

5.8: The Gamma Distribution

In this section we will study a family of distributions that has special importance in probability and statistics. In particular, the arrival times in the Poisson process have gamma distributions, and the chi-square distribution in statistics is a special case of the gamma distribution. Also, the gamma distribution is widely used to model physical quantities that take positive values.

The Gamma Function

Before we can study the gamma *distribution*, we need to introduce the gamma *function*, a special function whose values will play the role of the normalizing constants.

Definition

The *gamma function* Γ is defined as follows

$$\Gamma(k) = \int_0^{\infty} x^{k-1} e^{-x} dx, \quad k \in (0, \infty) \quad (5.8.1)$$

The function is well defined, that is, the integral converges for any $k > 0$. On the other hand, the integral diverges to ∞ for $k \leq 0$.

Proof

Note that

$$\int_0^{\infty} x^{k-1} e^{-x} dx = \int_0^1 x^{k-1} e^{-x} dx + \int_1^{\infty} x^{k-1} e^{-x} dx \quad (5.8.2)$$

For the first integral on the right,

$$\int_0^1 x^{k-1} e^{-x} dx \leq \int_0^1 x^{k-1} dx = \frac{1}{k} \quad (5.8.3)$$

For the second integral, let $n = \lceil k \rceil$. Then

$$\int_1^{\infty} x^{k-1} e^{-x} dx \leq \int_1^{\infty} x^{n-1} e^{-x} dx \quad (5.8.4)$$

The last integral can be evaluated explicitly by integrating by parts, and is finite for every $n \in \mathbb{N}_+$.

Finally, if $k \leq 0$, note that

$$\int_0^1 x^{k-1} e^{-x} dx \geq e^{-1} \int_0^1 x^{k-1} dx = \infty \quad (5.8.5)$$

The gamma function was first introduced by Leonhard Euler.

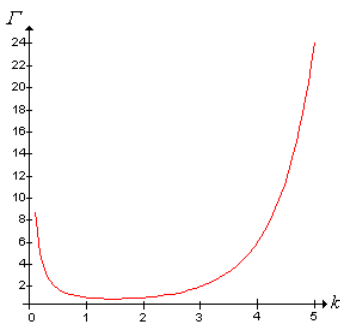


Figure 5.8.1: The graph of the gamma function on the interval $(0, 5)$

The (lower) *incomplete gamma function* is defined by

$$\Gamma(k, x) = \int_0^x t^{k-1} e^{-t} dt, \quad k, x \in (0, \infty) \quad (5.8.6)$$

Properties

Here are a few of the essential properties of the gamma function. The first is the fundamental identity.

$$\Gamma(k+1) = k \Gamma(k) \quad \text{for } k \in (0, \infty).$$

Proof

This follows from integrating by parts, with $u = x^k$ and $dv = e^{-x} dx$:

$$\Gamma(k+1) = \int_0^\infty x^k e^{-x} dx = (-x^k e^{-x})_0^\infty + \int_0^\infty kx^{k-1} e^{-x} dx = 0 + k \Gamma(k) \quad (5.8.7)$$

Applying this result repeatedly gives

$$\Gamma(k+n) = k(k+1) \cdots (k+n-1) \Gamma(k), \quad n \in \mathbb{N}_+ \quad (5.8.8)$$

It's clear that the gamma function is a continuous extension of the factorial function.

$$\Gamma(k+1) = k! \quad \text{for } k \in \mathbb{N}.$$

Proof

This follows from the [fundamental identity](#) and the fact that $\Gamma(1) = 1$.

The values of the gamma function for non-integer arguments generally cannot be expressed in simple, closed forms. However, there are exceptions.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Proof

By definition,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x} dx \quad (5.8.9)$$

Substituting $x = z^2/2$ gives

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \sqrt{2} e^{-z^2/2} dz = 2\sqrt{\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \quad (5.8.10)$$

But the last integrand is the PDF of the standard normal distribution, and so the integral evaluates to $\frac{1}{2}$

We can generalize the last result to odd multiples of $\frac{1}{2}$.

For $n \in \mathbb{N}$,

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \sqrt{\pi} = \frac{(2n)!}{4^n n!} \sqrt{\pi} \quad (5.8.11)$$

Proof

This follows from the [previous result](#) and the [fundamental identity](#).

Stirling's Approximation

One of the most famous asymptotic formulas for the gamma function is *Stirling's formula*, named for James Stirling. First we need to recall a definition.

Suppose that $f, g: D \rightarrow (0, \infty)$ where $D = (0, \infty)$ or $D = \mathbb{N}_+$. Then $f(x) \approx g(x)$ as $x \rightarrow \infty$ means that

$$\frac{f(x)}{g(x)} \rightarrow 1 \text{ as } x \rightarrow \infty \quad (5.8.12)$$

Stirling's formula

$$\Gamma(x+1) \approx \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \text{ as } x \rightarrow \infty \quad (5.8.13)$$

As a special case, Stirling's result gives an asymptotic formula for the factorial function:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \text{ as } n \rightarrow \infty \quad (5.8.14)$$

The Standard Gamma Distribution

Distribution Functions

The *standard gamma distribution* with *shape parameter* $k \in (0, \infty)$ is a continuous distribution on $(0, \infty)$ with probability density function f given by

$$f(x) = \frac{1}{\Gamma(k)} x^{k-1} e^{-x}, \quad x \in (0, \infty) \quad (5.8.15)$$

Clearly f is a valid probability density function, since $f(x) > 0$ for $x > 0$, and by definition, $\Gamma(k)$ is the normalizing constant for the function $x \mapsto x^{k-1} e^{-x}$ on $(0, \infty)$. The following theorem shows that the gamma density has a rich variety of shapes, and shows why k is called the shape parameter.

The gamma probability density function f with shape parameter $k \in (0, \infty)$ satisfies the following properties:

1. If $0 < k < 1$, f is decreasing with $f(x) \rightarrow \infty$ as $x \downarrow 0$.
2. If $k = 1$, f is decreasing with $f(0) = 1$.
3. If $k > 1$, f increases and then decreases, with mode at $k - 1$.
4. If $0 < k \leq 1$, f is concave upward.
5. If $1 < k \leq 2$, f is concave downward and then upward, with inflection point at $k - 1 + \sqrt{k - 1}$.
6. If $k > 2$, f is concave upward, then downward, then upward again, with inflection points at $k - 1 \pm \sqrt{k - 1}$.

Proof

These results follow from standard calculus. For $x > 0$,

$$\begin{aligned} f'(x) &= \frac{1}{\Gamma(k)} x^{k-2} e^{-x} [(k-1) - x] \\ f''(x) &= \frac{1}{\Gamma(k)} x^{k-3} e^{-x} [(k-1)(k-2) - 2(k-1)x + x^2] \end{aligned}$$

The special case $k = 1$ gives the *standard exponential distribution*. When $k \geq 1$, the distribution is unimodal.

In the simulation of the special distribution simulator, select the gamma distribution. Vary the shape parameter and note the shape of the density function. For various values of k , run the simulation 1000 times and compare the empirical density function to the true probability density function.

The distribution function and the quantile function do not have simple, closed representations for most values of the shape parameter. However, the distribution function has a trivial representation in terms of the incomplete and complete gamma functions.

The distribution function F of the standard gamma distribution with shape parameter $k \in (0, \infty)$ is given by

$$F(x) = \frac{\Gamma(k, x)}{\Gamma(k)}, \quad x \in (0, \infty) \quad (5.8.16)$$

Approximate values of the distribution and quantile functions can be obtained from special distribution calculator, and from most mathematical and statistical software packages.

Using the special distribution calculator, find the median, the first and third quartiles, and the interquartile range in each of the following cases:

1. $k = 1$
2. $k = 2$
3. $k = 3$

Moments

Suppose that X has the standard gamma distribution with shape parameter $k \in (0, \infty)$. The mean and variance are both simply the shape parameter.

The mean and variance of X are

1. $\mathbb{E}(X) = k$
2. $\text{var}(X) = k$

Proof

1. From the [fundamental identity](#),

$$\mathbb{E}(X) = \int_0^{\infty} x \frac{1}{\Gamma(k)} x^{k-1} e^{-x} dx = \frac{\Gamma(k+1)}{\Gamma(k)} = k \quad (5.8.17)$$

2. From the [fundamental identity](#) again

$$\mathbb{E}(X^2) = \int_0^{\infty} x^2 \frac{1}{\Gamma(k)} x^{k-1} e^{-x} dx = \frac{\Gamma(k+2)}{\Gamma(k)} = (k+1)k \quad (5.8.18)$$

and hence $\text{var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = k$

In the simulation of the special distribution simulator, select the gamma distribution. Vary the shape parameter and note the size and location of the mean \pm standard deviation bar. For selected values of k , run the simulation 1000 times and compare the empirical mean and standard deviation to the distribution mean and standard deviation.

More generally, the moments can be expressed easily in terms of the gamma function:

The moments of X are

1. $\mathbb{E}(X^a) = \Gamma(a+k)/\Gamma(k)$ if $a > -k$
2. $\mathbb{E}(X^n) = k^{[n]} = k(k+1) \cdots (k+n-1)$ if $n \in \mathbb{N}$

Proof

1. For $a > -k$,

$$\mathbb{E}(X^a) = \int_0^{\infty} x^a \frac{1}{\Gamma(k)} x^{k-1} e^{-x} dx = \frac{1}{\Gamma(k)} \int_0^{\infty} x^{a+k} e^{-x} dx = \frac{\Gamma(a+k)}{\Gamma(k)} \quad (5.8.19)$$

2. If $n \in \mathbb{N}$, then by the [fundamental identity](#), $\Gamma(k+n) = k(k+1) \cdots (k+n-1)\Gamma(k)$, so the result follows from (a).

Note also that $\mathbb{E}(X^a) = \infty$ if $a \leq -k$. We can now also compute the skewness and the kurtosis.

The skewness and kurtosis of X are

1. $\text{skew}(X) = \frac{2}{\sqrt{k}}$
2. $\text{kurt}(X) = 3 + \frac{6}{k}$

Proof

These results follows from the previous [moment results](#) and the computational formulas for skewness and kurtosis.

In particular, note that $\text{skew}(X) \rightarrow 0$ and $\text{kurt}(X) \rightarrow 3$ as $k \rightarrow \infty$. Note also that the *excess kurtosis* $\text{kurt}(X) - 3 \rightarrow 0$ as $k \rightarrow \infty$.

In the simulation of the special distribution simulator, select the gamma distribution. Increase the shape parameter and note the shape of the density function in light of the previous results on skewness and kurtosis. For various values of k , run the simulation 1000 times and compare the empirical density function to the true probability density function.

The following theorem gives the moment generating function.

The moment generating function of X is given by

$$\mathbb{E}(e^{tX}) = \frac{1}{(1-t)^k}, \quad t < 1 \quad (5.8.20)$$

Proof

For $t < 1$,

$$\mathbb{E}(e^{tX}) = \int_0^\infty e^{tx} \frac{1}{\Gamma(k)} x^{k-1} e^{-x} dx = \int_0^\infty \frac{1}{\Gamma(k)} x^{k-1} e^{-x(1-t)} dx \quad (5.8.21)$$

Substituting $u = x(1-t)$ so that $x = u/(1-t)$ and $dx = du/(1-t)$ gives

$$\mathbb{E}(e^{tX}) = \frac{1}{(1-t)^k} \int_0^\infty \frac{1}{\Gamma(k)} u^{k-1} e^{-u} du = \frac{1}{(1-t)^k} \quad (5.8.22)$$

The General Gamma Distribution

The gamma distribution is usually generalized by adding a scale parameter.

If Z has the standard gamma distribution with shape parameter $k \in (0, \infty)$ and if $b \in (0, \infty)$, then $X = bZ$ has the *gamma distribution* with *shape parameter* k and *scale parameter* b .

The reciprocal of the scale parameter, $r = 1/b$ is known as the *rate parameter*, particularly in the context of the Poisson process. The gamma distribution with parameters $k = 1$ and b is called the exponential distribution with scale parameter b (or rate parameter $r = 1/b$). More generally, when the shape parameter k is a positive integer, the gamma distribution is known as the *Erlang distribution*, named for the Danish mathematician Agner Erlang. The exponential distribution governs the time between arrivals in the Poisson model, while the Erlang distribution governs the actual arrival times.

Basic properties of the general gamma distribution follow easily from corresponding properties of the standard distribution and basic results for scale transformations.

Distribution Functions

Suppose that X has the gamma distribution with shape parameter $k \in (0, \infty)$ and scale parameter $b \in (0, \infty)$.

X has probability density function f given by

$$f(x) = \frac{1}{\Gamma(k)b^k} x^{k-1} e^{-x/b}, \quad x \in (0, \infty) \quad (5.8.23)$$

Proof

Recall that if g is the [PDF of the standard gamma distribution](#) with shape parameter k then $f(x) = \frac{1}{b} g\left(\frac{x}{b}\right)$ for $x > 0$.

Recall that the inclusion of a scale parameter does not change the shape of the density function, but simply scales the graph horizontally and vertically. In particular, we have the same basic shapes as for the [standard gamma density function](#).

The probability density function f of X satisfies the following properties:

1. If $0 < k < 1$, f is decreasing with $f(x) \rightarrow \infty$ as $x \downarrow 0$.
2. If $k = 1$, f is decreasing with $f(0) = 1$.
3. If $k > 1$, f increases and then decreases, with mode at $(k-1)b$.
4. If $0 < k \leq 1$, f is concave upward.
5. If $1 < k \leq 2$, f is concave downward and then upward, with inflection point at $b(k-1+\sqrt{k-1})$.
6. If $k > 2$, f is concave upward, then downward, then upward again, with inflection points at $b(k-1 \pm \sqrt{k-1})$.

In the simulation of the special distribution simulator, select the gamma distribution. Vary the shape and scale parameters and note the shape and location of the probability density function. For various values of the parameters, run the simulation 1000 times and compare the empirical density function to the true probability density function.

Once again, the distribution function and the quantile function do not have simple, closed representations for most values of the shape parameter. However, the distribution function has a simple representation in terms of the incomplete and complete gamma functions.

The distribution function F of X is given by

$$F(x) = \frac{\Gamma(k, x/b)}{\Gamma(k)}, \quad x \in (0, \infty) \quad (5.8.24)$$

Proof

From the [definition](#) we can take $X = bZ$ where Z has the standard gamma distribution with shape parameter k . Then $\mathbb{P}(X \leq x) = \mathbb{P}(Z \leq x/b)$ for $x \in (0, \infty)$, so the result follows from the [distribution function of \$Z\$](#) .

Approximate values of the distribution and quantile functions can be obtained from special distribution calculator, and from most mathematical and statistical software packages.

Open the special distribution calculator. Vary the shape and scale parameters and note the shape and location of the distribution and quantile functions. For selected values of the parameters, find the median and the first and third quartiles.

Moments

Suppose again that X has the gamma distribution with shape parameter $k \in (0, \infty)$ and scale parameter $b \in (0, \infty)$.

The mean and variance of X are

1. $\mathbb{E}(X) = bk$
2. $\text{var}(X) = b^2k$

Proof

From the [definition](#), we can take $X = bZ$ where Z has the standard gamma distribution with shape parameter k . Then using the [mean and variance of \$Z\$](#) ,

1. $\mathbb{E}(X) = b\mathbb{E}(Z) = bk$
2. $\text{var}(X) = b^2\text{var}(Z) = b^2k$

In the special distribution simulator, select the gamma distribution. Vary the parameters and note the shape and location of the mean \pm standard deviation bar. For selected values of the parameters, run the simulation 1000 times and compare the empirical mean and standard deviation to the distribution mean and standard deviation.

The moments of X are

1. $\mathbb{E}(X^a) = b^a \Gamma(a+k) / \Gamma(k)$ for $a > -k$
2. $\mathbb{E}(X^n) = b^n k^{[n]} = b^n k(k+1) \cdots (k+n-1)$ if $n \in \mathbb{N}$

Proof

Again, from the [definition](#), we can take $X = bZ$ where Z has the standard gamma distribution with shape parameter k . The results follow from the [moment results for \$Z\$](#) , since $E(X^a) = b^a E(Z^a)$.

Note also that $E(X^a) = \infty$ if $a \leq -k$. Recall that skewness and kurtosis are defined in terms of the standard score, and hence are unchanged by the addition of a scale parameter.

The skewness and kurtosis of X are

1. $\text{skew}(X) = \frac{2}{\sqrt{k}}$
2. $\text{kurt}(X) = 3 + \frac{6}{k}$

The moment generating function of X is given by

$$\mathbb{E}(e^{tX}) = \frac{1}{(1-bt)^k}, \quad t < \frac{1}{b} \quad (5.8.25)$$

Proof

From the [definition](#), we can take $X = bZ$ where Z has the standard gamma distribution with shape parameter k . Then $\mathbb{E}(e^{tX}) = \mathbb{E}[e^{(tb)Z}]$, so the result follows from the [moment generating function of \$Z\$](#) .

Relations

Our first result is simply a restatement of the meaning of the term *scale parameter*.

Suppose that X has the gamma distribution with shape parameter $k \in (0, \infty)$ and scale parameter $b \in (0, \infty)$. If $c \in (0, \infty)$, then cX has the gamma distribution with shape parameter k and scale parameter bc .

Proof

From the [definition](#), we can take $X = bZ$ where Z has the standard gamma distribution with shape parameter k . Then $cX = cbZ$.

More importantly, if the scale parameter is fixed, the gamma family is closed with respect to sums of independent variables.

Suppose that X_1 and X_2 are independent random variables, and that X_i has the gamma distribution with shape parameter $k_i \in (0, \infty)$ and scale parameter $b \in (0, \infty)$ for $i \in \{1, 2\}$. Then $X_1 + X_2$ has the gamma distribution with shape parameter $k_1 + k_2$ and scale parameter b .

Proof

Recall that the MGF of $X = X_1 + X_2$ is the product of the MGFs of X_1 and X_2 , so

$$\mathbb{E}(e^{tX}) = \frac{1}{(1-bt)^{k_1}} \frac{1}{(1-bt)^{k_2}} = \frac{1}{(1-bt)^{k_1+k_2}}, \quad t < \frac{1}{b} \quad (5.8.26)$$

From the previous result, it follows that the gamma distribution is infinitely divisible:

Suppose that X has the gamma distribution with shape parameter $k \in (0, \infty)$ and scale parameter $b \in (0, \infty)$. For $n \in \mathbb{N}_+$, X has the same distribution as $\sum_{i=1}^n X_i$, where (X_1, X_2, \dots, X_n) is a sequence of independent random variables, each with the gamma distribution with shape parameter k/n and scale parameter b .

From the sum result and the central limit theorem, it follows that if k is large, the gamma distribution with shape parameter k and scale parameter b can be approximated by the normal distribution with mean kb and variance kb^2 . Here is the precise statement:

Suppose that X_k has the gamma distribution with shape parameter $k \in (0, \infty)$ and fixed scale parameter $b \in (0, \infty)$. Then the distribution of the standardized variable below converges to the standard normal distribution as $k \rightarrow \infty$:

$$Z_k = \frac{X_k - kb}{\sqrt{kb}} \quad (5.8.27)$$

In the special distribution simulator, select the gamma distribution. For various values of the scale parameter, increase the shape parameter and note the increasingly “normal” shape of the density function. For selected values of the parameters, run the experiment 1000 times and compare the empirical density function to the true probability density function.

The gamma distribution is a member of the general exponential family of distributions:

The gamma distribution with shape parameter $k \in (0, \infty)$ and scale parameter $b \in (0, \infty)$ is a two-parameter exponential family with natural parameters $(k-1, -1/b)$, and natural statistics $(\ln X, X)$.

Proof

This follows from the definition of the general exponential family. The gamma PDF can be written as

$$f(x) = \frac{1}{b^k \Gamma(k)} \exp \left[(k-1) \ln x - \frac{1}{b} x \right], \quad x \in (0, \infty) \quad (5.8.28)$$

For $n \in (0, \infty)$, the gamma distribution with shape parameter $n/2$ and scale parameter 2 is known as the *chi-square distribution* with n degrees of freedom. The chi-square distribution is important enough to deserve a separate section.

Computational Exercise

Suppose that the lifetime of a device (in hours) has the gamma distribution with shape parameter $k = 4$ and scale parameter $b = 100$.

1. Find the probability that the device will last more than 300 hours.
2. Find the mean and standard deviation of the lifetime.

Answer

Let X denote the lifetime in hours.

1. $\mathbb{P}(X > 300) = 13e^{-3} \approx 0.6472$
2. $\mathbb{E}(X) = 400, \text{sd}(X) = 200$

Suppose that Y has the gamma distribution with parameters $k = 10$ and $b = 2$. For each of the following, compute the true value using the special distribution calculator and then compute the normal approximation. Compare the results.

1. $\mathbb{P}(18 < X < 25)$
2. The 80th percentile of Y

Answer

1. $\mathbb{P}(18 < X < 25) = 0.3860, \mathbb{P}(18 < X < 25) \approx 0.4095$
2. $y_{0.8} = 25.038, y_{0.8} \approx 25.325$

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