

## 6.1.1: The Standard Normal Distribution

### Z-Scores

The standard normal distribution is a normal distribution of standardized values called *z-scores*. A *z-score* is measured in units of the standard deviation.

#### Definition: Z-Score

If  $X$  is a normally distributed random variable and  $X \sim N(\mu, \sigma)$ , then the *z-score* is:

$$z = \frac{x - \mu}{\sigma} \quad (6.1.1.1)$$

**The *z-score* tells you how many standard deviations the value  $x$  is above (to the right of) or below (to the left of) the mean,  $\mu$ .** Values of  $x$  that are larger than the mean have positive *z-scores*, and values of  $x$  that are smaller than the mean have negative *z-scores*. If  $x$  equals the mean, then  $x$  has a *z-score* of zero. For example, if the mean of a normal distribution is five and the standard deviation is two, the value 11 is three standard deviations above (or to the right of) the mean. The calculation is as follows:

$$\begin{aligned} x &= \mu + (z)(\sigma) \\ &= 5 + (3)(2) = 11 \end{aligned}$$

The *z-score* is three.

Since the mean for the standard normal distribution is zero and the standard deviation is one, then the transformation in Equation 6.1.1.1 produces the distribution  $Z \sim N(0, 1)$ . The value  $x$  comes from a normal distribution with mean  $\mu$  and standard deviation  $\sigma$ .

*A *z-score* is measured in units of the standard deviation.*

#### ✓ Example 6.1.1.1

Suppose  $X \sim N(5, 6)$ . This says that  $x$  is a normally distributed random variable with mean  $\mu = 5$  and standard deviation  $\sigma = 6$ . Suppose  $x = 17$ . Then (via Equation 6.1.1.1):

$$z = \frac{x - \mu}{\sigma} = \frac{17 - 5}{6} = 2$$

This means that  $x = 17$  is **two** standard deviations ( $2\sigma$ ) above or to the right of the mean  $\mu = 5$ . The standard deviation is  $\sigma = 6$ .

Notice that:  $5 + (2)(6) = 17$  (The pattern is  $\mu + z\sigma = x$ )

Now suppose  $x = 1$ . Then:

$$z = \frac{x - \mu}{\sigma} = \frac{1 - 5}{6} = -0.67$$

(rounded to two decimal places)

This means that  $x = 1$  is 0.67 standard deviations ( $-0.67\sigma$ ) below or to the left of the mean  $\mu = 5$ . Notice that:  $5 + (-0.67)(6)$  is approximately equal to one (This has the pattern  $\mu + (-0.67)\sigma = 1$ )

Summarizing, when  $z$  is positive,  $x$  is above or to the right of  $\mu$  and when  $z$  is negative,  $x$  is to the left of or below  $\mu$ . Or, when  $z$  is positive,  $x$  is greater than  $\mu$ , and when  $z$  is negative  $x$  is less than  $\mu$ .

#### ? Exercise 6.1.1.1

What is the *z-score* of  $x$ , when  $x = 1$  and  $X \sim N(12, 3)$ ?

**Answer**

$$z = \frac{1 - 12}{3} \approx -3.67$$

### ✓ Example 6.1.1.2

Some doctors believe that a person can lose five pounds, on the average, in a month by reducing his or her fat intake and by exercising consistently. Suppose weight loss has a normal distribution. Let  $X$  = the amount of weight lost(in pounds) by a person in a month. Use a standard deviation of two pounds.  $X \sim N(5, 2)$ . Fill in the blanks.

- Suppose a person **lost** ten pounds in a month. The  $z$ -score when  $x = 10$  pounds is  $z = 2.5$  (verify). This  $z$ -score tells you that  $x = 10$  is \_\_\_\_\_ standard deviations to the \_\_\_\_\_ (right or left) of the mean \_\_\_\_\_. (What is the mean?).
- Suppose a person **gained** three pounds (a negative weight loss). Then  $z =$  \_\_\_\_\_. This  $z$ -score tells you that  $x = -3$  is \_\_\_\_\_ standard deviations to the \_\_\_\_\_ (right or left) of the mean.

### Answers

- This  $z$ -score tells you that  $x = 10$  is 2.5 standard deviations to the right of the mean five.
- Suppose the random variables  $X$  and  $Y$  have the following normal distributions:  $X \sim N(5, 6)$  and  $Y \sim N(2, 1)$ . If  $x = 17$ , then  $z = 2$ . (This was previously shown.) If  $y = 4$ , what is  $z$ ?

$$z = \frac{y - \mu}{\sigma} = \frac{4 - 2}{1} = 2$$

where  $\mu = 2$  and  $\sigma = 1$ .

The  $z$ -score for  $y = 4$  is  $z = 2$ . This means that four is  $z = 2$  standard deviations to the right of the mean. Therefore,  $x = 17$  and  $y = 4$  are both two (of their own) standard deviations to the right of their respective means.

The  $z$ -score allows us to compare data that are scaled differently. To understand the concept, suppose  $X \sim N(5, 6)$  represents weight gains for one group of people who are trying to gain weight in a six week period and  $Y \sim N(2, 1)$  measures the same weight gain for a second group of people. A negative weight gain would be a weight loss. Since  $x = 17$  and  $y = 4$  are each two standard deviations to the right of their means, they represent the same, standardized weight gain **relative to their means**.

### ? Exercise 6.1.1.2

Fill in the blanks.

Jerome averages 16 points a game with a standard deviation of four points.  $X \sim N(16, 4)$ . Suppose Jerome scores ten points in a game. The  $z$ -score when  $x = 10$  is  $-1.5$ . This score tells you that  $x = 10$  is \_\_\_\_\_ standard deviations to the \_\_\_\_\_ (right or left) of the mean \_\_\_\_\_. (What is the mean?).

### Answer

1.5, left, 16

## The Empirical Rule

If  $X$  is a random variable and has a normal distribution with mean  $\mu$  and standard deviation  $\sigma$ , then the *Empirical Rule* says the following:

- About 68% of the  $x$  values lie between  $-1\sigma$  and  $+1\sigma$  of the mean  $\mu$  (within one standard deviation of the mean).
- About 95% of the  $x$  values lie between  $-2\sigma$  and  $+2\sigma$  of the mean  $\mu$  (within two standard deviations of the mean).
- About 99.7% of the  $x$  values lie between  $-3\sigma$  and  $+3\sigma$  of the mean  $\mu$  (within three standard deviations of the mean). Notice that almost all the  $x$  values lie within three standard deviations of the mean.
- The  $z$ -scores for  $+1\sigma$  and  $-1\sigma$  are  $+1$  and  $-1$ , respectively.
- The  $z$ -scores for  $+2\sigma$  and  $-2\sigma$  are  $+2$  and  $-2$ , respectively.
- The  $z$ -scores for  $+3\sigma$  and  $-3\sigma$  are  $+3$  and  $-3$  respectively.

The empirical rule is also known as the 68-95-99.7 rule.

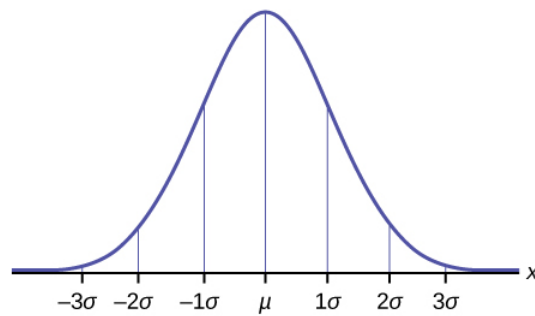


Figure 6.1.1.1

### ✓ Example 6.1.1.3

The mean height of 15 to 18-year-old males from Chile from 2009 to 2010 was 170 cm with a standard deviation of 6.28 cm. Male heights are known to follow a normal distribution. Let  $X$  = the height of a 15 to 18-year-old male from Chile in 2009 to 2010. Then  $X \sim N(170, 6.28)$ .

- Suppose a 15 to 18-year-old male from Chile was 168 cm tall from 2009 to 2010. The  $z$ -score when  $x = 168$  cm is  $z =$  \_\_\_\_\_. This  $z$ -score tells you that  $x = 168$  is \_\_\_\_\_ standard deviations to the \_\_\_\_\_ (right or left) of the mean \_\_\_\_\_. (What is the mean?).
- Suppose that the height of a 15 to 18-year-old male from Chile from 2009 to 2010 has a  $z$ -score of  $z = 1.27$ . What is the male's height? The  $z$ -score ( $z = 1.27$ ) tells you that the male's height is \_\_\_\_\_ standard deviations to the \_\_\_\_\_ (right or left) of the mean.

### Answers

- 0.32, 0.32, left, 170
- 177.98, 1.27, right

### ? Exercise 6.1.1.3

Use the information in Example 6.1.1.3 to answer the following questions.

- Suppose a 15 to 18-year-old male from Chile was 176 cm tall from 2009 to 2010. The  $z$ -score when  $x = 176$  cm is  $z =$  \_\_\_\_\_. This  $z$ -score tells you that  $x = 176$  cm is \_\_\_\_\_ standard deviations to the \_\_\_\_\_ (right or left) of the mean \_\_\_\_\_. (What is the mean?).
- Suppose that the height of a 15 to 18-year-old male from Chile from 2009 to 2010 has a  $z$ -score of  $z = -2$ . What is the male's height? The  $z$ -score ( $z = -2$ ) tells you that the male's height is \_\_\_\_\_ standard deviations to the \_\_\_\_\_ (right or left) of the mean.

### Answer

Solve the equation  $z = \frac{x - \mu}{\sigma}$  for  $z$ .  $x = \mu + (z)(\sigma)$

$z = \frac{176 - 170}{6.28}$ , This  $z$ -score tells you that  $x = 176$  cm is 0.96 standard deviations to the right of the mean 170 cm.

### Answer

Solve the equation  $z = \frac{x - \mu}{\sigma}$  for  $z$ .  $x = \mu + (z)(\sigma)$

$X = 157.44$  cm, The  $z$ -score ( $z = -2$ ) tells you that the male's height is two standard deviations to the left of the mean.

## ✓ Example 6.1.1.4

From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was 172.36 cm, and the standard deviation was 6.34 cm. Let  $Y$  = the height of 15 to 18-year-old males from 1984 to 1985. Then  $Y \sim N(172.36, 6.34)$ .

The mean height of 15 to 18-year-old males from Chile from 2009 to 2010 was 170 cm with a standard deviation of 6.28 cm. Male heights are known to follow a normal distribution. Let  $X$  = the height of a 15 to 18-year-old male from Chile in 2009 to 2010. Then  $X \sim N(170, 6.28)$ .

Find the  $z$ -scores for  $x = 160.58$  cm and  $y = 162.85$  cm. Interpret each  $z$ -score. What can you say about  $x = 160.58$  cm and  $y = 162.85$  cm?

**Answer**

- The  $z$ -score (Equation 6.1.1.1) for  $x = 160.58$  is  $z = -1.5$ .
- The  $z$ -score for  $y = 162.85$  is  $z = -1.5$ .

Both  $x = 160.58$  and  $y = 162.85$  deviate the same number of standard deviations from their respective means and in the same direction.

## ? Exercise 6.1.1.4

In 2012, 1,664,479 students took the SAT exam. The distribution of scores in the verbal section of the SAT had a mean  $\mu = 496$  and a standard deviation  $\sigma = 114$ . Let  $X$  = a SAT exam verbal section score in 2012. Then  $X \sim N(496, 114)$ .

Find the  $z$ -scores for  $x_1 = 325$  and  $x_2 = 366.21$ . Interpret each  $z$ -score. What can you say about  $x_1 = 325$  and  $x_2 = 366.21$ ?

**Answer**

The  $z$ -score (Equation 6.1.1.1) for  $x_1 = 325$  is  $z_1 = -1.15$ .

The  $z$ -score (Equation 6.1.1.1) for  $x_2 = 366.21$  is  $z_2 = -1.14$ .

Student 2 scored closer to the mean than Student 1 and, since they both had negative  $z$ -scores, Student 2 had the better score.

## ✓ Example 6.1.1.5

Suppose  $x$  has a normal distribution with mean 50 and standard deviation 6.

- About 68% of the  $x$  values lie within one standard deviation of the mean. Therefore, about 68% of the  $x$  values lie between  $-1\sigma = (-1)(6) = -6$  and  $1\sigma = (1)(6) = 6$  of the mean 50. The values  $50 - 6 = 44$  and  $50 + 6 = 56$  are within one standard deviation from the mean 50. The  $z$ -scores are  $-1$  and  $+1$  for 44 and 56, respectively.
- About 95% of the  $x$  values lie within two standard deviations of the mean. Therefore, about 95% of the  $x$  values lie between  $-2\sigma = (-2)(6) = -12$  and  $2\sigma = (2)(6) = 12$ . The values  $50 - 12 = 38$  and  $50 + 12 = 62$  are within two standard deviations from the mean 50. The  $z$ -scores are  $-2$  and  $+2$  for 38 and 62, respectively.
- About 99.7% of the  $x$  values lie within three standard deviations of the mean. Therefore, about 99.7% of the  $x$  values lie between  $-3\sigma = (-3)(6) = -18$  and  $3\sigma = (3)(6) = 18$  from the mean 50. The values  $50 - 18 = 32$  and  $50 + 18 = 68$  are within three standard deviations of the mean 50. The  $z$ -scores are  $-3$  and  $+3$  for 32 and 68, respectively.

## ? Exercise 6.1.1.5

Suppose  $X$  has a normal distribution with mean 25 and standard deviation five. Between what values of  $x$  do 68% of the values lie?

**Answer**

between 20 and 30.

### ✓ Example 6.1.1.6

From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was 172.36 cm, and the standard deviation was 6.34 cm. Let  $Y$  = the height of 15 to 18-year-old males in 1984 to 1985. Then  $Y \sim N(172.36, 6.34)$ .

- About 68% of the  $y$  values lie between what two values? These values are \_\_\_\_\_. The  $z$ -scores are \_\_\_\_\_, respectively.
- About 95% of the  $y$  values lie between what two values? These values are \_\_\_\_\_. The  $z$ -scores are \_\_\_\_\_, respectively.
- About 99.7% of the  $y$  values lie between what two values? These values are \_\_\_\_\_. The  $z$ -scores are \_\_\_\_\_, respectively.

#### Answer

- About 68% of the values lie between 166.02 and 178.7. The  $z$ -scores are  $-1$  and  $1$ .
- About 95% of the values lie between 159.68 and 185.04. The  $z$ -scores are  $-2$  and  $2$ .
- About 99.7% of the values lie between 153.34 and 191.38. The  $z$ -scores are  $-3$  and  $3$ .

### ? Exercise 6.1.1.6

The scores on a college entrance exam have an approximate normal distribution with mean,  $\mu = 52$  points and a standard deviation,  $\sigma = 11$  points.

- About 68% of the  $y$  values lie between what two values? These values are \_\_\_\_\_. The  $z$ -scores are \_\_\_\_\_, respectively.
- About 95% of the  $y$  values lie between what two values? These values are \_\_\_\_\_. The  $z$ -scores are \_\_\_\_\_, respectively.
- About 99.7% of the  $y$  values lie between what two values? These values are \_\_\_\_\_. The  $z$ -scores are \_\_\_\_\_, respectively.

#### Answer a

About 68% of the values lie between the values 41 and 63. The  $z$ -scores are  $-1$  and  $1$ , respectively.

#### Answer b

About 95% of the values lie between the values 30 and 74. The  $z$ -scores are  $-2$  and  $2$ , respectively.

#### Answer c

About 99.7% of the values lie between the values 19 and 85. The  $z$ -scores are  $-3$  and  $3$ , respectively.

## Summary

A  $z$ -score is a standardized value. Its distribution is the standard normal,  $Z \sim N(0, 1)$ . The mean of the  $z$ -scores is zero and the standard deviation is one. If  $y$  is the  $z$ -score for a value  $x$  from the normal distribution  $N(\mu, \sigma)$  then  $z$  tells you how many standard deviations  $x$  is above (greater than) or below (less than)  $\mu$ .

## Formula Review

$$Z \sim N(0, 1)$$

$z = a$  standardized value ( $z$ -score)

mean = 0; standard deviation = 1

To find the  $K^{\text{th}}$  percentile of  $X$  when the  $z$ -scores is known:

$$k = \mu + (z)\sigma$$

$$z\text{-score: } z = \frac{x - \mu}{\sigma}$$

$Z$  = the random variable for z-scores

$Z \sim N(0, 1)$

## Glossary

### Standard Normal Distribution

a continuous random variable (RV)  $X \sim N(0, 1)$ ; when  $X$  follows the standard normal distribution, it is often noted as  $(Z \sim N(0, 1))$ .

### z-score

the linear transformation of the form  $z = \frac{x - \mu}{\sigma}$ ; if this transformation is applied to any normal distribution  $X \sim N(\mu, \sigma)$  the result is the standard normal distribution  $Z \sim N(0, 1)$ . If this transformation is applied to any specific value  $x$  of the RV with mean  $\mu$  and standard deviation  $\sigma$ , the result is called the z-score of  $x$ . The z-score allows us to compare data that are normally distributed but scaled differently.

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