VECTORS

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Vectors

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CHAPTER OVERVIEW

1: Introduction

1.1: Prelude to Vectors

Vectors are essential to physics and engineering. Many fundamental physical quantities are vectors, including displacement, velocity, force, and electric and magnetic vector fields. Scalar products of vectors define other fundamental scalar physical quantities, such as energy. Vector products of vectors define still other fundamental vector physical quantities, such as torque and angular momentum. In other words, vectors are a component part of physics in much the same way as sentences are a component part of literature.

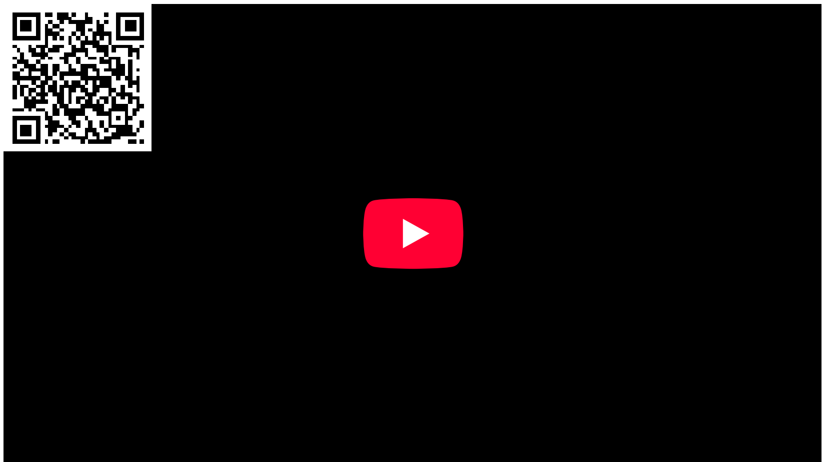
Figure 1.1.1: A signpost gives information about distances and directions to towns or to other locations relative to the location of the signpost. Distance is a scalar quantity. Knowing the distance alone is not enough to get to the town; we must also know the direction from the signpost to the town. The direction, together with the distance, is a vector quantity commonly called the displacement vector. A signpost, therefore, gives information about displacement vectors from the signpost to towns. (credit: modification of work by “studio tdes”/Flickr)

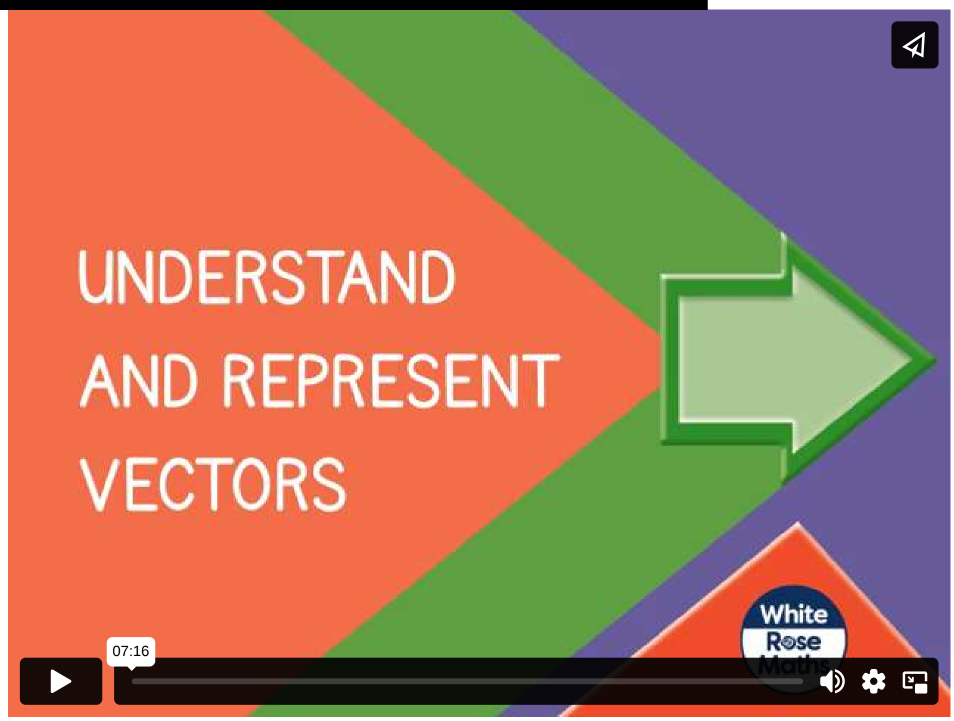
In introductory physics, vectors are Euclidean quantities that have geometric representations as arrows in one dimension (in a line), in two dimensions (in a plane), or in three dimensions (in space). They can be added, subtracted, or multiplied. In this chapter, we explore elements of vector algebra for applications in mechanics and in electricity and magnetism. Vector operations also have numerous generalizations in other branches of physics.

1.2: Scalars and Vectors Part 1

*To better understand the science of propulsion it is necessary to use some mathematical ideas from **vector analysis**. Most people are introduced to vectors in high school or college, but for those who were not introduced let's start with some videos*

DON'T PANIC!.





Spr10.3.1 - Understand and represent vectors from White Rose Education on Vimeo.

There are many complex parts to vector analysis and we aren't going there. We are going to limit ourselves to the very basics. Vectors allow us to look at complex, multi-dimensional problems as a simpler group of one-dimensional problems. We will be concerned mostly with definitions The words are a bit strange, but the ideas are very powerful as you will see. If you want to find out a lot more about vectors you can download this report on vector analysis.

Math and science were invented by humans to describe and understand the world around us. We live in a (at least) four-dimensional world governed by the passing of time and three space dimensions; up and down, left and right, and back and forth. We observe that there are some quantities and processes in our world that depend on the **direction** in which they occur, and there are some quantities that do not depend on direction. For example, the volume of an object, the three-dimensional space that an object occupies, does not depend on direction. If we have a 5 cubic foot block of iron and we move it up and down and then left and right, we still have a 5 cubic foot block of iron. On the other hand, the location, of an object does depend on direction. If we move the 5 cubic foot block 5 miles to the north, the resulting location is very different than if we moved it 5 miles to the east. Mathematicians and scientists call a quantity which depends on direction a **vector quantity**. A quantity which does not depend on direction is called a **scalar quantity**.

Vector quantities have two characteristics, a magnitude and a direction. Scalar quantities have only a magnitude. When comparing two vector quantities of the same type, you have to compare both the magnitude and the direction. For scalars, you only have to compare the magnitude. When doing any mathematical operation on a vector quantity (like adding, subtracting, multiplying ..) you have to consider both the magnitude and the direction. This makes dealing with vector quantities a little more complicated than scalars.

On the slide we list some of the physical quantities discussed in the Beginner's Guide to Aeronautics and group them into either vector or scalar quantities. Of particular interest, the forces which operate on a flying aircraft, the weight, thrust, and aerodynmaic forces, are all vector quantities. The resulting motion of the aircraft in terms of displacement, velocity, and acceleration are also vector quantities. These quantities can be determined by application of Newton's laws for vectors. The scalar quantities include most of the thermodynamic state variables involved with the propulsion system, such as the density, pressure, and temperature of the propellants. The energy, work, and entropy associated with the engines are also scalar quantities.

Vectors have magnitude and direction, scalars only have magnitude. The fact that **magnitude** occurs for both scalars and vectors can lead to some confusion. There are some quantities, like **speed**, which have very special definitions for scientists. By definition, speed is the scalar magnitude of a **velocity** vector. A car going down the road has a speed of 50 mph. Its velocity is 50 mph in the northeast direction. It can get very confusing when the terms are used interchangeably! Another example is **mass** and **weight**. Weight is a force which is a vector and has a magnitude and direction. Mass is a scalar. Weight and mass are related to one another, but they are not the same quantity.`

While Newton's laws describe the resulting motion of a solid, there are special equations which describe the motion of fluids, gases and liquids. For any physical system, the mass, momentum, and energy of the system must be conserved. Mass and energy are scalar quantities, while momentum is a vector quantity. This results in a **coupled** set of equations, called the Navier-Stokes equations, which describe how fluids behave when subjected to external forces. These equations are the fluid equivalent of Newton's laws of motion and are very difficult to solve and understand. A simplified version of the equations called the Euler equations can be solved for some fluids problems.

Here are some examples of scalar and vector quantities

Scalar Quantities	Vector Quantities
Length, area, volume	Displacement
Speed	Velocity
Mass, density	Acceleration
Pressure	Momentum
Temperature	Force
Energy	Lift, drag, thrust
Entropy	
Work, power	

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1.3: Scalars and Vectors Part 2

1.3: Scalars and Vectors Part 2

Learning Objectives

- Describe the difference between vector and scalar quantities.
- Identify the magnitude and direction of a vector.
- Explain the effect of multiplying a vector quantity by a scalar.
- Describe how one-dimensional vector quantities are added or subtracted.
- Explain the geometric construction for the addition or subtraction of vectors in a plane.
- Distinguish between a vector equation and a scalar equation.

Many familiar physical quantities can be specified completely by giving a single number and the appropriate unit. For example, “a class period lasts 50 min” or “the gas tank in my car holds 65 L” or “the distance between two posts is 100 m.” A physical quantity that can be specified completely in this manner is called a **scalar quantity**. Scalar is a synonym of “number.” Time, mass, distance, length, volume, temperature, and energy are examples of **scalar** quantities.

Scalar quantities that have the same physical units can be added or subtracted according to the usual rules of algebra for numbers. For example, a class ending 10 min earlier than 50 min lasts 50 min − 10 min = 40 min. Similarly, a 60-cal serving of corn followed by a 200-cal serving of donuts gives 60 cal + 200 cal = 260 cal of energy. When we multiply a scalar quantity by a number, we obtain the same scalar quantity but with a larger (or smaller) value. For example, if yesterday’s breakfast had 200 cal of energy and today’s breakfast has four times as much energy as it had yesterday, then today’s breakfast has 4(200 cal) = 800 cal of energy. Two scalar quantities can also be multiplied or divided by each other to form a derived scalar quantity. For example, if a train covers a distance of 100 km in 1.0 h, its speed is 100.0 km/1.0 h = 27.8 m/s, where the speed is a derived scalar quantity obtained by dividing distance by time.

Many physical quantities, however, cannot be described completely by just a single number of physical units. For example, when the U.S. Coast Guard dispatches a ship or a helicopter for a rescue mission, the rescue team must know not only the distance to the distress signal, but also the direction from which the signal is coming so they can get to its origin as quickly as possible. Physical quantities specified completely by giving a number of units (magnitude) and a direction are called **vector quantities**. Examples of vector quantities include displacement, velocity, position, force, and torque. In the language of mathematics, physical vector quantities are represented by mathematical objects called **vectors** (Figure $1.3.1$). We can add or subtract two vectors, and we can multiply a vector by a scalar or by another vector, but we cannot divide by a vector. The operation of division by a vector is not defined.

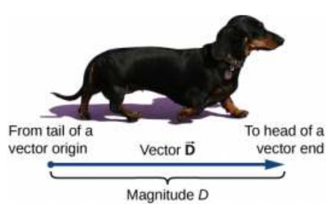


Figure $1.3.1$: We draw a vector from the initial point or origin (called the “tail” of a vector) to the end or terminal point (called the “head” of a vector), marked by an arrowhead. Magnitude is the length of a vector and is always a positive scalar quantity. (credit: modification of work by Cate Sevilla)

Let’s examine vector algebra using a graphical method to be aware of basic terms and to develop a qualitative understanding. In practice, however, when it comes to solving physics problems, we use analytical methods, which we’ll see in the next section. Analytical methods are more simple computationally and more accurate than graphical methods. From now on, to distinguish between a vector and a scalar quantity, we adopt the common convention that a letter in bold type with an arrow above it denotes a vector, and a letter without an arrow denotes a scalar. For example, a distance of 2.0 km, which is a scalar quantity, is denoted by d = 2.0 km, whereas a displacement of 2.0 km in some direction, which is a vector quantity, is denoted by $\vec{d}$.

Suppose you tell a friend on a camping trip that you have discovered a terrific fishing hole 6 km from your tent. It is unlikely your friend would be able to find the hole easily unless you also communicate the direction in which it can be found with respect to your campsite. You may say, for example, “Walk about 6 km northeast from my tent.” The key concept here is that you have to give not one but two pieces of information—namely, the distance or magnitude (6 km) **and** the direction (northeast).

Displacement is a general term used to describe a change in position, such as during a trip from the tent to the fishing hole. Displacement is an example of a vector quantity. If you walk from the tent (location A) to the hole (location B), as shown in Figure $1.3.2$, the vector $\vec{D}$, representing your **displacement**, is drawn as the arrow that originates at point A and ends at point B. The arrowhead marks the end of the vector. The direction of the displacement vector $\vec{D}$ is the direction of the arrow. The length of the arrow represents the **magnitude** D of vector $\vec{D}$. Here, D = 6 km. Since the magnitude of a vector is its length, which is a positive number, the magnitude is also indicated by placing the absolute value notation around the symbol that denotes the vector; so, we can write equivalently that D ≡ $|\vec{D}|$. To solve a vector problem graphically, we need to draw the vector $\vec{D}$ to scale. For example, if we assume 1 unit of distance (1 km) is represented in the drawing by a line segment of length u = 2 cm, then the total displacement in this example is represented by a vector of length d = 6u = 6(2 cm) = 12 cm , as shown in Figure $1.3.3$. Notice that here, to avoid confusion, we used D = 6 km to denote the magnitude of the actual displacement and d = 12 cm to denote the length of its representation in the drawing.

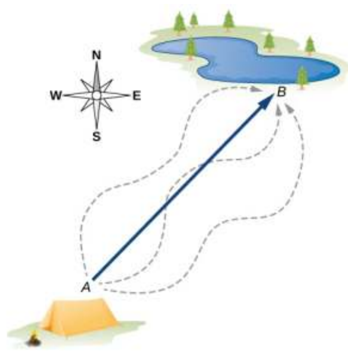


Figure $1.3.2$: The displacement vector from point A (the initial position at the campsite) to point B (the final position at the fishing hole) is indicated by an arrow with origin at point A and end at point B. The displacement is the same for any of the actual paths (dashed curves) that may be taken between points A and B.

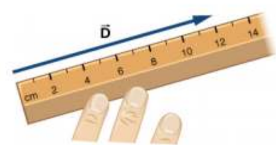


Figure $1.3.3$: A displacement $\vec{D}$of magnitude 6 km is drawn to scale as a vector of length 12 cm when the length of 2 cm represents 1 unit of displacement (which in this case is 1 km).

Suppose your friend walks from the campsite at A to the fishing pond at B and then walks back: from the fishing pond at B to the campsite at A. The magnitude of the displacement vector $\vec{D}_{AB}$ from A to B is the same as the magnitude of the displacement vector $\vec{D}_{BA}$ from B to A (it equals 6 km in both cases), so we can write $\vec{D}_{AB} = \vec{D}_{BA}$. However, vector $\vec{D}_{AB}$ is not equal to vector $\vec{D}_{BA}$ because these two vectors have different directions: $\vec{D}_{AB} \neq \vec{D}_{BA}$. In Figure 2.3, vector $\vec{D}_{BA}$ would be represented by a vector with an origin at point B and an end at point A, indicating vector $\vec{D}_{BA}$ points to the southwest, which is exactly 180° opposite to the direction of vector $\vec{D}_{AB}$. We say that vector $\vec{D}_{BA}$ is **antiparallel** to vector $\vec{D}_{AB}$ and write $\vec{D}_{AB} = -\vec{D}_{BA}$, where the minus sign indicates the antiparallel direction.

Two vectors that have identical directions are said to be **parallel vectors**—meaning, they are **parallel** to each other. Two parallel vectors $\vec{A}$ and $\vec{B}$ are equal, denoted by $\vec{A} = \vec{B}$, if and only if they have equal magnitudes $|\vec{A}| = |\vec{B}|$. Two vectors with directions perpendicular to each other are said to be **orthogonal vectors**. These relations between vectors are illustrated in Figure $1.3.4$.

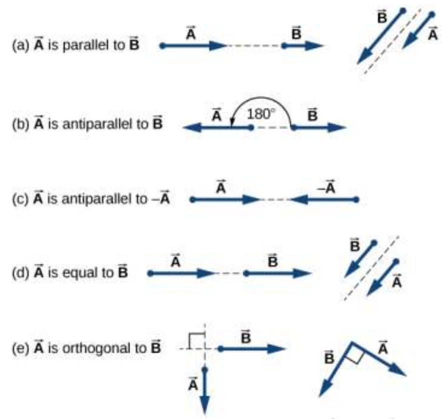


Figure 1.3.4: Various relations between two vectors $\vec{A}$ and $\vec{B}$. (a) $\vec{A} \neq \vec{B}$ because A ≠ B . (b) $\vec{A} \neq \vec{B}$ because they are not parallel and A ≠ B . (c) $\vec{A} \neq -\vec{A}$ because they have different directions (even though $|\vec{A}| = |-\vec{A}| = A$) . (d) $\vec{A} = \vec{B}$ because they are parallel and have identical magnitudes A = B. (e) $\vec{A} \neq \vec{B}$ because they have different directions (are not parallel); here, their directions differ by 90° —meaning, they are orthogonal.

? Exercise 2.1

Two motorboats named **Alice** and **Bob** are moving on a lake. Given the information about their velocity vectors in each of the following situations, indicate whether their velocity vectors are equal or otherwise.

a. **Alice** moves north at 6 knots and **Bob** moves west at 6 knots.
b. **Alice** moves west at 6 knots and **Bob** moves west at 3 knots.
c. **Alice** moves northeast at 6 knots and **Bob** moves south at 3 knots.
d. **Alice** moves northeast at 6 knots and **Bob** moves southwest at 6 knots.
e. **Alice** moves northeast at 2 knots and **Bob** moves closer to the shore northeast at 2 knots.

Algebra of Vectors in One Dimension

Vectors can be multiplied by scalars, added to other vectors, or subtracted from other vectors. We can illustrate these vector concepts using an example of the fishing trip seen in Figure $1.3.5$.

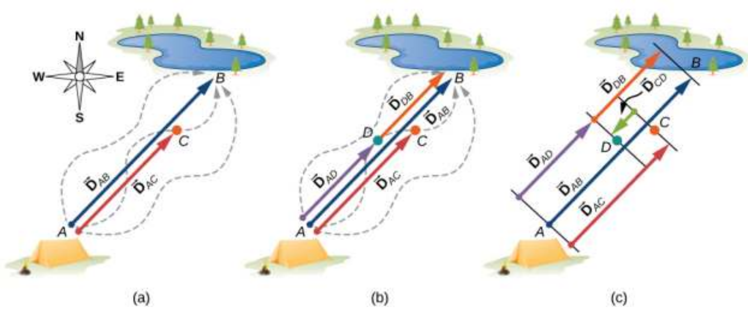


Figure $1.3.5$: Displacement vectors for a fishing trip. (a) Stopping to rest at point C while walking from camp (point A) to the pond (point B). (b) Going back for the dropped tackle box (point D). (c) Finishing up at the fishing pond.

Suppose your friend departs from point A (the campsite) and walks in the direction to point B (the fishing pond), but, along the way, stops to rest at some point C located three-quarters of the distance between A and B, beginning from point A (Figure $1.3.5a$). What is his displacement vector $\vec{D}_{AC}$ when he reaches point C? We know that if he walks all the way to B, his displacement vector relative to A is $\vec{D}_{AB}$, which has magnitude D_{AB} = 6 km and a direction of northeast. If he walks only a 0.75 fraction of the total distance, maintaining the northeasterly direction, at point C he must be 0.75 D_{AB} = 4.5 km away from the campsite at A. So, his displacement vector at the rest point C has magnitude D_{AC} = 4.5 km = 0.75 D_{AB} and is parallel to the displacement vector $\vec{D}_{AB}$. All of this can be stated succinctly in the form of the following **vector equation**:

$$\vec{D}_{AC} = 0.75\ \vec{D}_{AB}.$$

In a vector equation, both sides of the equation are vectors. The previous equation is an example of a vector multiplied by a positive scalar (number) α = 0.75. The result, $\vec{D}_{AC}$, of such a multiplication is a new vector with a direction parallel to the direction of the original vector $\vec{D}_{AB}$. In general, when a vector $\vec{D}_{A}$ is multiplied by a positive scalar α, the result is a new vector $\vec{D}_{B}$ that is parallel to $\vec{D}_{A}$:

$$\vec{B} = \alpha\vec{A} \tag{1.3.1}$$

The magnitude $|\vec{B}|$ of this new vector is obtained by multiplying the magnitude $|\vec{A}|$ of the original vector, as expressed by the **scalar equation**:

$$B = |\alpha|A. \tag{1.3.2}$$

In a scalar equation, both sides of the equation are numbers. Equation 1.3.2 is a scalar equation because the magnitudes of vectors are scalar quantities (and positive numbers). If the scalar α is **negative** in the vector equation Equation 1.3.1, then the magnitude $|\vec{B}|$ of the new vector is still given by Equation 1.3.2, but the direction of the new vector $\vec{B}$ is **antiparallel** to the direction of $\vec{A}$. These principles are illustrated in Figure $1.3.6a$ by two examples where the length of vector $\vec{A}$ is 1.5 units. When α = 2, the new vector $\vec{B} = 2\vec{A}$ has length B = 2A = 3.0 units (twice as long as the original vector) and is parallel to the original vector. When α = −2, the new vector $\vec{C} = -2\vec{A}$ has length C = |−2| A = 3.0 units (twice as long as the original vector) and is antiparallel to the original vector.

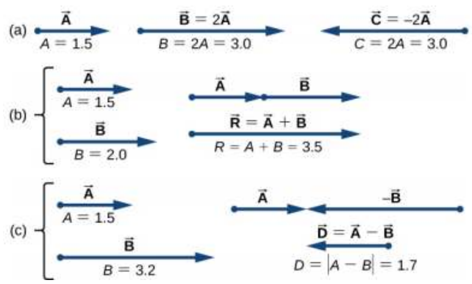


Figure 1.3.6: Algebra of vectors in one dimension. (a) Multiplication by a scalar. (b) Addition of two vectors ($\vec{R}$ is called the resultant of vectors ($\vec{A}$ and ($\vec{B}$). (c) Subtraction of two vectors ($\vec{D}$ is the difference of vectors ($\vec{A}$ and $\vec{B}$).

Now suppose your fishing buddy departs from point A (the campsite), walking in the direction to point B (the fishing hole), but he realizes he lost his tackle box when he stopped to rest at point C (located three-quarters of the distance between A and B, beginning from point A). So, he turns back and retraces his steps in the direction toward the campsite and finds the box lying on the path at some point D only 1.2 km away from point C (see Figure $1.3.5b$). What is his displacement vector $\vec{D}_{AD}$ when he finds the box at point D? What is his displacement vector $\vec{D}_{DB}$ from point D to the hole? We have already established that at rest point C his displacement vector is $\vec{D}_{AC}$ = 0.75 $\vec{D}_{AB}$. Starting at point C, he walks southwest (toward the campsite), which means his new displacement vector $\vec{D}_{CD}$ from point C to point D is antiparallel to $\vec{D}_{AB}$. Its magnitude $|\vec{D}_{CD}|$ is D_{CD} = 1.2 km = 0.2 D_{AB}, so his second displacement vector is $\vec{D}_{CD}$ = −0.2 $\vec{D}_{AB}$. His total displacement $\vec{D}_{AD}$ relative to the campsite is the vector sum of the two displacement vectors: vector $\vec{D}_{AC}$ (from the campsite to the rest point) and vector $\vec{D}_{CD}$ (from the rest point to the point where he finds his box):

$$\vec{D}_{AD} = \vec{D}_{AC} + \vec{D}_{CD}. \tag{1.3.3}$$

The vector sum of two (or more vectors is called the **resultant vector** or, for short, the **resultant**. When the vectors on the right-hand-side of Equation 1.3.3 are known, we can find the resultant $\vec{D}_{AD}$ as follows:

$$\vec{D}_{AD} = \vec{D}_{AC} + \vec{D}_{CD} = 0.75\ \vec{D}_{AB} - 0.2\ \vec{D}_{AB} = (0.75 - 0.2)\vec{D}_{AB} = 0.55\vec{D}_{AB}. \tag{1.3.4}$$

When your friend finally reaches the pond at B, his displacement vector $\vec{D}_{AB}$ from point A is the vector sum of his displacement vector $\vec{D}_{AD}$ from point A to point D and his displacement vector $\vec{D}_{DB}$ from point D to the fishing hole: $\vec{D}_{AB} = \vec{D}_{AD} + \vec{D}_{DB}$ (see Figure $1.3.5c$). This means his displacement vector $\vec{D}_{DB}$ is the difference of two vectors:

$$\vec{D}_{DB} = \vec{D}_{AB} - \vec{D}_{AD} = \vec{D}_{AB} + (-\vec{D}_{AD}). \tag{1.3.5}$$

Notice that a difference of two vectors is nothing more than a vector sum of two vectors because the second term in Equation 1.3.5 is vector $-\vec{D}_{AD}$ (which is antiparallel to $\vec{D}_{AD}$). When we substitute Equation 1.3.4 into Equation 1.3.5, we obtain the second displacement vector:

$$\vec{D}_{DB} = \vec{D}_{AB} - \vec{D}_{AD} = \vec{D}_{AB} - 0.55\ \vec{D}_{AB} = (1.0 - 0.55)\ \vec{D}_{AB} = 0.45\ \vec{D}_{AB}. \tag{1.3.6}$$

This result means your friend walked D_{DB} = 0.45 D_{AB} = 0.45(6.0 km) = 2.7 km from the point where he finds his tackle box to the fishing hole.

When vectors $\vec{A}$ and $\vec{B}$ lie along a line (that is, in one dimension), such as in the camping example, their resultant $\vec{R}$ = $\vec{A}$ + $\vec{B}$ and their difference $\vec{D}$ = $\vec{A}$ − $\vec{B}$ both lie along the same direction. We can illustrate the addition or subtraction of vectors by drawing the corresponding vectors to scale in one dimension, as shown in Figure 1.3.6.

To illustrate the resultant when $\vec{A}$ and $\vec{B}$ are two parallel vectors, we draw them along one line by placing the origin of one vector at the end of the other vector in head-to-tail fashion (see Figure (\PageIndex{6b}\)). The magnitude of this resultant is the sum of their magnitudes: R = A + B. The direction of the resultant is parallel to both vectors. When vector $\vec{A}$ is antiparallel to vector $\vec{B}$, we draw them along one line in either head-to-head fashion (Figure (\PageIndex{6c}\)) or tail-to-tail fashion. The magnitude of the vector difference, then, is the **absolute value** D = |A − B| of the difference of their magnitudes. The direction of the difference vector $\vec{D}$ is parallel to the direction of the longer vector.

In general, in one dimension—as well as in higher dimensions, such as in a plane or in space—we can add any number of vectors and we can do so in any order because the addition of vectors is **commutative**,

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}. \tag{1.3.7}$$

and **associative,**

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}). \tag{1.3.8}$$

Moreover, multiplication by a scalar is **distributive**:

$$\alpha_1 \vec{A} + \alpha_2 \vec{A} = (\alpha_1 + \alpha_2)\vec{A}. \tag{1.3.9}$$

We used the distributive property in Equation 1.3.4 and Equation 1.3.6.

When adding many vectors in one dimension, it is convenient to use the concept of a **unit vector**. A unit vector, which is denoted by a letter symbol with a hat, such as $\hat{u}$, has a magnitude of one and does not have any physical unit so that $|\hat{u}|$ ≡ u = 1. The only role of a unit vector is to specify direction. For example, instead of saying vector $\vec{D}_{AB}$ has a magnitude of 6.0 km and a direction of northeast, we can introduce a unit vector $\hat{u}$ that points to the northeast and say succinctly that $\vec{D}_{AB}$ = (6.0 km) $\hat{u}$. Then the southwesterly direction is simply given by the unit vector $-\hat{u}$. In this way, the displacement of 6.0 km in the southwesterly direction is expressed by the vector

$$\vec{D}_{BA} = (-6.0\ km)\ \hat{u}.$$

1.4: One Dimensional Motion (Video) Khan Academy

1.5: Scalars and Vectors (Video) Bozman Science

1.6: Scalars and Vectors (Video) Organic Chemistry Tutor

1.7: Additional Materials

Scalar and Vector Quantities NASA
Link: https://www.grc.nasa.gov/www/k-12/airplane/vectors.html#:~:text=Mathematicians%20and%20scientists%20call%20a,quantities%20have%20only%20a%20magnitude.

Scalars and Vectors Physics Classroom

CHAPTER OVERVIEW

2: Addition of Vectors- Graphical Method

2.1: Vector Addition in Two Dimensions

✓ Example 2.1.1: A Ladybug Walker

A long measuring stick rests against a wall in a physics laboratory with its 200-cm end at the floor. A ladybug lands on the 100-cm mark and crawls randomly along the stick. It first walks 15 cm toward the floor, then it walks 56 cm toward the wall, then it walks 3 cm toward the floor again. Then, after a brief stop, it continues for 25 cm toward the floor and then, again, it crawls up 19 cm toward the wall before coming to a complete rest (Figure 2.1.1). Find the vector of its total displacement and its final resting position on the stick.

Strategy

If we choose the direction along the stick toward the floor as the direction of unit vector $\hat{u}$, then the direction toward the floor is $+\hat{u}$ and the direction toward the wall is $-\hat{u}$. The ladybug makes a total of five displacements:

$$\begin{aligned}\vec{D}_1 &= (15\ cm)(+\hat{u}), \\ \vec{D}_2 &= (56\ cm)(-\hat{u}), \\ \vec{D}_3 &= (3\ cm)(+\hat{u}), \\ \vec{D}_4 &= (25\ cm)(+\hat{u}),\ and \\ \vec{D}_5 &= (19\ cm)(-\hat{u}).\end{aligned}$$

The total displacement $\vec{D}$ is the resultant of all its displacement vectors.

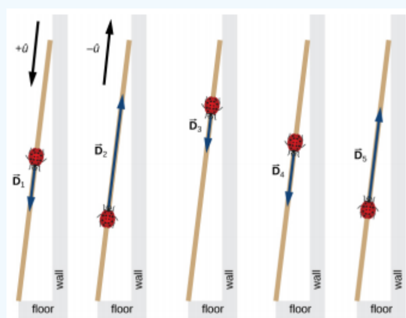


Figure 2.1.1: Five displacements of the ladybug. Note that in this schematic drawing, magnitudes of displacements are not drawn to scale. (credit: modification of work by "Persian Poet Gal"/Wikimedia Commons)

Solution

The resultant of all the displacement vectors is

$$\begin{aligned}\vec{D} &= \vec{D}_1 + \vec{D}_2 + \vec{D}_3 + \vec{D}_4 + \vec{D}_5 \\ &= (15\ cm)(+\hat{u}) + (56\ cm)(-\hat{u}) + (3\ cm)(+\hat{u}) + (25\ cm)(+\hat{u}) + (19\ cm)(-\hat{u}) \\ &= (15 - 56 + 3 + 25 - 19) cm\ \hat{u} \\ &= -32\ cm\ \hat{u}.\end{aligned}$$

In this calculation, we use the distributive law given by Equation 2.2.9. The result reads that the total displacement vector points away from the 100-cm mark (initial landing site) toward the end of the meter stick that touches the wall. The end that touches the wall is marked 0 cm, so the final position of the ladybug is at the (100 – 32) cm = 68-cm mark.

? Exercise 2.2

A cave diver enters a long underwater tunnel. When her displacement with respect to the entry point is 20 m, she accidentally drops her camera, but she doesn't notice it missing until she is some 6 m farther into the tunnel. She swims back 10 m but cannot find the camera, so she decides to end the dive. How far from the entry point is she? Taking the positive direction out of the tunnel, what is her displacement vector relative to the entry point?

Algebra of Vectors in Two Dimensions

When vectors lie in a plane—that is, when they are in two dimensions—they can be multiplied by scalars, added to other vectors, or subtracted from other vectors in accordance with the general laws expressed by Equation 2.2.1, Equation 2..2.2, Equation 2.2.7, and Equation 2.2.8. However, the addition rule for two vectors in a plane becomes more complicated than the rule for vector addition in one dimension. We have to use the laws of geometry to construct resultant vectors, followed by trigonometry to find vector magnitudes and directions. This geometric approach is commonly used in navigation (Figure $2.1.2$). In this section, we need to have at hand two rulers, a triangle, a protractor, a pencil, and an eraser for drawing vectors to scale by geometric constructions.

Figure $2.1.2$: In navigation, the laws of geometry are used to draw resultant displacements on nautical maps.

For a geometric construction of the sum of two vectors in a plane, we follow the **parallelogram rule**. Suppose two vectors $\vec{A}$ and $\vec{B}$ are at the arbitrary positions shown in Figure $2.1.3$. Translate either one of them in parallel to the beginning of the other vector, so that after the translation, both vectors have their origins at the same point. Now, at the end of vector $\vec{A}$ we draw a line parallel to vector $\vec{B}$ and at the end of vector $\vec{B}$ we draw a line parallel to vector $\vec{A}$ (the dashed lines in Figure $2.1.3$). In this way, we obtain a parallelogram. From the origin of the two vectors we draw a diagonal that is the resultant $\vec{R}$ of the two vectors: $\vec{R} = \vec{A} + \vec{B}$ (Figure $2.1.3a$). The other diagonal of this parallelogram is the vector difference of the two vectors $\vec{D} = \vec{A} - \vec{B}$, as shown in Figure $2.1.3b$. Notice that the end of the difference vector is placed at the end of vector $\vec{A}$.

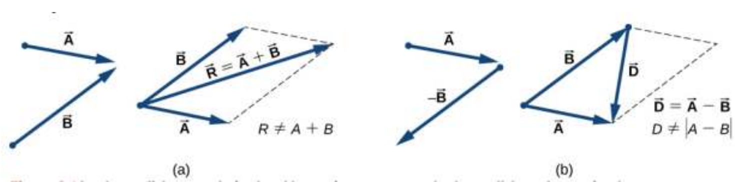


Figure $2.1.3$: The parallelogram rule for the addition of two vectors. Make the parallel translation of each vector to a point where their origins (marked by the dot) coincide and construct a parallelogram with two sides on the vectors and the other two sides (indicated by dashed lines) parallel to the vectors. (a) Draw the resultant vector $\vec{R}$ along the diagonal of the parallelogram from the common point to the opposite corner. Length R of the resultant vector is not equal to the sum of the magnitudes of the two vectors. (b) Draw the difference vector $\vec{D} = \vec{A} - \vec{B}$ along the diagonal connecting the ends of the vectors. Place the origin of vector $\vec{D}$ at the end of vector $\vec{B}$ and the end (arrowhead) of vector $\vec{D}$ at the end of vector $\vec{A}$. Length D of the difference vector is not equal to the difference of magnitudes of the two vectors.

It follows from the parallelogram rule that neither the magnitude of the resultant vector nor the magnitude of the difference vector can be expressed as a simple sum or difference of magnitudes A and B, because the length of a diagonal cannot be expressed as a simple sum of side lengths. When using a geometric construction to find magnitudes $|\vec{R}|$ and $|\vec{D}|$, we have to use trigonometry laws for triangles, which may lead to complicated algebra. There are two ways to circumvent this algebraic complexity. One way is to use the method of components, which we examine in the next section. The other way is to draw the vectors to scale, as is done in navigation, and read approximate vector lengths and angles (directions) from the graphs. In this section we examine the second approach.

If we need to add three or more vectors, we repeat the parallelogram rule for the pairs of vectors until we find the resultant of all of the resultants. For three vectors, for example, we first find the resultant of vector 1 and vector 2, and then we find the resultant of this resultant and vector 3. The order in which we select the pairs of vectors does not matter because the operation of vector addition is commutative and associative (see Equation 2.2.7 and Equation 2.2.8). Before we state a general rule that follows from repetitive applications of the parallelogram rule, let's look at the following example.

Suppose you plan a vacation trip in Florida. Departing from Tallahassee, the state capital, you plan to visit your uncle Joe in Jacksonville, see your cousin Vinny in Daytona Beach, stop for a little fun in Orlando, see a circus performance in Tampa, and visit the University of Florida in Gainesville. Your route may be represented by five displacement vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, and $\vec{E}$, which are indicated by the red vectors in Figure $2.1.4$. What is your total displacement when you reach Gainesville? The total displacement is the vector sum of all five displacement vectors, which may be found by using the parallelogram rule four times. Alternatively, recall that the displacement vector has its beginning at the initial position (Tallahassee) and its end at the final position (Gainesville), so the total displacement vector can be drawn directly as an arrow connecting Tallahassee with Gainesville (see the green vector in Figure $2.1.4$). When we use the parallelogram rule four times, the resultant $\vec{R}$ we obtain is exactly this green vector connecting Tallahassee with Gainesville: $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$.

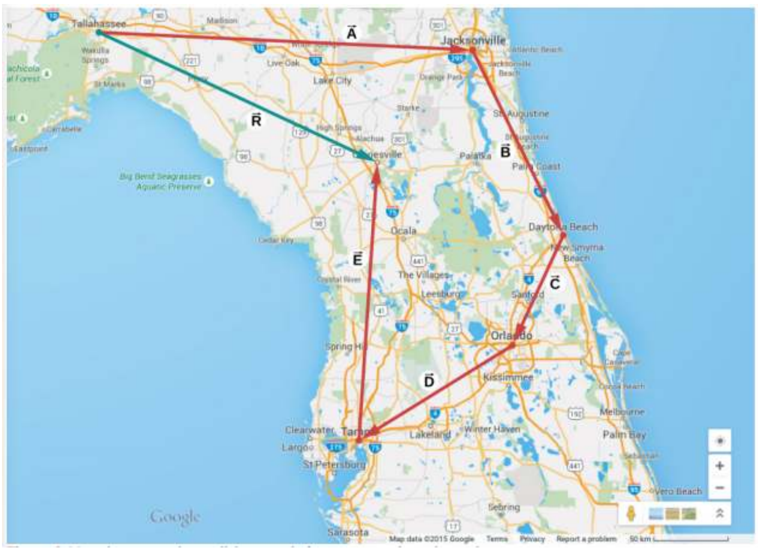


Figure $2.1.4$: When we use the parallelogram rule four times, we obtain the resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$, which is the green vector connecting Tallahassee with Gainesville.

Drawing the resultant vector of many vectors can be generalized by using the following tail-to-head geometric construction. Suppose we want to draw the resultant vector $\vec{R}$ of four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ (Figure $2.1.5a$). We select any one of the vectors as the first vector and make a parallel translation of a second vector to a position where the origin ("tail") of the second vector coincides with the end ("head") of the first vector. Then, we select a third vector and make a parallel translation of the third vector to a position where the origin of the third vector coincides with the end of the second vector. We repeat this procedure until all the vectors are in a head-to-tail arrangement like the one shown in Figure $2.1.5$. We draw the resultant vector $\vec{R}$ by connecting the origin ("tail") of the first vector with the end ("head") of the last vector. The end of the resultant vector is at the end of the last vector. Because the addition of vectors is associative and commutative, we obtain the same resultant vector regardless of which vector we choose to be first, second, third, or fourth in this construction.

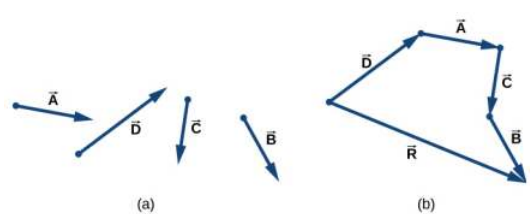


Figure $2.1.5$: Tail-to-head method for drawing the resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$. (a) Four vectors of different magnitudes and directions. (b) Vectors in (a) are translated to new positions where the origin ("tail") of one vector is at the end ("head") of another vector. The resultant vector is drawn from the origin ("tail") of the first vector to the end ("head") of the last vector in this arrangement.

Example 2.1.2: Geometric Construction of the Resultant

The three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Figure 2.1.6 are specified by their magnitudes A = 10.0, B = 7.0, and C = 8.0, respectively, and by their respective direction angles with the horizontal direction $\alpha = 35°$, $\beta = -110°$, and $\gamma = 30°$. The physical units of the magnitudes are centimeters. Choose a convenient scale and use a ruler and a protractor to find the following vector sums: (a) $\vec{R} = \vec{A} + \vec{B}$, (b) $\vec{D} = \vec{A} - \vec{B}$, and (c) $\vec{S} = \vec{A} - 3\vec{B} + \vec{C}$.

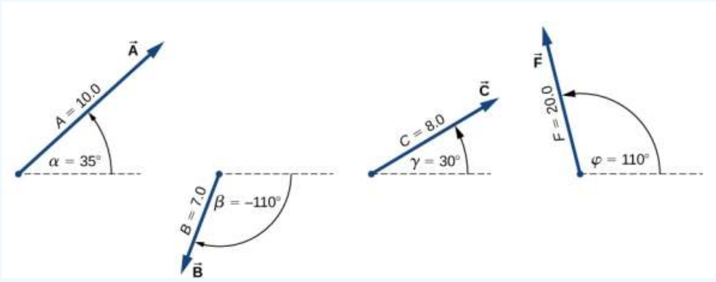


Figure 2.1.6: Vectors used in Example 2.1.2 and in the Exercise feature that follows.

Strategy

In geometric construction, to find a vector means to find its magnitude and its direction angle with the horizontal direction. The strategy is to draw to scale the vectors that appear on the right-hand side of the equation and construct the resultant vector. Then, use a ruler and a protractor to read the magnitude of the resultant and the direction angle. For parts (a) and (b) we use the parallelogram rule. For (c) we use the tail-to-head method.

Solution

For parts (a) and (b), we attach the origin of vector $\vec{B}$ to the origin of vector $\vec{A}$, as shown in Figure 2.1.7, and construct a parallelogram. The shorter diagonal of this parallelogram is the sum $\vec{A} + \vec{B}$. The longer of the diagonals is the difference $\vec{A} - \vec{B}$. We use a ruler to measure the lengths of the diagonals, and a protractor to measure the angles with the horizontal. For the resultant $\vec{R}$, we obtain R = 5.8 cm and $\theta_R \approx 0°$. For the difference $\vec{D}$, we obtain D = 16.2 cm and $\theta_D = 49.3°$, which are shown in Figure 2.1.7.

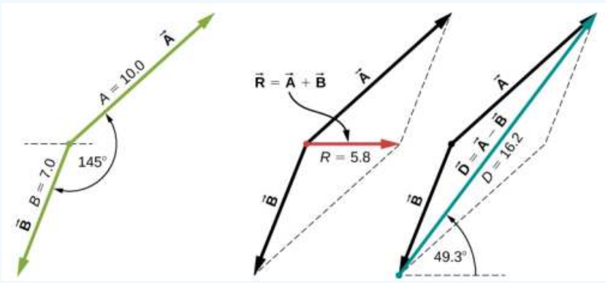


Figure 2.1.7: Using the parallelogram rule to solve (a) (finding the resultant, red) and (b) (finding the difference, blue).

For (c), we can start with vector $-3\vec{B}$ and draw the remaining vectors tail-to-head as shown in Figure 2.1.8. In vector addition, the order in which we draw the vectors is unimportant, but drawing the vectors to scale is very important. Next, we draw vector $\vec{S}$ from the origin of the first vector to the end of the last vector and place the arrowhead at the end of $\vec{S}$. We use a ruler to measure the length of $\vec{S}$, and find that its magnitude is S = 36.9 cm. We use a protractor and find that its direction angle is $\theta_S = 52.9°$. This solution is shown in Figure 2.1.8.

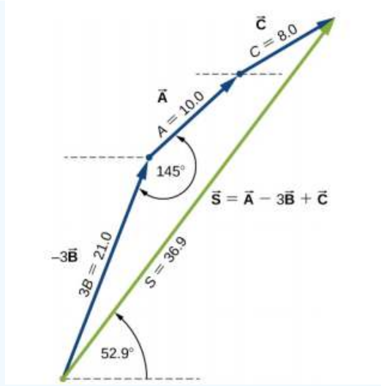


Figure 2.1.8: Using the tail-to-head method to solve (c) (finding vector $\vec{S}$, green).

Exercise 2.3

Using the three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{F}$ in Figure 2.1.6, choose a convenient scale, and use a ruler and a protractor to find vector $\vec{G}$ given by the vector equation $\vec{G} = \vec{A} + 2\vec{B} - \vec{F}$.

Simulation

Observe the addition of vectors in a plane by visiting this vector calculator and this PhET simulation.

2.2: Graphical Methods of Vector Addition

This man shooting a basketball requires a subconscious understanding of vectors

Figure 2.2.1: Copy and Paste Caption here. (Copyright; author via source)

Successfully shooting a basketball requires a subconscious understanding of the vectors involved in how the basketball moves through the air. The vertical and horizontal vectors must be perfectly organized if the ball is to pass through the basket.

Graphical Methods Vector Addition

In physics, a quantity, such as mass, length, or speed that is completely specified by its magnitude and has no direction is called a **scalar**. A **vector**, on the other hand, is a quantity possessing both magnitude and direction. A vector quantity is typically represented by an arrow-tipped line segment. The length of the line, drawn to scale, represents the magnitude of the quantity. The direction of the arrow indicates the direction of the vector. Not only can vectors be represented graphically, but they can also be added graphically.

For one dimensional **vector addition**, the first vector is placed on a number line with the tail of the vector on the origin. The second vector is placed with its tail exactly on the arrow head of the first vector. The sum of the two vectors is the vector that begins at the origin and ends at the arrow head of the final added vector.

Consider the following two vectors.

The top number line shows a vector with magnitude 11 in the positive direction and the bottom number line shows a vector with magnitude -3

Figure 2.2.2

The red vector has a magnitude of 11 in the positive direction on the number line. The blue vector has a magnitude of -3, indicating 3 units in the negative direction on the number line. In order to add these two vectors, we place one of the vectors on a number line and then the second vector is placed on the same number line such that its origin is on the arrow head of the first vector.

One number line with one vector with magnitude 11 in the positive direction and another vector overlapping with magnitude -3

Figure 2.2.3

The sum of these two vectors is the vector that begins at the origin of the first vector (the red one) and ends at the arrow head of the second (blue) vector. So the sum of these two vectors is the purple vector, as shown below.

One number line with one vector with magnitude 11 in the positive direction and another vector overlapping with magnitude -3, showing the sum of the two vectors with a third with magnitude 8

Figure 2.2.4

The vector sum of the first two vectors is a vector that begins at the origin and has a magnitude of 8 units in the positive direction. If we were adding three or four vectors all in one dimension, we would continue to place them head to toe in sequence on the number line. The sum would be the vector that begins at the beginning of the first vector and goes to the ending of the final vector.

Adding Vectors in Two Dimensions

In the following image, vectors A and B represent the two displacements of a person who walked 90. m east and then 50. m north. We want to add these two vectors to get the vector sum of the two movements.

Graph of two vectors, representing two displacements of a person who walked 90 mi east and then 50 mi north

Figure 2.2.5: Copy and Paste Caption here. (Copyright; author via source)

The graphical process for adding vectors in two dimensions is to place the tail of the second vector on the arrow head of the first vector as shown above.

The sum of the two vectors is the vector that begins at the origin of the first vector and goes to the ending of the second vector, as shown below.

Two vectors representing two displacements of a person who walked 90 mi east and then 50 mi north, with the third vector connecting the two, creating a triangle indicating the sum

Figure 2.2.6: Copy and Paste Caption here. (Copyright; author via source)

If we are using totally graphic means of adding these vectors, the magnitude of the sum would be determined by measuring the length of the sum vector and comparing it to the original standard. We would also use a compass to measure the angle of the summation vector.

If we are using calculation means, we can divide 50. m by 90. m and determine inverse tangent of the dividend. The result of 29.05 indicates the angle of 29° north of east. The length of the sum vector can also be determined mathematically by the Pythagorean theorem, a2+b2=c2. In this case, the length of the hypotenuse would be the square root of (8100 + 2500) or 103 m.

If three or four vectors are to be added by graphical means, we would continue to place each new vector head to toe with the vectors to be added until all the vectors were in the coordinate system. The sum vector is the vector from the origin of the first vector to the arrowhead of the last vector. The magnitude and direction of the sum vector can be measured.

alt

Figure 2.2.7: Copy and Paste Caption here. (Copyright; author via source)

Have you ever used a phone app that provides directions or a navigation system in your car? These programs help you get from Point A to Point B by breaking it down into a series of left and right turns that exemplify many of the graphical methods of vector addition described above. The navigation systems in self-driving cars are even more advanced. Continue to practice vector addition by helping a driverless car get to its destination in the following simulation:

Summary

- Scalars are quantities, such as mass, length, or speed, that are completely specified by magnitude and have no direction.
- Vectors are quantities possessing both magnitude and direction and can be represented by an arrow; the direction of the arrow indicates the direction of the quantity and the length of the arrow is proportional to the magnitude.
- Vectors that are in one dimension can be added arithmetically.
- Vectors that are in two dimensions are added geometrically.
- When vectors are added graphically, graphs must be done to scale and answers are only as accurate as the graphing.

Review

1. On the following number line, add the vector 7.5 m/s and the vector -2.0 m/s.

A number line

Figure 2.2.8

2. On a sheet of graph paper, add a vector that is 4.0 km due east and a vector that is 3.0 km due north.

Explore More

Use this resource to answer the questions that follow.

1. What is a resultant?
2. What are the steps necessary to add vectors in two dimensions?

Additional Resources

Real World Application: Drift

Real World Application: Threading the Needle

2.3: Adding Vectors (Algebraically and Graphically) (Video) Khan Academy

2.4: Adding Two Vectors Graphically (Video) Math and Science

2.5: Adding Vectors Graphically (Video) Michel van Biezen

2.6: Additional Materials

Vector Addition Physics Classroom
Link: https://www.physicsclassroom.com/cla...ector-Addition

Graphical Addition of Vectors Hyperphysics

CHAPTER OVERVIEW

3: Vector Addition - Algebraic Method

3.1: Vector Addition and Subtraction- Analytical Methods

Learning Objectives

By the end of this section, you will be able to:

- Understand the rules of vector addition and subtraction using analytical methods.
- Apply analytical methods to determine vertical and horizontal component vectors.
- Apply analytical methods to determine the magnitude and direction of a resultant vector.

Analytical methods of vector addition and subtraction employ geometry and simple trigonometry rather than the ruler and protractor of graphical methods. Part of the graphical technique is retained, because vectors are still represented by arrows for easy visualization. However, analytical methods are more concise, accurate, and precise than graphical methods, which are limited by the accuracy with which a drawing can be made. Analytical methods are limited only by the accuracy and precision with which physical quantities are known.

Resolving a Vector into Perpendicular Components

Analytical techniques and right triangles go hand-in-hand in physics because (among other things) motions along perpendicular directions are independent. We very often need to separate a vector into perpendicular components. For example, given a vector like A in Figure $3.1.1$, we may wish to find which two perpendicular vectors, A_x and A_y, add to produce it.

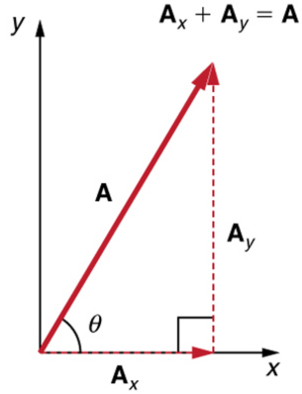


Figure 3.1.1:The vector A, with its tail at the origin of an x, y-coordinate system, is shown together with its x- and y-components, A_x and A_y. These vectors form a right triangle. The analytical relationships among these vectors are summarized below.

A_x and A_y are defined to be the components of A along the x- and y-axes. The three vectors A, A_x, and A_y form a right triangle:

$$A_x + A_y = A. \tag{3.1.1}$$

Note that this relationship between vector components and the resultant vector holds only for vector quantities (which include both magnitude and direction). The relationship does not apply for the magnitudes alone. For example, if $A_x = 3m$ east, $A_y = 4m$ north, and $A = 5m$north-east, then it is true that the vectors $A_x + A_y = A$. However, it is not true that the sum of the magnitudes of the vectors is also equal. That is,

$$3m + 4m \neq 5m \tag{3.1.2}$$

Thus,

$$A_x + A_y \neq A \tag{3.1.3}$$

If the vector A is known, then its magnitude A (its length) and its angle θ (its direction) are known. To find A_x and A_y, its x- and y-components, we use the following relationships for a right triangle.

$$A_x = Acos\theta \tag{3.1.4}$$

and

$$A_y = Asin\theta. \tag{3.1.5}$$

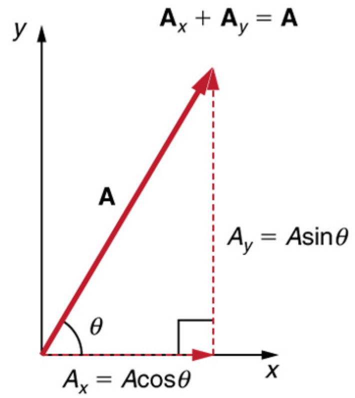


Figure 3.1.2: The magnitudes of the vector components A_x and A_y can be related to the resultant vector A and the angle θ with trigonometric identities. Here we see that $A_x = Acos\theta$ and $A_y = Asin\theta$.

Suppose, for example, that **A** is the vector representing the total displacement of the person walking in a city considered in Kinematics in Two Dimensions: An Introduction and Vector Addition and Subtraction: Graphical Methods.

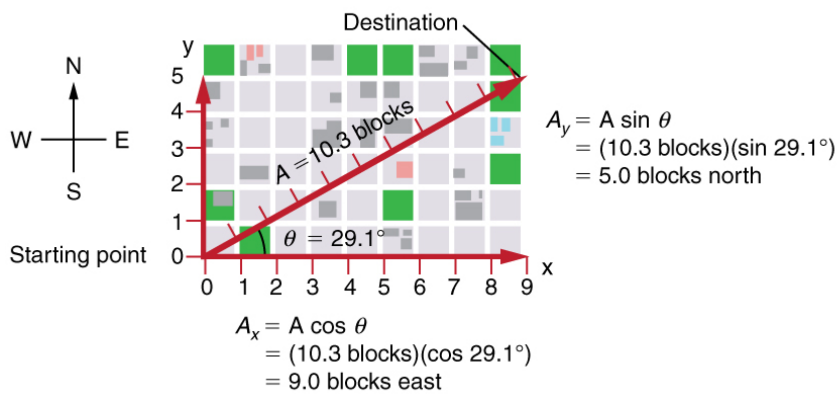


Figure 3.1.3: We can use the relationships $A_x = Acos\theta$ and $A_y = Asin\theta$ to determine the magnitude of the horizontal and vertical component vectors in this example.

Then $A = 10.3$ blocks and $\theta = 29.1^{\circ}$, so that

$$A_x = A\cos\theta = (10.3 blocks)(\cos 29.1^{\circ}) = 9.0 \quad blocks \tag{3.1.6}$$

$$A_y = A\sin\theta = (10.3 blocks)(\sin 29.1^{\circ}) = 5.0 \quad blocks \tag{3.1.7}$$

Calculating a Resultant Vector

If the perpendicular components A_x and A_y of a vector A are known, then A can also be found analytically. To find the magnitude A and direction θ of a vector from its perpendicular components A_x and A_y, we use the following relationships:

$$A = \sqrt{A_{x^2} + A_{y^2}} \tag{3.1.8}$$

$$\theta = tan^{-1}(A_y/A_x) \tag{3.1.9}$$

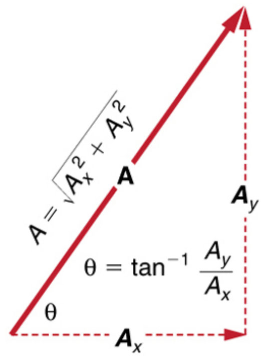


Figure 3.1.4: The magnitude and direction of the resultant vector can be determined once the horizontal and vertical components A_x and A_y have been determined.

Note that the equation $A = \sqrt{A_x^2 + A_y^2}$ is just the Pythagorean theorem relating the legs of a right triangle to the length of the hypotenuse. For example, if A_x and A_y are 9 and 5 blocks, respectively, then $A = \sqrt{9^2 + 5^2} = 10.3$ blocks, again consistent with the example of the person walking in a city. Finally, the direction is $\theta = tan^{-1}(5/9) = 29.1^{\circ}$, as before.

DETERMINING VECTORS AND VECTOR COMPONENTS WITH ANALYTICAL METHODS

Equations $A_x = Acos\theta$ and $A_y = Asin\theta$ are used to find the perpendicular components of a vector—that is, to go from A and θ to A_x and A_y. Equations $A = \sqrt{A_x^2 + A_y^2}$ and $\theta = tan^{-1}(A_y/A_x)$ are used to find a vector from its perpendicular components—that is, to go from A_x and A_y to A and θ. Both processes are crucial to analytical methods of vector addition and subtraction.

Adding Vectors Using Analytical Methods

To see how to add vectors using perpendicular components, consider Figure 3.1.5, in which the vectors A and B are added to produce the resultant R.

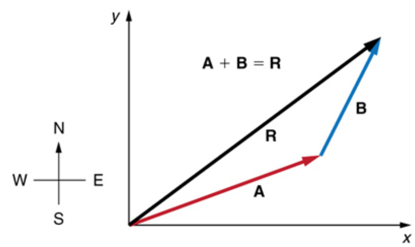


Figure 3.1.5: Vectors A and$B$ are two legs of a walk, and R is the resultant or total displacement. You can use analytical methods to determine the magnitude and direction of R.

If A and B represent two legs of a walk (two displacements), then R is the total displacement. The person taking the walk ends up at the tip of R. There are many ways to arrive at the same point. In particular, the person could have walked first in the x-direction and then in the y-direction. Those paths are the x- and y-components of the resultant, R_x and R_y. If we know R_x and R_y, we can find R and θ using the equations $A = \sqrt{A_x^2 + A_y^2}$ and $\theta = tan^{-1}(A_y/A_x)$. When you use the analytical method of vector addition, you can determine the components or the magnitude and direction of a vector.

Step 1. *Identify the x- and y-axes that will be used in the problem. Then, find the components of each vector to be added along the chosen perpendicular axes.* Use the equations $A_x = Acos\theta$ and $A_y = Asin\theta$ to find the components. In Figure, these components are A_x, A_y, B_x, and B_y. The angles that vectors A and B make with the x-axis are θ_A and θ_B, respectively.

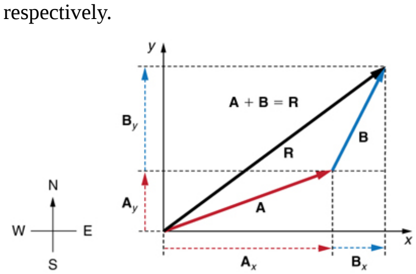


Figure 3.1.6: To add vectors A and B, first determine the horizontal and vertical components of each vector. These are the dotted vectors A_x, A_y, B_x and $B - y$ shown in the image.

Step 2. *Find the components of the resultant along each axis by adding the components of the individual vectors along that axis. That is, as shown in Figure* \(\PageIndex{7}\,

$$R_x = A_x + B_x \tag{3.1.10}$$

and

$$R_y = A_y + B_y. \tag{3.1.11}$$

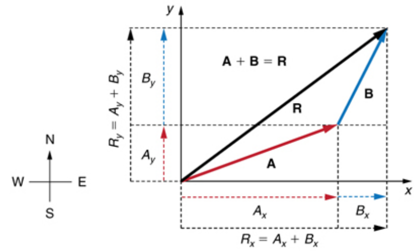


Figure 3.1.7: The magnitude of the vectors A_x and B_x add to give the magnitude R_x of the resultant vector in the horizontal direction. Similarly, the magnitudes of the vectors A_y and B_y add to give the magnitude R_y of the resultant vector in the vertical direction.

Components along the same axis, say the *x*-axis, are vectors along the same line and, thus, can be added to one another like ordinary numbers. The same is true for components along the *y*-axis. (For example, a 9-block eastward walk could be taken in two legs, the first 3 blocks east and the second 6 blocks east, for a total of 9, because they are along the same direction.) So resolving vectors into components along common axes makes it easier to add them. Now that the components of *R* are known, its magnitude and direction can be found.

Step 3. *To get the magnitude R of the resultant, use the Pythagorean theorem:*

$$R = \sqrt{R_x^2 + R_y^2} \tag{3.1.12}$$

Step 4. *To get the direction of the resultant:*

$$\theta = tan^{-1}(R_y/R_x) \tag{3.1.13}$$

The following example illustrates this technique for adding vectors using perpendicular components.

Example 3.1.1: Adding Vectors Using Analytical Methods

Add the vector A to the vector B shown in Figure, using perpendicular components along the *x*- and *y*-axes. The *x*- and *y*-axes are along the east–west and north–south directions, respectively. Vector A represents the first leg of a walk in which a person walks $53.0m$ in a direction $20.0°$ north of east. Vector B represents the second leg, a displacement of $34.0m$ in a direction $63.0°$ north of east.

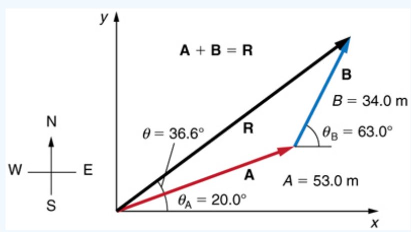


Figure 3.1.8: Vector A has magnitude $53.0m$ and direction $20.0°$ north of the *x*-axis. Vector B has magnitude $34.0m$ and direction $63.0°$ north of the x-axis. You can use analytical methods to determine the magnitude and direction of R.

Strategy

The components of A and B along the *x*- and *y*-axes represent walking due east and due north to get to the same ending point. Once found, they are combined to produce the resultant.

Solution

Following the method outlined above, we first find the components of A and B along the *x*- and *y*-axes. Note that $A = 53.0m, \theta_A = 20.0°$, $B = 34.0m$, and $\theta_B = 63.0°$. We find the *x*-components by using $A_x = Acos\theta$, which gives

$$A_x = Acos\theta_A = (53.0m)(cos20.0°)(53.0m)(0.940) = 49.8m \tag{3.1.14}$$

and

$$B_x = Bcos\theta_B = (34.0m)(cos63.0°)(34.0m)(0.454) = 15.4m. \tag{3.1.15}$$

Similarly, the *y*-components are found using $A_y = Asin\theta_A$:

$$A_y = Asin\theta_A = (53.0m)(sin20.0°)(53.0m)(0.342) = 18.1m \tag{3.1.16}$$

and

$$B_y = B sin\theta_B = (34.0m)(sin 63.0^{\circ})(34.0m)(0.891) = 30.3m. \tag{3.1.17}$$

The *x*- and *y*-components of the resultant are thus

$$R_x = A_x + B_x = 49.8m + 15.4m = 65.2m \tag{3.1.18}$$

and

$$R_y = A_y + B_y = 18.1m + 30.3m = 48.4m. \tag{3.1.19}$$

Now we can find the magnitude of the resultant by using the Pythagorean theorem:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(65.2)^2 + (48.4)^2 m} \tag{3.1.20}$$

so that

$$R = 81.2m. \tag{3.1.21}$$

Finally, we find the direction of the resultant:

$$\theta = tan^{-1}(R_y/R_x) = +tan^{-1}(48.4/65.2). \tag{3.1.22}$$

Thus,

$$\theta = tan^{-1}(0.742) = 36.6^{\circ}. \tag{3.1.23}$$

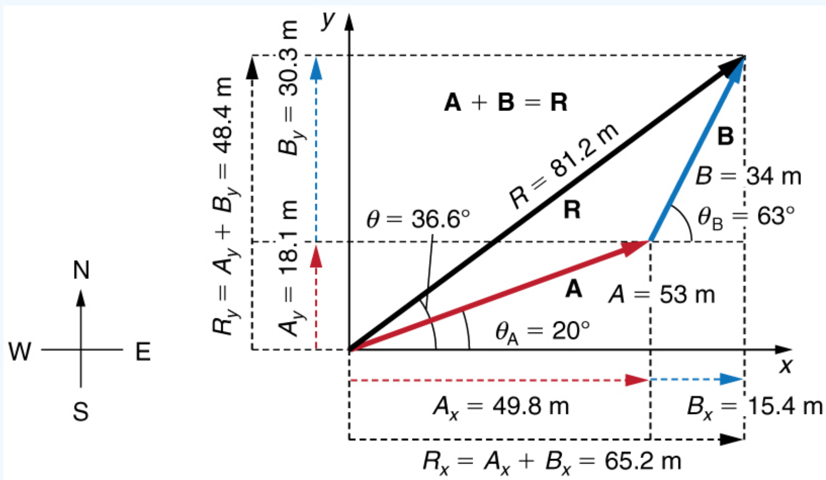


Figure 3.1.9: Using analytical methods, we see that the magnitude of R is $81.2m$ and its direction is 36.6° north of east.

Discussion

This example illustrates the addition of vectors using perpendicular components. Vector subtraction using perpendicular components is very similar—it is just the addition of a negative vector.

Subtraction of vectors is accomplished by the addition of a negative vector. That is, $A - B \equiv A + (-B)$. Thus, *the method for the subtraction of vectors using perpendicular components is identical to that for addition.* The components of $-B$ are the negatives of the components of B. The *x*- and *y*-components of the resultant $A - B = R$ are thus

$$R_x = A_x + (-B_x) \tag{3.1.24}$$

and

$$R_y = A_y + (-B_y) \tag{3.1.25}$$

and the rest of the method outlined above is identical to that for addition. (See Figure $3.1.10$)

Analyzing vectors using perpendicular components is very useful in many areas of physics, because perpendicular quantities are often independent of one another. The next module, Projectile Motion, is one of many in which using perpendicular components helps make the picture clear and simplifies the physics.

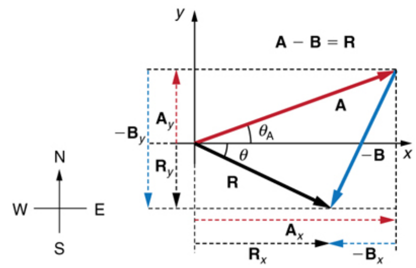
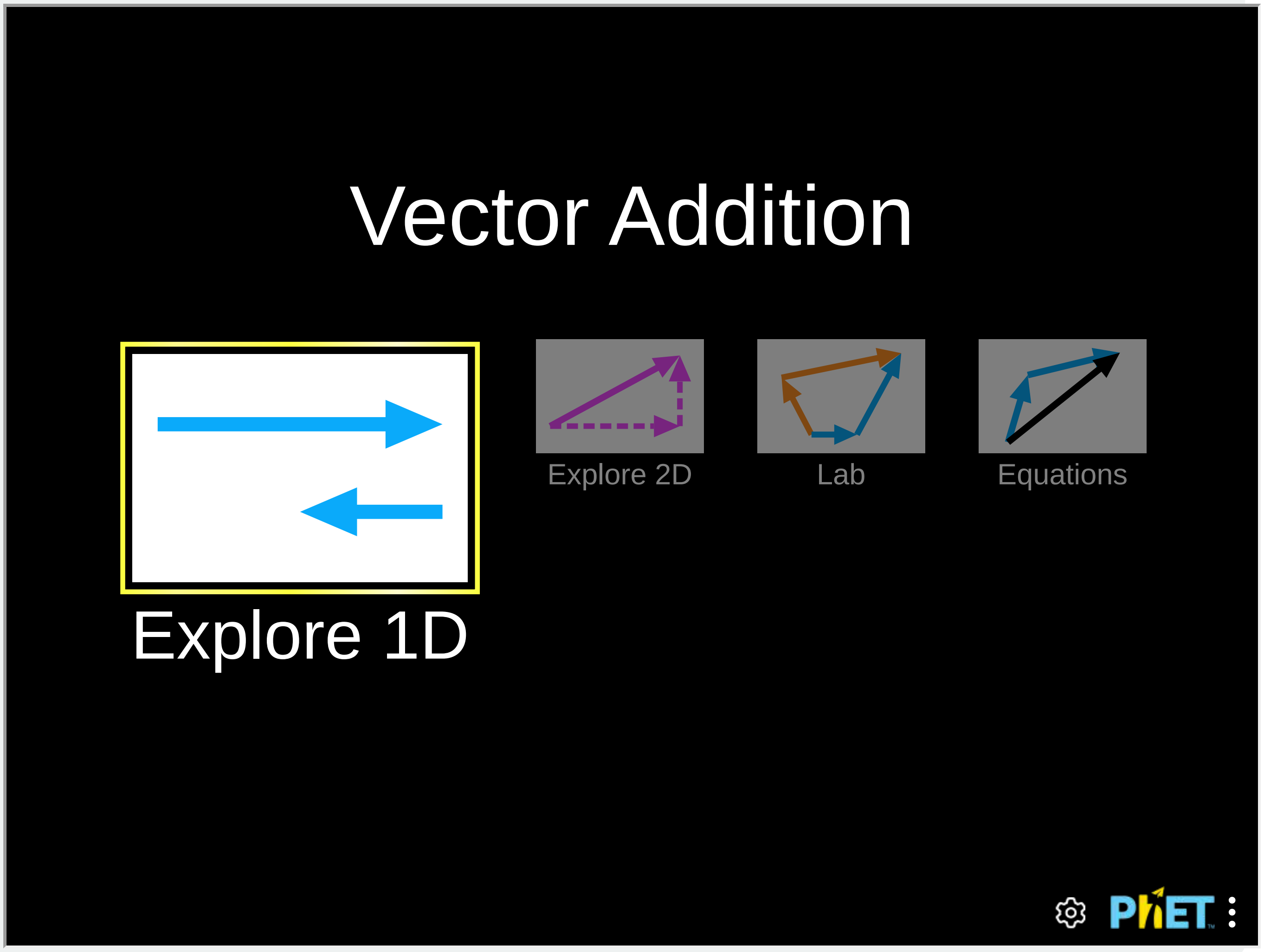


Figure 3.1.10. The components of $-B$ are the negatives of the components of B. The method of subtraction is the same as that for addition.

PHET EXPLORATIONS: VECTOR ADDITION

Learn how to add vectors. Drag vectors onto a graph, change their length and angle, and sum them together. The magnitude, angle, and components of each vector can be displayed in several formats.



Summary

- The analytical method of vector addition and subtraction involves using the Pythagorean theorem and trigonometric identities to determine the magnitude and direction of a resultant vector.
- The steps to add vectors A and B using the analytical method are as follows:

Step 1: Determine the coordinate system for the vectors. Then, determine the horizontal and vertical components of each vector using the equations

$$A_x = A cos\theta$$
$$B_x = B cos\theta$$

and

$$A_y = Asin\theta$$
$$B_y = Bsin\theta.$$

Step 2: Add the horizontal and vertical components of each vector to determine the components Rx and Ry of the resultant vector, R:

$$R_x = A_x + B_x$$

and

$$R_y = A_y + B_y\ .$$

Step 3: Use the Pythagorean theorem to determine the magnitude, R, of the resultant vector R:

$$R = \sqrt{R_x^2 + R_y^2}\ .$$

Step 4: Use a trigonometric identity to determine the direction, θ, of **R**:

$$\theta = tan^{-1}(R_y / R_x)\,.$$

Glossary

analytical method

the method of determining the magnitude and direction of a resultant vector using the Pythagorean theorem and trigonometric identities

3.2: Adding Vectors Mathematically and Graphically (Video) Physics on Line

3.3: Adding Vectors in Magnitude and Direction Form (Video) Khan Academy

3.4: Adding Vectors by Means of Components (Video) Organic Chemistry Tutor

3.5: Head to Tail Method of Vector Addition (Video)

3.6: Additional Materials

Vector Addition Hyperphysics

Vector Addition CUEMATH

Vector Addition and Subtraction BYJU’s Learning

CHAPTER OVERVIEW

4: Dot (Scalar) Product of Vectors

4.1: The Dot Product

Learning Objectives

- Calculate the dot product of two given vectors.
- Determine whether two given vectors are perpendicular.
- Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
- Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that work is done by the force. Previously, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the dot product, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

Definition: dot product

The **dot product** (also called the **scalar product**) of two vectors $\vec{\mathbf{u}}=\langle u_1,u_2,u_3\rangle$ and $\vec{\mathbf{v}}=\langle v_1,v_2,v_3\rangle$ is given by the sum of the products of the components

$$\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}=u_1v_1+u_2v_2+u_3v_3.$$

When vectors are written in standard unit vector form, $\vec{\mathbf{u}}=u_1\,\hat{\mathbf{i}}+u_2\hat{\mathbf{j}}+u_3\hat{\mathbf{k}}$ and $\vec{\mathbf{v}}=v_1\,\hat{\mathbf{i}}+v_2\hat{\mathbf{j}}+v_3\hat{\mathbf{k}}$ the dot product is the same sum of products. Notice the standard unit vectors themselves are not included in the dot product.

$$\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}=u_1v_1+u_2v_2+u_3v_3.$$

Note that if u and v are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\vec{\mathbf{u}}=\langle u_1,u_2\rangle$ and $\vec{\mathbf{v}}=\langle v_1,v_2\rangle$, then

$$\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}=u_1v_1+u_2v_2.$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the ***scalar product***. It may also be called the ***inner product***.

Example 4.1.1: Calculate Dot Products

a. Find the dot product of $\vec{\mathbf{u}}=\langle 3,5,2\rangle$ and $\vec{\mathbf{v}}=\langle -1,3,0\rangle$.
b. Find the scalar product of $\vec{\mathbf{p}}=10\,\hat{\mathbf{i}}-4\hat{\mathbf{j}}+7\hat{\mathbf{k}}$ and $\vec{\mathbf{q}}=-2\,\hat{\mathbf{i}}+\hat{\mathbf{j}}+6\hat{\mathbf{k}}$.

Solution:

a. Substitute the vector components into the formula for the dot product:

$$\begin{aligned}\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}&=u_1v_1+u_2v_2+u_3v_3\\&=3(-1)+5(3)+2(0)\\&=-3+15+0=12.\end{aligned}$$

b. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:

$$
\begin{aligned}
\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\
&= 10(-2) + (-4)(1) + (7)(6) \\
&= -20 - 4 + 42 = 18.
\end{aligned}
$$

Try It 4.1.1

a. Find $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$, where $\vec{\mathbf{u}} = \langle 2, 9, -1 \rangle$ and $\vec{\mathbf{v}} = \langle -3, 1, -4 \rangle$.
b. Find $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$, where $\vec{\mathbf{u}} = -8\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\vec{\mathbf{v}} = 2\hat{\mathbf{i}} + 9\hat{\mathbf{j}}$.

Hint

Multiply corresponding components and then add their products.

Answer

a. 7 b. 11

Like vector addition and subtraction, the dot product has several algebraic properties.

Magnitude of a vector and the Dot Product

The dot product of a vector $\vec{\mathbf{v}}$ with itself is the square of its magnitude:

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2 \tag{4.1.1}$$

Example 4.1.2: Dot Product Expressions

Let $\vec{\mathbf{a}} = \langle 1, 2, -3 \rangle$, $\vec{\mathbf{b}} = \langle 0, 2, 4 \rangle$, and $\vec{\mathbf{c}} = \langle 5, -1, 3 \rangle$.

Find each of the following products.

a. $(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$
b. $\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}})$
c. $\|\vec{\mathbf{b}}\|^2$

Solution

a. Note that this expression asks for the scalar multiple of $\vec{\mathbf{c}}$ by $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$:

$$
\begin{aligned}
(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}} &= (\langle 1, 2, -3 \rangle \cdot \langle 0, 2, 4 \rangle)\langle 5, -1, 3 \rangle \\
&= (1(0) + 2(2) + (-3)(4))\langle 5, -1, 3 \rangle \\
&= (-8)\langle 5, -1, 3 \rangle = \langle -40, 8, -24 \rangle.
\end{aligned}
$$

b. This expression is a dot product of vector $\vec{\mathbf{a}}$ and scalar multiple $2\vec{\mathbf{c}}$:

$$
\begin{aligned}
\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}}) &= \langle 1, 2, -3 \rangle \cdot 2(\langle 5, -1, 3 \rangle) \\
&= \langle 1, 2, -3 \rangle \cdot \langle 10, -2, 6 \rangle \\
&= (1)(10) + 2(-2) + (-3)(6)) \\
&= 10 - 4 - 18 = -12.
\end{aligned}
$$

c. Simplifying this expression is a straightforward application of the dot product:

$$
\begin{aligned}
\|\vec{\mathbf{b}}\|^2 &= \vec{\mathbf{b}} \cdot \vec{\mathbf{b}} \\
&= \langle 0, 2, 4 \rangle \cdot \langle 0, 2, 4 \rangle \\
&= 0^2 + 2^2 + 4^2 \\
&= 0 + 4 + 16 = 20.
\end{aligned}
$$

Try It 4.1.2

Find the following products for $\vec{\mathbf{p}} = 7\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\vec{\mathbf{q}} = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, and $\vec{\mathbf{r}} = 0\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$.

a. $(\vec{\mathbf{r}} \cdot \vec{\mathbf{p}})\vec{\mathbf{q}}$
b. $\|\vec{\mathbf{p}}\|^2$

Hint

$\vec{\mathbf{r}} \cdot \vec{\mathbf{p}}$ is a scalar.

Answer

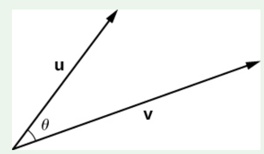
$$a. \quad (\vec{\mathbf{r}} \cdot \vec{\mathbf{p}})\vec{\mathbf{q}} = 12\hat{\mathbf{i}} - 12\hat{\mathbf{j}} + 12\hat{\mathbf{k}}; \quad b. \quad \|\vec{\mathbf{p}}\|^2 = 53$$

Find the Angle between Two Vectors

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them. The dot product provides a way to find the measure of this angle.

How to: Evaluate the dot product given the magnitude of 2 vectors and the angle between them

Given two non-zero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and the angle between them, θ, such that $0 \leq \theta \leq \pi$.

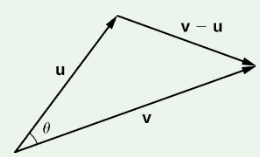


The dot product of the two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta.$$

Proof

Place vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in standard position and consider the vector $\vec{\mathbf{v}} - \vec{\mathbf{u}}$ (see the figure on the right). These three vectors form a triangle with side lengths $\|\vec{\mathbf{u}}\|$, $\|\vec{\mathbf{v}}\|$, and $\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|$.



Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|^2 = \|\vec{\mathbf{u}}\|^2 + \|\vec{\mathbf{v}}\|^2 - 2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta.$$

The dot product provides a way to rewrite the left side of Equation ???:

$$\begin{aligned} \|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|^2 &= (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \cdot (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \\ &= (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} - (\vec{\mathbf{v}} - \vec{\mathbf{u}}) \cdot \vec{\mathbf{u}} \\ &= \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{v}} \cdot \vec{\mathbf{u}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} \\ &= \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} \\ &= \|\vec{\mathbf{v}}\|^2 - 2\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \|\vec{\mathbf{u}}\|^2. \end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned} \|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|^2 &= \|\vec{\mathbf{u}}\|^2 + \|\vec{\mathbf{v}}\|^2 - 2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta \\ \|\vec{\mathbf{v}}\|^2 - 2\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \|\vec{\mathbf{u}}\|^2 &= \|\vec{\mathbf{u}}\|^2 + \|\vec{\mathbf{v}}\|^2 - 2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta \\ -2\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= -2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta \\ \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta. \qquad \square \end{aligned}$$

(Another proof uses the formula for the cosine of the sum of two angles instead of the Law of Cosines).

We can use the form of the dot product in Equation ??? to find the measure of the angle between two nonzero vectors by rearranging Equation ??? to solve for the cosine of the angle. Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0^\circ \leq \theta \leq 180^\circ$ (or $0 \leq \theta \leq \pi$ if we are working in radians). The inverse cosine is unique over this range, so we are then able to determine the measure of the angle θ.

How to: Find the angle between two vectors

Given two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then the smallest angle θ between them can be found using the equation:

$\cos\theta = \dfrac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}$.

Example 4.1.3: Find the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

a. $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}$
b. $\langle 2,5,6\rangle$ and $\langle -2,-4,4\rangle$

Solution

a. To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into Equation ???:

$$\begin{aligned}\cos\theta &= \frac{(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}})}{\left\|\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}\right\|\cdot\left\|2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}\right\|}\\ &= \frac{1(2)+(1)(-1)+(1)(-3)}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+(-1)^2+(-3)^2}} = \frac{-2}{\sqrt{3}\sqrt{14}} = \frac{-2}{\sqrt{42}}.\end{aligned}$$

Therefore, $\theta = \arccos\dfrac{-2}{\sqrt{42}} \approx 107.98^\circ$ or ≈ 1.88 rad.

b. Start by finding the value of the cosine of the angle between the vectors:

$$\begin{aligned}\cos\theta &= \frac{\langle 2,5,6\rangle\cdot\langle -2,-4,4\rangle}{\|\langle 2,5,6\rangle\|\cdot\|\langle -2,-4,4\rangle\|}\\ &= \frac{2(-2)+(5)(-4)+(6)(4)}{\sqrt{2^2+5^2+6^2}\sqrt{(-2)^2+(-4)^2+4^2}} = \frac{0}{\sqrt{65}\sqrt{36}} = 0.\end{aligned}$$

Now, $\cos\theta = 0$ and $0 \leq \theta \leq \pi$, so $\theta = 90^\circ$ or $\pi/2$ radians.

Try It 4.1.3a

Find the measure of the angle, in radians, formed by vectors $\vec{\mathbf{a}} = \langle 1,2,0\rangle$ and $\vec{\mathbf{b}} = \langle 2,4,1\rangle$. Round to the nearest hundredth.

Hint

Use the Equation ???.

Answer

$\theta \approx 12.61^\circ$ or $\theta \approx 0.22$ rad

Try It 4.1.3b

Let $\vec{\mathbf{v}} = \langle 3,-5,1\rangle$. Find the measure of the angles formed by each pair of vectors.

a. $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$

b. $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$
c. $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$

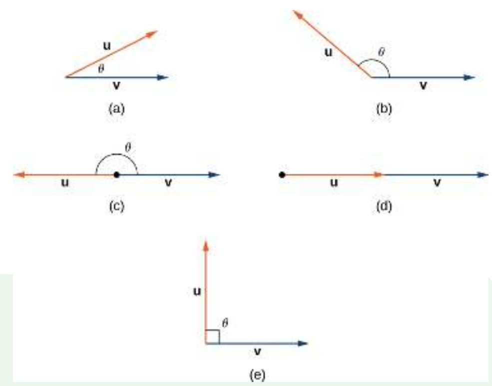
Hint

$\hat{\mathbf{i}} = \langle 1, 0, 0\rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0\rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1\rangle$

Answer

$a. \alpha \approx 1.04$ rad; b. $\beta \approx 2.58$ rad; c. $\gamma \approx 1.40$ rad

Figure 4.1.3Possible angles between vectors.



Possible angles between two vectors is illustrated in Figure 4.1.3. The angle between two vectors can be (a) acute $(0 < \cos\theta < 1)$, (b) obtuse $(-1 < \cos\theta < 0)$, or (c) straight $(\cos\theta = -1)$. If $\cos\theta = 1$, then (d) both vectors have the same direction. If $\cos\theta = 0$, then the vectors, when placed in standard position, form (e) a right angle. We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

Orthogonal Vectors

The nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are **orthogonal vectors** if and only if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

The terms **orthogonal, perpendicular,** and **normal** each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are **orthogonal** and lines are **perpendicular**. The term **normal** is used most often when measuring the angle made with a plane or other surface.

Example 4.1.4: Identifying Orthogonal Vectors

Determine whether $\vec{\mathbf{p}} = \langle 1, 0, 5\rangle$ and $\vec{\mathbf{q}} = \langle 10, 3, -2\rangle$ are orthogonal vectors.

Solution

Using the definition, we need only check the dot product of the vectors:

$$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Because $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 0,$ the vectors are orthogonal (Figure 4.1.4).

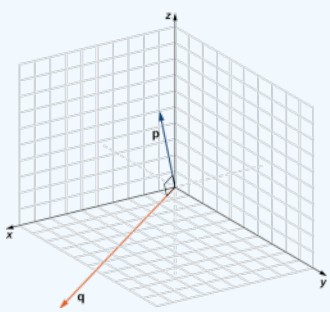


Figure 4.1.4 Vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ form a right angle when their initial points are aligned.

Try It 4.1.4

For which value of x is $\vec{\mathbf{p}} = \langle 2, 8, -1\rangle$ orthogonal to $\vec{\mathbf{q}} = \langle x, -1, 2\rangle$?

Hint

Vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are orthogonal if and only if $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 0$.

Answer

$x = 5$

Projections

As we have seen, addition combines two vectors to create a resultant vector. But what if we are given a vector and we need to find its component parts? We use vector projections to perform the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection. For example, if a child is pulling the handle of a wagon at a 55° angle, we can use projections to determine how much of the force on the handle is actually moving the wagon forward ($4.1.6$). We return to this example and learn how to solve it after we see how to calculate projections.

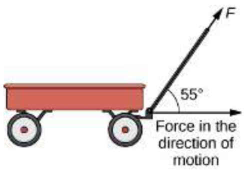


Figure $4.1.6$: When a child pulls a wagon, only the horizontal component of the force propels the wagon forward.

Definition: Vector Projection

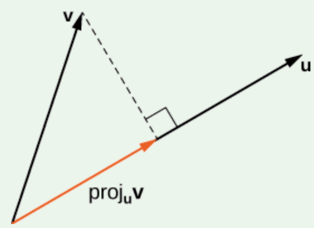


Figure $4.1.7$: Vector Projection.

The **vector projection** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ is the portion of $\vec{\mathbf{v}}$ that is in the direction of $\vec{\mathbf{u}}$. This is the vector labeled $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ in Figure $4.1.7$. It has the same initial point as $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and the same direction as $\vec{\mathbf{u}}$, and represents the component of $\vec{\mathbf{v}}$ that acts in the direction of $\vec{\mathbf{u}}$.

If θ represents the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then, by properties of triangles, we know the length of $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ is $\|\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\|\cos\theta$, which can also be expressed in terms of the dot product.

The magnitude of the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$:

$$\|\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\|\cos\theta = \|\vec{\mathbf{v}}\|\left(\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}\right) = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}. \tag{4.1.2}$$

We now multiply by a unit vector in the direction of $\vec{\mathbf{u}}$ to get $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$, the vector that is the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$:

$$\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}\left(\frac{1}{\|\vec{\mathbf{u}}\|}\vec{\mathbf{u}}\right) = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}}. \tag{4.1.3}$$

Example $4.1.5$: Finding Projections

Find the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

a. $\vec{\mathbf{v}} = \langle 3,5,1\rangle$ and $\vec{\mathbf{u}} = \langle -1,4,3\rangle$
b. $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ and $\vec{\mathbf{u}} = \hat{\mathbf{i}} + 6\hat{\mathbf{j}}$

Solution

a. Substitute the components of $\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}$ into the formula for the projection:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{\langle -1,4,3\rangle\cdot\langle 3,5,1\rangle}{\|\langle -1,4,3\rangle\|^2}\langle -1,4,3\rangle \\ &= \frac{-3+20+3}{(-1)^2+4^2+3^2}\langle -1,4,3\rangle \\ &= \frac{20}{26}\langle -1,4,3\rangle \\ &= \left\langle -\frac{10}{13},\frac{40}{13},\frac{30}{13}\right\rangle.\end{aligned}$$

b. To find the two-dimensional projection, simply adapt the formula to the two-dimensional case:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{(\hat{\mathbf{i}}+6\hat{\mathbf{j}})\cdot(3i-2j)}{\left\|\hat{\mathbf{i}}+6\hat{\mathbf{j}}\right\|^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}})j) \\ &= \frac{1(3)+6(-2)}{1^2+6^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}\hat{\mathbf{i}}-\frac{54}{37}\hat{\mathbf{j}}.\end{aligned}$$

Sometimes it is useful to decompose vectors—that is, to break a vector apart into a sum. This process is called the **resolution of a vector into components.** Projections allow us to identify two orthogonal vectors having a desired sum. For example, let $\vec{\mathbf{v}}=\langle 6,-4\rangle$ and let $\vec{\mathbf{u}}=\langle 3,1\rangle$. We want to decompose the vector $\vec{\mathbf{v}}$ into orthogonal components such that one of the component vectors has the same direction as $\vec{\mathbf{u}}$.

We first find the component that has the same direction as $\vec{\mathbf{u}}$ by projecting $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$. Let $\vec{\mathbf{p}}=\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$. Then, we have

$$\begin{aligned}\vec{\mathbf{p}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} = \frac{18-4}{9+1}\vec{\mathbf{u}} = \frac{7}{5}\vec{\mathbf{u}} \\ &= \frac{7}{5}\langle 3,1\rangle = \left\langle \frac{21}{5},\frac{7}{5}\right\rangle.\end{aligned}$$

Now consider the vector $\vec{\mathbf{q}}=\vec{\mathbf{v}}-\vec{\mathbf{p}}.$ We have

$$\begin{aligned}\vec{\mathbf{q}} &= \vec{\mathbf{v}}-\vec{\mathbf{p}} \\ &= \left\langle 6,-4\right\rangle - \left\langle \frac{21}{5},\frac{7}{5}\right\rangle \\ &= \left\langle \frac{9}{5},-\frac{27}{5}\right\rangle.\end{aligned}$$

Clearly, by the way we defined $\vec{\mathbf{q}}$, we have $\vec{\mathbf{v}}=\vec{\mathbf{q}}+\vec{\mathbf{p}},$ and

$$\begin{aligned}\vec{\mathbf{q}}\cdot\vec{\mathbf{p}} &= \left\langle\frac{9}{5},-\frac{27}{5}\right\rangle\cdot\left\langle\frac{21}{5},\frac{7}{5}\right\rangle\\ &=\frac{9(21)}{25}+-\frac{27(7)}{25}\\ &=\frac{189}{25}-\frac{189}{25}=0.\end{aligned}$$

Therefore, $\vec{\mathbf{q}}$ and $\vec{\mathbf{p}}$ are orthogonal.

Example 4.1.6: Resolving Vectors into Components

Express $\vec{\mathbf{v}}=\langle 8,-3,-3\rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}}=\langle 2,3,2\rangle$.

Solution

Let $\vec{\mathbf{p}}$ represent the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$:

$$\begin{aligned}\vec{\mathbf{p}}=\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &=\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}}\\ &=\frac{\langle 2,3,2\rangle\cdot\langle 8,-3,-3\rangle}{\|\langle 2,3,2\rangle\|^2}\langle 2,3,2\rangle\\ &=\frac{16-9-6}{2^2+3^2+2^2}\langle 2,3,2\rangle=\frac{1}{17}\langle 2,3,2\rangle=\left\langle\frac{2}{17},\frac{3}{17},\frac{2}{17}\right\rangle.\end{aligned}$$

Then,

$$\begin{aligned}\vec{\mathbf{q}}=\vec{\mathbf{v}}-\vec{\mathbf{p}} &=\langle 8,-3,-3\rangle-\left\langle\frac{2}{17},\frac{3}{17},\frac{2}{17}\right\rangle\\ &=\left\langle\frac{134}{17},-\frac{54}{17},-\frac{53}{17}\right\rangle.\end{aligned}$$

To check our work, we can use the dot product to verify that $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are orthogonal vectors:

$$\begin{aligned}\vec{\mathbf{p}}\cdot\vec{\mathbf{q}} &=\left\langle\frac{2}{17},\frac{3}{17},\frac{2}{17}\right\rangle\cdot\langle\left\langle\frac{134}{17},-\frac{54}{17},-\frac{53}{17}\right\rangle\\ &=\frac{268}{17}-\frac{162}{17}-\frac{106}{17}=0.\end{aligned}$$

Then,

$$\vec{\mathbf{v}}=\vec{\mathbf{p}}+\vec{\mathbf{q}}=\left\langle\frac{2}{17},\frac{3}{17},\frac{2}{17}\right\rangle+\left\langle\frac{134}{17},-\frac{54}{17},-\frac{53}{17}\right\rangle.$$

Try It 4.1.6

Express $\vec{\mathbf{v}}=5\,\hat{\mathbf{i}}-\hat{\mathbf{j}}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}}=4\,\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}$.

Hint

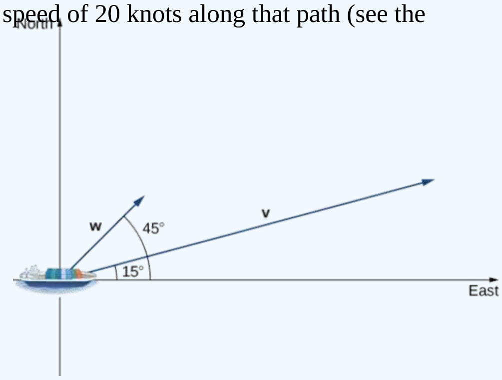
Start by finding the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

Answer

$\vec{\mathbf{v}}=\vec{\mathbf{p}}+\vec{\mathbf{q}}$, where $\vec{\mathbf{p}}=\frac{18}{5}\hat{\mathbf{i}}+\frac{9}{5}\hat{\mathbf{j}}$ and $\vec{\mathbf{q}}=\frac{7}{5}\hat{\mathbf{i}}-\frac{14}{5}\hat{\mathbf{j}}$

Example 4.1.7: Scalar Projection of Velocity

A container ship leaves port traveling 15° north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction 15° north of east? Round the answer to two decimal places.



Solution

Let $\vec{\mathbf{v}}$ be the velocity vector generated by the engine, and let w be the velocity vector of the current. We already know $\|\vec{\mathbf{v}}\| = 20$ along the desired route. We just need to add in the scalar projection of $\vec{\mathbf{w}}$ onto $\vec{\mathbf{v}}$. We get

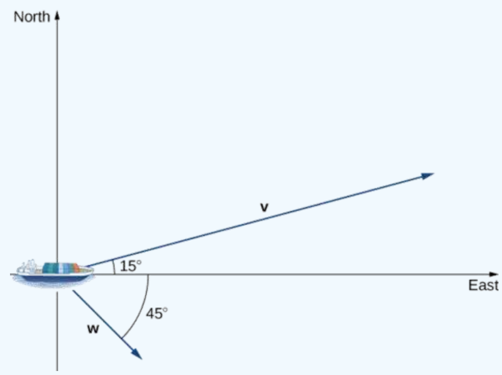
$$\text{magnitude of projection} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{\|\vec{\mathbf{v}}\|}$$

$$= \frac{\|\vec{\mathbf{v}}\|\|\vec{\mathbf{w}}\|\cos(30^\circ)}{\|\vec{\mathbf{v}}\|}[4pt] \qquad = \|\vec{\mathbf{w}}\|\cos(30^\circ) = 2\frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.73\,\text{knots}.$$

The ship is moving at 21.73 knots in the direction 15° north of east.

Try It 4.1.7

Repeat the previous example, but assume the ocean current is moving southeast instead of northeast, as shown in the following figure.



Hint

Compute the scalar projection of $\vec{\mathbf{w}}$ onto $\vec{\mathbf{v}}$.

Answer

21 knots

Work

Now that we understand dot products, we can see how to apply them to real-life situations. The most common application of the dot product of two vectors is in the calculation of work.

From physics, we know that work is done when an object is moved by a force. When the force is constant and applied in the same direction the object moves, then we define the work done as the product of the force and the distance the object travels: $W = Fd$. Now imagine the direction of the force is different from the direction of motion, as with the example of a child pulling a wagon. To find the work done, we need to multiply the component of the force that acts in the direction of the motion by the magnitude of the displacement. The dot product allows us to do just that. If we represent an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{\mathbf{s}}$, then the **work done by the force** is the dot product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$.

Definition: Constant Force

When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force $\vec{\mathbf{F}}$, acting at an angle θ from the line of motion, is given by

$$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta.$$

Let's revisit the problem of the child's wagon introduced earlier. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of *55°*. If the child pulls the wagon 50 ft, find the work done by the force (Figure 4.1.8).

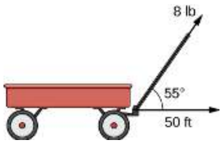


Figure 4.1.8 The horizontal component of the force is the projection of $\vec{\mathbf{F}}$ *onto the positive* x*-axis.*

We have

$$W = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta = 8(50)(\cos(55^\circ)) \approx 229 \text{ ft·lb}.$$

In U.S. standard units, we measure the magnitude of force $\left\|\vec{\mathbf{F}}\right\|$ in pounds. The magnitude of the displacement vector $\left\|\overrightarrow{PQ}\right\|$ tells us how far the object moved, and it is measured in feet. The customary unit of measure for work, then, is the foot-pound. One foot-pound is the amount of work required to move an object weighing 1 lb a distance of 1 ft straight up. In the metric system, the unit of measure for force is the newton (N), and the unit of measure of magnitude for work is a newton-meter (N·m), or a joule (J).

Example 4.1.8: Calculating Work

A conveyor belt generates a force $\vec{\mathbf{F}} = 5\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution

The displacement vector $\overrightarrow{PQ}$ has initial point $(1, 1, 1)$ and terminal point $(9, 4, 7)$:

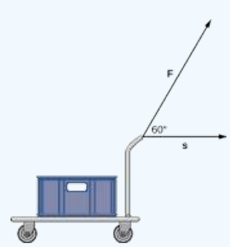
$$\overrightarrow{PQ} = \langle 9-1, 4-1, 7-1 \rangle = \langle 8, 3, 6 \rangle = 8\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 6\hat{\mathbf{k}}.$$

Work is the dot product of force and displacement:

$$\begin{aligned} W &= \vec{\mathbf{F}} \cdot \overrightarrow{PQ} \\ &= (5\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (8\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \\ &= 5(8) + (-3)(3) + 1(6) \\ &= 37 \text{ N·m} = 37 \text{ J} \end{aligned}$$

Try It 4.1.8

A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?



Hint

Use the definition of work as the dot product of force and distance.

Answer

150 ft-lb

Key Concepts

- The **dot product**, or scalar product, of two vectors $\vec{\mathbf{u}}=\langle u_1,u_2,u_3\rangle$ and $\vec{\mathbf{v}}=\langle v_1,v_2,v_3\rangle$ is $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}=u_1v_1+u_2v_2+u_3v_3$.
- The **dot product** of two vectors can be expressed, alternatively, as $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}=\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta.$ This form of the dot product is useful for finding the measure of the angle formed by two vectors.
- The cosine of the angle formed by $\vec{\mathbf{u}}$ **and** $\vec{\mathbf{v}}$ is $\cos\theta=\dfrac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}$
- The dot product satisfies the following properties:
  - $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}=\vec{\mathbf{v}}\cdot\vec{\mathbf{u}}$
  - $\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}+\vec{\mathbf{w}})=\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}+\vec{\mathbf{u}}\cdot\vec{\mathbf{w}}$
  - $c(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})=(c\vec{\mathbf{u}})\cdot\vec{\mathbf{v}}=\vec{\mathbf{u}}\cdot(c\vec{\mathbf{v}})$
  - $\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}=\|\vec{\mathbf{v}}\|^2$
- Vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are **orthogonal** if $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}=0$.
- The **vector projection** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ is the vector $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}=\dfrac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}}$.
- The **magnitude of the vector projection** is known as the ***scalar projection*** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$, given by $\text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}=\dfrac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}=\|\vec{\mathbf{v}}\|\cos\theta$.
- **Work** is done when a force is applied to an object, causing displacement. When the force is represented by the vector $\vec{\mathbf{F}}$ and the displacement is represented by the vector $\vec{\mathbf{s}}$, then the work done W is given by the formula $W=\vec{\mathbf{F}}\cdot\vec{\mathbf{s}}=\|\vec{\mathbf{F}}\|\|\vec{\mathbf{s}}\|\cos\theta$.

Contributors and Attributions

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- edited for vector notation by Paul Seeburger

4.2: Vector Dot Product

Learning Objectives

- Explain the difference between the scalar product and the vector product of two vectors.
- Determine the scalar product of two vectors.
- Determine the vector product of two vectors.
- Describe how the products of vectors are used in physics.

A vector can be multiplied by another vector but may not be divided by another vector. There are two kinds of products of vectors used broadly in physics and engineering. One kind of multiplication is a **scalar multiplication of two vectors**. Taking a scalar product of two vectors results in a number (a scalar), as its name indicates. Scalar products are used to define work and energy relations. For example, the work that a force (a vector) performs on an object while causing its displacement (a vector) is defined as a scalar product of the force vector with the displacement vector. A quite different kind of multiplication is a **vector multiplication of vectors**. Taking a vector product of two vectors returns as a result a vector, as its name suggests. Vector products are used to define other derived vector quantities. For example, in describing rotations, a vector quantity called **torque** is defined as a vector product of an applied force (a vector) and its lever arm (a vector). It is important to distinguish between these two kinds of vector multiplications because the scalar product is a scalar quantity and a vector product is a vector quantity.

The Scalar Product of Two Vectors (the Dot Product)

Scalar multiplication of two vectors yields a scalar product.

Definition: Scalar Product (Dot Product)

The scalar product $\vec{A} \cdot \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is a number defined by the equation

$$\vec{A} \cdot \vec{B} = AB\cos\varphi, \tag{4.2.1}$$

where ϕ is the angle between the vectors (shown in Figure 4.2.1). The scalar product is also called the **dot product** because of the dot notation that indicates it.

In the definition of the dot product, the direction of angle φ does not matter, and φ can be measured from either of the two vectors to the other because $\cos\varphi = \cos(-\varphi) = cos(2\pi - \varphi)$. The dot product is a negative number when $90° < \varphi \leq 180°$ and is a positive number when $0° \leq \phi < 90°$. Moreover, the dot product of two parallel vectors is $\vec{A} \cdot \vec{B}$ = AB cos 0° = AB, and the dot product of two antiparallel vectors is $\vec{A} \cdot \vec{B}$ = AB cos 180° = −AB. The scalar product of two orthogonal vectors vanishes: $\vec{A} \cdot \vec{B}$ = AB cos 90° = 0. The scalar product of a vector with itself is the square of its magnitude:

$$\vec{A}^2 \equiv \vec{A} \cdot \vec{A} = AA\cos 0^o = A^2 \tag{4.2.2}$$

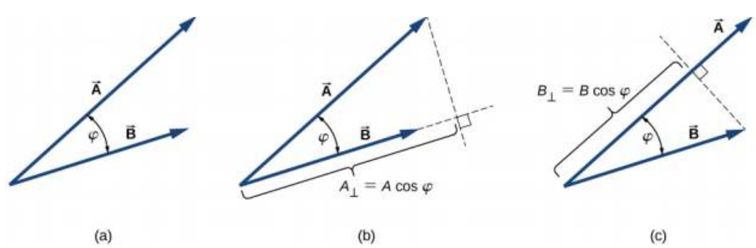

Figure 4.2.1: The scalar product of two vectors. (a) The angle between the two vectors. (b) The orthogonal projection $A_\perp$ of vector $\vec{A}$ onto the direction of vector $\vec{B}$. (c) The orthogonal projection $B_\perp$ of vector $\vec{B}$ onto the direction of vector $\vec{A}$.

Example 4.2.1: The Scalar Product

For the vectors shown in Figure 2.3.6, find the scalar product $\vec{A} \cdot \vec{F}$.

Strategy

From Figure 2.3.6, the magnitudes of vectors $\vec{A}$ and $\vec{B}$ are A = 10.0 and F = 20.0. Angle θ, between them, is the difference: $\theta=\varphi-\alpha$ = 110° − 35° = 75°. Substituting these values into Equation 4.2.1 gives the scalar product.

Solution

A straightforward calculation gives us

$$\vec{A}\cdot\vec{F}=AF\cos\theta=(10.0)(20.0)\cos 75^{o}=51.76. \tag{4.2.3}$$

? Exercise 2.11

For the vectors given in Figure 2.3.6, find the scalar products $\vec{A}\cdot\vec{B}$ and $\vec{B}\cdot\vec{C}$.

In the Cartesian coordinate system, scalar products of the unit vector of an axis with other unit vectors of axes always vanish because these unit vectors are orthogonal:

$$\hat{i}\cdot\ \hat{j}=|\hat{i}||\hat{j}|\cos 90^{o}=(1)(1)(0)=0, \tag{4.2.4}$$

$$\hat{i}\cdot\ \hat{k}=|\hat{i}||\hat{k}|\cos 90^{o}=(1)(1)(0)=0, \tag{4.2.5}$$

$$\hat{k}\cdot\ \hat{j}=|\hat{k}||\hat{j}|\cos 90^{o}=(1)(1)(0)=0. \tag{4.2.6}$$

In these equations, we use the fact that the magnitudes of all unit vectors are one: $|\hat{i}|=|\hat{j}|=|\hat{k}|$ = 1. For unit vectors of the axes, Equation 4.2.2 gives the following identities:

$$\hat{i}\cdot\ \hat{i}=i^2=\hat{j}\cdot\ \hat{j}=j^2=\hat{k}\cdot\ \hat{k}=1. \tag{4.2.7}$$

The scalar product $\vec{A}\cdot\vec{B}$ can also be interpreted as either the product of B with the projection $A_{\parallel}$ of vector $\vec{A}$ onto the direction of vector $\vec{B}$ (Figure 4.2.1(b)) or the product of A with the projection $B_{\parallel}$ of vector $\vec{B}$ onto the direction of vector $\vec{A}$ (Figure 4.2.1(c)):

$$\begin{aligned}\vec{A}\cdot\vec{B}&=AB\cos\varphi\\&=B(A\cos\varphi)=BA_{\parallel}\\&=A(B\cos\varphi)=AB_{\parallel}.\end{aligned}$$

For example, in the rectangular coordinate system in a plane, the scalar x-component of a vector is its dot product with the unit vector $\hat{i}$, and the scalar y-component of a vector is its dot product with the unit vector $\hat{j}$:

$$\begin{cases}\vec{A}\cdot\ \hat{i}=|\vec{A}||\hat{i}|\cos\theta_A=A\cos\theta_A=A\cos\theta_A=A_x\\ \vec{A}\cdot\ \hat{j}=|\vec{A}||\hat{j}|\cos(90^{o}-\theta_A)=A\sin\theta_A=A_y\end{cases} \tag{4.2.8}$$

Scalar multiplication of vectors is communtative,

$$\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}, \tag{4.2.9}$$

and obeys the distributive law:

$$\vec{A}\cdot(\vec{B}+\vec{C})=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}. \tag{4.2.10}$$

We can use the commutative and distributive laws to derive various relations for vectors, such as expressing the dot product of two vectors in terms of their scalar components.

? Exercise 2.12

For vector $\vec{A}=A_x\ \hat{i}+A_y\ \hat{j}+A_z\ \hat{k}$ in a rectangular coordinate system, use Equation 4.2.4 through Equation 4.2.10 to show that $\vec{A}\cdot\hat{i}=A_x\vec{A}\cdot\ \hat{j}=A_y$ and $\vec{A}\cdot\ \hat{k}=A_z$.

When the vectors in Equation 4.2.1 are given in their vector component forms,

$$\vec{A}=A_x\ \hat{i}+A_y\ \hat{j}+A_z\ \hat{k}\ and\vec{B}=B_x\ \hat{i}+B_y\ \hat{j}+B_z\ \hat{k}, \tag{4.2.11}$$

we can compute their scalar product as follows:

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\,\hat{i}+A_y\,\hat{j}+A_z\,\hat{k})\cdot(B_x\,\hat{i}+B_y\,\hat{j}+B_z\,\hat{k}) \\ &= A_xB_x\,\hat{i}\cdot\,\hat{i}+A_xB_y\,\hat{i}\cdot\,\hat{j}+A_xB_z\,\hat{i}\cdot\,\hat{k} \\ &+A_yB_x\,\hat{j}\cdot\,\hat{i}+A_yB_y\,\hat{j}\cdot\,\hat{j}+A_yB_z\,\hat{j}\cdot\,\hat{k} \\ &+A_zB_x\,\hat{k}\cdot\,\hat{i}+A_zB_y\,\hat{k}\cdot\,\hat{j}+A_zB_z\,\hat{k}\cdot\,\hat{k}.\end{aligned}$$

Since scalar products of two different unit vectors of axes give zero, and scalar products of unit vectors with themselves give one (see Equation 4.2.4 and Equation 4.2.7), there are only three nonzero terms in this expression. Thus, the scalar product simplifies to

$$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z. \tag{4.2.12}$$

We can use Equation 4.2.12 for the scalar product in terms of scalar components of vectors to find the angle between two vectors. When we divide Equation 4.2.1 by AB, we obtain the equation for cos φ, into which we substitute Equation 4.2.12:

$$\cos\varphi=\frac{\vec{A}\cdot\vec{B}}{AB}=\frac{A_xB_x+A_yB_y+A_zB_z}{AB}. \tag{4.2.13}$$

Angle φ between vectors $\vec{A}$ and $\vec{B}$ is obtained by taking the inverse cosine of the expression in Equation 4.2.13.

Example 4.2.2

Three dogs are pulling on a stick in different directions, as shown in Figure 4.2.2. The first dog pulls with force $\vec{F}_1$ = (10.0 $\hat{i}$ − 20.4 $\hat{j}$ + 2.0 $\hat{k}$)N, the second dog pulls with force $\vec{F}_2$ = (−15.0 $\hat{i}$ − 6.2 $\hat{k}$)N , and the third dog pulls with force $\vec{F}_3$ = (5.0 $\hat{i}$ + 12.5 $\hat{j}$)N . What is the angle between forces $\vec{F}_1$ and $\vec{F}_2$?

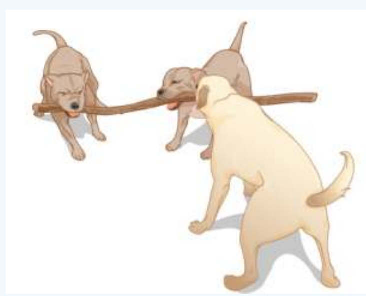

Figure 4.2.2: Three dogs are playing with a stick.

Strategy

The components of force vector $\vec{F}_1$ are F_{1x} = 10.0 N, F_{1y}= −20.4 N, and F_{1z} = 2.0 N, whereas those of force vector $\vec{F}_2$ are F_{2x} = −15.0 N, F_{2y} = 0.0 N, and F_{2z} = −6.2 N. Computing the scalar product of these vectors and their magnitudes, and substituting into Equation 4.2.13 gives the angle of interest.

Solution

The magnitudes of forces $\vec{F}_1$ and $\vec{F}_2$ are

$$F_1=\sqrt{F_{1x}^2+F_{1y}^2+F_{1z}^2}=\sqrt{10.0^2+20.4^2+2.0^2}\,N=22.8\,N \tag{4.2.14}$$

and

$$F_2=\sqrt{F_{2x}^2+F_{2y}^2+F_{2z}^2}=\sqrt{15.0^2+6.2^2}\,N=16.2\,N. \tag{4.2.15}$$

Substituting the scalar components into Equation 4.2.12 yields the scalar product

$$\begin{aligned}\vec{F}_1\cdot\vec{F}_2 &= F_{1x}F_{2x}+F_{1y}F_{2y}+F_{1z}F_{2z} \\ &=(10.0\,N)(-15.0\,N)+(-20.4\,N)(0.0\,N)+(2.0\,N)(-6.2\,N) \\ &=-162.4\,N^2.\end{aligned}$$

Finally, substituting everything into Equation 4.2.13 gives the angle

$$\cos\varphi = \frac{\vec{F}_1 \cdot \vec{F}_2}{F_1 F_2} = \frac{-162.4\ N^2}{(22.8\ N)(16.2\ N)} = -0.439 \Rightarrow \varphi = \cos^{-1}(-0.439) = 116.0^{\circ}. \tag{4.2.16}$$

Significance

Notice that when vectors are given in terms of the unit vectors of axes, we can find the angle between them without knowing the specifics about the geographic directions the unit vectors represent. Here, for example, the +x-direction might be to the east and the +y-direction might be to the north. But, the angle between the forces in the problem is the same if the +x-direction is to the west and the +y-direction is to the south.

Exercise 2.13

Find the angle between forces $\vec{F}_1$ and $\vec{F}_3$ in Example 4.2.2.

Example 4.2.3: The Work of a Force

When force $\vec{F}$ pulls on an object and when it causes its displacement $\vec{D}$, we say the force performs work. The amount of work the force does is the scalar product $\vec{F} \cdot \vec{D}$. If the stick in Example 4.2.2 moves momentarily and gets displaced by vector $\vec{D}$ = (−7.9 $\hat{j}$ − 4.2 $\hat{k}$)cm, how much work is done by the third dog in Example 4.2.2?

Strategy

We compute the scalar product of displacement vector $\vec{D}$ with force vector $\vec{F}_3$ = (5.0 $\hat{i}$ + 12.5 $\hat{j}$)N, which is the pull from the third dog. Let's use W_3 to denote the work done by force $\vec{F}_3$ on displacement $\vec{D}$.

Solution

Calculating the work is a straightforward application of the dot product:

$$\begin{aligned} W_3 &= \vec{F}_3 \cdot \vec{D} = F_{3x}D_x + F_{3y}D_y + F_{3z}D_z \\ &= (5.0\ N)(0.0\ cm) + (12.5\ N)(-7.9\ cm) + (0.0\ N)(-4.2\ cm) \\ &= -98.7\ N \cdot cm. \end{aligned}$$

Significance

The SI unit of work is called the joule (J) , where 1 J = 1 N · m. The unit cm · N can be written as 10^{-2} m · N = 10^{-2} J, so the answer can be expressed as W_3 = −0.9875 J ≈ −1.0 J.

Exercise 2.14

How much work is done by the first dog and by the second dog in Example 4.2.2 on the displacement in Example 4.2.3?

Contributors and Attributions

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4.3: Dot Product of Two Vectors (Video)

4.4: Vector Dot Product and Vector Length (Video)

4.5: The Vector Dot Product (Video)

4.6: Additional Materials

Scalar Product BYJU's Learning

Matrix Representation of Scalar Product BYJU's Learning

The Scalar Product The Math Centre, UK

CHAPTER OVERVIEW

5: Cross (Vector) Product of Vectors

5.1: The Cross Product

Learning Objectives

- Calculate the cross product of two given vectors.
- Use determinants to calculate a cross product.
- Find a vector orthogonal to two given vectors.
- Determine areas by using the cross product.

Imagine a mechanic turning a wrench to tighten a bolt. The mechanic applies a force at the end of the wrench. This creates rotation, or torque, which tightens the bolt. We can use vectors to represent the force applied by the mechanic, and the distance (radius) from the bolt to the end of the wrench. Then, we can represent torque by a vector oriented along the axis of rotation. Note that the torque vector is orthogonal to both the force vector and the radius vector.

In this section, we develop an operation called the **cross product**, which allows us to find a vector orthogonal to two given vectors. Calculating torque is an important application of cross products, and we examine torque in more detail later in the section.

The Cross Product and Its Properties

The dot product is a multiplication of two vectors that results in a scalar. In this section, we introduce a product of two vectors that generates a third vector orthogonal to the first two. Consider how we might find such a vector. Let $\vec{\mathbf{u}}=\langle u_1,u_2,u_3\rangle$ and $\vec{\mathbf{v}}=\langle v_1,v_2,v_3\rangle$ be nonzero vectors. We want to find a vector $\vec{\mathbf{w}}=\langle w_1,w_2,w_3\rangle$ orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$—that is, we want to find $\vec{\mathbf{w}}$ such that $\vec{\mathbf{u}}\cdot\vec{\mathbf{w}}=0$ and $\vec{\mathbf{v}}\cdot\vec{\mathbf{w}}=0$. Therefore, w_1, w_2, and w_3 must satisfy

$$u_1w_1+u_2w_2+u_3w_3=0 \tag{5.1.1}$$

$$v_1w_1+v_2w_2+v_3w_3=0. \tag{5.1.2}$$

If we multiply the top equation by v_3 and the bottom equation by u_3 and subtract, we can eliminate the variable w_3, which gives

$$(u_1v_3-v_1u_3)w_1+(u_2v_3-v_2u_3)w_2=0.$$

If we select

$$\begin{aligned} w_1 &= u_2v_3-u_3v_2 \\ w_2 &= -(u_1v_3-u_3v_1), \end{aligned}$$

we get a possible solution vector. Substituting these values back into the original equations (Equations 5.1.1 and 5.1.2) gives

$$w_3=u_1v_2-u_2v_1.$$

That is, vector

$$\vec{\mathbf{w}}=\langle u_2v_3-u_3v_2,-(u_1v_3-u_3v_1),u_1v_2-u_2v_1\rangle \tag{5.1.3}$$

is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Unfortunately, this formula is complicated and difficult to remember, so determinant notation can be used to simplify the process.

Determinants and the Cross Product

Using the formula in Equation 5.1.3 to find the cross product is difficult to remember. Fortunately, we have an alternative. We can calculate the cross product of two vectors usingdeterminant notation. Using determinants to evaluate a cross product is easier because there is fundamentally just a simple pattern to remember, rather than a complicated formula.

A 2×2 determinant is defined by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2-b_1a_2. \tag{5.1.4}$$

For example,

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3(1)-5(-2)=3+10=13. \tag{5.1.5}$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{5.1.6}$$

Equation 5.1.6 is referred to as the expansion of the determinant along the first row. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier.

$\begin{vmatrix} a_1 & \text{✗} & \text{✗} \\ \text{✗} & b_2 & b_3 \\ \text{✗} & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$ $= a_1(b_2c_3 - c_2b_3)$	$\begin{vmatrix} \text{✗} & a_2 & \text{✗} \\ b_1 & \text{✗} & b_3 \\ c_1 & \text{✗} & c_3 \end{vmatrix} = -a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$ $= -a_2(b_1c_3 - c_1b_3)$	$\begin{vmatrix} \text{✗} & \text{✗} & a_3 \\ b_1 & b_2 & \text{✗} \\ c_1 & c_2 & \text{✗} \end{vmatrix} = a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$ $= a_3(b_1c_2 - c_1b_2)$

Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

Observe that the expansion of each 2×2 determinant corresponds to the components listed in Equation 5.1.3 when $\vec{\mathbf{u}} = \langle b_1, b_2, b_3 \rangle$ and $\vec{\mathbf{v}} = \langle c_1, c_2, c_3 \rangle$. If the entries along the first row of the determinant are made the standard unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, we can obtain the determinant definition of the cross product.

Definition: Cross Product

The **cross product** $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is a vector that is perpendicular to two given vectors.

Given vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, the *cross product* is vector is defined by the formula

$$\begin{aligned} \vec{\mathbf{u}} \times \vec{\mathbf{v}} &= (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}} \\ &= \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle. \end{aligned} \tag{5.1.7}$$

The cross product is more easily calculated using determinant notation:

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{\mathbf{i}} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_4 \end{vmatrix}. \tag{5.1.8}$$

The next few examples provide practice evaluating determinants in general.

Example 5.1.1: Using Expansion Along the First Row to Compute a 3×3 Determinant

Evaluate the determinant $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$.

Solution

We have

$$\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5\begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix}$$

$$= 2(4-9) - 5(-4+6) - 1(-3+2)$$

$$= 2(-5) - 5(2) - 1(-1) = -10 - 10 + 1$$

$$= -19$$

Try It 5.1.1

Evaluate the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$.

Hint

Expand along the first row. Don't forget the second term is negative!

Answer

40

Cross Product Evaluation

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula. Now for some practice calculating cross products.

Example 5.1.2: Using Determinant Notation to find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$

Let $\vec{\mathbf{p}} = \langle -1, 2, 5\rangle$ and $\vec{\mathbf{q}} = \langle 4, 0, -3\rangle$. Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$.

Solution

We set up our determinant by putting the standard unit vectors across the first row, the components of $\vec{\mathbf{u}}$ in the second row, and the components of $\vec{\mathbf{v}}$ in the third row. Then, we have

$$\vec{\mathbf{p}} \times \vec{\mathbf{q}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix}\hat{\mathbf{k}}$$

$$= (-6-0)\hat{\mathbf{i}} - (3-20)\hat{\mathbf{j}} + (0-8)\hat{\mathbf{k}}$$

$$= -6\hat{\mathbf{i}} + 17\hat{\mathbf{j}} - 8\hat{\mathbf{k}}.$$

Notice that this answer confirms the calculation of the cross product in Example 5.1.1.

Try It 5.1.2

Use determinant notation to find $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$, where $\vec{\mathbf{a}} = \langle 8, 2, 3\rangle$ and $\vec{\mathbf{b}} = \langle -1, 0, 4\rangle$.

Hint

Calculate the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 8 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix}$.

Answer

$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 8\hat{\mathbf{i}} - 35\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Properties of the Cross Product

The Cross Product is a vector orthogonal to two other vectors

From the way we have developed $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, it should be clear that the cross product is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. However, it never hurts to check. To show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{u}}$, we calculate the dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1 \rangle \\ &= u_1(u_2v_3 - u_3v_2) + u_2(-u_1v_3 + u_3v_1) + u_3(u_1v_2 - u_2v_1) \\ &= u_1u_2v_3 - u_1u_3v_2 - u_1u_2v_3 + u_2u_3v_1 + u_1u_3v_2 - u_2u_3v_1 \\ &= (u_1u_2v_3 - u_1u_2v_3) + (-u_1u_3v_2 + u_1u_3v_2) + (u_2u_3v_1 - u_2u_3v_1) \\ &= 0 \end{aligned}$$

In a similar manner, we can show that the cross product is also orthogonal to $\vec{\mathbf{v}}$.

Although it may not be obvious from Equation 5.1.7, the direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is given by the right-hand rule. If we hold the right hand out with the fingers pointing in the direction of $\vec{\mathbf{u}}$, then curl the fingers toward vector $\vec{\mathbf{v}}$, the thumb points in the direction of the cross product, as shown in Figure 5.1.2.

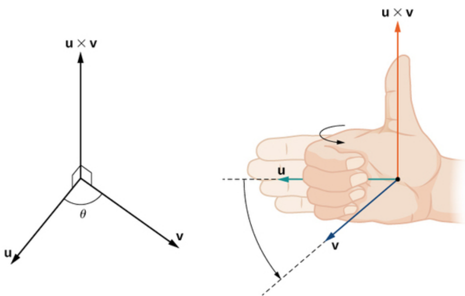

Figure 5.1.3 The direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ *is determined by the right-hand rule.*

Notice what this means for the direction of $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$. If we apply the right-hand rule to $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$, we start with our fingers pointed in the direction of $\vec{\mathbf{v}}$, then curl our fingers toward the vector $\vec{\mathbf{u}}$. In this case, the thumb points in the opposite direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$. (Try it!)

Example 5.1.3: Anticommutativity of the Cross Product

Let $\vec{\mathbf{u}} = \langle 0, 2, 1 \rangle$ and $\vec{\mathbf{v}} = \langle 3, -1, 0 \rangle$. Calculate $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ and graph them.

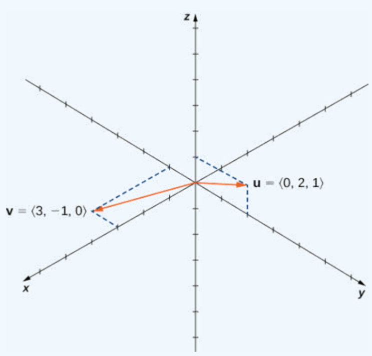

Figure 5.1.3 Are the cross products $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ in the same direction?

Solution

We have

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \langle (0+1), -(0-3), (0-6) \rangle = \langle 1, 3, -6 \rangle$$
$$\vec{\mathbf{v}} \times \vec{\mathbf{u}} = \langle (-1-0), -(3-0), (6-0) \rangle = \langle -1, -3, 6 \rangle.$$

We see that, in this case, $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$ (Figure 5.1.4). We prove this in general later in this section.

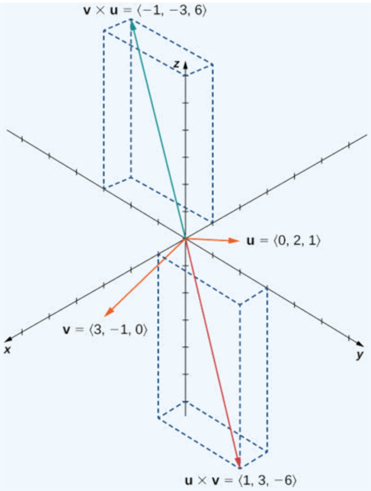

Figure 5.1.4 The cross products $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ *and* $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ *are both orthogonal to* $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$, *but in opposite directions.*

Try It 5.1.3

Suppose vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ lie in the xy-plane (the z-component of each vector is zero). Now suppose the x- and y-components of $\vec{\mathbf{u}}$ and the y-component of $\vec{\mathbf{v}}$ are all positive, whereas the x-component of $\vec{\mathbf{v}}$ is negative. Assuming the coordinate axes are oriented in the usual positions, in which direction does $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ point?

Hint

Remember the right-hand rule (Figure 5.1.2).

Answer

Up (the positive z-direction)

Magnitude of the Cross Product

So far in this section, we have been concerned with the direction of the vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, but we have not discussed its magnitude. It turns out there is a simple expression for the magnitude of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ involving the magnitudes of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, and the sine of the angle between them.

Magnitude of the Cross Product

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be vectors, and let θ be the angle between them. Then,

$$\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\| \cdot \|\vec{\mathbf{v}}\| \cdot \sin\theta.$$

Proof

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ denote the angle between them. Then

$$\begin{aligned} \|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2 \\ &= u_2^2v_3^2-2u_2u_3v_2v_3+u_3^2v_2^2+u_3^2v_1^2-2u_1u_3v_1v_3+u_1^2v_3^2+u_1^2v_2^2-2u_1u_2v_1v_2+u_2^2v_1^2 \\ &= u_1^2v_1^2+u_1^2v_2^2+u_1^2v_3^2+u_2^2v_1^2+u_2^2v_2^2+u_2^2v_3^2+u_3^2v_1^2+u_3^2v_2^2+u_3^2v_3^2 \\ &\quad -(u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2+2u_1u_2v_1v_2+2u_1u_3v_1v_3+2u_2u_3v_2v_3) \\ &= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2 \\ &= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2-(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})^2 \\ &= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2-\|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2\cos^2\theta \\ &= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2(1-\cos^2\theta) \\ &= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2(\sin^2\theta). \end{aligned}$$

Taking square roots and noting that $\sqrt{\sin^2\theta}=\sin\theta$ for $0\leq\theta\leq 180^\circ$, we have the desired result:

$$\|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\|=\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\sin\theta. \tag{5.1.9}$$

This definition of the cross product allows us to visualize or interpret the product geometrically. It is clear, for example, that the cross product is defined only for vectors in three dimensions, not for vectors in two dimensions. In two dimensions, it is impossible to generate a vector simultaneously orthogonal to two nonparallel vectors.

Example 5.1.4: Calculating the Cross Product

Find the magnitude of the cross product of $\vec{\mathbf{u}}=\langle 0,4,0\rangle$ and $\vec{\mathbf{v}}=\langle 0,0,-3\rangle$.

Solution

We have

$$\begin{aligned} \|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\| &= \|\vec{\mathbf{u}}\|\cdot\|\vec{\mathbf{v}}\|\cdot\sin\theta \\ &= \sqrt{0^2+4^2+0^2}\cdot\sqrt{0^2+0^2+(-3)^2}\cdot\sin\frac{\pi}{2} \\ &= 4(3)(1)=12 \end{aligned}$$

Try It 5.1.4

Find the magnitude of $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$, where $\vec{\mathbf{u}}=\langle -8,0,0\rangle$ and $\vec{\mathbf{v}}=\langle 0,2,0\rangle$.

Hint

Vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal.

Answer

16

The Unit Vector Orthogonal to Two Other Vectors

The cross product is very useful for several types of calculations, including finding a vector orthogonal to two given vectors, computing areas of triangles and parallelograms, and even determining the volume of the three-dimensional geometric shape made of parallelograms known as a *parallelepiped*. The following examples illustrate these calculations.

Example 5.1.5: Finding a Unit Vector Orthogonal to Two Given Vectors

Let $\vec{\mathbf{a}}=\langle 5,2,-1\rangle$ and $\vec{\mathbf{b}}=\langle 0,-1,4\rangle$. Find a unit vector orthogonal to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Solution

The cross product $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is orthogonal to both vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. We can calculate it with a determinant:

$$\begin{aligned}\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix} \hat{\mathbf{k}} \\ &= (8-1)\hat{\mathbf{i}} - (20-0)\hat{\mathbf{j}} + (-5-0)\hat{\mathbf{k}} \\ &= 7\hat{\mathbf{i}} - 20\hat{\mathbf{j}} - 5\hat{\mathbf{k}}. \end{aligned}$$

Normalize this vector to find a unit vector in the same direction:

$$\|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\| = \sqrt{(7)^2 + (-20)^2 + (-5)^2} = \sqrt{474}\,.$$

Thus, $\left\langle \frac{7}{\sqrt{474}}, \frac{-20}{\sqrt{474}}, \frac{-5}{\sqrt{474}} \right\rangle$ is a unit vector orthogonal to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Simplified, this vector becomes $\left\langle \frac{7\sqrt{474}}{474}, \frac{-10\sqrt{474}}{237}, \frac{-5\sqrt{474}}{474} \right\rangle$.

Try It 5.1.5

Find a unit vector orthogonal to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, where $\vec{\mathbf{a}} = \langle 4, 0, 3 \rangle$ and $\vec{\mathbf{b}} = \langle 1, 1, 4 \rangle$.

Hint

Normalize the cross product.

Answer

$\left\langle \frac{-3}{\sqrt{194}}, \frac{-13}{\sqrt{194}}, \frac{4}{\sqrt{194}} \right\rangle$ or, simplified as $\left\langle \frac{-3\sqrt{194}}{194}, \frac{-13\sqrt{194}}{194}, \frac{2\sqrt{194}}{97} \right\rangle$

Area of a Parallelogram

To use the cross product for calculating areas, we state and prove the following theorem.

Area of a Parallelogram

If we locate vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$ (Figure 5.1.5).

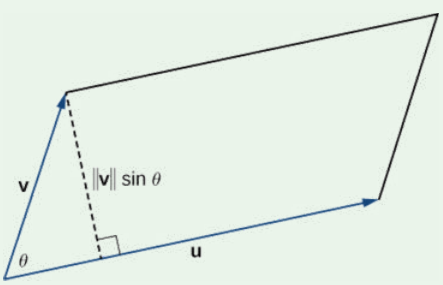

Figure 5.1.6: The parallelogram with adjacent sides $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ has base $\|\vec{\mathbf{u}}\|$ and height $\|\vec{\mathbf{v}}\| \sin\theta$.

Proof

We show that the magnitude of the cross product is equal to the base times height of the parallelogram.

$$\begin{aligned}\text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= \|\vec{\mathbf{u}}\|(\|\vec{\mathbf{v}}\| \sin\theta) \\ &= \|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| \end{aligned}$$

Example 5.1.6: Finding the Area of a Triangle

Let $P=(1,0,0), Q=(0,1,0)$, and $R=(0,0,1)$ be the vertices of a triangle (Figure $5.1.6a$). Find its area.

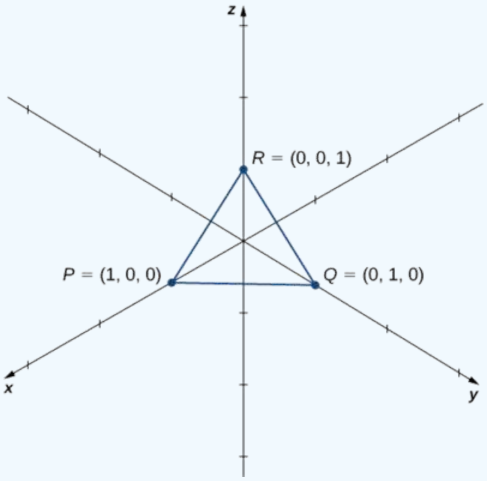

Figure $5.1.6a$ Finding the area of a triangle by using the cross product.

Solution

We have $\overrightarrow{PQ}=\langle 0-1,1-0,0-0\rangle=\langle -1,1,0\rangle$ and $\overrightarrow{PR}=\langle 0-1,0-0,1-0\rangle=\langle -1,0,1\rangle$. The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by $\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|$:

$$\begin{aligned}\overrightarrow{PQ}\times\overrightarrow{PR}&=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -1 & 1 & 0\\ -1 & 0 & 1\end{vmatrix}\\ &=(1-0)\hat{\mathbf{i}}-(-1-0)\hat{\mathbf{j}}+(0-(-1))\hat{\mathbf{k}}\\ &=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}\end{aligned}$$

$$\begin{aligned}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\|&=\|\langle 1,1,1\rangle\|\\ &=\sqrt{1^2+1^2+1^2}\\ &=\sqrt{3}.\end{aligned}$$

The area of ΔPQR is half the area of the parallelogram or $\sqrt{3}/2\ \text{units}^2$.

Try It 5.1.6

Find the area of the parallelogram $PQRS$ with vertices $P(1,1,0)$, $Q(7,1,0)$, $R(9,4,2)$, and $S(3,4,2)$.

Hint

Sketch the parallelogram and identify two vectors that form adjacent sides of the parallelogram.

Answer

$6\sqrt{13}\ \text{units}^2$

Key Concepts

- The cross product $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$ of two vectors $\vec{\mathbf{u}}=\langle u_1,u_2,u_3\rangle$ and $\vec{\mathbf{v}}=\langle v_1,v_2,v_3\rangle$ is a vector orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Its length is given by $\|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\|=\|\vec{\mathbf{u}}\|\cdot\|\vec{\mathbf{v}}\|\cdot\sin\theta,$ where θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Its direction is given by the right-hand rule.
- The algebraic formula for calculating the cross product of two vectors,

$\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, is

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}.$

- The cross product of vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- If vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$.
- The cross product can be used to identify a vector orthogonal to two given vectors or to a plane.

Key Equations

- **The cross product of two vectors in terms of the unit vectors**

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}} \tag{5.1.10}$$

Glossary

cross product

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}},$ where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$

determinant

a real number associated with a square matrix

vector product

the cross product of two vectors

Contributors and Attributions

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5.2: Vector Cross Product

The Vector Products of Two Vectors (the Cross Product)

Vector multiplication of two vectors yields a vector product.

Vector Product (Cross Product)

The vector product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \times \vec{B}$ and is often referred to as a cross product. The vector product is a vector that has its direction perpendicular to both vectors $\vec{A}$ and $\vec{B}$. In other words, vector $\vec{A} \times \vec{B}$ is perpendicular to the plane that contains vectors $\vec{A}$ and $\vec{B}$, as shown in Figure $5.2.1$. The magnitude of the vector product is defined as

$$|\vec{A} \times \vec{B}| = AB \sin \varphi, \tag{5.2.1}$$

where angle φ, between the two vectors, is measured from vector $\vec{A}$ (first vector in the product) to vector $\vec{B}$ (second vector in the product), as indicated in Figure $5.2.1$, and is between 0° and 180°.

According to Equation $5.2.1$, the vector product vanishes for pairs of vectors that are either parallel (φ = 0°) or antiparallel (φ = 180°) because sin 0° = sin 180° = 0.

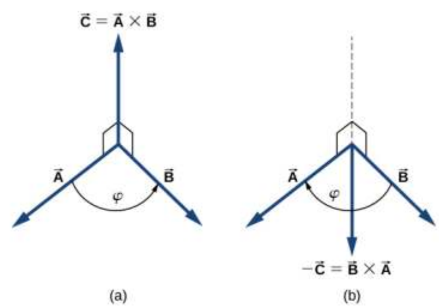

Figure $5.2.1$: The vector product of two vectors is drawn in three-dimensional space. (a) The vector product $\vec{A} \times \vec{B}$ is a vector perpendicular to the plane that contains vectors $\vec{A}$ and $\vec{B}$. Small squares drawn in perspective mark right angles between $\vec{A}$ and $\vec{C}$, and between $\vec{B}$ and $\vec{C}$ so that if $\vec{A}$ and $\vec{B}$ lie on the floor, vector $\vec{B}$ points vertically upward to the ceiling. (b) The vector product $\vec{B} \times \vec{A}$ is a vector antiparallel to vector $\vec{A} \times \vec{B}$.

On the line perpendicular to the plane that contains vectors $\vec{A}$ and $\vec{B}$ there are two alternative directions—either up or down, as shown in Figure $5.2.1$—and the direction of the vector product may be either one of them. In the standard right-handed orientation, where the angle between vectors is measured counterclockwise from the first vector, vector $\vec{A} \times \vec{B}$ points **upward**, as seen in Figure $5.2.1$(a). If we reverse the order of multiplication, so that now $\vec{B}$ comes first in the product, then vector $\vec{B} \times \vec{A}$ must point **downward**, as seen in Figure $5.2.1$(b). This means that vectors $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ are **antiparallel** to each other and that vector multiplication is **not** commutative but **anticommutative**. The **anticommutative property** means the vector product reverses the sign when the order of multiplication is reversed:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}. \tag{5.2.2}$$

The **corkscrew right-hand rule** is a common mnemonic used to determine the direction of the vector product. As shown in Figure $5.2.2$, a corkscrew is placed in a direction perpendicular to the plane that contains vectors $\vec{A}$ and $\vec{B}$, and its handle is turned in the direction from the first to the second vector in the product. The direction of the cross product is given by the progression of the corkscrew.

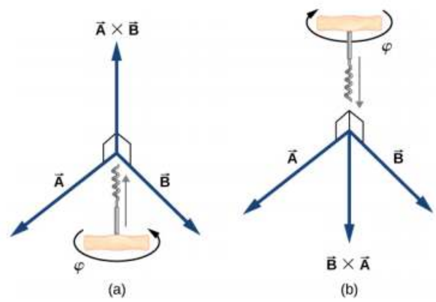

Figure 5.2.2: The corkscrew right-hand rule can be used to determine the direction of the cross product $\vec{A} \times \vec{B}$. Place a corkscrew in the direction perpendicular to the plane that contains vectors $\vec{A}$ and $\vec{B}$, and turn it in the direction from the first to the second vector in the product. The direction of the cross product is given by the progression of the corkscrew. (a) Upward movement means the cross-product vector points up. (b) Downward movement means the cross-product vector points downward.

✔ Example 5.2.1: The Torque of a Force

The mechanical advantage that a familiar tool called a **wrench** provides (Figure 5.2.3) depends on magnitude F of the applied force, on its direction with respect to the wrench handle, and on how far from the nut this force is applied. The distance R from the nut to the point where force vector $\vec{F}$ is attached is called the **lever arm** and is represented by the radial vector $\vec{R}$. The physical vector quantity that makes the nut turn is called **torque** (denoted by $\vec{\tau}$), and it is the vector product of the lever arm with the force: $\vec{\tau} = \vec{R} \times \vec{F}$.

To loosen a rusty nut, a 20.00-N force is applied to the wrench handle at angle φ = 40° and at a distance of 0.25 m from the nut, as shown in Figure 5.2.3(a). Find the magnitude and direction of the torque applied to the nut. What would the magnitude and direction of the torque be if the force were applied at angle φ = 45°, as shown in Figure 5.2.3(b)? For what value of angle φ does the torque have the largest magnitude?

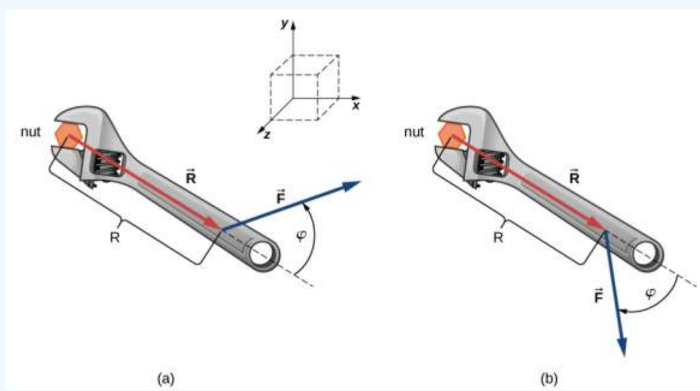

Figure 5.2.3: A wrench provides grip and mechanical advantage in applying torque to turn a nut. (a) Turn counterclockwise to loosen the nut. (b) Turn clockwise to tighten the nut.

Strategy

We adopt the frame of reference shown in Figure 5.2.3, where vectors $\vec{R}$ and $\vec{F}$ lie in the xy-plane and the origin is at the position of the nut. The radial direction along vector $\vec{R}$ (pointing away from the origin) is the reference direction for measuring the angle φ because $\vec{R}$ is the first vector in the vector product $\vec{\tau} = \vec{R} \times \vec{F}$. Vector $\vec{\tau}$ must lie along the z-axis because this is the axis that is perpendicular to the xy-plane, where both $\vec{R}$ and $\vec{F}$ lie. To compute the magnitude τ, we use Equation 5.2.1. To find the direction of $\vec{\tau}$, we use the corkscrew right-hand rule (Figure 5.2.2).

Solution

For the situation in (a), the corkscrew rule gives the direction of $\vec{R} \times \vec{F}$ in the positive direction of the z-axis. Physically, it means the torque vector $\vec{\tau}$ points out of the page, perpendicular to the wrench handle. We identify F = 20.00 N and R = 0.25 m, and compute the magnitude using Equation 5.2.1:

$$\tau = |\vec{R} \times \vec{F}| = RF \sin\varphi = (0.25\ m)(20.00\ N)\sin 40^o = 3.21\ N\cdot m. \tag{5.2.3}$$

For the situation in (b), the corkscrew rule gives the direction of $\vec{R} \times \vec{F}$ in the negative direction of the z-axis. Physically, it means the vector $\vec{\tau}$ points into the page, perpendicular to the wrench handle. The magnitude of this torque is

$$\tau = |\vec{R} \times \vec{F}| = RF \sin\varphi = (0.25\ m)(20.00\ N) \sin 45^o = 3.53\ N \cdot m. \tag{5.2.4}$$

The torque has the largest value when sin φ = 1, which happens when φ = 90°. Physically, it means the wrench is most effective—giving us the best mechanical advantage—when we apply the force perpendicular to the wrench handle. For the situation in this example, this best-torque value is τ_{best} = RF = (0.25 m)(20.00 N) = 5.00 N • m.

Significance

When solving mechanics problems, we often do not need to use the corkscrew rule at all, as we'll see now in the following equivalent solution. Notice that once we have identified that vector $\vec{R} \times \vec{F}$ lies along the z-axis, we can write this vector in terms of the unit vector $\hat{k}$ of the z-axis:

$$\vec{R} \times \vec{F} = RF \sin\varphi \hat{k}. \tag{5.2.5}$$

In this equation, the number that multiplies $\hat{k}$ is the scalar z-component of the vector $\vec{R} \times \vec{F}$. In the computation of this component, care must be taken that the angle φ is measured counterclockwise from $\vec{R}$ (first vector) to $\vec{F}$ (second vector) Following this principle for the angles, we obtain RF sin (+ 40°) = + 3.2 N • m for the situation in (a), and we obtain RF sin (−45°) = −3.5 N • m for the situation in (b). In the latter case, the angle is negative because the graph in Figure 5.2.3 indicates the angle is measured clockwise; but, the same result is obtained when this angle is measured counterclockwise because +(360° − 45°) = + 315° and sin (+ 315°) = sin (−45°). In this way, we obtain the solution without reference to the corkscrew rule. For the situation in (a), the solution is $\vec{R} \times \vec{F}$ = + 3.2 N • m $\hat{k}$; for the situation in (b), the solution is $\vec{R} \times \vec{F}$ = −3.5 N • m $\hat{k}$.

? Exercise 2.15

For the vectors given in Figure 2.3.6, find the vector products $\vec{A} \times \vec{B}$ and $\vec{C} \times \vec{F}$.

Similar to the dot product (Equation 2.8.10), the cross product has the following distributive property:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}. \tag{5.2.6}$$

The distributive property is applied frequently when vectors are expressed in their component forms, in terms of unit vectors of Cartesian axes. When we apply the definition of the cross product, Equation 5.2.1, to unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ that define the positive x-, y-, and z-directions in space, we find that

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0. \tag{5.2.7}$$

All other cross products of these three unit vectors must be vectors of unit magnitudes because $\hat{i}$, $\hat{j}$, and $\hat{k}$ are orthogonal. For example, for the pair $\hat{i}$ and $\hat{j}$, the magnitude is $|\hat{i} \times \hat{j}|$ = ij sin 90° = (1)(1)(1) = 1. The direction of the vector product $\hat{i} \times \hat{j}$ must be orthogonal to the xy-plane, which means it must be along the z-axis. The only unit vectors along the z-axis are $-\hat{k}$ or $+\hat{k}$. By the corkscrew rule, the direction of vector $\hat{i} \times \hat{j}$ must be parallel to the positive z-axis. Therefore, the result of the multiplication $\hat{i} \times \hat{j}$ is identical to + $\hat{k}$. We can repeat similar reasoning for the remaining pairs of unit vectors. The results of these multiplications are

$$\begin{cases} \hat{i} \times \hat{j} = +\hat{k}, \\ \hat{j} \times \hat{k} = +\hat{i}, \\ \hat{k} \times \hat{i} = +\hat{j}. \end{cases} \tag{5.2.8}$$

Notice that in Equation 5.2.8, the three unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ appear in the cyclic order shown in a diagram in Figure 5.2.4(a). The cyclic order means that in the product formula, $\hat{i}$ follows $\hat{k}$ and comes before $\hat{j}$, or $\hat{k}$ follows $\hat{j}$ and comes before $\hat{i}$, or $\hat{j}$ follows $\hat{i}$ and comes before $\hat{k}$. The cross product of two different unit vectors is always a third unit vector. When two unit vectors in the cross product appear in the cyclic order, the result of such a multiplication is the remaining unit vector, as illustrated in Figure 5.2.4(b). When unit vectors in the cross product appear in a different order, the result is a unit vector that is antiparallel to the remaining unit vector (i.e., the result is with the minus sign, as shown by the examples in Figure 5.2.4(c) and Figure 5.2.4(d).

In practice, when the task is to find cross products of vectors that are given in vector component form, this rule for the cross-multiplication of unit vectors is very useful.

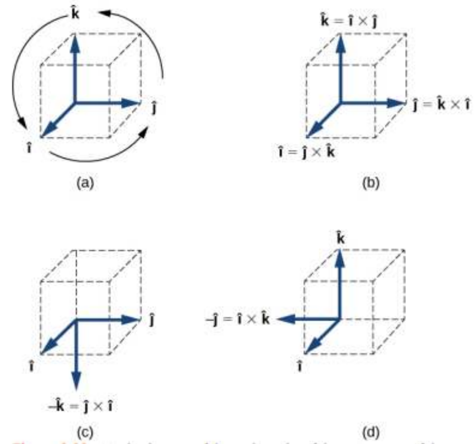

Figure 5.2.4: (a) The diagram of the cyclic order of the unit vectors of the axes. (b) The only cross products where the unit vectors appear in the cyclic order. These products have the positive sign. (c, d) Two examples of cross products where the unit vectors do not appear in the cyclic order. These products have the negative sign.

Suppose we want to find the cross product $\vec{A} \times \vec{B}$ for vectors $\vec{A}$= $A_x\,\hat{i}$+ $A_y\,\hat{j}$ + $A_z\,\hat{k}$ and $\vec{B}$ = $B_x\,\hat{i}$ + $B_y\,\hat{j}$ + $B_z\,\hat{k}$. We can use the distributive property (Equation 5.2.6), the anticommutative property (Equation 5.2.2), and the results in Equation 5.2.7 and Equation 5.2.8 for unit vectors to perform the following algebra:

$$\begin{aligned}\vec{A}\times\vec{B} &= (A_x\,\hat{i}+A_y\,\hat{j}+A_z\,\hat{k})\times(B_x\,\hat{i}+B_y\,\hat{j}+B_z\,\hat{k})\\ &= A_x\,\hat{i}\times(B_x\,\hat{i}+B_y\,\hat{j}+B_z\,\hat{k})+A_y\,\hat{j}\times(B_x\,\hat{i}+B_y\,\hat{j}+B_z\,\hat{k})+A_z\,\hat{k}\times(B_x\,\hat{i}+B_y\,\hat{j}+B_z\,\hat{k})\\ &= A_xB_x\,\hat{i}\times\hat{i}+A_xB_y\,\hat{i}\times\hat{j}+A_xB_z\,\hat{i}\times\hat{k}\\ &\quad +A_yB_x\,\hat{j}\times\hat{i}+A_yB_y\,\hat{j}\times\hat{j}+A_yB_z\,\hat{j}\times\hat{k}\\ &\quad +A_zB_x\,\hat{k}\times\hat{i}+A_zB_y\,\hat{k}\times\hat{j}+A_zB_z\,\hat{k}\times\hat{k}\\ &= A_xB_x(0)+A_xB_y(+\hat{k})+A_xB_z(-\hat{j})\\ &\quad +A_yB_x(-\hat{k})+A_yB_y(0)+A_yB_z(+\hat{i})\\ &\quad +A_zB_x(+\hat{j})+A_zB_y(-\hat{i})+A_zB_z(0).\end{aligned}$$

When performing algebraic operations involving the cross product, be very careful about keeping the correct order of multiplication because the cross product is anticommutative. The last two steps that we still have to do to complete our task are, first, grouping the terms that contain a common unit vector and, second, factoring. In this way we obtain the following very useful expression for the computation of the cross product:

$$\vec{C}=\vec{A}\times\vec{B}=(A_yB_z-A_zB_y)\,\hat{i}+(A_zB_x-A_xB_z)\,\hat{j}+(A_xB_y-A_yB_x)\,\hat{k}. \tag{5.2.9}$$

In this expression, the scalar components of the cross-product vector are

$$\begin{cases} C_x = A_yB_z-A_zB_y,\\ C_y = A_zB_x-A_xB_z,\\ C_z = A_xB_y-A_yB_x.\end{cases} \tag{5.2.10}$$

When finding the cross product, in practice, we can use either Equation 5.2.1 or Equation 5.2.9, depending on which one of them seems to be less complex computationally. They both lead to the same final result. One way to make sure if the final result is correct is to use them both.

Example 5.2.2: A Particle in a Magnetic Field

When moving in a magnetic field, some particles may experience a magnetic force. Without going into details—a detailed study of magnetic phenomena comes in later chapters—let's acknowledge that the magnetic field $\vec{B}$ is a vector, the magnetic force $\vec{F}$ is a vector, and the velocity $\vec{u}$ of the particle is a vector. The magnetic force vector is proportional to the vector product of the velocity vector with the magnetic field vector, which we express as $\vec{F} = \zeta\vec{u} \times \vec{B}$. In this equation, a constant ζ takes care of the consistency in physical units, so we can omit physical units on vectors $\vec{u}$ and $\vec{B}$. In this example, let's assume the constant ζ is positive. A particle moving in space with velocity vector $\vec{u} = -5.0\ \hat{i} - 2.0\ \hat{j} + 3.5\ \hat{k}$ enters a region with a magnetic field and experiences a magnetic force. Find the magnetic force $\vec{F}$ on this particle at the entry point to the region where the magnetic field vector is (a) $\vec{B} = 7.2\ \hat{i} - \hat{j} - 2.4\ \hat{k}$ and (b) $\vec{B} = 4.5\ \hat{k}$. In each case, find magnitude F of the magnetic force and angle θ the force vector $\vec{F}$ makes with the given magnetic field vector $\vec{B}$.

Strategy

First, we want to find the vector product $\vec{u} \times \vec{B}$, because then we can determine the magnetic force using $\vec{F} = \zeta\vec{u} \times \vec{B}$. Magnitude F can be found either by using components, F = $\sqrt{F_x^2 + F_y^2 + F_z^2}$, or by computing the magnitude $|\vec{u} \times \vec{B}|$ directly using Equation 5.2.1. In the latter approach, we would have to find the angle between vectors $\vec{u}$ and $\vec{B}$. When we have $\vec{F}$, the general method for finding the direction angle θ involves the computation of the scalar product $\vec{F} \cdot \vec{B}$ and substitution into Equation 2.8.13. To compute the vector product we can either use Equation 5.2.9 or compute the product directly, whichever way is simpler.

Solution

The components of the velocity vector are $u_x = -5.0$, $u_y = -2.0$, and $u_z = 3.5$. (a) The components of the magnetic field vector are $B_x = 7.2$, $B_y = -1.0$, and $B_z = -2.4$. Substituting them into Equation 5.2.10 gives the scalar components of vector $\vec{F} = \zeta\vec{u} \times \vec{B}$:

$$\begin{cases} F_x = \zeta(u_yB_z - u_zB_y) = \zeta[(-2.0)(-2.4) - (3.5)(-1.0)] = 8.3\zeta \\ F_y = \zeta(u_zB_x - u_xB_z) = \zeta[(3.5)(7.2) - (-5.0)(-2.4)] = 13.2\zeta \\ F_z = \zeta(u_xB_y - u_yB_x) = \zeta[(-5.0)(-1.0) - (-2.0)(7.2)] = 19.4\zeta \end{cases} \tag{5.2.11}$$

Thus, the magnetic force is $\vec{F} = \zeta(8.3\ \hat{i} + 13.2\ \hat{j} + 19.4\ \hat{k})$ and its magnitude is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \zeta\sqrt{(8.3)^2 + (13.2)^2 + (19.4)^2} = 24.9\zeta. \tag{5.2.12}$$

To compute angle θ, we may need to find the magnitude of the magnetic field vector

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(7.2)^2 + (-1.0)^2 + (-2.4)^2} = 7.6, \tag{5.2.13}$$

and the scalar product $\vec{F} \cdot \vec{B}$:

$$\vec{F} \cdot \vec{B} = F_xB_x + F_yB_y + F_zB_z = (8.3\zeta)(7.2) + (13.2\zeta)(-1.0) + (19.4\zeta)(-2.4) =. \tag{5.2.14}$$

Now, substituting into Equation 2.8.13 gives angle θ:

$$\cos\theta = \frac{\vec{F} \cdot \vec{B}}{FB} = \frac{0}{(18.2\zeta)(7.6)} = 0 \Rightarrow \theta = 90^o. \tag{5.2.15}$$

Hence, the magnetic force vector is perpendicular to the magnetic field vector. (We could have saved some time if we had computed the scalar product earlier.)

(b) Because vector $\vec{B} = 4.5\ \hat{k}$ has only one component, we can perform the algebra quickly and find the vector product directly:

$$\begin{aligned} \vec{F} &= \zeta\vec{u} \times \vec{B} = \zeta(-5.0\hat{i} - 2.0\hat{j} + 3.5\hat{k}) \times (4.5\hat{k}) \\ &= \zeta[(-5.0)(4.5)\hat{i} \times \hat{k} + (-2.0)(4.5)\hat{j} \times \hat{k} + (3.5)(4.5)\hat{k} \times \hat{k}] \\ &= \zeta[-22.5(-\hat{j}) - 9.0(+\hat{i}) + 0] = \zeta(-9.0\hat{i} + 22.5\hat{j}). \end{aligned}$$

The magnitude of the magnetic force is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \zeta\sqrt{(-9.0)^2 + (22.5)^2 + (0.0)^2} = 24.2\zeta. \tag{5.2.16}$$

Because the scalar product is

$$\vec{F} \cdot \vec{B} = F_x B_x + F_y B_y + F_z B_z = (-9.0\zeta)(90) + (22.5\zeta)(0) + (0)(4.5) = 0, \tag{5.2.17}$$

the magnetic force vector $\vec{F}$ is perpendicular to the magnetic field vector $\vec{B}$.

Significance

Even without actually computing the scalar product, we can predict that the magnetic force vector must always be perpendicular to the magnetic field vector because of the way this vector is constructed. Namely, the magnetic force vector is the vector product $\vec{F} = \zeta\vec{u} \times \vec{B}$ and, by the definition of the vector product (see Figure 5.2.1), vector $\vec{F}$ must be perpendicular to both vectors $\vec{u}$ and $\vec{B}$.

? Exercise 2.16

Given two vectors $\vec{A} = -\hat{i} + \hat{j}$ and $\vec{B} = 3\hat{i} - \hat{j}$, find (a) $\vec{A} \times \vec{B}$, (b) $|\vec{A} \times (\vec{B}|$, (c) the angle between $\vec{A}$ and $\vec{B}$, and (d) the angle between $\vec{A} \times \vec{B}$ and vector $\vec{C} = \hat{i} + \hat{k}$.

In conclusion to this section, we want to stress that "dot product" and "cross product" are entirely different mathematical objects that have different meanings. The dot product is a scalar; the cross product is a vector. Later chapters use the terms **dot product** and **scalar product** interchangeably. Similarly, the terms **cross product** and **vector product** are used interchangeably.

Contributors

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