21.16: Attenuation in Coaxial Cable

In this section, we consider the issue of attenuation in coaxial transmission line. Recall that attenuation can be interpreted in the context of the "lumped element" equivalent circuit transmission line model as the contributions of the resistance per unit length R' and conductance per unit length G'. In this model, R' represents the physical resistance in the inner and outer conductors, whereas G' represents loss due to current flowing directly between the conductors through the spacer material.

The parameters used to describe the relevant features of coaxial cable are shown in Figure $21.16.1$. In this figure, a and b are the radii of the inner and outer conductors, respectively. σ_{ic} and σ_{oc} are the conductivities (SI base units of S/m) of the inner and outer conductors, respectively. Conductors are assumed to be non-magnetic; i.e., having permeability μ equal to the free space value μ_0. The spacer material is assumed to be a lossy dielectric having relative permittivity ϵ_r and conductivity σ_s.

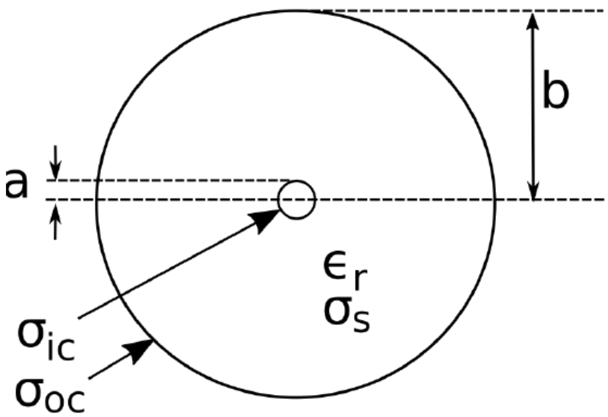

Figure $21.16.1$: Parameters defining the design of a coaxial cable.

Resistance per unit length

The resistance per unit length is the sum of the resistances of the inner and outer conductor per unit length. The resistance per unit length of the inner conductor is determined by σ_{ic} and the effective cross-sectional area through which the current flows. The latter is equal to the circumference $2\pi a$ times the skin depth δ_{ic} of the inner conductor, so:

$$R'_{ic} \approx \frac{1}{(2\pi a \cdot \delta_{ic})\,\sigma_{ic}} \quad \text{for } \delta_{ic} \ll a$$

This expression is only valid for $\delta_{ic} \ll a$ because otherwise the cross-sectional area through which the current flows is not well-modeled as a thin ring near the surface of the conductor. Similarly, we find the resistance per unit length of the outer conductor is

$$R'_{oc} \approx \frac{1}{(2\pi b \cdot \delta_{oc})\,\sigma_{oc}} \quad \text{for } \delta_{oc} \ll t$$

where δ_{oc} is the skin depth of the outer conductor and t is the thickness of the outer conductor. Therefore, the total resistance per unit length is

$$\begin{aligned} R' &= R'_{ic} + R'_{oc} \\ &\approx \frac{1}{(2\pi a \cdot \delta_{ic})\,\sigma_{ic}} + \frac{1}{(2\pi b \cdot \delta_{oc})\,\sigma_{oc}} \end{aligned} \tag{21.16.1}$$

Recall that skin depth depends on conductivity. Specifically:

$$\delta_{ic} = \sqrt{2/\omega\mu\sigma_{ic}} \tag{21.16.2}$$

$$\delta_{oc} = \sqrt{2/\omega\mu\sigma_{oc}} \tag{21.16.3}$$

Expanding Equation 21.16.1 to show explicitly the dependence on conductivity, we find:

$$R' \approx \frac{1}{2\pi\sqrt{2/\omega\mu_0}} \left[\frac{1}{a\sqrt{\sigma_{ic}}} + \frac{1}{b\sqrt{\sigma_{oc}}} \right]$$

At this point it is convenient to identify two particular cases for the design of the cable. In the first case, “Case I,” we assume $\sigma_{oc} \gg \sigma_{ic}$. Since $b > a$, we have in this case

$$\begin{aligned} R' &\approx \frac{1}{2\pi\sqrt{2/\omega\mu_0}} \left[\frac{1}{a\sqrt{\sigma_{ic}}}\right] \\ &= \frac{1}{2\pi\delta_{ic}\sigma_{ic}} \frac{1}{a} \quad \text{(Case~I)} \end{aligned} \tag{21.16.4}$$

In the second case, “Case II,” we assume $\sigma_{oc} = \sigma_{ic}$. In this case, we have

$$\begin{aligned} R' &\approx \frac{1}{2\pi\sqrt{2/\omega\mu_0}} \left[\frac{1}{a\sqrt{\sigma_{ic}}} + \frac{1}{b\sqrt{\sigma_{ic}}}\right] \\ &= \frac{1}{2\pi\delta_{ic}\sigma_{ic}} \left[\frac{1}{a} + \frac{1}{b}\right] \quad \text{(Case~II)} \end{aligned} \tag{21.16.5}$$

A simpler way to deal with these two cases is to represent them both using the single expression

$$R' \approx \frac{1}{2\pi\delta_{ic}\sigma_{ic}} \left[\frac{1}{a} + \frac{C}{b}\right]$$

where $C = 0$ in Case I and $C = 1$ in Case II.

Conductance per unit length

The conductance per unit length of coaxial cable is simply that of the associated coaxial structure at DC; i.e.,

$$G' = \frac{2\pi\sigma_s}{\ln(b/a)}$$

Unlike resistance, the conductance is independent of frequency, at least to the extent that σ_s is independent of frequency.

Attenuation

The attenuation of voltage and current waves as they propagate along the cable is represented by the factor $e^{-\alpha z}$, where z is distance traversed along the cable. It is possible to find an expression for α in terms of the material and geometry parameters using:

$$\gamma \triangleq \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta \tag{21.16.6}$$

where L' and C' are the inductance per unit length and capacitance per unit length, respectively. These are given by

$$L' = \frac{\mu}{2\pi} \ln(b/a)$$

and

$$C' = \frac{2\pi\epsilon_0\epsilon_r}{\ln(b/a)}$$

In principle we could solve Equation 21.16.6 for α. However, this course of action is quite tedious, and a simpler approximate approach facilitates some additional insights. In this approach, we define parameters α_R associated with R' and α_G associated with G' such that

$$e^{-\alpha_R z} e^{-\alpha_G z} = e^{-(\alpha_R + \alpha_G) z} = e^{-\alpha z}$$

which indicates

$$\alpha = \alpha_R + \alpha_G$$

Next we postulate

$$\alpha_R \approx K_R \frac{R'}{Z_0} \tag{21.16.7}$$

where Z_0 is the characteristic impedance

$$Z_0 \approx \frac{\eta_0}{2\pi} \frac{1}{\sqrt{\epsilon_r}} \ln \frac{b}{a} \quad \text{(low loss)} \tag{21.16.8}$$

and where K_R is a unitless constant to be determined. The justification for Equation 21.16.7is as follows: First, α_R must increase monotonically with increasing R'. Second, R' must be divided by an impedance in order to obtain the correct units of 1/m. Using similar reasoning, we postulate

$$\alpha_G \approx K_G G' Z_0 \tag{21.16.9}$$

where K_G is a unitless constant to be determined. The following example demonstrates the validity of Equations 21.16.7 and 21.16.9, and will reveal the values of K_R and K_G.

Example 21.16.1: Attenuation constant for RG-59

RG-59 is a popular form of coaxial cable having the parameters $a \cong 0.292$ mm, $b \cong 1.855$ mm, $\sigma_{ic} \cong 2.28 \times 10^7$ S/m, $\sigma_s \cong 5.9 \times 10^{-5}$ S/m, and $\epsilon_r \cong 2.25$. The conductivity σ_{oc} of the outer conductor is difficult to quantify because it consists of a braid of thin metal strands. However, $\sigma_{oc} \gg \sigma_{ic}$, so we may assume Case I; i.e., $\sigma_{oc} \gg \sigma_{ic}$, and subsequently $C = 0$.

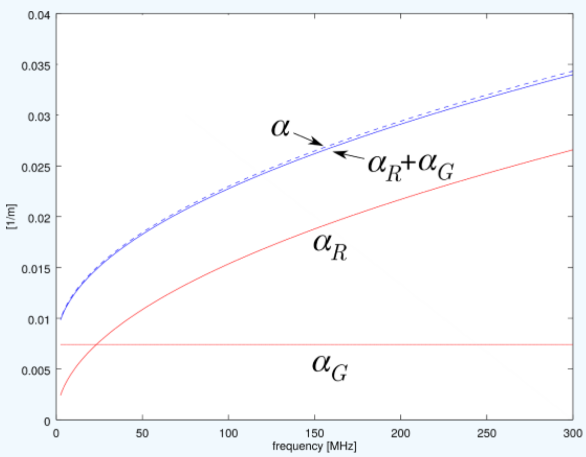

Figure 21.16.2 Comparison of $\alpha = \text{Re}\{\gamma\}$ to α_R, α_G, and $\alpha_R + \alpha_G$ for $K_R = K_G = 1/2$. The result for α has been multiplied by 1.01; otherwise the curves would be too close to tell apart.

Figure 21.16.2shows the components α_G and α_R computed for the particular choice $K_R = K_G = 1/2$. The figure also shows $\alpha_G + \alpha_R$, along with α computed using Equation 21.16.6. We find that the agreement between these values is very good, which is compelling evidence that the ansatz is valid and $K_R = K_G = 1/2$.

Note that there is nothing to indicate that the results demonstrated in the example are not generally true. Thus, we come to the following conclusion:

> The attenuation constant $\alpha \approx \alpha_G + \alpha_R$ where $\alpha_G \triangleq R'/2Z_0$ and $\alpha_R \triangleq G' Z_0 / 2$.

Minimizing attenuation

Let us now consider if there are design choices which minimize the attenuation of coaxial cable. Since $\alpha = \alpha_R + \alpha_G$, we may consider α_R and α_G independently. Let us first consider α_G:

$$\begin{aligned} \alpha_G &\triangleq \frac{1}{2} G' Z_0 \\ &\approx \frac{1}{2} \cdot \frac{2\pi\sigma_s}{\ln(b/a)} \cdot \frac{1}{2\pi} \frac{\eta_0}{\sqrt{\epsilon_r}} \ln(b/a) \\ &= \frac{\eta_0}{2} \frac{\sigma_s}{\sqrt{\epsilon_r}} \end{aligned} \tag{21.16.10}$$

It is clear from this result that α_G is minimized by minimizing $\sigma_s/\sqrt{\epsilon_r}$. Interestingly the physical dimensions a and b have no discernible effect on α_G. Now we consider α_R:

$$\begin{aligned} \alpha_R &\triangleq \frac{R'}{2Z_0} \\ &= \frac{1}{2} \frac{(1/2\pi\delta_{ic}\sigma_{ic})\left[1/a+C/b\right]}{(1/2\pi)\left(\eta_0/\sqrt{\epsilon_r}\right)\ln(b/a)} \\ &= \frac{\sqrt{\epsilon_r}}{2\eta_0\delta_{ic}\sigma_{ic}} \cdot \frac{\left[1/a+C/b\right]}{\ln(b/a)} \end{aligned} \tag{21.16.11}$$

Now making the substitution $\delta_{ic} = \sqrt{2/\omega\mu_0\sigma_{ic}}$ in order to make the dependences on the constitutive parameters explicit, we find:

$$\alpha_R = \frac{1}{2\sqrt{2}\cdot\eta_0}\sqrt{\frac{\omega\mu_0\epsilon_r}{\sigma_{ic}}} \cdot \frac{\left[1/a+C/b\right]}{\ln(b/a)}$$

Here we see that α_R is minimized by minimizing ϵ_r/σ_{ic}. It's not surprising to see that we should maximize σ_{ic}. However, it's a little surprising that we should minimize ϵ_r. Furthermore, this is in contrast to α_G, which is minimized by *maximizing* ϵ_r. Clearly there is a tradeoff to be made here. To determine the parameters of this tradeoff, first note that the result depends on frequency: Since α_R dominates over α_G at sufficiently high frequency (as demonstrated in Figure $21.16.2$), it seems we should minimize ϵ_r if the intended frequency of operation is sufficiently high; otherwise the optimum value is frequency-dependent. However, σ_s may vary as a function of ϵ_r, so a general conclusion about optimum values of σ_s and ϵ_r is not appropriate.

However, we also see that α_R – unlike α_G – depends on a and b. This implies the existence of a generally-optimum geometry. To find this geometry, we minimize α_R by taking the derivative with respect to a, setting the result equal to zero, and solving for a and/or b. Here we go:

$$\frac{\partial}{\partial a}\alpha_R = \frac{1}{2\sqrt{2}\cdot\eta_0}\sqrt{\frac{\omega\mu_0\epsilon_r}{\sigma_{ic}}} \cdot \frac{\partial}{\partial a}\frac{\left[1/a+C/b\right]}{\ln(b/a)} \tag{21.16.12}$$

This derivative is worked out in an addendum at the end of this section. Using the result from the addendum, the right side of Equation 21.16.12can be written as follows:

$$\frac{1}{2\sqrt{2}\cdot\eta_0}\sqrt{\frac{\omega\mu_0\epsilon_r}{\sigma_{ic}}} \cdot \left[\frac{-1}{a^2\ln(b/a)} + \frac{1/a+C/b}{a\ln^2(b/a)}\right] \tag{21.16.13}$$

In order for $\partial\alpha_R/\partial a = 0$, the factor in the square brackets above must be equal to zero. After a few steps of algebra, we find:

$$\ln(b/a) = 1 + \frac{C}{b/a}$$

In Case I ($\sigma_{oc} \gg \sigma_{ic}$), $C = 0$ so:

$$b/a = e \cong 2.72 \quad \text{(Case I)}$$

In Case II ($\sigma_{oc} = \sigma_{ic}$), $C = 1$. The resulting equation can be solved by plotting the function, or by a few iterations of trial and error; either way one quickly finds

$$b/a \cong 3.59 \quad \text{(Case II)}$$

Summarizing, we have found that α is minimized by choosing the ratio of the outer and inner radii to be somewhere between 2.72 and 3.59, with the precise value depending on the relative conductivity of the inner and outer conductors.

Substituting these values of b/a into Equation 21.16.8, we obtain:

$$Z_0 \approx \frac{59.9\ \Omega}{\sqrt{\epsilon_r}} \text{ to } \frac{76.6\ \Omega}{\sqrt{\epsilon_r}} \tag{21.16.14}$$

as the range of impedances of coaxial cable corresponding to physical designs that minimize attenuation.

Equation 21.16.14gives the range of characteristic impedances that minimize attenuation for coaxial transmission lines. The precise value within this range depends on the ratio of the conductivity of the outer conductor to that of the inner conductor.

Since $\epsilon_r \geq 1$, the impedance that minimizes attenuation is less for dielectric-filled cables than it is for air-filled cables. For example, let us once again consider the RG-59 from Example $21.16.1$ In that case, $\epsilon_r \cong 2.25$ and $C = 0$, indicating $Z_0 \approx 39.9\ \Omega$ is optimum for attenuation. The actual characteristic impedance of Z_0 is about $75\ \Omega$, so clearly RG-59 is not optimized for attenuation. This is simply because other considerations apply, including power handling capability (addressed in Section 7.4) and the convenience of standard values (addressed in Section 7.5).

Addendum: Derivative of $a^2 \ln(b/a)$

Evaluation of Equation 21.16.12requires finding the derivative of $a^2 \ln(b/a)$ with respect to a. Using the chain rule, we find:

$$\begin{aligned}\frac{\partial}{\partial a}\left[a^2 \ln\left(\frac{b}{a}\right)\right] = &\left[\frac{\partial}{\partial a}a^2\right]\ln\left(\frac{b}{a}\right) \\ &+a^2\left[\frac{\partial}{\partial a}\ln\left(\frac{b}{a}\right)\right]\end{aligned} \tag{21.16.15}$$

Note

$$\frac{\partial}{\partial a}a^2 = 2a$$

and

$$\begin{aligned}\frac{\partial}{\partial a}\ln\left(\frac{b}{a}\right) &= \frac{\partial}{\partial a}[\ln(b) - \ln(a)] \\ &= -\frac{\partial}{\partial a}\ln(a) \\ &= -\frac{1}{a}\end{aligned} \tag{21.16.16}$$

So:

$$\begin{aligned}\frac{\partial}{\partial a}\left[a^2 \ln\left(\frac{b}{a}\right)\right] &= [2a]\ln\left(\frac{b}{a}\right) + a^2\left[-\frac{1}{a}\right] \\ &= \boxed{2a\ln\left(\frac{b}{a}\right) - a}\end{aligned} \tag{21.16.17}$$

This result is substituted for $a^2 \ln(b/a)$ in Equation 21.16.12to obtain Equation 21.16.13.