5.34: The Gompertz Distribution

The *Gompertz distributon*, named for Benjamin Gompertz, is a continuous probability distribution on $[0, \infty)$ that has exponentially increasing failure rate. Unfortunately, the death rate of adult humans increases exponentially, so the Gompertz distribution is widely used in actuarial science.

The Basic Gompertz Distribution

Distribution Functions

We will start by giving the reliability function, since most applications of the Gompertz distribution deal with mortality.

The *basic Gompertz distribution* with shape parameter $a \in (0, \infty)$ is a continuous distribution on $[0, \infty)$ with reliability function G^c given by

$$G^c(x) = \exp[-a(e^x - 1)], \quad x \in [0, \infty) \tag{5.34.1}$$

The special case $a = 1$ gives the *standard Gompertz distribution*.

Proof

Note that G^c is continuous and decreasing on $[0, \infty)$ with $G^c(0) = 1$ and $G^c(x) \to 0$ as $x \to \infty$.

The distribution function G is given by

$$G(x) = 1 - \exp[-a(e^x - 1)], \quad x \in [0, \infty) \tag{5.34.2}$$

Proof

This follows trivially from the reliability function reliability function, since $G = 1 - G^c$.

The quantile function G^{-1} is given by

$$G^{-1}(p) = \ln\left[1 - \frac{1}{a}\ln(1-p)\right], \quad p \in [0, 1) \tag{5.34.3}$$

1. The first quartile is $q_1 = \ln[1 + (\ln 4 - \ln 3)/a]$.
2. The median is $q_2 = \ln(1 + \ln 2/a)$.
3. The third quartile is $q_3 = \ln(1 + \ln 4/a)$.

Proof

The formula for G^{-1} follows from the distribution function by solving $p = G(x)$ for x in terms of p.

For the standard Gompertz distribution ($a = 1$), the first quartile is $q_1 = \ln[1 + (\ln 4 - \ln 3)] \approx 0.2529$, the median is $q_2 = \ln(1 + \ln 2) \approx 0.5266$, and the third quartile is $q_3 = \ln(1 + \ln 4) \approx 0.8697$.

Open the special distribution calculator and select the Gompertz distribution. Vary the shape parameter and note the shape of the distribution function. For selected values of the shape parameter, computer a few values of the distribution function and the quantile function.

The probability density function g is given by

$$g(x) = ae^x \exp[-a(e^x - 1)], \quad x \in [0, \infty) \tag{5.34.4}$$

1. If $a < 1$ then g is increasing and then decreasing with mode $x = -\ln(a)$.
2. If $a \ge 1$ then g is decreasing with mode $x = 0$.
3. If $a < (3 - \sqrt{5})/2 \approx 0.382$ then g is concave up and then down then up again, with inflection points at $x = \ln[(3 \pm \sqrt{5})/2a]$.
4. If $(3 - \sqrt{5})/2 \le a < (3 + \sqrt{5})/2 \approx 2.618$ then g is concave down and then up, with inflection point at $x = \ln[(3 + \sqrt{5})/2a]$.

5. If $a \ge (3+\sqrt{5})/2$ then g is concave up.

Proof

The formula for g follows from the distribution function since $g = G'$ Parts (a)–(d) follow from

$$g'(x) = ae^x(1-ae^x)\exp[-a(e^x-1)] \tag{5.34.5}$$

$$g''(x) = ae^x(1-3ae^x+a^2e^{2x})\exp[-a(e^x-1)] \tag{5.34.6}$$

So for the standard Gompertz distribution ($a=1$), the inflection point is $x = \ln(3+\sqrt{5}) \approx 1.6555$.

Open the special distribution simulator and select the Gompertz distribution. Vary the shape parameter and note the shape of the probability density function. For selected values of the shape parameter, run the simulation 1000 times and compare the empirical density function to the probability density function.

Finally, as promised, the Gompertz distribution has exponentially increasing failure rate.

The failure rate function r is given by $r(x) = ae^x$ for $x \in [0,\infty)$

Proof

Recall that the is $r(x) = g(x)/G^c(x)$ so the result follows from the distribution function and the probability density function.

Moments

The moments of the basic Gompertz distribution cannot be given in simple closed form, but the mean and moment generating function can at least be expressed in terms of a special function known as the *exponential integral*. There are many variations on the exponential integral, but for our purposes, the following version is best:

The *exponential integral* with parameter $a \in (0,\infty)$ is the function $E_a : \mathbb{R} \to (0,\infty)$ defined by

$$E_a(t) = \int_1^\infty u^t e^{-au}\,du, \quad t \in \mathbb{R} \tag{5.34.7}$$

For the remainder of this discussion, we assume that X has the basic Gompertz distribution with shape parameter $a \in (0,\infty)$.

X has moment generating function m given by

$$m(t) = \mathbb{E}\left(e^{tX}\right) = ae^a E_a(t), \quad t \in \mathbb{R} \tag{5.34.8}$$

Proof

Using the substitution $u = e^x$ we have

$$m(t) = \int_0^\infty e^{tx} ae^x e^a \exp(-ae^x)\,dx = ae^a \int_1^\infty u^t e^{-au}\,du = ae^a E_a(t) \tag{5.34.9}$$

It follows that X has moments of all orders. Here is the mean:

X has mean $\mathbb{E}(X) = e^a E_a(-1)$.

Proof

First we use the substitution $y = e^x$ to get

$$\mathbb{E}(X) = \int_0^\infty x ae^x e^a \exp(-ae^x)\,dx = ae^a \int_1^\infty \ln(y) e^{-ay}\,dy \tag{5.34.10}$$

Next, integration by parts with $u = \ln y$, $dv = e^{-ay}dy$ gives

$$\mathbb{E}(X) = e^a \int_1^\infty \frac{1}{y} e^{-ay}\,dy = e^a E_a(-1) \tag{5.34.11}$$

If X has the standard Gompertz distribution, $\mathbb{E}(X) \approx 0.5963$.

Open the special distribution simulator and select the Gompertz distribution. Vary the shape parameter and note the size and location of the mean $\pm$ standard deviation bar. For selected values of the parameter, run the simulation 1000 times and compare the empirical mean and standard deviation to the distribution mean and standard deviation.

Related Distributions

The basic Gompertz distribution has the usual connections to the standard uniform distribution by means of the distribution function and quantile function computed above.

Suppose that $a \in (0, \infty)$.

1. If U has the standard uniform distribution then $X = \ln\left(1 - \frac{1}{a} \ln U\right)$ has the basic Gompertz distribution with shape parameter a.
2. If X has the basic Gompertz distribution with shape parameter a then $U = \exp\left[-a\left(e^X - 1\right)\right]$ has the standard uniform distribution.

Proof

1. Recall that if U has the standard uniform distribution, then $1 - U$ also has the standard uniform distribution, and hence $X = G^{-1}(1 - U)$ has the basic Gompertz distribution with shape parameter a.
2. If X has the basic Gompertz distribution with shape parameter a then $G(X)$ has the standard uniform distribution, and hence so does $U = 1 - G(X)$.

Since the quantile function of the basic Gompertz distribution has a simple closed form, the distribution can be simulated using the random quantile method.

Open the random quantile experiment and select the Gompertz distribution. Vary the shape parameter and note the shape of the distribution and probability density functions. For selected values of the parameter, run the simulation 1000 times and compare the empirical density function, mean, and standard deviation to their distributional counterparts.

The basic Gompertz distribution also has simple connections to the exponential distribution.

Suppose that $a \in (0, \infty)$.

1. If X has the basic Gompertz distribution with shape parameter a, then $Y = e^X - 1$ has the exponential distribution with rate parameter a.
2. If Y has the exponential distribution with rate parameter a, then $X = \ln(Y + 1)$ has the Gompertz distribution with shape parameter a.

Proof

These results follow from the standard change of variables formula. The transformations, which are inverses of each other, are $y = e^x - 1$ and $x = \ln(y + 1)$ for $x, y \in [0, \infty)$. Let g and h denote PDFs of X and Y respectively.

1. We start with $g(x) = a e^x \exp\left[-a\left(e^x - 1\right)\right]$ for $x \in [0, \infty)$ and then
$$h(y) = g(x) \frac{dx}{dy} = a \exp[\ln(y+1)] \exp\{-a[\exp(\ln(y+1)) - 1]\} \frac{1}{y+1} = a e^{-ay}, \quad y \in [0, \infty) \tag{5.34.12}$$
which is the PDF of the exponential distribution with rate parameter a.
2. We start with $h(y) = a e^{-ay}$ for $y \in [0, \infty)$ and then
$$g(x) = h(y) \frac{dy}{dx} = a \exp[-a(e^x - 1)] e^x, \quad x \in [0, \infty) \tag{5.34.13}$$
which is the PDF of the Gompertz distribution with shape parameter a.

In particular, if Y has the standard exponential distribution (rate parameter 1), then $X = \ln(Y + 1)$ has the standard Gompertz distribution (shape parameter 1). Since the exponential distribution is a scale family (the scale parameter is the reciprocal of the rate parameter), we can construct an arbitrary basic Gompertz variable from a standard exponential variable. Specifically, if Y has the standard exponential variable and $a \in (0, \infty)$, then

$$X = \ln\left(\frac{1}{a}Y + 1\right) \tag{5.34.14}$$

has the Gompertz distribution with shape parameter a.

The extreme value distribution (Gumbel distribution) is also related to the Gompertz distribution.

If X has the standard extreme value distribution for minimums, then the conditional distribution of X given $X \geq 0$ is the standard Gompertz distribution.

Proof

By definition, X has PDF f given by $f(x) = e^x \exp(-e^x)$ for $x \in \mathbb{R}$. The conditional PDF of X given $X \geq 0$ is

$$g(x) = \frac{f(x)}{\mathbb{P}(X \geq 0)} = \frac{e^x \exp(-e^x)}{e^{-1}} = e^x \exp[-(e^x - 1)], \quad x \in [0, \infty) \tag{5.34.15}$$

which is the PDF of the standard Gompertz distribution.

The General Gompertz Distribution

The basic Gompertz distribution is generalized, like so many distributions on $[0, \infty)$, by adding a scale parameter. Recall that scale transformations often correspond to a change of units (minutes to hours, for example) and thus are fundamental.

If Z has the basic Gompertz distribution with shape parameter $a \in (0, \infty)$ and $b \in (0, \infty)$ then $X = bZ$ has the *Gompertz distribution* with shape parameter a and scale parameter b.

Distribution Functions

Suppose that X has the Gompertz distribution with shape parameter $a \in (0, \infty)$ and scale parameter $b \in (0, \infty)$.

X has reliability function F^c given by

$$F^c(x) = \mathbb{P}(X > x) = \exp\left[-a\left(e^{x/b} - 1\right)\right], \quad x \in [0, \infty) \tag{5.34.16}$$

Proof

Recall that $F^c(x) = G^c(x/b)$ where G^c is the reliability function of the corresponding basic distribution.

X has distribution function F given by

$$F(x) = \mathbb{P}(X \leq x) = 1 - \exp\left[-a\left(e^{x/b} - 1\right)\right], \quad x \in [0, \infty) \tag{5.34.17}$$

Proof

As before, $F = 1 - F^c$. Also, $F(x) = G(x/b)$ where G is the CDF of the corresponding basic distribution.

X has quantile function F^{-1} given by

$$F^{-1}(p) = b\ln\left[1 - \frac{1}{a}\ln(1-p)\right], \quad p \in [0, 1) \tag{5.34.18}$$

1. The first quartile is $q_1 = b\ln[1 + (\ln 4 - \ln 3)/a]$.
2. The median is $q_2 = b\ln(1 + \ln 2/a)$.
3. The third quartile is $q_3 = b\ln(1 + \ln 4/a)$.

Proof

Recall that $F^{-1}(p) = bG^{-1}(p)$ where G^{-1} is the quantile function of the corresponding basic distribution.

Open the special distribution calculator and select the Gompertz distribution. Vary the shape and scale parameters and note the shape and location of the distribution function. For selected values of the parameters, computer a few values of the distribution function and the quantile function.

X has probability density function f given by

$$f(x) = \frac{a}{b} e^{x/b} \exp\left[-a\left(e^{x/b} - 1\right)\right], \quad x \in [0, \infty) \tag{5.34.19}$$

1. If $a < 1$ then f is increasing and then decreasing with mode $x = -b \ln(a)$.
2. If $a \ge 1$ then f is decreasing with mode $x = 0$.
3. If $a < (3 - \sqrt{5})/2 \approx 0.382$ then f is concave up and then down then up again, with inflection points at $x = b \ln\left[(3 \pm \sqrt{5})/2a\right]$.
4. If $(3 - \sqrt{5})/2 \le a < (3 + \sqrt{5})/2 \approx 2.618$ then f has is concave down and then up, with inflection point at $x = b \ln\left[(3 + \sqrt{5})/2a\right]$.
5. If $a \ge (3 + \sqrt{5})/2$ then f is concave up.

Proof

Recall that $f(x) = \frac{1}{b} g\left(\frac{x}{b}\right)$ where g is the PDF of the corresponding basic distribution.

Open the special distribution simulator and select the Gompertz distribution. Vary the shape and scale parameters and note the shape and location of the probability density function. For selected values of the parameters, run the simulation 1000 times and compare the empirical density function to the probability density function.

Once again, X has exponentially increasing failure rate.

X has failure rate function R given by

$$R(x) = \frac{a}{b} e^{x/b}, \quad x \in [0, \infty) \tag{5.34.20}$$

Proof

Recall that $R(x) = f(x) / F^c(x)$. Also, $R(x) = \frac{1}{b} r\left(\frac{x}{b}\right)$ where r is the failure rate function of the corresponding basic distribution.

Moments

As with the basic distribution, the moment generating function and mean of the general Gompertz distribution can be expressed in terms of the exponential integral. Suppose again that X has the Gompertz distribution with shape parameter $a \in (0, \infty)$ and scale parameter $b \in (0, \infty)$.

X has moment generating function M given by

$$M(t) = \mathbb{E}\left(e^{tX}\right) = a e^a E_a(bt), \quad t \in \mathbb{R} \tag{5.34.21}$$

Proof

Recall that $M(t) = m(bt)$ where m is the MGF of the corresponding basic distribution.

X has mean $\mathbb{E}(X) = b e^a E_a(-1)$.

Proof

This follows from the mean of the corresponding basic distribution, and the standard property $\mathbb{E}(X) = b\mathbb{E}(Z)$.

Open the special distribution simulator and select the Gompertz distribution. Vary the shape and scale parameters and note the size and location of the mean $\pm$ standard deviation bar. For selected values of the parameters, run the simulation 1000 times and compare the empirical mean and standard deviation to the distribution mean and standard deviation.

Related Distributions

Since the Gompertz distribution is a scale family for each value of the shape parameter, it is trivially closed under scale transformations.

Suppose that X has the Gompertz distribution with shape parameter $a \in (0, \infty)$ and scale parameter $b \in (0, \infty)$. If $c \in (0, \infty)$ then $Y = cX$ has the Gompertz distribution with shape parameter a and scale parameter bc.

Proof

By definition, we can take $X = bZ$ where Z has the standard Gompertz distribution with shape parameter a. But then $Y = cX = (bc)Z$.

As with the basic distribution, the Gompertz distribution has the usual connections with the standard uniform distribution by means of the distribution function and quantile function computed above.

Suppose that $a, b \in (0, \infty)$.

1. If U has the standard uniform distribution then $X = b \ln\left(1 - \frac{1}{a} \ln U\right)$ has the Gompertz distribution with shape parameter a and scale parameter b.
2. If X has the Gompertz distribution with shape parameter a and scale parameter b, then $U = \exp\left[-a\left(e^{X/b} - 1\right)\right]$ has the standard uniform distribution.

Proof

This follows from the corresponding result for the basic distribution and the definition of the general Gompertz variable as $X = bZ$ where Z has the basic Gompertz distribution with shape parameter a.

Again, since the quantile function of the Gompertz distribution has a simple closed form, the distribution can be simulated using the random quantile method.

Open the random quantile experiment and select the Gompertz distribution. Vary the shape and scale parameters and note the shape and location of the distribution and probability density functions. For selected values of the parameters, run the simulation 1000 times and note the agreement between the empirical density function and the probability density function.

The following result is a slight generalization of the connection above between the basic Gompertz distribution and the extreme value distribution.

If X has the extreme value distribution for minimums with scale parameter $b > 0$, then the conditional distribution of X given $X \ge 0$ is the Gompertz distribution with shape parameter 1 and scale parameter b.

Proof

We can take $X = bV$ where V has the standard extreme value distribution for minimums. Note that $X \ge 0$ if and only if $V \ge 0$. Hence the conditional distribution of X given $X \ge 0$ is the same as the conditional distribution of bV given $V \ge 0$. But by the result above the conditional distribution of V given $V \ge 0$ has the standard Gompertz distribution.

Finally, we give a slight generalization of the connection above between the Gompertz distribution and the exponential distribution.

Suppose that $a, b \in (0, \infty)$.

1. If X has the Gompertz distribution with shape parameter a and scale parameter b, then $Y = e^{X/b} - 1$ has the exponential distribution with rate parameter a.
2. If Y has the exponential distribution with rate parameter a, then $X = b\ln(Y + 1)$ has the Gompertz distribution with shape parameter a and scale parameter b.

Proof

These results follow from the corresponding result for the basic distribution.

1. If X has the Gompertz distribution with shape parameter a and scale parameter b, then X/b has the basic Gompertz distribution with shape parameter a. Hence $Y = e^{X/b} - 1$ has the exponential distribution with rate parameter a.

2. If Y has the exponential distribution with rate parameter a then $\ln(Y+1)$ has the basic Gompertz distribution with shape parameter a and hence $X = b\ln(Y+1)$ has the Gompertz distribution with shape parameter a and scale parameter b. (

As a corollary, we can construct a general Gompertz variable from a standard exponential variable. Specifically, if Y has the standard exponential distribution and if $a, b \in (0, \infty)$ then

$$X = b\ln\left(\frac{1}{a}Y + 1\right) \tag{5.34.22}$$

has the Gompertz distribution with shape parameter a and scale parameter b.