17.6: Appendix F to Applied Probability- Properties of conditional expectation, given a random vector

We suppose, without repeated assertion, that the random variables and functions of random vectors are integrable, as needed.

(CE1): **Defining condition**. $e(X) = E[g(Y)|X]$ a.s. iff $E[I_M(X)g(Y)] = E[I_M(X)e(X)]$ for each Borel set M on the codomain of X.

(CE1a): If $P(X \in M) > 0$, then $E[I_M(X)e(X)] = E[g(Y)|X \in M]P(X \in M)$

(CE1b): Law of total probability. $E[g(Y)] = E\{[g(Y)|X]\}$

(CE2): **Linearity**. For any constants a, b

$E[ag(Y) + bh(Z)|X] = aE[g(Y)|X] + bE[h(Z)|X]$ a.s.

(Extends to any finite linear combination)

(CE3): positivity; monotonicty.

a. $g(Y) \ge 0$ a.s. implies $E[g(Y)|X] \ge 0$ a.s.

b. $g(Y) \ge h(Z)$ a.s. implies $E[g(Y)|X] \ge E[h(Z)|X]$ a.s.

(CE4): **Monotone convergence**. $Y_n \to Y$ a.s. monotonically implies $E[Y_n|X] \to E[Y|X]$ a.s.

(CE5): **Independence**. $\{X, Y\}$ is an independent pair

a. iff $E[g(Y)|X] = E[g(Y)]$ a.s. for all Borel functions g

b. iff $E[I_N(Y)|X] = E[I_N(Y)]$ a.s. for all Borel sets N on the codomain of Y

(CE6): $e(X) = E[g(Y)|X]$ a.s. iff $E[h(X)g(Y)] = E[h(X)e(X)]$ a.s. for any Borel function h

(CE7): $E[h(X)|X] = h(X)$ a.s. for any Borel function h

(CE8): $E[h(X)g(Y)|X] = h(X)E[g(Y)|X]$ a.s. for any Borel function h

(CE9): If $X = h(W)$ and $W = k(X)$, with h, k Borel functions, then $E[g(Y)|X] = E[g(Y)|W]$ a.s.

(CE10): If $g$is a Borel function such that $E[g(t, Y)]$ is finite for all t on the range of X and $E[g(X, Y)]$ is finite, then

a. $E[g(X, Y)|X = t] = E[g(t, Y)|X = t]$ a.s. $[P_X]$

b. If $\{X, Y\}$ is independent, then $E[g(X, Y)|X = t] = E[g(t, Y)]$ a.s. $[P_X]$

(CE11): Suppose $\{X(t) : t \in T\}$ is a real-valued measurable random process whose parameter set T is a Borel subset of the real line and S is a random variable whose range is a subset of T, so that $X(S)$ is a random variable.

If $E[X(t)]$ is finite for all t in T and $E[X(S)]$ is finite, then

a. \9E[X(S)|S = t] = E[X(t)|S = t]\) a.s $[P_S]$

b. If, in addition, $\{S, X_T\}$ is independent, then $E[X(S)|S = t] = E[X(t)]$ a.s. $[P_S]$

(CE12): **Countable additivity and countable sums**.

a. If Y is integrable on A and $A = \bigvee_{n=1}^{\infty} A_n$.

then $E[I_A Y|X] = \sum_{n=1}^{\infty} E[I_A Y|X]$ a.s.

b. If $\sum_{n=1}^{\infty} E[|Y_n|] < \infty$, thne $E[\sum_{n=1}^{\infty} Y_n|X]$ a.s.

(CE13): **Triangle inequality**. $|E[g(Y)|X]| \le E[|g(Y)||X]$ a.s.

(CE14): **Jensen's inequality**. If g is a convex function on an interval I which contains the range of a real random variable Y, then $g\{E[Y|X]\} \le E[g(Y)|X]$ a.s.

(CE15): Suppose $E[|Y|^p] < \infty$ and $E[|Z|^p] < \infty$ for $1 \le p < \infty$. Then $E\{|E[Y|X] - E[Z|X]|^p\} \le E[|Y - Z|^p] < \infty$