11.3: Problems on Mathematical Expectation

Exercise 11.3.1

(See Exercise 1 from "Problems on Distribution and Density Functions", m-file npr07_01.m). The class $\{C_j : 1 \le j \le 10\}$ is a partition. Random variable X has values {1, 3, 2, 3, 4, 2, 1, 3, 5, 2} on C_1 through C_{10}, respectively, with probabilities 0.08, 0.13, 0.06, 0.09, 0.14, 0.11, 0.12, 0.07, 0.11, 0.09. Determine $E[X]$

Answer

```
% file npr07_01.m
% Data for Exercise 1 from "Problems on Distribution and Density Functions"
T = [1 3 2 3 4 2 1 3 5 2];
pc = 0.01*[8 13 6 9 14 11 12 7 11 9];
disp('Data are in T and pc')
npr07_01
Data are in T and pc
EX = T*pc'
EX = 2.7000
[X,PX] csort(T,pc): % Alternate using X, PX
ex = X*PX'
ex = 2.7000
```

Exercise 11.3.2

(See Exercise 2 from "Problems on Distribution and Density Functions", m-file npr07_02.m). A store has eight items for sale. The prices are \$3.50, \$5.00, \$3.50, \$7.50, \$5.00, \$5.00, \$3.50, and \$7.50, respectively. A customer comes in. She purchases one of the items with probabilities 0.10, 0.15, 0.15, 0.20, 0.10 0.05, 0.10 0.15. The random variable expressing the amount of her purchase may be written

$$X = 3.5I_{C_1} + 5.0I_{C_2} + 3.5I_{C_3} + 7.5I_{C_4} + 5.0I_{C_5} + 5.0I_{C_6} + 3.5I_{C_7} + 7.5I_{C_8}$$

Determine the expection $E[X]$ of the value of her purchase.

Answer

```
% file npr07_02.m
% Data for Exercise 2 from "Problems on Distribution and Density Functions"
T = [3.5 5.0 3.5 7.5 5.0 5.0 3.5 7.5];
pc = 0.01*[10 15 15 20 10 5 10 15];
disp('Data are in T and pc')
npr07_02
Data are in T and pc
EX = T*pc'
EX = 5.3500
[X,PX] csort(T,pc)
ex = X*PX'
ex = 5.3500
```

Exercise 11.3.3

See Exercise 12 from "Problems on Random Variables and Probabilities", and Exercise 3 from "Problems on Distribution and Density Functions," m-file npr06_12.m). The class $\{A, B, C, D\}$ has minterm probabilities

$$pm = 0.001 * [5\ 7\ 6\ 8\ 9\ 14\ 22\ 33\ 21\ 32\ 50\ 75\ 86\ 129\ 201\ 302]$$

Determine the mathematical expection for the random variable $X = I_A + I_B + I_C + I_D$, which counts the number of the events which occur on a trial.

Answer

```
% file npr06_12.m
% Data for Exercise 12 from "Problems on Random Variables and Probabilities"
pm = 0.001*[5 7 6 8 9 14 22 33 21 32 50 75 86 129 201 302];
c = [1 1 1 1 0];
disp('Minterm probabilities in pm, coefficients in c')
npr06_12
Minterm probabilities in pm, coefficients in c
canonic
 Enter row vector of coefficients c
 Enter row vector of minterm probabilities pm
Use row matrices X and PX for calculations
call for XDBN to view the distribution
EX = X*PX'
EX = 2.9890
T = sum(mintable(4));
[x,px] = csort(T,pm);
ex = x*px
ex = 2.9890
```

Exercise 11.3.4

(See Exercise 5 from "Problems on Distribution and Density Functions"). In a thunderstorm in a national park there are 127 lightning strikes. Experience shows that the probability of of a lightning strike starting a fire is about 0.0083. Determine the expected number of fires.

Answer

X ~ binomial (127, 0.0083), $E[X] = 127 \cdot 0.0083 = 1.0541$

Exercise 11.3.5

(See Exercise 8 from "Problems on Distribution and Density Functions"). Two coins are flipped twenty times. Let X be the number of matches (both heads or both tails). Determine $E[X]$

Answer

X ~ binomial (20, 1/2). $E[X] = 20 \cdot 0.5 = 10$

Exercise 11.3.6

(See Exercise 12 from "Problems on Distribution and Density Functions"). A residential College plans to raise money by selling "chances" on a board. Fifty chances are sold. A player pays \$10 to play; he or she wins \$30 with probability $p = 0.2$. The profit to the College is

$$X = 50 \cdot 10 - 30N \text{, where } N \text{ is the numbe of winners}$$

Determine the expected profit $E[X]$.

Answer

N ~ binomial (50, 0.2). $E[N] = 50 \cdot 0.2 = 10$. $E[X] = 500 - 30E[N] = 200$.

Exercise 11.3.7

(See Exercise 19 from "Problems on Distribution and Density Functions"). The number of noise pulses arriving on a power circuit in an hour is a random quantity having Poisson (7) distribution. What is the expected number of pulses in an hour?

Answer

X ~ Poisson (7). $E[X] = 7$.

Exercise 11.3.8

(See Exercise 24 and Exercise 25 from "Problems on Distribution and Density Functions"). The total operating time for the units in Exercise 24 is a random variable T ~ gamma (20, 0.0002). What is the expected operating time?

Answer

X ~ gamma (20, 0.0002). $E[X] = 20/0.0002 = 100,000$.

Exercise 11.3.9

(See Exercise 41 from "Problems on Distribution and Density Functions"). Random variable X has density function

$$f_X(t) = \begin{cases} (6/5)t^2 & \text{for } 0 \le t \le 1 \\ (6/5)(2-t) & \text{for } 1 \le t \le 2 \end{cases} = I_{[0,1]}(t)\frac{6}{5}t^2 + I_{(1,2]}(t)\frac{6}{5}(2-t) \ .$$

What is the expected value $E[X]$?

Answer

$$E[X] = \int t f_X(t)\, dt = \frac{6}{5}\int_0^1 t^3\, dt + \frac{6}{5}\int_1^2 (2t - t^2)\, dt = \frac{11}{10}$$

Exercise 11.3.10

Truncated exponential. Suppose X ~ exponential (λ) and $Y = I_{[0,a]}(X)X + I_{a,\infty}(X)a$.

a. Use the fact that

$$\int_0^\infty te^{-\lambda t}\, dt = \frac{1}{\lambda^2} \text{ and } \int_a^\infty te^{-\lambda t}\, dt = \frac{1}{\lambda^2}e^{-\lambda t}(1+\lambda a)$$

to determine an expression for $E[Y]$.

b. Use the approximation method, with $\lambda = 1/50$, $a = 30$. Approximate the exponential at 10,000 points for $0 \le t \le 1000$. Compare the approximate result with the theoretical result of part (a).

Answer

$$E[Y] = \int g(t) f_X(t)\, dt = \int_0^a t\lambda e^{-\lambda t}\, dt + aP(X > a) =$$

$$\frac{\lambda}{\lambda^2}[1-e^{-\lambda a}(1+\lambda a)]+ae^{-\lambda a}=\frac{1}{\lambda}(1-e^{-\lambda a})$$

```
tappr
Enter matrix [a b] of x-range endpoints [0 1000]
Enter number of x approximation points 10000
Enter density as a function of t (1/50)*exp(-t/50)
Use row matrices X and PX as in the simple case
G = X.*(X<=30) + 30*(X>30);
EZ = G8PX'
EZ = 22.5594
ez = 50*(1-exp(-30/50))     %Theoretical value
ez = 22.5594
```

Exercise 11.3.11

(See Exercise 1 from "Problems On Random Vectors and Joint Distributions", m-file npr08_01.m). Two cards are selected at random, without replacement, from a standard deck. Let X be the number of aces and Y be the number of spades. Under the usual assumptions, determine the joint distribution. Determine $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$, and $E[XY]$.

Answer

```
npr08_01
Data in Pn, P, X, Y
jcalc
Enter JOINT PROBABILITIES (as on the plane) P
Enter row marix of VALUES of X    X
Enter row marix of VALUES of Y    Y
 Use array operations on matrices X, Y, PX, PY, t, u, and P
EX = X*PX'
EX = 0.1538

ex = total(t.*P)            % Alternate
ex = 0.1538
EY = Y*PY'
EY = 0.5000
EX2 = (X.^2)*PX'
EX2 = 0.1629
EY2 = (Y.^2)*PY'
EY2 = 0.6176
EXY = total(t.*u.*P)
EXY = 0.0769
```

Exercise 11.3.12

(See Exercise 2 from "Problems On Random Vectors and Joint Distributions", m-file npr08_02.m). Two positions for campus jobs are open. Two sophomores, three juniors, and three seniors apply. It is decided to select two at random (each possible pair equally likely). Let X be the number of sophomores and Y be the number of juniors who are selected. Determine the joint distribution for $\{X, Y\}$ and $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$, and $E[XY]$.

Answer

```
npr08_02
Data are in X, Y, Pn, P
jcalc
-----------------------
EX = X*PX'
EX = 0.5000
EY = Y*PY'
EY = 0.7500
EX2 = (X.^2)*PX'
EX2 = 0.5714
EY2 = (Y.^2)*PY'
EY2 = 0.9643
EXY = total(t.*u.*P)
EXY = 0.2143
```

Exercise 11.3.13

(See Exercise 3 from "Problems On Random Vectors and Joint Distributions", m-file npr08_03.m). A die is rolled. Let X be the number of spots that turn up. A coin is flipped X times. Let Y be the number of heads that turn up. Determine the joint distribution for the pair $\{X, Y\}$. Assume $P(X=k) = 1/6$ for $1 \le k \le 6$ and for each k, $P(Y=j|X=k)$ has the binomial $(k, 1/2)$ distribution. Arrange the joint matrix as on the plane, with values of Y increasing upward. Determine the expected value $E[Y]$

Answer

```
npr08_03
Data are in X, Y, P, PY
jcalc
-----------------------
EX = X*PX'
EX = 3.5000
EY = Y*PY'
EY = 1.7500
EX2 = (X.^2)*PX'
EX2 = 15.1667
EY2 = (Y.^2)*PY'
EY2 = 4.6667
EXY = total(t.*u.*P)
EXY = 7.5833
```

Exercise 11.3.14

(See Exercise 4 from "Problems On Random Vectors and Joint Distributions", m-file npr08_04.m). As a variation of Exercise, suppose a pair of dice is rolled instead of a single die. Determine the joint distribution for $\{X, Y\}$ and determine $E[Y]$.

Answer

```
npr08_04
Data are in X, Y, P
jcalc
-----------------------
EX = X*PX'
EX = 7
EY = Y*PY'
EY = 3.5000
EX2 = (X.^2)*PX'
EX2 = 54.8333
EY2 = (Y.^2)*PY'
EY2 = 15.4583
```

Exercise 11.3.15

(See Exercise 5 from "Problems On Random Vectors and Joint Distributions", m-file npr08_05.m). Suppose a pair of dice is rolled. Let X be the total number of spots which turn up. Roll the pair an additional X times. Let Y be the number of sevens that are thrown on the X rolls. Determine the joint distribution for $\{X, Y\}$ and determine $E[Y]$

Answer

```
npr08_05
Data are in X, Y, P, PY
jcalc
-----------------------
EX = X*PX'
EX = 7.0000
EY = Y*PY'
EY = 1.1667
```

Exercise 11.3.16

(See Exercise 6 from "Problems On Random Vectors and Joint Distributions", m-file npr08_06.m). The pair $\{X, Y\}$ has the joint distribution:

$$X = \text{[-2.3 -0.7 1.1 3.9 5.1]} \; Y = \text{[1.3 2.5 4.1 5.3]}$$

$$P = \begin{bmatrix} 0.0483 & 0.0357 & 0.0420 & 0.0399 & 0.0441 \\ 0.0437 & 0.0323 & 0.0380 & 0.0361 & 0.0399 \\ 0.0713 & 0.0527 & 0.0620 & 0.0609 & 0.0551 \\ 0.0667 & 0.0493 & 0.0580 & 0.0651 & 0.0589 \end{bmatrix}$$

Determine $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$ and $E[XY]$.

Answer

```
npr08_06
Data are in X, Y, P
jcalc
---------------------
EX = X*PX'
```

```
EX = 1.3696
EY = Y*PY'
EY = 3.0344
EX2 = (X.^2)*PX'
EX2 = 9.7644
EY2 = (Y.^2)*PY'
EY2 = 11.4839
EXY = total(t.*u.*P)
EXY = 4.1423
```

Exercise 11.3.17

(See Exercise 7 from "Problems On Random Vectors and Joint Distributions", m-file npr08_07.m). The pair $\{X, Y\}$ has the joint distribution:

$$P(X=t, Y=u)$$

t =	-3.1	-0.5	1.2	2.4	3.7	4.9
u = 7.5	0.0090	0.0396	0.0594	0.0216	0.0440	0.0203
4.1	0.0495	0	0.1089	0.0528	0.0363	0.0231
-2.0	0.0405	0.1320	0.0891	0.0324	0.0297	0.0189
-3.8	0.0510	0.0484	0.0726	0.0132	0	0.0077

Determine $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$ and $E[XY]$.

Answer

```
npr08_07
Data are in X, Y, P
jcalc
---------------------
EX = X*PX'
EX = 0.8590
EY = Y*PY'
EY = 1.1455
EX2 = (X.^2)*PX'
EX2 = 5.8495
EY2 = (Y.^2)*PY'
EY2 = 19.6115
EXY = total(t.*u.*P)
EXY = 3.6803
```

Exercise 11.3.18

(See Exercise 8 from "Problems On Random Vectors and Joint Distributions", m-file npr08_08.m). The pair $\{X, Y\}$ has the joint distribution:

$$P(X=t, Y=u)$$

t=	1	3	5	7	9	11	13	15	17	19
u = 12	0.0156	0.0191	0.0081	0.0035	0.0091	0.0070	0.0098	0.0056	0.0091	0.0049
10	0.0064	0.0204	0.0108	0.0040	0.0054	0.0080	0.0112	0.0064	0.0104	0.0056
9	0.0196	0.0256	0.0126	0.0060	0.0156	0.0120	0.0168	0.0096	0.0056	0.0084
5	0.0112	0.0182	0.0108	0.0070	0.0182	0.0140	0.0196	0.0012	0.0182	0.0038
3	0.0060	0.0260	0.0162	0.0050	0.0160	0.0200	0.0280	0.0060	0.0160	0.0040
-1	0.0096	0.0056	0.0072	0.0060	0.0256	0.0120	0.0268	0.0096	0.0256	0.0084
-3	0.0044	0.0134	0.0180	0.0140	0.0234	0.0180	0.0252	0.0244	0.0234	0.0126
-5	0.0072	0.0017	0.0063	0.0045	0.0167	0.0090	0.0026	0.0172	0.0217	0.0223

Determine $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$ and $E[XY]$.

Answer

```
npr08_08
Data are in X, Y, P
jcalc
---------------------
EX = X*PX'
EX = 10.1000
EY = Y*PY'
EY = 3.0016
EX2 = (X.^2)*PX'
EX2 = 133.0800
EY2 = (Y.^2)*PY'
EY2 = 41.5564
EXY = total(t.*u.*P)
EXY = 22.2890
```

Exercise 11.3.19

(See Exercise 9 from "Problems On Random Vectors and Joint Distributions", m-file npr08_09.m). Data were kept on the effect of training time on the time to perform a job on a production line. X is the amount of training, in hours, and Y is the time to perform the task, in minutes. The data are as follows:

$$P(X=t, Y=u)$$

t =	1	1.5	2	2.5	3
u = 5	0.039	0.011	0.005	0.001	0.001
4	0.065	0.070	0.050	0.015	0.010
3	0.031	0.061	0.137	0.051	0.033
2	0.012	0.049	0.163	0.058	0.039
1	0.003	0.009	0.045	0.025	0.017

Determine $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$ and $E[XY]$.

Answer

```
npr08_09
Data are in X, Y, P
jcalc
---------------------
EX = X*PX'
EX = 1.9250
EY = Y*PY'
EY = 2.8050
EX2 = (X.^2)*PX'
EX2 = 4.0375
EY2 = (Y.^2)*PY'            EXY = total(t.*u.*P)
EY2 = 8.9850                EXY = 5.1410
```

For the joint densities in Exercise 20-32 below

a. Determine analytically $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$ and $E[XY]$.

b. Use a discrete approximation for $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$ and $E[XY]$.

Exercise 11.3.20

(See Exercise 10 from "Problems On Random Vectors and Joint Distributions"). $f_{XY}(t,u) = 1$ for $0 \le t \le 1$. $0 \le u \le 2(1-t)$.

$$f_X(t) = 2(1-t),\ 0 \le t \le 1,\ f_Y(u) = 1 - u/2,\ 0 \le u \le 2$$

Answer

$$E[X] = \int_0^1 2t(1-t)\,dt = 1/3,\ E[Y] = 2/3,\ E[X^2] = 1/6,\ E[Y^2] = 2/3$$

$$E[XY] = \int_0^1 \int_0^{2(1-t)} tu\,dudt = 1/6$$

```
tuappr: [0 1] [0 2] 200 400  u<=2*(1-t)
EX = 0.3333    EY = 0.6667    EX2 = 0.1667    EY2 = 0.6667
EXY = 0.1667 (use t, u, P)
```

Exercise 11.3.21

(See Exercise 11 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t,u) = 1/2$ on the square with vertices at (1, 0), (2, 1) (1, 2), (0, 1).

$$f_X(t) = f_Y(t) = I_{[0,1]}(t)t + I_{(1,2]}(t)(2-t)$$

Answer

$$E[X] = E[Y] = \int_0^1 t^2\,dt + \int_1^t (2t - t^2)\,dt = 1 \quad, E[X^2] = E[Y^2] = 7/6$$

$$E[XY] = (1/2)\int_0^1 \int_{1-t}^{1+t} dtdt + (1/2)\int_1^2 \int_{t-1}^{3-t} dudt = 1$$

```
tuappr: [0 2] [0 2] 200 200  0.5*(u<=min(t+1,3-t))&(u>=max(1-t,t-1))
EX = 1.0000    EY = 1.0002    EX2 = 1.1684    EY2 = 1.1687    EXY = 1.0002
```

Exercise 11.3.22

(See Exercise 12 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t, u) = 4t(1-u)$ for $0 \le t \le 1$, $0 \le u \le 1$

$$f_X(t) = 2t\,, 0 \le t \le 1\,, f_Y(u) = 2(1-u)\,, 0 \le u \le 1$$

Answer

$$E[X] = 2/3, E[Y] = 1/3, E[X^2] = 1/2, E[Y^2] = 1/6, E[XY] = 2/9$$

```
tuappr: [0 1] [0 1] 200 200  4*t.*(1-u)
EX = 0.6667    EY = 0.3333    EX2 = 0.5000    EY2 = 0.1667    EXY = 0.2222
```

Exercise 11.3.23

(See Exercise 13 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t, u) = \dfrac{1}{8}(t+u)$ for $0 \le t \le 2$, $0 \le u \le 2$

$$f_X(t) = f_Y(t) = \dfrac{1}{4}(t+1)\ , 0 \le t \le 2$$

Answer

E[X] = E[Y] = \dfrac{1}{4} \int_{0}^{2} (t^2 + t) \ dt = \dfrac{7}{6}, $E[X^2] = E[Y^2] = 5/3$

$$E[XY] = \dfrac{1}{8} \int_0^2 \int_0^2 (t^2 u + tu^2)\ dudt = \dfrac{4}{3}$$

```
tuappr: [0 1] [0 1] 200 200  4*t.*(1-u)
EX = 1.1667    EY = 1.1667    EX2 = 1.6667    EY2 = 1.6667    EXY = 1.3333
```

Exercise 11.3.24

(See Exercise 14 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t, u) = 4ue^{-2t}$ for $0 \le t, 0 \le u \le 1$

$$f_X(t) = 2e^{-2t}\,, 0 \le t,\, f_Y(u) = 2u,\, 0 \le u \le 1$$

Answer

$$E[X] = \int_0^\infty 2te^{-2t}\ dt = \dfrac{1}{2}\,, E[Y] = \dfrac{2}{3}\,, E[X^2] = \dfrac{1}{2}\,, E[Y^2] = \dfrac{1}{2}\,, E[XY] = \dfrac{1}{3}$$

```
tuappr: [0 6] [0 1] 600 200  4*u.*exp(-2*t)
EX = 0.5000    EY = 0.6667    EX2 = 0.4998    EY2 = 0.5000    EXY = 0.3333
```

Exercise 11.3.25

(See Exercise 15 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t, u) = \dfrac{3}{88}(2t + 3u^2)$ for $0 \le t \le 2$, $0 \le u \le 1 + t$.

$$f_X(t) = \dfrac{3}{88}(1+t)(1+4t+t^2) = \dfrac{3}{88}(1+5t+5t^2+t^3) \quad , 0 \le t \le 2$$

$$f_Y(t) = I_{[0,1]}(u)\dfrac{3}{88}(6u^2+4) + I_{(1,3]}(u)\dfrac{3}{88}(3+2u+8u^2-3u^3)$$

Answer

$$E[X] = \frac{313}{220}, E[Y] = \frac{1429}{880}, E[X^2] = \frac{49}{22}, E[Y^2] = \frac{172}{55}, E[XY] = \frac{2153}{880}$$

```
tuappr: [0 2] [0 3] 200 300  (3/88)*(2*t + 3*u.^2).*(u<1+t)
EX = 1.4229    EY = 1.6202    EX2 = 2.2277    EY2 = 3.1141    EXY = 2.4415
```

Exercise 11.3.26

(See Exercise 16 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t, u) = 12t^2 u$ on the parallelogram with vertices

(-1, 0), (0, 0), (1, 1), (0, 1)

$$f_X(t) = I_{[-1,0]}(t) 6t^2 (t+1)^2 + I_{(0,1]}(t) 6t^2 (1-t^2) \ \ , f_Y(u) 12u^3 - 12u^2 + 4u, 0 \le u \le 1$$

Answer

$$E[X] = \frac{2}{5}, E[Y] = \frac{11}{15}, E[X^2] = \frac{2}{5}, E[Y^2] = \frac{3}{5}, E[XY] = \frac{2}{5}$$

```
tuappr: [-1 1] [0 1] 400 300  12*t.^2.*u.*(u>=max(0,t)).*(u<=min(1+t,1))
EX = 0.4035    EY = 0.7342    EX2 = 0.4016    EY2 = 0.6009    EXY = 0.4021
```

Exercise 11.3.27

(See Exercise 17 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t, u) = \frac{24}{11} tu$ for $0 \le t \le 2$, $0 \le u \le \min\{1, 2-t\}$.

$$f_X(t) = I_{[0,1]}(t) \frac{12}{11} t + I_{(1,2]}(t) \frac{12}{11} t(2-t)^2 \ \ , f_Y(u) = \frac{12}{11} u(u-2)^2, 0 \le u \le 1$$

Answer

$$E[X] = \frac{52}{55}, E[Y] = \frac{32}{55}, E[X^2] = \frac{57}{55}, E[Y^2] = \frac{2}{5}, E[XY] = \frac{28}{55}$$

```
tuappr: [0 2] [0 1] 400 200  (24/11)*t.*u.*(u<=min(1,2-t))
EX = 0.9458    EY = 0.5822    EX2 = 1.0368    EY2 = 0.4004    EXY = 0.5098
```

Exercise 11.3.28

(See Exercise 18 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t, u) = \frac{3}{23}(t + 2u)$ for $0 \le t \le 2$, $0 \le u \le \max\{2-t, t\}$.

$$f_X(t) = I_{[0,1]}(t) \frac{6}{23}(2-t) + I_{(1,2]}(t) \frac{6}{23} t^2 \ \ , f_Y(u) = I_{[0,1]}(u) \frac{6}{23}(2u+1) + I_{(1,2]}(u) \frac{3}{23}(4 + 6u - 4u^2)$$

Answer

$$E[X] = \frac{53}{46}, E[Y] = \frac{22}{23}, E[X^2] = \frac{397}{230}, E[Y^2] = \frac{261}{230}, E[XY] = \frac{251}{230}$$

```
tuappr: [0 2] [0 2] 200 200  (3/23)*(t + 2*u).*(u<=max(2-t,t))
EX = 1.1518    EY = 0.9596    EX2 = 1.7251    EY2 = 1.1417    EXY = 1.0944
```

Exercise 11.3.29

(See Exercise 19 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t,u) = \frac{12}{179}(3t^2+u)$, for $0 \le t \le 2$, $0 \le u \le \min\{2, 3-t\}$.

$f_X(t) = I_{[0,1]}(t)\frac{24}{179}(3t^2+1) + I_{(1,2]}(t)\frac{6}{179}(9-6t+19t^2-6t^3)$

$f_Y(u) = I_{[0,1]}(t)\frac{24}{179}(4+u) + I_{(1,2]}(t)\frac{12}{179}(27-24u+8u^2-u^3)$

Answer

$$E[X] = \frac{2313}{1790}, E[Y] = \frac{778}{895}, E[X^2] = \frac{1711}{895}, E[Y^2] = \frac{916}{895}, E[XY] = \frac{1811}{1790}$$

```
tuappr: [0 2] [0 2] 400 400  (12/179)*(3*t.^2 + u).*(u<=min(2,3-t))
EX = 1.2923    EY = 0.8695    EX2 = 1.9119    EY2 = 1.0239    EXY = 1.0122
```

Exercise 11.3.30

(See Exercise 20 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t,u) = \frac{12}{227}(3t+2tu)$, for $0 \le t \le 2$, $0 \le u \le \min\{1+t, 2\}$.

$$f_X(t) = I_{[0,1]}(t)\frac{12}{227}(t^3+5t^2+4t) + I_{(1,2]}(t)\frac{120}{227}t$$

$$f_Y(u) = I_{[0,1]}(t)\frac{24}{227}(2u+3) + I_{(1,2]}(u)\frac{6}{227}(2u+3)(3+2u-u^2)$$

$$= I_{[0,1]}(u)\frac{24}{227}(2u+3) + I_{(1,2]}(u)\frac{6}{227}(9+12u+u^2-2u^3)$$

Answer

$$E[X] = \frac{1567}{1135}, E[Y] = \frac{2491}{2270}, E[X^2] = \frac{476}{227}, E[Y^2] = \frac{1716}{1135}, E[XY] = \frac{5261}{3405}$$

```
tuappr: [0 2] [0 2] 400 400  (12/227)*(3*t + 2*t.*u).*(u<=min(1+t,2))
EX = 1.3805    EY = 1.0974    EX2 = 2.0967    EY2 = 1.5120    EXY = 1.5450
```

Exercise 11.3.31

(See Exercise 21 from "Problems On Random Vectors and Joint Distribution"). $f_{XY}(t,u) = \frac{2}{13}(t+2u)$, for $0 \le t \le 2$, $0 \le u \le \min\{2t, 3-t\}$.

$$f_X(t) = I_{[0,1]}(t)\frac{12}{13}t^2 + I_{(1,2]}(t)\frac{6}{13}(3-t)$$

$$f_Y(u) = I_{[0,1]}(u)(\frac{4}{13} + \frac{8}{13}u - \frac{9}{52}u^2) + I_{(1,2]}(u)(\frac{9}{13} + \frac{6}{13}u - \frac{51}{52}u^2)$$

Answer

$$E[X] = \frac{16}{13}, E[Y] = \frac{11}{12}, E[X^2] = \frac{219}{130}, E[Y^2] = \frac{83}{78}, E[XY] = \frac{431}{390}$$

```
tuappr: [0 2] [0 2] 400 400  (2/13)*(t + 2*u).*(u<=min(2*t,3-t))
EX = 1.2309    EY = 0.9169    EX2 = 1.6849    EY2 = 1.0647    EXY = 1.1056
```

Exercise 11.3.32

(See Exercise 22 from "Problems On Random Vectors and Joint Distribution").

$$f_{XY}(t,u) = I_{[0,1]}(t)\frac{3}{8}(t^2+2u) + I_{(1,2]}(t)\frac{9}{14}t^2u^2 \text{ , for } 0 \le u \le 1.$$

$$f_X(t) = I_{[0,1]}(t)\frac{3}{8}(t^2+1) + I_{(1,2]}(t)\frac{3}{14}t^2 \text{ , } f_Y(u) = \frac{1}{8} + \frac{3}{4}u + \frac{3}{2}u^2 \quad \text{(0 \le u \le 1\)}$$

Answer

$$E[X] = \frac{243}{224}, E[Y] = \frac{11}{16}, E[X^2] = \frac{107}{70}, E[Y^2] = \frac{127}{240}, E[XY] = \frac{347}{448}$$

```
tuappr: [0 2] [0 1] 400 200  (3/8)*(t.^2 + 2*u).*(t<=1) + (9/14)*(t.^2.*u.^2).*(t
EX = 1.0848    EY = 0.6875    EX2 = 1.5286    EY2 = 0.5292    EXY = 0.7745
```

Exercise 11.3.33

The class $\{X, Y, Z\}$ of random variables is iid(independent, identically distributed) with common distribution

$$X = \text{[-5 -1 3 4 7] } PX = \text{0.01 * [15 20 30 25 10]}$$

Let $W = 3X - 4Y + 2Z$. Determine $E[W]$. Do this using icalc, then repeat with icalc3 and compare results.

Answer

Use x and px to prevent renaming.

```
x = [-5 -1 3 4 7];
px = 0.01*[15 20 30 25 10];
icalc
Enter row matrix of X-values  x
Enter row matrix of Y-values  x
Enter X probabilities px
Enter Y probabilities px
 Use array operations on matrices X, Y, PX, PY, t, u, and P
 G = 3*t - 4*u
 [R,PR] = csort(G,P);
 icalc
Enter row matrix of X-values  R
Enter row matrix of Y-values  x
Enter X probabilities  PR
Enter Y probabilities  px
 Use array operations on matrices X, Y, PX, PY, t, u, and P
H = t + 2*u;
EH = total(H.*P)
EH = 1.6500
[W,PW] = csort(H,P);  % Alternate
EW = W*PW'
EW = 1.6500
icalc3                % Solution with icalc3
```

```
Enter row matrix of X-values  x
Enter row matrix of Y-values  x
Enter row matrix of Z-values  x
Enter X probabilities  px
Enter Y probabilities  px
Enter Z probabilities  px
Use array operations on matrices X, Y, Z,
PX, PY, PZ, t, u, v, and P
K = 3*t - 4*u + 2*v;
EK = total(K.*P)
EK = 1.6500
```

Exercise 11.3.34

(See Exercise 5 from "Problems on Functions of Random Variables") The cultural committee of a student organization has arranged a special deal for tickets to a concert. The agreement is that the organization will purchase ten tickets at $20 each (regardless of the number of individual buyers). Additional tickets are available according to the following schedule:

11-20, $18 each; 21-30 $16 each; 31-50, $15 each; 51-100, $13 each

If the number of purchasers is a random variable X, the total cost (in dollars) is a random quantity $Z = g(X)$ described by

$$g(X) = 200 + 18I_{M1}(X)(X-10) + (16-18)I_{M2}(X)(X-20) +$$

$$(15-16)I_{M3}(X)(X-30) + (13-15)I_{M4}(X)(X-50)$$

$$\text{where } M1 = [10, \infty), M2 = [20, \infty), M3 = [30, \infty), M4 = [50, \infty)$$

Suppose X ~ Poisson (75). Approximate the Poisson distribution by truncating at 150. Determine $E[Z]$ and $E[Z^2]$.

Answer

```
X = 0:150;
PX = ipoisson(75, X);
G = 200 + 18*(X - 10).*(X>=10) + (16 - 18)*(X - 20).*(X>=20) + ...
      (15 - 16)*(X - 30).*(X>=30) + (13 - 15)*(X>=50);
[Z,PZ] = csort(G,PX);
EZ = Z*PZ'
EZ = 1.1650e+03
EZ2 = (Z.^2)*PZ'
EZ2 = 1/3699e+06
```

Exercise 11.3.35

The pair $\{X, Y\}$ has the joint distribution (in m-file npr08_07.m):

$P(X = t, Y = u)$

t =	-3.1	-0.5	1.2	2.4	3.7	4.9
u = 7.5	0.0090	0.0396	0.0594	0.0216	0.0440	0.0203
4.1	0.0495	0	0.1089	0.0528	0.0363	0.0231
-2.0	0.0405	0.1320	0.0891	0.0324	0.0297	0.0189

-3.8	0.0510	0.0484	0.0726	0.0132	0	0.0077

Let $Z = g(X,Y) = 3X^2 + 2XY - Y^2)$. Determine $E[Z]$ and $E[Z^2]$.

Answer

```
npr08_07
Data are in X, Y, P
jcalc
------------------
G = 3*t.^2 + 2*t.*u - u.^2;
EG = total(G.*P)
EG = 5.2975
ez2 = total(G.^2.*P)
EG2 = 1.0868e+03
[Z,PZ] = csort(G,P);        % Alternate
EZ = Z*PZ'
EZ = 5.2975
EZ2 = (Z.^2)*PZ'
EZ2 = 1.0868e+03
```

Exercise 11.3.36

For the pair $\{X,Y\}$ in Exercise 11.3.35, let

$$W = g(X,Y) = \begin{cases} X & \text{for } X+Y \le 4 \\ 2Y & \text{for } X+Y > 4 \end{cases} = I_M(X,Y)X + I_{M^c}(X,Y)2Y$$

Determine $E[W]$ and $E[W^2]$.

Answer

```
H = t.*(t+u<=4) + 2*u.*(t+u>4);
EH = total(H.*P)
EH = 4.7379
EH2 = total(H.^2.*P)
EH2 = 61.4351
[W,PW] = csort(H,P);    %Alternate
EW = W*PW'
EW = 4.7379
EW2 = (W.^2)*PW'
EW2 = 61.4351
```

For the distribution in Exercises 37-41 below

a. Determine analytically $E[Z]$ and $E[Z^2]$
b. Use a discrete approximation to calculate the same quantities.

Exercise 11.3.37

$f_{XY}(t,u) = \frac{3}{88}(2t+3u^2)$ for $0 \le t \le 2$, $0 \le u \le 1+t$ (see Exercise 25).

$$Z = I_{[0,1]}(X)4X + I_{(1,2]}(X)(X+Y)$$

Answer

$$E[Z] = \frac{3}{88}\int_0^1 \int_0^{1+t} 4t(2t+3u^2)\,dudt + \frac{3}{88}\int_1^2 \int_0^{1+t}(t+u)(2t+3u^2)\,dudt = \frac{5649}{1760}$$

$$E[Z^2] = \frac{3}{88}\int_0^1 \int_0^{1+t}(4t)^2(2t+3u^2)\,dudt + \frac{3}{88}\int_1^2 \int_0^{1+t}(t+u)^2(2t+3u^2)\,dudt = \frac{4881}{440}$$

```
tuappr: [0 2] [0 3] 200 300 (3/88)*(2*t+3*u.^2).*(u<=1+t)
G = 4*t.*(t<=1) + (t + u).*(t>1);
EG = total(G.*P)
EG = 3.2086
EG2 = total(G.^2.*P)
EG2 = 11.0872
```

Exercise 11.3.38

$f_{XY}(t,u) = \frac{24}{11}tu$ for $0 \le t \le 2$, $0 \le u \le \min\{1, 2-t\}$ (see Exercise 27)

$$Z = I_M(X,Y)\frac{1}{2}X + I_{M^c}(X,Y)Y^2\,,\ M = \{(t,u): u > t\}$$

Answer

$$E[Z] = \frac{12}{11}\int_0^1 \int_t^1 t^2u\,dudt + \frac{24}{11}\int_0^1 \int_0^t tu^3\,dudt + \frac{24}{11}\int_1^2 \int_0^{2-t} tu^3\,dudt = \frac{16}{55}$$

$$E[Z^2] = \frac{6}{11}\int_0^1 \int_t^1 t^3u\,dudt + \frac{24}{11}\int_0^1 \int_0^t tu^5\,dudt + \frac{24}{11}\int_1^2 \int_0^{2-t} tu^5\,dudt = \frac{39}{308}$$

```
tuappr: [0 2] [0 1] 400 200  (24/11)*t.*u.*(u<=min(1,2-t))
G = (1/2)*t.*(u>t) + u.^2.*(u<=t);
EZ = 0.2920 EZ2 = 0.1278
```

Exercise 11.3.39

$f_{XY}(t,u) = \frac{3}{23}(t+2u)$ for $0 \le t \le 2$, $0 \le u \le \max\{2-t, t\}$ (see Exercise 28)

$$Z = I_M(X,Y)(X+Y) + I_{M^c}(X,Y)2Y\,,\ M = \{(t,u): \max(t,u) \le 1\}$$

Answer

$$E[Z] = \frac{3}{23}\int_0^1 \int_0^1 (t+u)(t+2u)\,dudt + \frac{3}{23}\int_0^1 \int_1^{2-t} 2u(1+2u)\,dudt + \frac{3}{23}\int_1^2 \int_1^t 2u(t+2u)\,dudt = \frac{175}{92}$$

$$E[Z^2] = \frac{3}{23}\int_0^1 \int_0^1 (t+u)^2(t+2u)\,dudt + \frac{3}{23}\int_0^1 \int_1^{2-t} 4u^2(1+2u)\,dudt + \frac{3}{23}\int_1^2 \int_1^t 4u^2(t+2u)\,dudt =$$

```
tuappr: [0 2] [0 2] 400 400  (3/23)*(t+2*u).*(u<=max(2-t,t))
M = max(t,u)<=1;
G = (t+u).*M + 2*u.*(1-M);
EZ = total(G.*P)
EZ = 1.9048
EZ2 = total(G.^2.*P)
EZ2 = 4.4963
```

Exercise 11.3.40

$f_{XY}(t,u) = \frac{12}{179}(3t^2 + u)$, for $0 \le t \le 2$, $0 \le u \le \min\{2, 3-t\}$ (see Exercise 19)

$$Z = I_M(X,Y)(X+Y) + I_{M^c}(X,Y)2Y^2 \;, M = \{(t,u) : t \le 1, u \ge 1\}$$

Answer

$$E[Z] = \frac{12}{179}\int_0^1\int_1^2 (t+u)(3t^2+u)\,du dt + \frac{12}{179}\int_0^1\int_0^1 2u^2(3t^2+u)\,du dt + \frac{12}{179}\int_1^2\int_0^{3-t} 2u^2(3t^2+u)\,du dt = \frac{1422}{895}$$

$$E[Z^2] = \frac{12}{179}\int_0^1\int_1^2 (t+u)^2(3t^2+u)\,du dt + \frac{12}{179}\int_0^1\int_0^1 4u^4(3t^2+u)\,du dt + \frac{12}{179}\int_1^2\int_0^{3-t} 4u^4(3t^2+u)\,du dt = \frac{28296}{6265}$$

```
tuappr: [0 2] [0 2] 400 400  (12/179)*(3*t.^2 + u).*(u <= min(2,3-t))
M = (t<=1)&(u>=1);
G = (t + u).*M + 2*u.^2.*(1 - M);
EZ = total(G.*P)
EZ = 1.5898
EZ2 = total(G.^2.*P)
EZ2 = 4.5224
```

Exercise 11.3.41

$f_{XY}(t,u) = \frac{12}{227}(2t + 2tu)$, for $0 \le t \le 2$, $0 \le u \le \min\{1+t, 2\}$ (see Exercise 30).

$$Z = I_M(X,Y)X + I_{M^c}(X,Y)XY \;, M = \{(t,u) : u \le \min(1, 2-t)\}$$

Answer

$$E[Z] = \frac{12}{227}\int_0^1\int_0^1 t(3t+2tu)\,du dt + \frac{12}{227}\int_1^2\int_0^{2-t} t(3t+2tu)\,du dt +$$

$$\frac{12}{227}\int_0^1\int_1^{1+t} tu(3t+2tu)\,du dt + \frac{12}{227}\int_1^2\int_{2-t}^2 tu(3t+2tu)\,du dt = \frac{5774}{3405}$$

$$E[Z^2] = \frac{56673}{15890}$$

```
tuappr: [0 2] [0 2] 400 400  (12/227)*(3*t + 2*t.*u).*(u <= min(1+t,2))
M = u <= min(1,2-t);
G = t.*M + t.*u.*(1 - M);
EZ = total(G.*P)
EZ = 1.6955
```

```
EZ2 = total(G.^2.*P)
EZ2 = 3.5659
```

Exercise 11.3.42

The class $\{X, Y, Z\}$ is independent. (See Exercise 16 from "Problems on Functions of Random Variables", m-file npr10_16.m)

$X = -2I_A + I_B + 3I_C$. Minterm probabilities are (in the usual order)

0.255 0.025 0.375 0.045 0.108 0.012 0.162 0.018

$Y = I_D + 3I_E + I_F - 3$. The class $\{D, E, F\}$ is independent with

$$P(D) = 0.32 \; P(E) = 0.56 \; P(F) = 0.40$$

Z has distribution

Value	-1.3	1.2	2.7	3.4	5.8
Probability	0.12	0.24	0.43	0.13	0.08

$W = X^2 + 3XY^2 - 3Z$. Determine $E[W]$ and $E[W^2]$.

Answer

```
npr10_16
Data are in cx, pmx, cy, pmy, Z, PZ
[X,PX] = canonicf(cx,pmx);
[Y,PY] - canonicf(cy,pmy);
icalc3
input: X, Y, Z, PX, PY, PZ
-------------
Use array operations on matrices X, Y, Z.
PX, PY, PZ, t, u, v, and P
G = t.^2 + 3*t.*u.^2 - 3*v;
[W,PW] = csort(G,P);
EW = W*PW'
EW = -1.8673
EW2 = (W.^2)*PW'
EW2 = 426.8529
```