4.3: Conditional Probability and Independence

Conditional Probability

What is the probability of an event A "given that" we have some partial information about the outcome of the experiment? This is called *conditional probability*.

Definition: Conditional Probability

The conditional probability of the outcome of interest A given condition B is computed as the following:

$$P(A|B) = \frac{P(AandB)}{P(B)} \tag{4.3.1}$$

Independent Events

Two events are independent if the following are true:

- $P(\text{A}|\text{B}) = P(\text{A})$
- $P(\text{B}|\text{A}) = P(\text{B})$
- $P(\text{A AND B}) = P(\text{A})P(\text{B})$

Two events A and B are independent if the knowledge that one occurred does not affect the chance the other occurs. For example, the outcomes of two roles of a fair die are independent events. The outcome of the first roll does not change the probability for the outcome of the second roll. To show two events are independent, you must show only one of the above conditions. If two events are NOT independent, then we say that they are dependent.

Sampling a population

Sampling may be done with replacement or without replacement (Figure $4.3.1$):

- **With replacement:** If each member of a population is replaced after it is picked, then that member has the possibility of being chosen more than once. When sampling is done with replacement, then events are considered to be *independent*, meaning the result of the first pick will not change the probabilities for the second pick.
- **Without replacement:** When sampling is done without replacement, each member of a population may be chosen only once. In this case, the probabilities for the second pick are affected by the result of the first pick. The events are considered to be *dependent* or *not independent*.

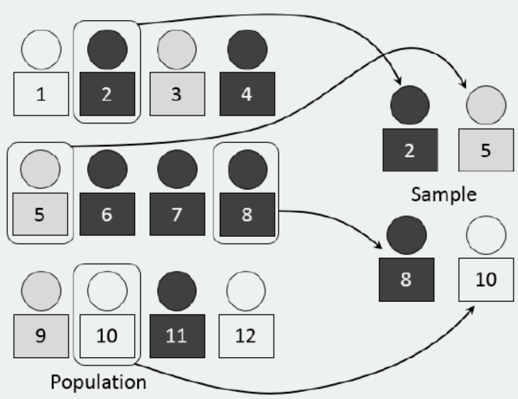

Figure $4.3.1$: A visual representation of the sampling process. If the sample items are replaced after each sampling event, then this is "sampling with replacement" if not, then it is "sampling without replacement". (CC BY-SA 4.0; Dan Kernler).

If it is not known whether A and B are independent or dependent, assume they are dependent until you can show otherwise.

Example 4.3.1: Sampling with and without replacement

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king) of that suit.

a. Sampling with replacement:

Suppose you pick three cards with replacement. The first card you pick out of the 52 cards is the Q of spades. You put this card back, reshuffle the cards and pick a second card from the 52-card deck. It is the ten of clubs. You put this card back, reshuffle the cards and pick a third card from the 52-card deck. This time, the card is the Q of spades again. Your picks are $\{\mathrm{Q}$ of spades, ten of clubs, Q of spades$\}$. You have picked the Q of spades twice. You pick each card from the 52-card deck.

b. Sampling without replacement:

Suppose you pick three cards without replacement. The first card you pick out of the 52 cards is the K of hearts. You put this card aside and pick the second card from the 51 cards remaining in the deck. It is the three of diamonds. You put this card aside and pick the third card from the remaining 50 cards in the deck. The third card is the J of spades. Your picks are $\{\mathrm{K}$ of hearts, three of diamonds, J of spades$\}$. Because you have picked the cards without replacement, you cannot pick the same card twice.

Example 4.3.2

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts, and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), and K (king) of that suit. $\mathrm{S} =$ spades, $\mathrm{H} =$ Hearts, $\mathrm{D} =$ Diamonds, $\mathrm{C} =$ Clubs.

a. Suppose you pick four cards, but do not put any cards back into the deck. Your cards are $\mathrm{QS}, 1\mathrm{D}, 1\mathrm{C}, \mathrm{QD}$.
b. Suppose you pick four cards and put each card back before you pick the next card. Your cards are $\mathrm{KH}, 7\mathrm{D}, 6\mathrm{D}, \mathrm{KH}$.

Which of a. or b. did you sample with replacement and which did you sample without replacement?

Answer a

Without replacement

Answer b

With replacement

Mutually Exclusive Events

A and B are mutually exclusive events if they **cannot** occur at the same time. This means that A and B do not share any outcomes and $P(\mathrm{A\ AND\ B}) = 0$.

For example, suppose the sample space

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Let $\mathrm{A} = \{1, 2, 3, 4, 5\}, \mathrm{B} = \{4, 5, 6, 7, 8\}$ and $\mathrm{C} = \{7, 9\}$. $\mathrm{A\ AND\ B} = \{4, 5\}$.

$$P(\mathrm{A\ AND\ B}) = \frac{2}{10}$$

and is not equal to zero. Therefore, A and B are not mutually exclusive. A and C do not have any numbers in common so $P(\mathrm{A\ AND\ C}) = 0$. Therefore, A and C are mutually exclusive.

If it is not known whether A and B are mutually exclusive, **assume they are not until you can show otherwise**. The following examples illustrate these definitions and terms.

Example 4.3.3

Flip two fair coins.

The sample space is $\{HH, HT, TH, TT\}$ where $T =$ tails and $H =$ heads. The outcomes are $HH, HT, TH,$ and TT. The outcomes HT and TH are different. The HT means that the first coin showed heads and the second coin showed tails. The

TH means that the first coin showed tails and the second coin showed heads.

- Let $\text{A} =$ the event of getting **at most one tail**. (At most one tail means zero or one tail.) Then A can be written as $\{HH, HT, TH\}$. The outcome HH shows zero tails. HT and TH each show one tail.
- Let $\text{B} =$ the event of getting all tails. B can be written as $\{TT\}$. B is the **complement** of A, so $\text{B} = \text{A}'$. Also, $P(\text{A}) + P(\text{B}) = P(\text{A}) + P(\text{A}') = 1$.
- The probabilities for A and for B are $P(\text{A}) = \frac{3}{4}$ and $P(\text{B}) = \frac{1}{4}$.
- Let $\text{C} =$ the event of getting all heads. $\text{C} = \{HH\}$. Since $\text{B} = \{TT\}$, $P(\text{B AND C}) = 0$. B and Care mutually exclusive. B and C have no members in common because you cannot have all tails and all heads at the same time.)
- Let $\text{D} =$ event of getting **more than one** tail. $\text{D} = \{TT\}$. $P(\text{D}) = \frac{1}{4}$
- Let $\text{E} =$ event of getting a head on the first roll. (This implies you can get either a head or tail on the second roll.) $\text{E} = \{HT, HH\}$. $P(\text{E}) = \frac{2}{4}$
- Find the probability of getting **at least one** (one or two) tail in two flips. Let $\text{F} =$ event of getting at least one tail in two flips. $\text{F} = \{HT, TH, TT\}$. $P(\text{F}) = \frac{3}{4}$

Example $4.3.4$

Flip two fair coins. Find the probabilities of the events.

a. Let $\text{F} =$ the event of getting at most one tail (zero or one tail).
b. Let $\text{G} =$ the event of getting two faces that are the same.
c. Let $\text{H} =$ the event of getting a head on the first flip followed by a head or tail on the second flip.
d. Are F and G mutually exclusive?
e. Let $\text{J} =$ the event of getting all tails. Are J and H mutually exclusive?

Solution

Look at the sample space in Example $4.3.3$.

1. Zero (0) or one (1) tails occur when the outcomes HH, TH, HT show up. $P(\text{F}) = \frac{3}{4}$
2. Two faces are the same if HH or TT show up. $P(\text{G}) = \frac{2}{4}$
3. A head on the first flip followed by a head or tail on the second flip occurs when HH or HT show up. $P(\text{H}) = \frac{2}{4}$
4. F and G share HH so $P(\text{F AND G})$ is not equal to zero (0). F and G are not mutually exclusive.
5. Getting all tails occurs when tails shows up on both coins (TT). H's outcomes are HH and HT.

J and H have nothing in common so $P(\text{J AND H}) = 0$. J and H are mutually exclusive.

Example $4.3.5$

Roll one fair, six-sided die. The sample space is {1, 2, 3, 4, 5, 6}. Let event $\text{A} =$ a face is odd. Then $\text{A} = \{1, 3, 5\}$. Let event $\text{B} =$ a face is even. Then $\text{B} = \{2, 4, 6\}$.

- Find the complement of A, A'. The complement of A, A', is B because A and B together make up the sample space. $P(\text{A}) + P(\text{B}) = P(\text{A}) + P(\text{A}') = 1$. Also, $P(\text{A}) = \frac{3}{6}$ and $P(\text{B}) = \frac{3}{6}$.
- Let event $\text{C} =$ odd faces larger than two. Then $\text{C} = \{3, 5\}$. Let event $\text{D} =$ all even faces smaller than five. Then $\text{D} = \{2, 4\}$. $P(\text{C AND D}) = 0$ because you cannot have an odd and even face at the same time. Therefore, C and D are mutually exclusive events.
- Let event $\text{E} =$ all faces less than five. $\text{E} = \{1, 2, 3, 4\}$.

Are C and E mutually exclusive events? (Answer yes or no.) Why or why not?

Answer

No. $\mathrm{C}=\{3,5\}$ and $\mathrm{E}=\{1,2,3,4\}$. $P(\mathrm{C\ AND\ E})=\frac{1}{6}$. To be mutually exclusive, $P(\mathrm{C\ AND\ E})$ must be zero.

- Find $P(\mathrm{C}|\mathrm{A})$. This is a conditional probability. Recall that the event C is {3, 5} and event A is {1, 3, 5}. To find $P(\mathrm{C}|\mathrm{A})$, find the probability of C using the sample space A. You have reduced the sample space from the original sample space {1, 2, 3, 4, 5, 6} to {1, 3, 5}. So, $P(\mathrm{C}|\mathrm{A})=\frac{2}{3}$.

Example 4.3.6

Let event $\mathrm{G}=$ taking a math class. Let event $\mathrm{H}=$ taking a science class. Then, $\mathrm{G\ AND\ H}=$ taking a math class and a science class. Suppose $P(\mathrm{G})=0.6$, $P(\mathrm{H})=0.5$, and $P(\mathrm{G\ AND\ H})=0.3$. Are G and H independent?

If G and H are independent, then you must show **ONE** of the following:

- $P(\mathrm{G}|\mathrm{H})=P(\mathrm{G})$
- $P(\mathrm{H}|\mathrm{G})=P(\mathrm{H})$
- $P(\mathrm{G\ AND\ H})=P(\mathrm{G})P(\mathrm{H})$

The choice you make depends on the information you have. You could choose any of the methods here because you have the necessary information.

a. a. Show that $P(\mathrm{G}|\mathrm{H})=P(\mathrm{G})$.
b. b. Show $P(\mathrm{G\ AND\ H})=P(\mathrm{G})P(\mathrm{H})$.

Solution

a. $P(\mathrm{G}|\mathrm{H})=\frac{P(\mathrm{G\ AND\ H})}{P(\mathrm{H})}=\frac{0.3}{0.5}=0.6=P(\mathrm{G})$
b. $P(\mathrm{G})P(\mathrm{H})=(0.6)(0.5)=0.3=P(\mathrm{G\ AND\ H})$

Since G and H are independent, knowing that a person is taking a science class does not change the chance that he or she is taking a math class. If the two events had not been independent (that is, they are dependent) then knowing that a person is taking a science class would change the chance he or she is taking math. For practice, show that $P(\mathrm{H}|\mathrm{G})=P(\mathrm{H})$ to show that G and H are independent events.

Example 4.3.7

Let event $\mathrm{C}=$ taking an English class. Let event $\mathrm{D}=$ taking a speech class.

Suppose $P(\mathrm{C})=0.75$, $P(\mathrm{D})=0.3$, $P(\mathrm{C}|\mathrm{D})=0.75$ and $P(\mathrm{C\ AND\ D})=0.225$.

Justify your answers to the following questions numerically.

a. Are C and D independent?
b. Are C and D mutually exclusive?
c. What is $P(\mathrm{D}|\mathrm{C})$?

Solution

a. Yes, because $P(\mathrm{C}|\mathrm{D})=P(\mathrm{C})$.
b. No, because $P(\mathrm{C\ AND\ D})$ is not equal to zero.
c. $P(\mathrm{D}|\mathrm{C})=\frac{P(\mathrm{C\ AND\ D})}{P(\mathrm{C})}=\frac{0.225}{0.75}=0.3$

Example 4.3.8

In a box there are three red cards and five blue cards. The red cards are marked with the numbers 1, 2, and 3, and the blue cards are marked with the numbers 1, 2, 3, 4, and 5. The cards are well-shuffled. You reach into the box (you cannot see into it) and draw one card.

Let

- $\text{R} =$ red card is drawn,
- $\text{B} =$ blue card is drawn,
- $\text{E} =$ even-numbered card is drawn.

The sample space $S = R1, R2, R3, B1, B2, B3, B4, B5$.

S has eight outcomes.

- $P(\text{R}) = \frac{3}{8}$. $P(\text{B}) = \frac{5}{8}$. $P(\text{R AND B}) = 0$. (You cannot draw one card that is both red and blue.)
- $P(\text{E}) = \frac{3}{8}$. (There are three even-numbered cards, $R2$, $B2$, and $B4$.)
- $P(\text{E}|\text{B}) = \frac{2}{5}$. (There are five blue cards: $B1$, $B2$, $B3$, $B4$, and $B5$. Out of the blue cards, there are two even cards; $B2$ and $B4$.)
- $P(\text{B}|\text{E}) = \frac{2}{3}$. (There are three even-numbered cards: $R2$, $B2$, and $B4$. Out of the even-numbered cards, to are blue; $B2$ and $B4$.)
- The events R and B are mutually exclusive because $P(\text{R AND B}) = 0$.
- Let $\text{G} =$ card with a number greater than 3. $\text{G} = \{B4, B5\}$. $P(\text{G}) = \frac{2}{8}$. Let $\text{H} =$ blue card numbered between one and four, inclusive. $\text{H} = \{B1, B2, B3, B4\}$. $P(\text{G}|\text{H}) = \frac{1}{4}$. (The only card in H that has a number greater than three is B4.) Since $\frac{2}{8} = \frac{1}{4}$, $P(\text{G}) = P(\text{G}|\text{H})$, which means that G and H are independent.

Example 4.3.9

In a particular college class, 60% of the students are female. Fifty percent of all students in the class have long hair. Forty-five percent of the students are female and have long hair. Of the female students, 75% have long hair. Let F be the event that a student is female. Let L be the event that a student has long hair. One student is picked randomly. Are the events of being female and having long hair independent?

- The following probabilities are given in this example:
- $P(\text{F}) = 0.60$; $P(\text{L}) = 0.50$
- $P(\text{F AND L}) = 0.45$
- $P(\text{L}|\text{F}) = 0.75$

The choice you make depends on the information you have. You could use the first or last condition on the list for this example. You do not know $P(\text{F}|\text{L})$ yet, so you cannot use the second condition.

Solution 1

Check whether $P(\text{F AND L}) = P(\text{F})P(\text{L})$. We are given that $P(\text{F AND L}) = 0.45$, but $P(\text{F})P(\text{L}) = (0.60)(0.50) = 0.30$. The events of being female and having long hair are not independent because $P(\text{F AND L})$ does not equal $P(\text{F})P(\text{L})$.

Solution 2

Check whether $P(\text{L}|\text{F})$ equals $P(\text{L})$. We are given that $P(\text{L}|\text{F}) = 0.75$, but $P(\text{L}) = 0.50$; they are not equal. The events of being female and having long hair are not independent.

Interpretation of Results

The events of being female and having long hair are not independent; knowing that a student is female changes the probability that a student has long hair.

Example 4.3.10

a. Toss one fair coin (the coin has two sides, H and T). The outcomes are ________. Count the outcomes. There are ____ outcomes.

b. Toss one fair, six-sided die (the die has 1, 2, 3, 4, 5 or 6 dots on a side). The outcomes are ________________. Count the outcomes. There are ___ outcomes.
c. Multiply the two numbers of outcomes. The answer is _______.
d. If you flip one fair coin and follow it with the toss of one fair, six-sided die, the answer in three is the number of outcomes (size of the sample space). What are the outcomes? (Hint: Two of the outcomes are $H1$ and $T6$.)
e. Event $\text{A} =$ heads (H) on the coin followed by an even number (2, 4, 6) on the die. A = {_________________}. Find $P(\text{A})$.
f. Event $\text{B} =$ heads on the coin followed by a three on the die. $\text{B} =$ {________}. Find $P(\text{B})$.
g. Are A and B mutually exclusive? (Hint: What is $P(\text{A AND B})$? If $P(\text{A AND B}) = 0$, then A and B are mutually exclusive.)
h. Are A and B independent? (Hint: Is $P(\text{A AND B}) = P(\text{A})P(\text{B})$? If $P(\text{A AND B}) = P(\text{A})P(\text{B})$, then A and B are independent. If not, then they are dependent).

Solution

a. H and T; 2
b. 1, 2, 3, 4, 5, 6; 6
c. 2(6) = 12
d. $T1, T2, T3, T4, T5, T6, H1, H2, H3, H4, H5, H6$
e. $\text{A} = \{H2, H4, H6\}$; $P(\text{A}) = \frac{3}{12}$
f. $\text{B} = \{H3\}$; $P(\text{B}) = \frac{1}{12}$
g. Yes, because $P(\text{A AND B}) = 0$
h. $P(\text{A AND B}) = 0$. $P(\text{A})P(\text{B}) = \left(\frac{3}{12}\right)\left(\frac{1}{12}\right)$. $P(\text{A AND B})$ does not equal $P(\text{A})P(\text{B})$, so A and B are dependent.

WeBWorK Problems

References

1. Lopez, Shane, Preety Sidhu. “U.S. Teachers Love Their Lives, but Struggle in the Workplace.” Gallup Wellbeing, 2013. http://www.gallup.com/poll/161516/te...workplace.aspx (accessed May 2, 2013).
2. Data from Gallup. Available online at www.gallup.com/ (accessed May 2, 2013).

Review

Two events A and B are independent if the knowledge that one occurred does not affect the chance the other occurs. If two events are not independent, then we say that they are dependent.

In sampling with replacement, each member of a population is replaced after it is picked, so that member has the possibility of being chosen more than once, and the events are considered to be independent. In sampling without replacement, each member of a population may be chosen only once, and the events are considered not to be independent. When events do not share outcomes, they are mutually exclusive of each other.

Formula Review

- If A and B are independent, $P(\text{A AND B}) = P(\text{A})P(\text{B}), P(\text{A}|\text{B}) = P(\text{A})$ and $P(\text{B}|\text{A}) = P(\text{B})$.
- If A and B are mutually exclusive, $P(\text{A OR B}) = P(\text{A}) + P(\text{B}) and P(\text{A AND B}) = 0$.

Contributors and Attributions

Exercise 4.3.11

E and F are mutually exclusive events. $P(\text{E})=0.4$; $P(\text{F})=0.5$. Find $P(\text{E}|\text{F})$.

Exercise 4.3.12

J and K are independent events. $P(\text{J}|\text{K})=0.3$. Find $P(\text{J})$.

Answer

$P(\text{J})=0.3$

Exercise 4.3.13

U and V are mutually exclusive events. $P(\text{U})=0.26$; $P(\text{V})=0.37$. Find:

a. $P(\text{U AND V})=$
b. $P(\text{U}|\text{V})=$
c. $P(\text{U OR V})=$

Exercise 4.3.14

Q and R are independent events. $P(\text{Q})=0.4$ and $P(\text{Q AND R})=0.1$. Find $P(\text{R})$.

Answer

$P(\text{Q AND R})=P(\text{Q})P(\text{R})$

$0.1=(0.4)P(\text{R})$

$P(\text{R})=0.25$

Bringing It Together

Exercise 4.3.16

A previous year, the weights of the members of the **San Francisco 49ers** and the **Dallas Cowboys** were published in the *San Jose Mercury News*. The factual data are compiled into Table.

Shirt#	≤ 210	211–250	251–290	290≤
1–33	21	5	0	0
34–66	6	18	7	4
66–99	6	12	22	5

For the following, suppose that you randomly select one player from the 49ers or Cowboys.

If having a shirt number from one to 33 and weighing at most 210 pounds were independent events, then what should be true about $P(\text{Shirt}\#1\text{–}33|\leq 210 \text{ pounds})$?

Exercise 4.3.17

The probability that a male develops some form of cancer in his lifetime is 0.4567. The probability that a male has at least one false positive test result (meaning the test comes back for cancer when the man does not have it) is 0.51. Some of the following questions do not have enough information for you to answer them. Write “not enough information” for those answers. Let $\text{C}=$ a man develops cancer in his lifetime and $\text{P}=$ man has at least one false positive.

a. $P(\text{C})=$ ______
b. $P(\text{P}|\text{C})=$ ______

c. $P(\mathrm{P} \mid \mathrm{C'}) =$ ______
d. If a test comes up positive, based upon numerical values, can you assume that man has cancer? Justify numerically and explain why or why not.

Answer

a. $P(\mathrm{C}) = 0.4567$
b. not enough information
c. not enough information
d. No, because over half (0.51) of men have at least one false positive text

Exercise 4.3.18

Given events G and H : $P(\mathrm{G}) = 0.43$; $P(\mathrm{H}) = 0.26$; $P(\mathrm{H\ AND\ G}) = 0.14$

a. Find $P(\mathrm{H\ OR\ G})$.
b. Find the probability of the complement of event ($\mathrm{H\ AND\ G}$).
c. Find the probability of the complement of event ($\mathrm{H\ OR\ G}$).

Exercise 4.3.19

Given events J and K : $P(\mathrm{J}) = 0.18$; $P(\mathrm{K}) = 0.37$; $P(\mathrm{J\ OR\ K}) = 0.45$

a. Find $P(\mathrm{J\ AND\ K})$.
b. Find the probability of the complement of event ($\mathrm{J\ AND\ K}$).
c. Find the probability of the complement of event ($\mathrm{J\ AND\ K}$).

Answer

a. $P(\mathrm{J\ OR\ K}) = P(\mathrm{J}) + P(\mathrm{K}) - P(\mathrm{J\ AND\ K}); 0.45 = 0.18 + 0.37 - P(\mathrm{J\ AND\ K})$; solve to find $P(\mathrm{J\ AND\ K}) = 0.10$
b. $P(\mathrm{NOT\ (J\ AND\ K)}) = 1 - P(\mathrm{J\ AND\ K}) = 1 - 0.10 = 0.90$
c. $P(\mathrm{NOT\ (J\ OR\ K)}) = 1 - P(\mathrm{J\ OR\ K}) = 1 - 0.45 = 0.55$

WeBWorK Problems

Glossary

Dependent Events

If two events are NOT independent, then we say that they are dependent.

Sampling with Replacement

If each member of a population is replaced after it is picked, then that member has the possibility of being chosen more than once.

Sampling without Replacement

When sampling is done without replacement, each member of a population may be chosen only once.

The Conditional Probability of One Event Given Another Event

$P(A|B)$ is the probability that event A will occur given that the event B has already occurred.

The OR of Two Events

An outcome is in the event A OR B if the outcome is in A, is in B, or is in both A and B.

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