5.18: The Beta Prime Distribution

Basic Theory

The beta prime distribution is the distribution of the odds ratio associated with a random variable with the beta distribution. Since variables with beta distributions are often used to model random probabilities and proportions, the corresponding odds ratios occur naturally as well.

Definition

Suppose that U has the beta distribution with shape parameters $a, b \in (0, \infty)$. Random variable $X = U \big/ (1-U)$ has the *beta prime distribution* with shape parameters a and b.

The special case $a = b = 1$ is known as the *standard beta prime distribution*. Since U has a continuous distribution on the interval $(0, 1)$, random variable X has a continuous distribution on the interval $(0, \infty)$.

Distribution Functions

Suppose that X has the beta prime distribution with shape parameters $a, b \in (0, \infty)$, and as usual, let B denote the beta function.

X has probability density function f given by

$$f(x) = \frac{1}{B(a,b)} \frac{x^{a-1}}{(1+x)^{a+b}}, \quad x \in (0, \infty) \tag{5.18.1}$$

Proof

First, recall that the beta PDF g with parameters a and b is

$$g(u) = u^{a-1}(1-u)^{b-1}, \quad u \in (0,1) \tag{5.18.2}$$

The transformation $x = u \big/ (1-u)$ maps $(0,1)$ onto $(0,\infty)$ and is increasing. The inverse transformation is $u = x \big/ (x+1)$, and $1 - u = 1 \big/ (x+1)$ and $du/dx = 1 \big/ (x+1)^2$. Thus, by the change of variables formula,

$$f(x) = g(u)\frac{du}{dx} = \frac{1}{B(a,b)} \left(\frac{x}{x+1}\right)^{a-1} \left(\frac{1}{x+1}\right)^{b-1} \frac{1}{(x+1)^2} = \frac{1}{B(a,b)} \frac{x^{a-1}}{(x+1)^{a+b}}, \quad x \in (0,\infty) \tag{5.18.3}$$

If $a \ge 1$, the probability density function is defined at $x = 0$, so in this case, it's customary add this endpoint to the domain. In particular, for the standard beta prime distribution,

$$f(x) = \frac{1}{(1+x)^2}, \quad x \in [0, \infty) \tag{5.18.4}$$

Qualitatively, the first order properties of the probability density function f depend only on a, and in particular on whether a is less than, equal to, or greater than 1.

The probability density function f satisfies the following properties:

1. If $0 < a < 1$, f is decreasing with $f(x) \to \infty$ as $x \downarrow 0$.
2. If $a = 1$, f is decreasing with mode at $x = 0$.
3. If $a > 1$, f increases and then decreases with mode at $x = (a-1) \big/ (b+1)$.

Proof

These properties follow from standard calculus. The first derivative of f is

$$f'(x) = \frac{1}{B(a,b)} \frac{x^{a-2}}{(1+x)^{a+b+1}} [(a-1) - x(b+1)], \quad x \in (0,\infty) \tag{5.18.5}$$

Qualitatively, the second order properties of f also depend only on a, with transitions at $a = 1$ and $a = 2$.

For $a > 1$, define

$$x_1 = \frac{(a-1)(b+2) - \sqrt{(a-1)(b+2)(a+b)}}{(b+1)(b+2)} \tag{5.18.6}$$

$$x_2 = \frac{(a-1)(b+2) + \sqrt{(a-1)(b+2)(a+b)}}{(b+1)(b+2)} \tag{5.18.7}$$

The probability density function f satisfies the following properties:

1. If $0 < a \le 1$, f is concave upward.
2. If $1 < a \le 2$, f is concave downward and then upward, with inflection point at x_2.
3. If $a > 2$, f is concave upward, then downward, then upward again, with inflection points at x_1 and x_2.

Proof

These results follow from standard calculus. The second derivative of f is

$$f''(x) = \frac{1}{B(a,b)} \frac{x^{a-3}}{(1+x)^{a+b+2}} \left[(a-1)(a-2) - 2(a-1)(b+2)x + (b+1)(b+2)x^2\right], \quad x \in (0, \infty) \tag{5.18.8}$$

Open the Special Distribution Simulator and select the beta prime distribution. Vary the parameters and note the shape of the probability density function. For selected values of the parameters, run the simulation 1000 times and compare the empirical density function to the probability density function.

Because of the definition of the beta prime variable, the distribution function of X has a simple expression in terms of the beta distribution function with the same parameters, which in turn is the regularized incomplete beta function. So let G denote the distribution function of the beta distribution with parameters $a, b \in (0, \infty)$, and recall that

$$G(x) = \frac{B(x; a, b)}{B(a,b)}, \quad x \in (0,1) \tag{5.18.9}$$

X has distribution function F given by

$$F(x) = G\left(\frac{x}{x+1}\right), \quad x \in [0, \infty) \tag{5.18.10}$$

Proof

As noted in the proof of the formula for the PDF, $x = u/(1-u)$ is strictly increasing with inverse $u = x/(x+1)$. Hence

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}\left(\frac{U}{U-1} \le x\right) = \mathbb{P}\left(U \le \frac{x}{x+1}\right) = G\left(\frac{x}{x+1}\right), \quad x \in [0, \infty) \tag{5.18.11}$$

Similarly, the quantile function of X has a simple expression in terms of the beta quantile function G^{-1} with the same parameters.

X has quantile function F^{-1} given by

$$F^{-1}(p) = \frac{G^{-1}(p)}{1 - G^{-1}(p)}, \quad p \in [0, 1) \tag{5.18.12}$$

Proof

This follows from the result for the CDF by solving $p = F(x) = G\left(\frac{x}{x+1}\right)$ for x in terms of p.

Open the Special Distribution Calculator and choose the beta prime distribution. Vary the parameters and note the shape of the distribution function. For selected values of the parameters, find the median and the first and third quartiles.

For certain values of the parameters, the distribution and quantile functions have simple, closed form expressions.

If $a \in (0, \infty)$ and $b = 1$ then

1. $F(x) = \left(\frac{x}{x+1}\right)^a$ for $x \in [0, \infty)$
2. $F^{-1}(p) = \frac{p^{1/a}}{1-p^{1/a}}$ for $p \in [0, 1)$

Proof

For $a > 0$ and $b = 1$, $G(u) = u^a$ for $u \in [0, 1]$ and $G^{-1}(p) = p^{1/a}$ for $p \in [0, 1]$

If $a=1$ and $b \in (0,\infty)$ then

1. $F(x)=1-\left(\frac{1}{x+1}\right)^b$ for $x \in [0,\infty)$
2. $F^{-1}(p)=\frac{1-(1-p)^{1/b}}{(1-p)^{1/b}}$ for $p \in [0,1)$

Proof

For $a=1$ and $b>0$, $G(u)=1-(1-u)^b$ for $u \in [0,1]$ and $G^{-1}(p)=1-(1-p)^{1/b}$ for $p \in [0,1]$.

If $a=b=\frac{1}{2}$ then

1. $F(x)=\frac{2}{\pi}\arcsin\left(\sqrt{\frac{x}{x+1}}\right)$ for $x \in [0,\infty)$
2. $F^{-1}(p)=\frac{\sin^2\left(\frac{\pi}{2}p\right)}{1-\sin^2\left(\frac{\pi}{2}p\right)}$ for $p \in [0,1)$

Proof

For $a=b=\frac{1}{2}$, $G(u)=\frac{2}{\pi}\arcsin(\sqrt{u})$ for $u \in (0,1)$ and $G^{-1}(p)=\sin^2\left(\frac{\pi}{2}p\right)$ for $p \in [0,1]$

When $a=b=\frac{1}{2}$, X is the odds ratio for a variable with the standard arcsine distribution.

Moments

As before, X denotes a random variable with the beta prime distribution, with parameters a, $b \in (0,\infty)$. The moments of X have a simple expression in terms of the beta function.

If $t \in (-a,b)$ then

$$\mathbb{E}\left(X^t\right)=\frac{B(a+t,b-t)}{B(a,b)} \tag{5.18.13}$$

If $t \in (-\infty,-a] \cup [b,\infty)$ then $\mathbb{E}(X^t)=\infty$.

Proof

Once again, let g denote the beta PDF with parameters a and b. With the transformation $x=u/(1-u)$, as in the proof PDF formula, we have $f(x)dx=g(u)du$. Hence

$$\int_0^\infty x^t f(x)dx=\int_0^1\left(\frac{u}{1-u}\right)^t g(u)du=\frac{1}{B(a,b)}\int_0^1 u^{a+t-1}(1-u)^{b-t-1}du \tag{5.18.14}$$

If $t \le -a$ the improper integral diverges to ∞ at 0. If $t \ge b$ the improper integral diverges to ∞ at 1. If $-a<t<b$ the integral is $B(a+t,b-t)$ by definition of the beta function.

Of course, we are usually most interested in the integer moments of X. Recall that for $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the *rising power* of x of order n is $x^{[n]}=x(x+1)\cdots(x+n-1)$.

Suppose that $n \in \mathbb{N}$. If $n<b$ Then

$$\mathbb{E}\left(X^n\right)=\prod_{k=1}^n\frac{a+k-1}{b-k} \tag{5.18.15}$$

If $n \ge b$ then $\mathbb{E}\left(X^n\right)=\infty$.

Proof

From the general moment result,

$$\mathbb{E}(X^n)=\frac{B(a+n,b-n)}{B(a,b)}=\frac{\Gamma(a+n)\Gamma(a-n)}{\Gamma(a+b)}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}=\frac{\Gamma(a+n)}{\Gamma(a)}\frac{\Gamma(b-n)}{\Gamma(b)}=\frac{a^{[n]}}{(b-n)^{[n]}} \tag{5.18.16}$$

by a basic property of the gamma function.

As a corollary, we have the mean and variance.

If $b>1$ then

$$\mathbb{E}(X) = \frac{a}{b-1} \tag{5.18.17}$$

If $b > 2$ then

$$\operatorname{var}(X) = \frac{a(a+b-1)}{(b-1)^2(b-2)} \tag{5.18.18}$$

Proof

This follows from the general moment result above and the computational formula $\operatorname{var}(X) = \mathbb{E}\left(X^2\right) = [\mathbb{E}(X)]^2$.

Open the Special Distribution Simulator and select the beta prime distribution. Vary the parameters and note the size and location of the mean$\pm$standard deviation bar. For selected values of the parameters, run the simulation 1000 times and compare the empirical mean and standard deviation to the distribution mean and standard deviation.

Finally, the general moment result leads to the skewness and kurtosis of X.

If $b > 3$ then

$$\operatorname{skew}(X) = \frac{2(2a+b-1)}{b-3}\sqrt{\frac{b-2}{a(a+b-1)}} \tag{5.18.19}$$

Proof

This follows from the usual computational formula for skewness in terms of the moments $\mathbb{E}(X^n)$ for $n \in \{1,2,3\}$and the general moment result above.

In particular, the distibution is positively skewed for all $a > 0$ and $b > 3$.

If $b > 4$ then

$$\begin{aligned} &\operatorname{kurt}(X) \\ &= \frac{3a^3b^2+69a^3b-30a^3+6a^2b^3+12a^2b^2-78a^2b+60a^2+3ab^4+9ab^3-69ab^2+99ab-42a+6b^4-30b^3+54b^2-42b+12}{(a+b-1)(b-3)(b-4)} \end{aligned} \tag{5.18.20}$$

Proof

This follows from the usual computational formula for kurtosis in terms of the moments $\mathbb{E}(X^n)$ for $n \in \{1,2,3,4\}$ and the general moment result above.

Related Distributions

The most important connection is the one between the beta prime distribution and the beta distribution given in the definition. We repeat this for emphasis.

Suppose that $a,\ b \in (0,\infty)$.

1. If U has the beta distribution with parameters a and b, then $X = U\big/(1-U)$ has the beta prime distribution with parameters a and b.
2. If X has the beta prime distribution with parameters a and b, then $U = X\big/(X+1)$ has the beta distribution with parameters a and b.

The beta prime family is closed under the reciprocal transformation.

If X has the beta prime distribution with parameters $a,\ b \in (0,\infty)$ then $1/X$ has the beta prime distribution with parameters b and a.

Proof

A direct proof using the change of variables formula is possible, of course, but a better proof uses a corresponding property of the beta distribution. By definition, we can take $X = U\big/(1-U)$ where U has the beta distribution with parameters a and b. But then $1/X = (1-U)\big/U$, and $1-U$ has the beta distribution with parameters b and a. By another application of the definition, $1/X$ has the beta prime distribution with parameters b and a.

The beta prime distribution is closely related to the F distribution by a simple scale transformation.

Connections with the F distributions.

1. If X has the beta prime distribution with parameters $a, b \in (0, \infty)$ then $Y = \frac{b}{a} X$ has the F distribution with $2a$ degrees of the freedom in the numerator and $2b$ degrees of freedom in the denominator.
2. If Y has the F distribution with $n \in (0, \infty)$ degrees of freedom in the numerator and $d \in (0, \infty)$ degrees of freedom in the denominator, then $X = \frac{n}{d} Y$ has the beta prime distribution with parameters $n/2$ and $d/2$.

Proof

Let f denote the PDF of X and g the PDF of Y.

1. By the change of variables formula,
$$g(y) = \frac{a}{b} f\left(\frac{a}{b} y\right), \quad x \in (0, \infty) \tag{5.18.21}$$
Substituting into the beta prime PDF shows that Y has the appropriate F distribution.
2. Again using the change of variables formula,
$$f(x) = \frac{d}{n} g\left(\frac{d}{n} x\right), \quad x \in (0, \infty) \tag{5.18.22}$$
Substituting into the F PDF shows that X has the appropriate beta prime PDF.

The beta prime is the distribution of the ratio of independent variables with standard gamma distributions. (Recall that *standard* here means that the scale parameter is 1.)

Suppose that Y and Z are independent and have standard gamma distributions with shape parameters $a \in (0, \infty)$ and $b \in (0, \infty)$, respectively. Then $X = Y/Z$ has the beta prime distribution with parameters a and b.

Proof

Of course, a direct proof can be constructed, but a better approach is to use the previous result. Thus suppose that Y and Z are as stated in the theorem. Then $2Y$ and $2Z$ are independent chi-square variables with $2a$ and $2b$ degrees of freedom, respectively. Hence
$$W = \frac{Y/2a}{Z/2b} \tag{5.18.23}$$
has the F distribution with $2a$ degrees of freedom in the numerator and $2b$ degrees of freedom in the denominator. By the previous result,
$$X = \frac{2a}{2b} W = \frac{Y}{Z} \tag{5.18.24}$$
has the beta prime distribution with parameters a and b.

The standard beta prime distribution is the same as the standard log-logistic distribution.

Proof

The PDF of the standard beta prime distribution is $f(x) = 1/(1+x)^2$ for $x \in [0, \infty)$, which is the same as the PDF of the standard log-logistic distribution.

Finally, the beta prime distribution is a member of the general exponential family of distributions.

Suppose that X has the beta prime distribution with parameters $a, b \in (0, \infty)$. Then X has a two-parameter general exponential distribution with natural parameters $a - 1$ and $-(a + b)$ and natural statistics $\ln(X)$ and $\ln(1 + X)$.

Proof

This follows from the definition of the general exponential family, since the PDF can be written in the form
$$f(x) = \frac{1}{B(a, b)} \exp[(a-1)\ln(x) - (a+b)\ln(1+x)], \quad x \in (0, \infty) \tag{5.18.25}$$