LibreTexts

12.4: Problems on Variance, Covariance, Linear Regression

Exercise 12.4.1

(See Exercise 1 from "Problems on Distribution and Density Functions ", and Exercise 1 from "Problems on Mathematical Expectation", m-file npr07_01.m). The class $\{C_j: 1 \le j \le 10\}$ is a partition. Random variable X has values {1, 3, 2, 3, 4, 2, 1, 3, 5, 2} on C_1 through C_{10}, respectively, with probabilities 0.08, 0.13, 0.06, 0.09, 0.14, 0.11, 0.12, 0.07, 0.11, 0.09. Determine $\text{Var}[X]$.

Answer

```
npr07_01
Data are in T and pc
EX = T*pc'
EX =  2.7000
VX = (T.^2)*pc' - EX^2
VX =  1.5500
[X,PX] = csort(T,pc);    % Alternate
Ex = X*PX'
Ex =  2.7000
Vx = (X.^2)*PX' - EX^2
Vx =  1.5500
```

Exercise 12.4.2

(See Exercise 2 from "Problems on Distribution and Density Functions ", and Exercise 2 from "Problems on Mathematical Expectation", m-file npr07_02.m). A store has eight items for sale. The prices are \$3.50, \$5.00, \$3.50, \$7.50, \$5.00, \$5.00, \$3.50, and \$7.50, respectively. A customer comes in. She purchases one of the items with probabilities 0.10, 0.15, 0.15, 0.20, 0.10 0.05, 0.10 0.15. The random variable expressing the amount of her purchase may be written

$$X = 3.5 I_{C_1} + 5.0 I_{C_2} + 3.5 I_{C_3} + 7.5 I_{C_4} + 5.0 I_{C_5} + 5.0 I_{C_6} + 3.5 I_{C_7} + 7.5 I_{C_8}$$

Determine $\text{Var}[X]$.

Answer

```
npr07_02
Data are in T, pc
EX = T*pc';
VX = (T.^2)*pc' - EX^2
VX =  2.8525
```

Exercise 12.4.3

(See Exercise 12 from "Problems on Random Variables and Probabilities", Exercise 3 from "Problems on Mathematical Expectation", m-file npr06_12.m). The class $\{A, B, C, D\}$ has minterm probabilities

$$pm = 0.001 * [5\ 7\ 6\ 8\ 9\ 14\ 22\ 33\ 21\ 32\ 50\ 75\ 86\ 129\ 201\ 302]$$

Consider $X = I_A + I_B + I_C + I_D$, which counts the number of these events which occur on a trial. Determine $\text{Var}[X]$.

Answer

```
npr06_12
Minterm probabilities in pm, coefficients in c
canonic
 Enter row vector of coefficients  c
 Enter row vector of minterm probabilities  pm
Use row matrices X and PX for calculations
Call for XDBN to view the distribution
VX = (X.^2)*PX' - (X*PX')^2
VX =  0.7309
```

Exercise 12.4.4

(See Exercise 4 from "Problems on Mathematical Expectation"). In a thunderstorm in a national park there are 127 lightning strikes. Experience shows that the probability of each lightning strike starting a fire is about 0.0083. Determine $\text{Var}[X]$.

Answer

X ~ binomial (127, 0.0083). $\text{Var}[X] = 127 \cdot 0.0083 \cdot (1 - 0.0083) = 1.0454$

Exercise 12.4.5

(See Exercise 5 from "Problems on Mathematical Expectation"). Two coins are flipped twenty times. Let X be the number of matches (both heads or both tails). Determine $\text{Var}[X]$.

Answer

X ~ binomial (20, 1/2). $\text{Var}[X] = 20 \cdot (1/2)^2 = 5$.

Exercise 12.4.6

(See Exercise 6 from "Problems on Mathematical Expectation"). A residential College plans to raise money by selling “chances” on a board. Fifty chances are sold. A player pays \$10 to play; he or she wins \$30 with probability $p = 0.2$. The profit to the College is

$$X = 50 \cdot 10 - 30N, \text{ where } N \text{ is the number of winners}$$

Determine $\text{Var}[X]$.

Answer

N ~ binomial (50, 0.2). $\text{Var}[N] = 50 \cdot 0.2 \cdot 0.8 = 8$. $\text{Var}[X] = 30^2 \text{ Var}[N] = 7200$.

Exercise 12.4.7

(See Exercise 7 from "Problems on Mathematical Expectation"). The number of noise pulses arriving on a power circuit in an hour is a random quantity X having Poisson (7) distribution. Determine $\text{Var}[X]$.

Answer

X ~ Poisson (7). $\text{Var}[X] = \mu = 7$.

Exercise 12.4.8

(See Exercise 24 from "Problems on Distribution and Density Functions", and Exercise 8 from "Problems on Mathematical Expectation"). The total operating time for the units in Exercise 24 from "Problems on Distribution and Density Functions" is a random variable T ~ gamma (20, 0.0002). Determine $\text{Var}[T]$.

Answer

T ~ gamma (20, 0.0002). $\text{Var}[T] = 20/0.0002^2 = 500,000,000$

Exercise 12.4.9

The class $\{A, B, C, D, E, F\}$ is independent, with respective probabilities

0.43, 0.53, 0.46, 0.37, 0.45, 0.39. Let

$$X = 6I_A + 13I_B - 8I_C \text{ , } Y = -3I_D + 4I_E + I_F - 7$$

a. Use properties of expectation and variance to obtain $E[X]$, $\text{Var}[X]$, $E[Y]$, and $\text{Var}[Y]$. *Note* that it is *not* necessary to obtain the distributions for X or Y.

b. Let $Z = 3Y - 2X$.

Determine $E[Z]$, and $\text{Var}[Z]$.

Answer

```
cx = [6 13 -8 0];
cy = [-3 4 1 -7];
px = 0.01*[43 53 46 100];
py = 0.01*[37 45 39 100];
EX = dot(cx,px)
EX =   5.7900
EY = dot(cy,py)
EY =  -5.9200
VX = sum(cx.^2.*px.*(1-px))
VX =  66.8191
VY = sum(cy.^2.*py.*(1-py))
VY =   6.2958
EZ = 3*EY - 2*EX
EZ = -29.3400
VZ = 9*VY + 4*VX
VZ = 323.9386
```

Exercise 12.4.10

Consider $X = -3.3I_A - 1.7I_B + 2.3I_C + 7.6I_D - 3.4$. The class $\{A, B, C, D\}$ has minterm probabilities (data are in m-file npr12_10.m)

$$\text{pmx} = [0.0475\ 0.0725\ 0.0120\ 0.0180\ 0.1125\ 0.1675\ 0.0280\ 0.0420 \cdots$$
$$0.0480\ 0.0720\ 0.0130\ 0.0170\ 0.1120\ 0.1680\ 0.0270\ 0.0430]$$

a. Calculate $E[X]$ and $\text{Var}[X]$.

b. Let $W = 2X^2 - 3X + 2$.
Calculate $E[W]$ and $\text{Var}[W]$

Answer

```
npr12_10
Data are in cx, cy, pmx and pmy
canonic
 Enter row vector of coefficients  cx
 Enter row vector of minterm probabilities  pmx
Use row matrices X and PX for calculations
Call for XDBN to view the distribution
EX = dot(X,PX)
EX =  -1.2200
VX = dot(X.^2,PX) - EX^2
VX =  18.0253
G = 2*X.^2 - 3*X + 2;
[W,PW] = csort(G,PX);
EW = dot(W,PW)
EW =  44.6874
VW = dot(W.^2,PW) - EW^2
VW =  2.8659e+03
```

Exercise 12.4.11

Consider a second random variable $Y = 10I_E + 17I_F + 20I_G - 10$ in addition to that in Exercise 12.4.10. The class $\{E, F, G\}$ has minterm probabilities (in mfile npr12_10.m)

$$\text{pmy} = [0.06\ 0.14\ 0.09\ 0.21\ 0.06\ 0.14\ 0.09\ 0.21]$$

The pair $\{X, Y\}$ is independent.

a. Calculate $E[Y]$ and $\text{Var}[Y]$.

b. Let $Z = X^2 + 2XY - Y$.
Calculate $E[Z]$ and $\text{Var}[Z]$.

Answer

(Continuation of Exercise 12.4.10)

```
[Y,PY] = canonicf(cy,pmy);
EY = dot(Y,PY)
EY =  19.2000
VY = dot(Y.^2,PY) - EY^2
VY = 178.3600
icalc
Enter row matrix of X-values  X
Enter row matrix of Y-values  Y
Enter X probabilities  PX
Enter Y probabilities  PY
 Use array operations on matrices X, Y, PX, PY, t, u, and P
H = t.^2 + 2*t.*u - u;
[Z,PZ] = csort(H,P);
EZ = dot(Z,PZ)
```

```
EZ = -46.5343
VZ = dot(Z.^2,PZ) - EZ^2
VZ =  3.7165e+04
```

Exercise 12.4.12

Suppose the pair $\{X, Y\}$ is independent, with X ~ gamma (3, 0.1) and Y ~ Poisson (13). Let $Z = 2X - 5Y$. Determine $E[Z]$ and $\text{Var}[Z]$.

Answer

X ~ gamma (3, 0.1) implies $E[X] = 30$ and $\text{Var}[X] = 300.$ Y ~ Poisson (13) implies $E[Y] = \text{Var}[Y] = 13$. Then

$$E[Z] = 2 \cdot 30 - 5 \cdot 13 = -5 \text{ , } \text{Var}[Z] = 4 \cdot 300 + 25 \cdot 13 = 1525.$$

Exercise 12.4.13

The pair $\{X, Y\}$ is jointly distributed with the following parameters:

$$E[X] = 3,\ E[Y] = 4,\ E[XY] = 15,\ E[X^2] = 11,\ \text{Var}[Y] = 5$$

Determine $\text{Var}[3X - 2Y]$.

Answer

```
EX = 3;
EY = 4;
EXY = 15;
EX2 = 11;
VY = 5;
VX = EX2 - EX^2
VX =  2
CV = EXY - EX*EY
CV =  3
VZ = 9*VX + 4*VY - 6*2*CV
VZ =  2
```

Exercise 12.4.14

The class $\{A, B, C, D, E, F\}$ is independent, with respective probabilities

0.47, 0.33, 0.46, 0.27, 0.41, 0.37

Let

$$X = 8I_A + 11I_B - 7I_C \text{ , } Y = -3I_D + 5I_E + I_F - 3 \text{ , and } Z = 3Y - 2X$$

a. Use properties of expectation and variance to obtain $E[X]$, $\text{Var}[X]$, $E[Y]$, and $\text{Var}[Y]$.

b. Determine $E[Z]$, and $\text{Var}[Z]$.

c. Use appropriate m-programs to obtain $E[X]$, $\text{Var}[X]$, $E[Y]$, $\text{Var}[Y]$, $E[Z]$, and $\text{Var}[Z]$. Compare with results of parts (a) and (b).

Answer

```
px = 0.01*[47 33 46 100];
py = 0.01*[27 41 37 100];
cx = [8 11 -7 0];
cy = [-3 5 1 -3];
ex = dot(cx,px)
ex =   4.1700
ey = dot(cy,py)
ey =  -1.3900
vx = sum(cx.^2.*px.*(1 - px))
vx =  54.8671
vy = sum(cy.^2.*py.*(1-py))
vy =   8.0545
[X,PX] = canonicf(cx,minprob(px(1:3)));
[Y,PY] = canonicf(cy,minprob(py(1:3)));
icalc
Enter row matrix of X-values  X
Enter row matrix of Y-values  Y
Enter X probabilities  PX
Enter Y probabilities  PY
 Use array operations on matrices X, Y, PX, PY, t, u, and P
EX = dot(X,PX)
EX =   4.1700
EY = dot(Y,PY)
EY =  -1.3900
VX = dot(X.^2,PX) - EX^2
VX =  54.8671
VY = dot(Y.^2,PY) - EY^2
VY =   8.0545
EZ = 3*EY - 2*EX
EZ = -12.5100
VZ = 9*VY + 4*VX
VZ = 291.9589
```

Exercise 12.4.15

For the Beta (r, s) distribution.

a. Determine $E[X^n]$, where n is a positive integer.

b. Use the result of part (a) to determine $E[X]$ and $\text{Var}[X]$.

Answer

$$E[X^n] = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 t^{r+n-1}\,dt = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \cdot \frac{\Gamma(r+n)\Gamma(s)}{\Gamma(r+s+n)} = \frac{\Gamma(r+n)\Gamma(r+s)}{\Gamma(r+s+n)\Gamma(r)}$$

Using $\Gamma(x+1) = x\Gamma(x)$ we have

$$E[X] = \frac{r}{r+s}, E[X^2] = \frac{r(r+1)}{(r+s)(r+s+1)}$$

Some algebraic manipulations show that

$$\text{Var}[X] = E[X^2] - E^2[X] = \frac{rs}{(r+s)^2(r+s+1)}$$

Exercise 12.4.16

The pair $\{X, Y\}$ has joint distribution. Suppose

$$E[X] = 3, E[X^2] = 11, E[Y] = 10, E[Y^2] = 101, E[XY] = 30$$

Determine $\text{Var}[15X - 2Y]$.

Answer

```
EX = 3;
EX2 = 11;
EY = 10;
EY2 = 101;
EXY = 30;
VX = EX2 - EX^2
VX =    2
VY = EY2 - EY^2
VY =    1
CV = EXY - EX*EY
CV =    0
VZ = 15^2*VX + 2^2*VY
VZ =  454
```

Exercise 12.4.17

The pair $\{X, Y\}$ has joint distribution. Suppose

$$E[X] = 2, E[X^2] = 5, E[Y] = 1, E[Y^2] = 2, E[XY] = 1$$

Determine $\text{Var}[3X + 2Y]$.

Answer

```
EX = 2;
EX2 = 5;
EY = 1;
EY2 = 2;
EXY = 1;
VX = EX2 - EX^2
VX =    1
VY = EY2 - EY^2
VY =    1
CV = EXY - EX*EY
CV =   -1
```

```
VZ = 9*VX + 4*VY + 2*6*CV
VZ =    1
```

Exercise 12.4.18

The pair $\{X, Y\}$ is independent, with

$$E[X] = 2,\, E[Y] = 1,\, \text{Var}[X] = 6,\, \text{Var}[Y] = 4$$

Let $Z = 2X^2 + XY^2 - 3Y + 4$.

Determine $E[Z]$.

Answer

```
EX = 2;
EY = 1;
VX = 6;
VY = 4;
EX2 = VX + EX^2
EX2 =  10
EY2 = VY + EY^2
EY2 =   5
EZ = 2*EX2 + EX*EY2 - 3*EY + 4
EZ =   31
```

Exercise 12.4.19

(See Exercise 9 from "Problems on Mathematical Expectation"). Random variable X has density function

$$f_X(t) = \begin{cases} (6/5)t^2 & \text{for } 0 \le t \le 1 \\ (6/5)(2-t) & \text{for } 1 < t \le 2 \end{cases} = I_{[0,1]}(t)\frac{6}{5}t^2 + I_{(1,2]}(t)\frac{6}{5}(2-t)$$

$E[X] = 11/10$. Determine $\text{Var}[X]$.

Answer

$$E[X^2] = \int t^2 f_X(t)\, dt = \frac{6}{5}\int_0^1 t^4\, dt + \frac{6}{5}\int_1^2 (2t^2 - t^3)\, dt = \frac{67}{50}$$

$$\text{Var}[X] = E[X^2] - E^2[X] = \frac{13}{100}$$

For the distributions in Exercises 20-22

Determine $\text{Var}[X]$, $\text{Cov}[X, Y]$, and the regression line of Y on X.

Exercise 12.4.20

(See Exercise 7 from "Problems On Random Vectors and Joint Distributions", and Exercise 17 from "Problems on Mathematical Expectation"). The pair $\{X, Y\}$ has the joint distribution (in file npr08_07.m):

$$P(X = t, Y = u)$$

t =	-3.1	-0.5	1.2	2.4	3.7	4.9
u = 7.5	0.0090	0.0396	0.0594	0.0216	0.0440	0.0203

4.1	0.0495	0	0.1089	0.0528	0.0363	0.0231
-2.0	0.0405	0.1320	0.0891	0.0324	0.0297	0.0189
-3.8	0.0510	0.0484	0.0726	0.0132	0	0.0077

Answer

```
npr08_07
Data are in X, Y, P
jcalc
- - - - - - - - - - -
EX = dot(X,PX);
EY = dot(Y,PY);
VX = dot(X.^2,PX) - EX^2
VX =   5.1116
CV = total(t.*u.*P) - EX*EY
CV =   2.6963
a = CV/VX
a =    0.5275
b = EY - a*EX
b =    0.6924       % Regression line: u = at + b
```

Exercise 12.4.21

(See Exercise 8 from "Problems On Random Vectors and Joint Distributions", and Exercise 18 from "Problems on Mathematical Expectation"). The pair $\{X, Y\}$ has the joint distribution (in file npr08_08.m):

$$P(X=t, Y=u)$$

t =	1	3	5	7	9	11	13	15	17	19
u = 12	0.0156	0.0191	0.0081	0.0035	0.0091	0.0070	0.0098	0.0056	0.0091	0.0049
10	0.0064	0.0204	0.0108	0.0040	0.0054	0.0080	0.0112	0.0064	0.0104	0.0056
9	0.0196	0.0256	0.0126	0.0060	0.0156	0.0120	0.0168	0.0096	0.0056	0.0084
5	0.0112	0.0182	0.0108	0.0070	0.0182	0.0140	0.0196	0.0012	0.0182	0.0038
3	0.0060	0.0260	0.0162	0.0050	0.0160	0.0200	0.0280	0.0060	0.0160	0.0040
-1	0.0096	0.0056	0.0072	0.0060	0.0256	0.0120	0.0268	0.0096	0.0256	0.0084
-3	0.0044	0.0134	0.0180	0.0140	0.0234	0.0180	0.0252	0.0244	0.0234	0.0126
-5	0.0072	0.0017	0.0063	0.0045	0.0167	0.0090	0.0026	0.0172	0.0217	0.0223

Answer

```
npr08_08
Data are in X, Y, P
jcalc
- - - - - - - - - - - -
EX = dot(X,PX);
```

```
EY = dot(Y,PY);
VX = dot(X.^2,PX) - EX^2
VX =  31.0700
CV = total(t.*u.*P) - EX*EY
CV =  -8.0272
a  = CV/VX
a  =  -0.2584
b = EY - a*EX
b  =   5.6110       % Regression line: u = at + b
```

Exercise 12.4.22

(See Exercise 9 from "Problems On Random Vectors and Joint Distributions", and Exercise 19 from "Problems on Mathematical Expectation"). Data were kept on the effect of training time on the time to perform a job on a production line. X is the amount of training, in hours, and Y is the time to perform the task, in minutes. The data are as follows (in file npr08_09.m):

$$P(X = t, Y = u)$$

t =	1	1.5	2	2.5	3
u = 5	0.039	0.011	0.005	0.001	0.001
4	0.065	0.070	0.050	0.015	0.010
3	0.031	0.061	0.137	0.051	0.033
2	0.012	0.049	0.163	0.058	0.039
1	0.003	0.009	0.045	0.025	0.017

Answer

```
npr08_09
Data are in X, Y, P
jcalc
- - - - - - - - - - - -
EX = dot(X,PX);
EY = dot(Y,PY);
VX = dot(X.^2,PX) - EX^2
VX =   0.3319
CV = total(t.*u.*P) - EX*EY
CV =  -0.2586
a  = CV/VX
a  =  -0.77937/6;
b = EY - a*EX
b  =   4.3051       % Regression line: u = at + b
```

For the joint densities in Exercises 23-30 below

a. Determine analytically $\text{Var}[X]$ $\text{Cov}[X, Y]$, and the regression line of Y on X.
b. Check these with a discrete approximation.

Exercise 12.4.23

(See Exercise 10 from "Problems On Random Vectors and Joint Distributions", and Exercise 20 from "Problems on Mathematical Expectation"). $f_{XY}(t,u) = 1$ for $0 \le t \le 1$, $0 \le u \le 2(1-t)$.

$$E[X] = \frac{1}{3},\ E[X^2] = \frac{1}{6},\ E[Y] = \frac{2}{3}$$

Answer

$$E[XY] = \int_0^1 \int_0^{2(1-t)} tu\, du dt = 1/6$$

$$\text{Cov}[X,Y] = \frac{1}{6} - \frac{1}{3}\cdot\frac{2}{3} = -1/18 \quad \text{Var}[X] = 1/6 - (1/3)^2 = 1/18$$

$$a = \text{Cov}[X,Y]/\text{Var}[X] = -1 \quad b = E[Y] - aE[X] = 1$$

```
tuappr: [0 1] [0 2] 200 400  u<=2*(1-t)
EX = dot(X,PX);
EY = dot(Y,PY);
VX = dot(X.^2,PX) - EX^2
VX =   0.0556
CV = total(t.*u.*P) - EX*EY
CV =  -0.0556
a = CV/VX
a =   -1.0000
b = EY - a*EX
b =    1.0000
```

Exercise 12.4.24

(See Exercise 13 from "Problems On Random Vectors and Joint Distributions", and Exercise 23 from "Problems on Mathematical Expectation"). $f_{XY}(t,u) = \frac{1}{8}(t+u)$ for $0 \le t \le 2$, $0 \le u \le 2$.

$$E[X] = E[Y] = \frac{7}{6},\ E[X^2] = \frac{5}{3}$$

Answer

$$E[XY] = \frac{1}{8}\int_0^2 \int_0^2 tu(t+u)\, du dt = 4/3,\ \text{Cov}[X,Y] = -1/36,\ \text{Var}[X] = 11/36$$

$$a = \text{Cov}[X,Y]/\text{Var}[X] = -1/11,\ b = E[Y] - aE[X] = 14/11$$

```
tuappr:  [0 2] [0 2] 200 200 (1/8)*(t+u)
VX =  0.3055  CV = -0.0278  a = -0.0909  b =  1.2727
```

Exercise 12.4.25

(See Exercise 15 from "Problems On Random Vectors and Joint Distributions", and Exercise 25 from "Problems on Mathematical Expectation"). $f_{XY}(t,u) = \frac{3}{88}(2t+3u^2)$ for $0 \le t \le 2$, $0 \le u \le 1+t$.

$$E[X] = \frac{313}{220},\ E[Y] = \frac{1429}{880},\ E[X^2] = \frac{49}{22}$$

Answer

$$E[XY] = \frac{3}{88}\int_0^2 \int_0^{1+t} tu(2t+3u^2)\,dudt = 2153/880,\ \text{Cov}[X,Y] = 26383/193600,\ \text{Var}[X] = 9831/48400$$

$$a = \text{Cov}[X,Y]/\text{Var}[X] = 26383/39324,\ b = E[Y] - aE[X] = 26321/39324$$

```
tuappr:  [0 2] [0 3] 200 300  (3/88)*(2*t + 3*u.^2).*(u<=1+t)
VX =  0.2036  CV = 0.1364 a = 0.6700  b = 0.6736
```

Exercise 12.4.26

(See Exercise 16 from "Problems On Random Vectors and Joint Distributions", and Exercise 26 from "Problems on Mathematical Expectation"). $f_{XY}(t,u) = 12t^2u$ on the parallelogram with vertices

(-1, 0), (0, 0), (1, 1), (0, 1)

$$E[X] = \frac{2}{5},\ E[Y] = \frac{11}{15},\ E[X^2] = \frac{2}{5}$$

Answer

$$E[XY] = 12\int_{-1}^0 \int_0^{t+1} t^3u^2\,dudt + 12\int_0^1 \int_t^1 t^3u^2\,dudt = \frac{2}{5}$$

$$\text{Cov}[X,Y] = \frac{8}{75},\ \text{Var}[X] = \frac{6}{25}$$

$$a = \text{Cov}[X,Y]/\text{Var}[X] = 4/9,\ b = E[Y] - aE[X] = 5/9$$

```
tuappr: [-1 1] [0 1] 400 200  12*t.^2.*u.*(u>= max(0,t)).*(u<= min(1+t,1))
VX = 0.2383  CV = 0.1056  a = 0.4432  b = 0.5553
```

Exercise 12.4.27

(See Exercise 17 from "Problems On Random Vectors and Joint Distributions", and Exercise 27 from "Problems on Mathematical Expectation"). $f_{XY}(t,u) = \frac{24}{11}tu$ for $0 \le t \le 2,\ 0 \le u \le \min\{1, 2-t\}$.

$$E[X] = \frac{52}{55},\ E[Y] = \frac{32}{55},\ E[X^2] = \frac{627}{605}$$

Answer

$$E[XY] = \frac{24}{11}\int_0^1 \int_0^1 t^2u^2\,dudt + \frac{24}{11}\int_1^2 \int_0^{2-t} t^2u^2\,dudt = \frac{28}{55}$$

$$\text{Cov}[X,Y] = -\frac{124}{3025},\ \text{Var}[X] = \frac{431}{3025}$$

$$a = \text{Cov}[X,Y]/\text{Var}[X] = -\frac{124}{431},\ b = E[Y] - aE[X] = \frac{368}{431}$$

```
tuappr: [0 2] [0 1] 400 200 (24/11)*t.*u.*(u<=min(1,2-t))
VX = 0.1425  CV =-0.0409  a = -0.2867 b = 0.8535
```

Exercise 12.4.28

(See Exercise 18 from "Problems On Random Vectors and Joint Distributions", and Exercise 28 from "Problems on Mathematical Expectation"). $f_{XY}(t,u) = \frac{3}{23}(t+2u)$, for $0 \le t \le 2,\ 0 \le u \le \max\{2-t, t\}$.

$$E[X] = \frac{53}{46}, E[Y] = \frac{22}{23}, E[X^2] = \frac{9131}{5290}$$

Answer

$$E[XY] = \frac{3}{23} \int_0^1 \int_0^{2-t} tu(t+2u)\, dudt + \frac{3}{23} \int_1^2 \int_0^t tu(t+2u)\, dudt = \frac{251}{230}$$

$$\text{Cov}[X, Y] = -\frac{57}{5290}, \text{Var}[X] = \frac{4217}{10580}$$

$$a = \text{Cov}[X, Y]/\text{Var}[X] = -\frac{114}{4217}, b = E[Y] - aE[X] = \frac{4165}{4217}$$

```
tuappr: [0 2] [0 2] 200 200 (3/23)*(t + 2*u).*(u<=max(2-t,t))
VX = 0.3984 CV = -0.0108  a = -0.0272  b = 0.9909
```

Exercise 12.4.29

(See Exercise 21 from "Problems On Random Vectors and Joint Distributions", and Exercise 31 from "Problems on Mathematical Expectation"). $f_{XY}(t, u) = \frac{2}{13}(t+2u)$, for $0 \le t \le 2$, $0 \le u \le \min\{2t, 3-t\}$.

$$E[X] = \frac{16}{13}, E[Y] = \frac{11}{12}, E[X^2] = \frac{2847}{1690}$$

Answer

$$E[XY] = \frac{2}{13} \int_0^1 \int_0^{3-t} tu(t+2u)\, dudt + \frac{2}{13} \int_1^2 \int_0^{2t} tu(t+2u)\, dudt = \frac{431}{390}$$

$$\text{Cov}[X, Y] = -\frac{3}{130}, \text{Var}[X] = \frac{287}{1690}$$

$$a = \text{Cov}[X, Y]/\text{Var}[X] = -\frac{39}{297}, b = E[Y] - aE[X] = \frac{3733}{3444}$$

```
tuappr: [0 2] [0 2] 400 400 (2/13)*(t + 2*u).*(u<=min(2*t,3-t))
VX = 0.1698  CV = -0.0229  a = -0.1350  b = 1.0839
```

Exercise 12.4.30

(See Exercise 22 from "Problems On Random Vectors and Joint Distributions", and Exercise 32 from "Problems on Mathematical Expectation"). $f_{XY}(t, u) = I_{[0,1]}(t)\frac{3}{8}(t^2+2u) + I_{(1,2]}(t)\frac{9}{14}t^2u^2$, for $0 \le u \le 1$.

$$E[X] = \frac{243}{224}, E[Y] = \frac{11}{16}, E[X^2] = \frac{107}{70}$$

Answer

$$E[XY] = \frac{3}{8} \int_0^1 \int_0^1 tu(t^2+2u)\, dudt + \frac{9}{14} \int_1^2 \int_0^1 t^3u^3\, dudt = \frac{347}{448}$$

$$\text{Cov}[X, Y] = -\frac{103}{3584}, \text{Var}[X] = \frac{88243}{250880}$$

$$a = \text{Cov}[X, Y]/\text{Var}[X] = -\frac{7210}{88243}, b = E[Y] - aE[X] = \frac{105691}{176486}$$

```
tuappr: [0 2] [0 1] 400 200 (3/8)*(t.^2 + 2*u).*(t<=1) + (9/14)*t.^2.*u.^2.*(t>1
VX = 0.3517  CV = 0.0287 a = 0.0817  b = 0.5989
```

Exercise 12.4.31

The class $\{X, Y, Z\}$ of random variables is iid (independent, identically distributed) with common distribution

$$X = \text{[-5 -1 3 4 7]} \; PX = 0.01 * \text{[15 20 30 25 10]}$$

Let $W = 3X - 4Y + 2Z$. Determine $E[W]$ and $\text{Var}[W]$. Do this using icalc, then repeat with icalc3 and compare results.

Answer

```
x = [-5 -1 3 4 7];
px = 0.01*[15 20 30 25 10];
EX = dot(x,px)                % Use of properties
EX =   1.6500
VX = dot(x.^2,px) - EX^2
VX =  12.8275
EW = (3 - 4+ 2)*EX
EW =   1.6500
VW = (3^2 + 4^2 + 2^2)*VX
VW = 371.9975
icalc                         % Iterated use of icalc
Enter row matrix of X-values  x
Enter row matrix of Y-values  x
Enter X probabilities  px
Enter Y probabilities  px
 Use array operations on matrices X, Y, PX, PY, t, u, and P
G = 3*t - 4*u;
[R,PR] = csort(G,P);
icalc
Enter row matrix of X-values  R
Enter row matrix of Y-values  x
Enter X probabilities  PR
Enter Y probabilities  px
 Use array operations on matrices X, Y, PX, PY, t, u, and P
H = t + 2*u;
[W,PW] = csort(H,P);
EW = dot(W,PW)
EW =   1.6500
VW = dot(W.^2,PW) - EW^2
VW = 371.9975
icalc3                        % Use of icalc3
Enter row matrix of X-values  x
Enter row matrix of Y-values  x
Enter row matrix of Z-values  x
Enter X probabilities  px
Enter Y probabilities  px
Enter Z probabilities  px
Use array operations on matrices X, Y, Z,
PX, PY, PZ, t, u, v, and P
```

```
S = 3*t - 4*u + 2*v;
[w,pw] = csort(S,P);
Ew = dot(w,pw)
Ew =   1.6500
Vw = dot(w.^2,pw) - Ew^2
Vw = 371.9975
```

Exercise 12.4.32

$f_{XY}(t,u) = \frac{3}{88}(2t+3u^2)$ for $0 \le t \le 2$, $0 \le u \le 1+t$ (see Exercise 25 and Exercise 37 from "Problems on Mathematical Expectation").

$$Z = I_{[0,1]}(X)4X + I_{(1,2]}(X)(X+Y)$$

$$E[X] = \frac{313}{220}, E[Z] = \frac{5649}{1760}, E[Z^2] = \frac{4881}{440}$$

Determine $\text{Var}[Z]$ and $\text{Cov}[X, Z]$. Check with discrete approximation.

Answer

$$E[XZ] = \frac{3}{88}\int_0^1\int_0^{1+t} 4t^2(2t+2u^2)\,du\,dt + \frac{3}{88}\int_1^2\int_0^{1+t} t(t+u)(2t+3u^2)\,du\,dt = \frac{16931}{3520}$$

$$\text{Var}[Z] = E[Z^2] - E^2[Z] = \frac{2451039}{3097600}\quad \text{Cov}[X,Z] = E[XZ] - E[X]E[Z] = \frac{94273}{387200}$$

```
tuappr: [0 2] [0 3] 200 300 (3/88)*(2*t+3*u.^2).*(u<=1+t)
G = 4*t.*(t<=1) + (t+u).*(t>1);
EZ = total(G.*P)
EZ = 3.2110
EX = dot(X,PX)
EX = 1.4220
CV = total(G.*t.*P) - EX*EZ
CV = 0.2445                         % Theoretical 0.2435
VZ = total(G.^2.*P) - EZ^2
VZ = 0.7934                         % Theoretical 0.7913
```

Exercise 12.4.33

$f_{XY}(t,u) = \frac{24}{11}tu$ for $0 \le t \le 2$, $0 \le u \le \min\{1, 2-t\}$ (see Exercise 27 and Exercise 38 from "Problems on Mathematical Expectation").

$$Z = I_M(X,Y)(X+Y) + I_{M^c}(X,Y)2Y\ ,\ M = \{(t,u) : \max(t,u) \le 1\}$$

$$E[X] = \frac{52}{55}, E[Z] = \frac{16}{55}, E[Z^2] = \frac{39}{308}$$

Determine $\text{Var}[Z]$ and $\text{Cov}[X, Z]$. Check with discrete approximation.

Answer

$$E[XZ] = \frac{24}{11}\int_0^1\int_t^1 t(t/2)tu\,du\,dt + \frac{24}{11}\int_0^1\int_0^t tu^2tu\,du\,dt\frac{24}{11}\int_1^2\int_0^{2-t} ttu^2tu\,du\,dt = \frac{211}{770}$$

$$\text{Var}[Z] = E[Z^2] - E^2[Z] = \frac{3557}{84700} \quad \text{Cov}[Z,X] = E[XZ] - E[X]E[Z] = -\frac{43}{42350}$$

```
tuappr:  [0 2] [0 1] 400 200 (24/11)*t.*u.*(u<=min(1,2-t))
G = (t/2).*(u>t) + u.^2.*(u<=t);
VZ = total(G.^2.*P) - EZ^2
VZ =   0.0425
CV = total(t.*G.*P) - EZ*dot(X,PX)
CV = -9.2940e-04
```

Exercise 12.4.34

$f_{XY}(t,u) = \frac{3}{23}(t+2u)$ for $0 \le t \le 2$, $0 \le u \le \max\{2-t, t\}$ (see Exercise 28 and Exercise 39 from "Problems on Mathematical Expectation").

$$Z = I_M(X,Y)(X+Y) + I_{M^c}(X,Y)2Y \;\; , M = \{(t,u) : \max(t,u) \le 1\}$$

$$E[X] = \frac{53}{46}, E[Z] = \frac{175}{92}, E[Z^2] = \frac{2063}{460}$$

Determine $\text{Var}[Z]$ and $\text{Cov}[Z]$. Check with discrete approximation.

Answer

$$E[ZX] = \frac{3}{23}\int_0^1\int_0^1 t(t+u)(t+2u)\,du\,dt + \frac{3}{23}\int_0^1\int_1^{2-t} 2tu(t+2u)\,du\,dt +$$

$$\frac{3}{23}\int_1^2\int_1^t 2tu(t+2u)\,du\,dt = \frac{1009}{460}$$

$$\text{Var}[Z] = E[Z^2] - E^2[Z] = \frac{36671}{42320} \quad \text{Cov}[Z,X] = E[ZX] - E[Z]E[X] = \frac{39}{21160}$$

```
tuappr:  [0 2] [0 2] 400 400 (3/23)*(t+2*u).*(u<=max(2-t,t))
M = max(t,u)<=1;
G = (t+u).*M + 2*u.*(1-M);
EZ = total(G.*P);
EX = dot(X,PX);
CV = total(t.*G.*P) - EX*EZ
CV =  0.0017
```

Exercise 12.4.35

$f_{XY}(t,u) = \frac{12}{179}(3t^2+u)$, for $0 \le t \le 2$, $0 \le u \le \min\{2, 3-t\}$ (see Exercise 29 and Exercise 40 from "Problems on Mathematical Expectation").

$$Z = I_M(X,Y)(X+Y) + I_{M^c}(X,Y)2Y^2 \;\; , M = \{(t,u) : t \le 1, u \ge 1\}$$

$$E[X] = \frac{2313}{1790}, E[Z] = \frac{1422}{895}, E[Z^2] = \frac{28296}{6265}$$

Determine $\text{Var}[Z]$ and $\text{Cov}[X,Z]$. Check with discrete approximation.

Answer

$$E[ZX] = \frac{12}{179}\int_0^1\int_1^2 t(t+u)(3t^2+u)\,du\,dt + \frac{12}{179}\int_0^1\int_0^1 2tu^2(3t^2+u)\,du\,dt +$$

$$\frac{12}{179}\int_1^2\int_0^{3-t}2tu^2(3t^2+u)\,dudt = \frac{24029}{12530}$$

$$\text{Var}[Z] = E[Z^2]-E^2[Z] = \frac{11170332}{5607175}\ \ \text{Cov}[Z,X] = E[ZX]-E[Z]E[X] = -\frac{1517647}{11214350}$$

```
tuappr:  [0 2] [0 2] 400 400 (12/179)*(3*t.^2 + u).*(u <= min(2,3-t))
M = (t<=1)&(u>=1);
G = (t + u).*M + 2*u.^2.*(1 - M);
EZ = total(G.*P);
EX = dot(X,PX);
CV = total(t.*G.*P) - EZ*EX
CV = -0.1347
```

Exercise 12.4.36

$f_{XY}(t,u) = \frac{12}{227}(3t+2tu)$, for $0 \le t \le 2$, $0 \le u \le \min\{1+t, 2\}$ (see Exercise 30 and Exercise 41 from "Problems on Mathematical Expectation").

$$Z = I_M(X,Y)X + I_{M^c}(X,Y)XY\ ,\ M = \{(t,u) : u \le \min(1, 2-t)\}$$

$$E[X] = \frac{1567}{1135},\ E[Z] = \frac{5774}{3405},\ E[Z^2] = \frac{56673}{15890}$$

Determine $\text{Var}[Z]$ and $\text{Cov}[X,Z]$. Check with discrete approximation.

Answer

$$E[ZX] = \frac{12}{227}\int_0^1\int_0^1 t^2(3t+2tu)\,dudt + \frac{12}{227}\int_1^2\int_0^{2-t} t^2(3t+2tu)\,dudt +$$

$$\frac{12}{227}\int_0^1\int_1^{1+t} t^2u(3t+2tu)\,dudt + \frac{12}{227}\int_1^2\int_{2-t}^2 t^2u(3t+2tu)\,dudt = \frac{20338}{7945}$$

$$\text{Var}[Z] = E[Z^2]-E^2[Z] = \frac{112167631}{162316350}\ \text{Cov}[Z,X] = E[ZX]-E[Z]E[X] = \frac{5915884}{27052725}$$

```
tuappr: [0 2] [0 2] 400 400 (12/227)*(3*t + 2*t.*u).*(u <= min(1+t,2))
EX = dot(X,PX);
M = u <= min(1,2-t);
G = t.*M + t.*u.*(1 - M);
EZ = total(G.*P);
EZX = total(t.*G.*P)
EZX =  2.5597
CV = EZX - EX*EZ
CV =   0.2188
VZ = total(G.^2.*P) - EZ^2
VZ =   0.6907
```

Exercise 12.4.37

(See Exercise 12.4.20, and Exercises 9 and 10 from "Problems on Functions of Random Variables"). For the pair $\{X,Y\}$ in Exercise 12.4.20, let

$$Z = g(X,Y) = 3X^2 + 2XY - Y^2$$

$$W = h(X,Y) = \begin{cases} X & \text{for } X+Y \le 4 \\ 2Y & \text{for } X+Y > 4 \end{cases} = I_M(X,Y)X + I_{M^c}(X,Y)2Y$$

Determine the joint distribution for the pair $\{Z, W\}$ and determine the regression line of W on Z.

Answer

```
npr08_07
Data are in X, Y, P
jointzw
Enter joint prob for (X,Y) P
Enter values for X X
Enter values for Y Y
Enter expression for g(t,u) 3*t.^2 + 2*t.*u - u.^2
Enter expression for h(t,u) t.*(t+u<=4) + 2*u.*(t+u>4)
Use array operations on Z, W, PZ, PW, v, w, PZW
EZ = dot(Z,PZ)
EZ =    5.2975
EW = dot(W,PW)
EW =    4.7379
VZ = dot(Z.^2,PZ) - EZ^2
VZ =    1.0588e+03
CZW = total(v.*w.*PZW) - EZ*EW
CZW = -12.1697
a = CZW/VZ
a =   -0.0115
b = EW - a*EZ
b =    4.7988                 % Regression line: w = av + b
```

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