

3.4: Electric Potential Energy of Point Charges

Learning Objectives

By the end of this section, you will be able to:

- Calculate the electric potential energy due to multiple point charges
- Prove that the electric force is a conservative force.
- Apply work and potential energy in systems with electric charges

In the previous section, we showed that we could calculate the electric potential energy from the work done by the electric force on a charge in a uniform field. We will now generalize that result to a system of point charges.

To begin, consider an electric charge $+q$ fixed at the origin and move another charge $+Q$ toward q in such a manner that, at each instant, the applied force \vec{F} exactly balances the electric force \vec{F}_e on Q (Figure 3.4.1). The work done by the applied force \vec{F} on the charge Q changes the potential energy of Q . We call this potential energy the **electrical potential energy** of Q .

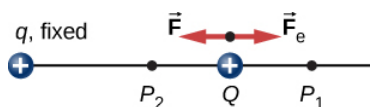


Figure 3.4.1: Displacement of “test” charge Q in the presence of fixed “source” charge q .

The work W_{12} done by the applied force \vec{F} when the particle moves from P_1 to P_2 may be calculated by

$$W_{12} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}. \quad (3.4.1)$$

Since the applied force \vec{F} balances the electric force \vec{F}_e on Q , the two forces have equal magnitude and opposite directions. Therefore, the applied force is

$$\vec{F} = -\vec{F}_e = -\frac{kqQ}{r^2} \hat{r}, \quad (3.4.2)$$

where we have defined positive to be pointing away from the origin and r is the distance from the origin. The directions of both the displacement and the applied force in the system in Figure 3.4.1 are parallel, and thus the work done on the system is positive.

We use the letter U to denote electric potential energy, which has units of joules (J). When a conservative force does negative work, the system gains potential energy. When a conservative force does positive work, the system loses potential energy, $\Delta U = -W$. In the system in Figure 3.4.1, the Coulomb force acts in the opposite direction to the displacement; therefore, the work is negative. However, we have increased the potential energy in the two-charge system.

✓ Example 3.4.1: Kinetic Energy of a Charged Particle

A $+3.0 \text{ nC}$ charge Q is initially at rest a distance of 10 cm (r_1) from a $+5.0 \text{ nC}$ charge q fixed at the origin (Figure 3.4.2). Naturally, the Coulomb force accelerates Q away from q , eventually reaching 15 cm (r_2).

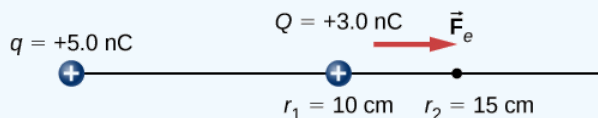


Figure 3.4.2: The charge Q is repelled by q , thus having work done on it and gaining kinetic energy.

- What is the work done by the electric field between r_1 and r_2 ?
- How much kinetic energy does Q have at r_2 ?

Strategy

Calculate the work with the usual definition. Since Q started from rest, this is the same as the kinetic energy.

Solution

Integrating force over distance, we obtain

$$\begin{aligned}
 W_{12} &= \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} \\
 &= \int_{r_1}^{r_2} \frac{kqQ}{r^2} dr \\
 &= -\frac{kqQ}{r} \Big|_{r_1}^{r_2} \\
 &= kqQ \left[\frac{-1}{r_2} + \frac{1}{r_1} \right] \\
 &= (8.99 \times 10^9 \text{ N m}^2/\text{C}^2)(5.0 \times 10^{-9} \text{ C})(3.0 \times 10^{-9} \text{ C}) \left[\frac{-1}{0.15 \text{ m}} + \frac{1}{0.10 \text{ m}} \right] \\
 &= 4.5 \times 10^{-7} \text{ J}.
 \end{aligned}$$

This is also the value of the kinetic energy at r_2 .

Significance

Charge Q was initially at rest; the electric field of q did work on Q , so now Q has kinetic energy equal to the work done by the electric field.

? Exercise 3.4.1

If Q has a mass of $4.00 \mu\text{g}$ what is the speed of Q at r_2 ?

Answer

$$K = \frac{1}{2}mv^2, v = \sqrt{2\frac{K}{m}} = \sqrt{2\frac{4.5 \times 10^{-7} \text{ J}}{4.00 \times 10^{-9} \text{ kg}}} = 15 \text{ m/s}.$$

In this example, the work W done to accelerate a positive charge from rest is positive and results from a loss in U , or a negative ΔU . A value for U can be found at any point by taking one point as a reference and calculating the work needed to move a charge to the other point.

📌 Electric Potential Energy

Work W done to accelerate a positive charge from rest is positive and results from a loss in U , or a negative ΔU . Mathematically,

$$W = -\Delta U. \quad (3.4.3)$$

Potential energy accounts for work done by a conservative force and gives added insight regarding energy and energy transformation without the necessity of dealing with the force directly. It is much more common, for example, to use the concept of electric potential energy than to deal with the Coulomb force directly in real-world applications.

The Conservative Electric Force

We have previously asserted that the electric force is a conservative force. In this section, we will provide justification for this claim. First let's return to the scenario in in Fig. 3.4.2. In polar coordinates with q at the origin and Q located at r , the displacement element vector is $d\vec{l} = \hat{r} dr$ and thus the work becomes

$$\begin{aligned}
 W_{12} &= kqQ \int_{r_1}^{r_2} \frac{1}{r^2} \hat{r} \cdot \hat{r} dr \\
 &= \underbrace{kqQ \frac{1}{r_2}}_{\text{final point}} - \underbrace{kqQ \frac{1}{r_1}}_{\text{initial point}}. \quad (3.4.4)
 \end{aligned}$$

Notice that this result only depends on the endpoints and is otherwise independent of the path taken. To explore this further, compare path P_1 to P_2 with path $P_1P_3P_4P_2$ in Figure 3.4.3.

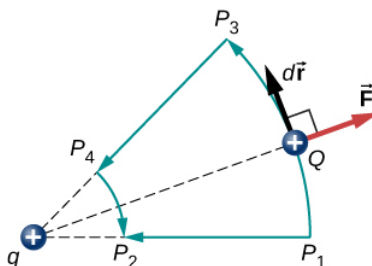


Figure 3.4.3: Two paths for displacement P_1 to P_2 . The work on segments P_1P_3 and P_4P_2 are zero due to the electrical force being perpendicular to the displacement along these paths. Therefore, work on paths P_1P_2 and $P_1P_3P_4P_2$ are equal.

The segments P_1P_3 and P_4P_2 are arcs of circles centered at q . Since the force on Q points either toward or away from q , no work is done by a force balancing the electric force, because it is perpendicular to the displacement along these arcs. Therefore, the only work done is along segment P_3P_4 which is identical to P_1P_2 .

One implication of this work calculation is that if we were to go around the path $P_1P_3P_4P_2P_1$, the net work would be zero (Figure 3.4.4). Recall that this is how we determine whether a force is **conservative** or not. Hence, because the electric force is related to the electric field by $\vec{F} = q\vec{E}$, the electric field is itself conservative. That is,

$$\oint \vec{E} \cdot d\vec{l} = 0. \quad (3.4.5)$$

Note that Q is a constant.

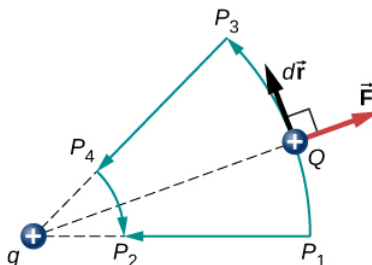


Figure 3.4.4: A closed path in an electric field. The net work around this path is zero.

Electric Potential Energy of Two Point Charges

Another implication is that we may define an electric potential energy for point charges. Recall that the work done by a conservative force is also expressed as the difference in the potential energy corresponding to that force. Therefore, the work W_{ref} to bring a charge from a reference point to a point of interest may be written as

$$W_{ref} = \int_{r_{ref}}^r \vec{F} \cdot d\vec{l} \quad (3.4.6)$$

and, by Equation 3.4.3, the difference in potential energy ($U_2 - U_1$) of the test charge Q between the two points is

$$\Delta U = - \int_{r_{ref}}^r \vec{F} \cdot d\vec{l}. \quad (3.4.7)$$

Therefore, we can write a general expression for the potential energy of two point charges (in spherical coordinates):

$$\Delta U = - \int_{r_{ref}}^r \frac{kqQ}{r^2} dr = - \left[-\frac{kqQ}{r} \right]_{r_{ref}}^r = kqQ \left[\frac{1}{r} - \frac{1}{r_{ref}} \right]. \quad (3.4.8)$$

We may take the second term to be an arbitrary constant reference level, which serves as the zero reference:

$$U(r) = k \frac{qQ}{r} - U_{ref}. \quad (3.4.9)$$

A convenient choice of reference that relies on our common sense is that when the two charges are infinitely far apart, there is no interaction between them. (Recall the discussion of reference potential energy in [Potential Energy and Conservation of Energy](#).) Taking the potential energy of this state to be zero removes the term U_{ref} from the equation (just like when we say the ground is zero potential energy in a gravitational potential energy problem), and the potential energy of Q when it is separated from q by a distance r assumes the form

$$\underbrace{U(r) = k \frac{qQ}{r}}_{\text{zero reference at } r=\infty} . \quad (3.4.10)$$

This formula is symmetrical with respect to q and Q , so it is best described as the potential energy of the two-charge system.

✓ Example 3.4.2: Potential Energy of a Charged Particle

A $+3.0$ nC charge Q is initially at rest a distance of 10 cm (r_1) from a $+5.0$ nC charge q fixed at the origin (Figure 3.4.5). Naturally, the Coulomb force accelerates Q away from q , eventually reaching 15 cm (r_2).

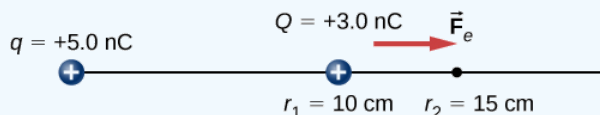


Figure 3.4.5: The charge Q is repelled by q , thus having work done on it and losing potential energy.

What is the change in the potential energy of the two-charge system from r_1 to r_2 ?

Strategy

Calculate the potential energy with the definition given above:

$\Delta U_{12} = - \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r}$. Since Q started from rest, this is the same as the kinetic energy.

Solution

We have

$$\begin{aligned} \Delta U_{12} &= - \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} \\ &= - \int_{r_1}^{r_2} \frac{kqQ}{r^2} dr \\ &= - \left[-\frac{kqQ}{r} \right]_{r_1}^{r_2} \\ &= kqQ \left[\frac{1}{r_2} - \frac{1}{r_1} \right] \\ &= (8.99 \times 10^9 \text{ N m}^2/\text{C}^2)(5.0 \times 10^{-9} \text{ C})(3.0 \times 10^{-9} \text{ C}) \left[\frac{1}{0.15 \text{ m}} - \frac{1}{0.10 \text{ m}} \right] \\ &= -4.5 \times 10^{-7} \text{ J}. \end{aligned} \quad (3.4.11)$$

Significance

The change in the potential energy is negative, as expected, and equal in magnitude to the change in kinetic energy in this system. Recall from Example 3.4.1 that the change in kinetic energy was positive.

? Exercise 3.4.2

What is the potential energy of Q relative to the zero reference at infinity at r_2 in the above example?

Answer

It has kinetic energy of $4.5 \times 10^{-7} \text{ J}$ at point r_2 and potential energy of $9.0 \times 10^{-7} \text{ J}$, which means that as Q approaches infinity, its kinetic energy totals three times the kinetic energy at r_2 , since all of the potential energy gets converted to kinetic.

Due to Coulomb's law, the forces due to multiple charges on a test charge Q superimpose; they may be calculated individually and then added. This implies that the work integrals and hence the resulting potential energies exhibit the same behavior. To demonstrate this, we consider an example of assembling a system of four charges.

✓ Example 3.4.3: Assembling Four Positive Charges

Find the amount of work an external agent must do in assembling four charges $+2.0 \mu\text{C}$, $+3.0 \mu\text{C}$, $+4.0 \mu\text{C}$, and $+5.0 \mu\text{C}$ at the vertices of a square of side 1.0 cm , starting each charge from infinity (Figure 3.4.6).

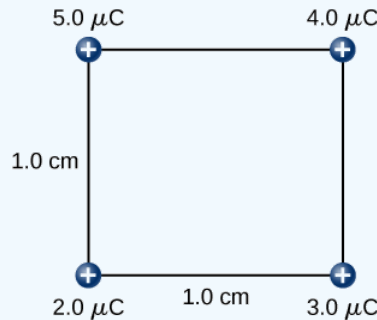


Figure 3.4.6: How much work is needed to assemble this charge configuration?

Strategy

We bring in the charges one at a time, giving them starting locations at infinity and calculating the work to bring them in from infinity to their final location. We do this in order of increasing charge.

Solution

Step 1. First bring the $+2.0 \mu\text{C}$ charge to the origin. Since there are no other charges at a finite distance from this charge yet, no work is done in bringing it from infinity,

$$W_1 = 0. \quad (3.4.12)$$

Step 2. While keeping the $+2.0 \mu\text{C}$ charge fixed at the origin, bring the $+3.0 \mu\text{C}$ charge to $(x, y, z) = (1.0 \text{ cm}, 0, 0)$ (Figure 3.4.7). Now, the applied force must do work against the force exerted by the $+2.0 \mu\text{C}$ charge fixed at the origin. The work done equals the change in the potential energy of the $+3.0 \mu\text{C}$ charge:

$$\begin{aligned} W_2 &= k \frac{q_1 q_2}{r_{12}} \\ &= \left(9.0 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right) \frac{(2.0 \times 10^{-6} \text{ C})(3.0 \times 10^{-6} \text{ C})}{1.0 \times 10^{-2} \text{ m}} \\ &= 5.4 \text{ J}. \end{aligned}$$

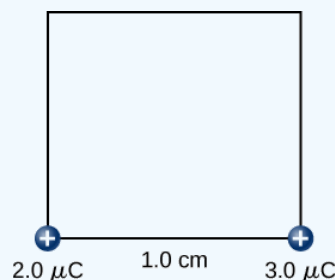


Figure 3.4.7: Step 2. Work W_2 to bring the $+3.0 \mu\text{C}$ charge from infinity.

Step 3. While keeping the charges of $+2.0 \mu\text{C}$ and $+3.0 \mu\text{C}$ fixed in their places, bring in the $+4.0 \mu\text{C}$ charge to $(x, y, z) = (1.0 \text{ cm}, 1.0 \text{ cm}, 0)$ (Figure 3.4.8). The work done in this step is

$$\begin{aligned} W_3 &= k \frac{q_1 q_3}{r_{13}} + k \frac{q_2 q_3}{r_{23}} \\ &= \left(9.0 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right) \left[\frac{(2.0 \times 10^{-6} \text{C})(4.0 \times 10^{-6} \text{C})}{\sqrt{2} \times 10^{-2} \text{m}} + \frac{(3.0 \times 10^{-6} \text{C})(4.0 \times 10^{-6} \text{C})}{1.0 \times 10^{-2} \text{m}} \right] \\ &= 15.9 \text{ J}. \end{aligned}$$

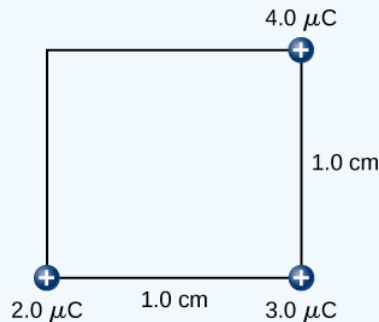


Figure 3.4.8: Step 3. The work W_3 to bring the $+4.0 \mu\text{C}$ charge from infinity.

Step 4. Finally, while keeping the first three charges in their places, bring the $+5.0 \mu\text{C}$ charge to $(x, y, z) = (0, 1.0 \text{ cm}, 0)$ (Figure 3.4.9). The work done here is

$$\begin{aligned} W_4 &= k q_4 \left[\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right], \\ &= \left(9.0 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right) (5.0 \times 10^{-6} \text{C}) \left[\frac{(2.0 \times 10^{-6} \text{C})}{1.0 \times 10^{-2} \text{m}} + \frac{(3.0 \times 10^{-6} \text{C})}{\sqrt{2} \times 10^{-2} \text{m}} + \frac{(4.0 \times 10^{-6} \text{C})}{1.0 \times 10^{-2} \text{m}} \right] \\ &= 36.5 \text{ J}. \end{aligned}$$

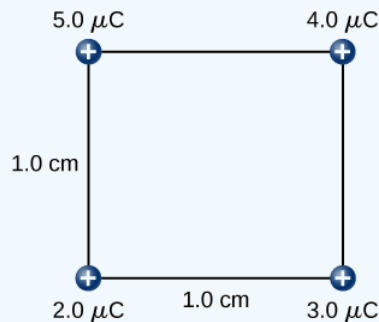


Figure 3.4.9: Step 4. The work W_4 to bring the $+5.0 \mu\text{C}$ charge from infinity.

Hence, the total work done by the applied force in assembling the four charges is equal to the sum of the work in bringing each charge from infinity to its final position:

$$\begin{aligned} W_T &= W_1 + W_2 + W_3 + W_4 \\ &= 0 + 5.4 \text{ J} + 15.9 \text{ J} + 36.5 \text{ J} \\ &= 57.8 \text{ J}. \end{aligned}$$

Significance

The work on each charge depends only on its pairwise interactions with the other charges. No more complicated interactions need to be considered; the work on the third charge only depends on its interaction with the first and second charges, the interaction between the first and second charge does not affect the third.

? Exercise 3.4.3

(1) Is the electrical potential energy of two point charges positive or negative if the charges are of the same sign? (2) Opposite signs? (3) How does this relate to the work necessary to bring the charges into proximity from infinity?

Answer

(1) positive, (2) negative, and (3) these quantities are the same as the work you would need to do to bring the charges in from infinity

Note that the electrical potential energy is positive if the two charges are of the same type, either positive or negative, and negative if the two charges are of opposite types. This makes sense if you think of the change in the potential energy ΔU as you bring the two charges closer or move them farther apart. Depending on the relative types of charges, you may have to work on the system or the system would do work on you, that is, your work is either positive or negative. If you have to do positive work on the system (actually push the charges closer), then the energy of the system should increase. If you bring two positive charges or two negative charges closer, you have to do positive work on the system, which raises their potential energy. Since potential energy is proportional to $1/r$, the potential energy goes up when r goes down between two positive or two negative charges.

On the other hand, if you bring a positive and a negative charge nearer, you have to do negative work on the system (the charges are pulling you), which means that you take energy away from the system. This reduces the potential energy. Since potential energy is negative in the case of a positive and a negative charge pair, the increase in $1/r$ makes the potential energy more negative, which is the same as a reduction in potential energy.

The result from Example 3.4.2 may be extended to systems with any arbitrary number of charges. In this case, it is most convenient to write the formula as

$$W_{12\dots N} = \frac{k}{2} \sum_i^N \sum_j^N \frac{q_i q_j}{r_{ij}} \text{ for } i \neq j. \quad (3.4.13)$$

The factor of $1/2$ accounts for adding each pair of charges twice.

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