1.4: Problems on Probability Systems

Exercise 1.4.1

Let Ω consist of the set of positive integers. Consider the subsets

$A = \{\omega : \omega \le 12\}$ $B = \{\omega : \omega < 8\}$ $C = \{\omega : \omega \text{ is even}\}$

$D = \{\omega : \omega \text{ is a multiple of } 3\}$ $E = \{\omega : \omega \text{ is a multiple of } 4\}$

Describe in terms of A, B, C, D, E and their complements the following sets:

a. {1, 3, 5, 7}
b. {3, 6, 9}
c. {8, 10}
d. The even integers greater than 12
e. The positive integers which are multiples of six.
f. The integers which are even and no greater than 6 or which are odd and greater than 12.

Answer

$a = BC^c$
$b = DAE^c$
$c = CAB^cD^c$
$d = CA^c$
$e = CD$
$f = BC \bigvee A^cC^c$

Exercise 1.4.2

Let Ω be the set of integers 0 through 10. Let $A = \{5, 6, 7, 8\}$, $B =$ the odd integers in Ω, and $C =$ the integers in Ω which are even or less than three. Describe the following sets by listing their elements.

a. AB
b. AC
c. $AB^c \cup C$
d. ABC^c
e. $A \cup B^c$
f. $A \cup BC^c$
g. ABC
h. A^cBC^c

Answer

a. $AB = 5, 7$
b. $AC = 6, 8$
c. $AB^c \cup C = C$
d. $ABC^c = AB$
e. $A \cup B^c = 0, 2, 4, 5, 6, 7, 8, 10$
f. $ABC = \emptyset$
g. $A^cBC^c = 3, 9$

Exercise 1.4.3

Consider fifteen-word messages in English. Let $A =$ the set of such messages which contain the word "bank" and let $B =$ the set of messages which contain the word "bank" *and* the word "credit." Which event has the greater probability? Why?

Answer

$B \subset A$ implies $P(B) \leq P(A)$.

Exercise 1.4.4

A group of five persons consists of two men and three women. They are selected one-by-one in a random manner. Let E_i be the event a man is selected on the ith selection. Write an expression for the event that both men have been selected by the third selection.

Answer

$A = E_1 E_2 \bigvee E_1 E_2^c E_3 \bigvee E_1^c E_2 E_3$

Exercise 1.4.5

Two persons play a game consecutively until one of them is successful or there are ten unsuccessful plays. Let E_i be the event of a success on the ith play of the game. Let A, B, C be the respective events that player one, player two, or neither wins. Write an expression for each of these events in terms of the events E_i, $1 \leq i \leq 10$.

Answer

$A = E_1 \bigvee E_1^c E_2^c E_3 \bigvee E_1^c E_2^c E_3^c E_4^c E_5 \bigvee E_1^c E_2^c E_3^c E_4^c E_5^c E_6^c E_7 \bigvee E_1^c E_2^c E_3^c E_4^c E_5^c E_6^c E_7^c E_8^c E_9$

$B = E_1^c E_2 \bigvee E_1^c E_2^c E_3^c E_4 \bigvee E_1^c E_2^c E_3^c E_4^c E_5^c E_6 \bigvee E_1^c E_2^c E_3^c E_4^c E_5^c E_6^c E_7^c E_8 \bigvee E_1^c E_2^c E_3^c E_4^c E_5^c E_6^c E_7^c E_8^c E_9^c E_{10}$

$C = \bigcap_{i=1}^{10} E_i^c$

Exercise 1.4.6

Suppose the game in Exercise 1.4.5 could, in principle, be played an unlimited number of times. Write an expression for the event D that the game will be terminated with a success in a finite number of times. Write an expression for the event F that the game will never terminate.

Answer

Let $F_0 = \Omega$ and $F_k = \bigcap_{i=1}^{k} E_i^c$ for $k \geq 1$. Then

$$D = \bigvee_{n=1}^{\infty} F_{n-1} E_n \text{ and } F = D^c = \bigcap_{i=1}^{\infty} E_i^c$$

Exercise 1.4.7

Find the (classical) probability that among three random digits, with each digit (0 through 9) being equally likely and each triple equally likely:

a. All three are alike.
b. No two are alike.
c. The first digit is 0.
d. Exactly two are alike.

Answer

Each triple has probability $1/10^3 = 1/1000$

a. Ten triples, all alike: $P = 10/1000$
b. $10 \times 9 \times 8$ triples all different: $P = 720/1000$
c. 100 triples with first one zero: $P = 100/1000$
d. $C(3, 2) = 3$ ways to pick two positions alike; 10 ways to pick the common value; 9 ways to pick the other. $P = 270/1000$

Exercise 1.4.8

The classical probability model is based on the assumption of equally likely outcomes. Some care must be shown in analysis to be certain that this assumption is good. A well known example is the following. Two coins are tossed. One of three outcomes is observed: Let ω_1 be the outcome both are "heads," ω_2 the outcome that both are "tails," and ω_3 be the outcome that they are different. Is it reasonable to suppose these three outcomes are equally likely? What probabilities would you assign?

Answer

$P(\{\omega_1\}) = P(\{\omega_2\}) = 1/4$, $P(\{\omega_3\}) = 1/2$

Exercise 1.4.9

A committee of five is chosen from a group of 20 people. What is the probability that a specified member of the group will be on the committee?

Answer

$C(20, 5)$ committees; $C(19, 4)$ have a designated member.

$$P = \frac{19!}{4!15!} \cdot \frac{5!15!}{20!} = 5/20 = 1/4$$

Exercise 1.4.10

Ten employees of a company drive their cars to the city each day and park randomly in ten spots. What is the (classical) probability that on a given day Jim will be in place three? There are $n!$ equally likely ways to arrange n items (order important).

Answer

10! permutations, $1 \times 9!$ permutations with Jim in place 3. $P = 9!/10! = 1/10$

Exercise 1.4.11

An extension of the classical model involves the use of areas. A certain region L (say of land) is taken as a reference. For any subregion A, define $P(A) = area(A)/area(L)$. Show that $P(\cdot)$ is a probability measure on the subregions of L.

Answer

Additivity follows from additivity of areas of disjoint regions.

Exercise 1.4.12

John thinks the probability the Houston Texans will win next Sunday is 0.3 and the probability the Dallas Cowboys will win is 0.7 (they are not playing each other). He thinks the probability both will win is somewhere between—say, 0.5. Is that a reasonable assumption? Justify your answer.

Answer

$P(AB) = 0.5$ is not reasonable. It must no greater than the minimum of $P(A) = 0.3$ and $P(B) = 0.7$.

Exercise 1.4.13

Suppose $P(A) = 0.5$ and $P(B) = 0.3$. What is the largest possible value of $P(AB)$? Using the maximum value of $P(AB)$, dertermine $P(AB^c)$, $P(A^cB)$, $P(A^cB^c)$ and $P(A \cup B)$. Are these values determined uniquely?

Answer

Draw a Venn diagram, or use algebraic expressions $P(AB^c) = P(A) - P(AB) = 0.2$

$P(A^cB) = P(B) - P(AB) = 0$ $P(A^cB^c) = P(A^c) - P(A^cB) = 0.5$ $P(A \cup B) = 0.5$

Exercise 1.4.14

For each of the following probability "assignments", fill out the table. Which assignments are not permissible? Explain why, in each case.

$P(A)$	$P(B)$	$P(AB)$	$P(A \cup B)$	$P(AB^c)$	$P(A^cB)$	$P(A) + P(B)$
0.3	0.7	0.4				
0.2	0.1	0.4				
0.3	0.7	0.2				
0.3	0.5	0				
0.3	0.8	0				

Answer

$P(A)$	$P(B)$	$P(AB)$	$P(A \cup B)$	$P(AB^c)$	$P(A^cB)$	$P(A) + P(B)$
0.3	0.7	0.4	0.6	-0.1	0.3	1.0
0.2	0.1	0.4	-0.1	-0.2	-0.3	0.3
0.3	0.7	0.2	0.8	0.1	0.5	1.0
0.3	0.5	0	0.8	0.3	0.5	0.8
0.3	0.8	0	1.1	0.3	0.8	1.1

Only the third and fourth assignments are permissible.

Exercise 1.4.15

The class $\{A, B, C\}$ of events is a partition. Event A is twice as likely as C and event B is as likely as the combination A or C. Determine the probabilities $P(A)$, $P(B)$, $P(C)$.

Answer

$P(A) + P(B) + P(C) = 1$, $P(A) = 2P(C)$, and $P(B) = P(A) + P(C) = 3P(C)$, which implies

$$P(C) = 1/6, P(A) = 1/3, P(B) = 1/2$$

Exercise 1.4.16

Determione the probability $P(A \cup B \cup C)$ in terms of the probabilities of the events A, B, C and their intersections.

Answer

$P(A \cup B \cup C) = P(A \cup B) + P(C) - P(AC \cup BC)$
$= P(A) + P(B) - P(AB) + P(C) - P(AC) - P(BC) + P(ABC)$

Exercise 1.4.17

If occurrence of event A implies occurrence of B, show that $P(A^cB) = P(B) - P(A)$.

Answer

$P(AB) = P(A)$ and $P(AB) + P(A^cB) = P(B)$ implies $P(A^cB) = P(B) - P(A)$.

Exercise 1.4.18

Show that $P(AB) \ge P(A) + P(B) - 1$.

Answer

Follows from $P(A) + P(B) - P(AB) = P(A \cup B) \le 1$.

Exercise 1.4.19

The set combination $A \oplus B = AB^c \bigvee A^cB$ is known as the *disjunctive union* or the *symetric difference* of A and B. This is the event that only one of the events A or B occurs on a trial. Determine $P(A \oplus B)$ in terms of $P(A)$, $P(B)$, and $P(AB)$

Answer

A Venn diagram shows $P(A \oplus B) = P(AB^c) + P(AB^c) = P(A) + P(B) - 2P(AB)$.

Exercise 1.4.20

Use fundamental properties of probability to show

a. $P(AB) \le P(A) \le P(A \cup B) \le P(A) + P(B)$
b. $P(\bigcap_{j=1}^{\infty} E_j) \le P(E_i) \le P(\bigcup_{j=1}^{\infty} E_j) \le \sum_{j=1}^{\infty} P(E_j)$

Answer

$AB \subset A \subset A \cup B$ implies $P(AB) \le P(A) \le P(A \cup B) = P(A) + P(B) - P(AB) \le P(A) + P(B)$. The general case follows similarly, with the last inequality determined by subadditivity.

Exercise 1.4.21

Suppose P_1, P_2 are probability measures and c_1, c_2 are positive numbers such that $c_1 + c_2 = 1$. Show that the assignment $P(E) = c_1 P_1(E) + c_2 P_2(E)$ to the class of events is a probability measure. Such a combination of probability measures is known as a *mixture*. Extend this to

$P(E) = \sum_{i=1}^{n} c_i P_i(E)$, where the P_i are probabilities measures, $c_i > 0$, and $\sum_{i=1}^{n} c_i = 1$

Answer

Clearly $P(E) \ge 0$. $P(\Omega) = c_1 P_1(\Omega) + c_2 P_2(\Omega) = 1$.

$E = \bigvee_{i=1}^{\infty} E_i$ implies $P(E) = c_1 \sum_{i=1}^{\infty} P_1(E_i) + c_2 \sum_{i=1}^{\infty} P_2(E_i) = \sum_{i=1}^{\infty} P(E_i)$

The pattern is the same for the general case, except that the sum of two terms is replaced by the sum of n terms $c_i P_i(E)$.

Exercise 1.4.22

Suppose $\{A_1, A_2, \cdots, A_n\}$ is a partition and $\{c_1, c_2, \cdots, c_n\}$ is a class of positive constants. For each event E, let

$$Q(E) = \sum_{i=1}^{n} c_i P(EA_i) / \sum_{i=1}^{n} c_i P(A_i)$$

Show that $Q(\cdot)$ us a probability measure.

Answer

Clearly $Q(E) \geq 0$ and since $A_i \Omega = A_i$ we have $Q(\Omega) = 1$. If

$$E = \bigvee_{k=1}^{\infty} E_k \text{, then } P(EA_i) = \sum_{k=1}^{\infty} P(E_k A_i) \ \ \forall i$$

Interchanging the order of summation shows that Q is countably additive.