

17.4: Inequalities

Basic Theory

In this section, we will study a number of interesting inequalities associated with martingales and their sub-martingale and super-martingale cousins. These turn out to be very important for both theoretical reasons and for applications. You may need to review infimums and supremums.

Basic Assumptions

As in the Introduction, we start with a stochastic process $\mathbf{X} = \{X_t : t \in T\}$ on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, having state space \mathbb{R} , and where the index set T (representing time) is either \mathbb{N} (discrete time) or $[0, \infty)$ (continuous time). Next, we have a filtration $\mathfrak{F} = \{\mathcal{F}_t : t \in T\}$, and we assume that \mathbf{X} is *adapted* to \mathfrak{F} . So \mathfrak{F} is an increasing family of sub σ -algebras of \mathcal{F} and X_t is measurable with respect to \mathcal{F}_t for $t \in T$. We think of \mathcal{F}_t as the collection of events up to time $t \in T$. We assume that $\mathbb{E}(|X_t|) < \infty$, so that the mean of X_t exists as a real number, for each $t \in T$. Finally, in continuous time where $T = [0, \infty)$, we make the standard assumptions that $t \mapsto X_t$ is right continuous and has left limits, and that the filtration \mathfrak{F} is right continuous and complete.

Maximal Inequalities

For motivation, let's review a modified version of Markov's inequality, named for Andrei Markov.

If X is a real-valued random variable then

$$\mathbb{P}(X \geq x) \leq \frac{1}{x} \mathbb{E}(X; X \geq x), \quad x \in (0, \infty) \quad (17.4.1)$$

Proof

The modified version has essentially the same elegant proof as the original. Clearly

$$x \mathbf{1}(X \geq x) \leq X \mathbf{1}(X \geq x), \quad x \in (0, \infty) \quad (17.4.2)$$

Taking expected values through the inequality gives $x \mathbb{P}(X \geq x) \leq \mathbb{E}(X; X \geq x)$. Dividing both sides by x gives the result (and it is at this point that we need $x > 0$).

So Markov's inequality gives an upper bound on the probability that X exceeds a given positive value x , in terms of a moment of X . Now let's return to our stochastic process $\mathbf{X} = \{X_t : t \in T\}$. To simplify the notation, let $T_t = \{s \in T : s \leq t\}$ for $t \in T$. Here is the main definition:

For the process \mathbf{X} , define the corresponding *maximal process* $\mathbf{U} = \{U_t : t \in T\}$ by

$$U_t = \sup\{X_s : s \in T_t\}, \quad t \in T \quad (17.4.3)$$

Clearly, the maximal process is increasing, so that if $s, t \in T$ with $s \leq t$ then $U_s \leq U_t$. A trivial application of Markov's inequality above would give

$$\mathbb{P}(U_t \geq x) \leq \frac{1}{x} \mathbb{E}(U_t; U_t \geq x), \quad x > 0 \quad (17.4.4)$$

But when \mathbf{X} is a sub-martingale, the following theorem gives a much stronger result by replacing the first occurrence of U_t on the right with X_t . The theorem is known as *Doob's sub-martingale maximal inequality* (or more simply as *Doob's inequality*), named once again for Joseph Doob who did much of the pioneering work on martingales. A sub-martingale has an *increasing property* of sorts in the sense that if $s, t \in T$ with $s \leq t$ then $\mathbb{E}(X_t | \mathcal{F}_s) \geq X_s$, so it's perhaps not entirely surprising that such a bound is possible.

Suppose that \mathbf{X} is a sub-martingale. For $t \in T$, let $U_t = \sup\{X_s : s \in T_t\}$. Then

$$\mathbb{P}(U_t \geq x) \leq \frac{1}{x} \mathbb{E}(X_t; U_t \geq x), \quad x \in (0, \infty) \quad (17.4.5)$$

Proof in the discrete time

So $T = \mathbb{N}$ and the maximal process is given by $U_n = \max\{X_k : k \in \mathbb{N}_n\}$ for $n \in \mathbb{N}$. Let $x \in (0, \infty)$, and define $\tau_x = \min\{k \in \mathbb{N} : X_k \geq x\}$ where as usual, $\min(\emptyset) = \infty$. The random time τ_x is a stopping time relative to \mathfrak{F} . Moreover, the processes $\{U_n : n \in \mathbb{N}\}$ and $\{\tau_x : x \in (0, \infty)\}$ are inverses in the sense that for $n \in \mathbb{N}$ and $x \in (0, \infty)$,

$$U_n \geq x \text{ if and only if } \tau_x \leq n \quad (17.4.6)$$

We have seen this type of duality before—in the Poisson process and more generally in renewal processes. Let $n \in \mathbb{N}$. First note that

$$\mathbb{E}(X_{\tau_x \wedge n}) = \mathbb{E}(X_{\tau_x \wedge n}; \tau_x \leq n) + \mathbb{E}(X_{\tau_x \wedge n}; \tau_x > n) \quad (17.4.7)$$

If $\tau_x \leq n$ then $X_{\tau_x \wedge n} = X_{\tau_x} \geq x$. On the other hand if $\tau_x > n$ then $X_{\tau_x \wedge n} = X_n$. So we have

$$\mathbb{E}(X_{\tau_x \wedge n}) \geq x\mathbb{P}(\tau_x \leq n) + \mathbb{E}(X_n; \tau_x > n) = x\mathbb{P}(U_t \geq x) + \mathbb{E}(X_n; \tau_x > n) \quad (17.4.8)$$

Similarly,

$$\mathbb{E}(X_n) = \mathbb{E}(X_n; \tau_x \leq n) + \mathbb{E}(X_n; \tau_x > n) = \mathbb{E}(X_n; U_t \geq x) + \mathbb{E}(X_n; \tau_x > n) \quad (17.4.9)$$

But by the optional stopping theorem, $\mathbb{E}(X_{\tau_x \wedge n}) \leq \mathbb{E}(X_n)$. Hence we have

$$x\mathbb{P}(U_t \geq x) + \mathbb{E}(X_n; \tau_x > n) \leq \mathbb{E}(X_n; U_t \geq x) + \mathbb{E}(X_n; \tau_x > n) \quad (17.4.10)$$

Subtracting the common term and then dividing both sides by x gives the result

Proof in continuous time

For $k \in \mathbb{N}$, let $\mathbb{D}_k^+ = \{j/2^k : j \in \mathbb{N}\}$ denote the set of nonnegative dyadic rationals (or *binary rationals*) of rank k or less. For $t \in [0, \infty)$ let $T_t^k = (\mathbb{D}_k^+ \cap [0, t]) \cup \{t\}$, so that T_t^k is the finite set of such dyadic rationals that are less than t , with t added to the set. Note that T_t^k has an ordered enumeration, so $\mathbf{X}^k = \{X_s : s \in T_t^k\}$ is a discrete-time sub-martingale for each $k \in \mathbb{N}$. Let $U_t^k = \sup\{X_s : s \in T_t^k\}$ for $k \in \mathbb{N}$. Note that $T_t^j \subset T_t^k \subset [0, t]$ for $t \in [0, \infty)$ and for $j, k \in \mathbb{N}$ with $j < k$ and therefore $U_t^j \leq U_t^k \leq U_t$. It follows that for $x \in (0, \infty)$,

$$\{U_t^j \geq x\} \subseteq \{U_t^k \geq x\} \subseteq \{U_t \geq x\} \quad (17.4.11)$$

The set \mathbb{D}^+ of all nonnegative dyadic rationals is dense in $[0, \infty)$ and so since \mathbf{X} is right continuous and has left limits, it follows that if $U_t \geq x$ then $U_t^k \geq x$ for some $k \in \mathbb{N}$. That is, we have

$$\{U_t \geq x\} = \bigcup_{k=0}^{\infty} \{U_t^k \geq x\} \quad (17.4.12)$$

The maximal inequality applies to the discrete-time sub-martingale \mathbf{X}^k and so

$$P(U_t^k \geq x) \leq \frac{1}{x} \mathbb{E}(X_t; U_t^k \geq x) \quad (17.4.13)$$

for each $k \in \mathbb{N}$. By the monotone convergence theorem, the left side converges to $\mathbb{P}(U_t \geq x)$ as $k \rightarrow \infty$ and the right side converges to $\mathbb{E}(X; U_t \geq x)$ as $k \rightarrow \infty$.

There are a number of simple corollaries of the maximal inequality. For the first, recall that the *positive part* of $x \in \mathbb{R}$ is $x^+ = x \vee 0$, so that $x^+ = x$ if $x > 0$ and $x^+ = 0$ if $x \leq 0$.

Suppose that \mathbf{X} is a sub-martingale. For $t \in T$, let $V_t = \sup\{X_s^+ : s \in T_t\}$. Then

$$\mathbb{P}(V_t \geq x) \leq \frac{1}{x} \mathbb{E}(X_t^+; V_t \geq x), \quad x \in (0, \infty) \quad (17.4.14)$$

Proof

Recall that since \mathbf{X} is a sub-martingale and $x \mapsto x^+$ is increasing and convex, $\mathbf{X}^+ = \{X_t^+ : t \in T\}$ is also a sub-martingale. Hence the result follows from the general maximal inequality for sub-martingales.

As a further simple corollary, note that

$$\mathbb{P}(V_t \geq x) \leq \frac{1}{x} \mathbb{E}(X_t^+), \quad x \in (0, \infty) \quad (17.4.15)$$

This is sometimes how the maximal inequality is given in the literature.

Suppose that \mathbf{X} is a martingale. For $t \in T$, let $W_t = \sup\{|X_s| : s \in T_t\}$. Then

$$\mathbb{P}(W_t \geq x) \leq \frac{1}{x} \mathbb{E}(|X_t|; W_t \geq x), \quad x \in (0, \infty) \quad (17.4.16)$$

Proof

Recall that since \mathbf{X} is a martingale, and $x \mapsto |x|$ is convex, $|\mathbf{X}| = \{|X_t| : t \in T\}$ is a sub-martingale. Hence the result follows from the general maximal inequality for sub-martingales.

Once again, a further simple corollary is

$$\mathbb{P}(W_t \geq x) \leq \frac{1}{x} \mathbb{E}(|X_t|), \quad x \in (0, \infty) \quad (17.4.17)$$

Next recall that for $k \in (1, \infty)$, the k -norm of a real-valued random variable X is $\|X\|_k = [\mathbb{E}(|X|^k)]^{1/k}$, and the vector space \mathcal{L}_k consists of all real-valued random variables for which this norm is finite. The following theorem is the *norm version* of the Doob's maximal inequality.

Suppose again that \mathbf{X} is a martingale. For $t \in T$, let $W_t = \sup\{|X_s| : s \in T_t\}$. Then for $k > 1$,

$$\|W_t\|_k \leq \frac{k}{k-1} \|X_t\|_k \quad (17.4.18)$$

Proof

Fix $t \in T$. If $\mathbb{E}(|X_t|^k) = \infty$, the inequality trivial holds, so assume that $\mathbb{E}(|X_t|^k) < \infty$, and thus that $X_t \in \mathcal{L}_k$. The proof relies fundamentally on Hölder's inequality, and for that inequality to work, we need to truncate the variable W_t and consider instead the bounded random variable $W_t \wedge c$ where $c \in (0, \infty)$. First we need to show that

$$\mathbb{P}(W_t \wedge c \geq x) \leq \frac{1}{x} \mathbb{E}(|X_t|; W_t \wedge c \geq x), \quad x \in (0, \infty) \quad (17.4.19)$$

If $c < x$, both sides are 0. If $c \geq x$, $\{W_t \wedge c \geq x\} = \{W_t \geq x\}$ and so from the [maximal inequality](#) above,

$$\mathbb{P}(W_t \wedge c \geq x) = \mathbb{P}(W_t \geq x) \leq \frac{1}{x} \mathbb{E}(|X_t|; W_t \geq x) = \mathbb{E}(|X_t|; W_t \wedge c \geq x) \quad (17.4.20)$$

Next recall that

$$\|W_t \wedge c\|_k^k = \mathbb{E}[(W_t \wedge c)^k] = \int_0^\infty kx^{k-1} \mathbb{P}(W_t \wedge c \geq x) dx \quad (17.4.21)$$

Applying the inequality gives

$$\mathbb{E}[(W_t \wedge c)^k] \leq \int_0^\infty kx^{k-2} \mathbb{E}[|X_t|; W_t \wedge c \geq x] dx \quad (17.4.22)$$

By Fubini's theorem we can interchange the expected value and the integral which gives

$$\mathbb{E}[(W_t \wedge c)^k] \leq \mathbb{E} \left[\int_0^{W_t \wedge c} kx^{k-2} |X_t| dx \right] = \frac{k}{k-1} \mathbb{E}[|X_t| (W_t \wedge c)^{k-1}] \quad (17.4.23)$$

But $X_t \in \mathcal{L}_k$ and $(W_t \wedge c)^{k-1} \in \mathcal{L}_j$ where $j = k/(k-1)$ is the exponent conjugate to k . So an application of Hölder's inequality gives

$$\|W_t \wedge c\|_k^k \leq \frac{k}{k-1} \|X_t\|_k \|(W_t \wedge c)^{k-1}\|_j = \frac{k}{k-1} \|X_t\|_k \|W_t \wedge c\|_k^{k-1} \quad (17.4.24)$$

where we have used the simple fact that $\|(W_t \wedge c)^{k-1}\|_j = \|W_t \wedge c\|_k^{k-1}$. Dividing by this factor gives

$$\|W_t \wedge c\|_k \leq \frac{k}{k-1} \|X_t\|_k \quad (17.4.25)$$

Finally, $\|W_t \wedge c\|_k \uparrow \|W_t\|_k$ as $c \rightarrow \infty$ by the monotone convergence theorem. So letting $c \rightarrow \infty$ in the last displayed equation gives

$$\|W_t\|_k \leq \frac{k}{k-1} \|X_t\|_k \quad (17.4.26)$$

Once again, $\mathbf{W} = \{W_t : t \in T\}$ is the maximal process associated with $|\mathbf{X}| = \{|X_t| : t \in T\}$. As noted in the proof, $j = k/(k-1)$ is the exponent *conjugate* to k , satisfying $1/j + 1/k = 1$. So this version of the maximal inequality states that the k norm of the maximum of the martingale \mathbf{X} on T_t is bounded by j times the k norm of X_t , where j and k are conjugate exponents. Stated just in terms of expected value, rather than norms, the \mathcal{L}_k maximal inequality is

$$\mathbb{E}(|W_t|^k) \leq \left(\frac{k}{k-1}\right)^k \mathbb{E}(|X_t|^k) \quad (17.4.27)$$

Our final result in this discussion is a variation of the maximal inequality for super-martingales.

Suppose that $\mathbf{X} = \{X_t : t \in T\}$ is a nonnegative super-martingale, and let $U_\infty = \sup\{X_t : t \in T\}$. Then

$$\mathbb{P}(U_\infty \geq x) \leq \frac{1}{x} \mathbb{E}(X_0), \quad x \in (0, \infty) \quad (17.4.28)$$

Proof

Let $Y_t = -X_t$ for $t \in T$. Since \mathbf{X} is a super-martingale, \mathbf{Y} is a sub-martingale. And since \mathbf{X} is nonnegative, $Y_t^+ = X_t$ for $t \in T$. Let $U_t = \sup\{X_s : s \in T_t\} = \sup\{Y_s^+ : s \in T_t\}$ for $t \in T$. By the maximal inequality for sub-martingales, and since \mathbf{X} is a super-martingale we have for $t \in T$,

$$\mathbb{P}(U_t \geq x) \leq \frac{1}{x} \mathbb{E}(Y_t^+) = \frac{1}{x} \mathbb{E}(X_t) \leq \frac{1}{x} \mathbb{E}(X_0), \quad x \in (0, \infty) \quad (17.4.29)$$

Next note that $U_t \uparrow U_\infty$ as $t \rightarrow \infty$. Let $x \in (0, \infty)$ and $\epsilon \in (0, x)$. If $U_\infty \geq x$ then $U_t \geq x - \epsilon$ for sufficiently large $t \in T$. Hence

$$\{U_\infty \geq x\} \subseteq \bigcup_{k=1}^{\infty} \{U_k \geq x - \epsilon\} \quad (17.4.30)$$

Using the continuity theorem for increasing events, and our result above we have

$$\mathbb{P}(U_\infty \geq x) \leq \lim_{k \rightarrow \infty} \mathbb{P}(U_k \geq x - \epsilon) \leq \frac{1}{x - \epsilon} \mathbb{E}(X_0) \quad (17.4.31)$$

Since this holds for all $\epsilon \in (0, x)$, it follows that $\mathbb{P}(U_\infty \geq x) \leq \frac{1}{x} \mathbb{E}(X_0)$.

The Up-Crossing Inequality

The up-crossing inequality gives a bound on how much a sub-martingale (or super-martingale) can oscillate, and is the main tool in the martingale convergence theorems that will be studied in the next section. It should come as no surprise by now that the inequality is due to Joseph Doob. We start with the discrete-time case.

Suppose that $\mathbf{x} = (x_n : n \in \mathbb{N})$ is a sequence of real numbers, and that $a, b \in \mathbb{R}$ with $a < b$. Define $t_0(\mathbf{x}) = 0$ and then recursively define

$$\begin{aligned} s_{k+1}(\mathbf{x}) &= \inf\{n \in \mathbb{N} : n \geq t_k(\mathbf{x}), x_n \leq a\}, \quad k \in \mathbb{N} \\ t_{k+1}(\mathbf{x}) &= \inf\{n \in \mathbb{N} : n \geq s_{k+1}(\mathbf{x}), x_n \geq b\}, \quad k \in \mathbb{N} \end{aligned}$$

1. The number of up-crossings of the interval $[a, b]$ by the sequence \mathbf{x} up to time $n \in \mathbb{N}$ is

$$u_n(a, b, \mathbf{x}) = \sup\{k \in \mathbb{N} : t_k(\mathbf{x}) \leq n\} \quad (17.4.32)$$

2. The total number of up-crossings of the interval $[a, b]$ by the sequence \mathbf{x} is

$$u_{\infty}(a, b, \mathbf{x}) = \sup\{k \in \mathbb{N} : t_k(\mathbf{x}) < \infty\} \quad (17.4.33)$$

Details

As usual, we define $\inf(\emptyset) = \infty$. Note that if $t_k(\mathbf{x}) < \infty$ for $k \in \mathbb{N}_+$, then $(x_n : n = s_k(\mathbf{x}), \dots, t_k(\mathbf{x}))$ is the k th up-crossing of the interval $[a, b]$ by the sequence \mathbf{x} .

So informally, as the name suggests, $u_n(a, b, \mathbf{x})$ is the number of times that the sequence (x_0, x_1, \dots, x_n) goes from a value below a to one above b , and $u(a, b, \mathbf{x})$ is the number of times the entire sequence \mathbf{x} goes from a value below a to one above b . Here are a few of simple properties:

Suppose again that $\mathbf{x} = (x_n : n \in \mathbb{N})$ is a sequence of real numbers and that $a, b \in \mathbb{R}$ with $a < b$.

1. $u_n(a, b, \mathbf{x})$ is increasing in $n \in \mathbb{N}$.
2. $u_n(a, b, \mathbf{x}) \rightarrow u(a, b, \mathbf{x})$ as $n \rightarrow \infty$.
3. If $c, d \in \mathbb{R}$ with $a < c < d < b$ then $u_n(c, d, \mathbf{x}) \geq u_n(a, b, \mathbf{x})$ for $n \in \mathbb{N}$, and $u(c, d, \mathbf{x}) \geq u(a, b, \mathbf{x})$.

Proof

1. Note that $\{k \in \mathbb{N} : t_k(\mathbf{x}) \leq n\} \subseteq \{k \in \mathbb{N} : t_k(\mathbf{x}) \leq n+1\}$.
2. Note that $\bigcup_{n=0}^{\infty} \{k \in \mathbb{N} : t_k(\mathbf{x}) \leq n\} = \{k \in \mathbb{N} : t_k(\mathbf{x}) < \infty\}$.
3. Every up-crossing of $[a, b]$ is also an up-crossing of $[c, d]$.

The importance of the definitions is found in the following theorem. Recall that $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ is the set of extended real numbers, and \mathbb{Q} is the set of rational real numbers.

Suppose again that $\mathbf{x} = (x_n : n \in \mathbb{N})$ is a sequence of real numbers. Then $\lim_{n \rightarrow \infty} x_n$ exists in \mathbb{R}^* is and only if $u_{\infty}(a, b, \mathbf{x}) < \infty$ for every $a, b \in \mathbb{Q}$ with $a < b$.

Proof

We prove the contrapositive. Note that the following statements are equivalent:

1. $\lim_{n \rightarrow \infty} x_n$ does not exist in \mathbb{R}^* .
2. $\liminf_{n \rightarrow \infty} x_n < \limsup_{n \rightarrow \infty} x_n$.
3. There exists $a, b \in \mathbb{Q}$ with $a < b$ and with $x_n \leq a$ for infinitely many $n \in \mathbb{N}$ and $x_n \geq b$ for infinitely many $n \in \mathbb{N}$.
4. There exists $a, b \in \mathbb{Q}$ with $a < b$ and $u_{\infty}(a, b, \mathbf{x}) = \infty$.

Clearly the theorem is true with \mathbb{Q} replaced with \mathbb{R} , but the countability of \mathbb{Q} will be important in the martingale convergence theorem. As a simple corollary, if \mathbf{x} is bounded and $u_{\infty}(a, b, \mathbf{x}) < \infty$ for every $a, b \in \mathbb{Q}$ with $a < b$, then \mathbf{x} converges in \mathbb{R} . The up-crossing inequality for a discrete-time martingale \mathbf{X} gives an upper bound on the expected number of up-crossings of \mathbf{X} up to time $n \in \mathbb{N}$ in terms of a moment of X_n .

Suppose that $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$ satisfies the basic assumptions with respect to the filtration $\mathfrak{F} = \{\mathcal{F}_n : n \in \mathbb{N}\}$, and let $a, b \in \mathbb{R}$ with $a < b$. Let $U_n = u_n(a, b, \mathbf{X})$, the random number of up-crossings of $[a, b]$ by \mathbf{X} up to time $n \in \mathbb{N}$.

1. If \mathbf{X} is a super-martingale relative to \mathfrak{F} then

$$\mathbb{E}(U_n) \leq \frac{1}{b-a} \mathbb{E}[(X_n - a)^-] \leq \frac{1}{b-a} [\mathbb{E}(X_n^-) + |a|] \leq \frac{1}{b-a} [\mathbb{E}(|X_n|) + |a|], \quad n \in \mathbb{N} \quad (17.4.34)$$

2. If \mathbf{X} is a sub-martingale relative to \mathfrak{F} then

$$\mathbb{E}(U_n) \leq \frac{1}{b-a} \mathbb{E}[(X_n - a)^+] \leq \frac{1}{b-a} [\mathbb{E}(X_n^+) + |a|] \leq \frac{1}{b-a} [\mathbb{E}(|X_n|) + |a|], \quad n \in \mathbb{N} \quad (17.4.35)$$

Proof

In the context of the up-crossing definition [above](#), let $\sigma_k = s_k(\mathbf{X})$ and $\tau_k = t_k(\mathbf{X})$. These are the random times that define the up-crossings of \mathbf{X} . Let $Y_k = X_{\tau_k \wedge n} - X_{\sigma_k \wedge n}$ and then define $Z_n = \sum_{k=1}^n Y_k$. To understand the sum, let's take cases for the k th term Y_k :

- If $\tau_k \leq n$ then $Y_k = X_{\tau_k} - X_{\sigma_k} \geq b - a$. By definition, the first U_n terms are of this form.
- If $\sigma_k \leq n < \tau_k$ then $Y_k = X_n - X_{\sigma_k} \geq X_n - a$. There is at most one such term, with index $k = U_n + 1$.
- If $\sigma_k > n$ then $Y_k = X_n - X_n = 0$.

Hence $Z_n \geq (b-a)U_n + (X_n - a)\mathbf{1}(\sigma_{U_n+1} \leq n)$ and so $(b-a)U_n \leq Z_n - (X_n - a)\mathbf{1}(\sigma_{U_n+1} \leq n)$. Next note that $\sigma_k \wedge n$ and $\tau_k \wedge n$ are bounded stopping times and of course $\sigma_k \wedge n \leq \tau_k \wedge n$.

1. If \mathbf{X} is a super-martingale, it follows from the optional stopping theorem that

$$\mathbb{E}(Y_k) = \mathbb{E}(X_{\tau_k \wedge n}) - \mathbb{E}(X_{\sigma_k \wedge n}) \leq 0 \quad (17.4.36)$$

and therefore $\mathbb{E}(Z_n) \leq 0$. Finally, $-(X_n - a)\mathbf{1}(\sigma_{U_n+1} \leq n) \leq (X_n - a)^-$. Taking expected values gives

$$(b-a)\mathbb{E}(U_n) \leq \mathbb{E}(Z_n) + \mathbb{E}[(X_n - a)^-] \leq \mathbb{E}[(X_n - a)^-] \quad (17.4.37)$$

The remaining parts of the inequality follow since $(x-a)^- \leq x^- + |a| \leq |x| + |a|$ for $x \in \mathbb{R}$.

Additional details

The process $\mathbf{Z} = \{Z_n : n \in \mathbb{N}\}$ in the proof can be viewed as a transform of $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$ by a predictable process. Specifically, for $n \in \mathbb{N}_+$, let $I_n = 1$ if $\sigma_k < n \leq \tau_k$ for some $k \in \mathbb{N}$, and let $I_n = 0$ otherwise. Since σ_k and τ_k are stopping times, note that $\{I_n = 1\} \in \mathcal{F}_{n-1}$ for $n \in \mathbb{N}_+$. Hence the process $\mathbf{I} = \{I_n : n \in \mathbb{N}_+\}$ is predictable with respect to \mathcal{F} . Moreover, the transform of \mathbf{X} by \mathbf{I} is

$$(\mathbf{I} \cdot \mathbf{X})_n = \sum_{j=1}^n I_j(X_j - X_{j-1}) = \sum_{k=1}^n (X_{\tau_k \wedge n} - X_{\sigma_k \wedge n}) = Z_n, \quad n \in \mathbb{N} \quad (17.4.38)$$

Since \mathbf{I} is a nonnegative process, if \mathbf{X} is a martingale (sub-martingale, super-martingale), then $\mathbf{I} \cdot \mathbf{X}$ is also a martingale (sub-martingale, super-martingale).

Of course if \mathbf{X} is a martingale with respect to \mathcal{F} then both inequalities apply. In continuous time, as usual, the concepts are more complicated and technical.

Suppose that $\mathbf{x} : [0, \infty) \rightarrow \mathbb{R}$ and that $a, b \in \mathbb{R}$ with $a < b$.

1. If $I \subset [0, \infty)$ is finite, define $t_0^I(\mathbf{x}) = 0$ and then recursively define

$$\begin{aligned} s_{k+1}^I(\mathbf{x}) &= \inf \{t \in I : t \geq t_k^I(\mathbf{x}), x_t \leq a\}, \quad k \in \mathbb{N} \\ t_{k+1}^I(\mathbf{x}) &= \inf \{t \in I : t \geq s_{k+1}^I(\mathbf{x}), x_t \geq b\}, \quad k \in \mathbb{N} \end{aligned}$$

The number of up-crossings of the interval $[a, b]$ by the function \mathbf{x} restricted to I is

$$u_I(a, b, \mathbf{x}) = \sup \{k \in \mathbb{N} : t_k^I(\mathbf{x}) < \infty\} \quad (17.4.39)$$

2. If $I \subseteq [0, \infty)$ is infinite, the number of up-crossings of the interval $[a, b]$ by \mathbf{x} restricted to I is

$$u_I(a, b, \mathbf{x}) = \sup \{u_J(a, b, \mathbf{x}) : J \text{ is finite and } J \subset I\} \quad (17.4.40)$$

To simplify the notation, we will let $u_t(a, b, \mathbf{x}) = u_{[0, t]}(a, b, \mathbf{x})$, the number of up-crossings of $[a, b]$ by \mathbf{x} on $[0, t]$, and $u_\infty(a, b, \mathbf{x}) = u_{[0, \infty)}(a, b, \mathbf{x})$, the total number of up-crossings of $[a, b]$ by \mathbf{x} . In continuous time, the definition of up-crossings is built out of finite subsets of $[0, \infty)$ for measurability concerns, which arise when we replace the deterministic function \mathbf{x} with a stochastic process \mathbf{X} . Here are the simple properties that are analogous to our previous ones.

Suppose again that $\mathbf{x} : [0, \infty) \rightarrow \mathbb{R}$ and that $a, b \in \mathbb{R}$ with $a < b$.

1. If $I, J \subseteq [0, \infty)$ with $I \subseteq J$, then $u_I(a, b, \mathbf{x}) \leq u_J(a, b, \mathbf{x})$.
2. If $(I_n : n \in \mathbb{N})$ is an increasing sequence of sets in $[0, \infty)$ and $J = \bigcup_{n=0}^\infty I_n$ then $u_{I_n}(a, b, \mathbf{x}) \rightarrow u_J(a, b, \mathbf{x})$ as $n \rightarrow \infty$.
3. If $c, d \in \mathbb{R}$ with $a < c < d < b$ and $I \subset [0, \infty)$ then $u_I(c, d, \mathbf{x}) \geq u_I(a, b, \mathbf{x})$.

Proof

1. The result follows easily from the definitions if I is finite (and J either finite or infinite). If I is infinite (and hence so is J), note that

$$\{u_K(a, b, \mathbf{x}) : K \text{ is finite and } K \subseteq I\} \subseteq \{u_K(a, b, \mathbf{x}) : K \text{ is finite and } K \subseteq J\} \quad (17.4.41)$$

2. Since I_n is increasing in $n \in \mathbb{N}$ (in the subset partial order), note that if $K \subset [0, \infty)$ is finite, then $K \subseteq J$ if and only if $K \subseteq I_n$ for some $n \in \mathbb{N}$.
3. Every up-crossing of $[a, b]$ is an up-crossing of $[c, d]$.

The following result is the reason for studying up-crossings in the first place. Note that the definition built from finite set is sufficient.

Suppose that $\mathbf{x} : [0, \infty) \rightarrow \mathbb{R}$. Then $\lim_{t \rightarrow \infty} x_t$ exists in \mathbb{R}^* if and only if $u_\infty(a, b, \mathbf{x}) < \infty$ for every $a, b \in \mathbb{Q}$ with $a < b$.

Proof

As in the discrete-time case, we prove the contrapositive. The proof is almost the same: The following statements are equivalent:

1. $\lim_{t \rightarrow \infty} x_t$ does not exist in \mathbb{R}^* .
2. $\liminf_{t \rightarrow \infty} x_t < \limsup_{t \rightarrow \infty} x_t$.
3. There exists $a, b \in \mathbb{Q}$ with $a < b$ and there exists $s_n, t_n \in [0, \infty)$ with $x_{s_n} \leq a$ for $n \in \mathbb{N}$ and $x_{t_n} \geq b$ for $n \in \mathbb{N}$.
4. There exists $a, b \in \mathbb{Q}$ with $a < b$ and $u_\infty(a, b, \mathbf{x}) = \infty$.

Finally, here is the up-crossing inequality for martingales in continuous time. Once again, the inequality gives a bound on the expected number of up-crossings.

Suppose that $\mathbf{X} = \{X_t : t \in [0, \infty)\}$ satisfies the basic assumptions with respect to the filtration $\mathfrak{F} = \{\mathcal{F}_t : t \in [0, \infty)\}$, and let $a, b \in \mathbb{R}$ with $a < b$. Let $U_t = u_t(a, b, \mathbf{X})$, the random number of up-crossings of $[a, b]$ by \mathbf{X} up to time $t \in [0, \infty)$.

1. If \mathbf{X} is a super-martingale relative to \mathfrak{F} then

$$\mathbb{E}(U_t) \leq \frac{1}{b-a} \mathbb{E}[(X_t - a)^-] \leq \frac{1}{b-a} [\mathbb{E}(X_t^-) + |a|] \leq \frac{1}{b-a} [\mathbb{E}(|X_t|) + |a|], \quad t \in [0, \infty) \quad (17.4.42)$$

2. If \mathbf{X} is a sub-martingale relative to \mathfrak{F} then

$$\mathbb{E}(U_t) \leq \frac{1}{b-a} \mathbb{E}[(X_t - a)^+] \leq \frac{1}{b-a} [\mathbb{E}(X_t^+) + |a|] \leq \frac{1}{b-a} [\mathbb{E}(|X_t|) + |a|], \quad t \in [0, \infty) \quad (17.4.43)$$

Proof

Suppose that \mathbf{X} is a sub-martingale; the proof for a super-martingale is analogous. Fix $t \in [0, \infty)$ and $a, b \in \mathbb{R}$ with $a < b$. For $I \subseteq [0, \infty)$ let $U_I = u_I(a, b, \mathbf{X})$, the number of up-crossings of $[a, b]$ by \mathbf{X} restricted to I . Suppose that I is finite and that $t \in I$ is the maximum of I . Since \mathbf{X} restricted to I is also a sub-martingale, the discrete-time up-crossing theorem applies and so

$$\mathbb{E}(U_I) \leq \frac{1}{b-a} \mathbb{E}[(X_t - a)^+] \quad (17.4.44)$$

Since $U_t = \sup\{U_I : I \text{ is finite and } I \subset [0, t]\}$, there exists finite I_n for $n \in \mathbb{N}$ with $U_{I_n} \uparrow U_t$ as $n \rightarrow \infty$. In particular, U_t is measurable. By property (a) in the theorem [above](#), there exists such a sequence with I_n increasing in n and $t \in I_n$ for each $n \in \mathbb{N}$. By the monotone convergence theorem, $\mathbb{E}(U_{I_n}) \rightarrow \mathbb{E}(U_t)$ as $n \rightarrow \infty$. So by the displayed equation above,

$$\mathbb{E}(U_t) \leq \frac{1}{b-a} \mathbb{E}[(X_t - a)^+] \quad (17.4.45)$$

Examples and Applications

Kolmogorov's Inequality

Suppose that $\mathbf{X} = \{X_n : n \in \mathbb{N}_+\}$ is a sequence of independent variables with $\mathbb{E}(X_n) = 0$ and $\text{var}(X_n) = \mathbb{E}(X_n^2) < \infty$ for $n \in \mathbb{N}_+$. Let $\mathbf{Y} = \{Y_n : n \in \mathbb{N}\}$ be the partial sum process associated with \mathbf{X} , so that

$$Y_n = \sum_{i=1}^n X_i, \quad n \in \mathbb{N} \quad (17.4.46)$$

From the Introduction we know that \mathbf{Y} is a martingale. A simple application of the maximal inequality gives the following result, which is known as *Kolmogorov's inequality*, named for Andrei Kolmogorov.

For $n \in \mathbb{N}$, let $U_n = \max\{|Y_i| : i \in \mathbb{N}_n\}$. Then

$$\mathbb{P}(U_n \geq x) \leq \frac{1}{x^2} \text{var}(Y_n) = \frac{1}{x^2} \sum_{i=1}^n \mathbb{E}(X_i^2), \quad x \in (0, \infty) \quad (17.4.47)$$

Proof

As noted above, \mathbf{Y} is a martingale. Since the function $x \mapsto x^2$ on \mathbb{R} is convex, $\mathbf{Y}^2 = \{Y_n^2 : n \in \mathbb{N}\}$ is a sub-martingale. Let $V_n = \max\{Y_i^2 : i \in \mathbb{N}_n\}$ for $n \in \mathbb{N}$, and let $x \in (0, \infty)$. Applying the maximal inequality for sub-martingales we have

$$\mathbb{P}(U_n \geq x) = \mathbb{P}(V_n \geq x^2) \leq \frac{1}{x^2} \mathbb{E}(Y_n^2) = \frac{1}{x^2} \text{var}(Y_n) \quad (17.4.48)$$

Finally, since \mathbf{X} is an independent sequence,

$$\text{var}(Y_n) = \sum_{i=1}^n \text{var}(X_i) = \sum_{i=1}^n \mathbb{E}(X_i^2) \quad (17.4.49)$$

Red and Black

In the game of red and black, a gambler plays a sequence of Bernoulli games with success parameter $p \in (0, 1)$ at even stakes. The gambler starts with an initial fortune x and plays until either she is ruined or reaches a specified target fortune a , where $x, a \in (0, \infty)$ with $x < a$. When $p \leq \frac{1}{2}$, so that the games are fair or unfair, an optimal strategy is *bold play*: on each game, the gambler bets her entire fortune or just what is needed to reach the target, whichever is smaller. In the section on bold play we showed that when $p = \frac{1}{2}$, so that the games are fair, the probability of winning (that is, reaching the target a starting with x) is x/a . We can use the maximal inequality for super-martingales to show that indeed, one cannot do better.

To set up the notation and review various concepts, let X_0 denote the gambler's initial fortune and let X_n denote the outcome of game $n \in \mathbb{N}_+$, where 1 denotes a win and -1 a loss. So $\{X_n : n \in \mathbb{N}\}$ is a sequence of independent variables with $\mathbb{P}(X_n = 1) = p$ and $\mathbb{P}(X_n = -1) = 1 - p$ for $n \in \mathbb{N}_+$. (The initial fortune X_0 has an unspecified distribution on $(0, \infty)$.) The gambler is at a casino after all, so of course $p \leq \frac{1}{2}$. Let

$$Y_n = \sum_{i=0}^n X_i, \quad n \in \mathbb{N} \quad (17.4.50)$$

so that $\mathbf{Y} = \{Y_n : n \in \mathbb{N}\}$ is the partial sum process associated with $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$. Recall that \mathbf{Y} is also known as the *simple random walk* with parameter p , and since $p \leq \frac{1}{2}$, is a super-martingale. The process $\{X_n : n \in \mathbb{N}_+\}$ is the *difference sequence* associated with \mathbf{Y} . Next let Z_n denote the amount that the gambler bets on game $n \in \mathbb{N}_+$. The process $\mathbf{Z} = \{Z_n : n \in \mathbb{N}_+\}$ is *predictable* with respect to $\mathbf{X} = \{X_n : n \in \mathbb{N}\}$, so that Z_n is measurable with respect to $\sigma\{X_0, X_1, \dots, X_{n-1}\}$ for $n \in \mathbb{N}_+$. So the gambler's fortune after n games is

$$W_n = X_0 + \sum_{i=1}^n Z_i X_i = X_0 + \sum_{i=1}^n Z_i (Y_i - Y_{i-1}) \quad (17.4.51)$$

Recall that $\mathbf{W} = \{W_n : n \in \mathbb{N}\}$ is the *transform* of \mathbf{Z} with \mathbf{Y} , denoted $\mathbf{W} = \mathbf{Z} \cdot \mathbf{Y}$. The gambler is not allowed to go into debt and so we must have $Z_n \leq W_{n-1}$ for $n \in \mathbb{N}_+$: the gambler's bet on game n cannot exceed her fortune after game $n - 1$. What's the probability that the gambler can ever reach or exceed the target a starting with fortune $x < a$?

Let $U_\infty = \sup\{W_n : n \in \mathbb{N}\}$. Suppose that $x, a \in (0, \infty)$ with $x < a$ and that $X_0 = x$. Then

$$\mathbb{P}(U_\infty \geq a) \leq \frac{x}{a} \quad (17.4.52)$$

Proof

Since Y is a super-martingale and Z is nonnegative, the transform $W = Z \cdot Y$ is also a super-martingale. By the inequality for nonnegative super-martingales [above](#):

$$\mathbb{P}(U_\infty \geq a) \leq \frac{1}{a} \mathbb{E}(W_0) = \frac{x}{a} \quad (17.4.53)$$

Note that the only assumptions made on the gambler's sequence of bets Z is that the sequence is predictable, so that the gambler cannot see into the future, and that gambler cannot go into debt. Under these basic assumptions, no strategy can do any better than bold play. However, there *are* strategies that do as well as bold play; these are variations on bold play.

Open the simulation of the red and black game. Select bold play and $p = \frac{1}{2}$. Play the game with various values of initial and target fortunes.

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