

11.2: Independent Sampling Models

We will first consider the case when we want to compare the population means of two populations using independent sampling.

Distribution of the difference of two sample means

Suppose we wanted to test the hypothesis $H_o : \mu_1 = \mu_2$. We have point estimators for both μ_1 and μ_2 , namely \bar{X}_1 and \bar{X}_2 , which have approximately Normal Distributions under the Central Limit Theorem, but it would be useful to combine them both into a single estimator. Fortunately it is known that if two random variables have a Normal Distribution, then so does the sum and difference. Therefore we can restate the hypothesis as $H_o : \mu_1 - \mu_2 = 0$ and use the difference of sample means $\bar{X}_1 - \bar{X}_2$ as a point estimator for the difference in population means $\mu_1 - \mu_2$.

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Distribution of $\bar{X}_1 - \bar{X}_2$ under the Central Limit Theorem

$$\mu_{\bar{X}_1 - \bar{X}_2} = \mu_1 - \mu_2$$

$$\sigma_{\bar{X}_1 - \bar{X}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad \text{if } n_1 \text{ and } n_2 \text{ are sufficiently large.}$$

Comparing two means, independent sampling: Model when population variances known

When the population variances are known, the test statistic for the Hypothesis $H_o : \mu_1 = \mu_2$ can be tested with Normal distribution Z test statistic shown above. Also, if both sample size n_1 and n_2 exceed 30, this model can also be used.

Example: Homes and pools

Are larger homes more likely to have pools? The square footage (size) data for single family homes in California was separated into two populations: Homes with pools and homes without pools. We have data from 130 homes with pools and 95 homes without pools.



Solution

Design

Research Hypotheses:

$H_o : \mu_1 \leq \mu_2$ (Homes with pools do not have more mean square footage)

$H_a : \mu_1 > \mu_2$ (Homes with pools do have more mean square footage)

Since both sample sizes are over 30, the model will be a **Large sample Z test comparing two population means with independent sampling**.

This model is appropriate since the sample sizes assures the distribution of the sample mean is approximately Normal from the Central Limit Theorem. We opt for a one-tailed test since we want to support the claim that homes with pools are larger. The

test statistic will be =
$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Type I error would be to reject the Null Hypothesis and claim home with pools are larger, when they are not larger. It was decided to limit this error by setting the level of significance (α) to 1%.

The decision rule under the critical value method would be to reject the Null Hypothesis when the value of the test statistic is in the rejection region. In other words, reject H_o when $Z > 2.326$. The decision under the p -value method is to reject H_o if the p -value is $< \alpha$.

Data/Results



Since the test statistic ($Z = 4.19$) is greater than the critical value (2.326), H_o is rejected. Also the p -value (0.000013) is less than $\alpha(0.01)$, the decision is to Reject H_o .

Conclusion

The researcher makes the strong statement that homes with pools have a significantly higher mean square footage than home without pools.

Model when population variances are unknown, but are assumed to be equal

In the case that the population standard deviations are unknown, it seems logical to simply replace the population standard deviations for each population with the sample standard deviations and use a t -distribution as we did for the one population case. However, this is not so simple when the sample size for either group is under 30

We will consider two models. This first model (which we prefer to use since it has more power) assumes the population variances are equal and is called the **pooled variance t -test**. In this model we combine or “pool” the two sample standard deviations into a single estimate called the pooled standard deviation, s_p . If the central limit theorem is working, we then can substitute s_p for s_1 and s_2 get a t -distribution with $n_1 + n_2 - 2$ degrees of freedom:

📌 Pooled variance t -test to compare the means for two independent populations

Model Assumptions

- Independent Sampling
- $\bar{X}_1 - \bar{X}_2$ approximately Normal
- $\sigma_1^2 = \sigma_2^2$

Test Statistic

- $$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$
- $$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$
- Degrees of freedom = $n_1 + n_2 - 2$

Example: Fuel economy

A recent EPA study compared the highway fuel economy of domestic and imported passenger cars. A sample of 15 domestic cars revealed a mean of 33.7 MPG (mile per gallon) with a standard deviation of 2.4 mpg. A sample of 12 imported cars revealed a mean of 35.7 mpg with a standard deviation of 3.9. At the .05 significance level can the EPA conclude that the MPG is higher for the imported cars?



Solution

Design

It is best to associate the subscript 2 with the control group; in this case we will let domestic cars be population 2.

Research Hypotheses:

$H_o : \mu_1 \leq \mu_2$ (Imported compact cars do not have a higher mean MPG)

$H_a : \mu_1 > \mu_2$ (Imported compact cars have a higher mean MPG)

We will assume the population variances are equal $\sigma_1^2 = \sigma_2^2$, so the model will be a **Pooled variance t -test**. This model is appropriate if the distribution of the differences of sample means is approximately Normal from the Central Limit Theorem. A one-tailed test is selected based on H_a .

Type I error would be to reject the Null Hypothesis and claim that imports have a higher mean MPG, when they do not have higher MPG. The test will be run at a level of significance (α) of 5%.

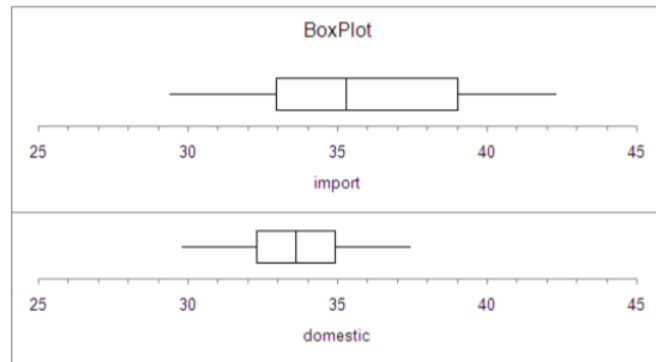
The degrees of freedom for this test is 25, so the decision rule under the critical value method would be to reject H_o when $t > 1.708$. The decision under the p -value method is to reject H_o if the p -value is $< \alpha$.

Data/Results

$$s_p = \sqrt{\frac{(12 - 1)3.86^2 + (12 - 1)2.16^2}{15 + 12 - 2}} = 3.03$$

$$t = \frac{(35.76 - 33.59) - 0}{3.03 \sqrt{\frac{1}{12} + \frac{1}{15}}} = 1.85$$

Since $1.85 > 1.708$, the decision would be to Reject H_0 . Also the p-value is calculated to be .0381 which again shows that the result is significant at the 5% level.



import	domestic	
35.76	33.59	mean
3.86	2.16	std. dev.
12	15	n

```

25 df
2.17000 difference (import - domestic)
9.16856 pooled variance
3.02796 pooled std. dev.
1.17273 standard error of difference
0 hypothesized difference

1.85 t
.0381 p-value (one-tailed, upper)

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Conclusion

Imported compact cars have a significantly higher mean MPG rating when compared to domestic cars

Model when population variances unknown, but assumed to be unequal

In the prior example, we assumed the population variances were equal. However, when looking at the box plot of the data or the sample standard deviations, it appears that the import cars have more variability MPG than domestic cars, which would violate the assumption of equal variances required for the Pooled Variance t -test.

Fortunately, there is an alternative model that has been developed for when population variances are unequal, called the Behrens-Fisher model⁸¹, or the **unequal variances t -test**.

Unequal variance t -test to compare the means for two independent populations

Model Assumptions

- Independent Sampling
- $\bar{X}_1 - \bar{X}_2$ approximately Normal
- $\sigma_1^2 \neq \sigma_2^2$

Test Statistic

$$t' = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\left[\frac{(s_1^2/n_1)^2}{(n_1-1)} + \frac{(s_2^2/n_2)^2}{(n_2-1)}\right]}$$

The degrees of freedom will be less than or equal to $n_1 + n_2 - 2$, so this test will usually have less power than the pooled variance t -test.

Example: Fuel economy

We will repeat the prior example to see if we can support the claim that imported compact cars have higher mean MPG when compared to domestic compact cars. This time we will assume that the population variances are not equal.

Solution

Design

Again we will let domestic cars be population 2.

Research Hypotheses:

$H_o : \mu_1 \leq \mu_2$ (Imported compact cars do not have a higher mean MPG)

$H_a : \mu_1 > \mu_2$ (Imported compact cars have a higher mean MPG)

We will assume the population variances are unequal $\sigma_1^2 \neq \sigma_2^2$, so the model will be an **unequal variance t-test**. This model is appropriate if the distribution of the differences of sample means is approximately Normal from the Central Limit Theorem. A one-tailed test is selected based on H_a .

Type I error would be to reject the Null Hypothesis and claim imports have a higher mean MPG, when they do not have higher MPG. The test will be run at a level of significance (α) of 5%. The degrees of freedom for this test is 16 (see calculation below), so the decision rule under the critical value method would be to reject H_o when $t > 1.746$. The decision under the p -value method is to reject H_o if the p -value is $< \alpha$

Data/Results

import	domestic	
35.76	33.59	mean
3.86	2.16	std. dev.
12	15	n

16	df
2.17000	difference (import - domestic)
1.24606	standard error of difference
0	hypothesized difference

1.74	t
.0504	p-value (one-tailed, upper)

$$df = \frac{\left(\frac{2.16^2}{15} + \frac{3.86^2}{12} \right)^2}{\left[\frac{(2.16^2/15)^2}{(15-1)} + \frac{(3.86^2/12)^2}{(12-1)} \right]} = 16$$

$$t = \frac{(35.76 - 33.59) - 0}{\sqrt{\frac{2.16^2}{15} + \frac{3.86^2}{12}}} = 1.74$$

Since $1.74 < 1.746$, the decision would be to Fail to Reject H_o . Also the p -value is calculated to be .0504 which again shows that the result is not significant (barely) at the 5% level.

Conclusion

Insufficient evidence to claim imported compact cars have a significantly higher mean MPG rating when compared to domestic cars.

You can see the lower power of this test when compared to the pooled variance t -test example where H_o was rejected. We always prefer to run the test with higher power when appropriate.

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