21.8: Wave Equation for a Transmission Line

Consider a TEM transmission line aligned along the z axis. The phasor form of the Telegrapher's Equations (Section 3.5) relate the potential phasor $\widetilde{V}(z)$ and the current phasor $\widetilde{I}(z)$ to each other and to the lumped-element model equivalent circuit parameters R', G', C', and L'. These equations are

$$-\frac{\partial}{\partial z}\widetilde{V}(z)=[R'+j\omega L']\ \widetilde{I}(z) \tag{21.8.1}$$

$$-\frac{\partial}{\partial z}\widetilde{I}(z)=[G'+j\omega C']\ \widetilde{V}(z) \tag{21.8.2}$$

An obstacle to using these equations is that we require both equations to solve for either the potential or the current. In this section, we reduce these equations to a single equation – a *wave equation* – that is more convenient to use and provides some additional physical insight.

We begin by differentiating both sides of Equation 21.8.1 with respect to z, yielding:

$$-\frac{\partial^2}{\partial z^2}\widetilde{V}(z)=[R'+j\omega L']\ \frac{\partial}{\partial z}\widetilde{I}(z)$$

Then using Equation 21.8.2 to eliminate $\widetilde{I}(z)$, we obtain

$$-\frac{\partial^2}{\partial z^2}\widetilde{V}(z)=-[R'+j\omega L'][G'+j\omega C']\ \widetilde{V}(z)$$

This equation is normally written as follows:

$$\boxed{\frac{\partial^2}{\partial z^2}\widetilde{V}(z)-\gamma^2\ \widetilde{V}(z)=0} \tag{21.8.3}$$

where we have made the substitution:

$$\gamma^2=(R'+j\omega L')(G'+j\omega C')$$

The principal square root of γ^2 is known as the *propagation constant*:

$$\gamma\triangleq\sqrt{(R'+j\omega L')(G'+j\omega C')} \tag{21.8.4}$$

> The *propagation constant* γ (units of m^{-1}) captures the effect of materials, geometry, and frequency in determining the variation in potential and current with distance on a TEM transmission line.

Following essentially the same procedure but beginning with Equation 21.8.2, we obtain

$$\boxed{\frac{\partial^2}{\partial z^2}\widetilde{I}(z)-\gamma^2\ \widetilde{I}(z)=0} \tag{21.8.5}$$

> Equations 21.8.3 and 21.8.5 are the *wave equations* for $\widetilde{V}(z)$ and $\widetilde{I}(z)$, respectively.

Note that both $\widetilde{V}(z)$ and $\widetilde{I}(z)$ satisfy the *same* linear homogeneous differential equation. This does *not* mean that $\widetilde{V}(z)$ and $\widetilde{I}(z)$ are equal. Rather, it means that $\widetilde{V}(z)$ and $\widetilde{I}(z)$ can differ by no more than a multiplicative constant. Since $\widetilde{V}(z)$ is potential and $\widetilde{I}(z)$ is current, that constant must be an impedance. This impedance is known as the *characteristic impedance* and is determined in Section 3.7.

The general solutions to Equations 21.8.3 and 21.8.5 are

$$\widetilde{V}(z)=V_0^+e^{-\gamma z}+V_0^-e^{+\gamma z} \tag{21.8.6}$$

$$\widetilde{I}(z)=I_0^+e^{-\gamma z}+I_0^-e^{+\gamma z} \tag{21.8.7}$$

where V_0^+, V_0^-, I_0^+, and I_0^- are complex-valued constants. It is shown in Section 3.8 that Equations 21.8.6 and 21.8.7 represent sinusoidal waves propagating in the $+z$ and $-z$ directions along the length of the line. The constants may represent sources, loads, or simply discontinuities in the materials and/or geometry of the line. The values of the constants are determined by boundary conditions; i.e., constraints on $\widetilde{V}(z)$ and $\tilde{I}(z)$ at some position(s) along the line.

The reader is encouraged to verify that the Equations 21.8.6 and 21.8.7 are in fact solutions to Equations 21.8.3 and 21.8.5, respectively, for any values of the constants V_0^+, V_0^-, I_0^+, and I_0^-.
