5.3: How to Calculate Classical Probability

We can use counting methods to determine classical probability. However, we need to be careful in our methods to be sure to get the correct answer.

An **Event** is a result of an experiment, usually referred to with a capital letter A, B, C, etc. Consider the experiment of flipping two coins. Then use the letter A to refer to the event of getting exactly one head.

An **Outcome** is a result of the experiment that cannot be broken down into smaller events. Consider event A, getting exactly one head. Note that there are two ways or outcomes to get one head in two tosses, by first getting a head then a tail, or by first getting a tail, then a head. Let's write these distinct outcomes as HT and TH.

The **Sample Space** is the set of all possible outcomes of an experiment. In the experiment of flipping two coins, there are 4 possible outcomes, which can be expressed in set notation.

$$\text{Sample Space } = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$$

We can now redefine an **Event** of an experiment to be a subset of the Sample Space. If event A is getting exactly one head in two coin tosses, then

$$\mathrm{A} = \{\mathrm{HT}, \mathrm{TH}\}$$

After carefully listing the outcomes of the Sample Space and the outcomes of the event, we can then calculate the **probability** the event occurs.

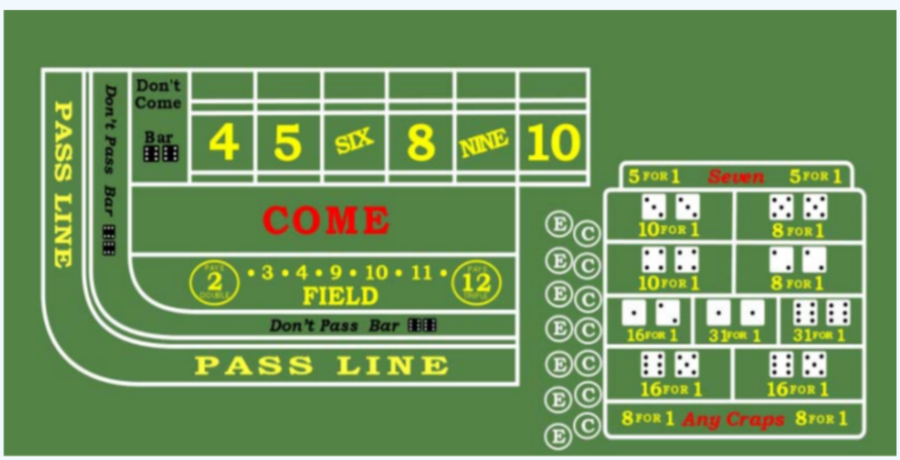
Probability Event Occurs = number of outcomes in Event / number of outcomes in Sample Space

We will use the notation P(A) to mean the probability event A occurs.

In the example, the probability of getting exactly 1 head in two coin tosses is 2 out of 4 or 50%.

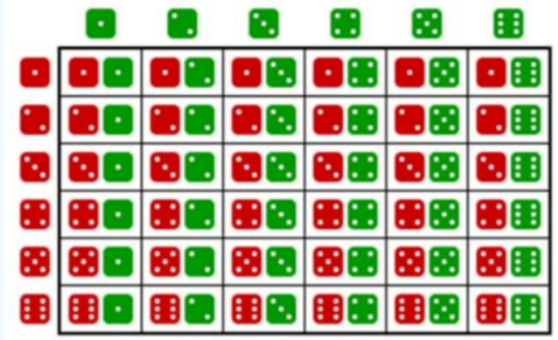
$$P(A) = 2/4 = 0.5 = 50\%$$

Example: Field Bet



In the casino game of craps, two dice are rolled at the same time and then the resulting two numbers are totaled. There are many bets in craps, so let us consider the Field bet. In this bet, the player will win even money if a total of 3, 4, 9, 10 or 11 is rolled. If a total of 2 is rolled, the player will win double the original bet, and if a total of 12 is rolled, the player will win triple the original bet. If a total of 5, 6, 7 or 8 is rolled, the player loses the original bet.

At first glance, this looks like a winning bet for the player since the player wins on 7 different numbers and the casino only wins on 4 different numbers. However, we know that a casino always designs games to give the casino the advantage. Let us carefully use counting methods to calculate the probability of a player winning the Field bet.



Let's first consider the task of listing the sample space of possible outcomes. Since there are two dice rolled, we can consider each outcome to be an ordered pair. There are 6 possible values for the first die and 6 possible values for the second die, meaning that there are 36 ordered pairs or outcomes. In the diagram, the red die is the first roll and the green die is the second roll.

$$\text{Sample Space } = \left\{\begin{array}{l}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\end{array}\right\}$$

Now define the event W to be the winning pairs of numbers in the Field bet, the pairs that add up to 2, 3, 4, 9, 10, 11 or 12. The winning pairs of numbers are shown in blue and the losing pairs are shown in red.

$$\text{Sample Space } = \left\{\begin{array}{l}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\end{array}\right\} \quad W = \left\{\begin{array}{l}(1,1),(1,2),(1,3), \\ (2,1),(2,2), \\ (3,1),(3,6), \\ (4,5),(4,6), \\ (5,4),(5,5),(5,6), \\ (6,3),(6,4),(6,5),(6,6)\end{array}\right\}$$

This means that there are 16 outcomes out of 36 in which the player wins. It's now easy to see the that probability of winning is less than 50%, as the casino took the numbers that occur the most frequently.

$$P(W)=\frac{16}{36}=\frac{4}{9} \approx 44.4\%$$

As a final note on this example, you might recall that the casino pays double if the player rolls (1,1) or triple if the player rolls (6,6). Even taking this extra bonus into account, if a player makes 36 \$100 bets, the casino will expect to win \$2000 (20 numbers x \$100), and the player will expect to win \$1900 (16 numbers x \$100, plus \$100 extra for the 2 and \$200 extra for the 12), meaning the player loses \$100 for every \$3600 bet, a house (casino) advantage of 2.78%.

Field Bet – Summary of 36 possible rolls	Amount won on \$100 bets
$(1,1)$ (pays double)	+\$200
$(6,6)$ (pays triple)	+\$300
$(1,2),(1,3),(2,1),(2,2),(3,1),(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5)$	+\$1400
$(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(5,1),(5,2),(5,3),(6,1),(6,2)$	-\$2000
Overall expected result of 36 rolls (\$3600 bet)	-\$100

Just remember, in the long run, the casino always wins.