13.4: Problems on Transform Methods

Exercise 13.4.1

Calculate directly the generating function $g_X(s)$ for the geometric (p) distribution.

Answer

$$g_X(s) = E[s^2] = \sum_{k=0}^{\infty} p_k s^k = p \sum_{k=0}^{\infty} q^k s^k = \frac{p}{1-qs} \quad \text{(geometric series)}$$

Exercise 13.4.2

Calculate directly the generating function $g_X(s)$ for the Poisson (μ) distribution.

Answer

$$g_X(s) = E[s^X] = \sum_{k=0}^{\infty} p_k s^k = e^{-\mu} \sum_{k=0}^{\infty} \frac{\mu^k s^k}{k!} = e^{-\mu} e^{\mu s} = e^{\mu(s-1)}$$

Exercise 13.4.3

A projection bulb has life (in hours) represented by X ~ exponential (1/50). The unit will be replaced immediately upon failure or at 60 hours, whichever comes first. Determine the moment generating function for the time Y to replacement.

Answer

$$Y = I_{[0,a]}(X)X + I_{(a,\infty)}(X)a \quad e^{sY} = I_{[0, a)}(X) e^{sX} + I_{(a, \infty)}(X) e^{as}$$

$$M_Y(s) = \int_0^a e^{st} \lambda e^{-\lambda t}\, dt + s^{sa} \int_a^{\infty} \lambda e^{-\lambda t}\, dt$$

$$= \frac{\lambda}{\lambda - s}[1 - e^{(\lambda - s)a}] + e^{-(\lambda - s)a}$$

Exercise 13.4.4

Simple random variable X has distribution

$X =$ [-3 -2 0 1 4] $PX =$ [0.15 0.20 0.30 0.25 0.10]

a. Determine the moment generating function for X
b. Show by direct calculation the $M_X'(0) = E[X]$ and $M_X''(0) = E[X^2]$.

Answer

$$M_X(s) = 0.15e^{-3s} + 0.20e^{-2s} + 0.30 + 0.25e^s + 0.10e^{4s}$$

$$M_X'(s) = -3 \cdot 0.15e^{-3s} - 2 \cdot 0.20e^{-2s} + 0 + 0.25e^s + 4 \cdot 0.10e^{4s}$$

$$M_X''(s) = (-3)^2 \cdot 0.15e^{-3s} + (-2)^2 \cdot 0.20e^{-2s} + 0 + 0.25e^s + 4^2 \cdot 0.10e^{4s}$$

Setting $s = 0$ and using $e^0 = 1$ give the desired results.

Exercise 13.4.5

Use the moment generating function to obtain the variances for the following distributions

EXponential (λ) Gamma (α, λ) Normal (μ, σ^2)

Answer

a. Exponential:

$$M_X(s) = \frac{\lambda}{\lambda - s} \quad M_X'(s) = \frac{\lambda}{(\lambda - s)^2} \quad M_X''(s) = \frac{2\lambda}{(\lambda - s)^3}$$

$$E[X] = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda} \quad E[X^2] = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2} \quad \text{Var}[X] = \frac{2}{\lambda^2} - (\frac{1}{\lambda})^2 = \frac{1}{\lambda^2}$$

b. Gamma (α, λ):

$$M_X(s) = (\frac{\lambda}{\lambda - s})^\alpha \quad M_X'(s) = \alpha(\frac{\lambda}{\lambda - s})^{\alpha - 1} \frac{\lambda}{(\lambda - s)^2} = \alpha(\frac{\lambda}{\lambda - s})^\alpha \frac{1}{\lambda - s}$$

$$M_X''(s) = \alpha^2(\frac{\lambda}{\lambda - s})^\alpha \frac{1}{\lambda - s} \frac{1}{\lambda - s} + \alpha(\frac{\lambda}{\lambda - s})^\alpha \frac{1}{(\lambda - s)^2}$$

$$E[X] = \frac{\alpha}{\lambda} \quad E[X^2] = \frac{\alpha^2 + \alpha}{\lambda^2} \quad \text{Var}[X] = \frac{\alpha}{\lambda^2}$$

c. Normal(μ, σ):

$$M_X(s) = \exp(\frac{\sigma^2 s^2}{2} + \mu s) \quad M_X'(s) = M_X(s) \cdot (\sigma^2 s + \mu)$$

$$M_X''(s) = M_X(s) \cdot (\sigma^2 s + \mu)^2 + M_X(s)\sigma^2$$

$$E[X] = \mu \quad E[X^2] = \mu^2 + \sigma^2 \quad \text{Var}[X] = \sigma^2$$

Exercise 13.4.6

The pair $\{X, Y\}$ is iid with common moment generating function $\frac{\lambda^3}{(\lambda - s)^3}$. Determine the moment generating function for $Z = 2X - 4Y + 3$.

Answer

$$M_Z(s) = e^{3s}(\frac{\lambda}{\lambda - 2s})^3(\frac{\lambda}{\lambda + 4s})^3$$

Exercise 13.4.7

The pair $\{X, Y\}$ is iid with common moment generating function $M_X(s) = (0.6 + 0.4e^s)$. Determine the moment generating function for $Z = 5X + 2Y$.

Answer

\(M_Z (s) = (0.6 + 0.4e^{5s})(0.6 + 0.4e^{2s})

Exercise 13.4.8

Use the moment generating function for the symmetric triangular distribution on $(-c, c)$ as derived in the section "Three Basic Transforms".

a. Obtain an expression for the symmetric triangular distribution on (a, b) for any $a < b$.
b. Use the result of part (a) to show that the sum of two independent random variables uniform on (a, b) has symmetric triangular distribution on $(2a, 2b)$.

Answer

Let $m = (a + b)/2$ and $c = (b - a)/2$. If Y ~ symetric triangular on $(-c, c)$, then $X = Y + m$ is symmetric triangular on $(m - c, m + c) = (a, b)$ and

$$M_X(s) = e^{ms} M_Y(s) = \frac{e^{cs} + e^{-cs} - 2}{c^2 s^2} e^{ms} = \frac{e^{(m+c)s} + e^{(m-c)s} - 2e^{ms}}{c^2 s^2} = \frac{e^{hs} + e^{as} - 2e^{\frac{a+b}{2}s}}{(\frac{b-a}{2})^2 s^2}$$

$$M_{X+Y}(s) = [\frac{e^{sb} - e^{sa}}{s(b-a)}]^2 = \frac{e^{s2b} + e^{s2a} - 2e^{s(b+a)}}{s^2(b-a)^2}$$

Exercise 13.4.9

Random variable X has moment generating function $\frac{p^2}{(1-qe^s)^2}$.

a. Use derivatives to determine $E[X]$ and $\text{Var}[X]$.

b. Recognize the distribution from the form and compare $E[X]$ and $\text{Var}[X]$ with the result of part (a).

Answer

$$[p^2(1-qe^s)^{-2}]' = \frac{2p^2qe^s}{(1-qe^s)^3} \quad \text{so that } E[X] = 2q/p$$

$$[p^2(1-qe^s)^{-2}]'' = \frac{6p^2q^2e^s}{(1-qe^s)^4} + \frac{2p^2qe^s}{(1-qe^s)^3} \quad \text{so that } E[X^2] = \frac{6q^2}{p^2} + \frac{2q}{p}$$

$$\text{Var}[X] = \frac{2q^2}{p^2} + \frac{2q}{p} = \frac{2(q^2+pq)}{p^2} = \frac{2q}{p^2}$$

X ~ negative binomial $(2, p)$, which has $E[X] = 2q/p$ and $\text{Var}[X] = 2q/p^2$.

Exercise 13.4.10

The pair $\{X, Y\}$ is independent. X ~ Poisson (4) and Y ~ geometric (0, 3). Determine the generating function g_Z for $Z = 3X + 2Y$.

Answer

$$g_Z(s) = g_X(s^3) g_Y(s^2) = e^{4(s^3-1)} \cdot \frac{0.3}{1-qs^2}$$

Exercise 13.4.11

Random variable X has moment generating function

$$M_X(s) = \frac{1}{1-3s} \cdot \exp(16s^2/2 + 3s)$$

By recognizing forms and using rules of combinations, determine $E[X]$ and $\text{Var}[X]$.

Answer

$$X = X_1 + X_2 \quad \text{with } X_1 \sim \text{exponential (1/3) } X_2 \sim N(3,\ 16)$$

$$E[X] = 3 + 3 = 6 \quad \text{Var}[X] = 9 + 16 = 25$$

Exercise 13.4.12

Random variable X has moment generating function

$$M_X(s) = \frac{\exp(3(e^s - 1))}{1-5s} \cdot \exp(16s^2/2 + 3s)$$

By recognizing forms and using rules of combinations, determine $E[X]$ and $\text{Var}[X]$.

Answer

$$X = X_1 + X_2 + X_3$$, with X_1 ~ Poisson (3), X_2 ~ exponential (1/5), X_3\0 \(N(3, 16)

$$E[X] = 3 + 5 + 3 = 11 \quad \text{Var}[X] = 3 + 25 + 16 = 44$$

Exercise 13.4.13

Suppose the class $\{A, B, C\}$ of events is independent, with respective probabilities 0.3, 0.5, 0.2. Consider

$$X = -3I_A + 2I_B + 4I_C$$

a. Determine the moment generating functions for and use properties of moment generating functions to determine the moment generating function for X.
b. Use the moment generating function to determine the distribution for X.
c. Use canonic to determine the distribution. Compare with result (b).
d. Use distributions for the separate terms; determine the distribution for the sum with mgsum3. Compare with result (b).

Answer

$$M_X(s) = (0.7 + 0.3e^{-3s})(0.5 + 0.5e^{2s})(0.8 + 0.2e^{4s}) =$$

$$0.12e^{-3s} + 0.12e^{-s} + 0.28 + 0.03e^{s} + 0.28e^{2s} + 0.03e^{3s} + 0.07e^{4s} + 0.07e^{6s}$$

The distribution is

$X =$ [-3 -1 0 1 2 3 4 6] $PX =$ [0.12 0.12 0.28 0.03 0.28 0.03 0.07 0.07]

```
c = [-3 2 4 0];
P = 0.1*[3 5 2];
canonic
 Enter row vector of coefficients  c
 Enter row vector of minterm probabilities  minprob(P)
Use row matrices X and PX for calculations
Call for XDBN to view the distribution
P1 = [0.7 0.3];
P2 = [0.5 0.5];
P3 = [0.8 0.2];
X1 = [0 -3];
X2 = [0 2];
X3 = [0 4];
[x,px] = mgsum3(X1,X2,X3,P1,P2,P3);
disp([X;PX;x;px]')
   -3.0000    0.1200   -3.0000    0.1200
   -1.0000    0.1200   -1.0000    0.1200
         0    0.2800         0    0.2800
    1.0000    0.0300    1.0000    0.0300
    2.0000    0.2800    2.0000    0.2800
    3.0000    0.0300    3.0000    0.0300
    4.0000    0.0700    4.0000    0.0700
    6.0000    0.0700    6.0000    0.0700
```

Exercise 13.4.14

Suppose the pair $\{X, Y\}$ is independent, with both X and Y binomial. Use generating functions to show under what condition, if any, $X+Y$ is binomial.

Answer

Binomial iff both have same p, as shown below.

$$g_{X+Y}(s) = (q_1 + p_1 s)^n (q_2 + p_2 s)^m = (q + ps)^{n+m} \quad \text{iff } p_1 = p_2$$

Exercise 13.4.15

Suppose the pair $\{X, Y\}$ is independent, with both X and Y Poisson.

a. Use generating functions to show under what condition $X+Y$ is Poisson.
b. What about $X-Y$? Justify your answer.

Answer

Always Poisson, as the argument below shows.

$$g_{X+Y}(s) = e^{\mu(s-1)} e^{v(s-1)} = e^{(\mu+v)(s-1)}$$

However, Y ~ X could have negative values.

Exercise 13.4.16

Suppose the pair $\{X, Y\}$ is independent, Y is nonnegative integer-valued, X is Poisson and $X+Y$ is Poisson. Use the generating functions to show that Y is Poisson.

Answer

$E[X+Y] = \mu + v$, where $v = E[Y] > 0$, $g_X(s) = e^{\mu(s-1)}$ and \(g_{X + Y} (s) = g_X (s) g_Y (s) = e^{(\mu + s) (s - 1)\). Division by $g_X(s)$ gives $g_Y(s) = e^{v(s-1)}$.

Exercise 13.4.17

Suppose the pair $\{X, Y\}$ is iid, binomial (6, 0.51). By the result of Exercise 13.4.14

$X+Y$ is binomial. Use mgsum to obtain the distribution for $Z = 2X + 4Y$. Does Z have the binomial distribution? Is the result surprising? Examine the first few possible values for Z. Write the generating function for Z; does it have the form for the binomial distribution?

Answer

```
x  = 0:6;
px = ibinom(6,0.51,x);
[Z,PZ] = mgsum(2*x,4*x,px,px);
disp([Z(1:5);PZ(1:5)]')
         0    0.0002       % Cannot be binomial, since odd values missing
    2.0000    0.0012
    4.0000    0.0043
    6.0000    0.0118
    8.0000    0.0259
    - - - - - - - -
```

$$g_X(s) = g_Y(s) = (0.49 + 0.51s)^6 \quad g_Z(s) = (0.49 + 0.51s^2)^6(0.49 + 0.51s^4)^6$$

Exercise 13.4.18

Suppose the pair $\{X, Y\}$ is independent, with X ~ binomial (5, 0.33) and Y ~ binomial (7, 0.47).

Let $G = g(X) = 3X^2 - 2X$ and $H = h(Y) = 2Y^2 + Y + 3$.

a. Use the mgsum to obtain the distribution for $G + H$.
b. Use icalc and csort to obtain the distribution for $G + H$ and compare with the result of part (a).

Answer

```
X = 0:5;
Y = 0:7;
PX = ibinom(5,0.33,X);
PY = ibinom(7,0.47,Y);
G = 3*X.^2 - 2*X;
H = 2*Y.^2 + Y + 3;
[Z,PZ] = mgsum(G,H,PX,PY);



icalc
Enter row matrix of X-values  X
Enter row matrix of Y-values  Y
Enter X probabilities  PX
Enter Y probabilities  PY
 Use array operations on matrices X, Y, PX, PY, t, u, and P
M = 3*t.^2 - 2*t + 2*u.^2 + u + 3;
[z,pz] = csort(M,P);
e = max(abs(pz - PZ))  % Comparison of p values
e =  0
```

Exercise 13.4.19

Suppose the pair $\{X, Y\}$ is independent, with X ~ binomial (8, 0.39) and Y ~ uniform on {-1.3, -0.5, 1.3, 2.2, 3.5}. Let

$U = 3X^2 - 2X + 1$ and $V = Y^3 + 2Y - 3$

a. Use mgsum to obtain the distribution for $U + V$.
b. Use icalc and csort to obtain the distribution for $U + V$ and compare with the result of part (a).

Answer

```
X = 0:8;
Y = [-1.3 -0.5 1.3 2.2 3.5];
PX = ibinom(8,0.39,X);
PY = (1/5)*ones(1,5);
U  = 3*X.^2 - 2*X + 1;
V  = Y.^3 + 2*Y - 3;
[Z,PZ] = mgsum(U,V,PX,PY);
```

```
icalc
Enter row matrix of X-values  X
Enter row matrix of Y-values  Y
Enter X probabilities  PX
Enter Y probabilities  PY
 Use array operations on matrices X, Y, PX, PY, t, u, and P
M = 3*t.^2 - 2*t + 1 + u.^3 + 2*u - 3;
[z,pz] = csort(M,P);
e = max(abs(pz - PZ))
e = 0
```

Exercise 13.4.20

If X is a nonnegative integer-valued random variable, express the generating function as a power series.

a. Show that the kth derivative at $s = 1$ is

$$g_X^{(k)}(1) = E[X(X-1)(X-2)\cdots(X-k+1)]$$

b. Use this to show the $\text{Var}[X] = g_X''(1) + g_X'(1) - [g_X'(1)]^2$.

Answer

Since power series may be differentiated term by term

$$g_X^{(n)}(s) = \sum_{k=0}^{\infty} k(k-1)\cdot(k-n+1)p_k s^{k-n} \quad \text{so that}$$

$$g_X^{(n)}(1) = \sum_{k=0}^{\infty} k(k-1)\cdot(k-n+1)p_k = E[X(X-1)\cdots(X-n+1)]$$

$$\text{Var}[X] = E[X^2] - E^2[X] = E[X(X-1)] + E[X] - E^2[X] = g_X''(1) + g_X'(1) - [g_X'(1)]^2$$

Exercise 13.4.21

Let $M_X(\cdot)$ be the moment generating function for X.

a. Show that $\text{Var}[X]$ is the second derivative of $e^{-s\mu}M_X(s)$ evaluated at $s = 0$.
b. Use this fact to show that $X \sim N(\mu, \sigma^2)$, then $\text{Var}[X] = \sigma^2$.

Answer

$$f(s) = e^{-s\mu}M_X(s) \quad f''(s) = e^{-s\mu}[-\mu M_X'(s) + \mu^2 M_X(s) + M_X''(s) - \mu M_X'(s)]$$

Setting $s = 0$ and using the result on moments gives

$$f''(0) = -\mu^2 + \mu^2 + E[X^2] - \mu^2 = \text{Var}[X]$$

Exercise 13.4.22

Use derivatives of $M_{M_m}(s)$ to obtain the mean and variance of the negative binomial (m, p) distribution.

Answer

To simplify writing use $f(s)$ for $M_X(S)$.

$$f(s) = \frac{p^m}{(1-qe^s)^m} \quad f'(s) = \frac{mp^m qe^s}{(1-qe^s)^{m+1}} \quad f''(s) = \frac{mp^m qe^s}{1-qe^s)^{m+1}} + \frac{m(m+1)p^m q^2 e^{2s}}{1-qe^s)^{m+2}}$$

$$E[X] = \frac{mp^m q}{(1-q)^{m+1}} = \frac{mq}{p} \quad E[X^2] = \frac{mq}{p} + \frac{m(m+1)p^m q^2}{(1-q)^{m+2}}$$

$$\text{Var}[X] = \frac{mq}{p} + \frac{m(m+1)q^2}{p^2} - \frac{m^2q^2}{p^2} = \frac{mq}{p^2}$$

Exercise 13.4.23

Use moment generating functions to show that variances add for the sum or difference of independent random variables.

Answer

To simplify writing, set $f(s) = M_X(s)$, $g(s) = M_Y(s)$, and $h(s) = M_X(s)M_Y(s)$

$$h'(s) = f'(s)g(s) + f(s)g'(s) \quad h''(s) = f''(s)g(s) + f'(s)g'(s) + f'(s)g'(s) + f(s)g''(s)$$

Setting $s = 0$ yields

$$E[X+Y] = E[X] + E[Y] \;\; E[(X+Y)^2] = E[X^2] + 2E[X]E[Y] + E[Y^2]$$
$$E^2[X+Y] = E^2[X] + 2E[X]E[Y] + E^2[Y]$$

Taking the difference gives $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$. A similar treatment with $g(s)$ replaced by $g(-s)$ shows $\text{Var}[X-Y] = \text{Var}[X] + \text{Var}[Y]$.

Exercise 13.4.24

The pair $\{X, Y\}$ is iid $N(3,5)$. Use the moment generating function to show that $Z = 2X - 2Y + 3$ is normal (see Example 3 from "Transform Methods" for general result).

Answer

$$M_{3X}(s) = M_X(3s) = \exp(\frac{9 \cdot 5s^2}{2} + 3 \cdot 3s) \quad M_{-2Y}(s) = M_Y(-2s) = \exp(\frac{4 \cdot 5s^2}{2} - 2 \cdot 3s)$$
$$M_Z(s) = e^{3s}\exp(\frac{(45+20)s^2}{2} + (9-6)s) = \exp(\frac{65s^2}{2} + 6s)$$

Exercise 13.4.25

Use the central limit theorem to show that for large enough sample size (usually 20 or more), the sample average

$$A_n = \frac{1}{n}\sum_{i=1}^n X_i$$

is approximately $N(\mu, \sigma^2/n)$ for any reasonable population distribution having mean value μ and variance σ^2.

Answer

$$E[A_n] = \frac{1}{n}\sum_{i=1}^n \mu = \mu \;\; \text{Var}[A_n] = \frac{1}{n^2}\sum_{i=1}^n \sigma^2 = \frac{\sigma^2}{n}$$

By the central limit theorem, A_n is approximately normal, with the mean and variance above.

Exercise 13.4.26

A population has standard deviation approximately three. It is desired to determine the sample size n needed to ensure that with probability 0.95 the sample average will be within 0.5 of the mean value.

a. Use the Chebyshev inequality to estimate the needed sample size.
b. Use the normal approximation to estimate n (see Example 1 from "Simple Random Samples and Statistics").

Answer

Chevyshev inequality:

$$P\left(\frac{|A_n - \mu|}{\sigma/\sqrt{n}} \geq \frac{0.5\sqrt{n}}{3}\right) \leq \frac{3^2}{0.5^2 n} \leq 0.05 \text{ implies } n \geq 720$$

Normal approximation: Use of the table in Example 1 from "Simple Random Samples and Statistics" shows

$$n \geq (3/0.5)^2 3.84 = 128$$