8.3: Problems on Random Vectors and Joint Distributions

Exercise 8.3.1

Two cards are selected at random, without replacement, from a standard deck. Let X be the number of aces and Y be the number of spades. Under the usual assumptions, determine the joint distribution and the marginals.

Answer

Let X be the number of aces and Y be the number of spades. Define the events AS_i, A_i, S_i, and N_i, $i = 1, 2$ of drawing ace of spades, other ace, spade (other than the ace), and neither on the *i* selection. Let $P(i, k) = P(X = i, Y = k)$.

$$P(0,0) = P(N_1 N_2) = \dfrac{36}{52} \cdot \dfrac{35}{51} = \dfrac{1260}{2652}$$

$$P(0,1) = P(N_1 S_2 \bigvee S_1 N_2) = \dfrac{36}{52} \cdot \dfrac{12}{51} + \dfrac{12}{52} \cdot \dfrac{36}{51} = \dfrac{864}{2652}$$

$$P(0,2) = P(S_1 S_2) = \dfrac{12}{52} \cdot \dfrac{11}{51} = \dfrac{132}{2652}$$

P(1, 0) = P(A_N_2 \bigvee N_1 S_2) = \dfrac{3}{52} \cdot \dfrac{36}{51} + \dfrac{36}{52} \cdot \dfrac{3}{51} = \dfrac{216}{2652}

$$P(1,1) = P(A_1 S_2 \bigvee S_1 A_2 \bigvee AS_1 N_2 \bigvee N_1 AS_2) = \dfrac{3}{52} \cdot \dfrac{12}{51} + \dfrac{12}{52} \cdot \dfrac{3}{51} + \dfrac{1}{52} \cdot \dfrac{36}{51} + \dfrac{36}{52} \cdot \dfrac{1}{51} = \dfrac{144}{2652}$$

$$P(1,2) = P(AS_1 S_2 \bigvee S_1 AS_2) = \dfrac{1}{52} \cdot \dfrac{12}{51} + \dfrac{12}{52} \cdot \dfrac{1}{51} = \dfrac{24}{2652}$$

$$P(2,0) = P(A_1 A_2) = \dfrac{3}{52} \cdot \dfrac{2}{51} = \dfrac{6}{2652}$$

$$P(2,1) = P(AS_1 A_2 \bigvee A_1 AS_2) = \dfrac{1}{52} \cdot \dfrac{3}{51} + \dfrac{3}{52} \cdot \dfrac{1}{51} = \dfrac{6}{2652}$$

$$P(2,2) = P(\emptyset) = 0$$

```
% type npr08_01
% file npr08_01.m
% Solution for Exercise 8.3.1.
X = 0:2;
Y = 0:2;
Pn = [132  24   0; 864 144  6; 1260 216 6];
P = Pn/(52*51);
disp('Data in Pn, P, X, Y')

npr08_01         % Call for mfile
Data in Pn, P, X, Y    % Result
PX = sum(P)
PX =  0.8507    0.1448    0.0045
PY = fliplr(sum(P'))
PY =  0.5588    0.3824    0.0588
```

Exercise 8.3.2

Two positions for campus jobs are open. Two sophomores, three juniors, and three seniors apply. It is decided to select two at random (each possible pair equally likely). Let X be the number of sophomores and Y be the number of juniors who are selected. Determine the joint distribution for the pair $\{X, Y\}$ and from this determine the marginals for each.

Answer

Let A_i, B_i, C_i be the events of selecting a sophomore, junior, or senior, respectively, on the ith trial. Let X be the number of sophomores and Y be the number of juniors selected.

Set $P(i,k) = P(X = i, Y = k)$

$P(0,0) = P(C_1 C_2) = \frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56}$

$P(0,1) = P(B_1 C_2) + P(C_1 B_2) = \frac{3}{8} \cdot \frac{3}{7} + \frac{3}{8} \cdot \frac{3}{7} = \frac{18}{56}$

$P(0,2) = P(B_1 B_2) = \frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56}$

$P(1,0) = P(A_1 C_2) + P(C_1 A_2) = \frac{2}{8} \cdot \frac{3}{7} + \frac{3}{8} \cdot \frac{2}{7} = \frac{12}{56}$

$P(1,1) = P(A_1 B_2) + P(B_1 A_2) = \frac{2}{8} \cdot \frac{3}{7} + \frac{3}{8} \cdot \frac{2}{7} = \frac{12}{56}$

$P(2,0) = P(A_1 A_2) = \frac{2}{8} \cdot \frac{1}{7} = \frac{2}{56}$

$P(1,2) = P(2,1) = P(2,2) = 0$

$PX =$ [30/56 24/56 2/56] $PY =$ [20/56 30/56 6/56]

```
% file npr08_02.m
% Solution for Exercise 8.3.2.
X = 0:2;
Y = 0:2;
Pn = [6 0 0; 18 12 0; 6 12 2];
P = Pn/56;
disp('Data are in X, Y,Pn, P')
npr08_02
Data are in X, Y,Pn, P
PX = sum(P)
PX =  0.5357    0.4286    0.0357
PY = fliplr(sum(P'))
PY =  0.3571    0.5357    0.1071
```

Exercise 8.3.3

A die is rolled. Let X be the number that turns up. A coin is flipped X times. Let Y be the number of heads that turn up. Determine the joint distribution for the pair $\{X, Y\}$. Assume $P(X = k) = 1/6$ for $1 \le k \le 6$ and for each k, $P(Y = j|X = k)$ has the binomial (k, 1/2) distribution. Arrange the joint matrix as on the plane, with values of Y increasing upward. Determine the marginal distribution for Y. (For a MATLAB based way to determine the joint distribution see Example 14.1.7 from "Conditional Expectation, Regression")

Answer

$P(X = i, Y = k) = P(X = i) P(Y = k|X = i) = (1/6) P(Y = k|X = i)$.

```
% file npr08_03.m
% Solution for Exercise 8.3.3.
X = 1:6;
Y = 0:6;
```

```
P0 = zeros(6,7);       % Initialize
for i = 1:6            % Calculate rows of Y probabilities
    P0(i,1:i+1) = (1/6)*ibinom(i,1/2,0:i);
end
P = rot90(P0);         % Rotate to orient as on the plane
PY = fliplr(sum(P'));  % Reverse to put in normal order
disp('Answers are in X, Y, P, PY')
npr08_03            % Call for solution m-file
Answers are in X, Y, P, PY
disp(P)
         0         0         0         0         0    0.0026
         0         0         0         0    0.0052    0.0156
         0         0         0    0.0104    0.0260    0.0391
         0         0    0.0208    0.0417    0.0521    0.0521
         0    0.0417    0.0625    0.0625    0.0521    0.0391
    0.0833    0.0833    0.0625    0.0417    0.0260    0.0156
    0.0833    0.0417    0.0208    0.0104    0.0052    0.0026
disp(PY)
     0.1641  0.3125  0.2578  0.1667  0.0755  0.0208  0.0026
```

Exercise 8.3.4

As a variation of Exercise 8.3.3., Suppose a pair of dice is rolled instead of a single die. Determine the joint distribution for the pair $\{X, Y\}$ and from this determine the marginal distribution for Y.

Answer

```
% file npr08_04.m
% Solution for Exercise 8.3.4.
X = 2:12;
Y = 0:12;
PX = (1/36)*[1 2 3 4 5 6 5 4 3 2 1];
P0 = zeros(11,13);
for i = 1:11
    P0(i,1:i+2) = PX(i)*ibinom(i+1,1/2,0:i+1);
end
P = rot90(P0);
PY = fliplr(sum(P'));
disp('Answers are in X, Y, PY, P')
npr08_04
Answers are in X, Y, PY, P
disp(P)
  Columns 1 through 7
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
```

```
          0          0          0          0          0          0     0.0005
          0          0          0          0          0     0.0013     0.0043
          0          0          0          0     0.0022     0.0091     0.0152
          0          0          0     0.0035     0.0130     0.0273     0.0304
          0          0     0.0052     0.0174     0.0326     0.0456     0.0380
          0     0.0069     0.0208     0.0347     0.0434     0.0456     0.0304
     0.0069     0.0208     0.0312     0.0347     0.0326     0.0273     0.0152
     0.0139     0.0208     0.0208     0.0174     0.0130     0.0091     0.0043
     0.0069     0.0069     0.0052     0.0035     0.0022     0.0013     0.0005
  Columns 8 through 11
          0          0          0     0.0000
          0          0     0.0000     0.0001
          0     0.0001     0.0003     0.0004
     0.0002     0.0008     0.0015     0.0015
     0.0020     0.0037     0.0045     0.0034
     0.0078     0.0098     0.0090     0.0054
     0.0182     0.0171     0.0125     0.0063
     0.0273     0.0205     0.0125     0.0054
     0.0273     0.0171     0.0090     0.0034
     0.0182     0.0098     0.0045     0.0015
     0.0078     0.0037     0.0015     0.0004
     0.0020     0.0008     0.0003     0.0001
     0.0002     0.0001     0.0000     0.0000
disp(PY)
  Columns 1 through 7
     0.0269     0.1025     0.1823     0.2158     0.1954     0.1400     0.0806
  Columns 8 through 13
     0.0375     0.0140     0.0040     0.0008     0.0001     0.0000
```

Exercise 8.3.5

Suppose a pair of dice is rolled. Let X be the total number of spots which turn up. Roll the pair an additional X times. Let Y be the number of sevens that are thrown on the X rolls. Determine the joint distribution for the pair $\{X, Y\}$ and from this determine the marginal distribution for Y. What is the probability of three or more sevens?

Answer

```
% file npr08_05.m
% Data and basic calculations for Exercise 8.3.5.
PX = (1/36)*[1 2 3 4 5 6 5 4 3 2 1];
X = 2:12;
Y = 0:12;
P0 = zeros(11,13);
for i = 1:11
  P0(i,1:i+2) = PX(i)*ibinom(i+1,1/6,0:i+1);
end
P = rot90(P0);
```

```
PY = fliplr(sum(P'));
disp('Answers are in X, Y, P, PY')
npr08_05
Answers are in X, Y, P, PY
disp(PY)
  Columns 1 through 7
    0.3072    0.3660    0.2152    0.0828    0.0230    0.0048    0.0008
  Columns 8 through 13
    0.0001    0.0000    0.0000    0.0000    0.0000    0.0000
```

Exercise 8.3.6

The pair $\{X, Y\}$ has the joint distribution (in m-file npr08_06.m):

$X =$ [-2.3 -0.7 1.1 3.9 5.1] $Y =$ = [1.3 2.5 4.1 5.3]

$$P = \begin{bmatrix} 0.0483 & 0.0357 & 0.0420 & 0.0399 & 0.0441 \\ 0.0437 & 0.0323 & 0.0380 & 0.0361 & 0.0399 \\ 0.0713 & 0.0527 & 0.0620 & 0.0609 & 0.0551 \\ 0.0667 & 0.0493 & 0.0580 & 0.0651 & 0.0589 \end{bmatrix}$$

Determine the marginal distribution and the corner values for F_{XY}. Determine $P(X+Y>2)$ and $P(X \geq Y)$.

Answer

```
npr08_06
Data are in X, Y, P
jcalc
Enter JOINT PROBABILITIES (as on the plane)  P
Enter row matrix of VALUES of X  X
Enter row matrix of VALUES of Y  Y
 Use array operations on matrices X, Y, PX, PY, t, u, and P
disp([X;PX]')
   -2.3000    0.2300
   -0.7000    0.1700
    1.1000    0.2000
    3.9000    0.2020
    5.1000    0.1980

disp([Y;PY]')
    1.3000    0.2980
    2.5000    0.3020
    4.1000    0.1900
    5.3000    0.2100
jddbn
Enter joint probability matrix (as on the plane)  P
To view joint distribution function, call for FXY
disp(FXY)
    0.2300    0.4000    0.6000    0.8020    1.0000
```

```
    0.1817    0.3160    0.4740    0.6361    0.7900
    0.1380    0.2400    0.3600    0.4860    0.6000
    0.0667    0.1160    0.1740    0.2391    0.2980
P1 = total((t+u>2).*P)
P1 =  0.7163
P2 = total((t>=u).*P)
P2 =  0.2799
```

Exercise 8.3.7

The pair $\{X, Y\}$ has the joint distribution (in m-file npr08_07.m):

$$P(X = i, Y = u)$$

t =	-3.1	-0.5	1.2	2.4	3.7	4.9
u = 7.5	0.0090	0.0396	0.0594	0.0216	0.0440	0.0203
4.1	0.0495	0	0.1089	0.0528	0.0363	0.0231
-2.0	0.0405	0.1320	0.0891	0.0324	0.0297	0.0189
-3.8	0.0510	0.0484	0.0726	0.0132	0	0.0077

Determine the marginal distributions and the corner values for F_{XY}. Determine $P(1 \le X \le 4, Y > 4)$ and $P(|X - Y| \le 2)$.

Answer

```
npr08_07
Data are in X, Y, P
jcalc
Enter JOINT PROBABILITIES (as on the plane)  P
Enter row matrix of VALUES of X  X
Enter row matrix of VALUES of Y  Y
 Use array operations on matrices X, Y, PX, PY, t, u, and P
disp([X;PX]')
   -3.1000    0.1500
   -0.5000    0.2200
    1.2000    0.3300
    2.4000    0.1200
    3.7000    0.1100
    4.9000    0.0700
disp([Y;PY]')
   -3.8000    0.1929
   -2.0000    0.3426
    4.1000    0.2706
    7.5000    0.1939
jddbn
Enter joint probability matrix (as on the plane)  P
To view joint distribution function, call for FXY
disp(FXY)
```

```
    0.1500    0.3700    0.7000    0.8200    0.9300    1.0000
    0.1410    0.3214    0.5920    0.6904    0.7564    0.8061
    0.0915    0.2719    0.4336    0.4792    0.5089    0.5355
    0.0510    0.0994    0.1720    0.1852    0.1852    0.1929
M = (1<=t)&(t<=4)&(u>4);
P1 = total(M.*P)
P1 =  0.3230
P2 = total((abs(t-u)<=2).*P)
P2 =  0.3357
```

Exercise 8.3.8

The pair $\{X, Y\}$ has the joint distribution (in m-file npr08_08.m):

$$P(X=t, Y=u)$$

t =	1	3	5	7	9	11	13	15	17	19
u = 12	0.0156	0.0191	0.0081	0.0035	0.0091	0.0070	0.0098	0.0056	0.0091	0.0049
10	0.0064	0.0204	0.0108	0.0040	0.0054	0.0080	0.0112	0.0064	0.0104	0.0056
9	0.0196	0.0256	0.0126	0.0060	0.0156	0.0120	0.0168	0.0096	0.0056	0.0084
5	0.0112	0.0182	0.0108	0.0070	0.0182	0.0140	0.0196	0.0012	0.0182	0.0038
3	0.0060	0.0260	0.0162	0.0050	0.0160	0.0200	0.0280	0.0060	0.0160	0.0040
-1	0.0096	0.0056	0.0072	0.0060	0.0256	0.0120	0.0268	0.0096	0.0256	0.0084
-3	0.0044	0.0134	0.0180	0.0140	0.0234	0.0180	0.0252	0.0244	0.0234	0.0126
-5	0.0072	0.0017	0.0063	0.0045	0.0167	0.0090	0.0026	0.0172	0.0217	0.0223

Determine the marginal distributions. Determine $F_{XY}(10, 6)$ and $P(X > Y)$.

Answer

```
npr08_08
Data are in X, Y, P
jcalc
- - - - - - - - - -
 Use array operations on matrices X, Y, PX, PY, t, u, and P
disp([X;PX]')
    1.0000    0.0800
    3.0000    0.1300
    5.0000    0.0900
    7.0000    0.0500
    9.0000    0.1300
   11.0000    0.1000
   13.0000    0.1400
   15.0000    0.0800
   17.0000    0.1300
   19.0000    0.0700
```

```
disp([Y;PY]')
   -5.0000    0.1092
   -3.0000    0.1768
   -1.0000    0.1364
    3.0000    0.1432
    5.0000    0.1222
    9.0000    0.1318
   10.0000    0.0886
   12.0000    0.0918
F = total(((t<=10)&(u<=6)).*P)
F =   0.2982
P = total((t>u).*P)
P =   0.7390
```

Exercise 8.3.9

Data were kept on the effect of training time on the time to perform a job on a production line. X is the amount of training, in hours, and Y is the time to perform the task, in minutes. The data are as follows (in m-file npr08_09.m):

$$P(X=t, Y=u)$$

t =	1	1.5	2	2.5	3
u = 5	0.039	0.011	0.005	0.001	0.001
4	0.065	0.070	0.050	0.015	0.010
3	0.031	0.061	0.137	0.051	0.033
2	0.012	0.049	0.163	0.058	0.039
1	0.003	0.009	0.045	0.025	0.017

Determine the marginal distributions. Determine $F_{XY}(2,3)$ and $P(Y/X \geq 1.25)$.

Answer

```
npr08_09
Data are in X, Y, P
jcalc
- - - - - - - - - - - -
 Use array operations on matrices X, Y, PX, PY, t, u, and P
disp([X;PX]')
    1.0000    0.1500
    1.5000    0.2000
    2.0000    0.4000
    2.5000    0.1500
    3.0000    0.1000
disp([Y;PY]')
    1.0000    0.0990
    2.0000    0.3210
    3.0000    0.3130
```

```
    4.0000    0.2100
    5.0000    0.0570
F = total(((t<=2)&(u<=3)).*P)
F =   0.5100
P = total((u./t>=1.25).*P)
P =   0.5570
```

For the joint densities in Exercises 10-22 below

a. Sketch the region of definition and determine analytically the marginal density functions f_X and f_Y.
b. Use a discrete approximation to plot the marginal density f_X and the marginal distribution function F_X.
c. Calculate analytically the indicated probabilities.
d. Determine by discrete approximation the indicated probabilities.

Exercise 8.3.10

$f_{XY}(t, u) = 1$ for $0 \le t \le 1$, $0 \le u \le 2(1-t)$.

$$P(X > 1/2, Y > 1), P(0 \le X \le 1/2, Y > 1/2), P(Y \le X)$$

Answer

Region is triangle with vertices (0, 0), (1, 0), (0, 2).

$$f_X(t) = \int_0^{2(1-t)} du = 2(1-t) \ \ , 0 \le t \le 1$$

$$f_Y(u) = \int_0^{1-u/2} dt = 1 - u/2 \ , 0 \le u \le 2$$

$$M1 = \{(t, u) : t > 1/2, u > 1\} \text{ lies outside the trianlge } P((X, Y) \in M1) = 0$$

$$M2 = \{(t, u) : 0 \le t \le 1/2, u > 1/2\} \text{ has area in the triangle = 1/2}$$

$$M3 \text{ = the region in the triangle under } u = t \text{, which has area 1/3}$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 1]
Enter matrix [c d] of Y-range endpoints  [0 2]
Enter number of X approximation points  200
Enter number of Y approximation points  400
Enter expression for joint density  (t<=1)&(u<=2*(1-t))
Use array operations on X, Y, PX, PY, t, u, and P
fx = PX/dx;
FX = cumsum(PX);
plot(X,fx,X,FX)          % Figure not reproduced
M1 = (t>0.5)&(u>1);
P1 = total(M1.*P)
P1 =  0                  % Theoretical = 0
M2 = (t<=0.5)&(u>0.5);
P2 = total(M2.*P)
P2 =  0.5000             % Theoretical = 1/2
P3 = total((u<=t).*P)
P3 =  0.3350             % Theoretical = 1/3
```

Exercise 8.3.11

$f_{XY}(t,u) = 1/2$ on the square with vertices at (1, 0), (2, 1), (1, 2), (0, 1).

$$P(X > 1, Y > 1), P(X \le 1/2, 1 < Y), P(Y \le X)$$

Answer

The region is bounded by lines $u = 1 + t$, $u = 1 - t$, $u = 3 - t$, and $u = t - 1$

$f_X(t) = I_{[0,1]}(t) 0.5 \int_{1-t}^{1+t} du + I_{(1,2]}(t) 0.5 \int_{t-1}^{3-t} du = I_{(1,2]}(t)(2-t) = f_Y(t)$ by symmetry

$M1 = \{(t,u) : t > 1, u > 1\}$ has area in the trangle = 1/2, so $PM1 = 1/4$

$M2 = \{(t,u) : t \le 1/2, u > 1\}$ has area in the trangle = 1/8\), so $PM2 = 1/16$

$M3 = \{(t,u) : u \le t\}$ has area in the trangle = 1, so $PM3 = 1/2$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 2]
Enter number of X approximation points  200
Enter number of Y approximation points  200
Enter expression for joint density  0.5*(u<=min(1+t,3-t))& ...
  (u>=max(1-t,t-1))
Use array operations on X, Y, PX, PY, t, u, and P
fx = PX/dx;
FX = cumsum(PX);
plot(X,fx,X,FX)          % Plot not shown
M1 = (t>1)&(u>1);
PM1 = total(M1.*P)
PM1 =  0.2501            % Theoretical = 1/4
M2 = (t<=1/2)&(u>1);
PM2 = total(M2.*P)
PM2 =  0.0631            % Theoretical = 1/16 = 0.0625
M3 = u<=t;
PM3 = total(M3.*P)
PM3 =  0.5023            % Theoretical = 1/2
```

Exercise 8.3.12

$f_{XY}(t,u) = 4t(1-u)$ for $0 \le t \le 1$, $0 \le u \le 1$.

$$P(1/2 < X < 3/4, Y > 1/2), P(X \le 1/2, Y > 1/2), P(Y \le X)$$

Answer

Region is the unit square,

$$f_X(t) = \int_0^1 4t(1-u)du = 2t \text{ , } 0 \le t \le 1$$

$$f_Y(u) = \int_0^1 4t(1-u)dt = 2(1-u) \text{ , } 0 \le u \le 1$$

$$P1 = \int_{1/2}^{3/4} \int_{1/2}^{1} 4t(1-u)dudt = 5/64 \quad P2 = \int_0^{1/2} \int_{1/2}^{1} 4t(1-u)dudt = 1/16$$

$$P3 = \int_0^1 \int_0^t 4t(1-u)dudt = 5/6$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 1]
Enter matrix [c d] of Y-range endpoints  [0 1]
Enter number of X approximation points  200
Enter number of Y approximation points  200
Enter expression for joint density  4*t.*(1 - u)
Use array operations on X, Y, PX, PY, t, u, and P
fx = PX/dx;
FX = cumsum(PX);
plot(X,fx,X,FX)           % Plot not shown
M1 = (1/2<t)&(t<3/4)&(u>1/2);
P1 = total(M1.*P)
P1 =  0.0781              % Theoretical = 5/64 = 0.0781
M2 = (t<=1/2)&(u>1/2);
P2 = total(M2.*P)
P2 =  0.0625              % Theoretical = 1/16 = 0.0625
M3 = (u<=t);
P3 = total(M3.*P)
P3 =  0.8350              % Theoretical = 5/6 = 0.8333
```

Exercise 8.3.13

$f_{XY}(t,u) = \dfrac{1}{8}(t+u)$ for $0 \le t \le 2$, $0 \le u \le 2$.

$$P(X > 1/2, Y > 1/2), P(0 \le X \le 1, Y > 1), P(Y \le X)$$

Answer

Region is the square $0 \le t \le 2$, $0 \le u \le 2$

$$f_X(t) = \frac{1}{8}\int_0^2 (t+u) = \frac{1}{4}(t+1) = f_Y(t) \quad , 0 \le t \le 2$$

$$P1 = \int_{1/2}^2 \int_{1/2}^2 (t+u)dudt = 45/64 \;\; P2 = \int_0^1 \int_1^2 (t+u)dudt = 1/4$$

$$P3 = \int_0^2 \int_0^1 (t+u)dudt = 1/2$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 2]
Enter number of X approximation points  200
Enter number of Y approximation points  200
Enter expression for joint density  (1/8)*(t+u)
Use array operations on X, Y, PX, PY, t, u, and P
fx = PX/dx;
FX = cumsum(PX);
plot(X,fx,X,FX)
M1 = (t>1/2)&(u>1/2);
P1 = total(M1.*P)
P1 =  0.7031              % Theoretical = 45/64 = 0.7031
```

```
M2 = (t<=1)&(u>1);
P2 = total(M2.*P)
P2 =  0.2500              % Theoretical = 1/4
M3 = u<=t;
P3 = total(M3.*P)
P3 =  0.5025              % Theoretical = 1/2
```

Exercise 8.3.14

$f_{XY}(t,u) = 4ue^{-2t}$ for $0 \le t, 0 \le u \le 1$

$$P(X \le 1, Y > 1), P(X > 0, 1/2 < Y < 3/4), P(X < Y)$$

Answer

Region is strip by $t = 0, u = 0, u = 1$

$$f_X(t) = 2e^{-2t}, 0 \le t, f_Y(u) = 2u, 0 \le u \le 1, f_{XY} = f_X f_Y$$

$$P1 = 0, P2 = \int_{0.5}^{\infty} 2e^{-2t} dt \int_{1/2}^{3/4} 2udu = e^{-1} 5/16$$

$$P3 = 4 \int_0^1 \int_t^1 ue^{-2t} dudt = \frac{3}{2} e^{-2} + \frac{1}{2} = 0.7030$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 3]
Enter matrix [c d] of Y-range endpoints  [0 1]
Enter number of X approximation points  400
Enter number of Y approximation points  200
Enter expression for joint density  4*u.*exp(-2*t)
Use array operations on X, Y, PX, PY, t, u, and P
M2 = (t > 0.5)&(u > 0.5)&(u<3/4);
p2 = total(M2.*P)
p2 =  0.1139            % Theoretical = (5/16)exp(-1) = 0.1150
p3 = total((t<u).*P)
p3 =  0.7047            % Theoretical = 0.7030
```

Exercise 8.3.15

$f_{XY}(t,u) = \frac{3}{88}(2t + 3u^2)$ for $0 \le t \le 2, 0 \le u \le 1 + t$.

$$F_{XY}(1,1), P(X \le 1, Y > 1), P(|X - Y| < 1)$$

Answer

Region bounded by $t = 0, t = 2, u = 0, u = 1 + t$

$$f_X(t) = \frac{3}{88} \int_0^{1+t} (2t + 3u^2) du = \frac{3}{88}(1+t)(1+4t+t^2) = \frac{3}{88}(1+5t+5t^2+t^3) \quad , 0 \le t \le 2$$

$$f_Y(u) = I_{[0,1]}(u) \frac{3}{88} \int_0^2 (2t + 3u^2) dt + I_{(1,3]}(u) \frac{3}{88} \int_{u-1}^2 (2t + 3u^2) dt =$$

$$I_{[0,1]}(u) \frac{3}{88}(6u^2 + 4) + I_{(1,3]}(t) \frac{3}{88}(3 + 2u + 8u^2 - 3u^3)$$

$$F_{XY}(1,1) = \int_0^1 \int_0^1 f_{XY}(t,u)dudt = 3/44$$

$$P1 = \int_0^1 \int_1^{1+t} f_{XY}(t,u)dudt = 41/352 \; P2 = \int_0^1 \int_1^{1+t} f_{XY}(t,u)dudt = 329/352$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 3]
Enter number of X approximation points  200
Enter number of Y approximation points  300
Enter expression for joint density  (3/88)*(2*t+3*u.^2).*(u<=1+t)
Use array operations on X, Y, PX, PY, t, u, and P
fx = PX/dx;
FX = cumsum(PX);
plot(X,fx,X,FX)
MF = (t<=1)&(u<=1);
F = total(MF.*P)
F =   0.0681            % Theoretical = 3/44 = 0.0682
M1 = (t<=1)&(u>1);
P1 = total(M1.*P)
P1 =  0.1172            % Theoretical = 41/352 = 0.1165
M2 = abs(t-u)<1;
P2 = total(M2.*P)
P2 =  0.9297           % Theoretical = 329/352 = 0.9347
```

Exercise 8.3.16

$f_{XY}(t,u) = 12t^2u$ on the parallelogram with vertices (-1, 0), (0, 0), (1, 1), (0, 1).

$$P(X \le 1/2, Y > 0), P(X < 1/2, Y \le 1/2), P(Y \ge 1/2)$$

Answer

Region bounded by $u = 0$, $u = t$, $u = 1$, $u = t + 1$

$$f_X(t) = I_{[-1,0]}(t) 12 \int_0^{t+1} t^2 u du + I_{(0,1]}(t) 12 \int_t^1 t^2 u du = I_{[-1,0]}(t) 6t^2 (t+1)^2 + I_{(0,1]}(t) 6t^2 (1-t^2)$$

$$f_Y(u) = 12 \int_{u-1}^t t^2 u du + 12u^3 - 12u^2 + 4u \;\; , 0 \le u \le 1$$

$$P1 = 1 - 12 \int_{1/2}^1 \int_t^1 t^2 ududt = 33/80, \; P2 = 12 \int_0^{1/2} \int_{u-1}^u t^2 udtdu = 3/16$$

$$P3 = 1 - P2 = 13/16$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [-1 1]
Enter matrix [c d] of Y-range endpoints  [0 1]
Enter number of X approximation points  400
Enter number of Y approximation points  200
Enter expression for joint density  12*u.*t.^2.*((u<=t+1)&(u>=t))
Use array operations on X, Y, PX, PY, t, u, and P
p1 = total((t<=1/2).*P)
p1 =  0.4098                % Theoretical = 33/80 = 0.4125
M2 = (t<1/2)&(u<=1/2);
```

```
p2 = total(M2.*P)
p2 =  0.1856                % Theoretical = 3/16  = 0.1875
P3 = total((u>=1/2).*P)
P3 =  0.8144                % Theoretical = 13/16 = 0.8125
```

Exercise 8.3.17

$f_{XY}(t,u) = \frac{24}{11}tu$ for $0 \le t \le 2$, $0 \le u \le \min\{1, 2-t\}$

$$P(X \le 1, Y \le 1), P(X > 1), P(X < Y)$$

Answer

Region is bounded by $t = 0, u = 0, u = 2, u = 2-t$

$$f_X(t) = I_{[0,1]}(t)\frac{24}{11}\int_0^1 tu du + I_{(1,2]}(t)\frac{24}{11}\int_0^{2-t} tu du =$$

$$I_{[0,1]}(t)\frac{12}{11}t + I_{(1,2]}(t)\frac{12}{11}t(2-t)^2$$

$$f_Y(u) = \frac{24}{11}\int_0^{2-u} tu dt = \frac{12}{11}u(u-2)^2 \ , 0 \le u \le 1$$

$$P1 = \frac{24}{11}\int_0^1\int_0^1 tu du dt = 6/11 \ P2 = \frac{24}{11}\int_1^2\int_0^{2-t} tu du dt = 5/11$$

$$P3 = \frac{24}{11}\int_0^1\int_t^1 tu du dt = 3/11$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 1]
Enter number of X approximation points  400
Enter number of Y approximation points  200
Enter expression for joint density  (24/11)*t.*u.*(u<=2-t)
Use array operations on X, Y, PX, PY, t, u, and P
M1 = (t<=1)&(u<=1);
P1 = total(M1.*P)
P1 = 0.5447              % Theoretical = 6/11 = 0.5455
P2 = total((t>1).*P)
P2 =  0.4553             % Theoretical = 5/11 = 0.4545
P3 = total((t<u).*P)
P3 =  0.2705             % Theoretical = 3/11 = 0.2727
```

Exercise 8.3.18

$f_{XY}(t,u) = \frac{3}{23}(t+2u)$ for $0 \le t \le 2$, $0 \le u \le \max\{2-t, t\}$

$$P(X \ge 1, Y \ge 1), P(Y \le 1), P(Y \le X)$$

Answer

Region is bounded by $t = 0, t = 2, u = 0, u = 2-t \ \ (0 \le t \le 1)$, $u = t(1 < t \le 2)$

$$f_X(t) = I_{[0,1]}(t)\frac{3}{23}\int_0^{2-t}(t+2u)du + I_{(1,2]}(t)\frac{3}{23}\int_0^t(t+2u)du = I_{[0,1]}(t)\frac{6}{23}(2-t) + I_{(1,2]}(t)\frac{6}{23}t^2$$

$$f_Y(u) = I_{[0,1]}(u)\frac{3}{23}\int_0^2(t+2u)du + I_{(1,2]}(u)[\frac{3}{23}\int_0^{2-u}(t+2u)dt + \frac{3}{23}\int_u^2(t+2u)dt] =$$

$$I_{[0,1]}(u)\frac{6}{23}(2u+1) + I_{(1,2]}(u)\frac{3}{23}(4+6u-4u^2)$$

$$P1 = \frac{3}{23}\int_1^2\int_1^t(t+2u)dudt = 13/46, P2 = \frac{3}{23}\int_0^2\int_0^1(t+2u)dudt = 12/23$$

$$P3 = \frac{3}{23}\int_0^2\int_0^t(t+2u)dudt = 16/23$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 2]
Enter number of X approximation points  200
Enter number of Y approximation points  200
Enter expression for joint density  (3/23)*(t+2*u).*(u<=max(2-t,t))
Use array operations on X, Y, PX, PY, t, u, and P
M1 = (t>=1)&(u>=1);
P1 = total(M1.*P)
P1 =  0.2841
13/46                 % Theoretical = 13/46 = 0.2826
P2 = total((u<=1).*P)
P2 =  0.5190            % Theoretical = 12/23 = 0.5217
P3 = total((u<=t).*P)
P3 =  0.6959            % Theoretical = 16/23 = 0.6957
```

Exercise 8.3.19

$f_{XY}(t,u) = \frac{12}{179}(3t^2+u)$, for $0 \le t \le 2$, $0 \le u \le \min\{1+t, 2\}$

$$P(X \ge 1, Y \ge 1), P(X \le 1, Y \le 1), P(Y < X)$$

Answer

Region has two parts: (1) $0 \le t \le 1, 0 \le u \le 2$ (2) $1 < t \le 2, 0 \le u \le 3-t$

$$f_X(t) = I_{[0,1]}(t)\frac{12}{179}\int_0^2(3t^2+u)du + I_{(1,2]}(t)\frac{12}{179}\int_0^{3-t}(3t^2+u)du =$$

$$I_{[0,1]}(t)\frac{24}{179}(3t^2+1) + I_{(1,2]}(t)\frac{6}{179}(9-6t+19t^2-6t^3)$$

$$f_Y(u) = I_{[0,1]}(u)\frac{12}{179}\int_0^2(3t^2+u)dt + I_{(1,2]}(u)\frac{12}{179}\int_0^{3-u}(3t^2+u)dt =$$

$$I_{[0,1]}(u)\frac{24}{179}(4+u) + I_{(1,2]}(u)\frac{12}{179}(27-24u+8u^2-u^3)$$

$$P1 = \frac{12}{179}\int 1^2\int_1^{3-t}(3t^2+u)dudt = 41/179 \;\; P2 = \frac{12}{179}\int_0^1\int_0^1(3t^2+u)dudt = 18/179$$

$$P3 = \frac{12}{179}\int_0^{3/2}\int_0^t(3t^2+u)dudt + \frac{12}{179}\int_{3/2}^2\int_0^{3-t}(3t^2+u)dudt = 1001/1432$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 2]
Enter number of X approximation points  200
Enter number of Y approximation points  200
Enter expression for joint density  (12/179)*(3*t.^2+u).* ...
     (u<=min(2,3-t))
Use array operations on X, Y, PX, PY, t, u, and P
fx = PX/dx;
FX = cumsum(PX);
plot(X,fx,X,FX)
M1 = (t>=1)&(u>=1);
P1 = total(M1.*P)
P1 =  2312           % Theoretical = 41/179 = 0.2291
M2 = (t<=1)&(u<=1);
P2 = total(M2.*P)
P2 =  0.1003           % Theoretical = 18/179 = 0.1006
M3 = u<=min(t,3-t);
P3 = total(M3.*P)
P3 =  0.7003            % Theoretical = 1001/1432 = 0.6990
```

Exercise 8.3.20

$f_{XY}(t,u) = \dfrac{12}{227}(3t+2tu)$ for $0 \le t \le 2$, $0 \le u \le \min\{1+t, 2\}$

$$P(X \le 1/2, Y \le 3/2), P(X \le 1.5, Y > 1), P(Y < X)$$

Answer

Region is in two parts:

1. $0 \le t \le 1$, $0 \le u \le 1+t$
2. $1 < t \le 2$, $0 \le u \le 2$

$$f_X(t) = I_{[0,1]}(t)\int_0^{1+t} f_{XY}(t,u)du + I_{(1,2]}(t)\int_0^2 f_{XY}(t,u)du =$$

$$I_{[0,1]}(t)\frac{12}{227}(t^3+5t^2+4t) + I_{(1,2]}(t)\frac{120}{227}t$$

$$f_Y(u) = I_{[0,1]}(u)\int_0^2 f_{XY}(t,u)dt + I_{(1,2]}(u)\int_{u-1}^2 f_{XY}(t,u)dt =$$

$$I_{[0,1]}(u)\frac{24}{227}(2u+3) + I_{(1,2]}(u)\frac{6}{227}(2u+3)(3+2u-u^2)$$

$$= I_{[0,1]}(u)\frac{24}{227}(2u+3) + I_{(1,2]}(u)\frac{6}{227}(9+12u+u^2-2u^3)$$

$$P1 = \frac{12}{227}\int_0^{1/2}\int_0^{1+t}(3t+2tu)dudt = 139/3632$$

$$P2 = \frac{12}{227}\int_0^1\int_1^{1+t}(3t+2tu)dudt + \frac{12}{227}\int_1^{3/2}\int_1^2(3t+2tu)dudt = 68/227$$

$$P3 = \frac{12}{227}\int_0^2\int_1^t(3t+2tu)dudt = 144/227$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 2]
Enter number of X approximation points  200
Enter number of Y approximation points  200
Enter expression for joint density  (12/227)*(3*t+2*t.*u).* ...
(u<=min(1+t,2))
Use array operations on X, Y, PX, PY, t, u, and P
M1 = (t<=1/2)&(u<=3/2);
P1 = total(M1.*P)
P1 =  0.0384             % Theoretical = 139/3632 = 0.0383
M2 = (t<=3/2)&(u>1);
P2 = total(M2.*P)
P2 =  0.3001             % Theoretical = 68/227 = 0.2996
M3 = u<t;
P3 = total(M3.*P)
P3 =  0.6308             % Theoretical = 144/227 = 0.6344
```

Exercise 8.3.21

$f_{XY}(t,u) = \frac{2}{13}(t+2u)$ for $0 \le t \le 2$, $0 \le u \le \min\{2t, 3-t\}$

$$P(X<1), P(X \ge 1, Y \le 1), P(Y \le X/2)$$

Answer

Region bounded by $t=2, u=2t\ (0 \le t \le 1), 3-t\ (1 \le t \le 2)$

$$f_X(t) = I_{[0,1]}(t)\frac{2}{13}\int_0^{2t}(t+2u)du + I_{(1,2]}(t)\frac{2}{13}\int_0^{3-t}(t+2u)du = I_{[0,1]}(t)\frac{12}{13}t^2 + I_{(1,2]}(t)\frac{6}{13}(3-t)$$

$$f_Y(u) = I_{[0,1]}(u)\frac{2}{13}\int_{u/2}^{2}(t+2u)dt + I_{(1,2]}(u)\frac{2}{13}\int_{u/2}^{3-u}(t+2u)dt =$$

$$I_{[0,1]}(u)\left(\frac{4}{13}+\frac{8}{13}u-\frac{9}{52}u^2\right) + I_{(1,2]}(u)\left(\frac{9}{13}+\frac{6}{13}u-\frac{21}{52}u^2\right)$$

$$P1 = \int_0^1\int_0^{2t}(t+2u)dudt = 4/13 \quad P2 = \int_1^2\int_0^1(t+2u)dudt = 5/13$$

$$P3 = \int_0^2\int_0^{u/2}(t+2u)dudt = 4/13$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 2]
Enter number of X approximation points  400
Enter number of Y approximation points  400
Enter expression for joint density  (2/13)*(t+2*u).*(u<=min(2*t,3-t))
Use array operations on X, Y, PX, PY, t, u, and P
P1 = total((t<1).*P)
P1 = 0.3076             % Theoretical = 4/13 = 0.3077
M2 = (t>=1)&(u<=1);
P2 = total(M2.*P)
```

```
P2 =  0.3844            % Theoretical = 5/13 = 0.3846
P3 = total((u<=t/2).*P)
P3 =  0.3076             % Theoretical = 4/13 = 0.3077
```

Exercise 8.3.22

$f_{XY}(t,u) = I_{[0,1]}(t)\frac{3}{8}(t^2+2u) + I_{(1,2]}(t)\frac{9}{14}t^2u^2$ for $0 \le u \le 1$.

$$P(1/2 \le X \le 3/2, Y \le 1/2)$$

Answer

Region is rectangle bounded by $t=0$, $t=2$, $u=0$, $u=1$

$$f_{XY}(t,u) = I_{[0,1]}(t)\frac{3}{8}(t^2+2u) + I_{(1,2]}(t)\frac{9}{14}t^2u^2 \ , 0 \le u \le 1$$

$$f_X(t) = I_{[0,1]}(t)\frac{3}{8}\int_0^1 (t^2+2u)du + I_{(1,2]}(t)\frac{9}{14}\int_0^1 t^2u^2du = I_{[0,1]}(t)\frac{3}{8}(t^2+1) + I_{(1,2]}(t)\frac{3}{14}t^2$$

$$f_Y(u) = \frac{3}{8}\int_0^1 (t^2+2u0dt + \frac{9}{14}\int_1^2 t^2u^2dt = \frac{1}{8} + \frac{3}{4}u + \frac{3}{2}u^2 \quad 0 \le u \le 1$$

$$P1 = \frac{3}{8}\int_{1/2}^1\int_0^{1/2}(t^2+2u)dudt + \frac{9}{14}\int_1^{3/2}\int_0^{1/2} t^2u^2dudt = 55/448$$

```
tuappr
Enter matrix [a b] of X-range endpoints  [0 2]
Enter matrix [c d] of Y-range endpoints  [0 1]
Enter number of X approximation points  400
Enter number of Y approximation points  200
Enter expression for joint density  (3/8)*(t.^2+2*u).*(t<=1) ...
       + (9/14)*(t.^2.*u.^2).*(t > 1)
Use array operations on X, Y, PX, PY, t, u, and P
M = (1/2<=t)&(t<=3/2)&(u<=1/2);
P = total(M.*P)
P =  0.1228          % Theoretical = 55/448 = 0.1228
```