5.20: General Uniform Distributions

This section explores uniform distributions in an abstract setting. If you are a new student of probability, or are not familiar with measure theory, you may want to skip this section and read the sections on the uniform distribution on an interval and the discrete uniform distributions.

Basic Theory

Definition

Suppose that $(S, \mathscr{S}, \lambda)$ is a measure space. That is, S is a set, $\mathscr{S}$ a σ-algebra of subsets of S, and λ a positive measure on $\mathscr{S}$. Suppose also that $0 < \lambda(S) < \infty$, so that λ is a finite, positive measure.

Random variable X with values in S has the *uniform distribution* on S (with respect to λ) if

$$\mathbb{P}(X \in A) = \frac{\lambda(A)}{\lambda(S)}, \quad A \in \mathscr{S} \tag{5.20.1}$$

Thus, the probability assigned to a set $A \in \mathscr{S}$ depends only on the size of A (as measured by λ).

The most common special cases are as follows:

1. *Discrete*: The set S is finite and non-empty, $\mathscr{S}$ is the σ-algebra of all subsets of S, and $\lambda = \#$ (counting measure).
2. *Euclidean*: For $n \in \mathbb{N}_+$, let $\mathscr{R}_n$ denote the σ-algebra of Borel measureable subsets of $\mathbb{R}^n$ and let λ_n denote Lebesgue measure on $(\mathbb{R}^n, \mathscr{R}_n)$. In this setting, $S \in \mathscr{R}_n$ with $0 < \lambda_n(S) < \infty$, $\mathscr{S} = \{A \in \mathscr{R}_n : A \subseteq S\}$, and the measure is λ_n restricted to $(S, \mathscr{S})$.

In the Euclidean case, recall that λ_1 is length measure on $\mathbb{R}$, λ_2 is area measure on $\mathbb{R}^2$, λ_3 is volume measure on $\mathbb{R}^3$, and in general λ_n is sometimes referred to as n-dimensional volume. Thus, $S \in \mathscr{R}_n$ is a set with positive, finite volume.

Properties

Suppose $(S, \mathscr{S}, \lambda)$ is a finite, positive measure space, as above, and that X is uniformly distributed on S.

The probability density function f of X (with respect to λ) is

$$f(x) = \frac{1}{\lambda(S)}, \quad x \in S \tag{5.20.2}$$

Proof

This follows directly from the definition of probability density function:

$$\int_A \frac{1}{\lambda(S)} \, d\lambda(x) = \frac{\lambda(A)}{\lambda(S)}, \quad A \in \mathscr{S} \tag{5.20.3}$$

Thus, the defining property of the uniform distribution on a set is constant density on that set. Another basic property is that uniform distributions are preserved under conditioning.

Suppose that $R \in \mathscr{S}$ with $\lambda(R) > 0$. The conditional distribution of X given $X \in R$ is uniform on R.

Proof

For $A \in \mathscr{S}$ with $A \subseteq R$,

$$\mathbb{P}(X \in A \mid X \in R) = \frac{\mathbb{P}(X \in A)}{\mathbb{P}(X \in R)} = \frac{\lambda(A)/\lambda(S)}{\lambda(R)/\lambda(S)} = \frac{\lambda(A)}{\lambda(R)} \tag{5.20.4}$$

In the setting of previous result, suppose that $\boldsymbol{X} = (X_1, X_2, \ldots)$ is a sequence of independent variables, each uniformly distributed on S. Let $N = \min\{n \in \mathbb{N}_+ : X_n \in R\}$. Then N has the geometric distribution on $\mathbb{N}_+$ with success parameter $p = \mathbb{P}(X \in R)$. More importantly, the distribution of X_N is the same as the conditional distribution of X given $X \in R$, and hence is uniform on R. This is the basis of the *rejection method* of simulation. If we can simulate a uniform distribution on S, then we can simulate a uniform distribution on R.

If h is a real-valued function on S, then $\mathbb{E}[h(X)]$ is the average value of h on S, as measured by λ:

If $h : S \to \mathbb{R}$ is integrable with respect to λ Then

$$\mathbb{E}[h(X)] = \frac{1}{\lambda(S)} \int_S h(x)\, d\lambda(x) \tag{5.20.5}$$

Proof

This result follows from the change of variables theorem for expected value, since

$$\mathbb{E}[h(X)] = \int_S h(x) f(x)\, d\lambda(x) = \frac{1}{\lambda(S)} \int_S h(x)\, d\lambda(x) \tag{5.20.6}$$

The entropy of the uniform distribution on S depends only on the size of S, as measured by λ:

The entropy of X is $H(X) = \ln[\lambda(S)]$.

Proof

$$H(X) = \mathbb{E}\{-\ln[f(X)]\} = \int_S -\ln\left(\frac{1}{\lambda(S)}\right)\frac{1}{\lambda(S)} = -\ln\left(\frac{1}{\lambda(S)}\right) = \ln[\lambda(S)] \tag{5.20.7}$$

Product Spaces

Suppose now that $(S, \mathscr{S}, \lambda)$ and $(T, \mathscr{T}, \mu)$ are finite, positive measure spaces, so that $0 < \lambda(S) < \infty$ and $0 < \mu(T) < \infty$. Recall the product space $(S \times T, \mathscr{S} \otimes \mathscr{T}, \lambda \otimes \mu)$. The *product σ-algebra* $\mathscr{S} \otimes \mathscr{T}$ is the σ-algebra of subsets of $S \times T$ generated by product sets $A \times B$ where $A \in \mathscr{S}$ and $B \in \mathscr{T}$. The *product measure* $\lambda \otimes \mu$ is the unique positive measure on $(S \times T, \mathscr{S} \otimes \mathscr{T})$ that satisfies $(\lambda \otimes \mu)(A \times B) = \lambda(A)\mu(B)$ for $A \in \mathscr{S}$ and $B \in \mathscr{T}$.

(X, Y) is uniformly distributed on $S \times T$ if and only if X is uniformly distributed on S, Y is uniformly distributed on T, and X and Y are independent.

Proof

Suppose first that (X, Y) is uniformly distributed on $S \times T$. If $A \in \mathscr{S}$ and $B \in \mathscr{T}$ then

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}[(X, Y) \in A \times B] = \frac{(\lambda \otimes \mu)(A \times B)}{(\lambda \otimes \mu)(S \times T)} = \frac{\lambda(A)\mu(B)}{\lambda(S)\mu(T)} = \frac{\lambda(A)}{\lambda(S)}\frac{\mu(B)}{\mu(T)} \tag{5.20.8}$$

Taking $B = T$ in the displayed equation gives $\mathbb{P}(X \in A) = \lambda(A)/\lambda(S)$ for $A \in \mathscr{S}$, so X is uniformly distributed on S. Taking $A = S$ in the displayed equation gives $\mathbb{P}(Y \in B) = \mu(B)/\mu(T)$ for $B \in \mathscr{T}$, so Y is uniformly distributed on T. Returning to the displayed equation generally gives $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$ for $A \in \mathscr{S}$ and $B \in \mathscr{T}$, so X and Y are independent.

Conversely, suppose that X is uniformly distributed on S, Y is uniformly distributed on T, and X and Y are independent. Then for $A \in \mathscr{S}$ and $B \in \mathscr{T}$,

$$\mathbb{P}[(X, Y) \in A \times B] = \mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B) = \frac{\lambda(A)}{\lambda(S)}\frac{\mu(B)}{\mu(T)} = \frac{\lambda(A)\mu(B)}{\lambda(S)\mu(T)} = \frac{(\lambda \otimes \mu)(A \times B)}{(\lambda \otimes \mu)(S \times T)} \tag{5.20.9}$$

It then follows (see the section on existence and uniqueness of measures) that $\mathbb{P}[(X, Y) \in C] = (\lambda \otimes \mu)(C)/(\lambda \otimes \mu)(S \times T)$ for every $C \in \mathscr{S} \otimes \mathscr{T}$, so (X, Y) is uniformly distributed on $S \times T$.