14.7: Compound Poisson Processes

In a *compound Poisson process*, each arrival in an ordinary Poisson process comes with an associated real-valued random variable that represents the *value* of the arrival in a sense. These variables are independent and identically distributed, and are independent of the underlying Poisson process. Our interest centers on the sum of the random variables for all the arrivals up to a fixed time t, which thus is a Poisson-distributed random sum of random variables. Distributions of this type are said to be *compound Poisson distributions*, and are important in their own right, particularly since some surprising parametric distributions turn out to be compound Poisson.

Basic Theory

Definition

Suppose we have a Poisson process with rate $r \in (0, \infty)$. As usual, we wil denote the sequence of inter-arrival times by $\boldsymbol{X} = (X_1, X_2, \ldots)$, the sequence of arrival times by $\boldsymbol{T} = (T_0, T_1, T_2, \ldots)$, and the counting process by $\boldsymbol{N} = \{N_t : t \in [0, \infty)\}$. To review some of the most important facts briefly, recall that $\boldsymbol{X}$ is a sequence of independent random variables, each having the exponential distribution on $[0, \infty)$ with rate r. The sequence $\boldsymbol{T}$ is the partial sum sequence associated with $\boldsymbol{X}$, and has stationary independent increments. For $n \in \mathbb{N}_+$, the nth arrival time T_n has the gamma distribution with parameters n and r. The process $\boldsymbol{N}$ is the inverse of $\boldsymbol{T}$, in a certain sense, and also has stationary independent increments. For $t \in (0, \infty)$, the number of arrivals N_t in $(0, t]$ has the Poisson distribution with parameter rt.

Suppose now that each arrival has an associated real-valued random variable that represents the *value* of the arrival in a certain sense. Here are some typical examples:

- The arrivals are customers at a store. Each customer spends a random amount of money.
- The arrivals are visits to a website. Each visitor spends a random amount of time at the site.
- The arrivals are failure times of a complex system. Each failure requires a random repair time.
- The arrivals are earthquakes at a particular location. Each earthquake has a random severity, a measure of the energy released.

For $n \in \mathbb{N}_+$, let U_n denote the value of the nth arrival. We assume that $\boldsymbol{U} = (U_1, U_2, \ldots)$ is a sequence of independent, identically distributed, real-valued random variables, and that $\boldsymbol{U}$ is independent of the underlying Poisson process. The common distribution may be discrete or continuous, but in either case, we let f denote the common probability density function. We will let $\mu = \mathbb{E}(U_n)$ denote the common mean, $\sigma^2 = \text{var}(U_n)$ the common variance, and G the common moment generating function, so that $G(s) = \mathbb{E}[\exp(sU_n)]$ for s in some interval I about 0. Here is our main definition:

> The *compound Poisson process* associated with the given Poisson process $\boldsymbol{N}$ and the sequence $\boldsymbol{U}$ is the stochastic process $\boldsymbol{V} = \{V_t : t \in [0, \infty)\}$ where
>
> $$V_t = \sum_{n=1}^{N_t} U_n \tag{14.7.1}$$

Thus, V_t is the total value for all of the arrivals in $(0, t]$. For the examples above

- V_t is the total income to the store up to time t.
- V_t is the total time spent at the site by the customers who arrived up to time t.
- V_t is the total repair time for the failures up to time t.
- V_t is the total energy released up to time t.

Recall that a sum over an empty index set is 0, so $V_0 = 0$.

Properties

Note that for fixed t, V_t is a random sum of independent, identically distributed random variables, a topic that we have studied before. In this sense, we have a special case, since the number of terms N_t has the Poisson distribution with parameter rt. But we also have a new wrinkle, since the process is indexed by the continuous time parameter t, and so we can study its properties as a stochastic process. Our first result is a pair of properties shared by the underlying Poisson process.

> $\boldsymbol{V}$ has stationary, independent increments:

1. If $s, t \in [0, \infty)$ with $s < t$, then $V_t - V_s$ has the same distribution as V_{t-s}.
2. If $(t_1, t_2, \ldots, t_n)$ is a sequence of points in $[0, \infty)$ with $t_1 < t_2 < \cdots < t_n$ then $(V_{t_1}, V_{t_2} - V_{t_1}, \ldots, V_{t_n} - V_{t_{n-1}})$ is a sequence of independent variables.

Proof

1. For $0 \le s < t$,

$$V_t - V_s = \sum_{i=1}^{N_t} U_i - \sum_{i=1}^{N_s} U_i = \sum_{i=N_s+1}^{N_t} U_i \tag{14.7.2}$$

The number of terms in the last sum is $N_t - N_s$, which has the same distribution as N_{t-s}. Since the variables in the sequence $\boldsymbol{U}$ are identically distributed, it follows that $V_t - V_s$ has the same distribution as V_{t-s}.

2. Suppose that $0 \le t_1 < t_2 < \cdots < t_n$ and let $t_0 = 0$. Then for $i \in \{1, 2, \ldots, n\}$, as in (a)

$$V_{t_i} - V_{t_i - 1} = \sum_{j=N_{t_i-1}+1}^{N_{t_i}} U_j \tag{14.7.3}$$

The number of terms in this sum is $N_{t_i} - N_{t_{i-1}}$. Since $\boldsymbol{N}$ has independent increments, and the variables in $\boldsymbol{U}$ are independent, and since the indices between $N_{t_{i-1}+1}$ and N_{t_i} are disjoint over $i \in \{1, 2, \ldots n\}$, it follows that the random variables $V_{t_i} - V_{t_i - 1}$ are independent over $i \in \{1, 2, \ldots, n\}$.

Next we consider various moments of the compound process.

For $t \in [0, \infty)$, the mean and variance of V_t are

1. $\mathbb{E}(V_t) = \mu r t$
2. $\operatorname{var}(V_t) = (\mu^2 + \sigma^2) r t$

Proof

Again, these are special cases of general results for random sums of IID variables, but we give separate proofs for completeness. The basic tool is conditional expected value and conditional variance. Recall also that $\mathbb{E}(N_t) = \operatorname{var}(N_t) = rt$.

1. Note that $\mathbb{E}(V_t) = \mathbb{E}[\mathbb{E}(V_t \mid N_t)] = \mathbb{E}(\mu N_t) = \mu r t$.
2. Similarly, note that $\operatorname{var}(V_t \mid N_t) = \sigma^2 N_t$ and hence $\operatorname{var}(V_t) = \mathbb{E}[\operatorname{var}(V_t \mid N_t)] + \operatorname{var}[\mathbb{E}(V_t \mid N_t)] = \mathbb{E}(\sigma^2 N_t) + \operatorname{var}(\mu N_t) = \sigma^2 r t + \mu^2 r t$.

For $t \in [0, \infty)$, the moment generating function of V_t is given by

$$\mathbb{E}[\exp(s V_t)] = \exp(rt[G(s) - 1]), \quad s \in I \tag{14.7.4}$$

Proof

Again, this is a special case of the more general result for random sums of IID variables, but we give a another proof for completeness. As with the last theorem, the key is to condition on N_t and recall that the MGF of a sum of independent variables is the product of the MGFs. Thus

$$\mathbb{E}[\exp(s V_t)] = \mathbb{E}(\mathbb{E}[\exp(s V_t \mid N_t)]) = \mathbb{E}\left[G^{N_t}(s)\right] = P_t[G(s)] \tag{14.7.5}$$

where P_t is the probability generating function of N_t. But we know from our study of the Poisson distribution that $P_t(x) = \exp[rt(x-1)]$ for $x \in \mathbb{R}$.

By exactly the same argument, the same relationship holds for characteristic functions and, in the case that the variables in $\boldsymbol{U}$ take values in $\mathbb{N}$, for probability generating functions.. That is, if the variables in $\boldsymbol{U}$ have generating function G, then the generating function H of V_t is given by

$$H(s) = \exp(rt[G(s) - 1]) \tag{14.7.6}$$

for s in the domain of G, where *generating function* can be any of the three types we have discussed: probability, moment, or characteristic.

Examples and Special Cases

The Discrete Case

First we note that Thinning a Poisson process can be thought of as a special case of a compound Poisson process. Thus, suppose that $\boldsymbol{U} = (U_1, U_2, \ldots)$ is a Bernoulli trials sequence with success parameter $p \in (0, 1)$, and as above, that $\boldsymbol{U}$ is independent of the Poisson process $\boldsymbol{N}$. In the usual language of thinning, the arrivals are of two types (1 and 0), and U_i is the type of the ith arrival. Thus the compound process $\boldsymbol{V}$ constructed above is the thinned process, so that V_t is the number of type 1 points up to time t. We know that $\boldsymbol{V}$ is also a Poisson process, with rate rp.

The results above for thinning generalize to the case where the values of the arrivals have a discrete distribution. Thus, suppose U_i takes values in a countable set $S \subseteq \mathbb{R}$, and as before, let f denote the common probability density function so that $f(u) = \mathbb{P}(U_i = u)$ for $u \in S$ and $i \in \mathbb{N}_+$. For $u \in S$, let N_t^u denote the number of arrivals up to time t that have the value u, and let $\boldsymbol{N}^u = \{N_t^u : t \in [0, \infty)\}$ denote the corresponding stochastic process. Armed with this setup, here is the result:

The compound Poisson process $\boldsymbol{V}$ associated with $\boldsymbol{N}$ and $\boldsymbol{U}$ can be written in the form

$$V_t = \sum_{u \in S} u N_t^u, \quad t \in [0, \infty) \tag{14.7.7}$$

The processes $\{\boldsymbol{N}^u : u \in S\}$ are independent Poisson processes, and $\boldsymbol{N}^u$ has rate $r f(u)$ for $u \in S$.

Proof

Note that $U_i = \sum_{u \in S} u \boldsymbol{1}(U_i = u)$ and hence

$$V_t = \sum_{i=1}^{N_t} U_i = \sum_{i=1}^{N_t} \sum_{u \in S} u \boldsymbol{1}(U_i = u) = \sum_{u \in S} u \sum_{i=1}^{N_t} \boldsymbol{1}(U_i = u) = \sum_{u \in S} u N_t^u \tag{14.7.8}$$

The fact that $\{\boldsymbol{N}^u : u \in S\}$ are independent Poisson processes, and that $\boldsymbol{N}^u$ has rate $r f(u)$ for $u \in S$ follows from our result on thinning.

Compound Poisson Distributions

A compound Poisson random variable can be defined outside of the context of a Poisson process. Here is the formal definition:

Suppose that $\boldsymbol{U} = (U_1, U_2, \ldots)$ is a sequence of independent, identically distributed random variables, and that N is independent of $\boldsymbol{U}$ and has the Poisson distribution with parameter $\lambda \in (0, \infty)$. Then $V = \sum_{i=1}^N U_i$ has a *compound Poisson distribution*.

But in fact, compound Poisson variables usually *do* arise in the context of an underlying Poisson process. In any event, the results on the mean and variance above and the generating function above hold with rt replaced by λ. Compound Poisson distributions are infinitely divisible. A famous theorem of William Feller gives a partial converse: an infinitely divisible distribution on $\mathbb{N}$ must be compound Poisson.

The negative binomial distribution on $\mathbb{N}$ is infinitely divisible, and hence must be compound Poisson. Here is the construction:

Let $p, k \in (0, \infty)$. Suppose that $\boldsymbol{U} = (U_1, U_2, \ldots)$ is a sequence of independent variables, each having the logarithmic series distribution with shape parameter $1 - p$. Suppose also that N is independent of $\boldsymbol{U}$ and has the Poisson distribution with parameter $-k \ln(p)$. Then $V = \sum_{i=1}^N U_i$ has the negative binomial distribution on $\mathbb{N}$ with parameters k and p.

Proof

As noted above, the probability generating function of V is $P(t) = \exp(\lambda[Q(t) - 1])$ where λ is the parameter of the Poisson variable N and $Q(t)$ is the common PGF of the the terms in the sum. Using the PGF of the logarithmic series distribution, and the particular values of the parameters, we have

$$P(t) = \exp\left[-k \ln(p) \left(\frac{\ln[1 - (1 - p)t]}{\ln(p)} - 1\right)\right], \quad |t| < \frac{1}{1 - p} \tag{14.7.9}$$

Using properties of logarithms and simple algebra, this reduces to

$$P(t) = \left(\frac{p}{1 - (1 - p)t} \right)^k, \quad |t| < \frac{1}{1 - p} \tag{14.7.10}$$

which is the PGF of the negative binomial distribution with parameters k and p.

As a special case ($k = 1$), it follows that the geometric distribution on $\mathbb{N}$ is also compound Poisson.