

5.2: Binomial Probability Distribution

Section 5.1 introduced the concept of a probability distribution. The focus of the section was on discrete **probability distributions** (pdf). To find the pdf for a situation, you usually needed to actually conduct the experiment and collect data. Then you can calculate the experimental probabilities. Normally you cannot calculate the theoretical probabilities instead. However, there are certain types of experiment that allow you to calculate the theoretical probability. One of those types is called a **Binomial Experiment**.

Properties of a binomial experiment (or Bernoulli trial)

1. Fixed number of trials, n , which means that the experiment is repeated a specific number of times.
2. The n trials are independent, which means that what happens on one trial does not influence the outcomes of other trials.
3. There are only two outcomes, which are called a success and a failure.
4. The probability of a success doesn't change from trial to trial, where p = probability of success and q = probability of failure, $q = 1 - p$.

If you know you have a binomial experiment, then you can calculate binomial probabilities. This is important because binomial probabilities come up often in real life. Examples of binomial experiments are:

- Toss a fair coin ten times, and find the probability of getting two heads.
- Question twenty people in class, and look for the probability of more than half being women?
- Shoot five arrows at a target, and find the probability of hitting it five times?

To develop the process for calculating the probabilities in a binomial experiment, consider Example 5.2.1.

Example 5.2.1: Deriving the Binomial Probability Formula

Suppose you are given a 3 question multiple-choice test. Each question has 4 responses and only one is correct. Suppose you want to find the probability that you can just guess at the answers and get 2 questions right. (Teachers do this all the time when they make up a multiple-choice test to see if students can still pass without studying. In most cases the students can't.) To help with the idea that you are going to guess, suppose the test is in Martian.

- a. What is the random variable?
- b. Is this a binomial experiment?
- c. What is the probability of getting 2 questions right?
- d. What is the probability of getting zero right, one right, and all three right?

Solution

a. x = number of correct answers

b.

1. There are 3 questions, and each question is a trial, so there are a fixed number of trials. In this case, $n = 3$.
2. Getting the first question right has no affect on getting the second or third question right, thus the trials are independent.
3. Either you get the question right or you get it wrong, so there are only two outcomes. In this case, the success is getting the question right.
4. The probability of getting a question right is one out of four. This is the same for every trial since each question has 4 responses. In this case, $p = \frac{1}{4}$ and $q = 1 - \frac{1}{4} = \frac{3}{4}$

This is a binomial experiment, since all of the properties are met.

c. To answer this question, start with the sample space. $SS = \{RRR, RRW, RWR, WRR, WWR, WRW, RWW, WWW\}$, where RRW means you get the first question right, the second question right, and the third question wrong. The same is similar for the other outcomes.

Now the event space for getting 2 right is $\{RRW, RWR, WRR\}$. What you did in chapter four was just to find three divided by eight. However, this would not be right in this case. That is because the probability of getting a question right is different from getting a question wrong. What else can you do?

Look at just $P(RRW)$ for the moment. Again, that means $P(RRW) = P(R \text{ on } 1\text{st}, R \text{ on } 2\text{nd}, \text{ and } W \text{ on } 3\text{rd})$

Since the trials are independent, then $P(RRW) = P(R \text{ on 1st, R on 2nd, and W on 3rd}) = P(R \text{ on 1st}) * P(R \text{ on 2nd}) * P(W \text{ on 3rd})$

Just multiply $p * p * q$

$$P(RRW) = \frac{1}{4} * \frac{1}{4} * \frac{3}{4} = \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^1$$

The same is true for $P(RWR)$ and $P(WRR)$. To find the probability of 2 correct answers, just add these three probabilities together. You get

$$\begin{aligned} P(2 \text{ correct answers}) &= P(RRW) + P(RWR) + P(WRR) \\ &= \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^1 + \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^1 + \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^1 \\ &= 3 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^1 \end{aligned}$$

d. You could go through the same argument that you did above and come up with the following:

Table 5.2.1: Binomial pattern

r right	P(r right)
0 right	$1 * \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^3$
1 right	$3 * \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^2$
2 right	$3 * \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^1$
3 right	$1 * \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^0$

Hopefully you see the pattern that results. You can now write the general formula for the probabilities for a Binomial experiment

First, the random variable in a binomial experiment is x = number of successes. Be careful, a success is not always a good thing. Sometimes a success is something that is bad, like finding a defect. A success just means you observed the outcome you wanted to see happen.

Definition 5.2.1

Binomial Formula for the probability of r successes in n trials is

$$P(x = r) = {}_n C_r p^r q^{n-r} \text{ where } {}_n C_r = \frac{n!}{r!(n-r)!}$$

The ${}_n C_r$ is the number of combinations of n things taking r at a time. It is read “ n choose r ”. Some other common notations for n choose r are $C_{n,r}$, and $\binom{n}{r}$. $n!$ means you are multiplying $n * (n-1) * (n-2) * \dots * 2 * 1$. As an example, $5! = 5 * 4 * 3 * 2 * 1 = 120$.

When solving problems, make sure you define your random variable and state what n , p , q , and r are. Without doing this, the problems are a great deal harder.

Example 5.2.2: Calculating Binomial Probabilities

When looking at a person’s eye color, it turns out that 1% of people in the world has green eyes (“What percentage of,” 2013). Consider a group of 20 people.

- State the random variable.
- Argue that this is a binomial experiment.
- Find the probability that none have green eyes.
- Find the probability that nine have green eyes.
- Find the probability that at most three have green eyes.
- Find the probability that at most two have green eyes.
- Find the probability that at least four have green eyes.
- In Europe, four people out of twenty have green eyes. Is this unusual? What does that tell you?

Solution

a. x = number of people with green eyes

b.

- There are 20 people, and each person is a trial, so there are a fixed number of trials. In this case, $n = 20$.
- If you assume that each person in the group is chosen at random the eye color of one person doesn't affect the eye color of the next person, thus the trials are independent.
- Either a person has green eyes or they do not have green eyes, so there are only two outcomes. In this case, the success is a person has green eyes.
- The probability of a person having green eyes is 0.01. This is the same for every trial since each person has the same chance of having green eyes. $p = 0.01$ and $q = 1 - 0.01 = 0.99$

c. $P(x = 0) = {}_{20}C_0(0.01)^0(0.99)^{20-0} \approx 0.818$

d. $P(x = 9) = {}_{20}C_9(0.01)^9(0.99)^{20-9} \approx 1.50 \times 10^{-13} \approx 0.000$

e. At most three means that three is the highest value you will have. Find the probability of x is less than or equal to three.

$$\begin{aligned} P(x \leq 3) &= P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3) \\ &= {}_{20}C_0(0.01)^0(0.99)^{20} + {}_{20}C_1(0.01)^1(0.99)^{19} \\ &\quad + {}_{20}C_2(0.01)^2(0.99)^{18} + {}_{20}C_3(0.01)^3(0.99)^{17} \\ &\approx 0.818 + 0.165 + 0.016 + 0.001 > 0.999 \end{aligned}$$

The reason the answer is written as being greater than 0.999 is because the answer is actually 0.9999573791, and when that is rounded to three decimal places you get 1. But 1 means that the event will happen, when in reality there is a slight chance that it won't happen. It is best to write the answer as greater than 0.999 to represent that the number is very close to 1, but isn't 1.

f.

$$\begin{aligned} P(x \leq 2) &= P(x = 0) + P(x = 1) + P(x = 2) \\ &= {}_{20}C_0(0.01)^0(0.99)^{20} + {}_{20}C_1(0.01)^1(0.99)^{19} + {}_{20}C_2(0.01)^2(0.99)^{18} \\ &\approx 0.818 + 0.165 + 0.016 \approx 0.999 \end{aligned}$$

g. At least four means four or more. Find the probability of x being greater than or equal to four. That would mean adding up all the probabilities from four to twenty. This would take a long time, so it is better to use the idea of complement. The complement of being greater than or equal to four is being less than four. That would mean being less than or equal to three. Part (e) has the answer for the probability of being less than or equal to three. Just subtract that number from 1.

$$P(x \geq 4) = 1 - P(x \leq 3) = 1 - 0.999 = 0.001$$

Actually the answer is less than 0.001, but it is fine to write it this way.

h. Since the probability of finding four or more people with green eyes is much less than 0.05, it is unusual to find four people out of twenty with green eyes. That should make you wonder if the proportion of people in Europe with green eyes is more than the 1% for the general population. If this is true, then you may want to ask why Europeans have a higher proportion of green-eyed people. That of course could lead to more questions.

The binomial formula is cumbersome to use, so you can find the probabilities by using technology. On the TI-83/84 calculator, the commands on the TI-83/84 calculators when the number of trials is equal to n and the probability of a success is equal to p are $\text{binompdf}(n, p, r)$ when you want to find $P(x=r)$ and $\text{binomcdf}(n, p, r)$ when you want to find $P(x \leq r)$. If you want to find

$P(x \geq r)$, then you use the property that $P(x \geq r) = 1 - P(x \leq r - 1)$, since $x \geq r$ and $x < r$ or $x \leq r - 1$ are complementary events. Both `binompdf` and `binomcdf` commands are found in the DISTR menu. Using R, the commands are $P(x = r) = \text{dbinom}(r, n, p)$ and $P(x \leq r) = \text{pbinom}(r, n, p)$.

Example 5.2.3 using the binomial command on the ti-83/84

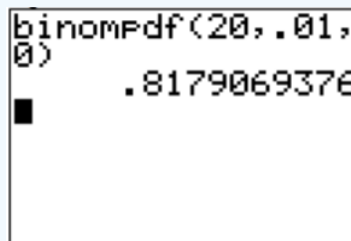
When looking at a person's eye color, it turns out that 1% of people in the world has green eyes ("What percentage of," 2013). Consider a group of 20 people.

- State the random variable.
- Find the probability that none have green eyes.
- Find the probability that nine have green eyes.
- Find the probability that at most three have green eyes.
- Find the probability that at most two have green eyes.
- Find the probability that at least four have green eyes.

Solution

a. x = number of people with green eyes

b. You are looking for $P(x=0)$. Since this problem is $x=0$, you use the `binompdf` command on the TI-83/84 or `dbinom` command on R. On the TI-83/84, you go to the DISTR menu, select the `binompdf`, and then type into the parenthesis your n , p , and r values into your calculator, making sure you use the comma to separate the values. The command will look like `binompdf(20, .01, 0)` and when you press ENTER you will be given the answer. (If you have the new software on the TI-84, the screen looks a bit different.)



```
binompdf(20, .01,
0)
.8179069376
```

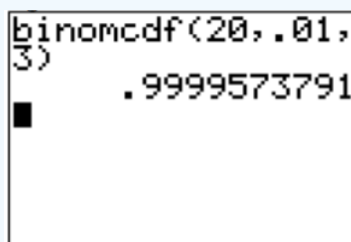
Figure 5.2.1: Calculator Results for binompdf

On R, the command would look like `dbinom(0, 20, 0.01)`

$P(x=0) = 0.8179$. Thus there is an 81.8% chance that in a group of 20 people none of them will have green eyes.

c. In this case you want to find the $P(x=9)$. Again, you will use the `binompdf` command or the `dbinom` command. Following the procedure above, you will have `binompdf(20, .01, 9)` on the TI-83/84 or `dbinom(9,20,0.01)` on R. Your answer is $P(x = 9) = 1.50 \times 10^{-13}$. (Remember when the calculators or computers gives you $1.50E-13$ or $1.50e-13$, this is how they display scientific notation.) The probability that out of twenty people, nine of them have green eyes is a very small chance.

d. At most three means that three is the highest value you will have. Find the probability of x being less than or equal to three, which is $P(x \leq 3)$. This uses the `binomcdf` command on the TI-83/84. You use the command on the TI-83/84 of `binomcdf(20, .01, 3)`.



```
binomcdf(20, .01,
3)
.9999573791
```

Figure 5.2.2: Calculator Results for binomcdf

Your answer is 0.99996. Thus there is a really good chance that in a group of 20 people at most three will have green eyes. (Note: don't round this to one, since one means that the event will happen, when in reality there is a slight chance that it won't)

happen. It is best to write the answer out to enough decimal points so it doesn't round off to one.

e. You are looking for $P(x \leq 2)$. Again use `binomcdf` or `pbinom`. Following the procedure above you will have `binomcdf(20, .01, 2)` on the TI-83/84 and `pbinom(2,20,0.01)`, with $P(x \leq 2) = 0.998996$. Again there is a really good chance that at most two people in the room will have green eyes.

f. At least four means four or more. Find the probability of x being greater than or equal to four. That would mean adding up all the probabilities from four to twenty. This would take a long time, so it is better to use the idea of complement. The complement of being greater than or equal to four is being less than four. That would mean being less than or equal to three. Part (e) has the answer for the probability of being less than or equal to three. Just subtract that number from 1.

$P(x \geq 4) = 1 - P(x \leq 3) = 1 - 0.999996 = 0.000004$ You can also find this answer by doing the following on TI-83/84:

$P(x \geq 4) = 1 - P(x \leq 3) = 1 - \text{binomcdf}(20, .01, 3) = 1 - 0.999996 = 0.000004$ on R:

$P(x \geq 4) = 1 - P(x \leq 3) = 1 - \text{pbinom}(3, 20, .01) = 1 - 0.999996 = 0.000004$ Again, this is very unlikely to happen.

There are other technologies that will compute binomial probabilities.

Example 5.2.4 calculating binomial probabilities

According to the Center for Disease Control (CDC), about 1 in 88 children in the U.S. have been diagnosed with autism ("CDC-data and statistics," 2013). Suppose you consider a group of 10 children.

- State the random variable.
- Argue that this is a binomial experiment.
- Find the probability that none have autism.
- Find the probability that seven have autism.
- Find the probability that at least five have autism.
- Find the probability that at most two have autism.
- Suppose five children out of ten have autism. Is this unusual? What does that tell you?

Solution

a. x = number of children with autism

b.

- There are 10 children, and each child is a trial, so there are a fixed number of trials. In this case, $n = 10$.
- If you assume that each child in the group is chosen at random, then whether a child has autism does not affect the chance that the next child has autism. Thus the trials are independent.
- Either a child has autism or they do not have autism, so there are two outcomes. In this case, the success is a child has autism.
- The probability of a child having autism is $1/88$. This is the same for every trial since each child has the same chance of having autism. $p = \frac{1}{88}$ and $q = 1 - \frac{1}{88} = \frac{87}{88}$.

c. Using the formula:

$$P(x = 0) = {}_{10}C_0 \left(\frac{1}{88}\right)^0 \left(\frac{87}{88}\right)^{10-0} \approx 0.892$$

Using the TI-83/84 Calculator:

$$P(x = 0) = \text{binompdf}(10, 1 \div 88, 0) \approx 0.892$$

Using R:

$$P(x = 0) = \text{pbinom}(0, 10, 1/88) \approx 0.892$$

d. Using the formula:

$$P(x = 7) = {}_{10}C_7 \left(\frac{1}{88}\right)^7 \left(\frac{87}{88}\right)^{10-7} \approx 0.000$$

Using the TI-83/84 Calculator:

$$P(x = 7) = \text{binompdf}(10, 1 \div 88, 7) \approx 2.84 \times 10^{-12}$$

Using R:

$$P(x = 7) = \text{dbinom}(7, 10, 1/88) \approx 2.84 \times 10^{-12}$$

e. Using the formula:

$$\begin{aligned} P(x \geq 5) &= P(x = 5) + P(x = 6) + P(x = 7) \\ &\quad + P(x = 8) + P(x = 9) + P(x = 10) \\ &= {}_{10}C_5 \left(\frac{1}{88}\right)^5 \left(\frac{78}{88}\right)^{10-5} + {}_{10}C_6 \left(\frac{1}{88}\right)^6 \left(\frac{78}{88}\right)^{10-6} \\ &\quad + {}_{10}C_7 \left(\frac{1}{88}\right)^7 \left(\frac{78}{88}\right)^{10-7} + {}_{10}C_8 \left(\frac{1}{88}\right)^8 \left(\frac{78}{88}\right)^{10-8} \\ &\quad + {}_{10}C_9 \left(\frac{1}{88}\right)^9 \left(\frac{78}{88}\right)^{10-9} + {}_{10}C_{10} \left(\frac{1}{88}\right)^{10} \left(\frac{78}{88}\right)^{10-10} \\ &= 0.000 + 0.000 + 0.000 + 0.000 + 0.000 + 0.000 \\ &= 0.000 \end{aligned}$$

Using the TI-83/84 Calculator:

To use the calculator you need to use the complement.

$$\begin{aligned} P(x \geq 5) &= 1 - P(x < 5) \\ &= 1 - P(x \leq 4) \\ &= 1 - \text{binomcdf}(10, 1 \div 88, 4) \\ &\approx 1 - 0.9999999 = 0.000 \end{aligned}$$

f. Using the formula:

$$\begin{aligned} P(x \leq 2) &= P(x = 0) + P(x = 1) + P(x = 2) \\ &= {}_{10}C_0 \left(\frac{1}{88}\right)^0 \left(\frac{78}{88}\right)^{10-0} + {}_{10}C_1 \left(\frac{1}{88}\right)^1 \left(\frac{78}{88}\right)^{10-1} \\ &\quad + {}_{10}C_2 \left(\frac{1}{88}\right)^2 \left(\frac{78}{88}\right)^{10-2} \\ &= 0.892 + 0.103 + 0.005 > 0.999 \end{aligned}$$

Using the TI-83/84 Calculator:

$$P(x \leq 2) = \text{binomcdf}(10, 1 \div 88, 2) \approx 0.9998$$

Using R:

$$P(x \leq 2) = \text{pbinom}(2, 10, 1/88) \approx 0.9998$$

g. Since the probability of five or more children in a group of ten having autism is much less than 5%, it is unusual to happen. If this does happen, then one may think that the proportion of children diagnosed with autism is actually more than 1/88.

Homework

Exercise 5.2.1

1. Suppose a random variable, x , arises from a binomial experiment. If $n = 14$, and $p = 0.13$, find the following probabilities using the binomial formula.
 - a. $P(x=5)$
 - b. $P(x=8)$
 - c. $P(x=12)$
 - d. $P(x \leq 4)$
 - e. $P(x \geq 8)$

- f. $P(x \leq 12)$
2. Suppose a random variable, x , arises from a binomial experiment. If $n = 22$, and $p = 0.85$, find the following probabilities using the binomial formula.
- $P(x=18)$
 - $P(x=5)$
 - $P(x=20)$
 - $P(x \leq 3)$
 - $P(x \geq 18)$
 - $P(x \leq 20)$
3. Suppose a random variable, x , arises from a binomial experiment. If $n = 10$, and $p = 0.70$, find the following probabilities using the binomial formula.
- $P(x=2)$
 - $P(x=8)$
 - $P(x=7)$
 - $P(x \leq 3)$
 - $P(x \geq 7)$
 - $P(x \leq 4)$
4. Suppose a random variable, x , arises from a binomial experiment. If $n = 6$, and $p = 0.30$, find the following probabilities using the binomial formula.
- $P(x=1)$
 - $P(x=5)$
 - $P(x=3)$
 - $P(x \leq 3)$
 - $P(x \geq 5)$
 - $P(x \leq 4)$
5. Suppose a random variable, x , arises from a binomial experiment. If $n = 17$, and $p = 0.63$, find the following probabilities using the binomial formula.
- $P(x=8)$
 - $P(x=15)$
 - $P(x=14)$
 - $P(x \leq 12)$
 - $P(x \geq 10)$
 - $P(x \leq 7)$
6. Suppose a random variable, x , arises from a binomial experiment. If $n = 23$, and $p = 0.22$, find the following probabilities using the binomial formula.
- $P(x=21)$
 - $P(x=6)$
 - $P(x=12)$
 - $P(x \leq 14)$
 - $P(x \geq 17)$
 - $P(x \leq 9)$
7. Approximately 10% of all people are left-handed ("11 little-known facts," 2013). Consider a grouping of fifteen people.
- State the random variable.
 - Argue that this is a binomial experiment Find the probability that
 - None are left-handed.
 - Seven are left-handed.
 - At least two are left-handed.
 - At most three are left-handed.
 - At least seven are left-handed.
 - Seven of the last 15 U.S. Presidents were left-handed. Is this unusual? What does that tell you?

8. According to an article in the American Heart Association's publication *Circulation*, 24% of patients who had been hospitalized for an acute myocardial infarction did not fill their cardiac medication by the seventh day of being discharged (Ho, Bryson & Rumsfeld, 2009). Suppose there are twelve people who have been hospitalized for an acute myocardial infarction.
 - a. State the random variable.
 - b. Argue that this is a binomial experiment Find the probability that
 - c. All filled their cardiac medication.
 - d. Seven did not fill their cardiac medication.
 - e. None filled their cardiac medication.
 - f. At most two did not fill their cardiac medication.
 - g. At least three did not fill their cardiac medication.
 - h. At least ten did not fill their cardiac medication.
 - i. Suppose of the next twelve patients discharged, ten did not fill their cardiac medication, would this be unusual? What does this tell you?
9. Eyeglassomatic manufactures eyeglasses for different retailers. In March 2010, they tested to see how many defective lenses they made, and there were 16.9% defective lenses due to scratches. Suppose Eyeglassomatic examined twenty eyeglasses.
 - a. State the random variable.
 - b. Argue that this is a binomial experiment Find the probability that
 - c. None are scratched.
 - d. All are scratched.
 - e. At least three are scratched.
 - f. At most five are scratched.
 - g. At least ten are scratched.
 - h. Is it unusual for ten lenses to be scratched? If it turns out that ten lenses out of twenty are scratched, what might that tell you about the manufacturing process?
10. The proportion of brown M&M's in a milk chocolate packet is approximately 14% (Madison, 2013). Suppose a package of M&M's typically contains 52 M&M's.
 - a. State the random variable.
 - b. Argue that this is a binomial experiment Find the probability that
 - c. Six M&M's are brown.
 - d. Twenty-five M&M's are brown.
 - e. All of the M&M's are brown.
 - f. Would it be unusual for a package to have only brown M&M's? If this were to happen, what would you think is the reason?

Answer

1. a. $P(x=5) = 0.0212$, b. $P(x=8) = 1.062 \times 10^{-4}$, c. $P(x=12) = 1.605 \times 10^{-9}$, d. $P(x \leq 4) = 0.973$, e. $P(x \geq 8) = 1.18 \times 10^{-4}$, f. $P(x \leq 12) = 0.99999$
3. a. $P(x=2) = 0.0014$, b. $P(x=8) = 0.2335$, c. $P(x=7) = 0.2668$, d. $P(x \leq 3) = 0.0106$, e. $P(x \geq 7) = 0.6496$, f. $P(x \leq 4) = 0.0473$
5. a. $P(x=8) = 0.0784$, b. $P(x=15) = 0.0182$, c. $P(x=14) = 0.0534$, d. $P(x \leq 12) = 0.8142$, e. $P(x \geq 10) = 0.7324$, f. $P(x \leq 7) = 0.0557$
7. a. See solutions, b. See solutions, c. $P(x=0) = 0.2059$, d. $P(x=7) = 2.770 \times 10^{-4}$, e. $P(x \geq 2) = 0.4510$, f. $P(x \leq 3) = 0.944$, g. $P(x \geq 7) = 3.106 \times 10^{-4}$, h. See solutions
9. a. See solutions, b. See solutions, c. $P(x=0) = 0.0247$, d. $P(x=20) = 3.612 \times 10^{-16}$, e. $P(x \geq 3) = 0.6812$, f. $P(x \leq 5) = 0.8926$, g. $P(x \geq 10) = 6.711 \times 10^{-4}$, h. See solutions

This page titled [5.2: Binomial Probability Distribution](#) is shared under a [CC BY-SA 4.0](#) license and was authored, remixed, and/or curated by [Kathryn Kozak](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform.

- **5.2: Binomial Probability Distribution** by [Kathryn Kozak](#) is licensed [CC BY-SA 4.0](#). Original source: <https://s3-us-west-2.amazonaws.com/oerfiles/statsusingtech2.pdf>.