5.26: The U-Power Distribution

The *U-power distribution* is a U-shaped family of distributions based on a simple family of power functions.

The Standard U-Power Distribution

Distribution Functions

The *standard U-power distribution* with shape parameter $k \in \mathbb{N}_+$ is a continuous distribution on $[-1, 1]$ with probability density function g given by

$$g(x) = \frac{2k+1}{2} x^{2k}, \quad x \in [-1, 1] \tag{5.26.1}$$

Proof

From simple calculus, g is a probability density function:

$$\int_{-1}^{1} x^{2k} dx = \frac{2}{2k+1} \tag{5.26.2}$$

The algebraic form of the probability density function explains the name of the distribution. The most common of the standard U-power distributions is the *U-quadratic distribution*, which corresponds to $k = 1$.

The standard U-power probability density function g satisfies the following properties:

1. g is symmetric about $x = 0$.
2. g decreases and then increases with minimum value at $x = 0$.
3. The modes are $x = \pm 1$.
4. g is concave upward.

Proof

Again, these properties follow from basic calculus since

$$g'(x) = \frac{1}{2}(2k+1)(2k)x^{2k-1}, \quad x \in [-1, 1] \tag{5.26.3}$$

$$g''(x) = \frac{1}{2}(2k+1)(2k)(2k-1)x^{2k-2}, \quad x \in [-1, 1] \tag{5.26.4}$$

Open the Special Distribution Simulator and select the U-power distribution. Vary the shape parameter but keep the default values for the other parameters. Note the graph of the probability density function. For selected values of the shape parameter, run the simulation 1000 times and compare the emprical density function to the probability density function.

The distribution function G given by

$$G(x) = \frac{1}{2}\left(1 + x^{2k+1}\right), \quad x \in [-1, 1] \tag{5.26.5}$$

Proof

This follows from the PDF above and simple calculus.

The quantile function G^{-1} given by $G^{-1}(p) = (2p-1)^{1/(2k+1)}$ for $p \in [0, 1]$.

1. $G^{-1}(1-p) = -G^{-1}(p)$ for $p \in [0, 1]$.
2. The first quartile is $q_1 = -\frac{1}{2^{1/(2k+1)}}$.
3. The median is 0.
4. The third quartile is $q_3 = \frac{1}{2^{1/(2k+1)}}$.

Proof

The formula for the quantile function follows immediately from the CDF above by solving $p = G(x)$ for x in terms of $p \in [0,1]$. Property (a) follows from the symmetry of the distribution about 0.

Open the Special Distribution Calculator and select the U-power distribution. Vary the shape parameter but keep the default values for the other parameters. Note the shape of the distribution function. For various values of the shape parameter, compute a few quantiles.

Moments

Suppose that Z has the standard U-power distribution with parameter $k \in \mathbb{N}_+$. The moments (about 0) are easy to compute.

Let $n \in \mathbb{N}$. The moment of order $2n+1$ is $\mathbb{E}(Z^{2n+1}) = 0$. The moment of order $2n$ is

$$\mathbb{E}\left(Z^{2n}\right) = \frac{2k+1}{2(n+k)+1} \tag{5.26.6}$$

Proof

This result follows from simple calculus. The fact that the even order moments are 0 also follows from the symmetry of the distribution about 0.

Since the mean is 0, the moments about 0 are also the central moments.

The mean and variance of Z are

1. $\mathbb{E}(Z) = 0$
2. $\text{var}(Z) = \frac{2k+1}{2k+3}$

Proof

These results follow from the previous general moment result.

Note that $\text{var}(Z) \to 1$ as $k \to \infty$.

Open the Special Distribution Simulator and select the U-power distribution. Vary the shape parameter but keep the default values for the other paramters. Note the position and size of the mean $\pm$ standard deviation bar. For selected values of the shape parameter, run the simulation 1000 times and compare the empirical mean and stadard deviation to the distribution mean and standard deviation.

The skewness and kurtosis of Z are

1. $\text{skew}(Z) = 0$
2. $\text{kurt}(Z) = \frac{(2k+3)^2}{(2k+5)(2k+1)}$

Proof

The skewness is 0 by the symmetry of the distribution. Since the mean is 0, the kurtosis is $\mathbb{E}(Z^4)/[\mathbb{E}(Z^2)]^2$ and so the result follows from the general moment result above

Note that $\text{kurt}(Z) \to 1$ as $k \to \infty$. The *excess kurtosis* is $\text{kurt}(Z) - 3 = \frac{(2k+3)^2}{(2k+5)(2k+1)} - 3$ and so $\text{kurt}(Z) - 3 \to -2$ as $k \to \infty$.

Related Distributions

The U-power probability density function g actually makes sense for $k = 0$ as well, and in this case the distribution reduces to the uniform distribution on the interval $[-1, 1]$. But of course, this distribution is not U-shaped, except in a degenerate sense. There are other connections to the uniform distribution. The first is a standard result since the U-power quantile function has a simple, closed representation:

Suppose that $k \in \mathbb{N}_+$.

1. If U has the standard uniform distribution then $Z = (2U - 1)^{1/(2k+1)}$ has the standard U-power distribution with parameter k.
2. If Z has the standard U-power distribution with parameter k then $U = \frac{1}{2}\left(1 + Z^{2k+1}\right)$ has the standard uniform distribution.

Part (a) of course leads to the random quantile method of simulation.

Open the random quantile simulator and select the U-power distribution. Vary the shape parameter but keep the default values for the other parameters. Note the shape of the distribution and density functions. For selected values of the parameter, run the simulation 1000 times and note the random quantiles. Compare the empirical density function to the probability density function.

The standard U-power distribution with shape parameter $k \in \mathbb{N}_+$ converges to the discrete uniform distribution on $\{-1, 1\}$ as $k \to \infty$.

Proof

This follows from the definition of convergence in distribution. The U-power distribution function G is 0 on $(-\infty, -1]$, is 1 on $[1, \infty)$, and is given by the formula above on $[-1, 1]$. As $k \to \infty$, $G(x) \to 0$ for $x \in (-\infty, -1)$, $G(x) \to \frac{1}{2}$ for $x \in (-1, 1)$, and $G(x) \to 1$ for $x \in (1, \infty)$. This agrees with the distribution function of the discrete uniform distribution on $\{-1, 1\}$ except at the points of discontinuity ± 1.

The General U-Power Distribution

Like so many standard distributions, the standard U-power distribution is generalized by adding location and scale parameters.

Definition

Suppose that Z has the standard U-power distribution with shape parameter $k \in \mathbb{N}_+$. If $\mu \in \mathbb{R}$ and $c \in (0, \infty)$ then $X = \mu + cZ$ has the *U-power distribution* with shape parameter k, location parameter μ and scale parameter c.

Note that X has a continuous distribution on the interval $[a, b]$ where $a = \mu - c$ and $b = \mu + c$, so the distribution can also be parameterized by the the shape parameter k and the endpoints a and b. With this parametrization, the location parameter is $\mu = \frac{a+b}{2}$ and the scale parameter is $c = \frac{b-a}{2}$.

Distribution Functions

Suppose that X has the U-power distribution with shape parameter $k \in \mathbb{N}_+$, location parameter $\mu \in \mathbb{R}$, and scale parameter $c \in (0, \infty)$.

X has probability density function f given by

$$f(x) = \frac{2k+1}{2c}\left(\frac{x-\mu}{c}\right)^{2k}, \quad x \in [\mu - c, \mu + c] \tag{5.26.7}$$

1. f is symmetric about μ.
2. f decreases and then increases with minimum value at $x = \mu$.
3. The modes are at $x = \mu \pm c$.
4. f is concave upward.

Proof

Recall that $f(x) = \frac{1}{c} g\left(\frac{x-\mu}{c}\right)$ where g is the PDF of Z.

Open the Special Distribution Simulator and select the U-power distribution. Vary the parameters and note the shape and location of the probability density function. For various values of the parameters, run the simulation 1000 times and compare the emprical density function to the probability density function.

X has distribution function F given by

$$F(x) = \frac{1}{2}\left[1 + \left(\frac{x-\mu}{c}\right)^{2k+1}\right], \quad x \in [\mu - c, \mu + c] \tag{5.26.8}$$

Proof

Recall that $F(x) = G\left(\frac{x-\mu}{c}\right)$ where G is the CDF of Z.

X has quantile function F^{-1} given by $F^{-1}(p) = \mu + c(2p-1)^{1/(2k+1)}$ for $p \in [0,1]$.

1. $F^{-1}(1-p) = \mu - cF^{-1}(p)$
2. The first quartile is $q_1 = \mu - c\frac{1}{2^{1/(2k+1)}}$
3. The median is μ.
4. The third quartile is $q_3 = \mu + c\frac{1}{2^{1/(2k+1)}}$

Proof

Recall that $F^{-1}(p) = \mu + cG^{-1}(p)$ where G^{-1} is the quantile function of Z.

Open the Special Distribution Calculator and select the U-power distribution. Vary the parameters and note the graph of the distribution function. For various values of the parameters, compute selected values of the distribution function and the quantile function.

Moments

Suppose again that X has the U-power distribution with shape parameter $k \in \mathbb{N}_+$, location parameter $\mu \in \mathbb{R}$, and scale parameter $c \in (0, \infty)$.

The mean and variance of X are

1. $\mathbb{E}(X) = \mu$
2. $\text{var}(X) = c^2 \frac{2k+1}{2k+3}$

Proof

These results follow from the representation $X = \mu + cZ$ where Z has the standard U-power distribution with shape parameter k, and from the mean and variance of Z.

Note that $\text{var}(Z) \to c^2$ as $k \to \infty$

Open the Special Distribution Simulator and select the U-power distribution. Vary the parameters and note the size and location of the mean $\pm$ standard deviation bar. For various values of the parameters, run the simulation 1000 times and compare the empirical mean and stadard deviation to the distribution mean and standard deviation.

The moments about 0 are messy, but the central moments are simple.

Let $n \in \mathbb{N}_+$. The central moment of order $2n+1$ is $\mathbb{E}\left[(X-\mu)^{2n+1}\right] = 0$. The moment of order $2n$ is

$$\mathbb{E}\left[(x-\mu)^{2n}\right] = c^{2n}\frac{2k+1}{2(n+k)+1} \tag{5.26.9}$$

Proof

This follows from the representation $X = \mu + cZ$ where Z has the standard U-power distribution with shape parameter k, and the central moments of Z.

The skewness and kurtosis of X are

1. $\text{skew}(X) = 0$

2. $\text{kurt}(X) = \frac{(2k+3)^2}{(2k+5)(2k+1)}$

Proof

Recall that the skewness and kurtosis are defined in terms of the standard score of X and hence are invariant under a location-scale transformation. Thus, the results are the same as for the standard distribution.

Again, $\text{kurt}(X) \to 1$ as $k \to \infty$ and the excess kurtosis is $\text{kurt}(X) - 3 = \frac{(2k+3)^2}{(2k+5)(2k+1)} - 3$

Related Distributions

Since the U-power distribution with a given shape parameter is a location-scale family, it is trivially closed under location-scale transformations.

Suppose that X has the U-power distribution with shape parameter $k \in \mathbb{N}_+$, location parameter $\mu \in \mathbb{R}$, and scale parameter $c \in (0, \infty)$. If $\alpha \in \mathbb{R}$ and $\beta \in (0, \infty)$, then $Y = \alpha + \beta X$ has the U-power distribution with shape parameter k, location parameter $\alpha + \beta\mu$, and scale parameter βc.

Proof

From the definition, we can take $X = \mu + cZ$ where Z has the standard U-power distribution with shape parameter k. Then $Y = \alpha + \beta X = (\alpha + \beta\mu) + (\beta c)Z$.

As before, since the U-power distribution function and the U-power quantile function have simple forms, we have the usual connections with the standard uniform distribution.

Suppose that $k \in \mathbb{N}_+$, $\mu \in \mathbb{R}$ and $c \in (0, \infty)$.

1. If U has the standard uniform distribution then $X = \mu + c(2U - 1)^{1/(2k+1)}$ has the U-power distribution with shape parameter k, location parameter μ, and scale parameter c.
2. If X has the U-power distribution with shape parameter k, location parameter μ, and scale parameter c, then $U = \frac{1}{2}\left[1 + \left(\frac{X-\mu}{c}\right)^{2k+1}\right]$ has the standard uniform distribution.

Again, part (a) of course leads to the random quantile method of simulation.

Open the random quantile simulator and select the U-power distribution. Vary the parameters and note the shape of the distribution and density functions. For selected values of the parameters, run the simulation 1000 times and note the random quantiles. Compare the empirical density function to the probability density function.

The U-power distribution with given location and scale parameters converges to the discrete uniform distribution at the endpoints as the shape parameter increases.

The U-power distribution with shape parameter $k \in \mathbb{N}_+$, location parameter $\mu \in \mathbb{R}$, and scale parameter $c \in (0, \infty)$ converges to the discrete uniform distribution on $\{\mu - c, \mu + c\}$ as $k \to \infty$.

Proof

This follows from the convergence result for the standard distribution and basic properties of convergence in distribution.

The U-power distribution is a general exponential family in the shape parameter, if the location and scale parameters are fixed.

Suppose that X has the U-power distribution with unspecified shape parameter $k \in \mathbb{N}_+$, but with specified location parameter $\mu \in \mathbb{R}$ and scale parameter $c \in (0, \infty)$. Then X has a one-parameter exponential distribution with natural parameter $2k$ and natural statistics $\ln\left(\frac{X-\mu}{c}\right)$.

Proof

This follows from the definition of the general exponential family, since the PDF of the U-power distribution can be written as

$$f(x) = \frac{2k+1}{2c} \exp\left[2k \ln\left(\frac{x-\mu}{c}\right)\right], \quad x \in [\mu - c, \mu + c] \tag{5.26.10}$$

Since the U-power distribution has a bounded probability density function on a bounded support interval, it can also be simulated via the rejection method.

Open the rejection method experiment and select the U-power distribution. Vary the parameters and note the shape of the probability density function. For selected values of the parameters, run the experiment 1000 times and watch the scatterplot. Compare the empirical density function, mean, and standard deviation to their distributional counterparts.