10.4: Problems on Functions of Random Variables

Exercise 10.4.1

Suppose X is a nonnegative, absolutely continuous random variable. Let $Z = g(X) = Ce^{-aX}$, where $a > 0$, $C > 0$. Then $0 < Z \le C$. Use properties of the exponential and natural log function to show that

$$F_Z(v) = 1 - F_X(-\frac{\text{In}\,(v/C)}{a}) \text{ for } 0 < v \le C$$

Answer

$Z = Ce^{-aX} \le v$ iff $e^{-aX} \le v/C$ iff $-aX \le \text{In}\,(v/C)$ iff $X \ge -\text{In}\,(v/C)/a$, so that

$$F_Z(v) = P(Z \le v) = P(X \ge -\text{In}\,(v/C)/a) = 1 - F_X(-\frac{\text{In}\,(v/C)}{a})$$

Exercise 10.4.2

Use the result of Exercise 10.4.1 to show that if X ~ exponential (λ), then

$$F_Z(v) = (\frac{v}{C})^{\lambda/a} \;\; 0 < v \le C$$

Answer

$$F_Z(v) = 1 - [1 - exp(-\frac{\lambda}{a} \cdot \text{In}\,(v/C))] = (\frac{v}{C})^{\lambda/a}$$

Exercise 10.4.3

Present value of future costs. Suppose money may be invested at an annual rate a, compounded continually. Then one dollar in hand now, has a value e^{ax} at the end of x years. Hence, one dollar spent x years in the future has a *present valuee*$^{-ax}$. Suppose a device put into operation has time to failure (in years) X ~ exponential (λ). If the cost of replacement at failure is C dollars, then the present value of the replacement is $Z = Ce^{-aX}$. Suppose $\lambda = 1/10$, $a = 0.07$, and $C = \$1000$.

a. Use the result of Exercise 10.4.2. to determine the probability $Z \le 700, 500, 200$.
b. Use a discrete approximation for the exponential density to approximate the probabilities in part (a). Truncate X at 1000 and use 10,000 approximation points.

Answer

$$P(Z \le v) = (\frac{v}{1000})^{10/7}$$

```
v = [700 500 200];
P = (v/1000).^(10/7)
P =  0.6008    0.3715    0.1003
tappr
Enter matrix [a b] of x-range endpoints  [0 1000]
Enter number of x approximation points  10000
Enter density as a function of t  0.1*exp(-t/10)
Use row matrices X and PX as in the simple case
G = 1000*exp(-0.07*t);
PM1 = (G<=700)*PX'
PM1 =  0.6005
PM2 = (G<=500)*PX'
```

```
PM2 =  0.3716
PM3 = (G<=200)*PX'
PM3 =  0.1003
```

Exercise 10.4.4

Optimal stocking of merchandise. A merchant is planning for the Christmas season. He intends to stock *m* units of a certain item at a cost of *c* per unit. Experience indicates demand can be represented by a random variable D ~ Poisson (μ). If units remain in stock at the end of the season, they may be returned with recovery of r per unit. If demand exceeds the number originally ordered, extra units may be ordered at a cost of *s* each. Units are sold at a price p per unit. If $Z = g(D)$ is the gain from the sales, then

- For $t \le m$, $g(t) = (p-c)t - (c-r)(m-t) = (p-r)t + (r-c)m$
- For $t > m$, $g(t) = (p-c)m + (t-m)(p-s) = (p-s)t + (s-c)m$

Let $M = (-\infty, m]$. Then

$g(t) = I_M(t)[(p-r)t + (r-c)m] + I_M(t)[(p-s)t + (s-c)m]$

Suppose $\mu = 50$ $m = 50$ $c = 30$ $p = 50$ $r = 20$ $s = 40$.

Approximate the Poisson random variable D by truncating at 100. Determine $P(500 \le Z \le 1100)$.

Answer

```
mu = 50;
D = 0:100;
c = 30;
p = 50;
r = 20;
s = 40;
m = 50;
PD = ipoisson(mu,D);
G = (p - s)*D + (s - c)*m +(s - r)*(D - m).*(D <= m);
M = (500<=G)&(G<=1100);
PM = M*PD'
PM =  0.9209

[Z,PZ] = csort(G,PD);          % Alternate: use dbn for Z
m = (500<=Z)&(Z<=1100);
pm = m*PZ'
pm =  0.9209
```

Exercise 10.4.5

(See Example 2 from "Functions of a Random Variable") The cultural committee of a student organization has arranged a special deal for tickets to a concert. The agreement is that the organization will purchase ten tickets at $20 each (regardless of the number of individual buyers). Additional tickets are available according to the following schedule:

- 11-20, $18 each
- 21-30, $16 each
- 31-50, $15 each
- 51-100, $13 each

If the number of purchasers is a random variable X, the total cost (in dollars) is a random quantity $Z = g(X)$ described by

$$g(X) = 200 + 18 I_{M1}(X)(X-10) + (16-18) I_{M2}(X)(X-20) +$$

$$(15-16) I_{M_3}(X)(X-30) + (13-15) I_{M4}(X)(X-50)$$

$$\text{where } M1 = [10, \infty), M2 = [20, \infty), M3 = [30, \infty), M4 = [50, \infty)$$

Suppose X~ Poisson (75). Approximate the Poisson distribution by truncating at 150. Determine $P(Z \ge 1000)$, $P(Z \ge 1300)$ and $P(900 \le Z \le 1400)$.

Answer

```
X = 0:150;
PX = ipoisson(75,X);
G = 200 + 18*(X - 10).*(X>=10) + (16 - 18)*(X - 20).*(X>=20) + ...
     (15 - 16)*(X- 30).*(X>=30) + (13 - 15)*(X - 50).*(X>=50);
P1 = (G>=1000)*PX'
P1 =  0.9288
P2 = (G>=1300)*PX'
P2 =  0.1142
P3 = ((900<=G)&(G<=1400))*PX'
P3 =  0.9742
[Z,PZ] = csort(G,PX);         % Alternate: use dbn for Z
p1 = (Z>=1000)*PZ'
p1 =  0.9288
```

Exercise 10.4.6

(See Exercise 6 from "Problems on Random Vectors and Joint Distributions", and Exercise 1 from "Problems on Independent Classes of Random Variables")) The pair $\{X, Y\}$ has the joint distribution

(in m-file npr08_06.m):

$$X = \text{[-2.3 -0.7 1.1 3.9 5.1]} \; Y = \text{[1.3 2.5 4.1 5.3]}$$

$$P = \begin{bmatrix} 0.0483 & 0.0357 & 0.0420 & 0.0399 & 0.0441 \\ 0.0437 & 0.0323 & 0.0380 & 0.0361 & 0.0399 \\ 0.0713 & 0.0527 & 0.0620 & 0.0609 & 0.0551 \\ 0.0667 & 0.0493 & 0.0580 & 0.0651 & 0.0589 \end{bmatrix}$$

Determine $P(\max\{X, Y\} \le 4)$. Let $Z = 3X^3 + 3X^2Y - Y^3$.

Determine $P(Z < 0)$ and $P(-5 < Z \le 300)$.

Answer

```
npr08_06
Data are in X, Y, P
jcalc
Enter JOINT PROBABILITIES (as on the plane)  P
Enter row matrix of VALUES of X  X
Enter row matrix of VALUES of Y  Y
 Use array operations on matrices X, Y, PX, PY, t, u, and P
P1 = total((max(t,u)<=4).*P)
```

```
P1 =  0.4860
P2 = total((abs(t-u)>3).*P)
P2 =  0.4516
G = 3*t.^3 + 3*t.^2.*u - u.^3;
P3 = total((G<0).*P)
P3 =  0.5420
P4 = total(((-5<G)&(G<=300)).*P)
P4 =  0.3713
[Z,PZ] = csort(G,P);          % Alternate: use dbn for Z
p4 = ((-5<Z)&(Z<=300))*PZ'
p4 =  0.3713
```

Exercise 10.4.7

(See Exercise 2 from "Problems on Independent Classes of Random Variables") The pair $\{X, Y\}$ has the joint distribution (in m-file npr09_02.m):

$X =$ [-3.9 -1.7 1.5 2 8 4.1] $Y =$ [-2 1 2.6 5.1]

$$P = \begin{bmatrix} 0.0589 & 0.0342 & 0.0304 & 0.0456 & 0.0209 \\ 0.0962 & 0.056 & 0.0498 & 0.0744 & 0.0341 \\ 0.0682 & 0.0398 & 0.0350 & 0.0528 & 0.0242 \\ 0.0868 & 0.0504 & 0.0448 & 0.0672 & 0.0308 \end{bmatrix}$$

Determine $P(\{X+Y \geq 5\} \cup \{Y \leq 2\})$, $P(X^2+Y^2 \leq 10)$.

Answer

```
npr09_02
Data are in X, Y, P
jcalc
Enter JOINT PROBABILITIES (as on the plane)  P
Enter row matrix of VALUES of X  X
Enter row matrix of VALUES of Y  Y
 Use array operations on matrices X, Y, PX, PY, t, u, and P
M1 = (t+u>=5)|(u<=2);
P1 = total(M1.*P)
P1 =  0.7054
M2 = t.^2 + u.^2 <= 10;
P2 = total(M2.*P)
P2 =  0.3282
```

Exercise 10.4.8

(See Exercsie 7 from "Problems on Random Vectors and Joint Distributions", and Exercise 3 from "Problems on Independent Classes of Random Variables") The pair has the joint distribution

(in m-file npr08_07.m):

$P(X = t, Y = u)$

t =	-3.1	-0.5	1.2	2.4	3.7	4.9

u = 7.5	0.0090	0.0396	0.0594	0.0216	0.0440	0.0203
4.1	0.0495	0	0.1089	0.0528	0.0363	0.0231
-2.0	0.0405	0.1320	0.0891	0.0324	0.0297	0.0189
-3.8	0.0510	0.0484	0.0726	0.0132	0	0.0077

Determine $P(X^2 - 3X \le 0)$, $P(X^3 - 3|Y| < 3)$.

Answer

```
npr08_07
Data are in X, Y, P
jcalc
Enter JOINT PROBABILITIES (as on the plane)  P
Enter row matrix of VALUES of X  X
Enter row matrix of VALUES of Y  Y
 Use array operations on matrices X, Y, PX, PY, t, u, and P
M1 = t.^2 - 3*t <=0;
P1 = total(M1.*P)
P1 =  0.4500
M2 = t.^3 - 3*abs(u) < 3;
P2 = total(M2.*P)
P2 =  0.7876
```

Exercise 10.4.9

For the pair $\{X, Y\}$ in Exercise 10.4.8, let $Z = g(X, Y) = 3X^2 + 2XY - Y^2$. Determine and plot the distribution function for Z.

Answer

```
G = 3*t.^2 + 2*t.*u - u.^2;  % Determine g(X,Y)
[Z,PZ] = csort(G,P);         % Obtain dbn for Z = g(X,Y)
ddbn                         % Call for plotting m-procedure
Enter row matrix of VALUES  Z
Enter row matrix of PROBABILITIES  PZ   % Plot not reproduced here
```

Exercise 10.4.10

For the pair $\{X, Y\}$ in Exercise 8, let

$$W = g(X, Y) = \begin{cases} X & \text{for } X + Y \le 4 \\ 2Y & \text{for } X + Y > 4 \end{cases} = I_M(X, Y)X + I_{M^c}(X, Y)2Y$$

Determine and plot the distribution function for W.

Answer

```
H = t.*(t+u<=4) + 2*u.*(t+u>4);
[W,PW] = csort(H,P);
ddbn
Enter row matrix of VALUES  W
Enter row matrix of PROBABILITIES  PW   % Plot not reproduced here
```

For the distributions in Exercises 10-15 below

a. Determine analytically the indicated probabilities.
b. Use a discrete approximation to calculate the same probablities.'

Exercise 10.4.11

$f_{XY}(t,u) = \frac{3}{88}(2t+3u^2)$ for $0 \le t \le 2$, $0 \le u \le 1+t$ (see Exercise 15 from "Problems on Random Vectors and Joint Distributions").

$$Z = I_{[0,1]}(X)4X + I_{(1,2]}(X)(X+Y)$$

Determine $P(Z \le 2)$

Answer

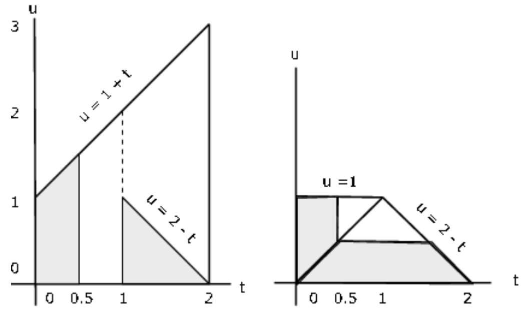
$$P(Z \le 2) = P(Z \in Q = Q1M1 \bigvee Q2M2) \text{, where } M1 = \{(t,u) : 0 \le t \le 1, 0 \le u \le 1+t\}$$

$$M2 = \{(t,u) : 1 < t \le 2, 0 \le u \le 1+t\}$$

$$Q1 = \{(t,u) : 0 \le t \le 1/2\}, Q2 = \{(t,u) : u \le 2-t\} \text{ (see figure)}$$

$$P = \frac{3}{88}\int_0^{1/2}\int_0^{1+t}(2t+3u^2)du\,dt + \frac{3}{88}\int_1^2\int_0^{2-t}(2t+3u^2)du\,dt = \frac{563}{5632}$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 3]
Enter number of X approximation points  200
Enter number of Y approximation points  300
Enter expression for joint density  (3/88)*(2*t + 3*u.^2).*(u<=1+t)
Use array operations on X, Y, PX, PY, t, u, and P
G = 4*t.*(t<=1) + (t+u).*(t>1);
[Z,PZ] = csort(G,P);
PZ2 = (Z<=2)*PZ'
PZ2 =  0.1010                     % Theoretical = 563/5632 = 0.1000
```



Problem P10-11   Problem P10-12

Figure 10.4.1

Exercise 10.4.12

$f_{XY}(t,u) = \frac{24}{11}$ for $0 \le t \le 2$, $0 \le u \le \min\{1, 2-t\}$ (see Exercise 17 from "Problems on Random Vectors and Joint Distributions").

$$Z = I_M(X,Y)\frac{1}{2}X + I_{M^c}(X,Y)Y^2 \text{ , } M = \{(t,u): u > t\}$$

Determine $P(Z \le 1/4)$.

Answer

$$P(Z \le 1/4) = P((X,Y) \in M_1Q_1 \bigvee M_2Q_2) \text{ , } M_1 = \{(t,u): 0 \le t \le u \le 1\}$$

$$M_2 = \{(t,u): 0 \le t \le 2, 0 \le t \le \min(t, 2-t)\}$$

$$Q_1 = \{(t,u): t \le 1/2\} \; Q_2 = \{(t,u): u \le 1/2\} \text{ (see figure)}$$

$$P = \frac{24}{11}\int_0^{1/2}\int_0^1 tu\,du\,dt + \frac{24}{11}\int_{1/2}^{3/2}\int_0^{1/2} tu\,du\,dt + \frac{24}{11}\int_{3/2}^{2}\int_0^{2-t} tu\,du\,dt = \frac{85}{176}$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 1]
Enter number of X approximation points  400
Enter number of Y approximation points  200
Enter expression for joint density  (24/11)*t.*u.*(u<=min(1,2-t))
Use array operations on X, Y, PX, PY, t, u, and P
G = 0.5*t.*(u>t) + u.^2.*(u<t);
[Z,PZ] = csort(G,P);
pp = (Z<=1/4)*PZ'
pp =  0.4844                        % Theoretical = 85/176 = 0.4830
```

Exercise 10.4.13

$f_{XY}(t,u) = \frac{3}{23}(t+2u)$ for $0 \le t \le 2$, $0 \le u \le \max\{2-t, t\}$ (see Exercise 18 from "Problems on Random Vectors and Joint Distributions").

$$Z = I_M(X,Y)(X+Y) + I_{M^c}(X,Y)2Y \text{ , } M = \{(t,u): \max(t,u) \le 1\}$$

Determine $P(Z \le 1)$

Answer

$$P(Z \le 1) = P((X,Y) \in M_1Q_1 \bigvee M_2Q_2) \text{ , } M_1 = \{(t,u): 0 \le t \le 1, 0 \le u \le 1-t\}$$

$$M_2 = \{(t,u): 1 \le t \le 2, 0 \le u \le t\}$$

$$Q_1 = \{(t,u): u \le 1-t\} \quad Q_2 = \{(t,u): u \le 1/2\} \text{ (see figure)}$$

$$P = \frac{3}{23}\int_0^1\int_0^{1-t}(t+2u)\,du\,dt + \frac{3}{23}\int_1^2\int_0^{1/2}(t+2u)\,du\,dt = \frac{9}{46}$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 2]
```

```
Enter number of X approximation points  300
Enter number of Y approximation points  300
Enter expression for joint density  (3/23)*(t + 2*u).*(u<=max(2-t,t))
Use array operations on X, Y, PX, PY, t, u, and P
M = max(t,u) <= 1;
G = M.*(t + u) + (1 - M)*2.*u;
p = total((G<=1).*P)
p =  0.1960                         % Theoretical = 9/46 = 0.1957
```

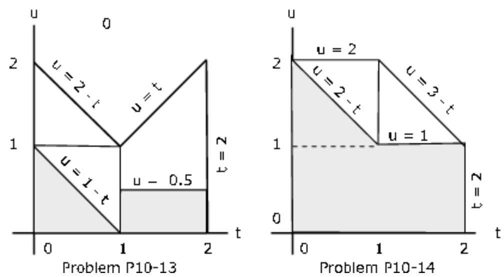


Figure 10.4.2

Exercise 10.4.14

$f_{XY}(t,u) = \frac{12}{179}(3t^2+u)$, for $0 \le t \le 2$, $0 \le u \le \min\{2, 3-t\}$ (see Exercise 19 from "Problems on Random Vectors and Joint Distributions").

$$Z = I_M(X,Y)(X+Y) + I_{M^c}(X,Y)2Y^2 \ \ , M = \{(t,u) : t \le 1, u \ge 1\}$$

Determine $P(Z \le 2)$.

Answer

$$P(Z \le 2) = P((X,Y) \in M_1Q_1 \bigvee (M_2 \bigvee M_3)Q_2) \ \ , M_1 = \{(t,u) : 0 \le t \le 1, 1 \le u \le 2\}$$

$$M_2 = \{(t,u) : 0 \le t \le 1, 0 \le u \le 1\} \ \ M_3 = \{(t,u) : 1 \le t \le 2, 0 \le u \le 3-t\}$$

$$Q_1 = \{(t,u) : u \le 1-t\} \ \ Q_2 = \{(t,u) : u \le 1/2\} \text{ (see figure)}$$

$$P = \frac{12}{179}\int_0^1\int_0^{2-t}(3t^2+u)du\,dt + \frac{12}{179}\int_1^2\int_0^1(3t^2+u)du\,dt = \frac{119}{179}$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 2]
Enter number of X approximation points  300
Enter number of Y approximation points  300
Enter expression for joint density  (12/179)*(3*t.^2 + u).*(u<=min(2,3-t))
Use array operations on X, Y, PX, PY, t, u, and P
M = (t<=1)&(u>=1);
Z = M.*(t + u) + (1 - M)*2.*u.^2;
G = M.*(t + u) + (1 - M)*2.*u.^2;
p = total((G<=2).*P)
p =  0.6662                          % Theoretical = 119/179 = 0.6648
```

Exercise 10.4.15

$f_{XY}(t,u) = \frac{12}{227}(3t+2tu)$, for $0 \le t \le 2$, $0 \le u \le \min\{1+t, 2\}$ (see Exercise 20 from "Problems on Random Variables and joint Distributions")

$Z = I_M(X,Y)X + I_{M^c}(X,Y)\frac{Y}{X}$, $M = \{(t,u) : u \le \min(1, 2-t)\}$

Determine $P(Z \le 1)$.

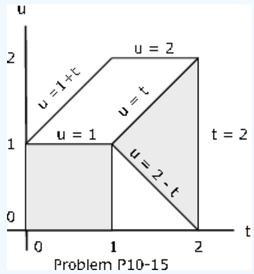


Figure 10.4.3

Answer

$$P(Z \le 1) = P((X,Y) \in M_1Q_1 \bigvee V_2Q_2) \;\; , M_1 = M \, , M_2 = M^c$$

$$Q_1 = \{(t,u) : 0 \le t \le\} \;\; Q_2 = \{(t,u) : u \le t\} \;\; \text{(see figure)}$$

$$P = \frac{12}{227}\int_0^1 \int_0^1 (3t+2tu)du\, dt + \frac{12}{227}\int_1^2 \int_{2-t}^t (3t+2tu)du\, dt = \frac{124}{227}$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 2]
Enter number of X approximation points  400
Enter number of Y approximation points  400
Enter expression for joint density  (12/227)*(3*t+2*t.*u).*(u<=min(1+t,2))
Use array operations on X, Y, PX, PY, t, u, and P
Q = (u<=1).*(t<=1) + (t>1).*(u>=2-t).*(u<=t);
P = total(Q.*P)
P =  0.5478                       % Theoretical = 124/227 = 0.5463
```

Exercise 10.4.16

The class $\{X, Y, Z\}$ is independent.

$X = -2I_A + I_B + 3I_C$. Minterm probabilities are (in the usual order)

0.255 0.025 0.375 0.045 0.108 0.012 0.162 0.018

$Y = I_D + 3I_E + I_F - 3$. The class $\{D, E, F\}$ is independent with

$$P(D) = 0.32 \; P(E) = 0.56 \; P(F) = 0.40$$

Z has distribution

Value	-1.3	1.2	2.7	3.4	5.8
Probability	0.12	0.24	0.43	0.13	0.08

Determine $P(X^2 + 3XY^2 > 3Z)$.

Answer

```
% file npr10_16.m  Data for Exercise 16.
cx = [-2 1 3 0];
pmx = 0.001*[255  25 375  45 108  12 162  18];
cy = [1 3 1 -3];
pmy = minprob(0.01*[32 56 40]);
Z = [-1.3 1.2 2.7 3.4 5.8];
PZ = 0.01*[12 24 43 13  8];
disp('Data are in cx, pmx, cy, pmy, Z, PZ')
npr10_16                % Call for data
Data are in cx, pmx, cy, pmy, Z, PZ
[X,PX] = canonicf(cx,pmx);
[Y,PY] = canonicf(cy,pmy);
icalc3
Enter row matrix of X-values  X
Enter row matrix of Y-values  Y
Enter row matrix of Z-values  Z
Enter X probabilities  PX
Enter Y probabilities  PY
Enter Z probabilities  PZ
Use array operations on matrices X, Y, Z,
PX, PY, PZ, t, u, v, and P
M = t.^2 + 3*t.*u.^2 > 3*v;
PM = total(M.*P)
PM =  0.3587
```

Exercise 10.4.17

The simple random variable X has distribution

$$X = \text{[-3.1 -0.5 1.2 2.4 3.7 4.9]} \; PX = \text{[0.15 0.22 0.33 0.12 0.11 0.07]}$$

a. Plot the distribution function F_X and the quantile function Q_X.
b. Take a random sample of size $n =$ 10,000. Compare the relative frequency for each value with the probability that value is taken on.

Answer

```
X = [-3.1 -0.5 1.2 2.4 3.7 4.9];
PX = 0.01*[15 22 33 12 11  7];
ddbn
Enter row matrix of VALUES  X
Enter row matrix of PROBABILITIES  PX  % Plot not reproduced here
dquanplot
Enter VALUES for X  X
Enter PROBABILITIES for X  PX          % Plot not reproduced here
rand('seed',0)                      % Reset random number generator
dsample                             % for comparison purposes
```

```
Enter row matrix of VALUES  X
Enter row matrix of PROBABILITIES  PX
Sample size n  10000
    Value      Prob    Rel freq
   -3.1000    0.1500    0.1490
   -0.5000    0.2200    0.2164
    1.2000    0.3300    0.3340
    2.4000    0.1200    0.1184
    3.7000    0.1100    0.1070
    4.9000    0.0700    0.0752
Sample average ex = 0.8792
Population mean E[X] = 0.859
Sample variance vx = 5.146
Population variance Var[X] = 5.112
```