12.2: Chi-square Test of Independence

In 2014, Colorado became the first state to legalize the recreational use of marijuana. Other states have joined Colorado, while some have decriminalized or authorized the medical use of marijuana. The question is should marijuana be legalized in all states. Suppose we took a poll of 1000 American adults and asked "Should marijuana be legal or not legal for recreational use" and got the following results:

Marijuana should be	Count	Percent
Legal	500	50%
Not Legal	450	45%
Don't know	50	5%
Total	**1000**	**100%**

The interpretation of this poll is that 50% of adults polled favored the legalization of marijuana for recreational use, while 45% opposed it. The remaining 5% were undecided.

At this time, you might have questions and want to explore this poll in more depth. For example, are younger people more likely to support legalization of marijuana? Do other demographic characteristics such as gender, ethnicity, sexual orientation, or religion affect people's opinions about legalization?

Let us explore the possibility of difference of opinion due to gender. Are men more likely (or less likely) to oppose legalization of marijuana compared to women?

In the example above, suppose we have exactly 500 men and 500 women in the survey. What would we expect to see in the data if there were no difference in opinion between men and women?

Two-way tables

Two-way or **contingency tables** are used to summarize two categorical variables, also known as bivariate categorical data. In order to create a two-way table, the researcher must **cross-tabulate** the two responses for each categorical questions.

In the example above, the two categorical variables are gender and opinion on marijuana legalization. Gender has two choices (male or female) while opinion on marijuana legalization has three choices (legal, not legal and unsure).

In the example above, suppose we have exactly 500 men and 500 women in the survey. What would we expect to see in the data if there were no difference in opinion between men and women? We could then simply apply the total percentages to each group.

To create a hypothetical two-way table if there was no difference in opinion between men and women, apply the total percentages for each choice of Opinion to the total number for each choice of Gender.

eg: Men/Legal would 50% of 500 or 250 people.

Marijuana should be	Men	Women	Total
Legal	50%	50%	50%
Not Legal	45%	45%	45%
Unsure	5%	5%	5%
Total	**100%**	**100%**	**100%**

Marijuana should be	Men	Women	Total
Legal	250	250	500
Not Legal	225	225	450
Unsure	25	25	50
Total	**500**	**500**	**1000**

Let's review from probability what independence means. If two events A an B are independent, then the following statements are true:

$$P(\text{A given B}) = P(A)$$
$$P(\text{B given A}) = P(B)$$
$$P(\text{A and B}) = P(A)P(B)$$

You can pick any two events in the table above to verify that Gender and Opinion of Legalization of Marijuana are independent events. For example, compare the events Not **Legal** and **Men**.

P(Not Legal given Men) = 225/500 = 45% same as P(Not Legal) = 45%

P(Men given Not Legal) = 225/450 = 50% same as P(Men) = 50%

P(Not Legal and Men) = 225/1000 = 22.5% same as P(Not Legal)P(Men) = (45%)(50%) = 22.5%

Based on these probability rules we can calculate the expected value of any pair of independent events by using the following formula:

Expected Value = (Row Total)(Column Total)/(Grand Total)

For example, looking at the events **Not Legal and Men**:

Expected Value = (450)(500)/(1000) = 225

What if the events are not independent? Let's review the same survey. What would we expect to see in the data if there was a difference in opinion between men and women? Let's say women were more likely to support legalization. In that case, we would expect the 450 people who supported legalization of marijuana to have a higher number of women (and a smaller number of men) compared to the first table. Note we only change the first six boxes (shaded below); the totals must remain the same.

This is an example of a hypothetical two-way table where women were more likely to support legalization.

Only the six boxes shaded in yellow change from the prior example

Marijuana should be	Men	Women	Total
Legal	40%	60%	50%
Not Legal	55%	35%	45%
Unsure	5%	5%	5%
Total	**100%**	**100%**	**100%**

Marijuana should be	Men	Women	Total
Legal	200	300	500
Not Legal	275	175	450
Unsure	25	25	50
Total	**500**	**500**	**1000**

Now let's see the actual results of this survey and see what is happening:

Actual Poll of 500 men and 500 women adults. Should marijuana be legal for recreational use?

Marijuana should be	Men	Women	Total
Legal	54%	46%	50%
Not Legal	41%	49%	45%
Unsure	5%	5%	5%
Total	**100%**	**100%**	**100%**

Marijuana should be	Men	Women	Total
Legal	270	230	500
Not Legal	205	245	450
Unsure	25	25	50
Total	**500**	**500**	**1000**

In this poll, a higher percentage of men support legalization of marijuana for recreational use compared to women. Question: Is this evidence strong enough to support the claim that gender and opinion about marijuana legalization are not independent events? This question can addressed by conducting a hypothesis test using with the **Chi-square Test for Independence** model.

Chi-square test of Independence

A Chi-square test of independence can be used to determine if there is a relationship between two randomized categorical variables. If the categorical variables are labeled A and B, the hypotheses are always written in this form:

H_o: A and B are independent events

H_a: A and B are dependent events.

If only one variable is randomized, then the test is called a Chi-square Test of Homogeneity, but the execution of the test is exactly the same. If A represents the randomized response variable and B represents the manipulated explanatory variable, then the hypotheses are written as:

H_o: There no difference in distribution of A due to B.

H_a: There is a difference in the distribution of A due to B.

Chi-square Test for Independence

Model Assumptions

- O_{ij} = Observed in category ij
- $E_{ij} = np_{ij} = \frac{(\text{ ColumnTotal })(\text{ RowTotal })}{\text{Grand Total}}$; $E_{ij} \geq 5$ for each ij

Test Statistic

- $\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$ $\quad \text{df} = (r-1)(c-1)$ where

r = number of row categories c = number of column categories n = sample size

Example: Legalization of marijuana

Are Gender and Opinion about legalization of marijuana for recreational use independent events? Conduct a hypothesis test with a significance level of 5%.

Solution

Research Hypotheses:

H_o: Gender and Opinion about legalization of marijuana for recreational use are independent events.

H_a: Gender and Opinion about legalization of marijuana for recreational use are dependent events.

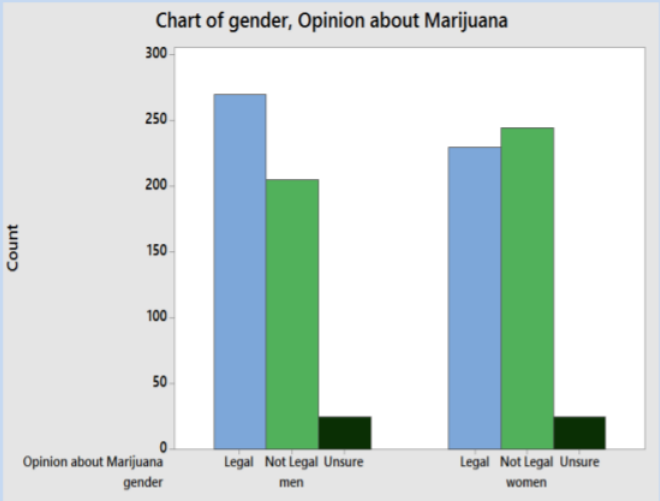
Statistical Model: Chi-square Test of Independence. The two categorical variables in this example are Gender and Opinion.

Results:

Rows: Opinion about Marijuana
Columns: gender

1st value = Observed
2nd Value = Expected
3rd Value = Contribution to Chi-square

	men	women	All
Legal	270	230	500
	250	250	
	1.600	1.600	
Not Legal	205	245	450
	225	225	
	1.778	1.778	
Unsure	25	25	50
	25	25	
	0.000	0.000	
All	500	500	1000



Important Assumption: The Expected Value of Each Category needs to be greater than or equal to 5. In this example, the lowest expected value is 225 (Men, not legal) so the assumption is met.

Test Statistic: $\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$ $\quad \text{df} = (3-1)(2-1) = 2$

Decision Rule (Critical Value Method): Reject H_o if $\chi^2 > 5.991 (\alpha = .05, 2\text{df})$

$\chi^2 = 1.600 + 1.600 + 1.778 + 1.778 = 6.756$

Since the Test Statistic exceeds the critical value, the decision is to **Reject H_o**. Under the p-value method, H_o is also rejected since the p-value $= p\left(\chi^2 > 6.756\right) = 0.034$, which is less than the Significance Level α of 5%.

Conclusion:

Gender and Opinion about legalization of marijuana for recreational use are dependent events. Men are more likely to support legalization of marijuana for recreational use.
