16.20: Chains Subordinate to the Poisson Process

Basic Theory

Introduction

Recall that the standard Poisson process with rate parameter $r \in (0, \infty)$ involves three interrelated stochastic processes. First the sequence of *interarrival times* $\boldsymbol{T} = (T_1, T_2, \ldots)$ is independent, and each variable has the exponential distribution with parameter r. Next, the sequence of *arrival times* $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots)$ is the partial sum sequence associated with the interrival sequence $\boldsymbol{T}$:

$$\tau_n = \sum_{i=1}^n T_i, \quad n \in \mathbb{N} \tag{16.20.1}$$

For $n \in \mathbb{N}_+$, the arrival time τ_n has the gamma distribution with parameters n and r. Finally, the Poisson counting process $\boldsymbol{N} = \{N_t : t \in [0, \infty)\}$ is defined by

$$N_t = \max\{n \in \mathbb{N} : \tau_n \le t\}, \quad t \in [0, \infty) \tag{16.20.2}$$

so that N_t is the number of arrivals in $(0, t]$ for $t \in [0, \infty)$. The counting variable N_t has the Poisson distribution with parameter rt for $t \in [0, \infty)$. The counting process $\boldsymbol{N}$ and the arrival time process $\boldsymbol{\tau}$ are inverses in the sense that $\tau_n \le t$ if and only if $N_t \ge n$ for $t \in [0, \infty)$ and $n \in \mathbb{N}$. The Poisson counting process can be viewed as a continuous-time Markov chain.

Suppose that X_0 takes values in $\mathbb{N}$ and is independent of $\boldsymbol{N}$. Define $X_t = X_0 + N_t$ for $t \in [0, \infty)$. Then $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ is a continuous-time Markov chain on $\mathbb{N}$ with exponential parameter function given by $\lambda(x) = r$ for $x \in \mathbb{N}$ and jump transition matrix Q given by $Q(x, x+1) = 1$ for $x \in S$.

Proof

This follows directly from the basic structure of a continuous-time Markov chain. Given $X_t = x$, the holding time in state $x \in \mathbb{N}$ is exponential with parameter r, and the next state is deterministically $x + 1$. Note that the addition of the variable X_0 is just to allow us the freedom of arbitrary initial distributions on the state space, as is routine with Markov processes.

Note that the Poisson process, viewed as a Markov chain is a pure birth chain. Clearly we can generalize this continuous-time Markov chain in a simple way by allowing a general embedded jump chain.

Suppose that $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ is a Markov chain with (countable) state space S, and with constant exponential parameter $\lambda(x) = r \in (0, \infty)$ for $x \in S$, and jump transition matrix Q. Then $\boldsymbol{X}$ is said to be *subordinate* to the Poisson process with rate parameter r.

1. The transition times $(\tau_1, \tau_2, \ldots)$ are the arrival times of the Poisson process with rate r.
2. The inter-transition times $(\tau_1, \tau_2 - \tau_1, \ldots)$ are the inter-arrival times of the Poisson process with rate r (independent, and each with the exponential distribution with rate r).
3. $\boldsymbol{N} = \{N_t : t \in [0, \infty)\}$ is the Poisson counting process, where N_t is the number of transitions in (0, t] for $t \in [0, \infty)$.
4. The Poisson process and the jump chain $\boldsymbol{Y} = (Y_0, Y_1, \ldots)$ are independent, and $X_t = Y_{N_t}$ for $t \in [0, \infty)$.

Proof

These results all follow from the basic structure of a continuous-time Markov chain.

Since all states are stable, note that we must have $Q(x, x) = 0$ for $x \in S$. Note also that for $x, y \in S$ with $x \ne y$, the exponential rate parameter for the transition from x to y is $\mu(x, y) = rQ(x, y)$. Conversely suppose that $\mu : S^2 \to (0, \infty)$ satisfies $\mu(x, x) = 0$ and $\sum_{y \in S} \mu(x, y) = r$ for every $x \in S$. Then the Markov chain with transition rates given by μ is subordinate to the Poisson process with rate r. It's easy to construct a Markov chain subordinate to the Poisson process.

Suppose that $\boldsymbol{N} = \{N_t : t \in [0, \infty)\}$ is a Poisson counting process with rate $r \in (0, \infty)$ and that $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}\}$ is a discrete-time Markov chain on S, independent of $\boldsymbol{N}$, whose transition matrix satisfies $Q(x, x) = 0$ for every $x \in S$. Let $X_t = Y_{N_t}$ for $t \in [0, \infty)$. Then $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ is a continuous-time Markov chain subordinate to the Poisson process.

Generator and Transition Matrices

Next let's find the generator matrix and the transition semigroup. Suppose again that $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ is a continuous-time Markov chain on S subordinate to the Poisson process with rate $r \in (0, \infty)$ and with jump transition matrix Q. As usual, let $\boldsymbol{P} = \{P_t : t \in [0, \infty)\}$ denote the transition semigroup and G the infinitesimal generator.

The generator matrix G of $\boldsymbol{X}$ is $G = r(Q - I)$. Hence for $t \in [0, \infty)$

1. The Kolmogorov backward equation is $P_t' = r(Q - I)P_t$
2. The Kolmogorov forward equation is $P_t' = rP_t(Q - I)$

Proof

This follows directly from the general theory since $G(x, x) = -\lambda(x) = -r$ for $x \in S$ and $G(x, y) = \lambda(x)Q(x, y) = rQ(x, y)$ for distinct $x, y \in S$.

There are several ways to find the transition semigroup $\boldsymbol{P} = \{P_t : t \in [0, \infty)\}$. The best way is a probabilistic argument using the underlying Poisson process.

For $t \in [0, \infty)$, the transition matrix P_t is given by

$$P_t = \sum_{n=0}^\infty e^{-rt} \frac{(rt)^n}{n!} Q^n \tag{16.20.3}$$

Proof from the underlying Poisson process

Let N_t denote the number of transitions in $(0, t]$ for $t \in [0, \infty)$, so that $\boldsymbol{N} = \{N_t : t \in [0, \infty)\}$ is the Poisson counting process. Let $\boldsymbol{Y} = (Y_0, Y_1, \ldots)$ denote the jump chain, with transition matrix Q. Then $\boldsymbol{N}$ and $\boldsymbol{Y}$ are independent, and $X_t = Y_{N_t}$ for $t \in [0, \infty)$. Conditioning we have

$$\begin{aligned} P_t(x, y) &= \mathbb{P}(X_t = y \mid X_0 = x) = \mathbb{P}\left(Y_{N_t} = y \mid Y_0 = x\right) \\ &= \sum_{n=0}^\infty \mathbb{P}\left(Y_{N_t} = y \mid N_t = n, Y_0 = y\right) \mathbb{P}(N_t = n \mid Y_0 = y) \\ &= \sum_{n=0}^\infty \mathbb{P}(Y_n = y \mid Y_0 = x) \mathbb{P}(N_t = n) = \sum_{n=0}^\infty e^{-rt} \frac{(rt)^n}{n!} Q^n(x, y) \end{aligned}$$

Proof using the generator matrix

Note first that for $n \in \mathbb{N}$,

$$G^n = [r(Q - I)]^n = r^n \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} Q^k \tag{16.20.4}$$

Hence

$$\begin{aligned} P_t &= e^{tG} = \sum_{n=0}^\infty \frac{t^n}{n!} G^n = \sum_{n=0}^\infty \frac{t^n}{n!} r^n \sum_{k=0}^\infty \binom{n}{k} (-1)^{n-k} Q^k \\ &= \sum_{n=0}^\infty \sum_{k=0}^n \frac{(rt)^n}{k!(n-k)!} (-1)^{n-k} Q^k = \sum_{k=0}^\infty \sum_{n=k}^\infty \frac{(rt)^n}{k!(n-k)!} (-1)^{n-k} Q^k \\ &= \sum_{k=0}^\infty \frac{(rt)^k}{k!} Q^k \sum_{n=k}^\infty \frac{1}{(n-k)!} (-rt)^{n-k} = \sum_{k=0}^\infty e^{-rt} \frac{(rt)^k}{k!} Q^k \end{aligned}$$

Potential Matrices

Next let's find the potential matrices. As with the transition matrices, we can do this in (at least) two different ways.

Suppose again that $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ is a continuous-time Markov chain on S subordinate to the Poisson process with rate $r \in (0, \infty)$ and with jump transition matrix Q. For $\alpha \in (0, \infty)$, the potential matrix U_α of $\boldsymbol{X}$ is

$$U_\alpha = \frac{1}{\alpha + r}\sum_{n=0}^\infty \left(\frac{r}{\alpha + r}\right)^n Q^n \tag{16.20.5}$$

Proof from the definition

Using the previous result,

$$\begin{aligned} U_\alpha(x, y) &= \int_0^\infty e^{-\alpha t} P_t(x, y)\,dt = \int_0^\infty e^{-\alpha t} \sum_{n=0}^\infty e^{-rt} \frac{(rt)^n}{n!} Q^n(x, y)\,dt \\ &= \sum_{n=0}^\infty Q^n(x, y) \frac{r^n}{n!} \int_0^\infty e^{-(r + \alpha)t} t^n\,dt \end{aligned}$$

The interchange of sum and integral is justified since the terms are nonnegative. Using the change of variables $s = (r + \alpha)t$ gives

$$U_\alpha(x, y) = \frac{1}{\alpha + r} \sum_{n=0}^\infty \left(\frac{r}{\alpha + r}\right)^n \frac{1}{n!} Q^n(x, y) \int_0^\infty e^{-s t} s^n\,ds \tag{16.20.6}$$

The last integral is $n!$.

Proof using the generator

From the result above,

$$\alpha I - G = \alpha I - r(Q - I) = (\alpha + r) I - r Q = (\alpha + r)\left(I - \frac{r}{\alpha + r} Q\right) \tag{16.20.7}$$

Since $\left\|\frac{r}{\alpha + r} Q\right\| = \frac{r}{\alpha + r} < 1$ we have

$$(\alpha I - G)^{-1} = \frac{1}{\alpha + r}\left(I - \frac{r}{\alpha + r} Q\right)^{-1} = \frac{1}{\alpha + r} \sum_{n=0}^\infty \left(\frac{r}{\alpha + r}\right)^n Q^n \tag{16.20.8}$$

Recall that for $p \in (0, 1)$, the p-potential matrix of the jump chain $\boldsymbol{Y}$ is $R_p = \sum_{n=0}^\infty p^n Q^n$. Hence we have the following nice relationship between the potential matrix of $\boldsymbol{X}$ and the potential matrix of $\boldsymbol{Y}$:

$$U_\alpha = \frac{1}{\alpha + r} R_{r/(\alpha + r)} \tag{16.20.9}$$

Next recall that $\alpha U_\alpha(x, \cdot)$ is the probability density function of X_T given $X_0 = x$, where T has the exponential distribution with parameter α and is independent of $\boldsymbol{X}$. On the other hand, $\alpha U_\alpha(x, \cdot) = (1 - p) R_p(x, \cdot)$ where $p = r \big/ (\alpha + r)$. We know from our study of discrete potentials that $(1 - p) R_p(x, \cdot)$ is the probability density function of Y_M where M has the geometric distribution on $\mathbb{N}$ with parameter $1 - p$ and is independent of $\boldsymbol{Y}$. But also $X_T = Y_{N_T}$. So it follows that if T has the exponential distribution with parameter α, $\boldsymbol{N} = \{N_t : t \in [0, \infty)\}$ is a Poisson process with rate r, and is independent of T, then N_T has the geometric distribution on $\mathbb{N}$ with parameter $\alpha \big/ (\alpha + r)$. Of course, we could easily verify this directly, but it's still fun to see such connections.

Limiting Behavior and Stationary Distributions

Once again, suppose that $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ is a continuous-time Markov chain on S subordinate to the Poisson process with rate $r \in (0, \infty)$ and with jump transition matrix Q. Let $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}\}$ denote the jump process. The limiting behavior and stationary distributions of $\boldsymbol{X}$ are closely related to those of $\boldsymbol{Y}$.

Suppose that $\boldsymbol{X}$ (and hence $\boldsymbol{Y}$) are irreducible and positive recurrent

1. $g : S \to (0, \infty)$ is invariant for $\boldsymbol{X}$ if and only if g is invariant for $\boldsymbol{Y}$.
2. f is an invariant probability density function for $\boldsymbol{X}$ if and only if f is an invariant probability density function for $\boldsymbol{Y}$.
3. $\boldsymbol{X}$ is null recurrent if and only if $\boldsymbol{Y}$ is null recurrent, and in this case, $\lim_{n \to \infty} Q^n(x, y) = \lim_{t \to \infty} P_t(x, y) = 0$ for $(x, y) \in S^2$.
4. $\boldsymbol{X}$ is positive recurrent if and only if $\boldsymbol{Y}$ is positive recurrent. If $\boldsymbol{Y}$ is aperiodic, then $\lim_{n \to \infty} Q^n(x, y) = \lim_{t \to \infty} P_t(x, y) = f(y)$ for $(x, y) \in S^2$, where f is the invariant probability density function.

Proof

All of these results follow from the basic theory of stationary and limiting distributions for continuous-time chains, and the fact that the exponential parameter function λ is constant.

Time Reversal

Once again, suppose that $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ is a continuous-time Markov chain on S subordinate to the Poisson process with rate $r \in (0, \infty)$ and with jump transition matrix Q. Let $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}\}$ denote the jump process. We assume that $\boldsymbol{X}$ (and hence $\boldsymbol{Y}$) are irreducible. The time reversal of $\boldsymbol{X}$ is closely related to that of $\boldsymbol{Y}$.

Suppose that $g : S \to (0, \infty)$ is invariant for $\boldsymbol{X}$. The time reversal $\hat{\boldsymbol{X}}$ with respect to g is also subordinate to the Poisson process with rate r. The jump chain $\hat{\boldsymbol{Y}}$ of $\hat{\boldsymbol{X}}$ is the (discrete) time reversal of $\boldsymbol{Y}$ with respect to g.

Proof

From the previous result, g is also invariant for $\boldsymbol{Y}$. From the general theory of time reversal, $\hat{\boldsymbol{X}}$ has the same exponential parameter function as $\boldsymbol{X}$ (namely the constant function r) and so is also subordinate to the Poisson process with rate r. Finally, the jump chain $\hat{\boldsymbol{Y}}$ of $\hat{\boldsymbol{X}}$ is the reversal of $\boldsymbol{Y}$ with respect to rg and hence also with respect to g.

In particular, $\boldsymbol{X}$ is reversible with respect to g if and only if $\boldsymbol{Y}$ is reversible with respect to g. As noted earlier, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are of the same type: both transient or both null recurrent or both positive recurrent. In the recurrent case, there exists a positive invariant function that is unique up to multiplication by constants. In this case, the reversal of $\boldsymbol{X}$ is unique, and is the chain subordinate to the Poisson process with rate r whose jump chain is the reversal of $\boldsymbol{Y}$.

Uniform Chains

In the construction above for a Markov chain $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ that is subordinate to the Poisson process with rate r and jump transition kernel Q, we assumed of course that $Q(x, x) = 0$ for every $x \in S$. So there are no absorbing states and the sequence $(\tau_1, \tau_2, \ldots)$ of arrival times of the Poisson process are the jump times of the chain $\boldsymbol{X}$. However in our introduction to continuous-time chains, we saw that the general construction of a chain starting with the function λ and the transition matrix Q works without this assumption on Q, although the exponential parameters and transition probabilities change. The same idea works here.

Suppose that $\boldsymbol{N} = \{N_t : t \in [0, \infty)\}$ is a counting Poisson process with rate $r \in (0, \infty)$ and that $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}\}$ is a discrete-time Markov chain with transition matrix Q on $S \times S$ satisfying $Q(x, x) < 1$ for $x \in S$. Assume also that $\boldsymbol{N}$ and $\boldsymbol{Y}$ are independent. Define $X_t = Y_{N_t}$ for $t \in [0, \infty)$. Then $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ is a continuous-Markov chain with exponential parameter function $\lambda(x) = r[1 - Q(x, x)]$ for $x \in S$ and jump transition matrix $\tilde{Q}$ given by

$$\tilde{Q}(x, y) = \frac{Q(x, y)}{1 - Q(x, x)}, \quad (x, y) \in S^2,\ x \neq y \tag{16.20.10}$$

Proof

This follows from the result in the introduction.

The Markov chain constructed above is no longer a chain subordinate to the Poisson process by our definition above, since the exponential parameter function is not constant, and the transition times of $\boldsymbol{X}$ are no longer the arrival times of the Poisson process. Nonetheless, many of the basic results above still apply.

Let $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ be the Markov chain constructed in the previous theorem. Then

1. For $t \in [0, \infty)$, the transition matrix P_t is given by

$$P_t = \sum_{n=0}^{\infty} e^{-rt} \frac{(rt)^n}{n!} Q^n \tag{16.20.11}$$

2. For $\alpha \in (0, \infty)$, the α potential matrix is given by

$$U_\alpha = \frac{1}{\alpha + r} \sum_{n=0}^\infty \left(\frac{r}{\alpha + r}\right)^n Q^n \quad (16.20.12)$$

3. The generator matrix is $G = r(Q - I)$
4. $g : S \to (0, \infty)$ is invariant for $\boldsymbol{X}$ if and only if g is invariant for $\boldsymbol{Y}$.

Proof

The proofs are just as before.

It's a remarkable fact that every continuous-time Markov chain with bounded exponential parameters can be constructed as in the last theorem, a process known as *uniformization*. The name comes from the fact that in the construction, the exponential parameters become constant, but at the expense of allowing the embedded discrete-time chain to jump from a state back to that state. To review the definition, suppose that $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ is a continuous-time Markov chain on S with transition semigroup $\boldsymbol{P} = \{P_t : t \in [0, \infty)\}$, exponential parameter function λ and jump transition matrix Q. Then $\boldsymbol{P}$ is *uniform* if $P_t(x, x) \to 1$ as $t \downarrow 0$ uniformly in x, or equivalently if λ is bounded.

Suppose that $\lambda : S \to (0, \infty)$ is bounded and that Q is a transition matrix on S with $Q(x, x) = 0$ for every $x \in S$. Let $r \in (0, \infty)$ be an upper bound on λ and $\boldsymbol{N} = \{N_t : t \in [0, \infty)\}$ a Poisson counting process with rate r. Define the transition matrix $\hat{Q}$ on S by

$$\begin{aligned} \hat{Q}(x, x) &= 1 - \frac{\lambda(x)}{r} \quad x \in S \\ \hat{Q}(x, y) &= \frac{\lambda(x)}{r} Q(x, y) \quad (x, y) \in S^2,\ x \ne y \end{aligned}$$

and let $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}\}$ be a discrete-time Markov chain with transition matrix $\hat{Q}$, independent of $\boldsymbol{N}$. Define $X_t = Y_{N_t}$ for $t \in [0, \infty)$. Then $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ is a continuous-time Markov chain with exponential parameter function λ and jump transition matrix Q.

Proof

Note that $\hat{Q}(x, y) \ge 0$ for every $(x, y) \in S^2$ and $\sum_{y \in S} \hat{Q}(x, y) = 1$ for every $x \in S$. Thus $\hat{Q}$ is a transition matrix on S. Note also that $\hat{Q}(x, x) < 1$ for every $x \in S$. By construction, $\lambda(x) = r[1 - \hat{Q}(x, x)]$ for $x \in S$ and

$$Q(x, y) = \frac{\hat{Q}(x, y)}{1 - \hat{Q}(x, x)}, \quad (x, y) \in S^2,\ x \ne y \quad (16.20.13)$$

So the result now follows from the theorem above.

Note in particular that if the state space S is finite then of course λ is bounded so the previous theorem applies. The theorem is useful for simulating a continuous-time Markov chain, since the Poisson process and discrete-time chains are simple to simulate. In addition, we have nice representations for the transition matrices, potential matrices, and the generator matrix.

Suppose that $\boldsymbol{X} = \{X_t : t \in [0, \infty\}$ is a continuous-time Markov chain on S with bounded exponential parameter function $\lambda : S \to (0, \infty)$ and jump transition matrix Q. Define r and $\hat{Q}$ as in the last theorem. Then

1. For $t \in [0, \infty)$, the transition matrix P_t is given by

$$P_t = \sum_{n=0}^\infty e^{-rt} \frac{(rt)^n}{n!} \hat{Q}^n \quad (16.20.14)$$

2. For $\alpha \in (0, \infty)$, the α potential matrix is given by

$$U_\alpha = \frac{1}{\alpha + r} \sum_{n=0}^\infty \left(\frac{r}{\alpha + r}\right)^n \hat{Q}^n \quad (16.20.15)$$

3. The generator matrix is $G = r(\hat{Q} - I)$
4. $g : S \to (0, \infty)$ is invariant for $\boldsymbol{X}$ if and only if g is invariant for $\hat{Q}$.

Proof

These results follow from the theorem above.

Examples

The Two-State Chain

The following exercise applies the uniformization method to the two-state chain.

Consider the continuous-time Markov chain $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ on $S = \{0, 1\}$ with exponential parameter function $\lambda = (a, b)$, where $a,\ b \in (0, \infty)$. Thus, states 0 and 1 are stable and the jump chain has transition matrix

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{16.20.16}$$

Let $r = a + b$, an upper bound on λ. Show that

1. $\hat{Q} = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix}$
2. $G = \begin{bmatrix} -a & a \\ b & -b \end{bmatrix}$
3. $P_t = \hat{Q} - \frac{1}{a+b} e^{-(a+b)t} G$ for $t \in [0, \infty)$
4. $U_\alpha = \frac{1}{\alpha} \hat{Q} - \frac{1}{(\alpha+a+b)(a+b)} G$ for $\alpha \in (0, \infty)$

Proof

The form of $\hat{Q}$ follows easily from the definition above. Note that the rows of $\hat{Q}$ are the invariant PDF. It then follows that $\hat{Q}^n = \hat{Q}$ for $n \in \mathbb{N}_+$. The results for the transition matrix P_t and the potential U_α then follow easily from the theorem above.

Although we have obtained all of these results for the two-state chain before, the derivation based on uniformization is the easiest.