

## 18.2: Brownian Motion with Drift and Scaling

### Basic Theory

#### Definition

We start with the assumptions that govern standard Brownian motion, except that we relax the restrictions on the parameters of the normal distribution.

Suppose that  $\mu \in \mathbb{R}$  and  $\sigma \in (0, \infty)$ . *Brownian motion* with drift parameter  $\mu$  and scale parameter  $\sigma$  is a random process  $\mathbf{X} = \{X_t : t \in [0, \infty)\}$  with state space  $\mathbb{R}$  that satisfies the following properties:

1.  $X_0 = 0$  (with probability 1).
2.  $\mathbf{X}$  has stationary increments. That is, for  $s, t \in [0, \infty)$  with  $s < t$ , the distribution of  $X_t - X_s$  is the same as the distribution of  $X_{t-s}$ .
3.  $\mathbf{X}$  has independent increments. That is, for  $t_1, t_2, \dots, t_n \in [0, \infty)$  with  $t_1 < t_2 < \dots < t_n$ , the random variables  $X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$  are independent.
4.  $X_t$  has the normal distribution with mean  $\mu t$  and variance  $\sigma^2 t$  for  $t \in [0, \infty)$ .
5. With probability 1,  $t \mapsto X_t$  is continuous on  $[0, \infty)$ .

Note that we cannot assign the parameters of the normal distribution of  $X_t$  arbitrarily. We know that since  $\mathbf{X}$  has stationary, independent increments,  $\mathbb{E}(X_t)$  and  $\text{var}(X_t)$  must be linear functions of  $t \in [0, \infty)$ .

Open the simulation of Brownian motion with drift and scaling. Run the simulation in single step mode several times for various values of the parameters. Note the behavior of the sample paths. For selected values of the parameters, run the simulation 1000 times and compare the empirical density function and moments to the true density function and moments.

It's easy to construct Brownian motion with drift and scaling from a standard Brownian motion, so we don't have to worry about the existence question.

Relation to standard Brownian motion.

1. Suppose that  $\mathbf{Z} = \{Z_t : t \in [0, \infty)\}$  is a standard Brownian motion, and that  $\mu \in \mathbb{R}$  and  $\sigma \in (0, \infty)$ . Let  $X_t = \mu t + \sigma Z_t$  for  $t \in [0, \infty)$ . Then  $\mathbf{X} = \{X_t : t \in [0, \infty)\}$  is a Brownian motion with drift parameter  $\mu$  and scale parameter  $\sigma$ .
2. Conversely, suppose that  $\mathbf{X} = \{X_t : t \in [0, \infty)\}$  is a Brownian motion with drift parameter  $\mu \in \mathbb{R}$  and scale parameter  $\sigma \in (0, \infty)$ . Let  $Z_t = (X_t - \mu t)/\sigma$  for  $t \in [0, \infty)$ . Then  $\mathbf{Z} = \{Z_t : t \in [0, \infty)\}$  is a standard Brownian motion.

Proof

It's straightforward to show that the processes  $\mathbf{X}$  and  $\mathbf{Z}$  satisfy the appropriate set of assumptions.

In differential form, part (a) can be written as

$$dX_t = \mu dt + \sigma dZ_t, \quad X_0 = 0 \quad (18.2.1)$$

#### Finite Dimensional Distributions

Suppose that  $\mathbf{X} = \{X_t : t \in [0, \infty)\}$  is Brownian motion with drift parameter  $\mu \in \mathbb{R}$  and scale parameter  $\sigma \in (0, \infty)$ . It follows from part (d) of the [definition](#) that  $X_t$  has probability density function  $f_t$  given by

$$f_t(x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left[-\frac{1}{2\sigma^2 t}(x - \mu t)^2\right], \quad x \in \mathbb{R} \quad (18.2.2)$$

This family of density functions determines the finite dimensional distributions of  $\mathbf{X}$ .

If  $t_1, t_2, \dots, t_n \in (0, \infty)$  with  $0 < t_1 < t_2 < \dots < t_n$  then  $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$  has probability density function  $f_{t_1, t_2, \dots, t_n}$  given by

$$f_{t_1, t_2, \dots, t_n}(x_1, x_2, \dots, x_n) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1}), \quad (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \quad (18.2.3)$$

Proof

This follows because  $\mathbf{X}$  has stationary, independent increments.

$\mathbf{X}$  is a Gaussian process with mean function mean function  $m$  and covariance function  $c$  given by

1.  $m(t) = \mu t$  for  $t \in [0, \infty)$
2.  $c(s, t) = \sigma^2 \min\{s, t\}$  for  $s, t \in [0, \infty)$ .

Proof

The fact that  $\mathbf{X}$  is a Gaussian process follows from the construction  $X_t = \mu t + \sigma Z_t$  for  $t \in [0, \infty)$ , where  $\mathbf{Z}$  is a standard Brownian motion. We know that  $\mathbf{Z}$  is a Gaussian process. The form of the mean and covariance functions follow because  $\mathbf{X}$  has stationary, independent increments. Note that  $\mu$  and  $\sigma^2$  are the mean and variance of  $X_1$ .

The correlation function is independent of the parameters, and thus is the same as for standard Brownian motion. This is hardly surprising since correlation is a standardized measure of association.

$$\text{cor}(X_s, X_t) \frac{\sigma^2 \min\{s, t\}}{\sigma s \sigma t} = \frac{\min\{s, t\}}{st} = \sqrt{\frac{\min\{s, t\}}{\max\{s, t\}}}, \quad (s, t) \in [0, \infty)^2 \quad (18.2.4)$$

## Transformations

There are a couple simple transformations that preserve Brownian motion, but perhaps change the drift and scale parameters. Our starting place is a Brownian motion  $\mathbf{X} = \{X_t : t \in [0, \infty)\}$  with drift parameter  $\mu \in \mathbb{R}$  and scale parameter  $\sigma \in (0, \infty)$ . Our first result involves scaling  $\mathbf{X}$  in time and space (and possibly reflecting in the spatial origin).

Let  $a \in \mathbb{R} \setminus \{0\}$  and  $b \in (0, \infty)$ . Define  $Y_t = aX_{bt}$  for  $t \geq 0$ . Then  $\mathbf{Y} = \{Y_t : t \geq 0\}$  is also a Brownian motion with drift parameter  $ab\mu$  and scale parameter  $|a|\sqrt{b}\sigma$ .

Proof

Clearly the new process is still a Gaussian process. The mean function is  $\mathbb{E}(Y_t) = a\mathbb{E}(X_{bt}) = ab\mu t$  for  $t \in [0, \infty)$ . The covariance function is  $\text{cov}(Y_s, Y_t) = a^2 \text{cov}(X_{bs}, X_{bt}) = a^2 \sigma^2 \min\{bs, bt\} = a^2 b \sigma^2 \min\{s, t\}$  for  $(s, t) \in [0, \infty)^2$ . Finally, since  $\mathbf{X}$  is continuous, so is  $\mathbf{Y}$ .

Suppose that  $a > 0$  in the previous theorem, so that we are scaling temporally and spatially. In order to preserve the original drift parameter  $\mu$  we must have  $ab = 1$  (if  $\mu \neq 0$ ). In order to preserve the original scale parameter  $\sigma$ , we must have  $a\sqrt{b} = 1$ . We can't have both unless  $\mu = 0$ , which leads to a slight generalization of one of our results for standard Brownian motion:

Suppose that  $\mathbf{X}$  is a Brownian motion with drift parameter  $\mu = 0$  and scale parameter  $\sigma > 0$ . Suppose also that  $c > 0$  and let  $Y_t = \frac{1}{c} X_{c^2 t}$  for  $t \geq 0$ . Then  $\mathbf{Y} = \{Y_t : t \in [0, \infty)\}$  is also a Brownian motion with drift parameter 0 and scale parameter  $\sigma$ .

Our next result is related to the Markov property, which we explore in more detail below. We return to the general case where  $\mathbf{X} = \{X_t : t \in [0, \infty)\}$  is a Brownian motion with drift parameter  $\mu \in \mathbb{R}$  and scale parameter  $\sigma \in (0, \infty)$ . If we “restart” Brownian motion at a fixed time  $s$ , and shift the origin to  $X_s$ , then we have another Brownian motion with the same parameters.

Fix  $s \in [0, \infty)$  and define  $Y_t = X_{s+t} - X_s$  for  $t \geq 0$ . Then  $\mathbf{Y} = \{Y_t : t \in [0, \infty)\}$  is also a Brownian motion with the same drift and scale parameters.

Proof

Clearly  $\mathbf{Y}$  is also a Gaussian process. Moreover,  $\mathbb{E}(Y_t) = \mathbb{E}(X_{s+t}) - \mathbb{E}(X_s) = \mu(s+t) - \mu s = \mu t$  for  $t \in [0, \infty)$ . Also, if  $r, t \in [0, \infty)$  with  $r \leq t$  then

$$\text{cov}(Y_r, Y_t) = \text{cov}(X_{s+r} - X_s, X_{s+t} - X_s) \quad (18.2.5)$$

$$= \text{cov}(X_{s+r}, X_{s+t}) - \text{cov}(X_{s+r}, X_s) - \text{cov}(X_s, X_{s+t}) + \text{cov}(X_s, X_s) \quad (18.2.6)$$

$$= \sigma^2(s+r) - \sigma^2 s - \sigma^2 s + \sigma^2 s = \sigma^2 r \quad (18.2.7)$$

Finally,  $\mathbf{Y}$  is continuous by the continuity of  $\mathbf{X}$ .

## The Markov Property and Stopping Times

As usual, we start with a Brownian motion  $\mathbf{X} = \{X_t : t \in [0, \infty)\}$  with drift parameter  $\mu$  and scale parameter  $\sigma$ . Recall again that a Markov process has the property that the future is independent of the past, given the present state. Because of the stationary, independent increments property, Brownian motion has the property. As a minor note, to view  $\mathbf{X}$  as a Markov process, we sometimes need to relax Assumption 1 and let  $X_0$  have an arbitrary value in  $\mathbb{R}$ . Let  $\mathcal{F}_t = \sigma\{X_s : 0 \leq s \leq t\}$ , the sigma-algebra generated by the process up to time  $t \in [0, \infty)$ . The family of  $\sigma$ -algebras  $\mathcal{F} = \{\mathcal{F}_t : t \in [0, \infty)\}$  is known as a filtration.

Brownian motion is a time-homogeneous Markov process with transition probability density  $p$  given by

$$p_t(x, y) = f_t(y - x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left[-\frac{1}{2\sigma^2 t}(y - x - \mu t)^2\right], \quad t \in (0, \infty); \quad x, y \in \mathbb{R} \quad (18.2.8)$$

Proof

Fix  $s \in [0, \infty)$ . The theorem follows from the fact that the process  $\{X_{s+t} - X_s : t \in [0, \infty)\}$  is another standard Brownian motion, as noted [above](#), and is independent of  $\mathcal{F}_s$ .

The transition density  $p$  satisfies the following *diffusion equations*. The first is known as the *forward equation* and the second as the *backward equation*.

$$\frac{\partial}{\partial t} p_t(x, y) = -\mu \frac{\partial}{\partial y} p_t(x, y) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial y^2} p_t(x, y) \quad (18.2.9)$$

$$\frac{\partial}{\partial t} p_t(x, y) = \mu \frac{\partial}{\partial x} p_t(x, y) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} p_t(x, y) \quad (18.2.10)$$

Proof

These results follow from standard calculus.

The diffusion equations are so named, because the spatial derivative in the first equation is with respect to  $y$ , the state *forward* at time  $t$ , while the spatial derivative in the second equation is with respect to  $x$ , the state *backward* at time 0.

Recall again that a random time  $\tau$  taking values in  $[0, \infty]$  is a *stopping time* with respect to the process  $\mathbf{X}$  if  $\{\tau \leq t\} \in \mathcal{F}_t$  for every  $t \in [0, \infty)$ . The  $\sigma$ -algebra associated with  $\tau$  is

$$\mathcal{F}_\tau = \{B \in \mathcal{F} : B \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\} \quad (18.2.11)$$

See the section on Filtrations and Stopping Times for more information on filtrations, stopping times, and the  $\sigma$ -algebra associated with a stopping time. Brownian motion  $\mathbf{X}$  is also a strong Markov process.

Suppose that  $\tau$  is a stopping time and define  $Y_t = X_{\tau+t} - X_\tau$  for  $t \in [0, \infty)$ . Then  $\mathbf{Y} = \{Y_t : t \in [0, \infty)\}$  is a Brownian motion with the same drift and scale parameters, and is independent of  $\mathcal{F}_\tau$ .

This page titled [18.2: Brownian Motion with Drift and Scaling](#) is shared under a [CC BY 2.0](#) license and was authored, remixed, and/or curated by [Kyle Siegrist \(Random Services\)](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform.