

4.2: The Standard Normal Distribution

Z-Scores

The standard normal distribution is a normal distribution of standardized values called *z-scores*. A *z-score* is measured in units of the standard deviation.

Definition: Z-Score

If X is a normally distributed random variable and $X \sim N(\mu, \sigma)$, then the *z-score* is:

$$z = \frac{x - \mu}{\sigma} \quad (4.2.1)$$

The *z-score* tells you how many standard deviations the value x is above (to the right of) or below (to the left of) the mean, μ . Values of x that are larger than the mean have positive *z-scores*, and values of x that are smaller than the mean have negative *z-scores*. If x equals the mean, then x has a *z-score* of zero. For example, if the mean of a normal distribution is five and the standard deviation is two, the value 11 is three standard deviations above (or to the right of) the mean. The calculation is as follows:

$$\begin{aligned} x &= \mu + (z)(\sigma) \\ &= 5 + (3)(2) = 11 \end{aligned}$$

The *z-score* is three.

Since the mean for the standard normal distribution is zero and the standard deviation is one, then the transformation in Equation 4.2.1 produces the distribution $Z \sim N(0, 1)$. The value x comes from a normal distribution with mean μ and standard deviation σ .

*A *z-score* is measured in units of the standard deviation.*

✓ Example 4.2.1

Suppose $X \sim N(5, 6)$. This says that x is a normally distributed random variable with mean $\mu = 5$ and standard deviation $\sigma = 6$. Suppose $x = 17$. Then (via Equation 4.2.1):

$$z = \frac{x - \mu}{\sigma} = \frac{17 - 5}{6} = 2$$

This means that $x = 17$ is **two** standard deviations (2σ) above or to the right of the mean $\mu = 5$. The standard deviation is $\sigma = 6$.

Notice that: $5 + (2)(6) = 17$ (The pattern is $\mu + z\sigma = x$)

Now suppose $x = 1$. Then:

$$z = \frac{x - \mu}{\sigma} = \frac{1 - 5}{6} = -0.67$$

(rounded to two decimal places)

This means that $x = 1$ is 0.67 standard deviations (-0.67σ) below or to the left of the mean $\mu = 5$. Notice that: $5 + (-0.67)(6)$ is approximately equal to one (This has the pattern $\mu + (-0.67)\sigma = 1$)

Summarizing, when z is positive, x is above or to the right of μ and when z is negative, x is to the left of or below μ . Or, when z is positive, x is greater than μ , and when z is negative x is less than μ .

? Exercise 4.2.1

What is the *z-score* of x , when $x = 1$ and $X \sim N(12, 3)$?

Answer

$$z = \frac{1 - 12}{3} \approx -3.67$$

✓ Example 4.2.2

Some doctors believe that a person can lose five pounds, on the average, in a month by reducing his or her fat intake and by exercising consistently. Suppose weight loss has a normal distribution. Let X = the amount of weight lost(in pounds) by a person in a month. Use a standard deviation of two pounds. $X \sim N(5, 2)$. Fill in the blanks.

- Suppose a person **lost** ten pounds in a month. The z -score when $x = 10$ pounds is $z = 2.5$ (verify). This z -score tells you that $x = 10$ is _____ standard deviations to the _____ (right or left) of the mean _____. (What is the mean?).
- Suppose a person **gained** three pounds (a negative weight loss). Then $z =$ _____. This z -score tells you that $x = -3$ is _____ standard deviations to the _____ (right or left) of the mean.

Answers

- This z -score tells you that $x = 10$ is 2.5 standard deviations to the right of the mean five.
- Suppose the random variables X and Y have the following normal distributions: $X \sim N(5, 6)$ and $Y \sim N(2, 1)$. If $x = 17$, then $z = 2$. (This was previously shown.) If $y = 4$, what is z ?

$$z = \frac{y - \mu}{\sigma} = \frac{4 - 2}{1} = 2$$

where $\mu = 2$ and $\sigma = 1$.

The z -score for $y = 4$ is $z = 2$. This means that four is $z = 2$ standard deviations to the right of the mean. Therefore, $x = 17$ and $y = 4$ are both two (of their own) standard deviations to the right of their respective means.

The z -score allows us to compare data that are scaled differently. To understand the concept, suppose $X \sim N(5, 6)$ represents weight gains for one group of people who are trying to gain weight in a six week period and $Y \sim N(2, 1)$ measures the same weight gain for a second group of people. A negative weight gain would be a weight loss. Since $x = 17$ and $y = 4$ are each two standard deviations to the right of their means, they represent the same, standardized weight gain **relative to their means**.

? Exercise 4.2.2

Fill in the blanks.

Jerome averages 16 points a game with a standard deviation of four points. $X \sim N(16, 4)$. Suppose Jerome scores ten points in a game. The z -score when $x = 10$ is -1.5 . This score tells you that $x = 10$ is _____ standard deviations to the _____ (right or left) of the mean _____. (What is the mean?).

Answer

1.5, left, 16

The Empirical Rule

If X is a random variable and has a normal distribution with mean μ and standard deviation σ , then the *Empirical Rule* says the following:

- About 68% of the x values lie between -1σ and $+1\sigma$ of the mean μ (within one standard deviation of the mean).
- About 95% of the x values lie between -2σ and $+2\sigma$ of the mean μ (within two standard deviations of the mean).
- About 99.7% of the x values lie between -3σ and $+3\sigma$ of the mean μ (within three standard deviations of the mean). Notice that almost all the x values lie within three standard deviations of the mean.
- The z -scores for $+1\sigma$ and -1σ are $+1$ and -1 , respectively.
- The z -scores for $+2\sigma$ and -2σ are $+2$ and -2 , respectively.
- The z -scores for $+3\sigma$ and -3σ are $+3$ and -3 respectively.

The empirical rule is also known as the 68-95-99.7 rule.

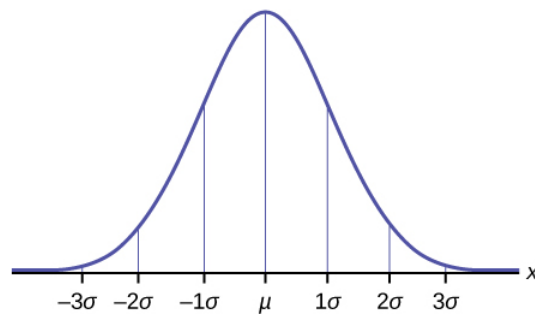


Figure 4.2.1

✓ Example 4.2.3

The mean height of 15 to 18-year-old males from Chile from 2009 to 2010 was 170 cm with a standard deviation of 6.28 cm. Male heights are known to follow a normal distribution. Let X = the height of a 15 to 18-year-old male from Chile in 2009 to 2010. Then $X \sim N(170, 6.28)$.

- Suppose a 15 to 18-year-old male from Chile was 168 cm tall from 2009 to 2010. The z -score when $x = 168$ cm is $z =$ _____. This z -score tells you that $x = 168$ is _____ standard deviations to the _____ (right or left) of the mean _____. (What is the mean?).
- Suppose that the height of a 15 to 18-year-old male from Chile from 2009 to 2010 has a z -score of $z = 1.27$. What is the male's height? The z -score ($z = 1.27$) tells you that the male's height is _____ standard deviations to the _____ (right or left) of the mean.

Answers

- 0.32, 0.32, left, 170
- 177.98, 1.27, right

? Exercise 4.2.3

Use the information in Example 4.2.3 to answer the following questions.

- Suppose a 15 to 18-year-old male from Chile was 176 cm tall from 2009 to 2010. The z -score when $x = 176$ cm is $z =$ _____. This z -score tells you that $x = 176$ cm is _____ standard deviations to the _____ (right or left) of the mean _____. (What is the mean?).
- Suppose that the height of a 15 to 18-year-old male from Chile from 2009 to 2010 has a z -score of $z = -2$. What is the male's height? The z -score ($z = -2$) tells you that the male's height is _____ standard deviations to the _____ (right or left) of the mean.

Answer

Solve the equation $z = \frac{x - \mu}{\sigma}$ for z . $x = \mu + (z)(\sigma)$

$z = \frac{176 - 170}{6.28}$, This z -score tells you that $x = 176$ cm is 0.96 standard deviations to the right of the mean 170 cm.

Answer

Solve the equation $z = \frac{x - \mu}{\sigma}$ for z . $x = \mu + (z)(\sigma)$

$X = 157.44$ cm, The z -score ($z = -2$) tells you that the male's height is two standard deviations to the left of the mean.

✓ Example 4.2.4

From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was 172.36 cm, and the standard deviation was 6.34 cm. Let Y = the height of 15 to 18-year-old males from 1984 to 1985. Then $Y \sim N(172.36, 6.34)$.

The mean height of 15 to 18-year-old males from Chile from 2009 to 2010 was 170 cm with a standard deviation of 6.28 cm. Male heights are known to follow a normal distribution. Let X = the height of a 15 to 18-year-old male from Chile in 2009 to 2010. Then $X \sim N(170, 6.28)$.

Find the z -scores for $x = 160.58$ cm and $y = 162.85$ cm. Interpret each z -score. What can you say about $x = 160.58$ cm and $y = 162.85$ cm?

Answer

- The z -score (Equation 4.2.1) for $x = 160.58$ is $z = -1.5$.
- The z -score for $y = 162.85$ is $z = -1.5$.

Both $x = 160.58$ and $y = 162.85$ deviate the same number of standard deviations from their respective means and in the same direction.

? Exercise 4.2.4

In 2012, 1,664,479 students took the SAT exam. The distribution of scores in the verbal section of the SAT had a mean $\mu = 496$ and a standard deviation $\sigma = 114$. Let X = a SAT exam verbal section score in 2012. Then $X \sim N(496, 114)$.

Find the z -scores for $x_1 = 325$ and $x_2 = 366.21$. Interpret each z -score. What can you say about $x_1 = 325$ and $x_2 = 366.21$?

Answer

The z -score (Equation 4.2.1) for $x_1 = 325$ is $z_1 = -1.15$.

The z -score (Equation 4.2.1) for $x_2 = 366.21$ is $z_2 = -1.14$.

Student 2 scored closer to the mean than Student 1 and, since they both had negative z -scores, Student 2 had the better score.

✓ Example 4.2.5

Suppose x has a normal distribution with mean 50 and standard deviation 6.

- About 68% of the x values lie within one standard deviation of the mean. Therefore, about 68% of the x values lie between $-1\sigma = (-1)(6) = -6$ and $1\sigma = (1)(6) = 6$ of the mean 50. The values $50 - 6 = 44$ and $50 + 6 = 56$ are within one standard deviation from the mean 50. The z -scores are -1 and $+1$ for 44 and 56, respectively.
- About 95% of the x values lie within two standard deviations of the mean. Therefore, about 95% of the x values lie between $-2\sigma = (-2)(6) = -12$ and $2\sigma = (2)(6) = 12$. The values $50 - 12 = 38$ and $50 + 12 = 62$ are within two standard deviations from the mean 50. The z -scores are -2 and $+2$ for 38 and 62, respectively.
- About 99.7% of the x values lie within three standard deviations of the mean. Therefore, about 99.7% of the x values lie between $-3\sigma = (-3)(6) = -18$ and $3\sigma = (3)(6) = 18$ from the mean 50. The values $50 - 18 = 32$ and $50 + 18 = 68$ are within three standard deviations of the mean 50. The z -scores are -3 and $+3$ for 32 and 68, respectively.

? Exercise 4.2.5

Suppose X has a normal distribution with mean 25 and standard deviation five. Between what values of x do 68% of the values lie?

Answer

between 20 and 30.

✓ Example 4.2.6

From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was 172.36 cm, and the standard deviation was 6.34 cm. Let Y = the height of 15 to 18-year-old males in 1984 to 1985. Then $Y \sim N(172.36, 6.34)$.

- About 68% of the y values lie between what two values? These values are _____. The z -scores are _____, respectively.
- About 95% of the y values lie between what two values? These values are _____. The z -scores are _____, respectively.
- About 99.7% of the y values lie between what two values? These values are _____. The z -scores are _____, respectively.

Answer

- About 68% of the values lie between 166.02 and 178.7. The z -scores are -1 and 1 .
- About 95% of the values lie between 159.68 and 185.04. The z -scores are -2 and 2 .
- About 99.7% of the values lie between 153.34 and 191.38. The z -scores are -3 and 3 .

? Exercise 4.2.6

The scores on a college entrance exam have an approximate normal distribution with mean, $\mu = 52$ points and a standard deviation, $\sigma = 11$ points.

- About 68% of the y values lie between what two values? These values are _____. The z -scores are _____, respectively.
- About 95% of the y values lie between what two values? These values are _____. The z -scores are _____, respectively.
- About 99.7% of the y values lie between what two values? These values are _____. The z -scores are _____, respectively.

Answer a

About 68% of the values lie between the values 41 and 63. The z -scores are -1 and 1 , respectively.

Answer b

About 95% of the values lie between the values 30 and 74. The z -scores are -2 and 2 , respectively.

Answer c

About 99.7% of the values lie between the values 19 and 85. The z -scores are -3 and 3 , respectively.

Summary

A z -score is a standardized value. Its distribution is the standard normal, $Z \sim N(0, 1)$. The mean of the z -scores is zero and the standard deviation is one. If y is the z -score for a value x from the normal distribution $N(\mu, \sigma)$ then z tells you how many standard deviations x is above (greater than) or below (less than) μ .

Formula Review

$$Z \sim N(0, 1)$$

$z = a$ standardized value (z -score)

mean = 0; standard deviation = 1

To find the K^{th} percentile of X when the z -scores is known:

$$k = \mu + (z)\sigma$$

$$z\text{-score: } z = \frac{x - \mu}{\sigma}$$

Z = the random variable for z-scores

$Z \sim N(0, 1)$

Glossary

Standard Normal Distribution

a continuous random variable (RV) $X \sim N(0, 1)$; when X follows the standard normal distribution, it is often noted as $(Z \sim N(0, 1))$.

z-score

the linear transformation of the form $z = \frac{x - \mu}{\sigma}$; if this transformation is applied to any normal distribution $X \sim N(\mu, \sigma)$ the result is the standard normal distribution $Z \sim N(0, 1)$. If this transformation is applied to any specific value x of the RV with mean μ and standard deviation σ , the result is called the z-score of x . The z-score allows us to compare data that are normally distributed but scaled differently.

References

1. "Blood Pressure of Males and Females." StatCrunch, 2013. Available online at <http://www.statcrunch.com/5.0/viewre...reportid=11960> (accessed May 14, 2013).
2. "The Use of Epidemiological Tools in Conflict-affected populations: Open-access educational resources for policy-makers: Calculation of z-scores." London School of Hygiene and Tropical Medicine, 2009. Available online at http://conflict.lshtm.ac.uk/page_125.htm (accessed May 14, 2013).
3. "2012 College-Bound Seniors Total Group Profile Report." CollegeBoard, 2012. Available online at media.collegeboard.com/digita...Group-2012.pdf (accessed May 14, 2013).
4. "Digest of Education Statistics: ACT score average and standard deviations by sex and race/ethnicity and percentage of ACT test takers, by selected composite score ranges and planned fields of study: Selected years, 1995 through 2009." National Center for Education Statistics. Available online at nces.ed.gov/programs/digest/d...s/dt09_147.asp (accessed May 14, 2013).
5. Data from the *San Jose Mercury News*.
6. Data from *The World Almanac and Book of Facts*.
7. "List of stadiums by capacity." Wikipedia. Available online at en.Wikipedia.org/wiki/List_o...ms_by_capacity (accessed May 14, 2013).
8. Data from the National Basketball Association. Available online at www.nba.com (accessed May 14, 2013).

This page titled [4.2: The Standard Normal Distribution](#) is shared under a [CC BY 4.0](#) license and was authored, remixed, and/or curated by [OpenStax](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform.

- [6.2: The Standard Normal Distribution](#) by [OpenStax](#) is licensed [CC BY 4.0](#). Original source: <https://openstax.org/details/books/introductory-statistics>.