16.10: Discrete-Time Reliability Chains

The Success-Runs Chain

Suppose that we have a sequence of *trials*, each of which results in either *success* or *failure*. Our basic assumption is that if there have been $x \in \mathbb{N}$ consecutive successes, then the probability of success on the next trial is $p(x)$, independently of the past, where $p : \mathbb{N} \to (0, 1)$. Whenever there is a failure, we start over, independently, with a new sequence of trials. Appropriately enough, p is called the *success function*. Let X_n denote the length of the run of successes after n trials.

$\boldsymbol{X} = (X_0, X_1, X_2, \ldots)$ is a discrete-time Markov chain with state space $\mathbb{N}$ and transition probability matrix P given by

$$P(x, x+1) = p(x), \; P(x, 0) = 1 - p(x); \quad x \in \mathbb{N} \tag{16.10.1}$$

The Markov chain $\boldsymbol{X}$ is called the *success-runs chain*.

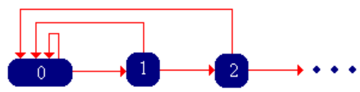

Figure 16.10.1: State graph of the success-runs chain

Now let T denote the trial number of the first failure, starting with a fresh sequence of trials. Note that in the context of the success-runs chain $\boldsymbol{X}$, $T = \tau_0$, the first return time to state 0, starting in 0. Note that T takes values in $\mathbb{N}_+ \cup \{\infty\}$, since presumably, it is possible that no failure occurs. Let $r(n) = \mathbb{P}(T > n)$ for $n \in \mathbb{N}$, the probability of at least n consecutive successes, starting with a fresh set of trials. Let $f(n) = \mathbb{P}(T = n + 1)$ for $n \in \mathbb{N}$, the probability of exactly n consecutive successes, starting with a fresh set of trails.

The functions p, r, and f are related as follows:

1. $p(x) = r(x+1)/r(x)$ for $x \in \mathbb{N}$
2. $r(n) = \prod_{x=0}^{n-1} p(x)$ for $n \in \mathbb{N}$
3. $f(n) = [1 - p(n)] \prod_{x=0}^{n-1} p(x)$ for $n \in \mathbb{N}$
4. $r(n) = 1 - \sum_{x=0}^{n-1} f(x)$ for $n \in \mathbb{N}$
5. $f(n) = r(n) - r(n+1)$ for $n \in \mathbb{N}$

Thus, the functions p, r, and f give equivalent information. If we know one of the functions, we can construct the other two, and hence any of the functions can be used to define the success-runs chain. The function r is the reliability function associated with T.

The function r is characterized by the following properties:

1. r is positive.
2. $r(0) = 1$
3. r is strictly decreasing.

The function f is characterized by the following properties:

1. f is positive.
2. $\sum_{x=0}^{\infty} f(x) \le 1$

Essentially, f is the probability density function of $T - 1$, except that it may be *defective* in the sense that the sum of its values may be less than 1. The leftover probability, of course, is the probability that $T = \infty$. This is the critical consideration in the classification of the success-runs chain, which we will consider shortly.

Verify that each of the following functions has the appropriate properties, and then find the other two functions:

1. p is a constant in $(0, 1)$.
2. $r(n) = 1/(n+1)$ for $n \in \mathbb{N}$.
3. $r(n) = (n+1)/(2n+1)$ for $n \in \mathbb{N}$.

4. $p(x) = 1/(x+2)$ for $x \in \mathbb{N}$.

Answer

1. $p(x) = p$ for $x \in \mathbb{N}$. $r(n) = p^n$ for $n \in \mathbb{N}$. $f(n) = (1-p)p^n$ for $n \in \mathbb{N}$.
2. $p(x) = \frac{x+1}{x+2}$ for $x \in \mathbb{N}$. $r(n) = \frac{1}{n+1}$ for $n \in \mathbb{N}$. $f(n) = \frac{1}{n+1} - \frac{1}{n}$ for $n \in \mathbb{N}$.
3. $p(x) = \frac{(x+2)(2x+1)}{(x+1)(2x+3)}$ for $x \in \mathbb{N}$. $r(n) = \frac{n+1}{2n+1}$ for $n \in \mathbb{N}$. $f(n) = \frac{n+1}{2n+1} - \frac{n+2}{2n+3}$ for $n \in \mathbb{N}$.
4. $p(x) = \frac{1}{x+2}$ for $x \in \mathbb{N}$. $r(n) = \frac{1}{(n+1)!}$ for $n \in \mathbb{N}$. $f(n) = \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$ for $n \in \mathbb{N}$.

In part (a), note that the trials are Bernoulli trials. We have an app for this case.

The success-runs app is a simulation of the success-runs chain based on Bernoulli trials. Run the simulation 1000 times for various values of p and various initial states, and note the general behavior of the chain.

The success-runs chain is irreducible and aperiodic.

Proof

The chain is irreducible, since 0 leads to every other state, and every state leads back to 0. The chain is aperiodic since $P(0,0) > 0$.

Recall that T has the same distribution as τ_0, the first return time to 0 starting at state 0. Thus, the classification of the chain as recurrent or transient depends on $\alpha = \mathbb{P}(T = \infty)$. Specifically, the success-runs chain is transient if $\alpha > 0$ and recurrent if $\alpha = 0$. Thus, we see that the chain is recurrent if and only if a failure is sure to occur. We can compute the parameter α in terms of each of the three functions that define the chain.

In terms of p, r, and f,

$$\alpha = \prod_{x=0}^{\infty} p(x) = \lim_{n \to \infty} r(n) = 1 - \sum_{x=0}^{\infty} f(x) \tag{16.10.2}$$

Compute α and determine whether the success-runs chain $\boldsymbol{X}$ is transient or recurrent for each of the examples above.

Answer

1. $\alpha = 0$, recurrent.
2. $\alpha = 0$, recurrent.
3. $\alpha = \frac{1}{2}$, transient.
4. $\alpha = 0$, recurrent.

Run the simulation of the success-runs chain 1000 times for various values of p, starting in state 0. Note the return times to state 0.

Let $\mu = \mathbb{E}(T)$, the expected trial number of the first failure, starting with a fresh sequence of trials.

μ is related to α, f, and r as follows:

1. If $\alpha > 0$ then $\mu = \infty$
2. If $\alpha = 0$ then $\mu = 1 + \sum_{n=0}^{\infty} n f(n)$
3. $\mu = \sum_{n=0}^{\infty} r(n)$

Proof

1. If $\alpha = \mathbb{P}(T = \infty) > 0$ then $\mu = \mathbb{E}(T) = \infty$.
2. If $\alpha = 0$, so that T takes values in $\mathbb{N}_+$, then f is the PDF of $T - 1$, so $\mu = 1 + \mathbb{E}(T-1)$.
3. This is a basic result from the general theory of expected value: $\mathbb{E}(T) = \sum_{n=0}^{\infty} \mathbb{P}(T > n)$.

The success-runs chain $\boldsymbol{X}$ is positive recurrent if and only if $\mu < \infty$.

Proof

Since T is the return time to 0, starting at 0, and since the chain is irreducible, it follows from the general theory that the chain is positive recurrent if and only if $\mu = \mathbb{E}(T) < \infty$.

If $\boldsymbol{X}$ is recurrent, then r is invariant for $\boldsymbol{X}$. In the positive recurrent case, when $\mu < \infty$, the invariant distribution has probability density function g given by

$$g(x) = \frac{r(x)}{\mu}, \quad x \in \mathbb{N} \tag{16.10.3}$$

Proof

If $y \in \mathbb{N}_+$ then from the result above,

$$(rP)(y) = \sum_{x=0}^\infty r(x) P(x, y) = r(y-1) p(y-1) = r(y) \tag{16.10.4}$$

For $y = 0$, using the result above again,

$$(rP)(0) = \sum_{x=0}^\infty r(x) P(x, 0) = \sum_{x=0}^\infty r(x)[1 - p(x)] = \sum_{x=0}^\infty [r(x) - r(x) p(x)] = \sum_{x=0}^\infty [r(x) - r(x+1)] \tag{16.10.5}$$

If the chain is recurrent, $r(n) \to 0$ as $n \to \infty$ so the last sum collapses to $r(0) = 1$. Recall that $\mu = \sum_{n=0}^\infty r(n)$. Hence if $\mu < \infty$, so that the chain is positive recurrent, the function g (which is just r normalized) is the invariant PDF.

When $\boldsymbol{X}$ is recurrent, we know from the general theory that every other nonnegative left invariant function is a nonnegative multiple of r

Determine whether the success-runs chain $\boldsymbol{X}$ is transient, null recurrent, or positive recurrent for each of the examples above. If the chain is positive recurrent, find the invariant probability density function.

Answer

1. $\mu = \frac{1}{1-p}$, positive recurrent. $g(x) = (1-p) p^x$ for $x \in \mathbb{N}$.
2. $\alpha = 0$, $\mu = \infty$, null recurrent.
3. $\alpha = \frac{1}{2}$, transient.
4. $\mu = e - 1$, positive recurrent. $g(x) = \frac{1}{(e-1)(x+1)!}$ for $x \in \mathbb{N}$.

From (a), the success-runs chain corresponding to Bernoulli trials with success probability $p \in (0, 1)$ has the geometric distribution on $\mathbb{N}$, with parameter $1 - p$, as the invariant distribution.

Run the simulation of the success-runs chain 1000 times for various values of p and various initial states. Compare the empirical distribution to the invariant distribution.

The Remaining Life Chain

Consider a device whose (discrete) time to failure U takes values in $\mathbb{N}$, with probability density function f. We assume that $f(n) > 0$ for $n \in \mathbb{N}$. When the device fails, it is immediately (and independently) replaced by an identical device. For $n \in \mathbb{N}$, let Y_n denote the time to failure of the device that is in service at time n.

$\boldsymbol{Y} = (Y_0, Y_1, Y_2, \ldots)$ is a discrete-time Markov chain with state space $\mathbb{N}$ and transition probability matrix Q given by

$$Q(0, x) = f(x), \; Q(x+1, x) = 1; \quad x \in \mathbb{N} \tag{16.10.6}$$

The Markov chain $\boldsymbol{Y}$ is called the *remaining life chain* with lifetime probability density function f, and has the state graph below.

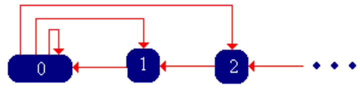

Figure 16.10.2: State graph of the remaining life chain

We have an app for the remaining life chain whose lifetime distribution is the geometric distribution on $\mathbb{N}$, with parameter $1-p \in (0,1)$.

Run the simulation of the remaining-life chain 1000 times for various values of p and various initial states. Note the general behavior of the chain.

If U denotes the lifetime of a device, as before, note that $T = 1 + U$ is the return time to 0 for the chain $\boldsymbol{Y}$, starting at 0.

$\boldsymbol{Y}$ is irreducible, aperiodic, and recurrent.

Proof

From the assumptions on f, state 0 leads to every other state (including 0), and every positive state leads (deterministically) to 0. Thus the chain is irreducible and aperiodic. By assumption, $\mathbb{P}(U \in \mathbb{N}) = 1$ so $\mathbb{P}(T < \infty) = 1$ and hence the chain is recurrent.

Now let $r(n) = \mathbb{P}(U \ge n) = \mathbb{P}(T > n)$ for $n \in \mathbb{N}$ and let $\mu = \mathbb{E}(T) = 1 + \mathbb{E}(U)$. Note that $r(n) = \sum_{x=n}^\infty f(x)$ and $\mu = 1 + \sum_{x=0}^\infty f(x) = \sum_{n=0}^\infty r(n)$.

The success-runs chain $\boldsymbol{X}$ is positive recurrent if and only if $\mu < \infty$, in which case the invariant distribution has probability density function g given by

$$g(x) = \frac{r(x)}{\mu}, \quad x \in \mathbb{N} \tag{16.10.7}$$

Proof

Since the chain is irreducible, it is positive recurrent if and only if $\mu = E(T) < \infty$. The function r is invariant for Q: for $y \in \mathbb{N}$

$$\begin{aligned}(rQ)(y) &= \sum_{x \in \mathbb{N}} r(x) Q(x,y) = r(0) Q(0,y) + r(y+1) Q(y+1,y) \\ &= f(y) + r(y+1) = r(y)\end{aligned}$$

In the positive recurrent case, μ is the normalizing constant for r, so g is the invariant PDF.

Suppose that $\boldsymbol{Y}$ is the remaining life chain whose lifetime distribution is the geometric distribution on $\mathbb{N}$ with parameter $1-p \in (0,1)$. Then this distribution is also the invariant distribution.

Proof

By assumption, $f(x) = (1-p)p^x$ for $x \in \mathbb{N}$, and the mean of this distribution is $p/(1-p)$. Hence $\mu = 1 + p/(1-p) = 1/(1-p)$, and $r(x) = \sum_{y=x}^\infty f(y) = p^x$ for $x \in \mathbb{N}$. Hence $g = f$.

Run the simulation of the success-runs chain 1000 times for various values of p and various initial states. Compare the empirical distribution to the invariant distribution.

Time Reversal

You probably have already noticed similarities, in notation and results, between the success-runs chain and the remaining-life chain. There are deeper connections.

Suppose that f is a probability density function on $\mathbb{N}$ with $f(n) > 0$ for $n \in \mathbb{N}$. Let $\boldsymbol{X}$ be the success-runs chain associated with f and $\boldsymbol{Y}$ the remaining life chain associated with f. Then $\boldsymbol{X}$ and $\boldsymbol{Y}$ are time reversals of each other.

Proof

https://stats.libretexts.org/@go/page/10297

Under the assumptions on f, both chains are recurrent and irreducible. Hence it suffices to show that

$$r(x)P(x,y) = r(y)Q(y,x), \quad x, y \in \mathbb{N} \tag{16.10.8}$$

It will then follow that the chains are time reversals of each other, and that r is a common invariant function (unique up to multiplication by positive constants). In the case that $\mu = \sum_{n=0}^\infty r(n) < \infty$, the function $g = r/\mu$ is the common invariant PDF. There are only two cases to consider. With $y = 0$, we have $r(x)P(x,0) = r(x)[1-p(x)]$ and $r(0)Q(y,0) = f(x)$. But $r(x)[1-p(x)] = f(x)$ by the result above. When $x \in \mathbb{N}$ and $y = x+1$, we have $r(x)P(x,x+1) = r(x)p(x)$ and $r(x+1)Q(x+1,x) = r(x+1)$. But $r(x)p(x) = r(x+1)$ by the result above.

In the context of reliability, it is also easy to see that the chains are time reversals of each other. Consider again a device whose random lifetime takes values in $\mathbb{N}$, with the device immediately replaced by an identical device upon failure. For $n \in \mathbb{N}$, we can think of X_n as the age of the device in service at time n and Y_n as the time remaining until failure for that device.

Run the simulation of the success-runs chain 1000 times for various values of p, starting in state 0. This is the time reversal of the simulation in the next exercise

Run the simulation of the remaining-life chain 1000 times for various values of p, starting in state 0. This is the time reversal of the simulation in the previous exercise.

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