3.7E: Exercises for Section 3.6

In exercises 1 - 6, given $y=f(u)$ and $u=g(x)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$.

1) $y=3u-6,\quad u=2x^2$

2) $y=6u^3,\quad u=7x-4$

Answer

$$\frac{dy}{dx}=18u^2\cdot 7=18(7x-4)^2\cdot 7=126(7x-4)^2$$

3) $y=\sin u,\quad u=5x-1$

4) $y=\cos u,\quad u=-\frac{x}{8}$

Answer

$$\frac{dy}{dx}=-\sin u\cdot\left(-\frac{1}{8}\right)=\frac{1}{8}\sin(-\frac{x}{8})$$

5) $y=\tan u,\quad u=9x+2$

6) $y=\sqrt{4u+3},\quad u=x^2-6x$

Answer

$$\frac{dy}{dx}=\frac{8x-24}{2\sqrt{4u+3}}=\frac{4x-12}{\sqrt{4x^2-24x+3}}$$

For each of the following exercises,

a. decompose each function in the form $y=f(u)$ and $u=g(x)$, and

b. find $\frac{dy}{dx}$ as a function of x.

7) $y=(3x-2)^6$

8) $y=(3x^2+1)^3$

Answer

a. $f(u)=u^3,\quad u=3x^2+1$;

b. $\frac{dy}{dx}=18x(3x^2+1)^2$

9) $y=\sin^5(x)$

10) $y=\left(\frac{x}{7}+\frac{7}{x}\right)^7$

Answer

a. $f(u)=u^7,\quad u=\frac{x}{7}+\frac{7}{x}$;

b. $\frac{dy}{dx}=7\left(\frac{x}{7}+\frac{7}{x}\right)^6\cdot\left(\frac{1}{7}-\frac{7}{x^2}\right)$

11) $y=\tan(\sec x)$

12) $y=\csc(\pi x+1)$

Answer

a. $f(u)=\csc u,\quad u=\pi x+1$;

b. $\dfrac{dy}{dx} = -\pi \csc(\pi x + 1) \cdot \cot(\pi x + 1)$

13) $y = \cot^2 x$

14) $y = -6\sin^{-3} x$

Answer

a. $f(u) = -6u^{-3}, \quad u = \sin x$;

b. $\dfrac{dy}{dx} = 18\sin^{-4} x \cdot \cos x$

In exercises 15 - 24, find $\dfrac{dy}{dx}$ for each function.

15) $y = (3x^2 + 3x - 1)^4$

16) $y = (5 - 2x)^{-2}$

Answer

$\dfrac{dy}{dx} = \dfrac{4}{(5-2x)^3}$

17) $y = \cos^3(\pi x)$

18) $y = (2x^3 - x^2 + 6x + 1)^3$

Answer

$\dfrac{dy}{dx} = 6(2x^3 - x^2 + 6x + 1)^2 \cdot (3x^2 - x + 3)$

19) $y = \dfrac{1}{\sin^2(x)}$

20) $y = \left(\tan x + \sin x\right)^{-3}$

Answer

$\dfrac{dy}{dx} = -3\left(\tan x + \sin x\right)^{-4} \cdot (\sec^2 x + \cos x)$

21) $y = x^2 \cos^4 x$

22) $y = \sin(\cos 7x)$

Answer

$\dfrac{dy}{dx} = -7\cos(\cos 7x) \cdot \sin 7x$

23) $y = \sqrt{6 + \sec \pi x^2}$

24) $y = \cot^3(4x + 1)$

Answer

$\dfrac{dy}{dx} = -12\cot^2(4x+1) \cdot \csc^2(4x+1)$

25) Let $y = \left[f(x)\right]^3$ and suppose that $f'(1) = 4$ and $\frac{dy}{dx} = 10$ for $x = 1$. Find $f(1)$.

26) Let $y = \left(f(x) + 5x^2\right)^4$ and suppose that $f(-1) = -4$ and $\frac{dy}{dx} = 3$ when $x = -1$. Find $f'(-1)$

Answer

$f'(-1) = 10\frac{3}{4}$

27) Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x$. If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2$, find $f'(4)$.

28) [T] Find the equation of the tangent line to $y=-\sin(\frac{x}{2})$ at the origin. Use a calculator to graph the function and the tangent line together.

Answer

$y=-\frac{1}{2}x$

29) [T] Find the equation of the tangent line to $y=\left(3x+\frac{1}{x}\right)^2$ at the point $(1,16)$. Use a calculator to graph the function and the tangent line together.

30) Find the x -coordinates at which the tangent line to $y=\left(x-\frac{6}{x}\right)^8$ is horizontal.

Answer

$x=\pm\sqrt{6}$

31) [T] Find an equation of the line that is normal to $g(\theta)=\sin^2(\pi\theta)$ at the point $\left(\frac{1}{4},\frac{1}{2}\right)$. Use a calculator to graph the function and the normal line together.

For exercises 32 - 39, use the information in the following table to find $h'(a)$ at the given value for a.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2
1	1	−2	3	0
2	4	4	1	−1
3	3	−3	2	3

32) $h(x)=f\big(g(x)\big);\quad a=0$

Answer

$h'(0)=10$

33) $h(x)=g\big(f(x)\big);\quad a=0$

34) $h(x)=\left(x^4+g(x)\right)^{-2};\quad a=1$

Answer

$h'(1)=-\frac{1}{8}$

35) $h(x)=\left(\dfrac{f(x)}{g(x)}\right)^2;\quad a=3$

36) $h(x)=f\big(x+f(x)\big);\quad a=1$

Answer

$h'(1)=-4$

37) $h(x)=\big(1+g(x)\big)^3;\quad a=2$

38) $h(x)=g\big(2+f(x^2)\big);\quad a=1$

Answer

$h'(1)=-12$

39) $h(x)=f\big(g(\sin x)\big);\quad a=0$

40) [T] The position function of a freight train is given by $s(t)=100(t+1)^{-2}$, with s in meters and t in seconds. At time $t=6$ s, find the train’s

a. velocity and

b. acceleration.

c. Considering your results in parts a. and b., is the train speeding up or slowing down?

Answer

a. $v(6) = -\frac{200}{343}$ m/s,

b. $a(6) = \frac{600}{2401}$ m/s^2,

c. The train is slowing down since velocity and acceleration have opposite signs.

41) [T] A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches:

$$s(t) = -3\cos\left(\pi t + \frac{\pi}{4}\right).$$

a. Determine the position of the spring at $t = 1.5$ s.

b. Find the velocity of the spring at $t = 1.5$ s.

42) [T] The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C = 0.0001x^3 - 0.02x^2 + 3x + 300$. In t weeks production is estimated to be $x = 1600 + 100t$ boxes.

a. Find the marginal cost $C'(x)$.

b. Use Leibniz's notation for the chain rule, $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.

c. Use your result in part b. to determine how fast costs are increasing when $t = 2$ weeks. Include units with the answer.

Answer

a. $C'(x) = 0.0003x^2 - 0.04x + 3$

b. $\frac{dC}{dt} = 100 \cdot (0.0003x^2 - 0.04x + 3) = 100 \cdot (0.0003(1600 + 100t)^2 - 0.04(1600 + 100t) + 3) = 300t^2 + 9200t + 70700$

c. Approximately $90,300 per week

43) [T] The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

a. Suppose $r = 2 - \frac{100}{(t+7)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the area is expanding.

b. Use your result in part a. to find the rate at which the area is expanding at $t = 4$ s.

44) [T] The formula for the volume of a sphere is $S = \frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

a. Suppose $r = \frac{1}{(t+1)^2} - \frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the snowball is melting.

b. Use your result in part a. to find the rate at which the volume is changing at $t = 1$ min.

Answer

a. $\frac{dS}{dt} = -\frac{8\pi r^2}{(t+1)^3} = -\frac{8\pi\left(\frac{1}{(t+1)^2} - \frac{1}{12}\right)^2}{(t+1)^3}$

b. The volume is decreasing at a rate of $-\frac{\pi}{36}$ ft^3/min

45) [T] The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function $T(x) = 94 - 10\cos\left[\frac{\pi}{12}(x-2)\right]$, where x is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

46) [T] The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t)=5\sin\left(\frac{\pi}{6}t-\frac{7\pi}{6}\right)+8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

Answer

2.3 ft/hr
