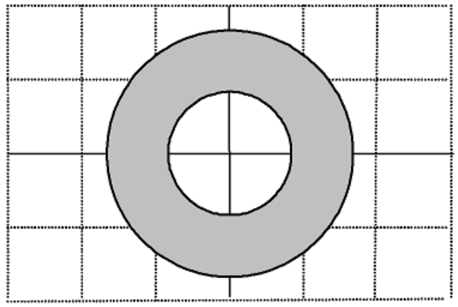
1.6: Analysis Tools - Gauss's Law

The long, hollow plastic cyclinder at right has inner radius a*, outer radius* b*, and uniform charge density* ρ. Find *the electric field at all points in a plane perpendicular to the cyclinder near its midpoint.*



For certain situations, typically ones with a high degree of symmetry, Gauss' Lae allows you to calculate the electric field relatively easily. Gauss's Law, mathematically, states:

$$\oint E \cdot dA = \frac{q_{enclosed}}{\epsilon_o} \tag{1.6.1}$$

Let's describe what this means in English. The left side of the equation involves the vector dot product between the electric field and an infinitesimally small area that is a piece of a larger closed surface (termed the *Gaussian surface*). This dot product between electric field and area is termed *electric flux*, and is often visualized as the amount of field that "passes through" the little piece of area. The integral simply tells us to add up all of these infinitesimal electric fluxes to get the total flux through the entire closed surface.

The gist of Gauss' Law is that this total electric flux is exactly equal to the total amount of electric charge enclosed within the gaussian surface, divided by a constant, e0. (e0 is the *permittivity of free space,* a constant equal to 8.85 x 10^{-12} C^2/Nm^2.)

Somewhat counter intuitively, the key to applying Gauss' Law is to choose a gaussian surface such that you never really have to do the integral on the left side of the equation! To try to help you understand what I'm talking about, let's walk through the solution of the above problem. The following sequence of steps will help you understand the process of applying Gauss's law:

1. *Choose the appropriate gaussian surface.*
2. *Carefully draw the hypothetical gaussian surface at the location of interest.*
3. *Carefully draw the electric field at all points on the gaussian surface.*
4. *Write an expression for the surface area parallel to the electric field.*
5. *Write an expression for* $q_{enclosed}$*, the charge enclosed within the gaussian surface.*
6. *Apply Gauss' Law and determine the electric field at all points on this hypothetical surface.*

Since we want to find the electric field at all points in space, there are three distinct regions we will have to investigate:

- the region inside the "hole" in the cylinder ($r < a$),
- the region within the actual material of the cylinder ($a < r < b$),
- and the region outside of the cylinder ($r > b$).

Let's start outside of the cylinder.

Outside of the cylinder: r > b

1. *Choose the appropriate gaussian surface.*

Gauss' law is primarily useful when the objects under investigation have a high degree of symmetry. Guass' law basically exploits that symmetry to make the calculation of the electric field relatively painless. To exploit the symmetry of the situation, you should always choose a gaussian surface to mimic the symmetry of the object you are investigating. In this example, since the object is a cylinder, my gaussian surfaces will all be cylinders.

2. *Carefully draw the hypothetical gaussian surface at the location of interest.*

Since we are trying to determine the electric field for all points outside of the cylinder, draw a cylindrical gaussian surface with radius r. The value of r is *variable*, and can take on any value greater than b, the radius of the *real* cylinder. Remember, the gaussian surface is *hypothetical*; it's a mathematical "object" that only exists to help you solve the problem. Try not to confuse it with the real cylinder of radius b.

The gaussian surface is represented from two separate viewpoints below. (The gray cylinder is the actual, charged cylinder while the dashed cylinder is the gaussian surface.) The gaussian surface has radius r, where r can be any value greater than b, and length L. It is located near the midpoint of the actual cylinder. The gaussian surface also includes the circular "end caps" of the cylinder since a gaussian surface must be a *closed* surface.

pic

3. *Carefully draw the electric field at all points on the gaussian surface.*

Although I have no idea what the *magnitude* of the electric field is at any point on my gaussian surface, the symmetry of the situation tells me that the *direction* of the electric field must be either radially away from or radially toward the central axis of the cylinder. If the charge density is positive, the field will be directed radially outward from the cylinder axis.

Moreover, even though I don't know the magnitude of the field, I *do* know that the magnitude is the same at every point on my surface.

pic

4. *Write an expression for the surface area parallel to the electric field.*

The left side of Gauss' law requires us to evaluate an integral over the surface of our gaussian cylinder. The integral requires us to find the dot product between the electric field and the area, and the integrate this dot product ove the entire surface. I mentioned earlier that you should never have to actually do this integral (assuming you chose the "correct" gaussian surface). So why don't we have to do this integral?

The vector dot product can be re-written as:

pic

where is the angle between the electric field and the area of the gaussian surface. You nay recall from calculus that the "direction" associated with area is perpendicular to its surface. Thus, the direction of each infinitesimal area is indicated on the diagrams below.

pic

Notice that the electric field vector and the area vector are parallel at every point on the cylindrical portion of the gaussian surface and perpendicular at every point on the circular end caps. Thus, breaking the integral into two parts, one over the cylindrical portion of the gaussian surface and one over the end caps, yields:

pic

Now note that the magnitude of the electric field is the same at every point on the cylindrical portion of the gaussian surface since every point is equal distance from the charge distribution. Thus, the electric field is constant and can be brought outside of the integral, leaving a pretty easy integral to evaluate.

pic

Notice that the entire left-hand side of Gauss' law reduces to the product of the electric field magnitude and the magnitude of the surface area parallel to this field. Thus, because of our wise choice of gaussian surface, all we really need to calculate is the magnitude of the surface area parallel to the electric field. In this case, the parallel area is given by:

pic

5. *Write an expression for $q_{enclosed}$ the charge enclosed within the gaussian surface.*

Since the gaussian surface is outside of the real cylinder, *all* of the charge on the cylinder within the length L is enclosed. Since we know the volume charge density on the cylinder, p, the total charge on the cylinder within the length L is the product of the charge density and the volume of the cylinder.

pic

6. *Apply Gauss' Law and determine the electric field at all points on this hypothetical surface.*

pic

Thus, the electric field outside of the cylinder is inversely proportional to the distance from the cylinder.

Now we have to repeat this analysis for the other two regions.

Within the cylinder: a < r < b

1. *Choose the appropriate gaussian surface.*

We will again use a cylindrical surface.

2. *Carefully draw the hypothetical gaussian surface at the location of interest.*

Since we are interested in the electric field within the actual cylinder, the radius of our gaussian surface is larger than a but less than b.

pic

3. *Carefully draw the electric field at all points on the gaussian surface.*

As before, the magnitude of the electric field must be constant on the gaussian surface and directed radially outward due to the symmetry of the situation.

pic

4. *Write an expression for the surface area parallel to the electric field.*

The area parallel to the electric field is again the surface area of the cylindrical portion of the gaussian surface:

pic

5. *Write an expression for $q_{enclosed}$ the charge enclosed within the gaussian surface.*

Since the gaussian surface is inside the real cylinder, not all of the charge on the real cylinder is enclosed by the gaussian surface. All of the charge between a and r is enclosed, but the charge between r and b is not enclosed by the gaussian surface. Thus, the amount enclosed is the product of the volume charge density and the volume of the portion of the cylinder enclosed by the gaussian surface.

pic

6. *Apply Gauss' Law and determine the electric field at all points on this hypothetical surface.*

pic

Notice that this electric field increases with increasing r, since as r increases, more and more charge is availablre to produce the electric field.

The last region we have to investigate is inside of the "hole" in the cylinder.

Inside the "hole": r < a

Since we must choose our gaussian surface to have a radius less than a, it is located inside the hollow center of the cylinder. Since there is *no* charge enclosed by this gaussian surface, the electric field in this region must be zero.

pic

Paul D'Alessandris (Monroe Community College)