6.8: Poisson Distribution

Random variables that can be thought of as "How many occurrences per time period", or "How many occurrences per region" have many practical applications. For example:

The number of strong earthquakes per year in California.

The number of customers per hour at a restaurant.

The number of accidents per week at a manufacturing plant.

The number of errors per page in a manuscript.

If the rate is constant, these random variables will follow a Poisson distribution.

The Poisson Distribution is actually derived from a Binomial Distribution in which the sample size n gets very large and the probability of success p is very small. A good example of this is the Powerball Lottery.

Example: Powerball Lottery

The odds of winning the Powerball Lottery jackpot with a single ticket are 292,000,000 to 1. Suppose the jackpot gets large and 292,000,000 tickets are sold.

Solution

Let X = Number of jackpot winning tickets sold.

Under the Binomial distribution, $n = 292,000,000$ and $p = 1/292,000,000$ Note that p is very close to zero, so 1-p is very close to 1.

$\mu = np = 1$

$\sigma^2 = np(1-p) \approx np = \mu = 1$

The number of winners can be modeled by the Poisson Distribution, in which the single parameter \(μ is the expected number of winners; in this case $\mu = 1$. There could theoretically be millions of winners, so the possible values of the Poisson is designed so there is no theoretical limit for the value of X (although there are practical limits in real life problems).

The important features of the Poisson Distribution are shown here:

Poisson Probability Distribution (parameter= μ)

μ = expected occurrences per given time period or region. This rate must be constant.

X = number of occurrence per given time period or region Possible values of X {0, 1, 2, …} (no upper limit)

$\sigma^2 = \mu$

$\sigma = \sqrt{\mu}$

$$P(x)=\frac{e^{-\mu}\mu^{x}}{x!}$$

Example: Continuation of Powerball Lottery

Find the probability of no jackpot winners.

$$P(0)=\frac{e^{-1}2^{0}}{0!}=0.368$$

Find the probability of at least one jackpot winner. The answer calculated directly is an infinite sum, so instead use the Rule of Complement

$$P(X\geq 1)=P(1)+P(2)+\cdots$$

$$P(X\geq 1)=1-P(0)=1-\frac{e^{-1}2^{0}}{0!}=0.632$$

There is a 63.2% chance that at least one winning ticket is sold.

Example: Earthquakes

Earthquakes of Richter magnitude 3 or greater occur on a certain fault at a rate of two times per every year. Assume this rate is constant.

Solution

Find the probability of at least one earthquake of RM 3 or greater in the next year.

$\mu=2$ per year.

$$P(X\geq 1)=1-P(0)=1-\frac{e^{-2}2^{0}}{0!}=0.865$$

Find the probability of exactly 6 earthquakes of RM 3 or greater in the next 2 years.

When determining the parameter m for the Poisson Distribution, make sure that the expected value is over the time period or region given in the problem. Since these earthquakes occur at a rate of 2 per year, we would expect 4 earthquakes in 2 years.

μ= (2 per year)(2 years) = 4

$$P(X=6)=\frac{e^{-6}4^{0}}{6!}=0.104$$

Counting methods that are modeled by random variables that follow a Poisson Distribution are also called a **Poisson Process**.