17.7: Appendix G to Applied Probability- Properties of conditional independence, given a random vector

Definition

The pair $\{X, Y\}$ is *conditionally independent, givenZ*, denoted $\{X, Y\}$ ci $|Z$ iff

$$E[I_M(X)I_N(Y)|Z] = E[I_M(X)|Z]E[I_N(Y)|Z] \text{ a.s. for all Borel sets } M, N$$

An arbitrary class $\{X_t : t \in T\}$ of random vectors is conditionally independent, give Z, iff such a product rule holds for each finite subclass or two or more members of the class.

Remark. The expression “for all Borel sets M, N," here and elsewhere, implies the sets are on the appropriate codomains. Also, the expressions below “for all Borel functions g,” etc., imply that the functions are real-valued, such that the indicated expectations are finite.

The following are equivalent. Each is necessary and sufficient that $\{X, Y\}$ ci $|Z$.

(CI1): $E[I_M(X)I_N(Y)|Z] = E[I_M(X)|Z]E[I_N(Y)|Z]$ a.s. for all Borel sets M, N
(CI2): $E[I_M(X)|Z, Y] = E[I_M(X)|Z]$ a.s. for all Borel sets M
(CI3): $E[I_M(X)I_Q(Z)|Z, Y] = E[I_M(X)I_Q(Z)|Z]$ a.s. for all Borel sets M, Q
(CI4): $E[I_M(X)I_Q(Z)|Y] = E\{E[I_M(X)I_Q(Z)|Z]|Y\}$ a.s. for all Borel sets M, Q

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(CI5): $E[g(X, Z)h(Y, Z)|Z] = E[g(X, Z)|Z]E[h(Y, Z)|Z]$ a.s. for all Borel functions g, h
(CI6): $E[g(X, Z)|Z, Y] = E[g(X, Z)|Z]$ a.s. for all Borel function g
(CI7): For any Borel function g, there exists a Borel function e_g such that

$$E[g(X, Z)|Z, Y] = e_g(Z) \text{ a.s.}$$

(CI8): $E[g(X, Z)|Y] = E\{E[g(X, Z)|Z]|Y\}$ a.s. for all Borel functions g

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(CI9): $\{U, V\}$ ci $|Z$, where $U = g(X, Z)$ and $V = h(Y, Z)$, for any Borel functions g, h.

Additional properties of conditional independence

(CI10): $\{X, Y\}$ ci $|Z$ implies $\{X, Y\}$ ci $|(Z, U)$, $\{X, Y\}$ ci $|(Z, V)$, and $\{X, Y\}$ ci $|(Z, U, V)$, where $U = h(X)$ and $V = k(Y)$, with h, k Borel.
(CI11): $\{X, Z\}$ ci $|Y$ and $\{X, W\}$ ci $|(Y, Z)$ iff $\{X, (Z, W)\}$ ci $|Y$.
(CI12): $\{X, Z\}$ ci $|Y$ and $\{(X, Y), W\}$ ci $|Z$ implies $\{X, (Z, W)\}$ is independent.
(CI13): $\{X, Y\}$ is independent and $\{X, Y\}$ ci $|Y$ iff $\{X, (Y, Z)\}$ is independent.
(CI14): $\{X, Y\}$ ci $|Z$ implies $E[g(X, Y)|Y = u, Z = v] = E[g(X, u)|Z = v]$ a.s. $[P_{YZ}]$
(CI15): $\{X, Y\}$ ci $|Z$ implies
a. $E[g(X, Z)h(Y, Z)] = E\{E[g(X, Z)|Z]E[h(Y, Z)|Z]\} = E[e_1(Z)e_2(Z)]$
b. $E[g(Y)|X \in M]P(X \in M) = E\{E[I_M(X)|Z]E[g(Y)|Z]\}$
(CI16): $\{(X, Y), Z\}$ ci $|W$ iff $E[I_M(X)I_N(Y)I_Q(Z)|W] = E[I_M(X)I_N(Y)|W]E[I_Q(Z)|W]$ a.s. for all Borel sets M, N, Q

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