5.16: The Lévy Distribution

The *Lévy distribution*, named for the French mathematician Paul Lévy, is important in the study of Brownian motion, and is one of only three stable distributions whose probability density function can be expressed in a simple, closed form.

The Standard Lévy Distribution

Definition

If Z has the standard normal distribution then $U = 1/Z^2$ has the *standard Lévy distribution*.

So the standard Lévy distribution is a continuous distribution on $(0, \infty)$.

Distribution Functions

We assume that U has the standard Lévy distribution. The distribution function of U has a simple expression in terms of the standard normal distribution function Φ, not surprising given the definition.

U has distribution function G given by

$$G(u) = 2\left[1 - \Phi\left(\frac{1}{\sqrt{u}}\right)\right], \quad u \in (0, \infty) \tag{5.16.1}$$

Proof

For $u \in (0, \infty)$,

$$\mathbb{P}\left(\frac{1}{Z^2} \le u\right) = \mathbb{P}\left(Z^2 \ge \frac{1}{u}\right) = \mathbb{P}\left(Z \ge \frac{1}{\sqrt{u}}\right) + \mathbb{P}\left(Z \le -\frac{1}{\sqrt{u}}\right) = 2\left[1 - \Phi\left(\frac{1}{\sqrt{u}}\right)\right] \tag{5.16.2}$$

Similarly, the quantile function of U has a simple expression in terms of the standard normal quantile function Φ^{-1}.

U has quantile function G^{-1} given by

$$G^{-1}(p) = \frac{1}{\left[\Phi^{-1}(1 - p/2)\right]^2}, \quad p \in [0, 1) \tag{5.16.3}$$

The quartiles of U are

1. $q_1 = \left[\Phi^{-1}\left(\frac{7}{8}\right)\right]^{-2} \approx 0.7557$, the first quartile.
2. $q_2 = \left[\Phi^{-1}\left(\frac{3}{4}\right)\right]^{-2} \approx 2.1980$, the median.
3. $q_3 = \left[\Phi^{-1}\left(\frac{5}{8}\right)\right]^{-2} \approx 9.8516$, the third quartile.

Proof

The quantile function can be obtained from the distribution function by solving $p = G(u)$ for $u = G^{-1}(p)$.

Open the Special Distribution Calculator and select the Lévy distribution. Keep the default parameter values. Note the shape and location of the distribution function. Compute a few values of the distribution function and the quantile function.

Finally, the probability density function of U has a simple closed expression.

U has probability density function g given by

$$g(u) = \frac{1}{\sqrt{2\pi}} \frac{1}{u^{3/2}} \exp\left(-\frac{1}{2u}\right), \quad u \in (0, \infty) \tag{5.16.4}$$

1. g increases and then decreasing with mode at $x = \frac{1}{3}$.

2. g is concave upward, then downward, then upward again, with inflection points at $x = \frac{1}{3} - \frac{\sqrt{10}}{15} \approx 0.1225$ and at $x = \frac{1}{3} + \frac{\sqrt{10}}{15} \approx 0.5442$.

Proof

The formula for g follows from differentiating the CDF given above:

$$g(u) = -2\Phi'(u^{-1/2})\left(-\frac{1}{2}u^{-3/2}\right), \quad u \in (0, \infty) \tag{5.16.5}$$

But $\Phi'(z) = \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, the standard normal PDF. Substitution and simplification then gives the results. Parts (a) and (b) also follow from standard calculus:

$$g'(u) = \frac{1}{2\sqrt{2\pi}} u^{-7/2} e^{-u^{-1}/2}(-3u+1) \tag{5.16.6}$$

$$g''(u) = \frac{1}{4\sqrt{2\pi}} u^{-11/2} e^{-u^{-1}/2}(15u^2 - 10u + 1) \tag{5.16.7}$$

Open the Special Distribtion Simulator and select the Lévy distribution. Keep the default parameter values. Note the shape of the probability density function. Run the simulation 1000 times and compare the empirical density function to the probability density function.

Moments

We assume again that U has the standard Lévy distribution. After exploring the graphs of the probability density function and distribution function above, you probably noticed that the Lévy distribution has a very heavy tail. The 99th percentile is about 6400, for example. The following result is not surprising.

$\mathbb{E}(U) = \infty$

Proof

Note that $u \mapsto e^{-1/2u}$ is increasing. Hence

$$\mathbb{E}(U) = \int_0^\infty u \frac{1}{\sqrt{2\pi} u^{3/2}} e^{-1/2u} du > \int_1^\infty \frac{1}{\sqrt{2\pi}} u^{-1/2} e^{-1/2} du = \infty \tag{5.16.8}$$

Of course, the higher-order moments are infinite as well, and the variance, skewness, and kurtosis do not exist. The moment generating function is infinite at every positive value, and so is of no use. On the other hand, the characteristic function of the standard Lévy distribution is very useful. For the following result, recall that the *sign* function sgn is given by $\text{sgn}(t) = 1$ for $t > 0$, $\text{sgn}(t) = -1$ for $t < 0$, and $\text{sgn}(0) = 0$.

U has characteristic function χ_0 given by

$$\chi_0(t) = \mathbb{E}\left(e^{itU}\right) = \exp\left(-|t|^{1/2}\left[1 + i\,\text{sgn}(t)\right]\right), \quad t \in \mathbb{R} \tag{5.16.9}$$

Related Distributions

The most important relationship is the one in the definition: If Z has the standard normal distribution then $U = 1/Z^2$ has the standard Lévy distribution. The following result is bascially the converse.

If U has the standard Lévy distribution, then $V = 1/\sqrt{U}$ has the standard half-normal distribution.

Proof

From the definition, we can take $U = 1/Z^2$ where Z has the standard normal distribution. Then $1/\sqrt{U} = |Z|$, and $|Z|$ has the standard half-normal distribution.

The General Lévy Distribution

Like so many other "standard distributions", the standard Lévy distribution is generalized by adding location and scale parameters.

Definition

Suppose that U has the standard Lévy distribution, and $a \in \mathbb{R}$ and $b \in (0, \infty)$. Then $X = a + bU$ has the *Lévy distribution* with *location parameter* a and *scale parameter* b.

Note that X has a continuous distribution on the interval (a, ∞).

Distribution Functions

Suppose that X has the Lévy distribution with location parameter $a \in \mathbb{R}$ and scale parameter $b \in (0, \infty)$. As before, the distribution function of X has a simple expression in terms of the standard normal distribution function Φ.

X has distribution function G given by

$$F(x) = 2\left[1 - \Phi\left(\sqrt{\frac{b}{(x-a)}}\right)\right], \quad x \in (a, \infty) \tag{5.16.10}$$

Proof

Recall that $F(x) = G\left(\frac{x-a}{b}\right)$ where G is the standard Lévy CDF.

Similarly, the quantile function of X has a simple expression in terms of the standard normal quantile function Φ^{-1}.

X has quantile function F^{-1} given by

$$F^{-1}(p) = a + \frac{b}{\left[\Phi^{-1}(1-p/2)\right]^2}, \quad p \in [0, 1) \tag{5.16.11}$$

The quartiles of X are

1. $q_1 = a + b\left[\Phi^{-1}\left(\frac{7}{8}\right)\right]^{-2}$, the first quartile.
2. $q_2 = a + b\left[\Phi^{-1}\left(\frac{3}{4}\right)\right]^{-2}$, the median.
3. $q_3 = a + b\left[\Phi^{-1}\left(\frac{5}{8}\right)\right]^{-2}$, the third quartile.

Proof

Recall that $F^{-1}(p) = a + bG^{-1}(p)$, where G^{-1} is the standard Lévy quantile function.

Open the Special Distribution Calculator and select the Lévy distribution. Vary the parameter values and note the shape of the graph of the distribution function function. For various values of the parameters, compute a few values of the distribution function and the quantile function.

Finally, the probability density function of X has a simple closed expression.

X has probability density function f given by

$$f(x) = \sqrt{\frac{b}{2\pi}} \frac{1}{(x-a)^{3/2}} \exp\left[-\frac{b}{2(x-a)}\right], \quad x \in (a, \infty) \tag{5.16.12}$$

1. f increases and then decreases with mode at $x = a + \frac{1}{3}b$.
2. f is concave upward, then downward, then upward again with inflection points at $x = a + \left(\frac{1}{3} \pm \frac{\sqrt{10}}{15}\right)b$.

Proof

Recall that $f(x) = \frac{1}{b} g\left(\frac{x-a}{b}\right)$ where g is the standard Lévy PDF, so the formula for f follow from the definition of g and simple algebra. Parts (a) and (b) follow from the corresponding results for g.

Open the Special Distribtion Simulator and select the Lévy distribution. Vary the parameters and note the shape and location of the probability density function. For various parameter values, run the simulation 1000 times and compare the empirical density function to the probability density function.

Moments

Assume again that X has the Lévy distribution with location parameter $a \in \mathbb{R}$ and scale parameter $b \in (0, \infty)$. Of course, since the standard Lévy distribution has infinite mean, so does the general Lévy distribution.

$\mathbb{E}(X) = \infty$

Also as before, the variance, skewness, and kurtosis of X are undefined. On the other hand, the characteristic function of X is very important.

X has characteristic function χ given by

$$\chi(t) = \mathbb{E}\left(e^{itX}\right) = \exp\left(ita - b^{1/2}|t|^{1/2}[1 + i\,\mathrm{sgn}(t)]\right), \quad t \in \mathbb{R} \tag{5.16.13}$$

Proof

This follows from the standard characteristic function since $\chi(t) = e^{ita}\chi_0(bt)$. Note that $\mathrm{sgn}(bt) = \mathrm{sgn}(t)$ since $b > 0$.

Related Distributions

Since the Lévy distribution is a location-scale family, it is trivially closed under location-scale transformations.

Suppose that X has the Lévy distribution with location parameter $a \in \mathbb{R}$ and scale parameter $b \in (0, \infty)$, and that $c \in \mathbb{R}$ and $d \in (0, \infty)$. Then $Y = c + dX$ has the Lévy distribution with location parameter $c + ad$ and scale parameter bd.

Proof

From the definition, we can take $X = a + bU$ where U has the standard Lévy distribution. Hence $Y = c + dX = (c + ad) + (bd)U$ has the Lévy distribution with location parameter $c + ad$ and scale parameter bd.

Of more interest is the fact that the Lévy distribution is closed under convolution (corresponding to sums of independent variables).

Suppose that X_1 and X_2 are independent, and that, X_k has the Lévy distribution with location parameter $a_k \in \mathbb{R}$ and scale parameter $b_k \in (0, \infty)$ for $k \in \{1, 2\}$. Then $X_1 + X_2$ has the Lévy distribution with location parameter $a_1 + a_2$ and scale parameter $(b_1^{1/2} + b_2^{1/2})^2$.

Proof

The characteristic function of X_k is

$$\chi_k(t) = \exp\left(ita_k - b_k^{1/2}|t|^{1/2}[1 + i\,\mathrm{sgn}(t)]\right), \quad t \in \mathbb{R} \tag{5.16.14}$$

for $k \in \{1, 2\}$. Hence the characteristic function of $X_1 + X_2$ is

$$\begin{aligned} \chi(t) = \chi_1(t)\chi_2(t) &= \exp\left[it(a_1 + a_2) - \left(b_1^{1/2} + b_2^{1/2}\right)|t|^{1/2}[1 + i\,\mathrm{sgn}(t)]\right] \\ &= \exp\left[itA - B^{1/2}|t|^{1/2}[1 + i\,\mathrm{sgn}(t)]\right], \quad t \in \mathbb{R} \end{aligned}$$

where $A = a_1 + a_2$ is the location parameter and $B = \left(b_1^{1/2} + b_2^{1/2}\right)^2$ is the scale parameter.

As a corollary, the Lévy distribution is a stable distribution with index $\alpha = \frac{1}{2}$:

Suppose that $n \in \mathbb{N}_+$ and that $(X_1, X_2, \ldots, X_n)$ is a sequence of independent random variables, each having the Lévy distribution with location parameter $a \in \mathbb{R}$ and scale parameter $b \in (0, \infty)$. Then $X_1 + X_2 + \cdots + X_n$ has the Lévy distribution with location parameter na and scale parameter $n^2 b$.

Stability is one of the reasons for the importance of the Lévy distribution. From the characteristic function, it follows that the skewness parameter is $\beta = 1$.
