17.2: Properties and Constructions

Basic Theory

Preliminaries

As in the Introduction, we start with a stochastic process $\boldsymbol{X} = \{X_t : t \in T\}$ on an underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$, having state space $\mathbb{R}$, and where the index set T (representing time) is either $\mathbb{N}$ (discrete time) or $[0, \infty)$ (continuous time). Next, we have a filtration $\mathfrak{F} = \{\mathscr{F}_t : t \in T\}$, and we assume that $\boldsymbol{X}$ is *adapted* to $\mathfrak{F}$. So $\mathfrak{F}$ is an increasing family of sub σ-algebras of $\mathscr{F}$ and X_t is measurable with respect to $\mathscr{F}_t$ for $t \in T$. We think of $\mathscr{F}_t$ as the collection of events up to time $t \in T$. We assume that $\mathbb{E}(|X_t|) < \infty$, so that the mean of X_t exists as a real number, for each $t \in T$. Finally, in continuous time where $T = [0, \infty)$, we make the standard assumptions that $\boldsymbol{X}$ is right continuous and has left limits, and that the filtration $\mathfrak{F}$ is right continuous and complete. Please recall the following from the Introduction:

Definitions

1. $\boldsymbol{X}$ is a *martingale* with respect to $\mathfrak{F}$ if $\mathbb{E}(X_t \mid \mathscr{F}_s) = X_s$ for all $s, t \in T$ with $s \le t$.
2. $\boldsymbol{X}$ is a *sub-martingale* with respect to $\mathfrak{F}$ if $\mathbb{E}(X_t \mid \mathscr{F}_s) \ge X_s$ for all $s, t \in T$ with $s \le t$.
3. $\boldsymbol{X}$ is a *super-martingale* with respect to $\mathfrak{F}$ if $\mathbb{E}(X_t \mid \mathscr{F}_s) \le X_s$ for all $s, t \in T$ with $s \le t$.

Our goal in this section is to give a number of basic properties of martingales and to give ways of constructing martingales from other types of processes. The deeper, fundamental theorems will be studied in the following sections.

Basic Properties

Our first result is that the martingale property is preserved under a coarser filtration.

Suppose that the process $\boldsymbol{X}$ and the filtration $\mathfrak{F}$ satisfy the basic assumptions above and that $\mathfrak{G}$ is a filtration coarser than $\mathfrak{F}$ so that $\mathscr{G}_t \subseteq \mathscr{F}_t$ for $t \in T$. If $\boldsymbol{X}$ is a martingale (sub-martingale, super-martingale) with respect to $\mathfrak{F}$ and $\boldsymbol{X}$ is adapted to $\mathfrak{G}$ then $\boldsymbol{X}$ is a martingale (sub-martingale, super-martingale) with respect to $\mathfrak{G}$.

Proof

Suppose that $s, t \in T$ with $s \le t$. The proof uses the tower and increasing properties of conditional expected value, and the fact that $\boldsymbol{X}$ is adapted to $\mathfrak{G}$

1. If $\boldsymbol{X}$ is a martingale with respect to $\mathfrak{F}$ then
$$\mathbb{E}(X_t \mid \mathscr{G}_s) = \mathbb{E}[\mathbb{E}(X_t \mid \mathscr{F}_s) \mid \mathscr{G}_s] = \mathbb{E}(X_s \mid \mathscr{G}_s) = X_s \tag{17.2.1}$$
2. If $\boldsymbol{X}$ is a sub-martinagle with respect to $\mathfrak{F}$ then
$$\mathbb{E}(X_t \mid \mathscr{G}_s) = \mathbb{E}[\mathbb{E}(X_t \mid \mathscr{F}_s) \mid \mathscr{G}_s] \ge \mathbb{E}(X_s \mid \mathscr{G}_s) = X_s \tag{17.2.2}$$
3. If $\boldsymbol{X}$ is a super-martinagle with respect to $\mathfrak{F}$ then
$$\mathbb{E}(X_t \mid \mathscr{G}_s) = \mathbb{E}[\mathbb{E}(X_t \mid \mathscr{F}_s) \mid \mathscr{G}_s] \le \mathbb{E}(X_s \mid \mathscr{G}_s) = X_s \tag{17.2.3}$$

In particular, if $\boldsymbol{X}$ is a martingale (sub-martingale, super-martingale) with respect to *some* filtration, then it is a martingale (sub-martingale, super-martingale) with respect to its own natural filtration.

The relations that define martingales, sub-martingales, and super-martingales hold for the ordinary (unconditional) expected values. We had this result in the last section, but it's worth repeating.

Suppose that $s, t \in T$ with $s \le t$.

1. If $\boldsymbol{X}$ is a martingale with respect to $\mathfrak{F}$ then $\mathbb{E}(X_s) = \mathbb{E}(X_t)$.
2. If $\boldsymbol{X}$ is a sub-martingale with respect to $\mathfrak{F}$ then $\mathbb{E}(X_s) \le \mathbb{E}(X_t)$.
3. If $\boldsymbol{X}$ is a super-martingale with respect to $\mathfrak{F}$ then $\mathbb{E}(X_s) \ge \mathbb{E}(X_t)$.

Proof

The results follow directly from the definitions, and the critical fact that $\mathbb{E}\left[\mathbb{E}\left(X_t \mid \mathscr{F}_s\right)\right] = \mathbb{E}(X_t)$ for $s, t \in T$.

So if $\boldsymbol{X}$ is a martingale then $\boldsymbol{X}$ has constant expected value, and this value is referred to as the *mean* of $\boldsymbol{X}$. The martingale properties are preserved under sums of the stochastic processes.

For the processes $\boldsymbol{X} = \{X_t : t \in T\}$ and $\boldsymbol{Y} = \{Y_t : t \in T\}$, let $\boldsymbol{X} + \boldsymbol{Y} = \{X_t + Y_t : t \in T\}$. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are martingales (sub-martingales, super-martinagles) with respect to $\mathfrak{F}$ then $\boldsymbol{X} + \boldsymbol{Y}$ is a martingale (sub-martingale, super-martinagle) with respect to $\mathfrak{F}$.

Proof

The results follow easily from basic properties of expected value and conditional expected value. First note that $\mathbb{E}\left(|X_t + Y_t|\right) \le \mathbb{E}\left(|X_t|\right) + \mathbb{E}\left(|Y_t|\right) < \infty$ for $t \in T$. Next $\mathbb{E}(X_t + Y_t \mid \mathscr{F}_s) = \mathbb{E}(X_t \mid \mathscr{F}_s) + \mathbb{E}(Y_t \mid \mathscr{F}_s)$ for $s, t \in T$ with $s \le t$.

The sub-martingale and super-martingale properties are preserved under multiplication by a positive constant and are reversed under multiplication by a negative constant.

For the process $\boldsymbol{X} = \{X_t : t \in T\}$ and the constant $c \in \mathbb{R}$, let $c\boldsymbol{X} = \{cX_t : t \in T\}$.

1. If $\boldsymbol{X}$ is a martingale with respect to $\mathfrak{F}$ then $c\boldsymbol{X}$ is also a martingale with respect to $\mathfrak{F}$
2. If $\boldsymbol{X}$ is a sub-martingale with respect to $\mathfrak{F}$, then $c\boldsymbol{X}$ is a sub-martingale if $c > 0$, a super-martingale if $c < 0$, and a martingale if $c = 0$.
3. If $\boldsymbol{X}$ is a super-martingale with respect to $\mathfrak{F}$, then $c\boldsymbol{X}$ is a super-martingale if $c > 0$, a sub-martingale if $c < 0$, and a martingale if $c = 0$.

Proof

The results follow easily from basic properties of expected value and conditional expected value. First note that $\mathbb{E}\left(|cX_t|\right) = |c|\mathbb{E}\left(|X_t|\right) < \infty$ for $t \in T$. Next $\mathbb{E}(cX_t \mid \mathscr{F}_s) = c\mathbb{E}(X_t \mid \mathscr{F}_s)$ for $s, t \in T$ with $s \le t$.

Property (a), together with the previous additive property, means that the collection of martingales with respect to a fixed filtration $\mathfrak{F}$ forms a vector space. Here is a class of transformations that turns martingales into sub-martingales.

Suppose that $\boldsymbol{X}$ takes values in an interval $S \subseteq \mathbb{R}$ and that $g : S \to \mathbb{R}$ is convex with $\mathbb{E}\left[|g(X_t)|\right] < \infty$ for $t \in T$. If either of the following conditions holds then $g(\boldsymbol{X}) = \{g(X_t) : t \in T\}$ is a sub-martingale with respect to $\mathfrak{F}$:

1. $\boldsymbol{X}$ is a martingale.
2. $\boldsymbol{X}$ is a sub-martingale and g is also increasing.

Proof

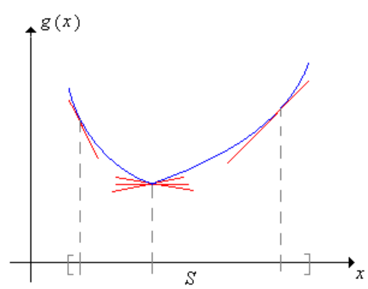

Figure 17.2.1: A convex function and several supporting lines

Here is the most important special case of the previous result:

Suppose again that $\boldsymbol{X}$ is a martingale with respect to $\mathfrak{F}$. Let $k \in [1, \infty)$ and suppose that $\mathbb{E}\left(|X_t|^k\right) < \infty$ for $t \in T$. Then the process $|\boldsymbol{X}|^k = \left\{|X_t|^k : t \in T\right\}$ is a sub-martingale with respect to $\mathfrak{F}$

Proof

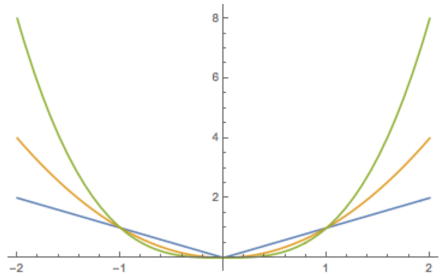

Figure 17.2.2: The graphs of $x \mapsto |x|$, $x \mapsto |x|^2$ and $x \mapsto |x|^3$ on the interval $[-2, 2]$

In particular, if $\boldsymbol{X}$ is a martingale relative to $\mathfrak{F}$ then $|\boldsymbol{X}| = \{|X_t| : t \in T\}$ is a sub-martingale relative to $\mathfrak{F}$. Here is a related result that we will need later. First recall that the *positive and negative parts* of $x \in \mathbb{R}$ are $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$, so that $x^+ \ge 0$, $x^- \ge 0$, $x = x^+ - x^-$, and $|x| = x^+ + x^-$.

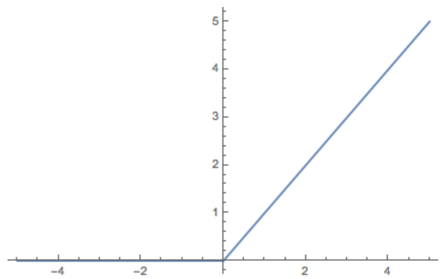

Figure 17.2.3: The graph of $x \mapsto x^+$ on the interval $[-5, 5]$

If $\boldsymbol{X} = \{X_t : t \in T\}$ is a sub-martingale relative to $\mathfrak{F} = \{\mathscr{F}_t : t \in T\}$ then $\boldsymbol{X}^+ = \{X_t^+ : t \in T\}$ is also a sub-martingale relative to $\mathfrak{F}$.

Proof

As shown in the graph above, the function $x \mapsto x^+$ is increasing and convex on $\mathbb{R}$.

Our last result of this discussion is that if we sample a continuous-time martingale at an increasing sequence of time points, we get a discrete-time martingale.

Suppose again that the process $\boldsymbol{X} = \{X_t : t \in [0, \infty)\}$ and the filtration $\mathfrak{F} = \{\mathscr{F}_t : t \in [0, \infty)\}$ satisfy the basic assumptions above. Suppose also that $\{t_n : n \in \mathbb{N}\} \subset [0, \infty)$ is a strictly increasing sequence of time points with $t_0 = 0$, and define $Y_n = X_{t_n}$ and $\mathscr{G}_n = \mathscr{F}_{t_n}$ for $n \in \mathbb{N}$. If $\boldsymbol{X}$ is a martingale (sub-martingale, super-martingale) with respect to $\mathfrak{F}$ then $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}\}$ is a martingale (sub-martingale, super-martingale) with respect to $\mathfrak{G}$.

Proof

Since the time points are increasing, it's clear that $\mathfrak{G}$ is a discrete-time filtration. Next, $\mathbb{E}(|Y_n|) = \mathbb{E}\left(|X_{t_n}|\right) < \infty$. Finally, suppose that $\boldsymbol{X}$ is a martingale and n, $k \in \mathbb{N}$ with $k < n$. Then $t_k < t_n$ so

$$\mathbb{E}(Y_n \mid \mathscr{G}_k) = \mathbb{E}\left(X_{t_n} \mid \mathscr{F}_{t_k}\right) = X_{t_k} = Y_k \tag{17.2.4}$$

Hence $\boldsymbol{Y}$ is also a martingale. The proofs for sub and super-martingales are similar, but with inequalities replacing the second equality.

This result is often useful for extending proofs of theorems in discrete time to continuous time.

The Martingale Transform

Our next discussion, in discrete time, shows how to build a new martingale from an existing martingale and an predictable process. This construction turns out to be very useful, and has an interesting gambling interpretation. To review the definition, recall that $\{Y_n : n \in \mathbb{N}_+\}$ is *predictable* relative to the filtration $\mathfrak{F} = \{\mathscr{F}_n : n \in \mathbb{N}\}$ if Y_n is measurable with respect to $\mathscr{F}_{n-1}$ for $n \in \mathbb{N}_+$. Think of Y_n as the bet that a gambler makes on game $n \in \mathbb{N}_+$. The gambler can base the bet on all of the information she has at that point, including the outcomes of the previous $n-1$ games. That is, she can base the bet on the information encoded in $\mathscr{F}_{n-1}$.

Suppose that $\boldsymbol{X} = \{X_n : n \in \mathbb{N}\}$ is adapted to the filtration $\mathfrak{F} = \{\mathscr{F}_n : n \in \mathbb{N}\}$ and that $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}_+\}$ is predictable relative to $\mathfrak{F}$. The *transform* of $\boldsymbol{X}$ by $\boldsymbol{Y}$ is the process $\boldsymbol{Y} \cdot \boldsymbol{X}$ defined by

$$(\boldsymbol{Y} \cdot \boldsymbol{X})_n = X_0 + \sum_{k=1}^n Y_k (X_k - X_{k-1}), \quad n \in \mathbb{N} \tag{17.2.5}$$

The motivating example behind the transfrom, in terms of a gambler making a sequence of bets, is given in an example below. Note that $\boldsymbol{Y} \cdot \boldsymbol{X}$ is also adapted to $\mathfrak{F}$. Note also that the transform depends on $\boldsymbol{X}$ only through X_0 and $\{X_n - X_{n-1} : n \in \mathbb{N}_+\}$. If $\boldsymbol{X}$ is a martingale, this sequence is the martingale difference sequence studied Introduction.

Suppose $\boldsymbol{X} = \{X_n : n \in \mathbb{N}\}$ is adapted to the filtration $\mathfrak{F} = \{\mathscr{F}_n : n \in \mathbb{N}\}$ and that $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}\}$ is a bounded process, predictable relative to $\mathfrak{F}$.

1. If $\boldsymbol{X}$ is a martingale relative to $\mathfrak{F}$ then $\boldsymbol{Y} \cdot \boldsymbol{X}$ is also a martingale relative to $\mathfrak{F}$.
2. If $\boldsymbol{X}$ is a sub-martingle relative to $\mathfrak{F}$ and $\boldsymbol{Y}$ is nonnegative, then $\boldsymbol{Y} \cdot \boldsymbol{X}$ is also a sub-martingale relative to $\mathfrak{F}$.
3. If $\boldsymbol{X}$ is a super-martingle relative to $\mathfrak{F}$ and $\boldsymbol{Y}$ is nonnegative, then $\boldsymbol{Y} \cdot \boldsymbol{X}$ is also a super-martingale relative to $\mathfrak{F}$.

Proof

Suppose that $|Y_n| \le c$ for $n \in \mathbb{N}$ where $c \in (0, \infty)$. Then

$$\mathbb{E}(|(\boldsymbol{Y} \cdot \boldsymbol{X})_n|) \le \mathbb{E}(|X_0|) + c \sum_{k=1}^n [\mathbb{E}(|X_k|) + \mathbb{E}(|X_{k-1}|)] < \infty, \quad n \in \mathbb{N} \tag{17.2.6}$$

Next, for $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}\left[(\boldsymbol{Y} \cdot \boldsymbol{X})_{n+1} \mid \mathscr{F}_n\right] &= \mathbb{E}\left[(\boldsymbol{Y} \cdot \boldsymbol{X})_n + Y_{n+1}(X_{n+1} - X_n) \mid \mathscr{F}_n\right] = (\boldsymbol{Y} \cdot \boldsymbol{X})_n + Y_{n+1} \mathbb{E}\left(X_{n+1} - X_n \mid \mathscr{F}_n\right) \\ &= (\boldsymbol{Y} \cdot \boldsymbol{X})_n + Y_{n+1}[\mathbb{E}(X_{n+1} \mid \mathscr{F}_n) - X_n] \end{aligned}$$

since $(\boldsymbol{Y} \cdot \boldsymbol{X})_n$, Y_{n+1} and X_n are $\mathscr{F}_n$-measurable. The results now follow from the definitions of martingle, sub-martingale, and super-martingale.

This construction is known as a *martingale transform*, and is a discrete version of the stochastic integral that we will study in the chapter on Brownian motion. The result also holds if instead of $\boldsymbol{Y}$ being bounded, we have $\boldsymbol{X}$ bounded and $\mathbb{E}(|Y_n|) < \infty$ for $n \in \mathbb{N}_+$

The Doob Decomposition

The next result, in discrete time, shows how to decompose a basic stochastic process into a martingale and a predictable process. The result is known as the *Doob decomposition theorem* and is named for Joseph Doob who developed much of the modern theory of martingales.

Suppose that $\boldsymbol{X} = \{X_n : n \in \mathbb{N}\}$ satisfies the basic assumptions above relative to the filtration $\mathfrak{F} = \{\mathscr{F}_n : n \in \mathbb{N}\}$. Then $X_n = Y_n + Z_n$ for $n \in \mathbb{N}$ where $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}\}$ is a martingale relative to $\mathfrak{F}$ and $\boldsymbol{Z} = \{Z_n : n \in \mathbb{N}\}$ is predictable relative to $\mathfrak{F}$. The decomposition is unique.

1. If $\boldsymbol{X}$ is a sub-martingale relative to $\mathfrak{F}$ then $\boldsymbol{Z}$ is increasing.
2. If $\boldsymbol{X}$ is a super-martingale relative to $\mathfrak{F}$ then $\boldsymbol{Z}$ is decreasing.

Proof

Recall that the basic assumptions mean that $\boldsymbol{X} = \{X_n : n \in \mathbb{N}\}$ is adapted to $\mathfrak{F}$ and $\mathbb{E}(|X_n|) < \infty$ for $n \in \mathbb{N}$. Define $Z_0 = 0$ and

$$Z_n = \sum_{k=1}^n \left[\mathbb{E}(X_k \mid \mathscr{F}_{k-1}) - X_{k-1}\right], \quad n \in \mathbb{N}_+ \tag{17.2.7}$$

Then Z_n is measurable with respect to $\mathscr{F}_{n-1}$ for $n \in \mathbb{N}_+$ so $\boldsymbol{Z}$ is predictable with respect to $\mathfrak{F}$. Now define

$$Y_n = X_n - Z_n = X_n - \sum_{k=1}^n \left[\mathbb{E}(X_k \mid \mathscr{F}_{k-1}) - X_{k-1}\right], \quad n \in \mathbb{N} \tag{17.2.8}$$

Then $\mathbb{E}(|Y_n|) < \infty$ and trivially $X_n = Y_n + Z_n$ for $n \in \mathbb{N}$. Next,

$$\begin{aligned}\mathbb{E}(Y_{n+1} \mid \mathscr{F}_n) &= \mathbb{E}(X_{n+1} \mid \mathscr{F}_n) - Z_{n+1} = \mathbb{E}(X_{n+1} \mid \mathscr{F}_n) - \sum_{k=1}^{n+1} [\mathbb{E}(X_k \mid \mathscr{F}_{k-1}) - X_{k-1}] \\ &= X_n - \sum_{k=1}^{n} [\mathbb{E}(X_k \mid \mathscr{F}_{k-1}) - X_{k-1}] = Y_n, \quad n \in \mathbb{N}\end{aligned}$$

Hence $\boldsymbol{Y}$ is a martingale. Conversely, suppose that $\boldsymbol{X}$ has the decomposition in terms of $\boldsymbol{Y}$ and $\boldsymbol{Z}$ given in the theorem. Since $\boldsymbol{Y}$ is a martingale and $\boldsymbol{Z}$ is predictable,

$$\begin{aligned}\mathbb{E}(X_n - X_{n-1} \mid \mathscr{F}_{n-1}) &= \mathbb{E}(Y_n \mid \mathscr{F}_{n-1}) - \mathbb{E}(Y_{n-1} \mid \mathscr{F}_{n-1}) + \mathbb{E}(Z_n \mid \mathscr{F}_{n-1}) - \mathbb{E}(Z_{n-1} \mid \mathscr{F}_{n-1}) \\ &= Y_{n-1} - Y_{n-1} + Z_n - Z_{n-1} = Z_n - Z_{n-1}, \quad n \in \mathbb{N}_+\end{aligned}$$

Also $Z_0 = 0$ so $\boldsymbol{X}$ uniquely determines $\boldsymbol{Z}$. But $Y_n = X_n - Z_n$ for $n \in \mathbb{N}$, so $\boldsymbol{X}$ uniquely determines $\boldsymbol{Y}$ also.

1. If $\boldsymbol{X}$ is a sub-martingale then $\mathbb{E}(X_n \mid \mathscr{F}_{n-1}) - X_{n-1} \ge 0$ for $n \in \mathbb{N}_+$ so $\boldsymbol{Z}$ is increasing.
2. If $\boldsymbol{X}$ is a super-martingale then $\mathbb{E}(X_n \mid \mathscr{F}_{n-1}) - X_{n-1} \le 0$ for $n \in \mathbb{N}_+$ so $\boldsymbol{Z}$ is decreasing.

A decomposition of this form is more complicated in continuous time, in part because the definition of a predictable process is more subtle and complex. The decomposition theorem holds in continuous time, with our basic assumptions and the additional assumption that the collection of random variables $\{X_\tau : \tau \text{ is a finite-valued stopping time}\}$ is uniformly integrable. The result is known as the *Doob-Meyer decomposition theorem*, named additionally for Paul Meyer.

Markov Processes

As you might guess, there are important connections between Markov processes and martingales. Suppose that $\boldsymbol{X} = \{X_t : t \in T\}$ is a (homogeneous) Markov process with state space $(S, \mathscr{S})$, relative to the filtration $\mathfrak{F} = \{\mathscr{F}_t : t \in T\}$. Let $\boldsymbol{P} = \{P_t : t \in T\}$ denote the collection of transition kernesl of $\boldsymbol{X}$, so that

$$P_t(x, A) = \mathbb{P}(X_t \in A \mid X_0 = x), \quad x \in S, A \in \mathscr{S} \tag{17.2.9}$$

Recall that (like all probability kernels), P_t operates (on the right) on (measurable) functions $h : S \to \mathbb{R}$ by the rule

$$P_t h(x) = \int_S P_t(x, dy) h(y) = \mathbb{E}[h(X_t) \mid X_0 = x], \quad x \in S \tag{17.2.10}$$

assuming as usual that the expected value exists. Here is the critical definition that we will need.

Suppose that $h : S \to \mathbb{R}$ and that $\mathbb{E}[|h(X_t)|] < \infty$ for $t \in T$.

1. h is *harmonic* for $\boldsymbol{X}$ if $P_t h = h$ for $t \in T$.
2. h is *sub-harmonic* for $\boldsymbol{X}$ if $P_t h \ge h$ for $t \in T$.
3. h is *super-harmonic* for $\boldsymbol{X}$ if $P_t h \le h$ for $t \in T$.

The following theorem gives the fundamental connection between the two types of stochastic processes. Given the similarity in the terminology, the result may not be a surprise.

Suppose that $h : S \to \mathbb{R}$ and $\mathbb{E}[|h(X_t)|] < \infty$ for $t \in T$. Define $h(\boldsymbol{X}) = \{h(X_t) : t \in T\}$.

1. h is harmonic for $\boldsymbol{X}$ if and only if $h(\boldsymbol{X})$ is a martingale with respect to $\mathfrak{F}$.
2. h is sub-harmonic for $\boldsymbol{X}$ if and only if $h(\boldsymbol{X})$ is a sub-martingale with respect to $\mathfrak{F}$.
3. h is super-harmonic for $\boldsymbol{X}$ if and only if $h(\boldsymbol{X})$ is a super-martingale with respect to $\mathfrak{F}$.

Proof

Suppose that $s, t \in T$ with $s \le t$. Then by the Markov property,

$$\mathbb{E}[h(X_t) \mid \mathscr{F}_s] = \mathbb{E}[h(X_t) \mid X_s] = P_{t-s} h(X_s) \tag{17.2.11}$$

So if h is harmonic, $\mathbb{E}[h(X_t) \mid \mathscr{F}_s] = h(X_s)$ so $\{h(X_t) : t \in T\}$ is a martingale. Conversely, if $\{h(X_t) : t \in T\}$ is a martingale, then $P_{t-s} h(X_s) = h(X_s)$. Letting $s = 0$ and $X_0 = x$ gives $P_t h(x) = h(x)$ so h is harmonic. The proofs for sub and super-martingales are similar, with inequalities replacing the equalities.

Several of the examples given in the Introduction can be re-interpreted in the context of harmonic functions of Markov chains. We explore some of these below.

Examples

Let $\mathscr{R}$ denote the usual set of Borel measurable subsets of $\mathbb{R}$, and for $A \in \mathscr{R}$ and $x \in \mathbb{R}$ let $A - x = \{y - x : y \in A\}$. Let I denote the identity function on $\mathbb{R}$, so that $I(x) = x$ for $x \in \mathbb{R}$. We will need this notation in a couple of our applications below.

Random Walks

Suppose that $\boldsymbol{V} = \{V_n : n \in \mathbb{N}\}$ is a sequence of independent, real-valued random variables, with $\{V_n : n \in \mathbb{N}_+\}$ identically distributed and having common probability measure Q on $(\mathbb{R}, \mathscr{R})$ and mean $a \in \mathbb{R}$. Recall from the Introduction that the partial sum process $\boldsymbol{X} = \{X_n : n \in \mathbb{N}\}$ associated with $\boldsymbol{V}$ is given by

$$X_n = \sum_{i=0}^{n} V_i, \quad n \in \mathbb{N} \tag{17.2.12}$$

and that $\boldsymbol{X}$ is a (discrete-time) *random walk*. But $\boldsymbol{X}$ is also a discrete-time Markov process with one-step transition kernel P given by $P(x, A) = Q(A - x)$ for $x \in \mathbb{R}$ and $A \in \mathscr{R}$.

The identity function I is

1. Harmonic for $\boldsymbol{X}$ if $a = 0$.
2. Sub-harmonic for $\boldsymbol{X}$ if $a \ge 0$.
3. Super-harmonic for $\boldsymbol{X}$ if $a \le 0$.

Proof

Note that

$$PI(x) = \mathbb{E}(X_1 \mid X_0 = x) = x + \mathbb{E}(X_1 - X_0 \mid X_0 = x) = x + \mathbb{E}(V_1 \mid X_0 = x) = I(x) + a \tag{17.2.13}$$

Since V_1 and $X_0 = V_0$ are independent. The results now follow from the definitions.

It now follows from our theorem above that $\boldsymbol{X}$ is a martingale if $a = 0$, a sub-martingale if $a > 0$, and a super-martingale if $a < 0$. We showed these results directly from the definitions in the Introduction.

The Simple Random Walk

Suppose now that that $\boldsymbol{V} = \{V_n : n \in \mathbb{N}\}$ is a sequence of independent random variables with $\mathbb{P}(V_i = 1) = p$ and $\mathbb{P}(V_i = -1) = 1 - p$ for $i \in \mathbb{N}_+$, where $p \in (0, 1)$. Let $\boldsymbol{X} = \{X_n : n \in \mathbb{N}\}$ be the partial sum process associated with $\boldsymbol{V}$ so that

$$X_n = \sum_{i=0}^{n} V_i, \quad n \in \mathbb{N} \tag{17.2.14}$$

Then $\boldsymbol{X}$ is the simple random walk with parameter p. In terms of gambling, our gambler plays a sequence of independent and identical games, and on each game, wins €1 with probability p and loses €1 with probability $1 - p$. So if X_0 is the gambler's initial fortune, then X_n is her net fortune after n games. In the Introduction we showed that $\boldsymbol{X}$ is a martingale if $p = \frac{1}{2}$, a super-martingale if $p < \frac{1}{2}$, and a sub-martingale if $p > \frac{1}{2}$. But suppose now that instead of making constant unit bets, the gambler makes bets that depend on the outcomes of previous games. This leads to a martingale transform as studied above.

Suppose that the gambler bets Y_n on game $n \in \mathbb{N}_+$ (at even stakes), where $Y_n \in [0, \infty)$ depends on $(V_0, V_1, V_2, \ldots, V_{n-1})$ and satisfies $\mathbb{E}(Y_n) < \infty$. So the process $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}_+\}$ is predictable with respect to $\boldsymbol{X}$, and the gambler's net winnings after n games is

$$(\boldsymbol{Y} \cdot \boldsymbol{X})_n = V_0 + \sum_{k=1}^{n} Y_k V_k = X_0 + \sum_{k=1}^{n} Y_k (X_k - X_{k-1}) \tag{17.2.15}$$

1. $\boldsymbol{Y} \cdot \boldsymbol{X}$ is a sub-martingale if $p > \frac{1}{2}$.
2. $\boldsymbol{Y} \cdot \boldsymbol{X}$ is a super-martingale if $p < \frac{1}{2}$.

3. $\boldsymbol{Y} \cdot \boldsymbol{X}$ is a martingale if $p = \frac{1}{2}$.

Proof

These result follow immediately the theorem for martingale transforms above.

The simple random walk $\boldsymbol{X}$ is also a discrete-time Markov chain on $\mathbb{Z}$ with one-step transition matrix P given by $P(x, x+1) = p$, $P(x, x-1) = 1 - p$.

The function h given by $h(x) = \left(\frac{1-p}{p}\right)^x$ for $x \in \mathbb{Z}$ is harmonic for $\boldsymbol{X}$.

Proof

For $x \in \mathbb{Z}$,

$$\begin{aligned} Ph(x) & = ph(x+1) + (1-p)h(x-1) = p\left(\frac{1-p}{p}\right)^{x+1} + (1-p)\left(\frac{1-p}{p}\right)^{x-1} \\ & \frac{(1-p)^{x+1}}{p^x} + \frac{(1-p)^x}{p^{x-1}} = \left(\frac{1-p}{p}\right)^x [(1-p)+p] = h(x) \end{aligned}$$

It now follows from our theorem above that the process $\boldsymbol{Z} = \{Z_n : n \in \mathbb{N}\}$ given by $Z_n = \left(\frac{1-p}{p}\right)^{X_n}$ for $n \in \mathbb{N}$ is a martingale. We showed this directly from the definition in the Introduction. As you may recall, this is *De Moivre's martingale* and named for Abraham De Moivre.

Branching Processes

Recall the discussion of the simple *branching process* from the Introduction. The fundamental assumption is that the particles act independently, each with the same offspring distribution on $\mathbb{N}$. As before, we will let f denote the (discrete) probability density function of the number of offspring of a particle, m the mean of the distribution, and ϕ the probability generating function of the distribution. We assume that $f(0) > 0$ and $f(0) + f(1) < 1$ so that a particle has a positive probability of dying without children and a positive probability of producing more than 1 child. Recall that q denotes the probability of extinction, starting with a single particle.

The stochastic process of interest is $\boldsymbol{X} = \{X_n : n \in \mathbb{N}\}$ where X_n is the number of particles in the nth generation for $n \in \mathbb{N}$. Recall that $\boldsymbol{X}$ is a discrete-time Markov chain on $\mathbb{N}$ with one-step transition matrix P given by $P(x, y) = f^{*x}(y)$ for $x, y \in \mathbb{N}$ where f^{*x} denotes the convolution power of order x of f.

The function h given by $h(x) = q^x$ for $x \in \mathbb{N}$ is harmonic for $\boldsymbol{X}$.

Proof

For $x \in \mathbb{N}$,

$$Ph(x) = \sum_{y \in \mathbb{N}} P(x, y)h(y) = \sum_{y \in \mathbb{N}} f^{*x}(y) q^y \tag{17.2.16}$$

The last expression is the probability generating function of f^{*x} evaluated at q. But this PGF is simply ϕ^x and q is a fixed point of ϕ so we have

$$Ph(x) = [\phi(q)]^x = q^x = h(x) \tag{17.2.17}$$

It now follows from our theorem above that the process $\boldsymbol{Z} = \{Z_n : n \in \mathbb{N}\}$ is a martingale where $Z_n = q^{X_n}$ for $n \in \mathbb{N}$. We showed this directly from the definition in the Introduction. We also showed that the process $\boldsymbol{Y} = \{Y_n : n \in \mathbb{N}\}$ is a martingale where $Y_n = X_n / m^n$ for $n \in \mathbb{N}$. But we can't write $Y_n = h(X_n)$ for a function h defined on the state space, so we can't interpret this martingale in terms of a harmonic function.

General Random Walks

Suppose that $\boldsymbol{X} = \{X_t : t \in T\}$ is a stochastic process satisfying the basic assumptions above relative to the filtration $\mathfrak{F} = \{\mathscr{F}_t : t \in T\}$. Recall from the Introduction that the term *increment* refers to a difference of the form $X_{s+t} - X_s$ for $s, t \in T$.

The process $\boldsymbol{X}$ has *independent increments* if this increment is always independent of $\mathscr{F}_s$, and has *stationary increments* this increment always has the same distribution as $X_t - X_0$. In discrete time, a process with stationary, independent increments is simply a random walk as discussed above. In continuous time, a process with stationary, independent increments (and with the continuity assumptions we have imposed) is called a *continuous-time random walk,* and also a *Lévy process,* named for Paul Lévy.

So suppose that $\boldsymbol{X}$ has stationary, independent increments. For $t \in T$ let Q_t denote the probability distribution of $X_t - X_0$ on $(\mathbb{R}, \mathscr{R})$, so that Q_t is also the probability distribution aof $X_{s+t} - X_s$ for every $s, t \in T$. From our previous study, we know that $\boldsymbol{X}$ is a Markov processes with transition kernel P_t at time $t \in T$ given by

$$P_t(x, A) = Q_t(A - x); \quad x \in \mathbb{R}, A \in \mathscr{R} \tag{17.2.18}$$

We also know that $\mathbb{E}(X_t - X_0) = at$ for $t \in T$ where $a = \mathbb{E}(X_1 - X_0)$ (assuming of course that the last expected value exists in $\mathbb{R}$).

The identity function I is .

1. Harmonic for $\boldsymbol{X}$ if $a = 0$.
2. Sub-harmonic for $\boldsymbol{X}$ if $a \ge 0$.
3. Super-harmonic for $\boldsymbol{X}$ if $a \le 0$.

Proof

Note that

$$P_t I(x) = \mathbb{E}(X_t \mid X_0 = x) = x + \mathbb{E}(X_t - X_0 \mid X_0 = x) = I(x) + at \tag{17.2.19}$$

since $X_t - X_0$ is independent of X_0. The results now follow from the definitions.

It now follows that $\boldsymbol{X}$ is a martingale if $a = 0$, a sub-martingale if $a \ge 0$, and a super-martingale if $a \le 0$. We showed this directly in the Introduction. Recall that in continuous time, the Poisson counting process has stationary, independent increments, as does standard Brownian motion