4.4.2: Permutations with Similar Elements

Learning Objectives

In this section you will learn to

1. Count the number of possible permutations when there are repeated items

In this section we will address the following problem.

1. In how many different ways can the letters of the word MISSISSIPPI be arranged?

This is an example of Permutations with Similar Elements.

Permutations with Similar Elements

Let us determine the number of distinguishable permutations of the letters ELEMENT.

Suppose we make all the letters different by labeling the letters as follows.

$$E_1 L E_2 M E_3 N T$$

Since all the letters are now different, there are 7! different permutations.

Let us now look at one such permutation, say

$$L E_1 M E_2 N E_3 T$$

Suppose we form new permutations from this arrangement by only moving the E's. Clearly, there are 3! or 6 such arrangements. We list them below.

$$\begin{array}{l} \mathrm{LE_1ME_2NE_3} \\ \mathrm{LE_1ME_3NE_2} \\ \mathrm{LE_2ME_1NE_3T} \\ \mathrm{LE_2ME_3NE_1T} \\ \mathrm{LE_3ME_2NE_1T} \\ \mathrm{LE_3ME_{\mathit{I}}NE_2T} \end{array}$$

Because the E's are not different, there is only one arrangement LEMENET and not six. This is true for every permutation.

Let us suppose there are n different permutations of the letters ELEMENT.

Then there are $n \cdot 3!$ permutations of the letters $E_1 L E_2 M E_3 N T$.

But we know there are 7! permutations of the letters $E_1 L E_2 M E_3 N T$.

Therefore, $n \cdot 3! = 7!$

Or $n = \frac{7!}{3!}$.

This gives us the method we are looking for.

Definition: Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, is

$$\frac{n!}{r_1! r_2! \ldots r_k!} \tag{4.4.2.1}$$

Example 4.4.2.3

Find the number of different permutations of the letters of the word MISSISSIPPI.

Solution

The word MISSISSIPPI has 11 letters. If the letters were all different there would have been 11! different permutations. But MISSISSIPPI has 4 S's, 4 I's, and 2 P's that are alike.

So the answer is $\frac{11!}{4!4!2!} = 34,650$.

Example 4.4.2.4

If a coin is tossed six times, how many different outcomes consisting of 4 heads and 2 tails are there?

Solution

Again, we have permutations with similar elements.

We are looking for permutations for the letters HHHHTT.

The answer is $\frac{6!}{4!2!} = 15$.

Example 4.4.2.5

In how many different ways can 4 nickels, 3 dimes, and 2 quarters be arranged in a row?

Solution

Assuming that all nickels are similar, all dimes are similar, and all quarters are similar, we have permutations with similar elements. Therefore, the answer is

$$\frac{9!}{4!3!2!} = 1260$$

Example 4.4.2.6

A stock broker wants to assign 20 new clients equally to 4 of its salespeople. In how many different ways can this be done?

Solution

This means that each sales person gets 5 clients. The problem can be thought of as an ordered partitions problem. In that case, using the formula we get

$$\frac{20!}{5!5!5!5!} = 11,732,745,024$$

Example 4.4.2.7

A shopping mall has a straight row of 5 flagpoles at its main entrance plaza. It has 3 identical green flags and 2 identical yellow flags. How many distinct arrangements of flags on the flagpoles are possible?

Solution

The problem can be thought of as distinct permutations of the letters GGGYY; that is arrangements of 5 letters, where 3 letters are similar, and the remaining 2 letters are similar:

$$\frac{5!}{3!2!} = 10$$

Just to provide a little more insight into the solution, we list all 10 distinct permutations:

GGGYY, GGYGY, GGYYG, GYGGY, GYGYG, GYYGG, YGGGY, YGGYG, YGYGY, YYGGG

We summarize.

Summary

Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, such that $\mathrm{n} = \mathrm{r}_1 + \mathrm{r}_2 + \ldots + \mathrm{r}_\mathrm{k}$ is

$$\frac{n!}{r_1! r_2! \ldots r_k!}$$

This is also referred to as **ordered partitions**.