4.10E: Exercises for Section 4.9

In exercises 1 - 5, write Newton's formula as $x_{n+1} = F(x_n)$ for solving $f(x) = 0$.

1) $f(x) = x^2 + 1$

2) $f(x) = x^3 + 2x + 1$

Answer

$$F(x_n) = x_n - \frac{x_n^3 + 2x_n + 1}{3x_n^2 + 2}$$

3) $f(x) = \sin x$

4) $f(x) = e^x$

Answer

$$F(x_n) = x_n - \frac{e^{x_n}}{e^{x_n}}$$

5) $f(x) = x^3 + 3xe^x$

In exercises 6 - 8, solve $f(x) = 0$ using the iteration $x_{n+1} = x_{n-c} f(x_n)$, which differs slightly from Newton's method. Find a c that works and a c that fails to converge, with the exception of $c = 0$.

6) $f(x) = x^2 - 4$, with $x_0 = 0$

Answer

$|c| > 0.5$ fails, $|c| \leq 0.5$ works

7) $f(x) = x^2 - 4x + 3$, with $x_0 = 2$

8) What is the value of "c" for Newton's method?

Answer

$$c = \frac{1}{f'(x_n)}$$

In exercises 9 - 16, compute x_1 and x_2 using the specified iterative method.

Start at

a. $x_0 = 0.6$ **and**

b. $x_0 = 2$.

9) $x_{n+1} = x_n^2 - \frac{1}{2}$

10) $x_{n+1} = 2x_n(1 - x_n)$

Answer

a. $x_1 = \frac{12}{25},\ x_2 = \frac{312}{625}$;
b. $x_1 = -4,\ x_2 = -40$

11) $x_{n+1} = \sqrt{x_n}$

12) $x_{n+1} = \frac{1}{\sqrt{x_n}}$

Answer

a. $x_1 = 1.291,\ x_2 = 0.8801$;
b. $x_1 = 0.7071,\ x_2 = 1.189$

13) $x_{n+1} = 3x_n(1 - x_n)$

14) $x_{n+1} = x_n^2 + x_{n-2}$

Answer

a. $x_1 = -\frac{26}{25},\ x_2 = -\frac{1224}{625}$;
b. $x_1 = 4,\ x_2 = 18$

15) $x_{n+1} = \frac{1}{2}x_n - 1$

16) $x_{n+1} = |x_n|$

Answer

a. $x_1 = \frac{6}{10},\ x_2 = \frac{6}{10}$;
b. $x_1 = 2,\ x_2 = 2$

In exercises 17 - 26, solve to four decimal places using Newton's method and a computer or calculator. Choose any initial guess x_0 that is not the exact root.

17) $x^2 - 10 = 0$

18) $x^4 - 100 = 0$

Answer

3.1623or -3.1623

19) $x^2 - x = 0$

20) $x^3 - x = 0$

Answer

$0, -1$ or 1

21) $x + 5\cos x = 0$

22) $x + \tan x = 0,$ choose $x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Answer

0

23) $\dfrac{1}{1-x} = 2$

24) $1 + x + x^2 + x^3 + x^4 = 2$

Answer

0.5188or -1.2906

25) $x^3 + (x+1)^3 = 10^3$

26) $x = \sin^2(x)$

Answer

0

In exercises 27 - 30, use Newton's method to find the fixed points of the function where $f(x) = x$; round to three decimals.

27) $\sin x$

28) $\tan x$ on $x = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Answer

4.493

29) $e^x - 2$

30) $\ln(x) + 2$

Answer

$0.159,\ 3.146$

Newton's method can be used to find maxima and minima of functions in addition to the roots. In this case apply Newton's method to the derivative function $f'(x)$ to find its roots, instead of the original function. In exercises 31 - 32, consider the formulation of the method.

31) To find candidates for maxima and minima, we need to find the critical points $f'(x) = 0$. Show that to solve for the critical points of a function $f(x)$, Newton's method is given by $x_{n+1} = x_n - \dfrac{f'(x_n)}{f''(x_n)}$.

32) What additional restrictions are necessary on the function f?

Answer

We need f to be twice continuously differentiable.

In exercises 33 - 40, use Newton's method to find the location of the local minima and/or maxima of the following functions; round to three decimals.

33) Minimum of $f(x) = x^2 + 2x + 4$

34) Minimum of $f(x) = 3x^3 + 2x^2 - 16$

Answer

$x = 0$

35) Minimum of $f(x) = x^2 e^x$

36) Maximum of $f(x) = x + \dfrac{1}{x}$

Answer

$x = -1$

37) Maximum of $f(x) = x^3 + 10x^2 + 15x - 2$

38) Maximum of $f(x) = \dfrac{\sqrt{x} - \sqrt[3]{x}}{x}$

Answer

$x = 5.619$

39) Minimum of $f(x) = x^2 \sin x$, closest non-zero minimum to $x = 0$

40) Minimum of $f(x) = x^4 + x^3 + 3x^2 + 12x + 6$

Answer

$x = -1.326$

In exercises 41 - 44, use the specified method to solve the equation. If it does not work, explain why it does not work.

41) Newton's method, $x^2 + 2 = 0$

42) Newton's method, $0 = e^x$

Answer

There is no solution to the equation.

43) Newton's method, $0=1+x^2$ starting at $x_0=0$

44) Solving $x_{n+1}=-x_n^3$ starting at $x_0=-1$

Answer

It enters a cycle.

In exercises 45 - 48, use the *secant method*, an alternative iterative method to Newton's method. The formula is given by

$$x_n=x_{n-1}-f(x_{n-1})\frac{x_{n-1}-x_{n-2}}{f(x_{n-1})-f(x_{n-2})}.$$

45) a root to $0=x^2-x-3$ accurate to three decimal places.

46) Find a root to $0=\sin x+3x$ accurate to four decimal places.

Answer

0

47) Find a root to $0=e^x-2$ accurate to four decimal places.

48) Find a root to $\ln(x+2)=\frac{1}{2}$ accurate to four decimal places.

Answer

-0.3513

49) Why would you use the secant method over Newton's method? What are the necessary restrictions on f?

In exercises 50 - 54, use both Newton's method and the secant method to calculate a root for the following equations. Use a calculator or computer to calculate how many iterations of each are needed to reach within three decimal places of the exact answer. For the secant method, use the first guess from Newton's method.

50) $f(x)=x^2+2x+1,\quad x_0=1$

Answer

Newton: 11 iterations, secant: 16 iterations

51) $f(x)=x^2,\quad x_0=1$

52) $f(x)=\sin x,\quad x_0=1$

Answer

Newton: three iterations, secant: six iterations

53) $f(x)=e^x-1,\quad x_0=2$

54) $f(x)=x^3+2x+4,\quad x_0=0$

Answer

Newton: five iterations, secant: eight iterations

In exercises 55 - 56, consider Kepler's equation regarding planetary orbits, $M=E-\varepsilon\sin(E)$, where M is the mean anomaly, E is eccentric anomaly, and ε measures eccentricity.

55) Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M=\frac{\pi}{3}$ and the eccentricity of the orbit $\varepsilon=0.25$; round to three decimals.

56) Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M=\frac{3\pi}{2}$ and the eccentricity of the orbit $\varepsilon=0.8$; round to three decimals.

Answer

$E = 4.071$

In exercises 57 - 58, consider a bank investment. The initial investment is $\$10,000$. After 25 years, the investment has tripled to $\$30,000$.

57) Use Newton's method to determine the interest rate if the interest was compounded annually.

58) Use Newton's method to determine the interest rate if the interest was compounded continuously.

Answer

4.394

59) The cost for printing a book can be given by the equation $C(x) = 1000 + 12x + \frac{1}{2}x^{2/3}$. Use Newton's method to find the break-even point if the printer sells each book for $\$20$.