3.1: Introduction to Random Variables

Now that we have formally defined probability and the underlying structure, we add another layer: *random variables*. Random variables allow characterization of outcomes, so that we do not need to focus on each outcome specifically. We begin with the formal definition.

Definition $3.1.1$

A ***random variable*** is a function from a sample space Ω to the real numbers $\mathbb{R}$. We denote random variables with capital letters, e.g.,

$$X : \Omega \to \mathbb{R}.$$

Informally, a random variable assigns numbers to outcomes in the sample space. So, instead of focusing on the outcomes themselves, we highlight a specific characteristic of the outcomes.

Example $3.1.1$

Consider again the context of Example 1.1.1, where we recorded the sequence of heads and tails in two tosses of a fair coin. The sample space for this random experiment is given by

$$\Omega = \{hh, ht, th, tt\}.$$

Suppose we are only interested in tosses that result in heads. We can define a random variable X that tracks the number of heads obtained in an outcome. So, if outcome hh is obtained, then X will equal 2. Formally, we denote this as follows:

$$\begin{aligned} X : \Omega &\to \mathbb{R} \\ \omega &\mapsto \text{number of } h\text{'s in } \omega \end{aligned}$$

Since there are only four outcomes in Ω, we can list the value of X for each outcome individually:

$$\begin{array}{ccc} \text{inputs: } \Omega & \xrightarrow{\text{function: } X} & \text{outputs: } \mathbb{R} \\ hh & \stackrel{X}{\mapsto} & 2 \\ th & \stackrel{X}{\mapsto} & 1 \\ ht & \stackrel{X}{\mapsto} & 1 \\ tt & \stackrel{X}{\mapsto} & 0 \end{array}$$

We can also write the above as follows:

$$X(hh) = 2, \quad X(ht) = X(th) = 1, \quad X(tt) = 0.$$

The advantage to defining the random variable X in this context is that the two outcomes ht and th are both assigned a value of 1, meaning we are not focused on the actual sequence of heads and tails that resulted in obtaining one heads.

In Example 3.1.1, note that the random variable we defined only equals one of three possible values: $0, 1, 2$. This is an example of what we call a *discrete* random variable. We will also encounter another type of random variable: *continuous*. The next definitions make precise what we mean by these two types.

Definition $3.1.2$

A ***discrete random variable*** is a random variable that has only a finite or countably infinite (think integers or whole numbers) number of possible values.

Definition $3.1.3$

A ***continuous random variable*** is a random variable with infinitely many possible values (think an interval of real numbers, e.g., $[0, 1]$).

In this chapter, we take a closer look at discrete random variables, then in Chapter 4 we consider continuous random variables.