16.7: Time Reversal in Discrete-Time Chains

The Markov property, stated in the form that the past and future are independent given the present, essentially treats the past and future symmetrically. However, there is a lack of symmetry in the fact that in the usual formulation, we have an *initial* time 0, but not a *terminal* time. If we introduce a terminal time, then we can run the process backwards in time. In this section, we are interested in the following questions:

- Is the new process still Markov?
- If so, how does the new transition probability matrix relate to the original one?
- Under what conditions are the forward and backward processes stochastically the same?

Consideration of these questions leads to *reversed chains*, an important and interesting part of the theory of Markov chains.

Basic Theory

Reversed Chains

Our starting point is a (homogeneous) discrete-time Markov chain $\boldsymbol{X} = (X_0, X_1, X_2, \ldots)$ with (countable) state space S and transition probability matrix P. Let m be a positive integer, which we will think of as the *terminal time* or *finite time horizon*. We won't bother to indicate the dependence on m notationally, since ultimately the terminal time will not matter. Define $\hat{X}_n = X_{m-n}$ for $n \in \{0, 1, \ldots, m\}$. Thus, the process *forward in time* is $\boldsymbol{X} = (X_0, X_1, \ldots, X_m)$ while the process *backwards in time* is

$$\hat{\boldsymbol{X}} = (\hat{X}_0, \hat{X}_1, \ldots, \hat{X}_m) = (X_m, X_{m-1}, \ldots, X_0) \tag{16.7.1}$$

For $n \in \{0, 1, \ldots, m\}$, let

$$\hat{\mathscr{F}}_n = \sigma\{\hat{X}_0, \hat{X}_1, \ldots, \hat{X}_n\} = \sigma\{X_{m-n}, X_{m-n+1}, \ldots, X_m\} \tag{16.7.2}$$

denote the σ algebra of the events of the process $\hat{\boldsymbol{X}}$ up to time n. So of course, an event for $\hat{\boldsymbol{X}}$ up to time n is an event for $\boldsymbol{X}$ from time $m-n$ forward. Our first result is that the reversed process is still a Markov chain, but not time homogeneous in general.

The process $\hat{\boldsymbol{X}} = (\hat{X}_0, \hat{X}_1, \ldots, \hat{X}_m)$ is a Markov chain, but is not time homogenous in general. The one-step transition matrix at time $n \in \{0, 1, \ldots, m-1\}$ is given by

$$\mathbb{P}(\hat{X}_{n+1} = y \mid \hat{X}_n = x) = \frac{\mathbb{P}(X_{m-n-1} = y)}{\mathbb{P}(X_{m-n} = x)} P(y, x), \quad (x, y) \in S^2 \tag{16.7.3}$$

Proof

Let $A \in \hat{\mathscr{F}}_n$ and $x,\ y \in S$. Then

$$\begin{aligned} \mathbb{P}(\hat{X}_{n+1} = y \mid \hat{X}_n = x, A) &= \frac{\mathbb{P}(\hat{X}_{n+1} = y, \hat{X}_n = x, A)}{\mathbb{P}(\hat{X}_n = x, A)} = \frac{\mathbb{P}(X_{m-n-1} = y, X_{m-n} = x, A)}{\mathbb{P}(X_{m-n} = x, A)} \\ &= \frac{\mathbb{P}(A \mid X_{m-n-1} = y, X_{m-n} = x)\mathbb{P}(X_{m-n} = x \mid X_{m-n-1} = y)\mathbb{P}(X_{m-n-1} = y)}{\mathbb{P}(A \mid X_{m-n} = x)\mathbb{P}(X_{m-n} = x)} \end{aligned}$$

But $A \in \sigma\{X_{m-n}, \ldots, X_m\}$ and so by the Markov property for $\boldsymbol{X}$,

$$\mathbb{P}(A \mid X_{m-n-1} = y, X_{m-n} = x) = \mathbb{P}(A \mid X_{m-n} = x) \tag{16.7.4}$$

By the time homogeneity of $\boldsymbol{X}$, $\mathbb{P}(X_{m-n} = x \mid X_{m-n-1} = y) = P(y, x)$. Substituting and simplifying gives

$$\mathbb{P}(\hat{X}_{n+1} = y \mid \hat{X}_n = x, A) = \frac{\mathbb{P}(X_{m-n-1} = y)}{\mathbb{P}(X_{m-n} = x)} P(y, x) \tag{16.7.5}$$

However, the backwards chain will be time homogeneous if X_0 has an invariant distribution.

Suppose that $\boldsymbol{X}$ is irreducible and positive recurrent, with (unique) invariant probability density function f. If X_0 has the invariant probability distribution, then $\hat{\boldsymbol{X}}$ is a time-homogeneous Markov chain with transition matrix $\hat{P}$ given by

$$\hat{P}(x, y) = \frac{f(y)}{f(x)} P(y, x), \quad (x, y) \in S^2 \tag{16.7.6}$$

Proof

This follows from the result above. Recall that if X_0 has PDF f, then X_k has PDF f for each $k \in \mathbb{N}$.

Recall that a discrete-time Markov chain is *ergodic* if it is irreducible, positive recurrent, and aperiodic. For an ergodic chain, the previous result holds in the limit of the terminal time.

Suppose that $\boldsymbol{X}$ is ergodic, with (unique) invariant probability density function f. Regardless of the distribution of X_0,

$$\mathbb{P}(\hat{X}_{n+1} = y \mid \hat{X}_n = x) \to \frac{f(y)}{f(x)} P(y, x) \text{ as } m \to \infty \tag{16.7.7}$$

Proof

This follows from the conditional probability above and our study of the limiting behavior of Markov chains. Since $\boldsymbol{X}$ is ergodic, $\mathbb{P}(X_k = x) \to f(x)$ as $k \to \infty$ for every $x \in S$.

These three results are motivation for the definition that follows. We can generalize by defining the reversal of an irreducible Markov chain, as long as there is a positive, invariant function. Recall that a positive invariant function defines a positive measure on S, but of course not in general a probability distribution.

Suppose that $\boldsymbol{X}$ is an irreducible Markov chain with transition matrix P, and that $g : S \to (0, \infty)$ is invariant for $\boldsymbol{X}$. The *reversal* of $\boldsymbol{X}$ with respect to g is the Markov chain $\hat{\boldsymbol{X}} = (\hat{X}_0, \hat{X}_1, \ldots)$ with transition probability matrix $\hat{P}$ defined by

$$\hat{P}(x, y) = \frac{g(y)}{g(x)} P(y, x), \quad (x, y) \in S^2 \tag{16.7.8}$$

Proof

We need to show that $\hat{P}$ is a valid transition probability matrix, so that the definition makes sense. Since g is invariant for $\boldsymbol{X}$,

$$\sum_{y \in S} \hat{P}(x, y) = \frac{1}{g(x)} \sum_{y \in S} g(y) P(y, x) = \frac{g(x)}{g(x)} = 1, \quad x \in S \tag{16.7.9}$$

Recall that if g is a positive invariant function for $\boldsymbol{X}$ then so is cg for every positive constant c. Note that g and cg generate the same reversed chain. So let's consider the cases:

Suppose that $\boldsymbol{X}$ is an irreducible Markov chain on S.

1. If $\boldsymbol{X}$ is recurrent, then $\boldsymbol{X}$ always has a positive invariant function that is unique up to multiplication by positive constants. Hence the reversal of a recurrent chain $\boldsymbol{X}$ always exists and is unique, and so we can refer to the reversal of $\boldsymbol{X}$ without reference to the invariant function.
2. Even better, if $\boldsymbol{X}$ is positive recurrent, then there exists a unique invariant probability density function, and the reversal of $\boldsymbol{X}$ can be interpreted as the *time* reversal (with respect to a terminal time) when $\boldsymbol{X}$ has the invariant distribution, as in the motivating exercises above.
3. If $\boldsymbol{X}$ is transient, then there may or may not exist a positive invariant function, and if one does exist, it may not be unique (up to multiplication by positive constants). So a transient chain may have no reversals or more than one.

Nonetheless, the general definition is natural, because most of the important properties of the reversed chain follow from the *balance equation* between the transition matrices P and $\hat{P}$, and the invariant function g:

$$g(x) \hat{P}(x, y) = g(y) P(y, x), \quad (x, y) \in S^2 \tag{16.7.10}$$

We will see this balance equation repeated with other objects related to the Markov chains.

Suppose that $\boldsymbol{X}$ is an irreducible Markov chain with invariant function $g : S \to (0, \infty)$, and that $\hat{\boldsymbol{X}}$ is the reversal of $\boldsymbol{X}$ with respect to g. For $x, y \in S$,

1. $\hat{P}(x, x) = P(x, x)$
2. $\hat{P}(x, y) > 0$ if and only if $P(y, x) > 0$

Proof

These results follow immediately from the balance equation $g(x)\hat{P}(x, y) = g(y)P(y, x)$ for $(x, y) \in S^2$.

From part (b) it follows that the state graphs of $\boldsymbol{X}$ and $\hat{\boldsymbol{X}}$ are reverses of each other. That is, to go from the state graph of one chain to the state graph of the other, simply reverse the direction of each edge. Here is a more complicated (but equivalent) version of the balance equation for chains of states:

Suppose again that $\boldsymbol{X}$ is an irreducible Markov chain with invariant function $g : S \to (0, \infty)$, and that $\hat{\boldsymbol{X}}$ is the reversal of $\boldsymbol{X}$ with respect to g. For every $n \in \mathbb{N}_+$ and every sequence of states $(x_1, x_2, \ldots, x_n, x_{n+1}) \in S^{n+1}$,

$$g(x_1)\hat{P}(x_1, x_2)\hat{P}(x_2, x_3)\cdots\hat{P}(x_n, x_{n+1}) = g(x_{n+1})P(x_{n+1}, x_n)\cdots P(x_3, x_2)P(x_2, x_1) \tag{16.7.11}$$

Proof

This follows from repeated applications of the basic equation. When $n = 1$, we have the balance equation itself:

$$g(x_1)\hat{P}(x_1, x_2) = g(x_2)P(x_2, x_1) \tag{16.7.12}$$

For $n = 2$,

$$g(x_1)\hat{P}(x_1, x_2)\hat{P}(x_2, x_3) = g(x_2)P(x_2, x_1)\hat{P}(x_2, x_3) = g(x_3)P(x_3, x_2)P(x_2, x_1) \tag{16.7.13}$$

Continuing in this manner (or using induction) gives the general result.

The balance equation holds for the powers of the transition matrix:

Suppose again that $\boldsymbol{X}$ is an irreducible Markov chain with invariant function $g : S \to (0, \infty)$, and that $\hat{\boldsymbol{X}}$ is the reversal of $\boldsymbol{X}$ with respect to g. For every $(x, y) \in S^2$ and $n \in \mathbb{N}$,

$$g(x)\hat{P}^n(x, y) = g(y)P^n(y, x) \tag{16.7.14}$$

Proof

When $n = 0$, the left and right sides are $g(x)$ if $x = y$ and are 0 otherwise. When $n = 1$, we have the basic balance equation: $g(x)\hat{P}(x, y) = g(y)P(y, x)$. In general, for $n \in \mathbb{N}_+$, by the previous result we have

$$\begin{aligned} g(x)\hat{P}^n(x, y) &= \sum_{(x_1, \ldots, x_{n-1}) \in S^{n-1}} g(x)\hat{P}(x, x_1)\hat{P}(x_1, x_2)\cdots\hat{P}(x_{n-1}, y) \\ &= \sum_{(x_1, \ldots, x_{n-1}) \in S^{n-1}} g(y)P(y, x_{n-1})P(x_{n-1}, x_{n-2})\cdots P(x_1, x) = g(y)P^n(y, x) \end{aligned}$$

We can now generalize the simple result above.

Suppose again that $\boldsymbol{X}$ is an irreducible Markov chain with invariant function $g : S \to (0, \infty)$, and that $\hat{\boldsymbol{X}}$ is the reversal of $\boldsymbol{X}$ with respect to g. For $n \in \mathbb{N}$ and $(x, y) \in S^2$,

1. $P^n(x, x) = \hat{P}^n(x, x)$
2. $\hat{P}^n(x, y) > 0$ if and only if $P^n(y, x) > 0$

In terms of the state graphs, part (b) has an obvious meaning: If there exists a path of length n from y to x in the original state graph, then there exists a path of length n from x to y in the reversed state graph. The time reversal definition is symmetric with respect to the two Markov chains.

Suppose again that $\boldsymbol{X}$ is an irreducible Markov chain with invariant function $g: S \to (0, \infty)$, and that $\hat{\boldsymbol{X}}$ is the reversal of $\boldsymbol{X}$ with respect to g. Then

1. g is also invariant for $\hat{\boldsymbol{X}}$.
2. $\hat{\boldsymbol{X}}$ is also irreducible.
3. $\boldsymbol{X}$ is the reversal of $\hat{\boldsymbol{X}}$ with respect to g.

Proof

1. For $y \in S$, using the balance equation,
$$\sum_{x \in S} g(x) \hat{P}(x, y) = \sum_{x \in S} g(y) P(y, x) = g(y) \tag{16.7.15}$$
2. Suppose $(x, y) \in S^2$. Since $\boldsymbol{X}$ is irreducible, there exist $n \in \mathbb{N}$ with $P^n(y, x) > 0$. But then from the previous result, $\hat{P}^n(x, y) > 0$. Hence $\hat{\boldsymbol{X}}$ is also irreducible.
3. This is clear from the symmetric relationship in the fundamental result.

The balance equation also holds for the potential matrices.

Suppose that $\boldsymbol{X}$ and $\hat{\boldsymbol{X}}$ are time reversals with respect to the invariant function $g: S \to (0, \infty)$. For $\alpha \in (0, 1]$, the α potential matrices are related by

$$g(x) \hat{R}_\alpha(x, y) = g(y) R_\alpha(y, x), \quad (x, y) \in S^2 \tag{16.7.16}$$

Proof

This follows easily from the result above and the definition of the potential matrices:

$$\begin{aligned} g(x) \hat{R}_\alpha(x, y) &= g(x) \sum_{n=0}^\infty \alpha^n \hat{P}^n(x, y) = \sum_{n=0}^\infty \alpha^n g(x) \hat{P}^n(x, y) \\ &= \sum_{n=0}^\infty \alpha^n g(y) P^n(y, x) = g(y) \sum_{n=0}^\infty \alpha^n P^n(y, x) = g(y) R_\alpha(y, x) \end{aligned}$$

Markov chains that are time reversals share many important properties:

Suppose that $\boldsymbol{X}$ and $\hat{\boldsymbol{X}}$ are time reversals. Then

1. $\boldsymbol{X}$ and $\hat{\boldsymbol{X}}$ are of the same type (transient, null recurrent, or positive recurrent).
2. $\boldsymbol{X}$ and $\hat{\boldsymbol{X}}$ have the same period.
3. $\boldsymbol{X}$ and $\hat{\boldsymbol{X}}$ have the same mean return time $\mu(x)$ for every $x \in S$.

Proof

Suppose that $\boldsymbol{X}$ and $\hat{\boldsymbol{X}}$ are time reversals with respect to the invariant function $g: S \to (0, \infty)$.

1. The expected number of visits to a state $x \in S$, starting in x, is the same for both chains: $\hat{R}(x, x) = R(x, x)$. Hence either both chains are transient (if the common potential is finite) or both chains are recurrent (if the common potential is infinite). If both chains are recurrent then the invariant function g is unique up to multiplication by positive constants, and both are null recurrent if $\sum_{x \in S} g(x) = \infty$ and both are positive recurrent if $\sum_{x \in S} g(x) < \infty$.
2. This follows since $P^n(x, x) = \hat{P}^n(x, x)$ for all $n \in \mathbb{N}$ and $x \in S$.
3. If both chains are transient or both are null recurrent, then $\mu(x) = \hat{\mu}(x) = \infty$ for all $x \in S$. If both chains are positive recurrent, then for all $n \in \mathbb{N}$ and $x \in S$, we have
$$\frac{1}{n} \sum_{k=1}^n P^k(x, x) = \frac{1}{n} \sum_{k=1}^n \hat{P}^k(x, x) \tag{16.7.17}$$
The left side converges to $1 / \mu(x)$ as $n \to \infty$ while the right side converges to $1 / \hat{\mu}(x)$ as $n \to \infty$.

The main point of the next result is that we don't need to know a-priori that g is invariant for $\boldsymbol{X}$, if we can guess g and $\hat{P}$.

Suppose again that $\boldsymbol{X}$ is irreducible with transition probability matrix P. If there exists a a function $g : S \to (0, \infty)$ and a transition probability matrix $\hat{P}$ such that $g(x)\hat{P}(x, y) = g(y)P(y, x)$ for all $(x, y) \in S^2$, then

1. g is invariant for $\boldsymbol{X}$.
2. $\hat{P}$ is the transition matrix of the reversal of $\boldsymbol{X}$ with respect to g.

Proof

1. Since $\hat{P}$ is a transition probability matrix, we have the same computation we have seen before:
$$gP(x) = \sum_{y \in S} g(y)P(y, x) = \sum_{y \in S} g(x)\hat{P}(x, y) = g(x), \quad x \in S \tag{16.7.18}$$
2. This follows from (a) and the definition.

As a corollary, if there exists a probability density function f on S and a transition probability matrix $\hat{P}$ such that $f(x)\hat{P}(x, y) = f(y)P(y, x)$ for all $(x, y) \in S^2$ then in addition to the conclusions above, we know that the chains $\boldsymbol{X}$ and $\hat{\boldsymbol{X}}$ are positive recurrent.

Reversible Chains

Clearly, an interesting special case occurs when the transition matrix of the reversed chain turns out to be the same as the original transition matrix. A chain of this type could be used to model a physical process that is stochastically the same, forward or backward in time.

Suppose again that $\boldsymbol{X} = (X_0, X_1, X_2, \ldots)$ is an irreducible Markov chain with transition matrix P and invariant function $g : S \to (0, \infty)$. If the reversal of $\boldsymbol{X}$ with respect to g also has transition matrix P, then $\boldsymbol{X}$ is said to be *reversible* with respect to g. That is, $\boldsymbol{X}$ is reversible with respect to g if and only if
$$g(x)P(x, y) = g(y)P(y, x), \quad (x, y) \in S^2 \tag{16.7.19}$$

Clearly if $\boldsymbol{X}$ is reversible with respect to the invariant function $g : S \to (0, \infty)$ then $\boldsymbol{X}$ is reversible with respect to the invariant function cg for every $c \in (0, \infty)$. So again, let's review the cases.

Suppose that $\boldsymbol{X}$ is an irreducible Markov chain on S.

1. If $\boldsymbol{X}$ is recurrent, there exists a positive invariant function that is unique up to multiplication by positive constants. So $\boldsymbol{X}$ is either reversible or not, and we don't have to reference the invariant function g.
2. If $\boldsymbol{X}$ is positive recurrent then there exists a unique invariant probability density function $f : S \to (0, 1)$, and again, either $\boldsymbol{X}$ is reversible or not. If $\boldsymbol{X}$ is reversible, then P is the transition matrix of $\boldsymbol{X}$ forward or backward in time, when the chain has the invariant distribution.
3. If $\boldsymbol{X}$ is transient, there may or may not exist positive invariant functions. If there are two or more positive invariant functions that are not multiples of one another, $\boldsymbol{X}$ might be reversible with respect to one function but not the others.

The non-symmetric simple random walk on $\mathbb{Z}$ falls into the last case. Using the last result in the previous subsection, we can tell whether $\boldsymbol{X}$ is reversible with respect to g without knowing a-priori that g is invariant.

Suppose again that $\boldsymbol{X}$ is irreducible with transition matrix P. If there exists a function $g : S \to (0, \infty)$ such that $g(x)P(x, y) = g(y)P(y, x)$ for all $(x, y) \in S^2$, then

1. g is invariant for $\boldsymbol{X}$.
2. $\boldsymbol{X}$ is reversible with respect to g

If we have reason to believe that a Markov chain is reversible (based on modeling considerations, for example), then the condition in the previous theorem can be used to find the invariant functions. This procedure is often easier than using the definition of invariance directly. The next two results are minor generalizations:

Suppose again that $\boldsymbol{X}$ is irreducible and that $g: S \to (0, \infty)$. Then g is invariant and $\boldsymbol{X}$ is reversible with respect to g if and only if for every $n \in \mathbb{N}_+$ and every sequence of states $(x_1, x_2, \ldots x_n, x_{n+1}) \in S^{n+1}$,

$$g(x_1)P(x_1,x_2)P(x_2,x_3)\cdots P(x_n,x_{n+1}) = g(x_{n+1})P(x_{n+1},x_n),\cdots P(x_3,x_2)P(x_2,x_1) \tag{16.7.20}$$

Suppose again that $\boldsymbol{X}$ is irreducible and that $g: S \to (0, \infty)$. Then g is invariant and $\boldsymbol{X}$ is reversible with respect to g if and only if for every $(x, y) \in S^2$ and $n \in \mathbb{N}_+$,

$$g(x)P^n(x,y) = g(y)P^n(y,x) \tag{16.7.21}$$

Here is the condition for reversibility in terms of the potential matrices.

Suppose again that $\boldsymbol{X}$ is irreducible and that $g: S \to (0, \infty)$. Then g is invariant and $\boldsymbol{X}$ is reversible with respect to g if and only if

$$g(x)R_\alpha(x,y) = g(y)R_\alpha(y,x), \quad \alpha \in (0,1], \ (x,y) \in S^2 \tag{16.7.22}$$

In the positive recurrent case (the most important case), the following theorem gives a condition for reversibility that does not directly reference the invariant distribution. The condition is known as the *Kolmogorov cycle condition*, and is named for Andrei Kolmogorov

Suppose that $\boldsymbol{X}$ is irreducible and positive recurrent. Then $\boldsymbol{X}$ is reversible if and only if for every sequence of states $(x_1, x_2, \ldots, x_n)$,

$$P(x_1,x_2)P(x_2,x_3)\cdots P(x_{n-1},x_n)P(x_n,x_1) = P(x_1,x_n)P(x_n,x_{n-1})\cdots P(x_3,x_2)P(x_2,x_1) \tag{16.7.23}$$

Proof

Suppose that $\boldsymbol{X}$ is reversible. Applying the chain result above to the sequence $(x_1, x_2, \ldots, x_n, x_1)$ gives the Kolmogorov cycle condition. Conversely, suppose that the Kolmogorov cycle condition holds, and let f denote the invariant probability density function of $\boldsymbol{X}$. From the cycle condition we have $P(x,y)P^k(y,x) = P(y,x)P^k(x,y)$ for every $(x,y) \in S$ and $k \in \mathbb{N}_+$. Averaging over k from 1 to n gives

$$P(x,y)\frac{1}{n}\sum_{k=1}^n P^k(y,x) = P(y,x)\frac{1}{n}\sum_{k=1}^n P^k(x,y), \quad (x,y) \in S^2, \ n \in \mathbb{N}_+ \tag{16.7.24}$$

Letting $n \to \infty$ gives $f(x)P(x,y) = f(y)P(y,x)$ for $(x,y) \in S^2$, so $\boldsymbol{X}$ is reversible.

Note that the Kolmogorov cycle condition states that the probability of visiting states $(x_2, x_3, \ldots, x_n, x_1)$ in sequence, starting in state x_1 is the same as the probability of visiting states $(x_n, x_{n-1}, \ldots, x_2, x_1)$ in sequence, starting in state x_1. The cycle condition is also known as the *balance equation for cycles*.

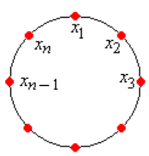

Figure 16.7.1: The Kolmogorov cycle condition

Examples and Applications

Finite Chains

Recall the general two-state chain $\boldsymbol{X}$ on $S = \{0, 1\}$ with the transition probability matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \tag{16.7.25}$$

where $p, q \in (0, 1)$ are parameters. The chain $\boldsymbol{X}$ is reversible and the invariant probability density function is $f = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$.

Proof

All we have to do is note that

$$\begin{bmatrix} q & p \end{bmatrix} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} q & p \end{bmatrix} \tag{16.7.26}$$

Suppose that $\boldsymbol{X}$ is a Markov chain on a finite state space S with symmetric transition probability matrix P. Thus $P(x,y) = P(y,x)$ for all $(x,y) \in S^2$. The chain $\boldsymbol{X}$ is reversible and that the uniform distribution on S is invariant.

Proof

All we have to do is note that $\mathbf{1}(x)P(x,y) = \mathbf{1}(y)P(y,x)$ where $\mathbf{1}$ is the constant function 1 on S.

Consider the Markov chain $\boldsymbol{X}$ on $S = \{a, b, c\}$ with transition probability matrix P given below:

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \tag{16.7.27}$$

1. Draw the state graph of $\boldsymbol{X}$ and note that the chain is irreducible.
2. Find the invariant probability density function f.
3. Find the mean return time to each state.
4. Find the transition probability matrix $\hat{P}$ of the time-reversed chain $\hat{\boldsymbol{X}}$.
5. Draw the state graph of $\hat{\boldsymbol{X}}$.

Answer

1. State graph of $\boldsymbol{X}$
   State1.png
2. $f = \left(\frac{6}{17}, \frac{6}{17}, \frac{5}{17}\right)$
3. $\mu = \left(\frac{17}{6}, \frac{17}{6}, \frac{17}{5}\right)$
4. $\hat{P} = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \\ \frac{3}{5} & \frac{2}{5} & 0 \end{bmatrix}$
5. State graph of $\hat{\boldsymbol{X}}$
   State1.png

Special Models

Read the discussion of reversibility for the Ehrenfest chains.

Read the discussion of reversibility for the Bernoulli-Laplace chain.

Read the discussion of reversibility for the random walks on graphs.

Read the discussion of time reversal for the reliability chains.

Read the discussion of reversibility for the birth-death chains.