2.3: Rules of Exponents and Scientific Notation

Learning Objectives

- Review the rules of exponents.
- Review the definition of negative exponents and zero as an exponent.
- Work with numbers using scientific notation.

Review of the Rules of Exponents

In this section, we review the rules of exponents. Recall that if a factor is repeated multiple times, then the product can be written in exponential form x^n. The positive integer exponent n indicates the number of times the base x is repeated as a factor.

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n\, times}$$

Consider the product of x^4 and x^6,

$$x^4 \cdot x^6 = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{10\ factors\ of\ x} = x^{10}$$

Expanding the expression using the definition produces multiple factors of the base which is quite cumbersome, particularly when n is large. For this reason, we have useful rules to help us simplify expressions with exponents. In this example, notice that we could obtain the same result by adding the exponents.

$$x^4 \cdot x^6 = x^{4+6} = x^{10} \quad \textit{Product rule for exponents}$$

In general, this describes the **product rule for exponents**[1]. In other words, when multiplying two expressions with the same base we add the exponents. Compare this to raising a factor involving an exponent to a power, such as $\left(x^6\right)^4$.

$$\begin{aligned} \left(x^6\right)^4 &= \underbrace{x^6 \cdot x^6 \cdot x^6 \cdot x^6}_{4\ factors\ of\ x^6} \\ &= x^{6+6+6+6} \\ &= x^{24} \end{aligned}$$

Here we have 4 factors of x^6, which is equivalent to multiplying the exponents.

$$\left(x^6\right)^4 = x^{6 \cdot 4} = x^{24} \quad \textit{Power rule for exponents}$$

This describes the **power rule for exponents**[2]. Now we consider raising grouped products to a power. For example,

$$\begin{aligned} \left(x^2y^3\right)^4 &= x^2y^3 \cdot x^2y^3 \cdot x^2y^3 \cdot x^2y^3 \\ &= x^2 \cdot x^2 \cdot x^2 \cdot x^2 \cdot y^3 \cdot y^3 \cdot y^3 \cdot y^3 \qquad \textit{Commutative property} \\ &= x^{2+2+2+2} \cdot y^{3+3+3+3} \\ &= x^8y^{12} \end{aligned}$$

After expanding, we are left with four factors of the product x^2y^3. This is equivalent to raising each of the original grouped factors to the fourth power and applying the power rule.

$$\left(x^2y^3\right)^4 = \left(x^2\right)^4\left(y^3\right)^4 = x^8y^{12}$$

In general, this describes the use of the power rule for a product as well as the power rule for exponents. In summary, the rules of exponents streamline the process of working with algebraic expressions and will be used extensively as we move through our study of algebra. Given any positive integers m and n where $x, y \neq 0$ we have

Table 2.3.1

Product rule for exponents:	$x^m \cdot x^n = x^{m+n}$
Quotient rule for exponents:	$\frac{x^m}{x^n} = x^{m-n}$
Power rule for exponents:	$\left(x^m\right)^n = x^{m \cdot n}$

Power rule for a product:[3]	$(xy)^n = x^n y^n$
Power rule for a quotient:[4]	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

These rules allow us to efficiently perform operations with exponents.

2.3.1 Example :

Simplify: $\frac{10^4 \cdot 10^{12}}{10^3}$

Answer

10^{13}

In the previous example, notice that we did not multiply the base 10 times itself. When applying the product rule, add the exponents and leave the base unchanged.

2.3.2 Example :

Simplify: $\left(x^5 \cdot x^4 \cdot x\right)^2$

Solution: Recall that the variable x is assumed to have an exponent of one, $x = x^1$.

Answer

x^{20}

The base could in fact be any algebraic expression.

2.3.3 Example :

Simplify: $(x+y)^9(x+y)^{13}$

Solution: Treat the expression $(x+y)$ as the base.

Answer

$(x+y)^{22}$

The commutative property of multiplication allows us to use the product rule for exponents to simplify factors of an algebraic expression.

2.3.4 Example :

Simplify: $-8x^5y \cdot 3x^7y^3$

Answer

$-24x^{12}y^4$

Division involves the quotient rule for exponents.

2.3.5 Example :

Simplify: $\frac{33x^7y^5(x-y)^{10}}{11x^6y(x-y)^3}$.

Answer

$3xy^4(x-y)^7$

The power rule for a quotient allows us to apply that exponent to the numerator and denominator. This rule requires that the denominator is nonzero and so we will make this assumption for the remainder of the section.

2.3.6 Example :

Simplify: $\left(\frac{-4a^2b}{c^4}\right)^3$

Answer

$-\frac{64a^6b^3}{c^{12}}$

Using the quotient rule for exponents, we can define what it means to have zero as an exponent. Consider the following calculation:

Twenty-five divided by twenty-five is clearly equal to one, and when the quotient rule for exponents is applied, we see that a zero exponent results. In general, given any nonzero real number x and integer n,

This leads us to the definition of **zero as an exponent**[5],

$$x^0 = 1 \qquad x \neq 0$$

It is important to note that 0^0 is indeterminate. If the base is negative, then the result is still positive one. In other words, any nonzero base raised to the zero power is defined to be equal to one. In the following examples assume all variables are nonzero.

2.3.7 Example :

Simplify:

a. $(-2x)^0$
b. $-2x^0$

Answer

a. Any nonzero quantity raised to the zero power is equal to 1.

b. In the example, $-2x^0$, the base is x, not $-2x$.

In general, given any nonzero real number x and integer n, the definition of **negative exponents**[6] is:

$$x^{-n} = \frac{1}{x^n} \qquad x \neq 0$$

An expression is completely simplified if it does not contain any negative exponents.

2.3.8 Example :

Simplify: $\left(-4x^2y\right)^{-2}$

Solution

Rewrite the entire quantity in the denominator with an exponent of 2 and then simplify further.

Answer

$\frac{1}{16x^4y^2}$

Sometimes negative exponents appear in the denominator.

2.3.9 Example :

Simplify: $\frac{x^{-3}}{y^{-4}}$

Answer

$\frac{y^4}{x^3}$

The previous example suggests a property of **quotients with negative exponents**[7]. Given any integers m and n where $x \neq 0$ and $y \neq 0$, then

$$\frac{x^{-n}}{y^{-m}} = \frac{\frac{1}{x^n}}{\frac{1}{y^m}} = \frac{1}{x^n} \cdot \frac{y^m}{1} = \frac{y^m}{x^n}$$

This leads us to the property

$$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$$

In other words, negative exponents in the numerator can be written as positive exponents in the denominator and negative exponents in the denominator can be written as positive exponents in the numerator.

2.3.10 Example :

Simplify: $\frac{-5x^{-3}y^3}{z^{-4}}$

Solution

Take care with the coefficient -5, recognize that this is the base and that the exponent is actually positive one: $-5 = (-5)^1$. Hence, the rules of negative exponents do not apply to this coefficient; leave it in the numerator.

Answer

$\frac{-5y^3z^4}{x^3}$

2.3.11 Exercise

Simplify: $\left(\frac{2x^{-2}y^3}{z}\right)^{-4}$

Answer

$\frac{x^8z^4}{16y^{12}}$

Scientific Notation

Real numbers expressed using **scientific notation**[8] have the form,

$$a \times 10^n$$

where n is an integer and $1 \leq a < 10$ .This form is particularly useful when the numbers are very large or very small. For example the number of atoms in a mole is 602200000000000000000000 $\frac{atoms}{mole}$ or Planck constant $\hbar$ which is 0.000000000000000000000000000000000663 $J \cdot sec$. It is cumbersome to write all the zeros in both of these cases. Scientific notation is an alternative, compact representation of these numbers. For scientific notation, the factor 10^n indicates the power of ten to multiply the coefficient by to convert back to decimal form:

$$9.46 \times 10^{15} = 9.46 \times 1,\overbrace{000,000,000,000,000}^{15\ zeros} = 9,460,000,000,000,000$$

This is equivalent to moving the decimal in the coefficient fifteen places to the right.

A negative exponent indicates that the number is very small:

$$2.5\times10^{-11} = 2.5\times\frac{1}{10^{11}} = \frac{2.5}{\underbrace{100,000,000,000}_{11\ zeros}} = 0.000000000025$$

This is equivalent to moving the decimal in the coefficient eleven places to the left.

We will leave it to you to convert Planck constant and number of atoms per mole into scientific notation (you can check the internet for the correct answer).

2.3.12 Example :

Write 0.000003045 using scientific notation.

Answer

3.045×10^{-6}

Often we will need to perform operations when using numbers in scientific notation. All the rules of exponents developed so far also apply to numbers in scientific notation.

2.3.13 Example :

Multiply: $\left(4.36 \times 10^{-5}\right)\left(5.3 \times 10^{12}\right)$

Answer

2.3108×10^{8}

2.3.14 Example :

Divide: $\left(3.24 \times 10^{8}\right) \div \left(9.0 \times 10^{-3}\right)$

Answer

3.6×10^{10}

2.3.15 Example :

The speed of light is approximately 6.7×10^{8} miles per hour. Express this speed in miles per second.

Answer

The speed of light is approximately 1.9×10^{5} miles per second.

2.3.16 Example :

The Sun moves around the center of the galaxy in a nearly circular orbit. The distance from the center of our galaxy to the Sun is approximately $26,000$ light years. What is the circumference of the orbit of the Sun around the galaxy in meters?

Solution

One light-year measures 9.46×10^{15} meters. Therefore, multiply this by $26,000$ or 2.60×10^{4} to find the length of $26,000$ light years in meters.

$$\begin{aligned}\left(9.46 \times 10^{15}\right)\left(2.60 \times 10^{4}\right) &= 9.46 \cdot 2.60 \times 10^{15} \cdot 10^{4} \\ &\approx 24.6 \times 10^{19} \\ &= 2.46 \times 10^{1} \cdot 10^{19} \\ &= 2.46 \times 10^{20}\end{aligned}$$

The radius r of this very large circle is approximately 2.46×10^{20} meters. Use the formula $C = 2\pi r$ to calculate the circumference of the orbit.

$$\begin{aligned} C &= 2\pi r \\ &\approx 2(3.14)\left(2.46 \times 10^{20}\right) \\ &= 15.4 \times 10^{20} \\ &= 1.54 \times 10^{1} \cdot 10^{20} \\ &= 1.54 \times 10^{21} \end{aligned}$$

Answer

The circumference of the Sun's orbit is approximately 1.54×10^{21} meters.

2.3.17 Exercise

Divide: $\left(3.15 \times 10^{-5}\right) \div \left(12 \times 10^{-13}\right)$.

Answer

2.625×10^{7}

Key Takeaways

- When multiplying two quantities with the same base, add exponents: $x^{m} \cdot x^{n} = x^{m+n}$.
- When dividing two quantities with the same base, subtract exponents: $\frac{x^{m}}{x^{n}} = x^{m-n}$.
- When raising powers to powers, multiply exponents: $(x^{m})^{n} = x^{m \cdot n}$.
- When a grouped quantity involving multiplication and division is raised to a power, apply that power to all of the factors in the numerator and the denominator: $(xy)^{n} = x^{n}y^{n}$ and $\left(\frac{x}{y}\right)^{n} = \frac{x^{n}}{y^{n}}$.
- Any nonzero quantity raised to the 0 power is defined to be equal to $1 : x^{0} = 1$.
- Expressions with negative exponents in the numerator can be rewritten as expressions with positive exponents in the denominator: $x^{-n} = \frac{1}{x^{n}}$.
- Expressions with negative exponents in the denominator can be rewritten as expressions with positive exponents in the numerator: $\frac{1}{x^{-m}} = x^{m}$.
- Take care to distinguish negative coefficients from negative exponents.
- Scientific notation is particularly useful when working with numbers that are very large or very small.

2.3.3 Exercise

Simplify. (Assume all variables represent nonzero numbers.)

1. $10^{4} \cdot 10^{7}$
2. $7^{3} \cdot 7^{2}$
3. $\dfrac{10^{2} \cdot 10^{4}}{10^{5}}$
4. $\dfrac{7^{5} \cdot 7^{9}}{7^{2}}$
5. $x^{3} \cdot x^{2}$
6. $y^{5} \cdot y^{3}$
7. $\dfrac{a^{8} \cdot a^{6}}{a^{5}}$
8. $\dfrac{b^{4} \cdot b^{10}}{b^{8}}$
9. $\dfrac{x^{2n} \cdot x^{3n}}{x^{n}}$
10. $\dfrac{x^{n} \cdot x^{8n}}{x^{3n}}$

11. $\left(x^{5}\right)^{3}$
12. $\left(y^{4}\right)^{3}$
13. $\left(x^{4} y^{5}\right)^{3}$
14. $\left(x^{7} y\right)^{5}$
15. $\left(x^{2} y^{3} z^{4}\right)^{4}$
16. $\left(x y^{2} z^{3}\right)^{2}$
17. $\left(-5 x^{2} y z^{3}\right)^{2}$
18. $\left(-2 x y^{3} z^{4}\right)^{5}$
19. $\left(x^{2} y z^{5}\right)^{n}$
20. $\left(x y^{2} z^{3}\right)^{2 n}$
21. $\left(x \cdot x^{3} \cdot x^{2}\right)^{3}$
22. $\left(y^{2} \cdot y^{5} \cdot y\right)^{2}$
23. $\frac{a^{2} \cdot\left(a^{4}\right)^{2}}{a^{3}}$
24. $\frac{a \cdot a^{3} \cdot a^{2}}{\left(a^{2}\right)^{3}}$
25. $(2 x+3)^{4}(2 x+3)^{9}$
26. $(3 y-1)^{7}(3 y-1)^{2}$
27. $(a+b)^{3}(a+b)^{5}$
28. $(x-2 y)^{7}(x-2 y)^{3}$
29. $5 x^{2} y \cdot 3 x y^{2}$
30. $-10 x^{3} y^{2} \cdot 2 x y$
31. $-6 x^{2} y z^{3} \cdot 3 x y z^{4}$
32. $2 x y z^{2}\left(-4 x^{2} y^{2} z\right)$
33. $3 x^{n} y^{2 n} \cdot 5 x^{2} y$
34. $8 x^{5 n} y^{n} \cdot 2 x^{2 n} y$
35. $\frac{40 x^{5} y^{3} z}{4 x^{2} y^{2} z}$
36. $\frac{8 x^{2} y^{5} z^{3}}{16 x^{2} y z}$
37. $\frac{24 a^{8} b^{3}(a-5 b)^{10}}{8 a^{5} b^{3}(a-5 b)^{2}}$
38. $\frac{175 m^{9} n^{5}(m+n)^{7}}{25 m^{8} n(m+n)^{3}}$
39. $\left(-2 x^{4} y^{2} z\right)^{6}$
40. $\left(-3 x y^{4} z^{7}\right)^{5}$
41. $\left(\frac{-3 a b^{2}}{2 c^{3}}\right)^{3}$
42. $\left(\frac{-10 a^{3} b}{3 c^{2}}\right)^{2}$
43. $\left(\frac{-2 x y^{4}}{z^{3}}\right)^{4}$
44. $\left(\frac{-7 x^{9} y}{z^{4}}\right)^{3}$
45. $\left(\frac{x y^{2}}{z^{3}}\right)^{n}$
46. $\left(\frac{2 x^{2} y^{3}}{z}\right)^{n}$
47. $(-5 x)^{0}$

48. $\left(3x^2y\right)^0$
49. $-5x^0$
50. $3x^2y^0$
51. $\left(-2a^2b^0c^3\right)^5$
52. $\left(-3a^4b^2c^0\right)^4$
53. $\frac{\left(9x^3y^2z^0\right)^2}{3xy^2}$
54. $\frac{\left(-5x^0y^5z\right)^3}{25y^2z^0}$
55. $-2x^{-3}$
56. $(-2x)^{-2}$
57. $a^4\cdot a^{-5}\cdot a^2$
58. $b^{-8}\cdot b^3\cdot b^4$
59. $\frac{a^8\cdot a^{-3}}{a^{-6}}$
60. $\frac{b^{-10}\cdot b^4}{b^{-2}}$
61. $10x^{-3}y^2$
62. $-3x^{-5}y^{-2}$
63. $3x^{-2}y^2z^{-1}$
64. $-5x^{-4}y^{-2}z^2$
65. $\frac{25x^{-3}y^2}{5x^{-1}y^{-3}}$
66. $\frac{-9x^{-1}y^3z^{-5}}{3x^{-2}y^2z^{-1}}$
67. $\left(-5x^{-3}y^2z\right)^{-3}$
68. $\left(-7x^2y^{-5}z^{-2}\right)^{-2}$
69. $\left(\frac{2x^{-3}z}{y^2}\right)^{-5}$
70. $\left(\frac{5x^5z^{-2}}{2y^{-3}}\right)^{-3}$
71. $\left(\frac{12x^3y^2z}{2x^7yz^8}\right)^3$
72. $\left(\frac{150xy^8z^2}{90x^7y^2z}\right)^2$
73. $\left(\frac{-9a^{-3}b^4c^{-2}}{3a^3b^5c^{-7}}\right)^{-4}$
74. $\left(\frac{-15a^7b^5c^{-8}}{3a^{-6}b^2c^3}\right)^{-3}$

Answer

1. 10^{11}
3. 10
5. x^5
7. a^9
9. x^{4n}
11. x^{15}
13. $x^{12}y^{15}$
15. $x^8y^{12}z^{16}$

17. $25x^4y^2z^6$

19. $x^{2n}y^nz^{5n}$

21. x^{18}

23. a^7

25. $(2x+3)^{13}$

27. $(a+b)^8$

29. $15x^3y^3$

31. $-18x^3y^2z^7$

33. $15x^{n+2}y^{2n+1}$

35. $10x^3y$

37. $3a^3(a-5b)^8$

39. $64x^{24}y^{12}z^6$

41. $-\frac{27a^3b^6}{8c^9}$

43. $\frac{16x^4y^{16}}{z^{12}}$

45. $\frac{x^ny^{2n}}{z^{3n}}$

47. 1

49. -5

51. $-32a^{10}c^{15}$

53. $27x^5y^2$

55. $-\frac{2}{x^3}$

57. a

59. a^{11}

61. $\frac{10y^2}{x^3}$

63. $\frac{3y^2}{x^2z}$

65. $\frac{5y^5}{x^2}$

67. $-\frac{x^9}{125y^6z^3}$

69. $\frac{x^{15}y^{10}}{32z^5}$

71. $\frac{216y^3}{x^{12}z^{21}}$

73. $\frac{a^{24}b^4}{81c^{20}}$

2.3.4 Exercise

The value in dollars of a new mobile phone can be estimated by using the formula $V=210(2t+1)^{-1}$, where t is the number of years after purchase.

1. How much was the phone worth new?
2. How much will the phone be worth in 1 year?
3. How much will the phone be worth in 3 years?
4. How much will the phone be worth in 10 years?
5. How much will the phone be worth in 100 years?
6. According to the formula, will the phone ever be worthless? Explain.
7. The height of a particular right circular cone is equal to the square of the radius of the base, $h=r^2$. Find a formula for the volume in terms of r.
8. A sphere has a radius $r=3x^2$ .Find the volume in terms of x.

Answer

1. $\$210$
3. $\$30$
5. $\$1.04$
7. $V=\frac{1}{3}\pi r^4$

2.3.5 Exercise

Convert to a decimal number.

1. 5.2×10^8
2. 6.02×10^9
3. 1.02×10^{-6}
4. 7.44×10^{-5}

Answer

1. $520,000,000$
3. 0.00000102

2.3.6 Exercise

Rewrite using scientific notation.

1. $7,050,000$
2. $430,000,000,000$
3. 0.00005001
4. 0.000000231

Answer

1. 7.05×10^6
3. 5.001×10^{-5}

2.3.7 Exercise

Perform the operations.

1. $\left(1.2\times10^9\right)\left(3\times10^5\right)$
2. $\left(4.8\times10^{-5}\right)\left(1.6\times10^{20}\right)$

3. $\left(9.1 \times 10^{23}\right)\left(3 \times 10^{10}\right)$
4. $\left(5.5 \times 10^{12}\right)\left(7 \times 10^{-25}\right)$
5. $\frac{9.6 \times 10^{16}}{1.2 \times 10^{-4}}$
6. $\frac{4.8 \times 10^{-14}}{2.4 \times 10^{-6}}$
7. $\frac{4 \times 10^{-8}}{8 \times 10^{10}}$
8. $\frac{2.3 \times 10^{23}}{9.2 \times 10^{-3}}$
9. $987,000,000,000,000 \times 23,000,000$
10. $0.00000000024 \times 0.00000004$
11. $0.000000000522 \div 0.0000009$
12. $81,000,000,000 \div 0.0000648$
13. The population density of Earth refers to the number of people per square mile of land area. If the total land area on Earth is 5.751×10^{7} square miles and the population in 2007 was estimated to be 6.67×10^{9} people, then calculate the population density of Earth at that time.
14. In 2008 the population of New York City was estimated to be 8.364 million people. The total land area is 305 square miles. Calculate the population density of New York City.
15. The mass of Earth is 5.97×10^{24} kilograms and the mass of the Moon is 7.35×10^{22} kilograms. By what factor is the mass of Earth greater than the mass of the Moon?
16. The mass of the Sun is 1.99×10^{30} kilograms and the mass of Earth is 5.97×10^{24} kilograms. By what factor is the mass of the Sun greater than the mass of Earth? Express your answer in scientific notation.
17. The radius of the Sun is 4.322×10^{5} miles and the average distance from Earth to the Moon is 2.392×10^{5} miles. By what factor is the radius of the Sun larger than the average distance from Earth to the Moon?
18. One light year, 9.461×10^{15} meters, is the distance that light travels in a vacuum in one year. If the distance from our Sun to the nearest star, Proxima Centauri, is estimated to be 3.991×10^{16} meters, then calculate the number of years it would take light to travel that distance.
19. It is estimated that there are about 1 million ants per person on the planet. If the world population was estimated to be 6.67 billion people in 2007, then estimate the world ant population at that time.
20. The radius of the earth is 6.3×10^{6} meters and the radius of the sun is 7.0×10^{8} meters. By what factor is the radius of the Sun larger than the radius of the Earth?
21. A gigabyte is 1×10^{9} bytes and a megabyte is 1×10^{6} bytes. If the average song in the MP3 format consumes about 4.5 megabytes of storage, then how many songs will fit on a 4-gigabyte memory card?
22. Water weighs approximately 18 grams per mole. If one mole is about 6×10^{23} molecules, then approximate the weight of each molecule of water.

Answer

1. 3.6×10^{14}

3. 2.73×10^{34}

5. 8×10^{20}

7. 5×10^{-19}

9. 2.2701×10^{22}

11. 5.8×10^{-4}

13. About 116 people per square mile

15. 81.2

17. 1.807

19. $6.67 \times 10^{15} ants$

21. Approximately 889 songs

2.3.8 Exercise

1. Use numbers to show that $(x+y)^n \neq x^n + y^n$.
2. Why is 0^0 indeterminate?
3. Explain to a beginning algebra student why $2^2 \cdot 2^3 \neq 4^5$.
4. René Descartes (1637) established the usage of exponential form: a^2, a^3, and so on. Before this, how were exponents denoted?

Answer

1. Answer may vary

3. Answer may vary

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[1]$x^m \cdot x^n = x^{m+n}$; the product of two expressions with the same base can be simplified by adding the exponents.

[2]$(x^m)^n = x^{mn}$; a power raised to a power can be simplified by multiplying the exponents.

[3]$(xy)^n = x^n y^n$; if a product is raised to a power, then apply that power to each factor in the product.

[4]$(xy)^n = x^n y^n$; if a quotient is raised to a power, then apply that power to the numerator and the denominator.

[5]$x^0 = 1$; any nonzero base raised to the 0 power is defined to be 1.

[6]$x^{-n} = \frac{1}{x^n}$, given any integer n, where x is nonzero.

[7] $\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$, given any integers m and n, where $x \neq 0$ and $y \neq 0$.

[8]Real numbers expressed the form $a \times 10^n$, where n is an integer and $1 \leq a < 10$.

[9] $\frac{x^m}{x^n} = x^{m-n}$; the quotient of two expressions with the same base can be simplified by subtracting the exponents.
