

### 3.4: General Multiplication Probability

It is a hot summery day. You are looking forward to an unopened pint of (cow milk based) ice cream that you have waiting for you in the freezer. When you reach for the carton and take off the lid, you see that there are only two spoonfuls left. Who is the ice cream thief in your household?

You live with your family, which consists of your grandma, both two parents, and three siblings; a total of six suspects. At this point, you have a probability of  $\frac{1}{6}$  of guessing correctly who is the thief.

However, you know your grandma's gums are sensitive to cold food. Also, one parent and two siblings are lactose intolerant. That leaves only two possible suspects: the other parent and one sibling. At this point, you have a probability of  $\frac{1}{2}$  of guessing correctly who is the thief. This is an example of **conditional probability**, where instead of looking at the entire sample space, we look at a smaller group to find the correct probability.

#### Conditional Probability

The conditional probability of A given B is written  $P(A|B)$ .  $P(A|B)$  is the probability that event A will occur given that the event B has already occurred. **A conditional reduces the sample space.** We calculate the probability of A from the reduced sample space B. The formula to calculate  $P(A|B)$  is

$$P(A|B) = \frac{P(A \text{ AND } B)}{P(B)}$$

where  $P(B)$  is greater than zero.

Although the formula can be intimidating, there are times when we can find conditional probabilities without it.

#### Example

Suppose we toss one fair, six-sided die. The sample space  $S = \{1, 2, 3, 4, 5, 6\}$ . Let event A be rolling a 2 or 3,  $A = \{2, 3\}$ . Let event B be rolling an even number,  $B = \{2, 4, 6\}$ .

What is the probability of getting event A, given we know event B happened? In other words, what is  $P(A|B)$ ? Since there three outcomes in event B. Both events A and B only share one outcome, "2". Then  $P(A|B) = \frac{1}{3}$ .

#### Example

Let's do the example above, but with using the formula.

Remember that S has six outcomes.

$$\begin{aligned} P(A|B) &= \frac{P(A \text{ AND } B)}{P(B)} \\ &= \frac{\text{the number of outcomes that are 2 or 3 AND even in S}}{\text{the number of outcomes that are even in S}} \\ &= \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3} \end{aligned}$$

Notice, we get the same answer as in Example 1A.

Notation:  $A|B$  is equivalent to A GIVEN B

### ✓ Example 3.4.2

A fair, six-sided die is rolled. Describe the sample space  $S$ , identify each of the following events with a subset of  $S$  and compute its probability (an outcome is the number of dots that show up).

- Event  $A$  = the outcome is an even number.
- Event  $B$  = the outcome is less than four.
- $A$  GIVEN  $B$
- $B$  GIVEN  $A$

**Answer**

- $A = \{2, 4, 6\}$ ,  $P(A) = \frac{1}{2}$
- $B = \{1, 2, 3\}$ ,  $P(B) = \frac{1}{2}$
- $A|B = \{2\}$ ,  $P(A|B) = \frac{1}{3}$
- $B|A = \{2\}$ ,  $P(B|A) = \frac{1}{3}$

### 📌 With or Without Replacement

Sampling may be done with replacement or without replacement (Figure 3.4.1):

- With replacement:** If each member of a population is replaced after it is picked, then that member has the possibility of being chosen more than once. When sampling is done with replacement, then events are considered to be *independent*, meaning the result of the first pick will not change the probabilities for the second pick.
- Without replacement:** When sampling is done without replacement, each member of a population may be chosen only once. In this case, the probabilities for the second pick are affected by the result of the first pick. The events are considered to be *dependent* or *not independent*.

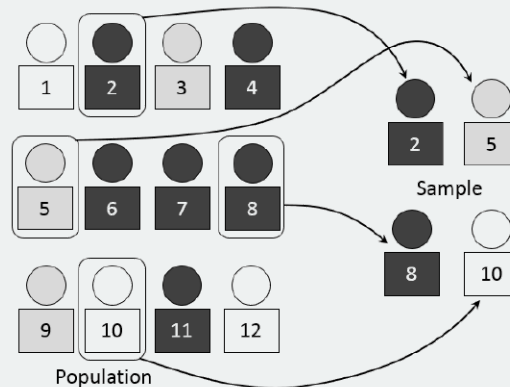


Figure 3.4.1 : A visual representation of the sampling process. If the sample items are replaced after each sampling event, then this is "sampling with replacement" if not, then it is "sampling without replacement". (CC BY-SA 4.0; Dan Kernler).

If it is not known whether  $A$  and  $B$  are independent or dependent, assume they are dependent until you can show otherwise.

### ✓ Example 3.4.3

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king) of that suit.

- Sampling with replacement:

Suppose you pick three cards with replacement. *The first card you pick out of the 52 cards is the Q of spades. You put this card back, reshuffle the cards and pick a second card from the 52-card deck. It is the ten of clubs. You put this card back, reshuffle the cards and pick a third card from the 52-card deck. This time, the card is the Q of spades again. Each time you drew a card and replaced it before drawing another card is **sampling with replacement**.*

b. Sampling without replacement:

Suppose you pick three cards without replacement. *The first card you pick* out of the 52 cards is the K of hearts. You put this card aside and *pick the second card* from the 51 cards remaining in the deck. It is the three of diamonds. You put this card aside and *pick the third card* from the remaining 50 cards in the deck. *The third card* is the J of spades. Each time you drew a card and did not replace it into the deck before drawing another card is **sampling without replacement**.

✓ Example 3.4.4

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king) of that suit. Three cards are picked at random.

- Suppose you know that the picked cards are Q of spades, K of hearts and Q of spades. Can you decide if the sampling was with or without replacement?
- Suppose you know that the picked cards are Q of spades, K of hearts, and J of spades. Can you decide if the sampling was with or without replacement?

**Answer a**

With replacement

**Answer b**

No

✓ Example 3.4.5

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts, and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), and K (king) of that suit. S = spades, H = Hearts, D = Diamonds, C = Clubs.

- Suppose you pick four cards, but do not put any cards back into the deck. Your cards are QS, 1D, 1C, QD.
- Suppose you pick four cards and put each card back before you pick the next card. Your cards are KH, 7D, 6D, KH.

Which of a. or b. did you sample with replacement and which did you sample without replacement?

**Answer a**

Without replacement

**Answer b**

With replacement

✓ General Multiplication Rule

In Section 3.3, we learned the Multiplication Rule when events A and B are independent:

$$P(A \text{ AND } B) = P(A) \cdot P(B)$$

However, if events A and B are dependent, then

$$P(A \text{ AND } B) = P(A) \cdot P(B|A)$$

✓ Example 3.4.6

A gum ball machine has 5 red balls, 6 green balls, and 5 yellow balls. Suppose you Find the following probabilities.

- $P(\text{draw a red ball and then a green})$  **with** replacement
- $P(\text{draw a red ball and then a green})$  **without** replacement
- $P(\text{draw three red balls})$  **with** replacement

- d.  $P(\text{draw three red balls})$  **without** replacement
- e.  $P(\text{draw a red ball, two green balls, and one yellow ball})$  **with** replacement
- f.  $P(\text{draw a red ball, two green balls, and one yellow ball})$  **without** replacement

#### Answer a

$$P(\text{draw a red ball and then a green replacement}) = P(\text{red}) \cdot P(\text{green}) = \frac{5}{16} \cdot \frac{6}{16} = 0.1172$$

Since we are replacing this the ball after each draw, you recover the total number of gumballs each time, which was 16.

#### Answer b

$$P(\text{draw a red ball and then a green}) = P(\text{red}) \cdot P(\text{green}) = \frac{5}{16} \cdot \frac{6}{15} = 0.128$$

Since we are **not** replacing this the ball after each draw, the total number of gumballs each time decreases by one.

#### Answer c

$$P(\text{draw three red balls}) = P(\text{red}) \cdot P(\text{red}) \cdot P(\text{red}) = \frac{5}{16} \cdot \frac{5}{16} \cdot \frac{5}{16} = 0.3052$$

#### Answer d

$$P(\text{draw three red balls}) = P(\text{red}) \cdot P(\text{red}) \cdot P(\text{red}) = \frac{5}{16} \cdot \frac{4}{15} \cdot \frac{3}{14} = 0.0179$$

#### Answer e

$$\begin{aligned} P(\text{draw a red ball, two green balls, and one yellow ball}) &= P(\text{red}) \cdot P(\text{green}) \cdot P(\text{green}) \\ &\cdot P(\text{yellow}) = \frac{5}{16} \cdot \frac{6}{16} \cdot \frac{6}{16} \cdot \frac{5}{16} = 0.0137 \end{aligned}$$

#### Answer f

$$\begin{aligned} P(\text{draw a red ball, two green balls, and one yellow ball}) &= P(\text{red}) \cdot P(\text{green}) \cdot P(\text{green}) \\ &\cdot P(\text{yellow}) = \frac{5}{16} \cdot \frac{6}{15} \cdot \frac{5}{14} \cdot \frac{5}{13} = 0.0172 \end{aligned}$$

## Glossary

### Independent Events

If one event does not affect the other, then the two events are independent.

### Dependent Events

If two events are NOT independent, then we say that they are dependent.

### Sampling with Replacement

If each member of a population is replaced after it is picked, then that member has the possibility of being chosen more than once.

### Sampling without Replacement

When sampling is done without replacement, each member of a population may be chosen only once.

### The Conditional Probability of One Event Given Another Event

$P(A|B)$  is the probability that event  $A$  will occur given that the event  $B$  has already occurred.

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