

## 5.28: The Laplace Distribution

The *Laplace distribution*, named for Pierre Simon Laplace arises naturally as the distribution of the difference of two independent, identically distributed exponential variables. For this reason, it is also called the *double exponential distribution*.

### The Standard Laplace Distribution

#### Distribution Functions

The *standard Laplace distribution* is a continuous distribution on  $\mathbb{R}$  with probability density function  $g$  given by

$$g(u) = \frac{1}{2}e^{-|u|}, \quad u \in \mathbb{R} \quad (5.28.1)$$

Proof

It's easy to see that  $g$  is a valid PDF. By symmetry

$$\int_{-\infty}^{\infty} \frac{1}{2}e^{-|u|} du = \int_0^{\infty} e^{-u} du = 1 \quad (5.28.2)$$

The probability density function  $g$  satisfies the following properties:

1.  $g$  is symmetric about 0.
2.  $g$  increases on  $(-\infty, 0]$  and decreases on  $[0, \infty)$ , with mode  $u = 0$ .
3.  $g$  is concave upward on  $(-\infty, 0]$  and on  $[0, \infty)$  with a cusp at  $u = 0$

Proof

These results follow from standard calculus, since  $g(u) = \frac{1}{2}e^{-u}$  for  $u \in [0, \infty)$  and  $g(u) = \frac{1}{2}e^u$  for  $u \in (-\infty, 0]$ .

Open the Special Distribution Simulator and select the Laplace distribution. Keep the default parameter value and note the shape of the probability density function. Run the simulation 1000 times and compare the empirical density function and the probability density function.

The standard Laplace distribution function  $G$  is given by

$$G(u) = \begin{cases} \frac{1}{2}e^u, & u \in (-\infty, 0] \\ 1 - \frac{1}{2}e^{-u}, & u \in [0, \infty) \end{cases} \quad (5.28.3)$$

Proof

Again this follows from basic calculus, since  $g(u) = \frac{1}{2}e^u$  for  $u \leq 0$  and  $g(u) = \frac{1}{2}e^{-u}$  for  $u \geq 0$ . Of course  $G(u) = \int_{-\infty}^u g(t) dt$ .

The quantile function  $G^{-1}$  given by

$$G^{-1}(p) = \begin{cases} \ln(2p), & p \in [0, \frac{1}{2}] \\ -\ln[2(1-p)], & p \in [\frac{1}{2}, 1] \end{cases} \quad (5.28.4)$$

1.  $G^{-1}(1-p) = -G^{-1}(p)$  for  $p \in (0, 1)$
2. The first quartile is  $q_1 = -\ln 2 \approx -0.6931$ .
3. The median is  $q_2 = 0$
4. The third quartile is  $q_3 = \ln 2 \approx 0.6931$ .

Proof

The formula for the quantile function follows immediately from the [CDF](#) by solving  $p = G(u)$  for  $u$  in terms of  $p \in (0, 1)$ . Part (a) is due to the symmetry of  $g$  about 0.

Open the Special Distribution Calculator and select the Laplace distribution. Keep the default parameter value. Compute selected values of the distribution function and the quantile function.

## Moments

Suppose that  $U$  has the standard Laplace distribution.

$U$  has moment generating function  $m$  given by

$$m(t) = \mathbb{E}(e^{tU}) = \frac{1}{1-t^2}, \quad t \in (-1, 1) \quad (5.28.5)$$

Proof

For  $t \in (-1, 1)$ ,

$$m(t) = \int_{-\infty}^{\infty} e^{tu} g(u) du = \int_{-\infty}^0 \frac{1}{2} e^{(t+1)u} du + \int_0^{\infty} \frac{1}{2} e^{(t-1)u} du = \frac{1}{2(t+1)} - \frac{1}{2(t-1)} = \frac{1}{1-t^2} \quad (5.28.6)$$

The moments of  $U$  are

1.  $\mathbb{E}(U^n) = 0$  if  $n \in \mathbb{N}$  is odd.
2.  $\mathbb{E}(U^n) = n!$  if  $n \in \mathbb{N}$  is even.

Proof

This result can be obtained from the [moment generating function](#) or directly. That the odd order moments are 0 follows from the symmetry of the distribution. For the even order moments, symmetry and an integration by parts (or using the gamma function) gives

$$\mathbb{E}(U^n) = \frac{1}{2} \int_{-\infty}^0 u^n e^u du + \frac{1}{2} \int_0^{\infty} u^n e^{-u} du = \int_0^{\infty} u^n e^{-u} du = n! \quad (5.28.7)$$

The mean and variance of  $U$  are

1.  $\mathbb{E}(U) = 0$
2.  $\text{var}(U) = 2$

Open the [Special Distribution Simulator](#) and select the Laplace distribution. Keep the default parameter value. Run the simulation 1000 times and compare the empirical mean and standard deviation to the distribution mean and standard deviation.

The skewness and kurtosis of  $U$  are

1.  $\text{skew}(U) = 0$
2.  $\text{kurt}(U) = 6$

Proof

1. This follows from the symmetry of the distribution.
2. Since  $\mathbb{E}(U) = 0$ , we have

$$\text{kurt}(U) = \frac{\mathbb{E}(U^4)}{[\mathbb{E}(U^2)]^2} = \frac{4!}{(2!)^2} = 6 \quad (5.28.8)$$

It follows that the *excess kurtosis* is  $\text{kurt}(U) - 3 = 3$ .

## Related Distributions

Of course, the standard Laplace distribution has simple connections to the standard exponential distribution.

If  $U$  has the standard Laplace distribution then  $V = |U|$  has the standard exponential distribution.

Proof

Using the [CDF of U](#) we have  $\mathbb{P}(V \leq v) = \mathbb{P}(-v \leq U \leq v) = G(v) - G(-v) = 1 - e^{-v}$  for  $v \in [0, \infty)$ . This function is the CDF of the standard exponential distribution.

If  $V$  and  $W$  are independent and each has the standard exponential distribution, then  $U = V - W$  has the standard Laplace distribution.

Proof using PDFs

Let  $h$  denote the standard exponential PDF, extended to all of  $\mathbb{R}$ , so that  $h(v) = e^{-v}$  if  $v \geq 0$  and  $h(v) = 0$  if  $v < 0$ . Using convolution, the PDF of  $V - W$  is  $g(u) = \int_{\mathbb{R}} h(v)h(v-u)dv$ . If  $v \geq 0$ ,

$$g(u) = \int_u^{\infty} e^{-v} e^{-(v-u)} dv = e^u \int_u^{\infty} e^{-2v} dv = \frac{1}{2} e^{-u} \quad (5.28.9)$$

If  $u < 0$  then

$$g(u) = \int_0^{\infty} e^{-v} e^{-(v-u)} dv = e^u \int_0^{\infty} e^{-2v} dv = \frac{1}{2} e^u \quad (5.28.10)$$

Proof using MGFs

The MGF of  $V$  is  $t \mapsto 1/(1-t)$  for  $t < 1$ . The MGF of  $-W$  is  $t \mapsto 1/(1+t)$  for  $t > -1$ . Hence the MGF of  $U$  is  $t \mapsto 1/(1-t)(1+t) = 1/(1-t^2)$  for  $-1 < t < 1$ , which is the [standard Laplace MGF](#).

If  $V$  has the standard exponential distribution,  $I$  has the standard Bernoulli distribution, and  $V$  and  $I$  are independent, then  $U = (2I - 1)V$  has the standard Laplace distribution.

Proof

If  $u \geq 0$  then

$$\mathbb{P}(U \leq u) = \mathbb{P}(I = 0) + \mathbb{P}(I = 1, V \leq u) = \mathbb{P}(I = 0) + \mathbb{P}(I = 1)\mathbb{P}(V \leq u) = \frac{1}{2} + \frac{1}{2}(1 - e^{-u}) = 1 - \frac{1}{2}e^{-u} \quad (5.28.11)$$

If  $u < 0$ ,

$$\mathbb{P}(U \leq u) = \mathbb{P}(I = 0, V > -u) = \mathbb{P}(I = 0)\mathbb{P}(V > -u) = \frac{1}{2}e^u \quad (5.28.12)$$

The standard Laplace distribution has a curious connection to the standard normal distribution.

Suppose that  $(Z_1, Z_2, Z_3, Z_4)$  is a random sample of size 4 from the standard normal distribution. Then  $U = Z_1 Z_2 + Z_3 Z_4$  has the standard Laplace distribution.

Proof

$Z_1 Z_2$  and  $Z_3 Z_4$  are independent, and each has a distribution known as the *product normal distribution*. The MGF of this distribution is

$$m_0(t) = \mathbb{E}(e^{tZ_1 Z_2}) = \int_{\mathbb{R}^2} e^{txy} \frac{1}{2\pi} e^{-(x^2+y^2)/2} dx dy \quad (5.28.13)$$

Changing to polar coordinates gives

$$m_0(t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{tr^2 \cos \theta \sin \theta} e^{-r^2/2} r dr d\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \exp \left[ r^2 \left( t \cos \theta \sin \theta - \frac{1}{2} \right) \right] r dr d\theta \quad (5.28.14)$$

The inside integral can be done with a simple substitution for  $|t| < 1$ , yielding

$$m_0(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - t \sin(2\theta)} d\theta = \frac{1}{\sqrt{1-t^2}} \quad (5.28.15)$$

Hence  $U$  has MGF  $m_0^2(t) = \frac{1}{1-t^2}$  for  $|t| < 1$ , which again is the [standard Laplace MGF](#).

The standard Laplace distribution has the usual connections to the standard uniform distribution by means of the [distribution function](#) and the [quantile function](#) computed above.

Connections to the standard uniform distribution.

1. If  $V$  has the standard uniform distribution then  $U = \ln(2V)\mathbf{1}(V < \frac{1}{2}) - \ln[2(1-V)]\mathbf{1}(V \geq \frac{1}{2})$  has the standard Laplace distribution.
2. If  $U$  has the standard Laplace distribution then  $V = \frac{1}{2}e^U\mathbf{1}(U < 0) + (1 - \frac{1}{2}e^{-U})\mathbf{1}(U \geq 0)$  has the standard uniform distribution.

From part (a), the standard Laplace distribution can be simulated with the usual random quantile method.

Open the random quantile experiment and select the Laplace distribution. Keep the default parameter values and note the shape of the probability density and distribution functions. Run the simulation 1000 times and compare the empirical density function, mean, and standard deviation to their distributional counterparts.

## The General Laplace Distribution

The standard Laplace distribution is generalized by adding location and scale parameters.

Suppose that  $U$  has the standard Laplace distribution. If  $a \in \mathbb{R}$  and  $b \in (0, \infty)$ , then  $X = a + bU$  has the *Laplace distribution* with location parameter  $a$  and scale parameter  $b$ .

### Distribution Functions

Suppos that  $X$  has the Laplace distribution with location parameter  $a \in \mathbb{R}$  and scale parameter  $b \in (0, \infty)$ .

$X$  has probability density function  $f$  given by

$$f(x) = \frac{1}{2b} \exp\left(-\frac{|x-a|}{b}\right), \quad x \in \mathbb{R} \quad (5.28.16)$$

1.  $f$  is symmetric about  $a$ .
2.  $f$  increases on  $[0, a]$  and decreases on  $[a, \infty)$  with mode  $x = a$ .
3.  $f$  is concave upward on  $[0, a]$  and on  $[a, \infty)$  with a cusp at  $x = a$ .

Proof

Recall that  $f(x) = \frac{1}{b} g\left(\frac{x-a}{b}\right)$  where  $g$  is the [standard Laplace PDF](#).

Open the Special Distribution Simulator and select the Laplace distribution. Vary the parameters and note the shape and location of the probability density function. For various values of the parameters, run the simulation 1000 times and compare the emprical density function to the probability density function.

$X$  has distribution function  $F$  given by

$$F(x) = \begin{cases} \frac{1}{2} \exp\left(\frac{x-a}{b}\right), & x \in (-\infty, a] \\ 1 - \frac{1}{2} \exp\left(-\frac{x-a}{b}\right), & x \in [a, \infty) \end{cases} \quad (5.28.17)$$

Proof

Recall that  $F(x) = G\left(\frac{x-a}{b}\right)$  where  $G$  is the [standard Laplace CDF](#).

$X$  has quantile function  $F^{-1}$  given by

$$F^{-1}(p) = \begin{cases} a + b \ln(2p), & 0 \leq p \leq \frac{1}{2} \\ a - b \ln[2(1-p)], & \frac{1}{2} \leq p < 1 \end{cases} \quad (5.28.18)$$

1.  $F^{-1}(1-p) = a - bF^{-1}(p)$  for  $p \in (0, 1)$
2. The first quartile is  $q_1 = a - b \ln 2$ .
3. The median is  $q_2 = a$
4. The third quartile is  $q_3 = a + b \ln 2$ .

Proof

Recall that  $F^{-1}(p) = a + bG^{-1}(p)$  where  $G^{-1}$  is the [standard Laplace quantile function](#).

Open the Special Distribution Calculator and select the Laplace distribution. For various values of the scale parameter, compute selected values of the distribution function and the quantile function.

### Moments

Again, we assume that  $X$  has the Laplace distribution with location parameter  $a \in \mathbb{R}$  and scale parameter  $b \in (0, \infty)$ , so that by [definition](#),  $X = a + bU$  where  $U$  has the standard Lapalce distribution.

$X$  has moment generating function  $M$  given by

$$M(t) = \mathbb{E}(e^{tX}) = \frac{e^{at}}{1 - b^2 t^2}, \quad t \in (-1/b, 1/b) \quad (5.28.19)$$

Proof

Recall that  $M(t) = e^{at}m(bt)$  where  $m$  is the [standard Laplace MGF](#).

The moments of  $X$  about the location parameter have a simple form.

The moments of  $X$  about  $a$  are

1.  $\mathbb{E}[(X - a)^n] = 0$  if  $n \in \mathbb{N}$  is odd.
2.  $\mathbb{E}[(X - a)^n] = b^n n!$  if  $n \in \mathbb{N}$  is even.

Proof

Note that  $\mathbb{E}[(X - a)^n] = b^n \mathbb{E}(U^n)$  so the results follow the [moments of  \$U\$](#) .

The mean and variance of  $X$  are

1.  $\mathbb{E}(X) = a$
2.  $\text{var}(X) = 2b^2$

Proof

Recall that  $\mathbb{E}(X) = a + b\mathbb{E}(U)$  and  $\text{var}(X) = b^2 \text{var}(U)$ , so the results follow from the [mean and variance of  \$U\$](#) .

Open the Special Distribution Simulator and select the Laplace distribution. Vary the parameters and note the size and location of the mean  $\pm$  standard deviation bar. For various values of the scale parameter, run the simulation 1000 times and compare the empirical mean and standard deviation to the distribution mean and standard deviation.

The skewness and kurtosis of  $X$  are

1.  $\text{skew}(X) = 0$
2.  $\text{kurt}(X) = 6$

Proof

Recall that skewness and kurtosis are defined in terms of the standard score, and hence are unchanged by a location-scale transformation. Thus the results from the [skewness and kurtosis of  \$U\$](#) .

As before, the excess kurtosis is  $\text{kurt}(X) - 3 = 3$ .

### Related Distributions

By construction, the Laplace distribution is a location-scale family, and so is closed under location-scale transformations.

Suppose that  $X$  has the Laplace distribution with location parameter  $a \in \mathbb{R}$  and scale parameter  $b \in (0, \infty)$ , and that  $c \in \mathbb{R}$  and  $d \in (0, \infty)$ . Then  $Y = c + dX$  has the Laplace distribution with location parameter  $c + ad$  scale parameter  $bd$ .

Proof

Again by [definition](#), we can take  $X = a + bU$  where  $U$  has the standard Laplace distribution. Hence  $Y = c + dX = (c + ad) + (bd)U$ .

Once again, the Laplace distribution has the usual connections to the standard uniform distribution by means of the [distribution function](#) and the [quantile function](#) computed above. The latter leads to the usual random quantile method of simulation.

Suppose that  $a \in \mathbb{R}$  and  $b \in (0, \infty)$ .

1. If  $V$  has the standard uniform distribution then

$$U = [a + b \ln(2V)] \mathbf{1}\left(V < \frac{1}{2}\right) + (a - b \ln[2(1 - V)]) \mathbf{1}\left(V \geq \frac{1}{2}\right) \quad (5.28.20)$$

has the Laplace distribution with location parameter  $a$  and scale parameter  $b$ .

2. If  $X$  has the Laplace distribution with location parameter  $a$  and scale parameter  $b$ , then

$$V = \frac{1}{2} \exp\left(\frac{X-a}{b}\right) \mathbf{1}(X < a) + \left[1 - \frac{1}{2} \exp\left(-\frac{X-a}{b}\right)\right] \mathbf{1}(X \geq a) \quad (5.28.21)$$

has the standard uniform distribution.

Open the random quantile experiment and select the Laplace distribution. Vary the parameter values and note the shape of the probability density and distribution functions. For selected values of the parameters, run the simulation 1000 times and compare the empirical density function, mean, and standard deviation to their distributional counterparts.

The Laplace distribution is also a member of the general exponential family of distributions.

Suppose that  $X$  has the Laplace distribution with known location parameter  $a \in \mathbb{R}$  and unspecified scale parameter  $b \in (0, \infty)$ . Then  $X$  has a general exponential distribution in the scale parameter  $b$ , with natural parameter  $-1/b$  and natural statistics  $|X - a|$ .

Proof

This follows from the definition of the general exponential family and the form of the [probability density function](#)  $f$

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