before we explore this possibility, let's look more closely at how the curvature of the universe can affect its age.

C. A Curved Universe of Matter

Let's revisit the first Friedmann equation (Equation 8.2.10) for a matter-dominated universe

$$\left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\rho R^2 - kc^2 \tag{8.2.76}$$

$$= \frac{8}{3}\pi G\left(\frac{C}{R^3}\right)R^2 - kc^2 \tag{8.2.77}$$

$$= \frac{C}{R} - kc^2 \tag{8.2.78}$$

Ignoring the curvature term hopefully led to a scale factor of the form:

$$R = \frac{C}{t}^{2/3} \tag{8.2.79}$$

which is sketched in Figure $8.2.4$.

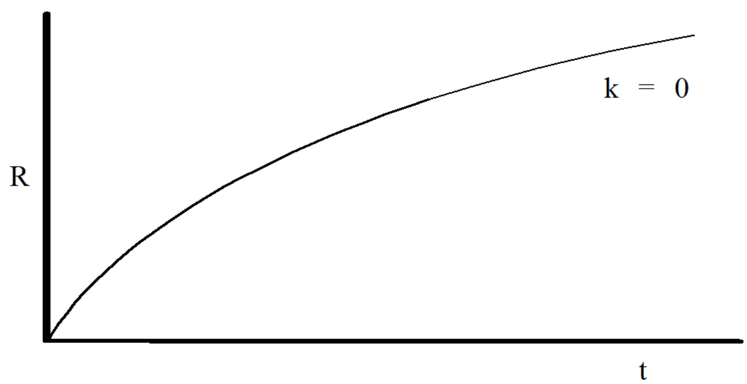


***Figure* $8.2.4$**

Q4

What is the expansion velocity for a flat ($k = 0$) matter-dominated universe as $t \to \infty$?

Now consider a hyperbolic ($k = -1$) universe. With $k = -1$, the right-hand side of the Friedmann equation (Equation 8.2.10) is now larger than it was for the $k = 0$ case. Since the left-side of the Friedmann equation is the square of the slope (dR/dt) of the expansion curve, this means that a $k = -1$ universe is always expanding quicker than a k = 0 universe. Thus, even at $t = \infty$ a hyperbolic universe is still expanding. **Sketch a curve representing a $k = -1$ universe on the graph above**.

What about a spherical (k = +1) universe? With $k = +1$, the right-hand side of the Friedmann equation is now smaller than it was for the $k = 0$ case, and thus a $k = +1$ universe is always expanding slower than a $k = 0$ universe. Thus, at some time the expansion velocity of a spherical universe must become negative, i.e. the universe must begin to collapse. **Sketch a curve representing a $k = +1$ universe on the graph above**.

The graph you've constructed above illustrates three different types of universes, all beginning their expansion at the same time. However, what we can directly measure about our universe is not when it began expanding but rather how quickly it is expanding right now. Thus, we know the slope of the expansion curve at the present time.

Matching the slope of the $k = 0$ universe to its presently measured value resulted in a universe about 9 billion years old. On the graph at below, carefully sketch the curves for $k = -1$ and $k = +1$ universes, giving them the correct slope at the present time.

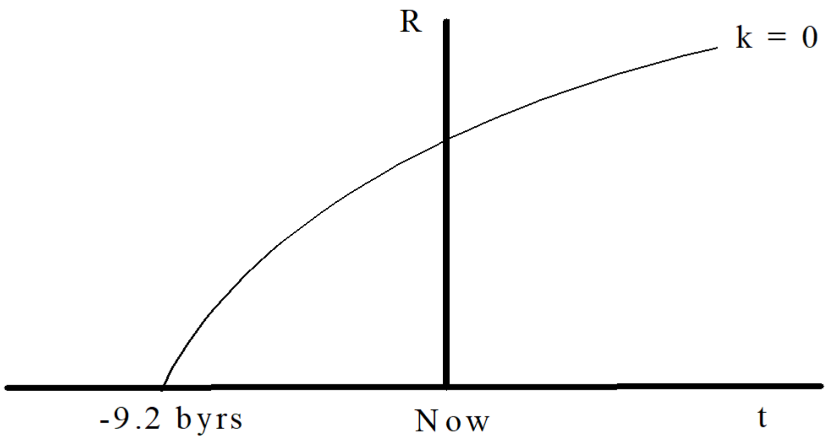


Figure $\mathbf{8.2.5}$

Q5

Given the current value of the Hubble constant, which type of universe ($k = \pm 1$ or 0) is the oldest? Carefully explain your answer.

V. Vacuum Energy

A. Justification

Let's assume empty space has a non-zero energy density. This is not the energy of something that occupies the space, but rather the energy of the space itself, an inherent property of the space. Why should we make this assumption? The best justification comes from quantum mechanics, where even apparently empty space should not have an energy density of precisely zero. An energy density of exactly zero, with no uncertainty, would seem to be in contradiction with the energy-form of Heisenberg's uncertainty principle. Quantum mechanics seems to predict that empty space is continually filled with virtual particles whose net energy density is non-zero.

Einstein, working before the advent of quantum mechanics and Hubble's discovery of an expanding universe, assumed space had a non-zero energy density, which he termed the *cosmological constant*, for totally different reasons. Because he realized that a universe of only matter and photons would have to be expanding or contracting, and there appeared to be no evidence for universal expansion or contraction, he examined the equations of general relativity and realized that energy of this form could counteract the effects of matter and photons and keep the universe static.

B. Equation of State for Vacuum Energy

The fundamental attribute of vacuum energy density is that it has a constant value, governed by quantum mechanics and independent of whatever else is happening in that region of space. Therefore, if we apply the cosmological work-energy relation (Equation 8.2.23) to vacuum energy:

$$R\left(\frac{d\rho}{dt}\right) = -3(\rho+p)\left(\frac{dR}{dt}\right) \tag{8.2.80}$$

$$R(0) = -3(\rho+p)\left(\frac{dR}{dt}\right) \tag{8.2.81}$$

$$0 = -3(\rho+p)\left(\frac{dR}{dt}\right) \tag{8.2.82}$$

Since the universe is expanding

$$\frac{dR}{dt} \neq 0, \tag{8.2.83}$$

so,

$$0 = \rho + p \tag{8.2.84}$$

$$\rho_V = -p_V \tag{8.2.85}$$

Thus, a positive vacuum energy density results in a *negative* pressure, and all the crazy results I alluded to earlier in this activity. Also, notice that unlike matter or photon energy densities, vacuum energy density *does not decrease as the universe expands*. Thus, if there is any non-zero value of vacuum energy in the universe, it will ultimately dominate the universe.

Before we describe how a vacuum-dominated universe behaves, let's see how Einstein used vacuum energy to "stabilize" the universe.

C. Vacuum Energy and a Static Universe

Examine the second Friedmann equation (Equation 8.2.32) for the acceleration of the universe:

$$\frac{d^R}{dt^2} = -\frac{4}{3}\pi GR(\rho + 3p) \tag{8.2.86}$$

For a matter-dominated or photon-dominated universe, the universe has a negative acceleration and is either slowing down during its expansion or speeding up during its collapse. Neither of these possibilities seemed possible to Einstein. Therefore, he concocted a universe with both matter and vacuum energy. This leads to a Friedmann equation:

$$\frac{d^2R}{dt^2} = -\frac{4}{3}\pi GR(\rho_M + 3(0)) - \frac{4}{3}\pi GR(\rho_V + 3(-p_V)) \tag{8.2.87}$$

$$= -\frac{4}{3}\pi GR(\rho_M - 2\rho_V) \tag{8.2.88}$$

In this combination universe, the acceleration can be zero if (and only if) the matter energy density is exactly equal to twice the vacuum energy density. This precise balance would seem to lead to the static universe that Einstein (and all of his contemporaries) believed in.

However, this balance is unstable. Although the energy densities could have this perfect proportion averaged over a large volume of the universe, there would almost certainly have to exist small regions in which, for example, the mass density was a tiny bit larger than twice the vacuum density. These regions would then start to collapse, which would further increase the mass density, which would accelerate the collapse, which would further increase the mass density, which … etc. In a region in which the mass density was slightly smaller than twice the vacuum density, the opposite would happen. The positive acceleration in these regions would cause them to expand, which would reduce the mass density, which would accelerate the expansion, etc. A perfectly static universe is *impossible*, it would immediately be filled with rapidly expanding and contracting sub-regions. Einstein's goal of a universe stabilized against collapse by the negative pressure of vacuum energy density is impossible.

Q6

Imagine a universe with both photons and vacuum energy. What relationship between photon energy density and vacuum energy density will result in a universal acceleration of zero? Is this universe stable or unstable?

D. The Vacuum-Dominated Universe

7. Solve the first Friedmann equation to determine the time dependence of the scale factor in a vacuum-dominated universe.

The Vacuum-Dominated universe is a seriously strange universe, termed the *deSitter universe* after the first physicist to clearly describe its properties. Perhaps the strangest aspect of this space is that it was never "created". At $t = 0$, the space has finite size, so there is no special Big Bang singularity of creation. Moreover, the space has a finite, positive size regardless of how far back in time you go (let $t \to -\infty$). The Hubble constant tells us nothing about this space's age (as it does for the photon and matter-dominated spaces) because the space has always existed. In fact, the Hubble constant is simply the argument in the exponential expansion of the scale factor,

$$R = Ae^{Ht} \tag{8.2.89}$$

and is not only a spatial constant, but a temporal constant as well. This means that not only does the deSitter universe look the same at every point in space it also looks the same at every point in time (unlike the photon and matter universes which clearly show signs of “aging”).

Of course, we don’t live in a pure deSitter universe, because obviously matter and photons exist in our universe, so how do we take the three models we’ve developed and use them to describe our universe?

E. Our Universe

Our universe consists of photons, matter and (almost assuredly) vacuum energy. Regardless of the exact proportions of these three types of energy, the story of the universe’s expansion neatly divides into three clear stages. In the early universe, photons dominate, even in the presence of vacuum energy, since photon energy density rises without limit as $R \to 0$:

$$\rho_V = \frac{C}{R^4} \tag{8.2.90}$$

while vacuum energy remains constant. Thus, during the early days of our universe (but maybe not the earliest fractions of a second) the universe behaves like a simple photon-only universe, expanding as $t^{1/2}$. Then, as the photons dilute, matter takes over, slightly altering the expansion rate, but the universe is still slowing in its expansion. Finally, ultimately, vacuum energy takes over. Our universe will begin to accelerate in its expansion, ultimately becoming indistinguishable from the exponentially expanding deSitter universe. The first evidence for this accelerated expansion was collected in the 1990s, and it now appears that several billion years ago our universe began to make its transition to the vacuum-dominated phase.

F. Vacuum Energy and the Creation of the Universe

Within the limits of observational measurements, our universe is flat. This flatness requires a very specific value of energy density, termed the *critical density* (ρ_C). The value of the critical density can be determined from the first Friedmann equation:

$$\left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\rho R^2 - kc^2 \tag{8.2.91}$$

$$\left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\rho_C R^2 - 0 \tag{8.2.92}$$

$$\frac{\left(\frac{dR}{dt}\right)^2}{R^2} = \frac{8}{3}\pi G\rho_C \tag{8.2.93}$$

$$H^2 = \frac{8}{3}\pi G\rho_C \tag{8.2.94}$$

$$\rho_C = \frac{3H^2}{8\pi G} \tag{8.2.95}$$

If the energy density of the universe is not equal to this precise value, the universe will not be flat.

$$\left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\rho R^2 - kc^2 \tag{8.2.96}$$

$$\frac{\left(\frac{dR}{dt}\right)^2}{R^2} = \frac{8}{3}\pi G\rho - \frac{kc^2}{R} \tag{8.2.97}$$

$$H^2 - \frac{8}{3}\pi G\rho = -\frac{kc^2}{R} \tag{8.2.98}$$

Now examine the Friedmann equation again, this time for arbitrary curvature:

$$H^2-\frac{8}{3}\pi G\rho=\frac{kc^2}{R^2} \tag{8.2.99}$$

Now multiply both sides by $3/8\pi G$and use the definition of critical density derived above:

$$\frac{3}{8\pi G}\left(H^2-\frac{8}{3}\pi G\rho\right)=-\frac{3kc^2}{8\pi GR^2} \tag{8.2.100}$$

$$\rho_C-\rho=-\frac{3kc^2}{8\pi GR^2} \tag{8.2.101}$$

and divide both sides by the energy density:

$$\frac{\rho_C-\rho}{\rho}=-\frac{3kc^2}{8\pi\rho GR^2} \tag{8.2.102}$$

$$\frac{\rho-\rho_C}{\rho}=\frac{C}{\rho R^2} \tag{8.2.103}$$

Equation 8.2.103 relates or the fractional difference between the energy density in our universe and the critical energy density needed for flatness. The important aspect of this equation is that this difference changes as the scale of the universe changes.

If matter or photons dominate the universe near creation, let's examine how this fractional difference changes. Since matter density scales with the expansion of the universe as:

$$\rho_M=\frac{C}{R^3} \tag{8.2.104}$$

the right-hand side of Equation 8.2.104grows as the universe expands:

$$\frac{\rho-\rho_C}{\rho}=\frac{C}{\left(\frac{C}{R^3}\right)R^2}\approx R \tag{8.2.105}$$

Since photon density scales with the expansion of the universe as:

$$\rho_\gamma=\frac{C}{R^4} \tag{8.2.106}$$

the right-hand side of Equation 8.2.106grows as the universe expands:

$$\frac{\rho-\rho_C}{\rho}=\frac{C}{\left(\frac{C}{R^4}\right)R^2}\approx R^2 \tag{8.2.107}$$

Thus, in a universe dominated by matter and photons, the difference between the density of the universe and the critical density will increase as the universe expands. This is equivalent to saying that the **flatness of the universe is unstable**, in that the universe becomes less flat as it expands. Thus, if the universe is observationally close to flat today it must have been extremely close to flat at creation. How close? Observationally, the universe is within $\pm 1\%$ of flatness today. Scaling backwards in time, it must have been within $\pm 10^{-60}\%$ of flatness near creation! This is an insanely precise value of energy density and, given natural variations in energy density in different regions of the universe, there is no feasible way in which a matter and photon dominated universe as large as ours can be flat. This is termed the *Flatness Problem* in cosmology and requires some sort of resolution.

I should point out that the vacuum dominated phase we are currently in does not resolve this problem. The curvature of the universe is measured by observing the microwave background radiation, emitted when the universe was about 400,000 years old. Thus, the universe was flat long before the current vacuum dominated phase.

However, the proposed resolution does involve vacuum energy. Imagine very early in the history of the universe (and I mean VERY early, like 10-34 s after "creation") a vacuum energy density larger than the matter or photon energy densities. (This is not the same vacuum energy density driving the current accelerated expansion of the universe. The current vacuum energy density does not dominate the universe until after matter and photon energies dilute over several billion years.) This vacuum energy density is either a short-term "release" of vacuum energy, analogous to the energy released when a system goes through a phase transformation, or the energy associated with the vacuum being in a higher energy state than it currently is in, analogous to the higher energy states allowed electrons in an atom. Either way, if vacuum energy dominates the very early universe, we can again examine our flatness equation. Since vacuum energy is constant during this time interval:

$$\rho_V = C \tag{8.2.108}$$

the right-hand side of the flatness equation (Equation 8.2.103decreases as the universe expands:

$$\frac{\rho - \rho_C}{\rho} = \frac{C}{(C)R^2} \approx \frac{1}{R^2} \tag{8.2.109}$$

Thus, *vacuum energy makes the universe more flat as it expands*! Regardless of the initial curvature of the universe, if it is dominated by vacuum energy near its birth it will emerge almost perfectly flat. This is, as mentioned before, in strong agreement with measurements of our universe. This process, in which a very early burst of accelerated expansion due to vacuum energy flattens our universe is termed inflation. (Inflation also explains a number of other features of our universe that more simple models cannot explain.)

In addition to solving the flatness problem, inflation (and vacuum energy) may also provide the answer to the most fundamental question in cosmology, “How was the universe created?” Remember that a pure, vacuum dominated space has no “time” of creation, it exists eternally. Sometimes termed eternal inflation, this model proposes an eternally inflating vacuum-energy dominated space, in which the vacuum in certain sub-regions occasionally decays into a lower energy state, creating universes such as ours (and probably many not so similar to ours). Obviously, the scope of this model is a bit beyond the level of this course.

CHAPTER OVERVIEW

Appendix

Topic hierarchy

Contributors and Attributions

- Paul D’Alessandris (Monroe Community College)

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