3.2: Problems on Conditional Probability

Exercise 3.2.1

Given the following data:

$P(A) = 0.55$, $P(AB) = 0.30$, $P(BC) = 0.20$, $P(A^c \cup BC) = 0.55$, $P(A^c BC^c) = 0.15$

Determine, if possible, the conditional probability $P(A^c|B) = P(A^c B)/P(B)$.

Answer

```
% file npr03_01.m
% Data for Exercise 3.2.1.
minvec3
DV = [A|Ac; A;  A&B; B&C; Ac|(B&C); Ac&B&Cc];
DP = [ 1   0.55 0.30 0.20   0.55     0.15  ];
TV = [Ac&B; B];
disp('Call for mincalc')
npr03_01
Variables are A, B, C, Ac, Bc, Cc
They may be renamed, if desired.
Call for mincalc
mincalc
Data vectors are linearly independent
 Computable target probabilities
    1.0000    0.2500
    2.0000    0.5500
The number of minterms is 8
The number of available minterms is 4
- - - - - - - - - - - -
P = 0.25/0.55
P =  0.4545
```

Exercise 3.2.2

In Exercise 11 from "Problems on Minterm Analysis," we have the following data: A survey of a represenative group of students yields the following information:

- 52 percent are male
- 85 percent live on campus
- 78 percent are male or are active in intramural sports (or both)
- 30 percent live on campus but are not active in sports
- 32 percent are male, live on campus, and are active in sports
- 8 percent are male and live off campus
- 17 percent are male students inactive in sports

Let A = male, B = on campus, C = active in sports.

a. A student is selected at random. He is male and lives on campus. What is the (conditional) probability that he is active in sports?
b. A student selected is active in sports. What is the(conditional) probability that she is a female who lives on campus?

Answer

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npr02_11
- - - - - - - - - - - -
mincalc
- - - - - - - - - - - -
mincalct
Enter matrix of target Boolean combinations  [A&B&C; A&B; Ac&B&C; C]
 Computable target probabilities
    1.0000    0.3200
    2.0000    0.4400
    3.0000    0.2300
    4.0000    0.6100
PC_AB = 0.32/0.44
PC_AB =  0.7273
PAcB_C = 0.23/0.61
PAcB_C = 0.3770
```

Exercise 3.2.3

In a certain population, the probability a woman lives to at least seventy years is 0.70 and is 0.55 that she will live to at least eighty years. If a woman is seventy years old, what is the conditional probability she will survive to eighty years? *Note* that if $A \subset B$ then $P(AB) = P(A)$.

Answer

Let $A =$ event she lives to seventy and $B =$ event she lives to eighty. Since $B \subset A$, $P(B|A) = P(AB)/P(A) = P(B)/P(A) = 55/70$.

Exercise 3.2.4

From 100 cards numbered 00, 01, 02, $\cdots$, 99, one card is drawn. Suppose A_i is the event the sum of the two digits on a card is i, $0 \le i \le 18$, and B_j is the event the product of the two digits is j. Determine $P(A_i|B_0)$ for each possible i.

Answer

B_0 is the event one of the first ten is draw. $A_i B_0$ is the event that the card with numbers $0i$ is drawn. $P(a_i|B_0) = (1/100)/(1/10) = 1/10$for each i, 0 through 9.

Exercise 3.2.5

Two fair dice are rolled.

a. What is the (conditional) probability that one turns up two spots, given they show different numbers?
b. What is the (conditional) probability that the first turns up six, given that the sum is k, for each k from two through 12?
c. What is the (conditional) probability that at least one turns up six, given that the sum is k, for each k from two through 12?

Answer

a. There are 6×5 ways to choose all different. There are 2×5 ways that they are different and one turns up two spots. The conditional probability is 2/6.

b. Let A_6 = event first is a six and $S_k =$ event the sum is k. Now $A_6 S_k = \emptyset$ for $k \le 6$. A table of sums shows $P(A_6 S_k) = 1/36$ and $P(S_k) = 6/36, 5/36, 4/36, 3/36, 2/36, 1/36$ for $k = 7$ through 12, respectively. Hence $P(A_6|S_k) = 1/6, 1/5.1/4, 1/3.1/2, 1$, respectively.

c. If AB_6 is the event at least one is a six, then $AB_6 S_k) = 2/36$ for $k = 7$ through 11 and $P(AB_6 S_1 2) = 1/36$. Thus, the conditional probabilities are 2/6, 2/5, 2/4, 2/3, 1, 1, respectively.

Exercise 3.2.6

Four persons are to be selected from a group of 12 people, 7 of whom are women.

a. What is the probability that the first and third selected are women?
b. What is the probability that three of those selected are women?
c. What is the (conditional) probability that the first and third selected are women, given that three of those selected are women?

Answer

$$P(W_1 W_3) = P(W_1 W_2 W_3) + P(W_1 W_2^c W_3) = \frac{7}{12} \cdot \frac{6}{11} \cdot \frac{5}{10} + \frac{7}{12} \cdot \frac{5}{11} \cdot \frac{6}{10} = \frac{7}{22}$$

Exercise 3.2.7

Twenty percent of the paintings in a gallery are not originals. A collector buys a painting. He has probability 0.10 of buying a fake for an original but never rejects an original as a fake, What is the (conditional) probability the painting he purchases is an original?

Answer

Let $B =$ the event the collector buys, and $G =$ the event the painting is original. Assume $P(B|G) = 1$ and $P(B|G^c) = 0.1$. If $P(G) = 0.8$, then

$$P(G|B) = \frac{P(GB)}{P(B)} = \frac{P(B|G)P(G)}{P(B|G)P(G) + P(B|G^c)P(G^c)} = \frac{0.8}{0.8 + 0.1 \cdot 0.2} = \frac{40}{41}$$

Exercise 3.2.8

Five percent of the units of a certain type of equipment brought in for service have a common defect. Experience shows that 93 percent of the units with this defect exhibit a certain behavioral characteristic, while only two percent of the units which do not have this defect exhibit that characteristic. A unit is examined and found to have the characteristic symptom. What is the conditional probability that the unit has the defect, given this behavior?

Answer

Let $D =$ the event the unit is defective and $C =$ the event it has the characteristic. Then $P(D) = 0.05$, $P(C|D) = 0.93$, and $P(C|D^c) = 0.02$.

$$P(D|C) = \frac{P(C|D)P(D)}{P(C|D)P(D) + P(C|D^c)P(D^c)} = \frac{0.93 \cdot 0.05}{0.93 \cdot 0.05 + 0.02 \cdot 0.95} = \frac{93}{131}$$

Exercise 3.2.9

A shipment of 1000 electronic units is received. There is an equally likely probability that there are 0, 1, 2, or 3 defective units in the lot. If one is selected at random and found to be good, what is the probability of no defective units in the lot?

Answer

Let $D_k =$ the event of k defective and G be the event a good one is chosen.

$$P(D_0|G) = \frac{P(G|D_0)P(D_0)}{P(G|D_0)P(D_0)+P(G|D_1)P(D_1)+P(G|D_2)P(D_2)+P(G|D_3)P(D_3)}$$

$$= \frac{1 \cdot 1/4}{(1/4)(1+999/1000+998/1000+997/1000)} = \frac{1000}{3994}$$

Exercise 3.2.10

Data on incomes and salary ranges for a certain population are analyzed as follows. S_1= event annual income is less than \$25,000; S_2= event annual income is between \$25,000 and \$100,000; S_3= event annual income is greater than \$100,000. E_1= event did not complete college education; E_2= event of completion of bachelor's degree; E_3= event of completion of graduate or professional degree program. Data may be tabulated as follows: $P(E_1) = 0.65$, $P(E_2) = 0.30$ and $P(E_3) = 0.05$.

$P(S_i|E_j)$

	S_1	S_2	S_3
E_1	0.85	0.10	0.05
E_2	0.10	0.80	0.10
E_3	0.05	0.50	0.45
$P(S_i)$	0.50	0.40	0.10

a. Determine $P(E_3S_3)$.
b. Suppose a person has a university education (no graduate study). What is the (conditional) probability that he or she will make \$25,000 or more?
c. Find the total probability that a person's income category is at least as high as his or her educational level.

Answer

a. $P(E_3S_3) = P(S_3|E_3)P(E_3) = 0.45 \cdot 0.05 = 0.0225$

b. $P(S_2 \vee S_3|E_2) = 0.80 + 0.10 = 0.90$

c. $p = (0.85 + 0.10 + 0.05) \cdot 0.65 + (0.80 + 0.10) \cdot 0.30 + 0.45 \cdot 0.05 = 0.9425$

Exercise 3.2.11

In a survey, 85 percent of the employees say they favor a certain company policy. Previous experience indicates that 20 percent of those who do not favor the policy say that they do, out of fear of reprisal. What is the probability that an employee picked at random really does favor the company policy? It is reasonable to assume that all who favor say so.

Answer

$P(S) = 0.85$, $P(S|F^c) = 0.20$. Also, reasonable to assume $P(S|F) = 1$.

$$P(S) = P(S|F)P(F) + P(S|F^c)[1 - P(F)] \quad \text{implies } P(F) = \frac{P(S) - P(S|F^c)}{1 - P(S|F^c)} = \frac{13}{16}$$

Exercise 3.2.12

A quality control group is designing an automatic test procedure for compact disk players coming from a production line. Experience shows that one percent of the units produced are defective. The automatic test procedure has probability 0.05 of giving a false positive indication and probability 0.02 of giving a false negative. That is, if D is the event a unit tested is defective, and T is the event that it tests satisfactory, then $P(T|D) = 0.05$ and $P(T^c|D^c) = 0.02$. Determine the probability $P(D^c|T)$ that a unit which tests good is, in fact, free of defects.

Answer

$$\frac{D^c|T}{P(D|T)} = \frac{P(T|D^c)P(D^c)}{P(T|D)P(D)} = \frac{0.98 \cdot 0.99}{0.05 \cdot 0.01} = \frac{9702}{5}$$

$$P(D^c|T) = \frac{9702}{9707} = 1 - \frac{5}{9707}$$

Exercise 3.2.13

Five boxes of random access memory chips have 100 units per box. They have respectively one, two, three, four, and five defective units. A box is selected at random, on an equally likely basis, and a unit is selected at random therefrom. It is defective. What are the (conditional) probabilities the unit was selected from each of the boxes?

Answer

$H_i =$ the event from box i. $P(H_i) = 1/5$ and $P(D|H_i) = i/100$.

$$P(H_i|D) = \frac{P(D|H_i)P(H_i)}{\sum P(D|H_i)P(H_j)} = i/15, 1 \le i \le 5$$

Exercise 3.2.14

Two percent of the units received at a warehouse are defective. A nondestructive test procedure gives two percent false positive indications and five percent false negative. Units which fail to pass the inspection are sold to a salvage firm. This firm applies a corrective procedure which does not affect any good unit and which corrects 90 percent of the defective units. A customer buys a unit from the salvage firm. It is good. What is the (conditional) probability the unit was originally defective?

Answer

Let T = event test indicates defective, D = event initially defective, and $G =$ event unit purchased is good. Data are

$$P(D) = 0.02,\ P(T^c|D) = 0.02,\ P(T|D^c) = 0.05,\ P(GT^c) = 0,$$

$$P(G|DT) = 0.90,\ P(G|D^cT) = 1$$

$$P(D|G) = \frac{P(GD)}{P(G)},\ P(GD) = P(GTD) = P(D)P(T|D)P(G|TD)$$

$$P(G) = P(GT) = P(GDT) + P(GD^cT) = P(D)P(T|D)P(G|TD) + P(D^c)P(T|D^c)P(G|TD^c)$$

$$P(D|G) = \frac{0.02 \cdot 0.98 \cdot 0.90}{0.02 \cdot 0.98 \cdot 0.90 + 0.98 \cdot 0.05 \cdot 1.00} = \frac{441}{1666}$$

Exercise 3.2.15

At a certain stage in a trial, the judge feels the odds are two to one the defendent is guilty. It is determined that the defendent is left handed. An investigator convinces the judge this is six times more likely if the defendent is guilty than if he were not. What is the likelihood, given this evidence, that the defendent is guilty?

Answer

Let G = event the defendent is guilty, L = the event the defendent is left handed. Prior odds: $P(G)/P(G^c) = 2$. Result of testimony: $P(L|G)/P(L|G^c) = 6$.

$$\frac{P(G|L)}{P(G^c|L)} = \frac{P(G)}{P(G^c)} \cdot \frac{P(L|G)}{P(L|G^c)} = 2 \cdot 6 = 12$$

$$P(G|L) = 12/13$$

Exercise 3.2.16

Show that if $P(A|C) > P(B|C)$ and $P(A|C^c) > P(B|C^c)$, then $P(A) > P(B)$. Is the converse true? Prove or give a counterexample.

Answer

$P(A) = P(A|C)P(C) + P(A|C^c)P(C^c) > P(B|C)P(C) + P(B|C^c)P(C^c) = P(B)$.

The converse is not true. Consider $P(C) = P(C^c) = 0.5$, $P(A|C) = 1/4$.

$P(A|C^c) = 3/4$, $P(B|C) = 1/2$, and $P(B|C^c) = 1/4$. Then

$$1/2 = P(A) = \frac{1}{2}(1/4 + 3/4) > \frac{1}{2}(1/2 + 1/4) = P(B) = 3/8$$

But $P(A|C) < P(B|C)$.

Exercise 3.2.17

Since $P(\cdot|B)$ is a probability measure for a given B, we must have $P(A|B) + P(A^c|B) = 1$. Construct an example to show that in general $P(A|B) + P(A|B^c) \neq 1$.

Answer

Suppose $A \subset B$ with $P(A) < P(B)$. Then $P(A|B) = P(A)/P(B) < 1$ and $P(A|B^c) = 0$ so the sum is less than one.

Exercise 3.2.18

Use property (CP4) to show

a. $P(A|B) > P(A)$ iff $P(A|B^c) < P(A)$

b. $P(A^c|B) > P(A^c)$ iff $P(A|B) < P(A)$

c. $P(A|B) > P(A)$ iff $P(A^c|B^c) > P(A^c)$

Answer

a. $P(A|B) > P(A)$ iff $P(AB) > P(A)P(B)$ iff $P(AB^c) < P(A)P(B^c)$ iff $P(A|B^c) < P(A)$

b. $P(A^c|B) > P(A^c)$ iff $P(A^cB) > P(A^c)P(B)$ iff $P(AB) < P(A)P(B)$ iff $P(A|B) < P(A)$

c. $P(A|B) > P(A)$ iff $P(AB) > P(A)P(B)$ iff $P(A^cB^c) > P(A^c)P(B^c)$ iff $P(A^c|B^c) > P(A^c)$

Exercise 3.2.19

Show that $P(A|B) \geq (P(A) + P(B) - 1)/P(B)$.

Answer

$1 \geq P(A \cup B) = P(A) + P(B) - P(AB) = P(A) + P(B) - P(A|B)P(B)$. Simple algebra gives the desired result.

Exercise 3.2.20

Show that $P(A|B) = P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B)$.

Answer

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(ABC) + P(ABC^c)}{P(B)}$$

$$= \frac{P(A|BC)P(BC) + P(A|BC^c)P(BC^c)}{P(B)} = P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B)$$

Exercise 3.2.21

An individual is to select from among n alternatives in an attempt to obtain a particular one. This might be selection from answers on a multiple choice question, when only one is correct. Let A be the event he makes a correct selection, and B be the event he knows which is correct before making the selection. We suppose $P(B) = p$ and $P(A|B^c) = 1/n$. Determine $P(B|A)$; show that $P(B|A) \geq P(B)$ and $P(B|A)$ increases with n for fixed p.

Answer

$P(A|B) = 1$, $P(A|B^c) = 1/n$, $P(B) = p$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)} = \frac{p}{p + \frac{1}{n}(1-p)} = \frac{np}{(n-1)p+1}$$

$$\frac{P(B|A)}{P(B)} = \frac{n}{np+1-p} \text{ increases from 1 to } 1/p \text{ as } n \to \infty$$

Exercise 3.2.22

Polya's urn scheme for a contagious disease. An urn contains initially b black balls and r red balls $(r + b = n)$. A ball is drawn on an equally likely basis from among those in the urn, then replaced along with c additional balls of the same color. The process is repeated. There are n balls on the first choice, $n + c$ balls on the second choice, etc. Let B_k be the event of a black ball on the kth draw and R_k be the event of a red ball on the kth draw. Determine

a. $P(B_2|R_1)$
b. $P(B_1B_2)$
c. $P(R_2)$
d. $P(B_1|R_2)$

Answer

a. $P(B_2|R_1) = \frac{b}{n+c}$

b. $P(B_1B_2) = P(B_2)P(B_2|B_1) = \frac{b}{n} \cdot \frac{b+c}{n+c}$

c. $P(R_2)P(R_2|R_1)P(R_1) + P(R_2|B_1)P(B_1)$

$$= \frac{r+c}{n+c} \cdot \frac{r}{n} + \frac{r}{n+c} \cdot \frac{b}{n} = \frac{r(r+c+b)}{n(n+c)}$$

d. $P(B_1|R_2) = \frac{P(R_2|B_1)P(B_1)}{P(R_2)}$ with $P(R_2|B_1)P(B_1) = \frac{r}{n+c} \cdot \frac{b}{n}$. Using (c), we have

$$P(B_1|R_2) = \frac{b}{r+b+c} = \frac{b}{n+c}$$