4.5E: Exercises for Section 4.4

1)Why do you need continuity to apply the Mean Value Theorem? Construct a counterexample.

2) Why do you need differentiability to apply the Mean Value Theorem? Find a counterexample.

Answer

One example is $f(x)=|x|+3, -2\leq x\leq 2$

3) When are Rolle's theorem and the Mean Value Theorem equivalent?

4) If you have a function with a discontinuity, is it still possible to have $f'(c)(b-a)=f(b)-f(a)$? Draw such an example or prove why not.

Answer

Yes, but the Mean Value Theorem still does not apply

In exercises 5 - 9, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.

5) $y=\sin(\pi x)$

6) $y=\dfrac{1}{x^3}$

Answer

$(-\infty,0),(0,\infty)$

7) $y=\sqrt{4-x^2}$

8) $y=\sqrt{x^2-4}$

Answer

$(-\infty,-2),(2,\infty)$

9) $y=\ln(3x-5)$

In exercises 10 - 13, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points c such that $f'(c)(b-a)=f(b)-f(a)$.

10) [T] $y=3x^3+2x+1$ over $[-1,1]$

Answer

2 points

11) [T] $y=\tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2},\frac{3}{2}\right]$

12) [T] $y=x^2\cos(\pi x)$ over $[-2,2]$

Answer

5 points

13) [T] $y=x^6-\frac{3}{4}x^5-\frac{9}{8}x^4+\frac{15}{16}x^3+\frac{3}{32}x^2+\frac{3}{16}x+\frac{1}{32}$ over $[-1,1]$

In exercises 14 - 19, use the Mean Value Theorem and find all points $0<c<2$ such that $f(2)-f(0)=f'(c)(2-0)$.

14) $f(x)=x^3$

Answer

$c=\frac{2\sqrt{3}}{3}$

15) $f(x)=\sin(\pi x)$

16) $f(x)=\cos(2\pi x)$

Answer

$c=\frac{1}{2},1,\frac{3}{2}$

17) $f(x)=1+x+x^2$

18) $f(x)=(x-1)^{10}$

Answer

$c=1$

19) $f(x)=(x-1)^9$

In exercises 20 - 23, show there is no c such that $f(1)-f(-1)=f'(c)(2)$. Explain why the Mean Value Theorem does not apply over the interval $[-1,1]$.

20) $f(x)=\left|x-\frac{1}{2}\right|$

Answer

Not differentiable

21) $f(x)=\dfrac{1}{x^2}$

22) $f(x)=\sqrt{|x|}$

Answer

Not differentiable

23) $f(x)=\lfloor x\rfloor$ (Hint: This is called the floor function and it is defined so that $f(x)$ is the largest integer less than or equal to x.)

In exercises 24 - 34, determine whether the Mean Value Theorem applies for the functions over the given interval $[a,b]$. Justify your answer.

24) $y=e^x$ over $[0,1]$

Answer

Yes

25) $y=\ln(2x+3)$ over $[-\frac{3}{2},0]$

26) $f(x)=\tan(2\pi x)$ over $[0,2]$

Answer

The Mean Value Theorem does not apply since the function is discontinuous at $x=\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4}$.

27) $y=\sqrt{9-x^2}$ over $[-3,3]$

28) $y=\dfrac{1}{|x+1|}$ over $[0,3]$

Answer

Yes

29) $y=x^3+2x+1$ over $[0,6]$

30) $y=\dfrac{x^2+3x+2}{x}$ over $[-1,1]$

Answer

The Mean Value Theorem does not apply; discontinuous at $x=0$.

31) $y=\dfrac{x}{\sin(\pi x)+1}$ over $[0,1]$

32) $y=\ln(x+1)$ over $[0,e-1]$

Answer

Yes

33) $y=x\sin(\pi x)$ over $[0,2]$

34) $y=5+|x|$ over $[-1,1]$

Answer

The Mean Value Theorem does not apply; not differentiable at $x=0$.

For exercises 35 - 37, consider the roots of each equation.

35) Show that the equation $y=x^3+3x^2+16$ has exactly one real root. What is it?

36) Find the conditions for exactly one root (double root) for the equation $y=x^2+bx+c$

Answer

$b=\pm 2\sqrt{c}$

37) Find the conditions for $y=e^x-b$ to have one root. Is it possible to have more than one root?

In exercises 38 - 42, use a calculator to graph the function over the interval $[a,b]$ and graph the secant line from a to b. Use the calculator to estimate all values of c as guaranteed by the Mean Value Theorem. Then, find the exact value of c, if possible, or write the final equation and use a calculator to estimate to four digits.

38) [T] $y=\tan(\pi x)$ over $\left[-\frac{1}{4},\frac{1}{4}\right]$

Answer

$c\approx\pm 0.1533$

$c=\pm\frac{1}{\pi}\cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right)$

39) [T] $y=\dfrac{1}{\sqrt{x+1}}$ over $[0,3]$

40) [T] $y=|x^2+2x-4|$ over $[-4,0]$

Answer

The Mean Value Theorem does not apply.

41) [T] $y=x+\dfrac{1}{x}$ over $\left[\frac{1}{2},4\right]$

42) [T] $y=\sqrt{x+1}+\dfrac{1}{x^2}$ over $[3,8]$

Answer

$\dfrac{1}{2\sqrt{c+1}}-\dfrac{2}{c^3}=\dfrac{521}{2880}$

$c\approx 3.133, 5.867$

43) At 10:17 a.m., you pass a police car at 55 mph that is stopped on the freeway. You pass a second police car at 55 mph at 10:53 a.m., which is located 39 mi from the first police car. If the speed limit is 60 mph, can the police cite you for speeding?

44) Two cars drive from one stoplight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

Answer

Yes

45) Show that $y = \sec^2 x$ and $y = \tan^2 x$ have the same derivative. What can you say about $y = \sec^2 x - \tan^2 x$?

46) Show that $y = \csc^2 x$ and $y = \cot^2 x$ have the same derivative. What can you say about $y = \csc^2 x - \cot^2 x$?

Answer

It is constant.
