8.2: The t-distribution

In practice, we rarely know the population standard deviation. In the past, when the sample size was large, this did not present a problem to statisticians. They used the sample standard deviation s as an estimate for σ and proceeded as before to calculate a confidence interval with close enough results. However, statisticians ran into problems when the sample size was small. A small sample size caused inaccuracies in the confidence interval.

William S. Goset (1876–1937) of the Guinness brewery in Dublin, Ireland ran into this problem. His experiments with hops and barley produced very few samples. Just replacing σ with s did not produce accurate results when he tried to calculate a confidence interval. He realized that he could not use a normal distribution for the calculation; he found that the actual distribution depends on the sample size. This problem led him to "discover" what is called the Student's t-distribution. The name comes from the fact that Gosset wrote under the pen name "Student."

Up until the mid-1970s, some statisticians used the normal distribution approximation for large sample sizes and only used the Student's t-distribution only for sample sizes of at most 30. With graphing calculators and computers, the practice now is to use the Student's t-distribution whenever s is used as an estimate for σ. If you draw a simple random sample of size n from a population that has an approximately a normal distribution with mean μ and unknown population standard deviation σ and calculate the t-score then the t-scores follow a Student's t-distribution with $n-1$ degrees of freedom. The t-score has the same interpretation as the z-score. It measures how far $\bar{x}$ is from its mean μ. For each sample size n, there is a different Student's t-distribution.

The degrees of freedom, $n-1$, come from the calculation of the sample standard deviation s. Previously, we used n deviations ($x-\bar{x}$ values) to calculate s. Because the sum of the deviations is zero, we can find the last deviation once we know the other $n-1$ deviations. The other $n-1$ deviations can change or vary freely. We call the number $n-1$ the degrees of freedom (df).

> *For each sample size n, there is a different Student's t-distribution.*

- The graph for the Student's t-distribution is similar to the standard normal curve.
- The mean for the Student's t-distribution is zero and the distribution is symmetric about zero.
- The Student's t-distribution has more probability in its tails than the standard normal distribution because the spread of the t-distribution is greater than the spread of the standard normal. So the graph of the Student's t-distribution will be thicker in the tails and shorter in the center than the graph of the standard normal distribution.
- The exact shape of the Student's t-distribution depends on the degrees of freedom. As the degrees of freedom increases, the graph of Student's t-distribution becomes more like the graph of the standard normal distribution.
- The underlying population of individual observations is assumed to be normally distributed with unknown population mean μ and unknown population standard deviation σ. The size of the underlying population is generally not relevant unless it is very small. If it is bell shaped (normal) then the assumption is met and doesn't need discussion. Random sampling is assumed, but that is a completely separate assumption from normality.

Calculators and computers can easily calculate any Student's t-probabilities. However for confidence intervals, we need to use **inverse** probability to find the value of *t* when we know the probability.

A probability table for the Student's t-distribution can also be used. The table gives t-scores that correspond to the confidence level (column) and degrees of freedom (row).

A Student's t-table gives t-scores given the degrees of freedom and the right-tailed probability. The table is very limited. **Calculators and computers can easily calculate any Student's t-probabilities.**

The notation for the Student's t-distribution (using *T* as the random variable) is:

- $T \sim t_{df}$ where $df = n-1$.
- For example, if we have a sample of size $n = 20$ items, then we calculate the degrees of freedom as $df = n - 1 = 20 - 1 = 19$ and we write the distribution as $T \sim t_{19}$.

If the population standard deviation is not known, the error bound for a population mean is:

- $EBM = \left(t_{\frac{\alpha}{2}}\right)\left(\frac{s}{\sqrt{n}}\right)$,
- $t_{\frac{\alpha}{2}}$ is the t-score with area to the right equal to $\frac{\alpha}{2}$,

- use $df = n - 1$ degrees of freedom, and
- $s =$ sample standard deviation.

Suppose you do a study of acupuncture to determine how effective it is in relieving pain. You measure sensory rates for 15 subjects with the results given. Use the sample data to construct a 95% confidence interval for the mean sensory rate for the population (assumed normal) from which you took the data.

The solution is shown step-by-step and by using the TI-83, 83+, or 84+ calculators.

8.6; 9.4; 7.9; 6.8; 8.3; 7.3; 9.2; 9.6; 8.7; 11.4; 10.3; 5.4; 8.1; 5.5; 6.9

Answer

- The first solution is step-by-step (Solution A).

Solution

To find the confidence interval, you need the sample mean, $\bar{x}$, and the EBM.

$$\bar{x} = 8.2267$$

$$s = 1.6722 \; n = 15$$

$$df = 15 - 1 = 14 CL so \alpha = 1 - CL = 1 - 0.95 = 0.05$$

$$\frac{\alpha}{2} = 0.025 t_{\frac{\alpha}{2}} = t_{0.025}$$

The area to the right of $t_{0.025}$ is 0.025, and the area to the left of $t_{0.025}$ is 1 – 0.025 = 0.975

$t_{\frac{\alpha}{2}} = t_{0.025} = 2.14$ using invT(.975,14) on the TI-84+ calculator.

$$EBM = \left(t_{\frac{\alpha}{2}}\right)\left(\frac{s}{\sqrt{n}}\right)$$

$$= (2.14)\left(\frac{1.6722}{\sqrt{15}}\right) = 0.924$$

Now it is just a direct application of Equation ???:

$$\bar{x} - EBM = 8.2267 - 0.9240 = 7.3$$

$$\bar{x} + EBM = 8.2267 + 0.9240 = 9.15$$

The 95% confidence interval is (7.30, 9.15).

We estimate with 95% confidence that the true population mean sensory rate is between 7.30 and 9.15.

When calculating the error bound, a probability table for the Student's t-distribution can also be used to find the value of t. The table gives t-scores that correspond to the confidence level (column) and degrees of freedom (row); the t-score is found where the row and column intersect in the table.

Example 8.2.2: The Human Toxome Project

The Human Toxome Project (HTP) is working to understand the scope of industrial pollution in the human body. Industrial chemicals may enter the body through pollution or as ingredients in consumer products. In October 2008, the scientists at HTP tested cord blood samples for 20 newborn infants in the United States. The cord blood of the "In utero/newborn" group was tested for 430 industrial compounds, pollutants, and other chemicals, including chemicals linked to brain and nervous system toxicity, immune system toxicity, and reproductive toxicity, and fertility problems. There are health concerns about the effects of some chemicals on the brain and nervous system. Table 8.2.1 shows how many of the targeted chemicals were found in each infant's cord blood.

Table 8.2.1

79	145	147	160	116	100	159	151	156	126

137	83	156	94	121	144	123	114	139	99

Use this sample data to construct a 90% confidence interval for the mean number of targeted industrial chemicals to be found in an in infant's blood.

Solution

From the sample, you can calculate $\bar{x} = 127.45$ and $s = 25.965$. There are 20 infants in the sample, so $n = 20$, and $df = 20 - 1 = 19$.

You are asked to calculate a 90% confidence interval: $CL = 0.90$, so

$$\alpha = 1 - CL = 1 - 0.90 = 0.10 \frac{\alpha}{2} = 0.05, t_{\frac{\alpha}{2}} = t_{0.05} \tag{8.2.1}$$

By definition, the area to the right of $t_{0.05}$ is 0.05 and so the area to the left of $t_{0.05}$ is $1 - 0.05 = 0.95$.

Use a table, calculator, or computer to find that $t_{0.05} = 1.729$.

$$EBM = t_{\frac{\alpha}{2}} \left(\frac{s}{\sqrt{n}} \right) = 1.729 \left(\frac{25.965}{\sqrt{20}} \right) \approx 10.038$$

$$\bar{x} - EBM = 127.45 - 10.038 = 117.412$$

$$\bar{x} + EBM = 127.45 + 10.038 = 137.488$$

We estimate with 90% confidence that the mean number of all targeted industrial chemicals found in cord blood in the United States is between 117.412 and 137.488.

Example 8.2.3

A random sample of statistics students were asked to estimate the total number of hours they spend watching television in an average week. The responses are recorded in Table 8.2.2. Use this sample data to construct a 98% confidence interval for the mean number of hours statistics students will spend watching television in one week.

Table 8.2.2

0	3	1	20	9
5	10	1	10	4
14	2	4	4	5

Solution A

- $\bar{x} = 6.133$,
- $s = 5.514$,
- $n = 15$, and
- $df = 15 - 1 = 14$.

$\frac{\alpha}{2} = 0.01 t_{\frac{\alpha}{2}} = t_{0.01} 2.624$

$\bar{x} - EBM = 6.133 - 3.736 = 2.397$

$$\bar{x} + EBM = 6.133 + 3.736 = 9.869$$

We estimate with 98% confidence that the mean number of all hours that statistics students spend watching television in one week is between 2.397 and 9.869.

WeBWorK Problems

Reference

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Glossary

Degrees of Freedom (df)

the number of objects in a sample that are free to vary

Normal Distribution

a continuous random variable (RV) with pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$, where μ is the mean of the distribution and σ is the standard deviation, notation: $X \sim N(\mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, the RV is called **the standard normal distribution**.

Standard Deviation

a number that is equal to the square root of the variance and measures how far data values are from their mean; notation: s for sample standard deviation and σ for population standard deviation

Student's t-Distribution

investigated and reported by William S. Gossett in 1908 and published under the pseudonym Student; the major characteristics of the random variable (RV) are:

- It is continuous and assumes any real values.
- The pdf is symmetrical about its mean of zero. However, it is more spread out and flatter at the apex than the normal distribution.
- It approaches the standard normal distribution as *n* get larger.
- There is a "family" of t–distributions: each representative of the family is completely defined by the number of degrees of freedom, which is one less than the number of data.

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