2.6E: Exercises for Section 2.5

In exercises 1 - 4, write the appropriate $\varepsilon - \delta$ definition for each of the given statements.

1) $\lim\limits_{x \to a} f(x) = N$

2) $\lim\limits_{t \to b} g(t) = M$

Answer

For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |t - b| < \delta$, then $|g(t) - M| < \varepsilon$

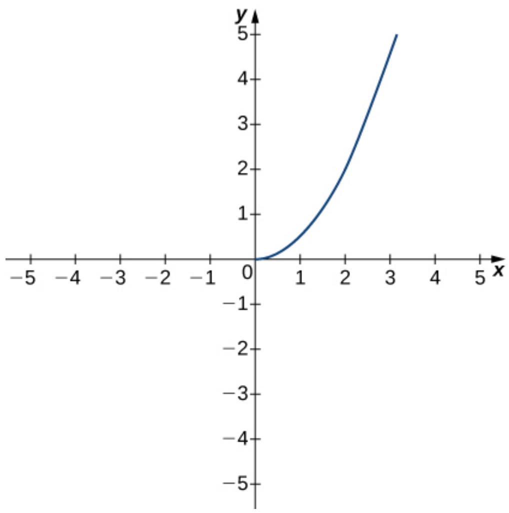
3) $\lim\limits_{x \to c} h(x) = L$

4) $\lim\limits_{x \to a} \varphi(x) = A$

Answer

For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |x - a| < \delta$, then $|\varphi(x) - A| < \varepsilon$

The following graph of the function f satisfies $\lim\limits_{x \to 2} f(x) = 2$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



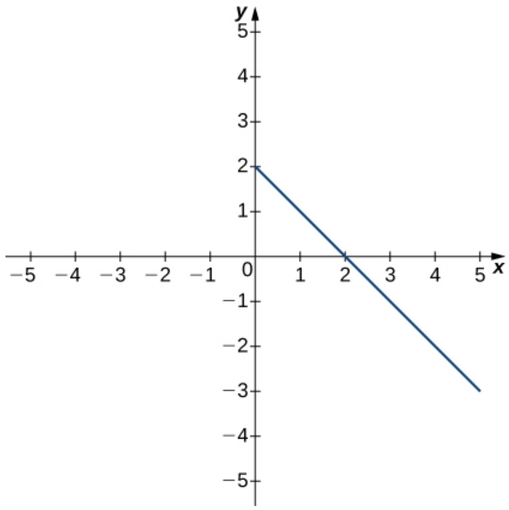
5) If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 1$.

6) If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 0.5$.

Answer

$\delta \leq 0.25$

The following graph of the function f satisfies $\lim\limits_{x \to 3} f(x) = -1$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



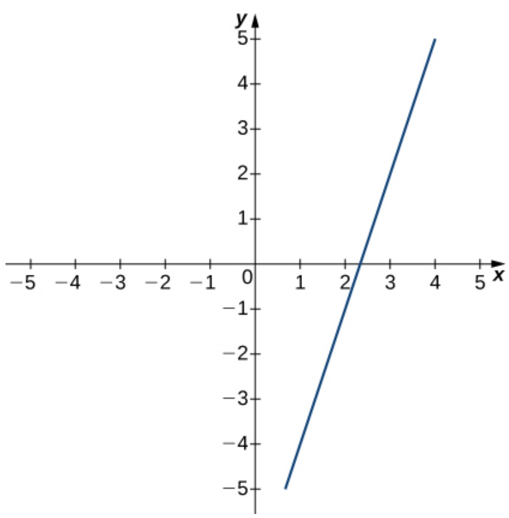
7) If $0 < |x-3| < \delta$, then $|f(x)+1| < 1$.

8) If $0 < |x-3| < \delta$, then $|f(x)+1| < 2$.

Answer

$\delta \leq 2$

The following graph of the function f satisfies $\lim_{x \to 3} f(x) = 2$. In the following exercises, for each value of ε, find a value of $\delta > 0$ such that the precise definition of limit holds true.



9) $\varepsilon = 1.5$

10) $\varepsilon = 3$

Answer

$\delta \leq 1$

[T] In exercises 11 - 12, use a graphing calculator to find a number δ such that the statements hold true.

11) $\left|\sin(2x) - \frac{1}{2}\right| < 0.1$, whenever $\left|x - \frac{\pi}{12}\right| < \delta$

12) $\left|\sqrt{x-4} - 2\right| < 0.1$, whenever $|x-8| < \delta$

Answer

$\delta < 0.3900$

In exercises 13 - 17, use the precise definition of limit to prove the given limits.

13) $\lim_{x\to 2}(5x+8)=18$

14) $\lim_{x\to 3}\frac{x^2-9}{x-3}=6$

Answer

Let $\delta=\varepsilon$. If $0<|x-3|<\varepsilon$, then $\left|\frac{x^2-9}{x-3}-6\right|=\left|\frac{(x+3)(x-3)}{x-3}-6\right|=|x+3-6|=|x-3|<\varepsilon$.

15) $\lim_{x\to 2}\frac{2x^2-3x-2}{x-2}=5$

16) $\lim_{x\to 0}x^4=0$

Answer

Let $\delta=\sqrt[4]{\varepsilon}$. If $0<|x|<\sqrt[4]{\varepsilon}$, then $|x^4-0|=x^4<\varepsilon$.

17) $\lim_{x\to 2}(x^2+2x)=8$

In exercises 18 - 20, use the precise definition of limit to prove the given one-sided limits.

18) $\lim_{x\to 5^-}\sqrt{5-x}=0$

Answer

Let $\delta=\varepsilon^2$. If $-\varepsilon^2<x-5<0,$ we can multiply through by -1 to get $0<5-x<\varepsilon^2$.
Then $|\sqrt{5-x}-0|=\sqrt{5-x}<\sqrt{\varepsilon^2}=\varepsilon$.

19) $\lim_{x\to 0^+}f(x)=-2$, where $f(x)=\begin{cases}8x-3, & \text{if } x<0\\ 4x-2, & \text{if } x\geq 0\end{cases}$.

20) $\lim_{x\to 1^-}f(x)=3$, where $f(x)=\begin{cases}5x-2, & \text{if } x<1\\ 7x-1, & \text{if } x\geq 1\end{cases}$.

Answer

Let $\delta=\varepsilon/5$. If $-\varepsilon/5<x-1<0,$ we can multiply through by -1 to get $0<1-x<\varepsilon/5$.
Then $|f(x)-3|=|5x-2-3|=|5x-5|=5(1-x),$ since $x<1$ here.
And $5(1-x)<5(\varepsilon/5)=\varepsilon$.

In exercises 21 - 23, use the precise definition of limit to prove the given infinite limits.

21) $\lim_{x\to 0}\frac{1}{x^2}=\infty$

22) $\lim_{x\to -1}\frac{3}{(x+1)^2}=\infty$

Answer

Let $\delta=\sqrt{\frac{3}{N}}$. If $0<|x+1|<\sqrt{\frac{3}{N}}$, then $f(x)=\frac{3}{(x+1)^2}>N$.

23) $\lim_{x\to 2}-\frac{1}{(x-2)^2}=-\infty$

24) An engineer is using a machine to cut a flat square of Aerogel of area $144\,\text{cm}^2$. If there is a maximum error tolerance in the area of $8\,\text{cm}^2$, how accurately must the engineer cut on the side, assuming all sides have the same length? How do these numbers relate to δ, ε, a, and L?

Answer

0.033 cm, $\varepsilon = 8$, $\delta = 0.33$, $a = 12$, $L = 144$

25) Use the precise definition of limit to prove that the following limit does not exist: $\lim\limits_{x \to 1} \frac{|x-1|}{x-1}$.

26) Using precise definitions of limits, prove that $\lim\limits_{x \to 0} f(x)$ does not exist, given that $f(x)$ is the ceiling function. (Hint: Try any $\delta < 1$.)

Answer

Answers may very.

27) Using precise definitions of limits, prove that $\lim\limits_{x \to 0} f(x)$ does not exist: $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$. (Hint: Think about how you can always choose a rational number $0 < d,>$

28) Using precise definitions of limits, determine $\lim\limits_{x \to 0} f(x)$ for $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$. (Hint: Break into two cases, x rational and x irrational.)

Answer

0

29) Using the function from the previous exercise, use the precise definition of limits to show that $\lim\limits_{x \to a} f(x)$ does not exist for $a \neq 0$

For exercises 30 - 32, suppose that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$ both exist. Use the precise definition of limits to prove the following limit laws:

30) $\lim\limits_{x \to a} (f(x) - g(x)) = L - M$

Answer

$f(x) - g(x) = f(x) + (-1)g(x)$

31) $\lim\limits_{x \to a} [cf(x)] = cL$ for any real constant c (Hint: Consider two cases: $c = 0$ and $c \neq 0$.)

32) $\lim\limits_{x \to a} [f(x)g(x)] = LM$. (Hint: $|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L|$.)

Answer

Answers may vary.
