3.4: Two Basic Rules of Probability

When calculating probability, there are two rules to consider when determining if two events are independent or dependent and if they are mutually exclusive or not.

The Multiplication Rule

If A and B are two events defined on a sample space, then:

$$P(A \text{ AND } B) = P(B)P(A|B) \tag{3.4.1}$$

This rule may also be written as:

$$P(A|B) = \frac{P(A \text{ AND } B)}{P(B)} \tag{3.4.2}$$

(The probability of A given B equals the probability of A and B divided by the probability of B.)

If A and B are *independent*, then

$$P(A|B) = P(A). \tag{3.4.3}$$

and Equation 3.4.1 becomes

$$P(A \text{ AND } B) = P(A)P(B). \tag{3.4.4}$$

The Addition Rule

If A and B are defined on a sample space, then:

$$P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B) \tag{3.4.5}$$

If A and B are **mutually exclusive**, then

$$P(A \text{ AND } B) = 0. \tag{3.4.6}$$

and Equation 3.4.5 becomes

$$P(A \text{ OR } B) = P(A) + P(B). \tag{3.4.7}$$

Example 3.4.1

Klaus is trying to choose where to go on vacation. His two choices are: $\mathrm{A} =$ New Zealand and $\mathrm{B} =$ Alaska

- Klaus can only afford one vacation. The probability that he chooses A is $P(\mathrm{A}) = 0.6$ and the probability that he chooses B is $P(\mathrm{B}) = 0.35$.
- $P(\mathrm{A \text{ AND } B}) = 0$ because Klaus can only afford to take one vacation
- Therefore, the probability that he chooses either New Zealand or Alaska is $P(\mathrm{A \text{ OR } B}) = P(\mathrm{A}) + P(\mathrm{B}) = 0.6 + 0.35 = 0.95$. Note that the probability that he does not choose to go anywhere on vacation must be 0.05.

Carlos plays college soccer. He makes a goal 65% of the time he shoots. Carlos is going to attempt two goals in a row in the next game. $\mathrm{A} =$ the event Carlos is successful on his first attempt. $P(\mathrm{A}) = 0.65$. $\mathrm{B} =$ the event Carlos is successful on his second attempt. $P(\mathrm{B}) = 0.65$. Carlos tends to shoot in streaks. The probability that he makes the second goal **GIVEN** that he made the first goal is 0.90.

a. What is the probability that he makes both goals?
b. What is the probability that Carlos makes either the first goal or the second goal?
c. Are A and B independent?
d. Are A and B mutually exclusive?

Solutions

a. The problem is asking you to find $P(\mathrm{A \text{ AND } B}) = P(\mathrm{B \text{ AND } A})$. Since $P(\mathrm{B|A}) = 0.90 : P(\mathrm{B \text{ AND } A}) = P(\mathrm{B|A})P(\mathrm{A}) = (0.90)(0.65) = 0.585$

Carlos makes the first and second goals with probability 0.585.

b. The problem is asking you to find $P(\text{A OR B})$.

$$P(\text{A OR B}) = P(\text{A}) + P(\text{B}) - P(\text{A AND B}) = 0.65 + 0.65 - 0.585 = 0.715 \tag{3.4.8}$$

Carlos makes either the first goal or the second goal with probability 0.715.

c. No, they are not, because $P(\text{B AND A}) = 0.585$.

$$P(\text{B})P(\text{A}) = (0.65)(0.65) = 0.423 \tag{3.4.9}$$

$$0.423 \neq 0.585 = P(\text{B AND A}) \tag{3.4.10}$$

So, $P(\text{B AND A})$ is **not** equal to $P(\text{B})P(\text{A})$.

d. No, they are not because $P(\text{A and B}) = 0.585$.

To be mutually exclusive, $P(\text{A AND B})$ must equal zero.

Exercise 3.4.1

Helen plays basketball. For free throws, she makes the shot 75% of the time. Helen must now attempt two free throws. $\text{C} =$ the event that Helen makes the first shot. $P(\text{C}) = 0.75$. $\text{D} =$ the event Helen makes the second shot. $P(\text{D}) = 0.75$. The probability that Helen makes the second free throw given that she made the first is 0.85. What is the probability that Helen makes both free throws?

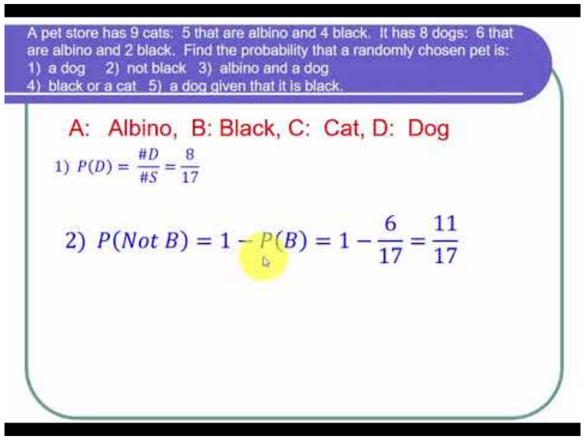
Answer

$$P(\text{D}|\text{C}) = 0.85 \tag{3.4.11}$$

$$P(\text{C AND D}) = P(\text{D AND C}) \tag{3.4.12}$$

$$P(\text{D AND C}) = P(\text{D}|\text{C})P(\text{C}) = (0.85)(0.75) = 0.6375 \tag{3.4.13}$$

Helen makes the first and second free throws with probability 0.6375.



Example 3.4.2

A community swim team has **150** members. **Seventy-five** of the members are advanced swimmers. **Forty-seven** of the members are intermediate swimmers. The remainder are novice swimmers. **Forty** of the advanced swimmers practice four times a week. **Thirty** of the intermediate swimmers practice four times a week. **Ten** of the novice swimmers practice four times a week. Suppose one member of the swim team is chosen randomly.

a. What is the probability that the member is a novice swimmer?
b. What is the probability that the member practices four times a week?
c. What is the probability that the member is an advanced swimmer and practices four times a week?
d. What is the probability that a member is an advanced swimmer and an intermediate swimmer? Are being an advanced swimmer and an intermediate swimmer mutually exclusive? Why or why not?
e. Are being a novice swimmer and practicing four times a week independent events? Why or why not?

Answer

a. $\frac{28}{150}$
b. $\frac{80}{150}$
c. $\frac{40}{150}$
d. $P(\text{advanced AND intermediate}) = 0$, so these are mutually exclusive events. A swimmer cannot be an advanced swimmer and an intermediate swimmer at the same time.
e. No, these are not independent events.

$$P(\text{novice AND practices four times per week}) = 0.0667 \tag{3.4.14}$$

$$P(\text{novice})P(\text{practices four times per week}) = 0.0996 \tag{3.4.15}$$

$$0.0667 \neq 0.0996 \tag{3.4.16}$$

Exercise 3.4.2

A school has 200 seniors of whom 140 will be going to college next year. Forty will be going directly to work. The remainder are taking a gap year. Fifty of the seniors going to college play sports. Thirty of the seniors going directly to work play sports. Five of the seniors taking a gap year play sports. What is the probability that a senior is taking a gap year?

Answer

$$P = \frac{200 - 140 - 40}{200} = \frac{20}{200} = 0.1 \tag{3.4.17}$$

Example 3.4.3

Felicity attends Modesto JC in Modesto, CA. The probability that Felicity enrolls in a math class is 0.2 and the probability that she enrolls in a speech class is 0.65. The probability that she enrolls in a math class GIVEN that she enrolls in speech class is 0.25.

Let: $\mathrm{M}=$ math class, $\mathrm{S}=$ speech class, $\mathrm{M}|\mathrm{S}=$ math given speech

a. What is the probability that Felicity enrolls in math and speech?
   Find $P(\mathrm{M\ AND\ S})=P(\mathrm{M}|\mathrm{S})P(\mathrm{S})$.
b. What is the probability that Felicity enrolls in math or speech classes?
   Find $P(\mathrm{M\ OR\ S})=P(\mathrm{M})+P(\mathrm{S})-P(\mathrm{M\ AND\ S})$.
c. Are M and S independent? Is $P(\mathrm{M}|\mathrm{S})=P(\mathrm{M})$?
d. Are M and S mutually exclusive? Is $P(\mathrm{M\ AND\ S})=0$?

Answer

a. 0.1625, b. 0.6875, c. No, d. No

Exercise 3.4.3

A student goes to the library. Let events $\mathrm{B}=$ the student checks out a book and $\mathrm{D}=$ the student check out a DVD. Suppose that $P(\mathrm{B})=0.40$, $P(\mathrm{D})=0.30$ and $P(\mathrm{D}|\mathrm{B})=0.5$.

a. Find $P(\mathrm{B\ AND\ D})$.
b. Find $P(\mathrm{B\ OR\ D})$.

Answer

a. $P(\mathrm{B\ AND\ D})=P(\mathrm{D}|\mathrm{B})P(\mathrm{B})=(0.5)(0.4)=0.20$.
b. $P(\mathrm{B\ OR\ D})=P(\mathrm{B})+P(\mathrm{D})-P(\mathrm{B\ AND\ D})=0.40+0.30-0.20=0.50$

Example 3.4.4

Studies show that about one woman in seven (approximately 14.3%) who live to be 90 will develop breast cancer. Suppose that of those women who develop breast cancer, a test is negative 2% of the time. Also suppose that in the general population of women, the test for breast cancer is negative about 85% of the time. Let $\mathrm{B}=$ woman develops breast cancer and let $\mathrm{N}=$ tests negative. Suppose one woman is selected at random.

a. What is the probability that the woman develops breast cancer? What is the probability that woman tests negative?
b. Given that the woman has breast cancer, what is the probability that she tests negative?
c. What is the probability that the woman has breast cancer AND tests negative?
d. What is the probability that the woman has breast cancer or tests negative?
e. Are having breast cancer and testing negative independent events?
f. Are having breast cancer and testing negative mutually exclusive?

Answers

a. $P(\mathrm{B})=0.143$; $P(\mathrm{N})=0.85$
b. $P(\mathrm{N}|\mathrm{B})=0.02$
c. $P(\mathrm{B\ AND\ N})=P(\mathrm{B})P(\mathrm{N}|\mathrm{B})=(0.143)(0.02)=0.0029$
d. $P(\mathrm{B\ OR\ N})=P(\mathrm{B})+P(\mathrm{N})-P(\mathrm{B\ AND\ N})=0.143+0.85-0.0029=0.9901$
e. No. $P(\mathrm{N})=0.85$; $P(\mathrm{N}|\mathrm{B})=0.02$. So, $P(\mathrm{N}|\mathrm{B})$ does not equal $P(\mathrm{N})$.
f. No. $P(\mathrm{B\ AND\ N})=0.0029$. For B and N to be mutually exclusive, $P(\mathrm{B\ AND\ N})$ must be zero

Exercise 3.4.4

A school has 200 seniors of whom 140 will be going to college next year. Forty will be going directly to work. The remainder are taking a gap year. Fifty of the seniors going to college play sports. Thirty of the seniors going directly to work play sports. Five of the seniors taking a gap year play sports. What is the probability that a senior is going to college and plays sports?

Answer

Let $\mathrm{A}=$ student is a senior going to college.

Let $\text{B} =$ student plays sports.

$$P(\text{B}) = \frac{140}{200}$$

$$P(\text{B}|\text{A}) = \frac{50}{140}$$

$$P(\text{A AND B}) = P(\text{B}|\text{A})P(\text{A})$$

$$P(\text{A AND B}) = (\frac{140}{200})(\frac{50}{140}) = \frac{1}{4}$$

Example 3.4.5

Refer to the information in Example $3.4.4$. $\text{P} =$ tests positive.

a. Given that a woman develops breast cancer, what is the probability that she tests positive. Find $P(\text{P}|\text{B}) = 1 - P(\text{N}|\text{B})$.
b. What is the probability that a woman develops breast cancer and tests positive. Find $P(\text{B AND P}) = P(\text{P}|\text{B})P(\text{B})$.
c. What is the probability that a woman does not develop breast cancer. Find $P(\text{B}') = 1 - P(\text{B})$.
d. What is the probability that a woman tests positive for breast cancer. Find $P(\text{P}) = 1 - P(\text{N})$.

Answer

a. 0.98; b. 0.1401; c. 0.857; d. 0.15

Exercise 3.4.5

A student goes to the library. Let events $\text{B} =$ the student checks out a book and $\text{D} =$ the student checks out a DVD. Suppose that $P(\text{B}) = 0.40$, $P(\text{D}) = 0.30$ and $P(\text{D}|\text{B}) = 0.5$.

a. Find $P(\text{B}')$.
b. Find $P(\text{D AND B})$.
c. Find $P(\text{B}|\text{D})$.
d. Find $P(\text{D AND B}')$.
e. Find $P(\text{D}|\text{B}')$.

Answer

a. $P(\text{B}') = 0.60$
b. $P(\text{D AND B}) = P(\text{D}|\text{B})P(\text{B}) = 0.20$
c. $P(\text{B}|\text{D}) = \frac{P(\text{B AND D})}{P(\text{D})} = \frac{(0.20)}{(0.30)} = 0.66$
d. $P(\text{D AND B}') = P(\text{D}) - P(\text{D AND B}) = 0.30 - 0.20 = 0.10$
e. $P(\text{D}|\text{B}') = P(\text{D AND B}')P(\text{B}') = (P(\text{D}) - P(\text{D AND B}))(0.60) = (0.10)(0.60) = 0.06$

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Chapter Review

The multiplication rule and the addition rule are used for computing the probability of A and B, as well as the probability of A or B for two given events A, B defined on the sample space. In sampling with replacement each member of a population is replaced after it is picked, so that member has the possibility of being chosen more than once, and the events are considered to be independent. In sampling without replacement, each member of a population may be chosen only once, and the events are considered to be not independent. The events A and B are mutually exclusive events when they do not have any outcomes in common.

Formula Review

The multiplication rule: $P(\mathrm{A \text{ AND } B}) = P(\mathrm{A|B})P(\mathrm{B})$

The addition rule: $P(\mathrm{A \text{ OR } B}) = P(\mathrm{A}) + P(\mathrm{B}) - P(\mathrm{A \text{ AND } B})$

Use the following information to answer the next ten exercises. Forty-eight percent of all Californians registered voters prefer life in prison without parole over the death penalty for a person convicted of first degree murder. Among Latino California registered voters, 55% prefer life in prison without parole over the death penalty for a person convicted of first degree murder. 37.6% of all Californians are Latino.

In this problem, let:

- $\mathrm{C} =$ Californians (registered voters) preferring life in prison without parole over the death penalty for a person convicted of first degree murder.
- $\mathrm{L} =$ Latino Californians

Suppose that one Californian is randomly selected.

Exercise 3.4.5

Find $P(\mathrm{C})$.

Exercise 3.4.6

Find $P(\mathrm{L})$.

Answer

0.376

Exercise 3.4.7

Find $P(\mathrm{C|L})$.

Exercise 3.4.8

In words, what is $\mathrm{C|L}$?

Answer

$\mathrm{C|L}$ means, given the person chosen is a Latino Californian, the person is a registered voter who prefers life in prison without parole for a person convicted of first degree murder.

Exercise 3.4.9

Find $P(\mathrm{L \text{ AND } C})$

Exercise 3.4.10

In words, what is $\mathrm{L \text{ AND } C}$?

Answer

$\mathrm{L \text{ AND } C}$ is the event that the person chosen is a Latino California registered voter who prefers life without parole over the death penalty for a person convicted of first degree murder.

Exercise 3.4.11

Are L and C independent events? Show why or why not.

Exercise 3.4.12

Find $P(\mathrm{L \text{ OR } C})$.

Answer

0.6492

Exercise 3.4.13

In words, what is $\mathrm{L \text{ OR } C}$?

Exercise 3.4.14

Are L and C mutually exclusive events? Show why or why not.

Answer

No, because $P(\mathrm{L \text{ AND } C})$ does not equal 0.

Glossary

Independent Events

The occurrence of one event has no effect on the probability of the occurrence of another event. Events A and B are independent if one of the following is true:

1. $P(\mathrm{A|B}) = P(\mathrm{A})$
2. $P(\mathrm{B|A}) = P(\mathrm{B})$
3. $P(\mathrm{A \text{ AND } B}) = P(\mathrm{A})P(\mathrm{B})$

Mutually Exclusive

Two events are mutually exclusive if the probability that they both happen at the same time is zero. If events A and B are mutually exclusive, then $P(\mathrm{A \text{ AND } B}) = 0$.