6.2: The Uniform and Other Simple Continuous Distributions

The uniform distribution is a continuous probability distribution and is concerned with events that are equally likely to occur. When working out problems that have a uniform distribution, be careful to note if the data is inclusive or exclusive.

Example 6.2.1

The data in Table $6.2.1$ are 55 smiling times, in seconds, of an eight-week-old baby.

Table 6.2.1

10.4	19.6	18.8	13.9	17.8	16.8	21.6	17.9	12.5	11.1	4.9
12.8	14.8	22.8	20.0	15.9	16.3	13.4	17.1	14.5	19.0	22.8
1.3	0.7	8.9	11.9	10.9	7.3	5.9	3.7	17.9	19.2	9.8
5.8	6.9	2.6	5.8	21.7	11.8	3.4	2.1	4.5	6.3	10.7
8.9	9.4	9.4	7.6	10.0	3.3	6.7	7.8	11.6	13.8	18.6

The sample mean = 11.49 and the sample standard deviation = 6.23.

We will assume that the smiling times, in seconds, follow a uniform distribution between zero and 23 seconds, inclusive. This means that any smiling time from zero to and including 23 seconds is equally likely. The histogram that could be constructed from the sample is an empirical distribution that closely matches the theoretical uniform distribution.

Let $X =$ length, in seconds, of an eight-week-old baby's smile.

The notation for the uniform distribution is

$X \sim U(a, b)$ where $a =$ the lowest value of x and $b =$ the highest value of x.

The probability density function is $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$.

For this example, $X \sim U(0, 23)$ and $f(x) = \frac{1}{23-0}$ for $0 \leq X \leq 23$.

Formulas for the theoretical mean and standard deviation are

$$\mu = \frac{a+b}{2}$$

and

$$\sigma = \sqrt{\frac{(b-a)^2}{12}}$$

For this problem, the theoretical mean and standard deviation are

$$\mu = \frac{0+23}{2} = 11.50\ seconds$$

and

$$\sigma = \frac{(23-0)^2}{12} = 6.64\ seconds.$$

Notice that the theoretical mean and standard deviation are close to the sample mean and standard deviation in this example.

Example 6.2.2

a. Refer to Example 6.2.1. What is the probability that a randomly chosen eight-week-old baby smiles between two and 18 seconds?

Answer

a. Find $P(2 < x < 18)$.

$P(2 < x < 18) = (\text{base})(\text{height}) = (18 - 2)\left(\frac{1}{23}\right) = \left(\frac{16}{23}\right)$.

This graph shows a uniform distribution. The horizontal axis ranges from 0 to 15. The distribution is modeled by a rectangle extending from x = 0 to x = 15. A region from x = 2 to x = 18 is shaded inside the rectangle.

Figure 6.2.1

Example 6.2.3

Suppose the time it takes a nine-year old to eat a donut is between 0.5 and 4 minutes, inclusive. Let $X =$ the time, in minutes, it takes a nine-year old child to eat a donut. Then $X \sim U(0.5, 4)$.

a. The probability that a randomly selected nine-year old child eats a donut in at least two minutes is _______.

Solution

a. 0.5714

Example 6.2.4

Ace Heating and Air Conditioning Service finds that the amount of time a repairman needs to fix a furnace is uniformly distributed between 1.5 and four hours. Let $x =$ the time needed to fix a furnace. Then $x \sim U(1.5, 4)$.

a. Find the probability that a randomly selected furnace repair requires more than two hours.
b. Find the probability that a randomly selected furnace repair requires less than three hours.
c. Find the 30$^{\text{th}}$ percentile of furnace repair times.
d. The longest 25% of furnace repair times take at least how long? (In other words: find the minimum time for the longest 25% of repair times.) What percentile does this represent?
e. Find the mean and standard deviation

Solution

a. To find $f(x): f(x) = \frac{1}{4-1.5} = \frac{1}{2.5}$ so $f(x) = 0.4$

$P(x > 2) = (\text{base})(\text{height}) = (4 - 2)(0.4) = 0.8$

This shows the graph of the function f(x) = 0.4. A horiztonal line ranges from the point (1.5, 0.4) to the point (4, 0.4). Vertical lines extend from the x-axis to the graph at x = 1.5 and x = 4 creating a rectangle. A region is shaded inside the rectangle from x = 2 to x = 4.

Figure 6.2.3. Uniform Distribution between 1.5 and four with shaded area between two and four representing the probability that the repair time x is greater than two

b. $P(x < 3) = (\text{base})(\text{height}) = (3 - 1.5)(0.4) = 0.6$

The graph of the rectangle showing the entire distribution would remain the same. However the graph should be shaded between $x = 1.5$ and $x = 3$. Note that the shaded area starts at $x = 1.5$ rather than at $x = 0$; since $X \sim U(1.5, 4)$, x can not be less than 1.5.

This shows the graph of the function f(x) = 0.4. A horiztonal line ranges from the point (1.5, 0.4) to the point (4, 0.4). Vertical lines extend from the x-axis to the graph at x = 1.5 and x = 4 creating a rectangle. A region is shaded inside the rectangle from x = 1.5 to x = 3.

Figure 6.2.4. Uniform Distribution between 1.5 and four with shaded area between 1.5 and three representing the probability that the repair time x is less than three

c.

This shows the graph of the function f(x) = 0.4. A horiztonal line ranges from the point (1.5, 0.4) to the point (4, 0.4). Vertical lines extend from the x-axis to the graph at x = 1.5 and x = 4 creating a rectangle. A region is shaded inside the rectangle from x = 1.5 to x = k. The shaded area represents P(x < k) = 0.3.

Figure 6.2.5. Uniform Distribution between 1.5 and 4 with an area of 0.30 shaded to the left, representing the shortest 30% of repair times.

$P(x < k) = 0.30$
$P(x < k) = (\text{base})(\text{height}) = (k - 1.5)(0.4)$

$\mathbf{0.3 = (k\text{–}1.5)(0.4)}$; Solve to find k:
$0.75 = k\text{–}1.5$, obtained by dividing both sides by 0.4
$\boldsymbol{k = 2.25}$, obtained by adding 1.5 to both sides

The 30th percentile of repair times is 2.25 hours. 30% of repair times are 2.25 hours or less.

d.

Figure 6.2.6. Uniform Distribution between 1.5 and 4 with an area of 0.25 shaded to the right representing the longest 25% of repair times.

$P(x > k) = 0.25$
$P(x > k) = (\text{base})(\text{height}) = (4\text{–}k)(0.4)$
$\mathbf{0.25 = (4\text{–}k)(0.4)}$; Solve for k:
$0.625 = 4 - k$,
obtained by dividing both sides by 0.4
$-3.375 = -k$,
obtained by subtracting four from both sides: $\boldsymbol{k = 3.375}$
The longest 25% of furnace repairs take at least 3.375 hours (3.375 hours or longer).

Note: Since 25% of repair times are 3.375 hours or longer, that means that 75% of repair times are 3.375 hours or less. 3.375 hours is the **75th percentile** of furnace repair times.

e. $\mu = \frac{a+b}{2}$ and $\sigma = \sqrt{\frac{(b-a)^2}{12}}$

$\mu = \frac{1.5+4}{2} = 2.75$ hours and $\sigma = \sqrt{\frac{(4-1.5)^2}{12}} = 0.7217$ hours

Review

If X has a uniform distribution where $a < x < b$ or $a \leq x \leq b$, then X takes on values between a and b (may include a and b). All values x are equally likely. We write $X \sim U(a, b)$. The mean of X is $\mu = \frac{a+b}{2}$. The standard deviation of X is $\sigma = \sqrt{\frac{(b-a)^2}{12}}$. The probability density function of X is $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$. The cumulative distribution function of X is $P(X \leq x) = \frac{x-a}{b-a}$. X is continuous.

The graph shows a rectangle with total area equal to 1. The rectangle extends from x = a to x = b on the x-axis and has a height of 1/(b-a).

Figure 6.2.8.

The probability $P(c < X < d)$ may be found by computing the area under $f(x)$, between c and d. Since the corresponding area is a rectangle, the area may be found simply by multiplying the width and the height.

Formula Review

$X =$ a real number between a and b (in some instances, X can take on the values a and b). $a =$ smallest X; $b =$ largest X

$X \sim U(a, b)$

The mean is $\mu = \frac{a+b}{2}$

The standard deviation is $\sigma = \sqrt{\frac{(b-a)^2}{12}}$

Probability density function: $f(x) = \frac{1}{b-a} \text{for} a \leq X \leq b$

Area to the Left of x: $P(X < x) = (x\text{–}a)\left(\frac{1}{b-a}\right)$

Area to the Right of x: P(X > x) = (b – x)$\left(\frac{1}{b-a}\right)$

Area Between c and d: $P(c < x < d) = (\text{base})(\text{height}) = (d\text{–}c)\left(\frac{1}{b-a}\right)$

Uniform: $X \sim U(a, b)$ where $a < x < b$

- pdf: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$

- cdf: $P(X \leq x) = \frac{x-a}{b-a}$
- mean $\mu = \frac{a+b}{2}$
- standard deviation $\sigma = \sqrt{\frac{(b-a)^2}{12}}$
- $P(c < X < d) = (d-c)\left(\frac{1}{b-a}\right)$

WeBWorK Problems

References

McDougall, John A. The McDougall Program for Maximum Weight Loss. Plume, 1995.

Use the following information to answer the next ten questions. The data that follow are the square footage (in 1,000 feet squared) of 28 homes.

1.5	2.4	3.6	2.6	1.6	2.4	2.0
3.5	2.5	1.8	2.4	2.5	3.5	4.0
2.6	1.6	2.2	1.8	3.8	2.5	1.5
2.8	1.8	4.5	1.9	1.9	3.1	1.6

The sample mean = 2.50 and the sample standard deviation = 0.8302.

The distribution can be written as $X \sim U(1.5, 4.5)$.

Glossary

Conditional Probability

the likelihood that an event will occur given that another event has already occurred

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