17.5: Appendix E to Applied Probability - Properties of Mathematical Expectation

$$E[g(X)] = \int g(X)\, dP$$

We suppose, without repeated assertion, that the random variables and Borel functions of random variables or random vectors are integrable. Use of an expression such as $I_M(X)$ involves the tacit assumption that M is a Borel set on the codomain of X.

(E1): $E[aI_A] = aP(A)$, any constant a, any event A

(E1a): $E[I_M(X)] = P(X \in M)$ and $E[I_M(X)I_N(Y)] - P(X \in M, Y \in N)$ for any Borel sets M, N (Extends to any finite product of such indicator functions of random vectors)

(E2): **Linearity**. For any constants a, b, $E[aX + bY) = aE[X] + bE[Y]$ (Extends to any finite linear combination)

(E3): **Positivity; monotonicity.**

a. $X \ge 0$ a.s. implies $E[X] \ge 0$, with equality iff $X = 0$ a.s.

b. $X \ge Y$ a.s. implies $E[X] \ge E[Y]$, with equality iff $X = Y$ a.s.

(E4): **Fundamental lemma**. If $X \ge 0$ is bounded, and $\{X_n : 1 \le n\}$ is a.s. nonnegative, nondecreasing, with $\lim_n X_n(\omega) \ge X(\omega)$ for a.e. ω, then $\lim_n E[X_n] \ge E[X]$

(E4a): **Monotone convergence**. If for all n, $0 \le X_n \le X_{n+1}$ a.s. and $X_n \to X$ a.s.,then $E[X_n] \to E[X]$ (The theorem also holds if $E[X] = \infty$)

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(E5): **Uniqueness**. * is to be read as one of the symbols $\le, =$, or $\ge$

a. $E[I_M(X)g(X)] * E[I_M(X)h(X)]$ for all M iff $g(X) * h(X)$ a.s.

b. $E[I_M(X)I_N(Z)g(X,Z)] = E[I_M(X)I_N(Z)h(X,Z)]$ for all M, N iff $g(X,Z) = h(X,Z)$ a.s.

(E6): Fatou's lemma. If $X_n \ge 0$ a.s., for all n, then $E[\liminf X_n] \le [\liminf E[X_n]$

(E7): Dominated convergence. If real or complex $X_n \to X$ a.s., $|X_n| \le Y$ a.s. for all n, and Y is integrable, then $\lim_n E[X_n] = E[X]$

(E8): **Countable additivity and countable sums**.

a. If X is integrable over E, and $E = \bigvee_{i=1}^{\infty} E_i$ (disjoint union), then $E[I_E X] = \sum_{i=1}^{\infty} E[I_{E_i} X]$

b. If $\sum_{n=1}^{\infty} E[|X_n|] < \infty$, then $\sum_{n=1}^{\infty} |X_n| < \infty$, a.s. and $E[\sum_{n=1}^{\infty} X_n] = \sum_{n=1}^{\infty} E[X_n]$

(E9): **Some integrability conditions**

a. X is integrable iff both X^+ and X^- are integrable iff $|X|$ is integrable.

b. X is integrable iff $E[I_{\{|X|>a\}}|X|] \to 0$ as $a \to \infty$

c. If X is integrable, then X is a.s. finite

d. If $E[X]$ exists and $P(A) = 0$, then $E[I_A X] = 0$

(E10): **Triangle inequality**. For integrable X, real or complex, $|E[X]| \le E[|X|]$

(E11): **Mean-value theorem**. If $a \le X \le b$ a.s. on A, then $aP(A) \le E[I_A X] \le bP(A)$

(E12): For nonnegative, Borel g, $E[g(X)] \ge aP(g(X) \ge a)$

(E13): **Markov's inequality**. If $g \ge 0$ and nondecreasing for $t \ge 0$ and $a \ge 0$, then

$$g(a)P(|X| \ge a) \le E[g(|X|)]$$

(E14): **Jensen's inequality**. If g is convex on an interval which contains the range of random variable X, then $g(E[X]) \le E[g(X)]$

(E15): **Schwarz' inequality**. For X, Y real or complex, $|E[XY]|^2 \le E[|X|^2]E[|Y|^2]$, with equality iff there is a constant c such that $X = cY$ a.s.

(E16): **Hölder's inequality**. For $1 \le p, q$, with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and X, Y real or complex.

$$E[|XY|] \le E[|X|^p]^{1/p} E[|Y|^q]^{1/q}$$

(E17): **Hölder's inequality**. For $1 < p$ and X, Y real or complex,

$$E[|X+Y|^p]^{1/p} \le E[|X|^p]^{1/p} + E[|Y|^p]^{1/p}$$

(E18): **Independence and expectation**. The following conditions are equivalent.

a. The pair $\{X, Y\}$ is independent

b. $E[I_M(X)I_N(Y)] = E[I_M(X)]E[I_N(Y)]$ for all Borel M, N

c. $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ for all Borel g, h such that $g(X)$, $h(Y)$ are integrable.

(E19): **Special case of the Radon-Nikodym theorem** If $g(Y)$ is integrable and X is a random vector, then there exists a real-valued Borel function $e(\cdot)$, defined on the range of X, unique a.s. $[P_X]$, such that $E[I_M(X)g(X)] = E[I_M(X)e(X)]$ for all Borel sets M on the codomain of X.

(E20): **Some special forms of expectation**

a. Suppose F is nondecreasing, right-continuous on $[0, \infty)$, with $F(0^-) = 0$. Let $F^*(t) = F(t-0)$. Consider $X \ge 0$ with $E[F(X)] < \infty$. Then,

(1) $E[F(X)] = \int_0^\infty P(X \ge t)F\ (dt)$ and (2) $E[F^*(X)] = \int_0^\infty P(X > t)F\ (dt)$

b. If X is integrable, then $E[X] = \int_{-\infty}^{\infty} [u(t) - F_X(t)]\ dt$

c. If X, Y are integrable, then $E[X - Y] = \int_{-\infty}^{\infty} [F_Y(t) - F_X(t)]\ dt$

d. if $X \ge 0$ is integrable, then

$\sum_{n=0}^{\infty} P(X \ge n+1) \le E[X] \le \sum_{n=0}^{\infty} P(X \ge n) \le N \sum_{k=0}^{\infty} P(X \ge kN)$, for all $N \ge 1$

e. If integrable $X \ge 0$ is integer-valued, then

$$E[X] = \sum_{n=1}^{\infty} P(X \ge n) = \sum_{n=0}^{\infty} P(X > n) E[X^2] = \sum_{n=1}^{\infty} (2n-1)P(X \ge n) = \sum_{n=0}^{\infty} (2n+1)P(X > n)$$

f. If Q is the quantile function for F_X, then $E[g(X)] = \int_0^1 g[Q(u)]\ du$