

## 13.3: Simple Dice Games

In this section, we will analyze several simple games played with dice—[poker dice](#), [chuck-a-luck](#), and [high-low](#). The casino game craps is more complicated and is studied in the next section.



Figure 13.3.1: The Dice Players by Georges de La Tour (c. 1651). For more on the influence of probability in painting, see the ancillary material on art.

### Poker Dice

#### Definition

The game of *poker dice* is a bit like standard poker, but played with dice instead of cards. In poker dice, 5 fair dice are rolled. We will record the outcome of our random experiment as the (ordered) sequence of scores:

$$\mathbf{X} = (X_1, X_2, X_3, X_4, X_5) \quad (13.3.1)$$

Thus, the sample space is  $S = \{1, 2, 3, 4, 5, 6\}^5$ . Since the dice are fair, our basic modeling assumption is that  $\mathbf{X}$  is a sequence of independent random variables and each is uniformly distributed on  $\{1, 2, 3, 4, 5, 6\}$

Equivalently,  $\mathbf{X}$  is uniformly distributed on  $S$ :

$$\mathbb{P}(\mathbf{X} \in A) = \frac{\#(A)}{\#(S)}, \quad A \subseteq S \quad (13.3.2)$$

In statistical terms, a poker dice hand is a random sample of size 5 drawn with replacement and with regard to order from the population  $D = \{1, 2, 3, 4, 5, 6\}$ . For more on this topic, see the chapter on Finite Sampling Models. In particular, in this chapter you will learn that the result of Exercise 1 would not be true if we recorded the outcome of the poker dice experiment as an unordered set instead of an ordered sequence.

#### The Value of the Hand

The value  $V$  of the poker dice hand is a random variable with support set  $\{0, 1, 2, 3, 4, 5, 6\}$ . The values are defined as follows:

0. *None alike*. Five distinct scores occur.
1. *One Pair*. Four distinct scores occur; one score occurs twice and the other three scores occur once each.
2. *Two Pair*. Three distinct scores occur; one score occurs twice and the other three scores occur once each.
3. *Three of a Kind*. Three distinct scores occur; one score occurs three times and the other two scores occur once each.
4. *Full House*. Two distinct scores occur; one score occurs three times and the other score occurs twice.
5. *Four of a kind*. Two distinct scores occur; one score occurs four times and the other score occurs once.
6. *Five of a kind*. Once score occurs five times.

Run the poker dice experiment 10 times in single-step mode. For each outcome, note that the value of the random variable corresponds to the type of hand, as given above.

### The Probability Density Function

Computing the probability density function of  $V$  is a good exercise in combinatorial probability. In the following exercises, we will need the two fundamental rules of combinatorics to count the number of dice sequences of a given type: the multiplication rule and the addition rule. We will also need some basic combinatorial structures, particularly combinations and permutations (with types of objects that are identical).

The number of different poker dice hands is  $\#(S) = 6^5 = 7776$ .

$$\mathbb{P}(V = 0) = \frac{720}{7776} = 0.09259.$$

Proof

Note that the dice scores form a permutation of size 5 from  $\{1, 2, 3, 4, 5\}$

$$\mathbb{P}(V = 1) = \frac{3600}{7776} \approx 0.46296.$$

Proof

The following steps form an algorithm for generating poker dice hands with one pair. The number of ways of performing each step is also given:

1. Select the score that will appear twice: 6
2. Select the 3 scores that will appear once each:  $\binom{5}{3}$
3. Select a permutation of the 5 numbers in parts (a) and (b):  $\binom{5}{2,1,1,1,1}$

$$\mathbb{P}(V = 2) = \frac{1800}{7776} \approx 0.23148.$$

Proof

The following steps form an algorithm for generating poker dice hands with two pair. The number of ways of performing each step is also given:

1. Select two scores that will appear twice each:  $\binom{6}{2}$
2. Select the score that will appear once: 4
3. Select a permutation of the 5 numbers in parts (a) and (b):  $\binom{5}{2,2,1}$

$$\mathbb{P}(V = 3) = \frac{1200}{7776} \approx 0.15432.$$

Proof

The following steps form an algorithm for generating poker dice hands with three of a kind. The number of ways of performing each step is also given:

1. Select the score that will appear 3 times: 6
2. Select the 2 scores that will appear once each:  $\binom{5}{2}$
3. Select a permutation of the 5 numbers in parts (a) and (b):  $\binom{5}{3,1,1}$

$$\mathbb{P}(V = 4) = \frac{300}{7776} \approx 0.03858.$$

Proof

The following steps form an algorithm for generating poker dice hands with a full house. The number of ways of performing each step is also given:

1. Select the score that will appear 3 times: 6
2. Select the score that will appear twice: 5
3. Select a permutation of the 5 numbers in parts (a) and (b):  $\binom{5}{3,2}$

$$\mathbb{P}(V = 5) = \frac{150}{7776} = 0.01929.$$

Proof

The following steps form an algorithm for generating poker dice hands with four of a kind. The number of ways of performing each step is also given:

1. Select the score that will appear 4 times: 6
2. Select the score that will appear once: 5
3. Select a permutation of the 5 numbers in parts (a) and (b):  $\binom{5}{4,1}$

$$\mathbb{P}(V = 6) = \frac{6}{7776} \approx 0.00077.$$

Proof

There are 6 choices for the score that will appear 5 times.

Run the poker dice experiment 1000 times and compare the relative frequency function to the density function.

Find the probability of rolling a hand that has 3 of a kind or better.

Answer

0.2130

In the poker dice experiment, set the stop criterion to the value of  $V$  given below. Note the number of hands required.

1.  $V = 3$
2.  $V = 4$
3.  $V = 5$
4.  $V = 6$

## Chuck-a-Luck

*Chuck-a-luck* is a popular carnival game, played with three dice. According to Richard Epstein, the original name was *Sweat Cloth*, and in British pubs, the game is known as *Crown and Anchor* (because the six sides of the dice are inscribed clubs, diamonds, hearts, spades, crown and anchor). The dice are over-sized and are kept in an hourglass-shaped cage known as the *bird cage*. The dice are rolled by spinning the bird cage.

Chuck-a-luck is very simple. The gambler selects an integer from 1 to 6, and then the three dice are rolled. If exactly  $k$  dice show the gambler's number, the payoff is  $k : 1$ . As with poker dice, our basic mathematical assumption is that the dice are fair, and therefore the outcome vector  $\mathbf{X} = (X_1, X_2, X_3)$  is uniformly distributed on the sample space  $S = \{1, 2, 3, 4, 5, 6\}^3$ .

Let  $Y$  denote the number of dice that show the gambler's number. Then  $Y$  has the binomial distribution with parameters  $n = 3$  and  $p = \frac{1}{6}$ :

$$\mathbb{P}(Y = k) = \binom{3}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{3-k}, \quad k \in \{0, 1, 2, 3\} \quad (13.3.3)$$

Let  $W$  denote the net winnings for a unit bet. Then

1.  $W = -1$  if  $Y = 0$
2.  $W = Y$  if  $Y > 0$

The probability density function of  $W$  is given by

1.  $\mathbb{P}(W = -1) = \frac{125}{216}$
2.  $\mathbb{P}(W = 1) = \frac{75}{216}$
3.  $\mathbb{P}(W = 2) = \frac{15}{216}$

$$4. \mathbb{P}(W = 3) = \frac{1}{216}$$

Run the chuck-a-luck experiment 1000 times and compare the empirical density function of  $W$  to the true probability density function.

The expected value and variance of  $W$  are

$$1. \mathbb{E}(W) = -\frac{17}{216} \approx 0.0787$$

$$2. \text{var}(W) = \frac{75815}{46656} \approx 1.239$$

Run the chuck-a-luck experiment 1000 times and compare the empirical mean and standard deviation of  $W$  to the true mean and standard deviation. Suppose you had bet \$1 on each of the 1000 games. What would your net winnings be?

## High-Low

In the game of *high-low*, a pair of fair dice are rolled. The outcome is

- *high* if the sum is 8, 9, 10, 11, or 12.
- *low* if the sum is 2, 3, 4, 5, or 6
- *seven* if the sum is 7

A player can bet on any of the three outcomes. The payoff for a bet of high or for a bet of low is 1 : 1. The payoff for a bet of seven is 4 : 1.

Let  $Z$  denote the outcome of a game of high-low. Find the probability density function of  $Z$ .

Answer

$$\mathbb{P}(Z = h) = \frac{15}{36}, \mathbb{P}(Z = l) = \frac{15}{36}, \mathbb{P}(Z = s) = \frac{6}{36}, \text{ where } h \text{ denotes high, } l \text{ denotes low, and } s \text{ denotes seven.}$$

Let  $W$  denote the net winnings for a unit bet. Find the expected value and variance of  $W$  for each of the three bets:

1. high
2. low
3. seven

Answer

Let  $W$  denote the net winnings on a unit bet in high-low.

1. Bet high:  $\mathbb{E}(W) = -\frac{1}{6}, \text{var}(W) = \frac{35}{36}$
2. Bet low:  $\mathbb{E}(W) = -\frac{1}{6}, \text{var}(W) = \frac{35}{36}$
3. Bet seven:  $\mathbb{E}(W) = -\frac{1}{6}, \text{var}(W) = \frac{7}{2}$

This page titled [13.3: Simple Dice Games](#) is shared under a [CC BY 2.0](#) license and was authored, remixed, and/or curated by [Kyle Siegrist \(Random Services\)](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform.