# MATH 100 LIBERAL ARTS MATH

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# FC: Math 100 Liberal Arts Math (Ikeda)

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# Licensing

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# CHAPTER OVERVIEW

## 1: Measurement

Measurement is a number that describes the size or amount of something. You can measure many things like length, area, capacity, weight, temperature and time. In the United States, two main systems of measurement are used: the metric system and the U.S. customary measurement system.

The material in this chapter is from Developmental Math by NROC.

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# 1.1: U.S. Customary Measurement System

## Learning Objectives

1. Define units of length, weight and capacity.
2. Convert from one unit to another.
3. Perform arithmetic calculations on units of length, weight and capacity.
4. Solve application problems involving units of length, weight and capacity.

## Introduction

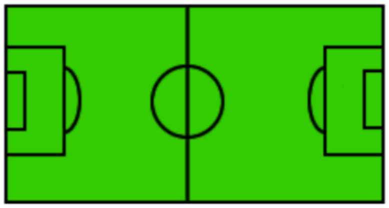
**Measurement** is a number that describes the size or amount of something. You can measure many things like length, area, capacity, weight, temperature and time. In the United States, two main systems of measurement are used: the **metric system** and the **U.S. customary measurement system**. This section addresses the measurements of length, weight and capacity using the U.S. customary measurement system.

Suppose you want to purchase tubing for a project, and you see two signs in a hardware store: *$1.88 for 2 feet* of tubing and *$5.49 for 3 yards* of tubing. If both types of tubing will work equally well for your project, which is the better price? You need to know about two **units of measurement**, yards and feet, in order to determine the answer.

## Units of Length

**Length** is the distance from one end of an object to the other end, or from one object to another. For example, the length of a letter-sized piece of paper is 11 inches. The system for measuring length in the United States is based on the four customary units of length: **inch, foot, yard,** and **mile**. Below are examples to show measurement in each of these units.

| Unit | Description | Image |
|---|---|---|
| Inch/Inches | Some people donate their hair to be made into wigs for cancer patients who have lost hair as a result of treatment. One company requires hair donations to be at least 8 inches long. |  |
| | Frame size of a bike: the distance from the center of the crank to the top of the seat tube. Frame size is usually measured in inches. This frame is 16 inches. |  |
| Foot/Feet | Rugs are typically sold in standard lengths. One typical size is a rug that is 8 feet wide and 11 feet long. This is often described as an 8 by 11 rug. |  |

| | | |
|---|---|---|
| Yard/Yards | Soccer fields vary some in their size. An official field can be any length between 100 and 130 yards. |  |
| Mile/Miles | A marathon is 26.2 miles long. One marathon route is shown in the map to the right. |  |

You can use any of these four U.S. customary measurement units to describe the length of something, but it makes more sense to use certain units for certain objects depending on their general size. For example, it makes more sense to describe the length of a rug in feet rather than miles, and to describe a marathon in miles rather than inches.

You may need to convert between units of measurement. For example, you might want to express your height using feet and inches (5 feet 4 inches) or using only inches (64 inches). You need to know the unit equivalents in order to make these conversions between units.

The table below shows equivalents and conversion factors for the four customary units of measurement of length.

| **Unit Equivalents** | **Conversion Factors (longer to shorter units of measurement)** | **Conversion Factors (shorter to longer units of measurement)** |
|---|---|---|
| 1 foot = 12 inches | $\frac{12\text{ inches}}{1\text{ foot}}$ | $\frac{1\text{ foot}}{12\text{ inches}}$ |
| 1 yard = 3 feet | $\frac{3\text{ feet}}{1\text{ yard}}$ | $\frac{1\text{ yard}}{3\text{ feet}}$ |
| 1 mile = 5280 feet | $\frac{5280\text{ feet}}{1\text{ mile}}$ | $\frac{1\text{ mile}}{5280\text{ feet}}$ |

Note that each of these conversion factors is a ratio of equal values, also called a unit fraction, so each conversion factor equals 1. Recall that multiplying a number by 1 does not change the number. Multiplying a measurement by a conversion factor does not change the size of the measurement at all since it is the same as multiplying by 1; it just changes the units that you are using to measure.

### Converting Between Units of Length

You can use the conversion factors to convert a measurement, such as feet, to another type of measurement, such as inches.

Note that there are many more inches for a measurement than there are feet for the same measurement, as feet is a longer unit of measurement. You could use the conversion factor $\frac{12\text{ inches}}{1\text{ foot}}$.

If a length is measured in feet, and you'd like to convert the length to yards, you can think, "I am converting from a shorter unit to a longer one, so the length in yards will be less than the length in feet." You could use the conversion factor $\frac{1\text{ yard}}{3\text{ feet}}$.

If a distance is measured in miles, and you want to know how many feet it is, you can think, "I am converting from a longer unit of measurement to a shorter one, so the number of feet would be greater than the number of miles." You could use the conversion factor $\frac{5280\text{ feet}}{1\text{ mile}}$.

Another way to determine which conversion factor, or unit fraction to use, is to check which unit will need to be cancelled. I.e. Converting inches to feet, you want the denominator of the unit fraction to cancel out the original units of inches. Hence the appropriate conversion factors has to be in $\frac{\_\_\_\_\_\_\_\text{ feet}}{\_\_\_\_\_\_\_\text{ inches}}$. Then you just need to fill in the appropriate numbers. Since 1 foot = 12 inches, the conversion factor is $\frac{1\text{ feet}}{12\text{ inches}}$.

You can use the **factor label method (dimensional analysis)** to convert a length from one unit of measure to another using the conversion factors. In the factor label method, you multiply by **unit fractions** to convert a measurement from one unit to another. Study the example below to see how the factor label method can be used to convert $3\frac{1}{2}$ feet into an equivalent number of inches.

### ✓ Example 1.1.1

How many inches are in $3\frac{1}{2}$ feet?

**Solution**

Begin by reasoning about your answer. Since a foot is longer than an inch, this means the answer would be greater than $3\frac{1}{2}$.

$$3\frac{1}{2}\text{ feet} = \_\_\_\_\_\_\_\text{ inches}$$

Find the conversion factor that compares inches and feet, with “inches” in the numerator, and "feet" in the denominator. Then multiply

$$3\frac{1}{2}\text{ feet}\cdot\frac{12\text{ inches}}{1\text{ foot}} = \_\_\_\_\_\_\_\text{ inches}$$

Rewrite the mixed number as an improper fraction before multiplying.

$$\frac{7\text{ feet}}{2}\cdot\frac{12\text{ inches}}{1\text{ foot}} = \_\_\_\_\_\_\_\text{ inches}$$

You can cancel the same units when they appear in the numerator *and* the denominator. Here “feet” and “foot” describe the same unit just in the singular and plural form. This eliminates the unit of "feet" from the problem and leaves the unit of "inches".

$$\frac{7\ \cancel{\text{feet}}}{2}\cdot\frac{12\text{ inches}}{1\ \cancel{\text{foot}}} = \_\_\_\_\_\_\_\text{ inches}$$

$$\frac{7}{2}\cdot\frac{12\text{ inches}}{1} = \_\_\_\_\_\_\_\text{ inches}$$

Rewrite as multiplication of numerators and denominators.

$$\frac{7\cdot 12\text{ inches}}{2\cdot 1} = \_\_\_\_\_\_\_\text{ inches}$$

Multiply.

$$\frac{84\text{ inches}}{2} = \_\_\_\_\_\_\_\text{ inches}$$

Divide.

$$\frac{84\text{ inches}}{2} = 42\text{ inches}$$

**Answer**: There are 42 inches in $3\frac{1}{2}$ feet.

Notice that by using the factor label method you can cancel the units out of the problem, just as if they were numbers. You can only cancel if the unit being canceled is in both the numerator and denominator of the fractions you are multiplying.

In the problem above, you canceled feet and foot leaving you with inches, which is what you were trying to find.

$$\frac{7\ \cancel{\text{feet}}}{2}\cdot\frac{12\text{ inches}}{1\ \cancel{\text{foot}}}=\rule{2em}{0.4pt}\text{ inches}$$

What if you had used the wrong conversion factor?

$$\frac{7\text{ feet}}{2}\cdot\frac{1\text{ foot}}{12\text{ inches}}=\rule{2em}{0.4pt}\text{ inches}$$

You could not cancel the feet because the unit is not the same in both the numerator and the denominator. So if you complete the computation, you would still have both feet and inches in the answer and no conversion would take place.

Here is another example of a length conversion using the factor label method.

**✓ Example 1.1.2**

How many yards is 7 feet?

**Solution**

Start by reasoning about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. So your answer will be less than 7.

$$7\text{ feet}=\rule{2em}{0.4pt}\text{ yards}$$

Find the conversion factor that compares feet and yards, with yards in the numerator.

$$7\text{ feet}\cdot\frac{1\text{ yard}}{3\text{ feet}}=\rule{2em}{0.4pt}\text{ yards}$$

Rewrite the whole number as a fraction in order to multiply.

$$\frac{7\text{ feet}}{1}\cdot\frac{1\text{ yard}}{3\text{ feet}}=\rule{2em}{0.4pt}\text{ yards}$$

Cancel the similar units “feet” and “feet” leaving only yards.

$$\frac{7\text{ feet}}{1}\cdot\frac{1\text{ yard}}{3\text{ feet}}=\rule{2em}{0.4pt}\text{ yards}$$

$$\frac{7}{1}\cdot\frac{1\text{ yard}}{3}=\rule{2em}{0.4pt}\text{ yards}$$

Multiply.

$$\frac{7\cdot 1\text{ yard}}{1\cdot 3}=\rule{2em}{0.4pt}\text{ yards}$$

Divide.

$$\frac{7\text{ yards}}{3}=2\frac{1}{3}\text{ yards}$$

**Answer**: 7 feet equals $2\frac{1}{3}$ yards.

Note that if the units do not cancel to give you the answer you are trying to find, you may not have used the correct conversion factor.

**✎ Try It 1.1.1**

How many feet are in $2\frac{1}{2}$ miles?

**Answer**

There are 5280 feet in a mile, so multiply $2\frac{1}{2}$ by $\frac{5280\text{feet}}{1\text{mile}}$ to get 13,200 feet.

## Applying Unit Conversions

There are times when you will need to perform computations on measurements that are given in different units. For example, consider the tubing problem given earlier. You must decide which of the two options is a better price, and you have to compare prices given in different unit measurements.

In order to compare, you need to convert the measurements into one single, common unit of measurement. To be sure you have made the computation accurately, think about whether the unit you are converting to is smaller or larger than the number you have. Its relative size will tell you whether the number you are trying to find is greater or lesser than the given number.

### ✓ Example 1.1.3

An interior decorator needs border trim for a home she is wallpapering. She needs 15 feet of border trim for the living room, 30 feet of border trim for the bedroom, and 26 feet of border trim for the dining room. How many yards of border trim does she need?

**Solution**

You need to find the total length of border trim that is needed for all three rooms in the house. You can only add or subtract numbers with the same units. Since the measurements for each room are given in feet, you can add the numbers.

$$15 \text{ feet} + 30 \text{ feet} + 26 \text{ feet} = 71 \text{ feet}$$

How many yards is 71 feet? Reason about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. Expect your answer to be less than 71.

$$71 \text{ feet} = \underline{\qquad\qquad} \text{ yards}$$

Use the conversion factor $\frac{1 \text{ yard}}{3 \text{ feet}}$.

$$\frac{71 \text{ feet}}{1} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = \underline{\qquad\quad} \text{ yards}$$

Since “feet” is in the numerator and denominator, you can cancel this unit.

$$\frac{71 \text{ feet}}{1} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = \underline{\qquad\quad} \text{ yards}$$

$$\frac{71}{1} \cdot \frac{1 \text{ yard}}{3} = \underline{\qquad\quad} \text{ yards}$$

Multiply.

$$\frac{71 \cdot 1 \text{ yards}}{1 \cdot 3} \underline{\qquad\quad} \text{ yards}$$

$$\frac{71 \text{ yards}}{3} = \underline{\qquad\quad} \text{ yards}$$

Divide, and write as a mixed number.

$$\frac{71 \text{ yards}}{3} = 23\frac{2}{3} \text{ yards}$$

**Answer**: The interior decorator needs $23\frac{2}{3}$ yards of border trim.

The next example uses the factor label method to solve a problem that requires converting from miles to feet.

### ✓ Example 1.1.4

Two runners were comparing how much they had trained earlier that day. Jo said, “According to my pedometer, I ran 8.3 miles.” Alex said, “That’s a little more than what I ran. I ran 8.1 miles.” How many more feet did Jo run than Alex?

**Solution**

You need to find the difference between the distance Jo ran and the distance Alex ran. Since both distances are given in the same unit, you can subtract and keep the unit the same.

$$8.3\text{ miles} – 8.1\text{ miles} = 0.2\text{ mile}$$

$$0.2\text{ mile} = \frac{2}{10}\text{ miles}$$

Since the problem asks for the difference in *feet*, you must convert from miles to feet. How many feet is 0.2 mile? Reason about the size of your answer. Since a mile is longer than a foot, the distance when expressed as feet will be a number greater than 0.2.

$$\frac{2}{10}\text{ miles} = \underline{\hspace{2cm}}\text{feet}$$

Use the conversion factor $\frac{5280\text{ feet}}{1\text{ mile}}$.

$$\frac{2\text{ miles}}{10} \cdot \frac{5280\text{ feet}}{1\text{ mile}} = \underline{\hspace{2cm}}\text{feet}$$

$$\frac{2}{10} \cdot \frac{5280\text{ feet}}{1} = \underline{\hspace{2cm}}\text{feet}$$

Multiply.

$$\frac{2 \cdot 5280\text{ feet}}{10 \cdot 1} = \underline{\hspace{2cm}}\text{feet}$$

$$\frac{10,560\text{ feet}}{10} = \underline{\hspace{2cm}}\text{feet}$$

Divide.

$$\frac{10,560\text{ feet}}{10} = 1056\text{ feet}$$

**Answer**: Jo ran 1056 feet more than Alex.

Now let’s revisit the question from earlier.

## Example 1.1.5

You are walking through a hardware store and notice two sales on tubing.

- 3 yards of Tubing A costs \$5.49.
- Tubing B sells for \$1.88 for 2 feet.

Either tubing is acceptable for your project. Which tubing is less expensive?

### Solution

Find the unit price for each tubing. This will make it easier to compare.

Tubing A: 3 yards = \$5.49

Find the cost per yard of Tubing A by dividing the cost of 3 yards of the tubing by 3.

$$\frac{\$5.49 \div 3}{3\text{ yards} \div 3} = \frac{\$1.83}{1\text{ yard}}$$

Tubing B is sold by the foot. Find the cost per foot by dividing \$1.88 by 2 feet.

Tubing B: 2 feet = \$1.88

$$\frac{\$1.88 \div 2}{2\text{ feet} \div 2} = \frac{\$0.94}{1\text{ foot}}$$

To compare the prices, you need to have the same unit of measure.

$$\frac{\$0.94}{1\text{ foot}}\cdot\frac{3\text{ feet}}{1\text{ yard}}=\frac{\$\_\_\_\_\_\_\_\_}{\_\_\_\_\_\_\_\_\text{ yard}}$$

Use the conversion factor $\frac{3\text{ feet}}{1\text{ yard}}$, cancel and multiply

$$\frac{\$0.94}{1}\cdot\frac{3}{1\text{ yard}}=\frac{\$2.82}{1\text{ yard}}$$

$$\therefore \$2.82\text{ per yard}$$

Compare prices for 1 yard of each tubing.

Tubing A: $1.83 per yard

Tubing B: $2.82 per yard

**Answer**: Tubing A is less expensive than Tubing B.

In the problem above, you could also have found the price per foot for each kind of tubing and compared the unit prices of each per foot.

**Try It 1.1.2**

A fence company is measuring a rectangular area in order to install a fence around its perimeter. If the length of the rectangular area is 130 yards and the width is 75 feet, what is the total length of the distance to be fenced?

**Answer**

130 yards is equivalent to 390 feet. To find the perimeter, add length + length + width + width: 390 feet + 390 feet + 75 feet + 75 feet = 930 feet.

### Summary

The four basic units of measurement that are used in the U.S. customary measurement system are: inch, foot, yard, and mile. Typically, people use yards, miles, and sometimes feet to describe long distances. Measurement in inches is common for shorter objects or lengths.
You need to convert from one unit of measure to another if you are solving problems that include measurements involving more than one type of measurement. Each of the units can be converted to one of the other units using the table of equivalents, the conversion factors, and/or the factor label method shown in this topic.

## Weight

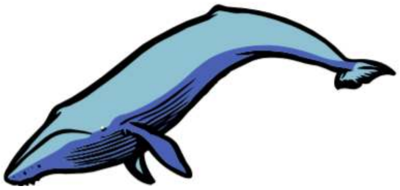
## Introduction

When you mention how heavy or light an object is, you are referring to its weight. In the U.S. customary system of measurement, weight is measured in ounces, pounds, and tons. Like other units of measurement that describe the same kind of quantity, you can convert between these units and you sometimes need to do this to solve problems.

In 2010, the post office charged $0.44 to mail an item that weighed an ounce or less. The post office charged $0.17 for each additional ounce, or fraction of an ounce, of weight. How much did it cost to mail a package that weighed two pounds three ounces? To answer this question, you need to understand the relationship between ounces and pounds.

### Units of Weight

You often use the word **weight** to describe how heavy or light an object or person is. Weight is measured in the U.S. customary system using three units: ounces, pounds, and tons. An **ounce** is the smallest unit for measuring weight, a **pound** is a larger unit, and a **ton** is the largest unit.

| | |
|---|---|
| Whales are some of the largest animals in the world. Some species can reach weights of up to 200 tons- that’s equal to 400,000 pounds |  |
| Meat is a product that is typically sold by the pound. One pound of ground beef makes about four hamburger patties. |  |
| Ounces are used to measure lighter objects. A stack of 11 pennies is equal to about one ounce. |  |

You can use any of the customary measurement units to describe the weight of something, but it makes more sense to use certain units depending on how heavy an object is. For example, it makes more sense to describe the weight of a human being in pounds rather than tons. It makes more sense to describe the weight of a car in tons rather than ounces.

1 pound = 16 ounces

1 ton = 2000 pounds

## Converting Between Units of Weight

Four ounces is a typical serving size of meat. Since meat is sold by the pound, you might want to convert the weight of a package of meat from pounds to ounces in order to determine how many servings are contained in a package of meat.

The weight capacity of a truck is often provided in tons. You might need to convert pounds into tons if you are trying to determine whether a truck can safely transport a big shipment of heavy materials.

The table below shows the unit conversions and conversion factors that are used to make conversions between customary units of weight.

| **Unit Equivalents** | **Conversion Factors (heavier to lighter units of measurement)** | **Conversion Factors (lighter to heavier units of measurement)** |
|---|---|---|
| 1 pound = 16 ounces | $\frac{16\text{ ounces}}{1\text{ pound}}$ | $\frac{1\text{ pound}}{16\text{ ounces}}$ |
| 1 ton = 2000 pounds | $\frac{2000\text{ pounds}}{1\text{ ton}}$ | $\frac{1\text{ ton}}{2000\text{ pounds}}$ |

You can use the *factor label method* to convert one customary unit of weight to another customary unit of weight. This method uses unit fractions conversion factor, which allow you to “cancel” units to end up with your desired unit of measurement.

Two examples illustrating the factor label method are shown below.

**Example 1.1.6**

How many ounces are in $2\frac{1}{4}$ pounds?

**Solution**

Begin by reasoning about your answer. Since a pound is heavier than an ounce, expect your answer to be a number greater than $2\frac{1}{4}$.

$$2\frac{1}{4}\text{ pounds} = \_\_\_\_\_\_\_\_\text{ounces}$$

Multiply by the conversion factor that relates ounces and pounds: $\frac{16\text{ ounces}}{1\text{ pound}}$.

$$2\frac{1}{4}\text{ pounds}\cdot\frac{16\text{ ounces}}{1\text{ pound}} = \_\_\_\_\_\_\_\_\text{ounces}$$

Write the mixed number as an improper fraction.

$$\frac{9\text{ pounds}}{4}\cdot\frac{16\text{ ounces}}{1\text{ pound}} = \_\_\_\_\_\_\_\_\text{ounces}$$

The common unit “pound” can be canceled because it appears in both the numerator and denominator.

$$\frac{9}{4}\cdot\frac{16\text{ ounces}}{1} = \_\_\_\_\_\_\_\_\text{ounces}$$

Multiply and simplify.

$$\frac{9\cdot 16\text{ ounces}}{4\cdot 1} = \_\_\_\_\_\_\_\_\text{ounces}$$

$$\frac{144\text{ ounces}}{4} = \_\_\_\_\_\_\_\_\text{ounces}$$

$$\frac{144\text{ ounces}}{4} = 36\text{ ounces}$$

**Answer**: There are 36 ounces in $2\frac{1}{4}$ pounds.

**Example 1.1.7**

How many tons is 6500 pounds?

**Solution**

Begin by reasoning about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 6500.

$$6500\text{ pounds} = \_\_\_\_\_\_\_\_\text{tons}$$

Multiply by the conversion factor that relates tons to pounds: $\frac{1\text{ ton}}{2000\text{ pounds}}$.

$$6500\text{ pounds}\cdot\frac{1\text{ ton}}{2000\text{ pounds}} = \_\_\_\_\_\_\_\_\text{tons}$$

$$\frac{6500\text{ pounds}}{1}\cdot\frac{1\text{ ton}}{2000\text{ pounds}} = \_\_\_\_\_\_\_\_\text{tons}$$

Apply the Factor Label method.

$$\frac{6500}{1}\cdot\frac{1\text{ ton}}{2000} = \_\_\_\_\_\_\_\_\text{tons}$$

Multiply and simplify.

$$\frac{6500\text{ tons}}{2000} = \_\_\_\_\_\_\_\_\text{tons}$$

$$\frac{6500\text{ tons}}{2000}=3\frac{1}{4}\text{ tons}$$

**Answer**: There are 6500 pounds in $3\frac{1}{4}$ tons.

**Try It 1.1.3**

How many pounds is 72 ounces?

**Answer**

There are 16 ounces in one pound, so $72\text{ ounces}\cdot\frac{1\text{ pound}}{16\text{ ounces}}=4\frac{1}{2}\text{ pounds}$ .

## Applying Unit Conversions

There are times when you need to perform calculations on measurements that are given in different units. To solve these problems, you need to convert one of the measurements to the same unit of measurement as the other measurement.

Think about whether the unit you are converting to is smaller or larger than the unit you are converting from. This will help you be sure that you are making the right computation. You can use the factor label method to make the conversion from one unit to another.

Here is an example of a problem that requires converting between units.

**Example 1.1.8**

A municipal trash facility allows a person to throw away a maximum of 30 pounds of trash per week. Last week, 140 people threw away the maximum allowable trash. How many tons of trash did this equal?

### Solution

Determine the total trash for the week expressed in pounds. If 140 people each throw away 30 pounds, you can find the total by multiplying.

$$140\cdot 30\text{ pounds}=4200\text{ pounds}$$

Then convert 4200 pounds to tons. Reason about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 4200.

$$4200\text{ pounds}=\underline{\qquad\quad}\text{ tons}$$

Find the conversion factor appropriate for the situation: $\frac{1\text{ ton}}{2000\text{ pounds}}$.

$$\frac{4200\text{ pounds}}{1}\cdot\frac{1\text{ ton}}{2000\text{ pounds}}=\underline{\qquad\quad}\text{tons}$$

$$\frac{4200}{1}\cdot\frac{1\text{ ton}}{2000}=\underline{\qquad\quad}\text{tons}$$

Multiply and simplify.

$$\frac{4200\cdot 1\text{ ton}}{1\cdot 2000}=\underline{\qquad\quad}\text{tons}$$

$$\frac{4200\text{ tons}}{2000}=\underline{\qquad\quad}\text{tons}$$

$$\frac{4200\text{ tons}}{2000}=2\frac{1}{10}\text{ tons}$$

**Answer**: The total amount of trash generated is $2\frac{1}{10}$ tons.

Let’s revisit the post office problem that was posed earlier. We can use unit conversion to solve this problem.

### Example 1.1.9

The post office charges $0.44 to mail something that weighs an ounce or less. The charge for each additional ounce, or fraction of an ounce, of weight is $0.17. At this rate, how much will it cost to mail a package that weighs 2 pounds 3 ounces?

**Solution**

Since the pricing is for ounces, convert the weight of the package from pounds and ounces into just ounces.

$$2 \text{ pounds } 3 \text{ ounces} = \_\_\_\_\_\_\_\_ \text{ounces}$$

First use the factor label method to convert 2 pounds to ounces.

$$\frac{2 \text{ pounds}}{1} \cdot \frac{16 \text{ ounces}}{\text{pound}} = \_\_\_\_\_\_\_\_ \text{ounces}$$

$$\frac{2}{1} \cdot \frac{16 \text{ ounces}}{1} = 32 \text{ ounces}$$

$$\therefore 2 \text{ pounds} = 32 \text{ ounces}$$

Add the additional 3 ounces to find the weight of the package. The package weighs 35 ounces.

$$32 \text{ ounces} + 3 \text{ ounces} = 35 \text{ ounces}$$

There are 34 additional ounces, since 35 – 1 = 34.

Apply the pricing formula. $0.44 for the first ounce and $0.17 for each additional ounce.

$$\$0.44(1) + \$0.17(34)$$

$$\$0.44 + \$5.78$$

$$\$0.44 + \$5.78 = \$6.22$$

**Answer**: It will cost $6.22 to mail a package that weighs 2 pounds 3 ounces.

### Try It 1.1.4

The average weight of a northern Bluefin tuna is 1800 pounds. The average weight of a great white shark is $2\frac{1}{2}$ tons. On average, how much more does a great white shark weigh, in pounds, than a northern Bluefin tuna?

**Answer**

$2\frac{1}{2}$ tons = 5000 pounds. 5000 pounds – 1800 pounds = 3200 pounds.

## Summary

In the U.S. customary system of measurement, weight is measured in three units: ounces, pounds, and tons. A pound is equivalent to 16 ounces, and a ton is equivalent to 2000 pounds. While an object's weight can be described using any of these units, it is typical to describe very heavy objects using tons and very light objects using an ounce. Pounds are used to describe the weight of many objects and people. Often, in order to compare the weights of two objects or people or to solve problems involving weight, you must convert from one unit of measurement to another unit of measurement. Using conversion factors with the factor label method is an effective strategy for converting units and solving problems.

## Capacity

## Introduction

**Capacity** is the amount of liquid (or other pourable substance) that an object can hold when it's full. When a liquid, such as milk, is being described in gallons or quarts, this is a measure of capacity.

Understanding units of capacity can help you solve problems like this: Sven and Johanna were hosting a potluck dinner. They did not ask their guests to tell them what they would be bringing, and three people ended up bringing soup. Erin brought 1 quart, Richard brought 3 pints, and LeVar brought 9 cups. How many cups of soup did they have all together?

## Units of Capacity

There are five main units for measuring capacity in the U.S. customary measurement system. The smallest unit of measurement is a **fluid ounce**. "Ounce" is also used as a measure of weight, so it is important to use the word "fluid" with ounce when you are talking about capacity. Sometimes the prefix "fluid" is not used when it is clear from the context that the measurement is capacity, not weight.

The other units of capacity in the customary system are the **cup**, **pint**, **quart**, and **gallon**. The table below describes each unit of capacity and provides an example to illustrate the size of the unit of measurement.

| | |
|---|---|
| **Fluid Ounce**<br>A unit of capacity equal to $\frac{1}{8}$ of a cup.<br>One fluid ounce of water at 62°F weighs about one ounce. The amount of liquid medicine is often measured in fluid ounces. |  |
| **Cup**<br>A unit equal to 8 fluid ounces. The capacity of a standard measuring cup is one cup. |  |
| **Pint**<br>A unit equal to 16 fluid ounces, or 2 cups. The capacity of a carton of ice cream is often measured in pints. |  |
| **Quart**<br>A unit equal to 32 fluid ounces, or 4 cups. You often see quarts of milk being sold in the supermarket. |  |
| **Gallon**<br>A unit equal to 4 quarts, or 128 fluid ounces. When you fill up your car with gasoline, the price of gas is often listed in dollars per gallon. |  |

You can use any of these five measurement units to describe the capacity of an object, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the capacity of a swimming pool in gallons and the capacity of an expensive perfume in fluid ounces. However, unlike the units for length and weight, the units of capacity are not that far apart and often multiple units would be appropriate for one situation.

Sometimes you will need to convert between units of measurement. For example, you might want to express 5 gallons of lemonade in cups if you are trying to determine how many 8-fluid ounce servings the amount of lemonade would yield.

The table below shows some of the most common equivalents and conversion factors for the five customary units of measurement of capacity.

| Unit Equivalents | Conversion Factors (heavier to lighter units of measurement) | Conversion Factors (lighter to heavier units of measurement) |
|---|---|---|
| 1 cup = 8 fluid ounces | $\frac{1\text{ cup}}{8\text{ fluid ounces}}$ | $\frac{8\text{ fluid ounces}}{1\text{ cup}}$ |
| 1 pint = 2 cups | $\frac{1\text{ pint}}{2\text{ cups}}$ | $\frac{2\text{ cups}}{1\text{ pint}}$ |
| 1 quart = 2 pints | $\frac{1\text{ quart}}{2\text{ pints}}$ | $\frac{2\text{ pints}}{1\text{ quart}}$ |
| 1 quart = 4 cups | $\frac{1\text{ quart}}{4\text{ cups}}$ | $\frac{4\text{ cups}}{1\text{ quart}}$ |
| 1 gallon = 4 quarts | $\frac{1\text{ gallon}}{4\text{ quarts}}$ | $\frac{4\text{ quarts}}{1\text{ gallon}}$ |
| 1 gallon = 16 cups | $\frac{1\text{ gallon}}{16\text{ cups}}$ | $\frac{16\text{ cups}}{1\text{ gallon}}$ |

## Converting Between Units of Capacity

As with converting units of length and weight, you can use the factor label method to convert from one unit of capacity to another. An example of this method is shown below.

### ✓ Example 1.1.10

How many pints is $2\frac{3}{4}$ gallons?

**Solution**

Begin by reasoning about your answer. Since a gallon is larger than a pint, expect the answer in pints to be a number greater than $2\frac{3}{4}$.

$$2\frac{3}{4}\text{ gallons} = \underline{\hspace{3em}}\text{ pints}$$

The table above does not contain a conversion factor for gallons and pints, so you cannot convert it in one step. However, you can use quarts as an intermediate unit, as shown here. Set up the equation so that two sets of labels cancel—gallons and quarts.

$$\frac{11\text{ gallons}}{4}\cdot\frac{4\text{ quarts}}{1\text{ gallon}}\cdot\frac{2\text{ pints}}{1\text{ quart}} = \underline{\hspace{3em}}\text{ pints}$$

$$\frac{11}{4}\cdot\frac{4}{1}\cdot\frac{2\text{ pints}}{1} = \underline{\hspace{3em}}\text{ pints}$$

Multiply and simplify.

$$\frac{11\cdot 4\cdot 2\text{ pints}}{4\cdot 1\cdot 1} = \underline{\hspace{3em}}\text{ pints}$$

$$\frac{88\text{ pints}}{4} = 22\text{ pints}$$

**Answer**: $2\frac{3}{4}$ gallons is 22 pints.

### ✓ Example 1.1.11

How many gallons is 32 fluid ounces?

**Solution**

Begin by reasoning about your answer. Since gallons is a larger unit than fluid ounces, expect the answer to be less than 32.

$$32 \text{ fluid ounces} = \underline{\hspace{3em}} \text{ gallons}$$

The table above does not contain a conversion factor for gallons and fluid ounces, so you cannot convert it in one step. Use a series of intermediate units, as shown here.

$$\frac{32 \text{ fluid ounces}}{1} \cdot \frac{1 \text{ cup}}{8 \text{ fluid ounces}} \cdot \frac{1 \text{ pint}}{2 \text{ cups}} \cdot \frac{1 \text{ quart}}{2 \text{ pints}} \cdot \frac{1 \text{ gallon}}{4 \text{ quarts}} \quad \underline{\hspace{3em}} \text{ gallons}$$

Cancel units that appear in both the numerator and denominator.

$$\frac{32}{1} \cdot \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1 \text{ gallon}}{4} = \underline{\hspace{3em}} \text{ gallons}$$

Multiply and simplify.

$$\frac{32 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \text{ gallon}}{1 \cdot 8 \cdot 2 \cdot 2 \cdot 4} = \underline{\hspace{3em}} \text{ gallons}$$

$$\frac{32 \text{ gallons}}{128} = \frac{1}{4} \text{ gallons}$$

**Answer**: 32 fluid ounces is the same as $\frac{1}{4}$ gallon.

## Try It 1.1.5

Find the sum of 4 gallons and 2 pints. Express your answer in cups.

**Answer**

Each gallon has 16 cups, so $4 \cdot 16 = 64$ will give you the number of cups in 4 gallons. Each pint has 2 cups, so $2 \cdot 2 = 4$ will give you the number of cups in 2 pints. $64 + 4 = 68$ cups.

## Applying Unit Conversions

There are times when you will need to combine measurements that are given in different units. In order to do this, you need to convert first so that the units are the same.

Consider the situation posed earlier in this topic.

## Example 1.1.12

Sven and Johanna were hosting a potluck dinner. They did not ask their guests to tell them what they would be bringing, and three people ended up bringing soup. Erin brought 1 quart, Richard brought 3 pints, and LeVar brought 9 cups. How much soup did they have total?

### Solution

Since the problem asks for the total amount of soup, you must add the three quantities. Before adding, you must convert the quantities to the same unit.

$$1 \text{ quart} + 3 \text{ pints} + 9 \text{ cups}$$

The problem does not require a particular unit, so you can choose. Cups might be the easiest computation because going to a smaller unit will multiply the numbers and not divide and yield fractions.

Converting 1 quart to cups is given in the table of equivalents.

$$1 \text{ quart} = 4 \text{ cups}$$

Use the factor label method to convert pints to cups.

$$\frac{3 \text{ pints}}{1} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} = \underline{\hspace{3em}} \text{cups}$$

$$\frac{3}{1} \cdot \frac{2 \text{ cups}}{1} = 6 \text{ cups}$$

Add the 3 quantities.

$$4 \text{ cups} + 6 \text{ cups} + 9 \text{ cups} = 19 \text{ cups}$$

**Answer**: There are 19 cups of soup for the dinner.

### ✓ Example 1.1.13

Natasha is making lemonade to bring to the beach. She has two containers. One holds one gallon and the other holds 2 quarts. If she fills both containers, how many cups of lemonade will she have?

**Solution**

This problem requires you to find the sum of the capacity of each container and then convert that sum to cups.

$$1 \text{ gallon} + 2 \text{ quarts} = \underline{\qquad\qquad} \text{ cups}$$

First, find the sum in quarts. 1 gallon is equal to 4 quarts.

$$4 \text{ quarts} + 2 \text{ quarts} = 6 \text{ quarts}$$

Since the problem asks for the capacity in cups, convert 6 quarts to cups.

$$\frac{6 \text{ quarts}}{1} \cdot \frac{2 \text{ pints}}{1 \text{ quart}} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} = \underline{\qquad\qquad} \text{ cups}$$

Cancel units that appear in both the numerator and denominator.

$$\frac{6}{1} \cdot \frac{2}{1} \cdot \frac{2 \text{ cups}}{1} = \underline{\qquad\qquad} \text{ cups}$$

Multiply.

$$6 \cdot 2 \cdot 2 = 24 \text{ cups}$$

**Answer**: Natasha will have 24 cups of lemonade.

Another way to work the problem above would be to first change 1 gallon to 16 cups and change 2 quarts to 8 cups. Then add: 16 + 8 = 24 cups.

### ✎ Try It 1.1.6

Alan is making chili. He is using a recipe that makes 24 cups of chili. He has a 5-quart pot and a 2- gallon pot and is trying to determine whether the chili will all fit in one of these pots. Which of the pots will fit the chili?

**Answer**

The chili will fit into the 2-gallon pot only. $5 \text{ quarts } = 5 \cdot 4 \text{ cups } = 20 \text{ cups}$ , so 24 cups of chili will not fit into the 5-quart pot. $2 \text{ gallons } = 32 \text{ cups}$ , so 24 cups of chili will fit in this pot.

## Summary

There are five basic units for measuring capacity in the U.S. customary measurement system. These are the fluid ounce, cup, pint, quart, and gallon. These measurement units are related to one another, and capacity can be described using any of the units. Typically, people use gallons to describe larger quantities and fluid ounces, cups, pints, or quarts to describe smaller quantities. Often, in order to compare or to solve problems involving the amount of liquid in a container, you need to convert from one unit of measurement to another.

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# 1.2: Metric Units of Measurement

**Learning Objectives**

1. Describe the general relationship between the U.S. customary units and metric units of length, weight/mass, and capacity.
2. Define the metric prefixes and use them to perform basic conversions among metric units.
3. Perform arithmetic calculations on metric units of length, mass, and capacity.
4. Solve application problems involving metric units of length, mass, and volume.

## Introduction

In the United States, both the **U.S. customary measurement system** and the **metric system** are used, especially in medical, scientific, and technical fields. In most other countries, the metric system is the primary system of measurement. If you travel to other countries, you will see that road signs list distances in kilometers and milk is sold in liters. People in many countries use words like “kilometer,” “liter,” and “milligram” to measure the length, capacity, and weight of different objects. These measurement units are part of the metric system.

Unlike the U.S. customary system of measurement, the metric system is based on 10s. For example, a liter is 10 times larger than a deciliter, and a centigram is 10 times larger than a milligram. This idea of “10” is not present in the U.S. customary system—there are 12 inches in a foot, and 3 feet in a yard…and 5280 feet in a mile! The benefit of the metric system is that the conversion works identically for any unit, i.e. converting grams to kilograms is the same as converting meters to kilometers. This system is not limited to units describing length, capacity, and weight, but also applies to units measuring time (seconds), current (amperes), power (watts) and many more.

So, what if you have to find out how many milligrams are in a decigram? Or, what if you want to convert meters to kilometers? Understanding how the metric system works is a good start.

## What is Metric?

The metric system uses units such as meter, liter, and gram to measure length, capacity, and mass, just as the U.S. customary system uses feet, quarts, and ounces to measure these.

In addition to the difference in the basic units, the metric system is based on factors of 10, and different measures for length include kilometer, meter, decimeter, centimeter, and millimeter. Notice that the word “meter” is part of all of these units. The prefix determines the "size" of the unit and indicates a fixed power of 10. For example "kilo" represents $10^3 = 1000$.

The metric system also applies the idea that units within the system get larger or smaller by a power of 10. This means that a meter is 100 times larger than a centimeter, and a kilogram is 1000 times heavier than a gram. You will explore this idea a bit later. For now, notice how this idea of “getting bigger or smaller by 10” is very different than the relationship between units in the U.S. customary system, where 3 feet equals 1 yard, and 16 ounces equals 1 pound.

## Length, Mass, and Capacity

The table below shows the basic units of the metric system. Note that the names of all metric units follow from these three basic units.

| Length | Mass | Capacity |
|---|---|---|
| | *basic units* | |
| meter | gram | liter |
| | *other units you may see* | |
| kilometer | kilogram | kiloliter |
| centimeter | centigram | centiliter |
| millimeter | milligram | milliliter |

In the metric system, the basic unit of length is the meter. A meter is slightly larger than a yardstick, or just over three feet.

The basic metric unit of mass is the gram. A regular-sized paperclip has a mass of about 1 gram.

Among scientists, one gram is defined as the mass of water that would fill a 1-centimeter cube. You may notice that the word "mass" is used here instead of "weight." In the sciences and technical fields, a distinction is made between weight and mass. Weight is a measure of the pull of gravity on an object. For this reason, an object's weight would be different if it was weighed on Earth or on the moon because of the difference in the gravitational forces. However, the object's mass would remain the same in both places because mass measures the amount of substance in an object. As long as you are planning on only measuring objects on Earth, you can use mass/weight fairly interchangeably—but it is worth noting that there is a difference!

Finally, the basic metric unit of capacity is the liter. A liter is slightly larger than a quart.

|  |  |  |
|---|---|---|
| The handle of a shovel is about 1 meter. | A paperclip weighs about 1 gram. | A medium-sized container of milk is about 1 liter. |

Though it is rarely necessary to convert between the customary and metric systems, sometimes it helps to have a mental image of how large or small some units are. The table below shows the relationship between some common units in both systems.

| | **Common Measurements in Customary and Metric Systems** |
|---|---|
| *Length* | 1 centimeter is a little less than half an inch. |
| | 1.6 kilometers is about 1 mile. |
| | 1 meter is about 3 inches longer than 1 yard. |
| *Mass* | 1 kilogram is a little more than 2 pounds |
| | 28 grams is about the same as 1 ounce. |
| *Capacity* | 1 liter is a little more than 1 quart. |
| | 4 liters is a little more than 1 gallon. |

## Prefixes in the Metric System

The metric system is a base-10 system. This means that each successive unit is 10 times larger than the previous one.

The names of metric units are formed by adding a prefix to the basic unit of measurement. To tell how large or small a unit is, you look at the prefix. To tell whether the unit is measuring length, mass, or capacity, you look at the base.

**Prefixes in the Metric System**

| *kilo-* | *hecto-* | *deka-* | *meter/gram/liter* | *deci-* | *centi-* | *milli-* |
|---|---|---|---|---|---|---|
| 1000 times **larger** than base unit | 100 times **larger** than base unit | 10 times **larger** than base unit | base units | 10 times **smaller** than base unit | 100 times **smaller** than base unit | 1000 times **smaller** than base unit |

Using this table as a reference, you can see the following:

- A kilogram is 1000 times larger than one gram (so 1 kilogram = 1000 grams).
- A centimeter is 100 times smaller than one meter (so 1 meter = 100 centimeters).
- A dekaliter is 10 times larger than one liter (so 1 dekaliter = 10 liters).

Here is a similar table that just shows the metric units of measurement for mass, along with their size relative to 1 gram (the base unit). The common abbreviations for these metric units have been included as well.

**Prefixes in the Metric System**

| kilogram (kg) | hectogram (hg) | dekagram (dag) | gram (g) | decigram (dg) | centigram (cg) | milligram (mg) |
|---|---|---|---|---|---|---|
| 1000 gram | 100 gram | 10 gram | gram | 0.1 gram | 0.01 gram | 0.001 gram |

Since the prefixes remain constant through the metric system, you could create similar charts for length and capacity. The prefixes have the same meanings whether they are attached to the units of length (meter), mass (gram), or capacity (liter).

### Try It 1.2.1

Which of the following sets of three units are all metric measurements of length?

A. inch, foot, yard
B. kilometer, centimeter, millimeter
C. kilogram, gram, centigram
D. kilometer, foot, decimeter

**Answer**

A. Incorrect. Although these units do measure length, they are all units of measurement from the U.S. customary system.
B. Correct. All of these measurements are from the metric system. You can tell they are measurements of length because they all contain the word “meter.”
C. Incorrect. These measurements are from the metric system, but they are measurements of mass, not length.
D. Incorrect. Kilometer and decimeter are metric units of length, but foot is not.

## Converting Units Up and Down the Metric Scale

Converting between metric units of measure requires knowledge of the metric prefixes and an understanding of the decimal system —that’s about it.

For instance, you can figure out how many centigrams are in one dekagram by using the table above. One dekagram is larger than one centigram, so you expect that one dekagram will equal many centigrams.

In the table, each unit is 10 times larger than the one to its immediate right. This means that 1 dekagram = 10 grams; 10 grams = 100 decigrams; and 100 decigrams = 1000 centigrams. So, 1 dekagram = 1000 centigrams.

### Example 1.2.1

How many milligrams are in one decigram?

**Solution**

Identify locations of milligrams and decigrams.

| | | | | | | |
|---|---|---|---|---|---|---|
| kg | hg | dag | g | **dg** | cg | **mg** |

Decigrams (dg) are larger than milligrams (mg), so you expect there to be many mg in one dg.

dg is 10 times larger than a cg, and a cg is 10 times larger than a mg.

| | | | | | | |
|---|---|---|---|---|---|---|
| kg | hg | dag | g | **dg** | cg | **mg** |
| | | | | → | → → → | → |
| | | | | | x10 | x10 |

Since you are going from a larger unit to a smaller unit, multiply.

Multiply: $1 \cdot 10 \cdot 10$, to find the number of milligrams in one decigram.

$$1 \text{ dg} \cdot 10 \cdot 10 = 100 \text{ mg}$$

**Answer**: There are 100 milligrams (mg) in 1 decigram (dg).

## Example 1.2.2

Convert 1 centimeter to kilometers.

### Solution

Identify locations of kilometers and centimeters.

| **km** | hm | dam | m | dm | **cm** | mm |
|---|---|---|---|---|---|---|

Kilometers (km) are larger than centimeters (cm), so you expect there to be less than one km in a cm.

cm is 10 times smaller than a dm; a dm is 10 times smaller than a m, etc.

| **km** | hm | dam | m | dm | **cm** | mm |
|---|---|---|---|---|---|---|
| ← | ← ← ← | ← ← ← | ← ← ← | ← ← ← | ← | |
| ÷10 | ÷10 | ÷10 | ÷10 | ÷10 | | |

Since you are going from a smaller unit to a larger unit, divide.

Divide: $1 \div 10 \div 10 \div 10 \div 10 \div 10$ , to find the number of kilometers in one centimeter.

$$1 \text{ cm} \div 10 \div 10 \div 10 \div 10 \div 10 = 0.00001 \text{ km}$$

**Answer**: 1 centimeter (cm) = 0.00001 kilometers (km).

Once you begin to understand the metric system, you can use a shortcut to convert among different metric units. The size of metric units increases tenfold as you go up the metric scale. The decimal system works the same way: a tenth is 10 times larger than a hundredth; a hundredth is 10 times larger than a thousandth, etc. By applying what you know about decimals to the metric system, converting among units is as simple as moving decimal points.

Here is the first problem from above: How many milligrams are in one decigram? You can recreate the order of the metric units as shown below:

kg hg dag g dg cg mg

This question asks you to start with 1 decigram and convert that to milligrams. As shown above, milligrams is two places to the right of decigrams. You can just move the decimal point two places to the right to convert decigrams to milligrams: $1.\text{ dg} = 100.\text{ mg}$ (Note the location of the decimal points).

The same method works when you are converting from a smaller to a larger unit, as in the problem: Convert 1 centimeter to kilometers.

km hm dam m dm cm mm

Note that instead of moving to the right, you are now moving to the left—so the decimal point must do the same: $1.\text{ cm} = 0.00001\text{ km}$.

## Try It 1.2.2

How many milliliters are in 1 liter?

**Answer**

There are 10 milliliters in a centiliter, 10 centiliters in a deciliter, and 10 deciliters in a liter. Multiply: $10 \cdot 10 \cdot 10$, to find the number of milliliters in a liter, which comes out 1000 milliliters in a liter.

## Summary

The metric system is an alternative system of measurement used in most countries, as well as in the United States. The metric system is based on joining one of a series of prefixes, including kilo- , hecto-, deka-, deci-, centi-, and milli-, with a base unit of measurement, such as meter, liter, or gram. Units in the metric system are all related by a power of 10, which means that each successive unit is 10 times larger than the previous one. This makes converting one metric measurement to another a straightforward process, and is often as simple as moving a decimal point. It is always important, though, to consider the direction of the conversion. If you are converting a smaller unit to a larger unit, then the decimal point has to move to the left (making your number smaller); if you are converting a larger unit to a smaller unit, then the decimal point has to move to the right (making your number larger).

# Converting within the Metric System

# Introduction

While knowing the different units used in the metric system is important, the real purpose behind learning the metric system is for you to be able to use these measurement units to calculate the size, mass, or capacity of different objects. In practice, it is often necessary to convert one metric measurement to another unit—this happens frequently in the medical, scientific, and technical fields, where the metric system is commonly used.

If you have a prescription for 5000 mg of medicine, and upon getting it filled, the dosage reads 5 g of medicine, did the pharmacist make a mistake?

For a moment, imagine that you are a pharmacist. You receive three prescriptions for liquid amoxicillin: one calls for 2.5 centiliters, one calls for 0.3 deciliters, and one calls for 450 milliliters. Amoxicillin is stored in the refrigerator in 1 liter, 1 deciliter, and 1 centiliter containers. Which container should you use to ensure you are not wasting any of the unused drug?

To solve this problem, you need to know how to convert from one measurement to another as well as how to add different quantities together. Let’s take a look at how to do this.

## Converting from Larger to Smaller Units

Converting between measurements in the metric system is simply a matter of identifying the unit that you have, the unit that you want to convert to, and then counting the number of units between them. A basic example of this is shown below.

### ✓ Example 1.2.3

Convert 1 kilometer to decimeters

**Solution**

Identify locations of kilometers and decimeters.

| **km** | hm | dam | m | **dm** | cm | mm |
|---|---|---|---|---|---|---|

Kilometers (km) are larger than decimeters (dm), so you expect there to be more than one dm in a km.

| **km** | hm | dam | m | **dm** | cm | mm |
|---|---|---|---|---|---|---|
| → | → → → | → → → | → → → | → | | |
| | x10 | x10 | x10 | x10 | | |

Count the intermediate units, multiplying by 10 as you go. (Since you are going from a larger unit to a smaller unit, you multiply.)

Multiply to find the number of decimeters in one kilometer.

$$1 \text{ km} \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000 \text{ dm}$$

**Answer**: 1 kilometer = 10,000 decimeters

This problem is straightforward because you are converting 1 kilometer to another unit. The example below shows how you would solve this problem if you were asked to convert 8.2 kilometers to decimeters. Notice that most steps are the same; the critical difference is that you multiply by 8.2 in the final step.

### ✓ Example 1.2.4

Convert 8.2 kilometers to decimeters

**Solution**

Identify locations of kilometers and decimeters.

| **km** | hm | dam | m | **dm** | cm | mm |
|---|---|---|---|---|---|---|

Kilometers (km) are larger than decimeters (dm), so you expect there to be more than one dm in a km.

| **km** | hm | dam | m | **dm** | cm | mm |
|---|---|---|---|---|---|---|
| → | → → → | → → → | → → → | → | | |
| | x10 | x10 | x10 | x10 | | |

Count the intermediate units, multiplying by 10 as you go. Since you are going from a larger unit to a smaller unit, multiply.

Multiply to find the number of decimeters in 8.2 kilometers.

$$8.2\text{ km}\cdot 10\cdot 10\cdot 10\cdot 10=82,000\text{ dm}$$

**Answer**: 8.2 kilometer = 82,000 decimeters

You can also apply the rules of base 10 to use the “move the decimal” shortcut method in this example. Notice how decimeters (dm) is four places to the right of kilometers (km); similarly, you move the decimal point four places to the right when converting 8.2 kilometers to decimeters.

km hm dam m dm cm mm

8.2000 km = 82000. dm

8.2 km = 82000 dm

### ✓ Example 1.2.5

Convert 0.55 liters to centiliters.

**Solution**

Count two places from liters to centiliters.

kl hl dal l dl cl ml

In 0.55 l, move the decimal point two places to the right.

0.55 l = 055.cl

0.55 l = 55 cl

**Answer**: 0.55 liters = 55 centiliters

### Try It 1.2.3

How many dekaliters are in 0.5 deciliters?

**Answer**

One deciliter is 100 times smaller than a dekaliter, so you move the decimal point two places to the left to convert 0.5 deciliters to 0.005 dekaliters.

### Converting from Smaller to Larger Units

You can use similar processes when converting from smaller to larger units. When converting a larger unit to a smaller one, you multiply; when you convert a smaller unit to a larger one, you divide. Here is an example.

### ✓ Example 1.2.6

Convert 739 centigrams to grams.

**Solution**

Identify locations of centigrams and grams.

| kg | hg | dag | **g** | dg | **cg** | mg |
|---|---|---|---|---|---|---|

Centigrams (cg) are smaller than grams (g), so you expect there to be less than 739 g in 739 cg.

Count the intermediate units, dividing by 10 as you go. Since you are going from a smaller unit to a larger unit, divide.

| kg | hg | dag | **g** | dg | **cg** | mg |
|---|---|---|---|---|---|---|
|  |  |  | ← | ← ← ← | ← |  |
|  |  |  | ÷10 | ÷10 |  |  |

Since you are going from a smaller unit to a larger unit, divide.

Divide to find the number of grams in 739 centigrams.

$$739 \div 10 \div 10 = 7.39 \text{ g}$$

**Answer**: 739 centigrams = 7.39 grams

Notice that the shortcut method of counting prefixes and moving the decimal the same number of places also works here. Just make sure you are moving the decimal point in the correct direction for the conversion.

### ✓ Example 1.2.7

Convert 205.5 milliliters to kiloliters.

**Solution**

Count six places from milliliters to kiloliters.

kl hl dal l dl cl ml

Milliliter is smaller than kiloliter, so you expect the number 205.5 to get smaller as you move up the metric chart.

In 205.5 ml, move the decimal point six places to the left.

205.5 ml = 0.0002055 kl

**Answer**: 205.5 milliliters = 0.0002055 kiloliters

### ✎ Try It 1.2.4

Convert 3085 milligrams to grams.

**Answer**

> One gram is 1000 times larger than a milligram, so you can move the decimal point in 3085 three places to the left to get 3.085 grams.

## Factor Label Method

There is yet another method that you can use to convert metric measurements—the **factor label method (dimensional analysis)**. You used this method when you were converting measurement units within the U.S. customary system.

The factor label method works the same in the metric system; it relies on the use of unit fractions and the reducing of intermediate units. The table below shows some of the **unit equivalents** and **unit fractions** for length in the metric system. (You should notice that all of the unit fractions contain a factor of 10. Remember that the metric system is based on the notion that each unit is 10 times larger than the one that came before it.)

Also, notice that two new prefixes have been added here: mega- (which is very big) and micro- (which is very small). Mega is abbbreviated with the prefix capital "M" (since lower case "m" is milli} and micro is represented with the prefix of the greek letter "mu" written with the symbol $\mu$

| Unit Equivalents | Conversion Factors | |
|---|---|---|
| 1 meter = 1,000,000 micrometers | $\frac{1\ \mathrm{m}}{1,000,000\ \mu\mathrm{m}}$ | $\frac{1,000,000\ \mu\mathrm{m}}{1\ \mathrm{m}}$ |
| 1 meter = 1000 millimeters | $\frac{1\ \mathrm{m}}{1000\ \mathrm{mm}}$ | $\frac{1000\ \mathrm{mm}}{1\ \mathrm{m}}$ |
| 1 meter = 100 centimeters | $\frac{1\ \mathrm{m}}{100\ \mathrm{cm}}$ | $\frac{100\ \mathrm{cm}}{1\ \mathrm{m}}$ |
| 1 meter = 10 decimeters | $\frac{1\ \mathrm{m}}{10\ \mathrm{dm}}$ | $\frac{10\ \mathrm{dm}}{1\ \mathrm{m}}$ |
| 1 dekameter = 10 meters | $\frac{1\ \mathrm{dam}}{10\ \mathrm{m}}$ | $\frac{10\ \mathrm{m}}{1\ \mathrm{dam}}$ |
| 1 hectometer = 100 meters | $\frac{1\ \mathrm{hm}}{100\ \mathrm{m}}$ | $\frac{100\ \mathrm{m}}{1\ \mathrm{hm}}$ |
| 1 kilometer = 1000 meters | $\frac{1\ \mathrm{km}}{1000\ \mathrm{m}}$ | $\frac{1000\ \mathrm{m}}{1\ \mathrm{km}}$ |
| 1 megameter = 1,000,000 meters | $\frac{1\ \mathrm{Mm}}{1,000,000\ \mathrm{m}}$ | $\frac{1,000,000\ \mathrm{m}}{1\ \mathrm{Mm}}$ |

When applying the factor label method in the metric system, be sure to check that you are not skipping over any intermediate units of measurement.

### ✓ Example 1.2.8

Convert 7225 centimeters to meters.

**Solution**

Meters is larger than centimeters, so you expect your answer to be less than 7225.

$$7225\ \mathrm{cm} = \underline{\hspace{2cm}}\mathrm{m}$$

Using the factor label method, write 7225 cm as a fraction and use unit fractions to convert it to m.

$$\frac{7225\ \mathrm{cm}}{1} \cdot \frac{1\ \mathrm{m}}{100\ \mathrm{cm}} = \underline{\hspace{2cm}}\mathrm{m}$$

Cancel similar units, multiply, and simplify.

$$\frac{7225}{1} \cdot \frac{1\ \mathrm{m}}{100} = \frac{7225}{100}\ \mathrm{m}$$

$$\frac{7225\text{ m}}{100}=72.25\text{ m}$$

**Answer**: 7225 centimeters = 72.25 meters

### Try It 1.2.5

Using whichever method you prefer, convert 32.5 kilometers to meters.

**Answer**

To find the number of meters in 32.5 kilometers, you can set up the following conversion: $\frac{32.5\text{ kilometers}}{1}\cdot\frac{1000\text{ meters}}{1\text{ kilometer}}=\frac{32,500\text{ meters}}{1}$. The kilometer units cancel, leaving the answer as 32,500 meters.

Now that you have seen how to convert among metric measurements in multiple ways, let's revisit the problem posed earlier.

### Example 1.2.9

If you have a prescription for 5000 mg of medicine, and upon getting it filled, the dosage reads 5 g of medicine, did the pharmacist make a mistake?

**Solution**

Need to convert mg to g.

$$5000\text{ mg}=\rule{2cm}{0.4pt}\text{g?}$$

$$\frac{5000\text{ mg}}{1}\cdot\frac{1\text{ g}}{1000\text{ mg}}=\rule{2cm}{0.4pt}\text{g}$$

$$\frac{5000}{1}\cdot\frac{1\text{ g}}{1000}=\_\_\_\_\_\_\_\_\text{ g}$$

$$\frac{5000\cdot 1\text{ g}}{1\cdot 1000}=\frac{5000\text{ g}}{1000}$$

$$\frac{5000\text{ g}}{1000}=5\text{ g}$$

**Answer**: 5 g = 5000 mg, so the pharmacist did not make a mistake.

### Summary

To convert among units in the metric system, identify the unit that you have, the unit that you want to convert to, and then count the number of units between them. If you are going from a larger unit to a smaller unit, you multiply by 10 successively. If you are going from a smaller unit to a larger unit, you divide by 10 successively. The factor label method can also be applied to conversions within the metric system. To use the factor label method, you multiply the original measurement by unit fractions; this allows you to represent the original measurement in a different measurement unit.

## Using Metric Conversions to Solve Problems

### Introduction

Learning how to solve real-world problems using metric conversions is as important as learning how to do the conversions themselves. Mathematicians, scientists, nurses, and even athletes are often confronted with situations where they are presented with information using metric measurements, and must then make informed decisions based on that data.

To solve these problems effectively, you need to understand the context of a problem, perform conversions, and then check the reasonableness of your answer. Do all three of these steps and you will succeed in whatever measurement system you find yourself using.

## Understanding Context and Performing Conversions

The first step in solving any real-world problem is to understand its context. This will help you figure out what kinds of solutions are reasonable (and the problem itself may give you clues about what types of conversions are necessary). Here is an example.

### ✓ Example 1.2.10

In the Summer Olympic Games, athletes compete in races of the following lengths: 100 meters, 200 meters, 400 meters, 800 meters, 1500 meters, 5000 meters and 10,000 meters. If a runner were to run in all these races, how many kilometers would he run?

**Solution**

To figure out how many kilometers he would run, you need to first add all of the lengths of the races together and *then* convert that measurement to kilometers.

$$10000\text{ m}+5000\text{ m}+1500\text{ m}+800\text{ m}+400\text{ m}+200\text{ m}+100\text{ m}=18000\text{ m}$$

Use the factor label method and unit fractions to convert from meters to kilometers.

$$\frac{18000\text{ m}}{1}\cdot\frac{1\text{ km}}{1000\text{ m}}=\underline{\hspace{2cm}}\text{ km}$$

Cancel, multiply, and solve.

$$\frac{18000}{1}\cdot\frac{1\text{ km}}{1000}=\frac{18000\text{ km}}{1000}$$

$$\frac{18000\text{ km}}{1000}=18\text{ km}$$

**Answer**: The runner would run 18 kilometers.

This may not be likely to happen (a runner would have to be quite an athlete to compete in all of these races) but it is an interesting question to consider. The problem required you to find the total distance that the runner would run (in kilometers). The example showed how to add the distances, in meters, and then convert that number to kilometers.

An example with a different context, but still requiring conversions, is shown below.

### ✓ Example 1.2.11

One bottle holds 295 dl while another one holds 28,000 ml. What is the difference in capacity between the two bottles?

**Solution**

The two measurements are in different units. You can convert both units to liters and then compare them

$$295\text{ dl}=\underline{\hspace{1.8cm}}\text{l}$$

$$28000\text{ ml}=\underline{\hspace{1.8cm}}\text{l}$$

Convert dl to liters.

$$\frac{295\text{ dl}}{1}\cdot\frac{1\text{ l}}{10\text{ dl}}=\underline{\hspace{2cm}}\text{l}$$

Cancel common units and multiply.

$$\frac{295}{1}\cdot\frac{1\text{ l}}{10}=\underline{\hspace{2cm}}\text{l}$$

295 dl = 29.5 liters.

$$\frac{295\text{ l}}{10}=29.5\text{ l}$$

Convert ml to liters.

$$\frac{28000\text{ ml}}{1}\cdot\frac{1\text{ l}}{1000\text{ ml}}=\underline{\hspace{1.8cm}}\text{l}$$

$$\frac{28000}{1}\cdot\frac{1\text{ l}}{1000}=\frac{28000\text{ l}}{1000}$$

28,000 ml = 28 liters

$$\frac{28000\text{ l}}{1000}=28\text{ l}$$

The question asks for “difference in capacity” between the bottles.

$$29.5\text{ liters}–28\text{ liters}=1.5\text{ liters}$$

**Answer**: There is a difference in capacity of 1.5 liters between the two bottles.

This problem asked for the difference between two quantities. The easiest way to find this is to convert one quantity so that both quantities are measured in the same unit, and then subtract one from the other. Here you could have also chosen to convert both numbers to deciliters or both numbers to milliliters instead of liters.

### Try It 1.2.6

One boxer weighs in at 85 kg. He is 80 dag heavier than his opponent. How much does his opponent weigh?

**Answer**

His opponent weights 84.2 kg since $80\text{ dekagrams }=0.8\text{ kilograms}$, and $85-0.8=84.2$.

## Checking your Conversions

Sometimes it is a good idea to check your conversions using a second method. This usually helps you catch any errors that you may make, such as using the wrong unit fractions or moving the decimal point the wrong way.

### Example 1.2.12

A two-liter bottle contains 87 centiliters of oil and 4.1 deciliters of water. How much more liquid is needed to fill the bottle?

**Solution**

You are looking for the amount of liquid needed to fill the bottle. Convert both measurements to liters and then solve the problem.

$$87\text{ cl}+4.1\text{ dl}+\underline{\qquad\quad}=2\text{ l}$$

Convert 87 cl to liters.

$$87\text{ cl}=\underline{\qquad\quad}\text{l}$$

$$\frac{87\text{ cl}}{1}\cdot\frac{1\text{ l}}{100\text{ cl}}=\underline{\qquad\quad}\text{l}$$

$$\frac{87}{1}\cdot\frac{1\text{ l}}{100}=\frac{87\text{ l}}{100}$$

$$\frac{87\text{ l}}{100}=0.87\text{ l}$$

Convert 4.1 dl to liters

$$4.1\text{ dl}=\underline{\qquad\quad}\text{l}$$

$$\frac{4.1\text{ dl}}{1}\cdot\frac{1\text{ l}}{10\text{ dl}}=\underline{\qquad\quad}\text{l}$$

$$\frac{4.1}{1}\cdot\frac{1\text{ l}}{10}=\frac{4.1\text{ l}}{10}$$

$$\frac{4.1\text{ l}}{10}=0.41\text{ l}$$

### Subtract to find how much more liquid is needed to fill the bottle.

$$87\text{ cl}+4.1\text{ dl}+\underline{\qquad\quad}=2\text{ l}$$

$$0.87\text{ liter}+0.41\text{ liter}+\underline{\qquad\quad}=2\text{ liters}$$

$$2\text{ liters}-0.87\text{ liter}-0.41\text{ liter}=0.72\text{ liter}$$

**Answer**: The amount of liquid needed to fill the bottle is 0.72 liter.

Having come up with the answer, you could also check your conversions using the quicker “move the decimal” method, shown below.

### ✓ Example 1.2.13

A two-liter bottle contains 87 centiliters of oil and 4.1 deciliters of water. How much more liquid is needed to fill the bottle?

**Solution**

You are looking for the amount of liquid needed to fill the bottle. Convert both measurements to liters and then solve the problem.

$$87\text{ cl}+4.1\text{ dl}+\underline{\qquad\quad}=2\text{ l}$$

Convert 87 cl to liters.

$$87\text{ cl}=\underline{\qquad\quad}\text{l}$$

On the chart, l is two places to the left of cl.

kl hl dal **l** dl **cl** ml

Move the decimal point two places to the left in 87 cl.

.87. cl

87 cl = 0.87 l

Convert 4.1 dl to liters.

$$4.1\text{ dl}=\underline{\qquad\quad}\text{l}$$

On the chart, l is one place to the left of dl.

kl hl dal **l** **dl** cl ml

Move the decimal point one place to the left in 4.1 dl.

.4.1 dl

4.1 dl = 0.41 l

### Subtract to find how much more liquid is needed to fill the bottle.

$$87\text{ cl}+4.1\text{ dl}+\underline{\qquad\quad}=2\text{ l}$$

$$0.87\text{ liter}+0.41\text{ liter}+\underline{\qquad\quad}=2\text{ liters}$$

$$2\text{ liters}-0.87\text{ liter}-0.41\text{ liter}=0.72\text{ liter}$$

**Answer**: The amount of liquid needed to fill the bottle is 0.72 liter.

The initial answer checks out—0.72 liter of liquid is needed to fill the bottle. Checking one conversion with another method is a good practice for catching any errors in scale.

## Summary

Understanding the context of real-life application problems is important. Look for words within the problem that help you identify what operations are needed, and then apply the correct unit conversions. Checking your final answer by using another conversion method (such as the “move the decimal” method, if you have used the factor label method to solve the problem) can cut down on errors in your calculations.

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# 1.3: Temperature and Time

**Learning Objectives**

1. State the freezing and boiling points of water on the Celsius and Fahrenheit temperature scales.
2. Convert temperatures and times using conversion formulas.
3. Perform arithmetic calculations on temperature and time.
4. Solve application problems using temperature and time.

## Introduction

Turn on the television any morning and you will see meteorologists talking about the day's weather forecast. In addition to telling you what the weather conditions will be like (sunny, cloudy, rainy, muggy), they also tell you the day's forecast for high and low temperatures. A hot summer day may reach 100° in Philadelphia, while a cool spring day may have a low of 40° in Seattle.

If you have been to other countries, though, you may notice that meteorologists measure heat and cold differently outside of the United States. For example, a TV weatherman in San Diego may forecast a high of 89°, but a similar forecaster in Tijuana, Mexico —which is only 20 miles south— may look at the same weather pattern and say that the day's high temperature is going to be 32°. What's going on here?

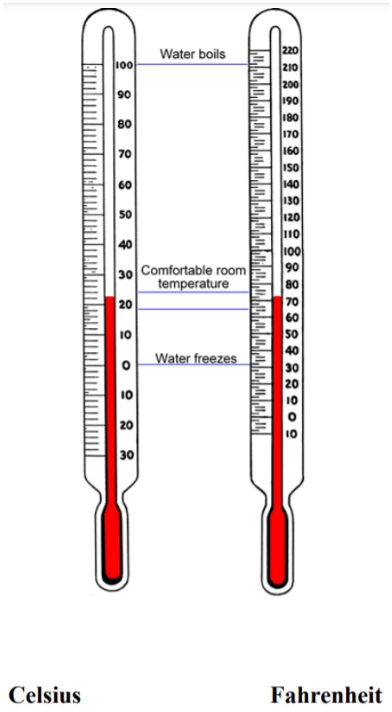
The difference is that the two countries use different temperature scales. In the United States, temperatures are usually measured using the **Fahrenheit** scale, while most countries that use the metric system use the **Celsius** scale to record temperatures. Learning about the different scales— including how to convert between them—will help you figure out what the weather is going to be like, no matter which country you find yourself in.

## Measuring Temperature on Two Scales

**Temperature**

**Temperature** is used to describe the level of heat or cold of an object or environment.

Fahrenheit and Celsius are two different scales for measuring temperature.

| | | |
|---|---|---|
| A thermometer measuring a temperature of 22° Celsius is shown here. |  | A thermometer measuring a temperature of 72° Fahrenheit is shown here. |
| On the Celsius scale, water freezes at 0° and boils at 100°. | | On the Fahrenheit scale, water freezes at 32° and boils at 212°. |
| If the United States were to adopt the Celsius scale, forecast temperatures would rarely go below -30° or above 45°. (A temperature of -18° may be forecast for a cold winter day in Michigan, while a temperature of 43° may be predicted for a hot summer day in Arizona.) | | In the United States, forecast temperatures measured in Fahrenheit rarely go below -20° or above 120°. (A temperature of 0° may be forecast for a cold winter day in Michigan, while a temperature of 110° may be predicted for a hot summer day in Arizona.) |
| Most office buildings maintain an indoor temperature between 18°C and 24°C to keep employees comfortable. | | Most office buildings maintain an indoor temperature between 65°F and 75°F to keep employees comfortable. |

### Try It 1.3.1

A cook puts a thermometer into a pot of water to see how hot it is. The thermometer reads 132°, but the water is not boiling yet. Which temperature scale is the thermometer measuring?

**Answer**

The thermometer is using Fahrenheit. Water boils at 212° on the Fahrenheit scale, so a measurement of 132° on a Fahrenheit scale is legitimate for hot (but non-boiling) water.

## Converting Between the Scales

By looking at the two thermometers shown, you can make some general comparisons between the scales. For example, many people tend to be comfortable in outdoor temperatures between 50°F and 80°F (or between 10°C and 25°C). If a meteorologist predicts an average temperature of 0°C (or 32°F), then it is a safe bet that you will need a winter jacket.

Sometimes, it is necessary to convert a Celsius measurement to its exact Fahrenheit measurement or vice versa. For example, what if you want to know the temperature of your child in Fahrenheit, and the only thermometer you have measures temperature in Celsius measurement? Converting temperature between the systems is a straightforward process as long as you use the formulas provided below.

## Temperature Conversion Formulas

To convert a Fahrenheit measurement to a Celsius measurement, use this formula.

$$C=\frac{5}{9}(F-32) \tag{1.3.1}$$

To convert a Celsius measurement to a Fahrenheit measurement, use this formula.

$$F=\frac{9}{5}C+32 \tag{1.3.2}$$

How were these formulas developed? They came from comparing the two scales. Since the freezing point is 0° in the Celsius scale and 32° on the Fahrenheit scale, we subtract 32 when converting from Fahrenheit to Celsius, and add 32 when converting from Celsius to Fahrenheit.

There is a reason for the fractions $\frac{5}{9}$ and $\frac{9}{5}$, also. There are 100 degrees between the freezing (0°) and boiling points (100°) of water on the Celsius scale and 180 degrees between the similar points (32° and 212°) on the Fahrenheit scale. Writing these two scales as a ratio, $\frac{F^{\circ}}{C^{\circ}}$, gives $\frac{180^{\circ}}{100^{\circ}}=\frac{180^{\circ}\div 20}{100^{\circ}\div 20}=\frac{9}{5}$. If you flip the ratio to be $\frac{C^{\circ}}{F^{\circ}}$, you get $\frac{100^{\circ}}{180^{\circ}}=\frac{100^{\circ}\div 20}{180^{\circ}\div 20}=\frac{5}{9}$. Notice how these fractions are used in the conversion formulas.

The example below illustrates the conversion of Celsius temperature to Fahrenheit temperature, using the boiling point of water, which is 100° C.

### Example 1.3.1

The boiling point of water is 100°C. What temperature does water boil at in the Fahrenheit scale?

**Solution**

A Celsius temperature is given. To convert it to the Fahrenheit scale, use Equation 1.3.2.

$$F=\frac{9}{5}C+32$$

Substitute 100 for $C$ and multiply.

$$F=\frac{9}{5}(100)+32$$

$$F=\frac{900}{5}+32$$

Simplify $\frac{900}{5}$ by dividing numerator and denominator by 5.

$$F=\frac{900\div 5}{5\div 5}+32$$

$$F=\frac{180}{1}+32$$

Add 180 + 32.

$$F=212$$

**Answer**: The boiling point of water is 212°F.

## ✓ Example 1.3.2

Water freezes at 32°F. On the Celsius scale, what temperature is this?

### Solution

A Fahrenheit temperature is given. To convert it to the Celsius scale, use Equation 1.3.1.

$$C=\frac{5}{9}(F-32)$$

Substitute 32 for $F$ and subtract.

$$C=\frac{5}{9}(32-32)$$

Any number multiplied by 0 is 0

$$C=\frac{5}{9}(0)$$

$$C=0$$

**Answer**: The freezing point of water is 0°C.

The two previous problems used the conversion formulas to verify some temperature conversions that were discussed earlier—the boiling and freezing points of water. The next example shows how these formulas can be used to solve a real-world problem using different temperature scales.

## ✓ Example 1.3.3

Two scientists are doing an experiment designed to identify the boiling point of an unknown liquid (remember different liquids have different boiling point). One scientist gets a result of 120°C; the other gets a result of 250°F. Which temperature is higher and by how much?

### Solution

One temperature is given in °C, and the other is given in °F. To find the difference between them, we need to measure them on the same scale.

What is the difference between 120°C and 250°F?

Use the conversion formula to convert 120°C to °F. (You could convert 250°F to °C instead; this is explained in the text after this example.)

$$F=\frac{9}{5}C+32$$

Substitute 120 for C.

$$F=\frac{9}{5}(120)+32$$

Multiply.

$$F=\frac{1080}{5}+32$$

Simplify $\frac{1080}{5}$ by dividing numerator and denominator by 5.

$$F=\frac{1080\div 5}{5\div 5}+32$$

Add 216 + 32.

$$F=\frac{216}{1}+32$$

You have found that 120°C = 248°F.

$$F=248$$

To find the difference between 248°F and 250°F, subtract.

$$250^\circ F – 248^\circ F=2^\circ F$$

**Answer**: 250°F is the higher temperature by 2°F

You could have converted 250°F to °C instead, and then found the difference in the two measurements. (Had you done it this way, you would have found that 250°F = 121.1°C, and that 121.1°C is 1.1°C higher than 120°C.) Whichever way you choose, it is important to compare the temperature measurements within the same scale, and to apply the conversion formulas accurately.

### Try It 1.3.2

Tatiana is researching vacation destinations, and she sees that the average summer temperature in Barcelona, Spain is around 26°C. What is the average temperature in degrees Fahrenheit?

**Answer**

Tatiana can find the Fahrenheit equivalent by solving the equation $F=\frac{9}{5}(26)+32$ . The result is 78.8° Fahrenheit, which rounds to 79° Fahrenheit.

## Time

### Time

**Time** is used to describe the amount of existence between events.

Time is universal--everyone uses the same measurements of time. Some of the common units (along with their abbreviations) for time are listed in the following table.

| Unit Conversion Table |
|---|
| 1 minute (min) = 60 seconds (sec)<br>1 hour ( hr) = 60 minutes<br>1 day (da) = 24 hours<br>1 week (wk) = 7 days<br>1 year (yr)= 52 weeks |

### Example 1.3.4

Convert 2,016 hr to weeks.

**Solution**

Looking in the unit conversion table under *time*, we see that $1 \text{ wk} = 7 \text{ da}$ and that $1 \text{ da} = 24 \text{ hr}$. To convert from hours to weeks, we must first convert from hours to days and then from days to weeks. We need two unit fractions.

The unit fraction needed for converting from hours to days is $\frac{1 \text{ da}}{24 \text{ hr}}$. The unit fraction needed for converting from days to weeks is $\frac{1 \text{ wk}}{7 \text{ da}}$.

$$\begin{aligned} 2{,}016 \text{ hr} &= \frac{2{,}016 \text{ hr}}{1} \cdot \frac{1 \text{ da}}{24 \text{ hr}} \cdot \frac{1 \text{ wk}}{7 \text{ da}} && \text{Divide out common units.} \\ &= \frac{2{,}016 \ \cancel{\text{hr}}}{1} \cdot \frac{1 \ \cancel{\text{da}}}{24 \ \cancel{\text{hr}}} \cdot \frac{1 \text{ wk}}{7 \ \cancel{\text{da}}} \\ &= \frac{2,016 \cdot 1 \text{ wk}}{24 \cdot 7} && \text{Simplify} \\ &= 12 \text{ wk} \end{aligned}$$

Thus, $2{,}016 \text{ hr} = 12 \text{ wk}$.

## Try It 1.3.3

Your employer tracks your accumulated vacation time in hours. You want to take a two week vacation and have accumulated 412 hours of paid time off. Do you have enough hours for your vacation?

**Answer**

To solve this we need to covert 412 hours to weeks.

The unit fraction needed for converting from hours to days is $\frac{1 \text{ da}}{24 \text{ hr}}$. The unit fraction needed for converting from days to weeks is $\frac{1 \text{ wk}}{7 \text{ da}}$.

$$\begin{aligned} 412 \text{ hr} &= \frac{412 \text{ hr}}{1} \cdot \frac{1 \text{ da}}{24 \text{ hr}} \cdot \frac{1 \text{ wk}}{7 \text{ da}} && \text{Divide out common units.} \\ &= \frac{412 \ \cancel{\text{hr}}}{1} \cdot \frac{1 \ \cancel{\text{da}}}{24 \ \cancel{\text{hr}}} \cdot \frac{1 \text{ wk}}{7 \ \cancel{\text{da}}} \\ &= \frac{412 \cdot 1 \text{ wk}}{24 \cdot 7} && \text{Simplify} \\ &= 2.5 \text{ wk} \end{aligned}$$

Thus, $412 \text{ hr} = 2.5 \text{ wk}$.

Yes, you have enough time for a 2.5 week vacation.

## Try It 1.3.4

According to the Global Web Index Survey, the average US resident between 16-64 years of age spent 144 minutes per day on social media in 2021. How many days is that per year? Assume the year has 365 days.

**Answer**

To solve this we need to multiply 144 minutes per day by 365 days to get the total minutes per year.

$144 * 365 = 52,560$minutes for the whole year.

The unit fraction needed for converting from minutes to hours is $\frac{1 \text{ hr}}{60 \text{ min}}$. The unit fraction needed for converting from hours to days is $\frac{1 \text{ da}}{24 \text{ hr}}$.

$$\begin{aligned} 52,560 \text{ min} &= \frac{52,560 \text{ min}}{1} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ da}}{24 \text{ hr}} && \text{Divide out common units.} \\ &= \frac{52,560 \ \cancel{\text{min}}}{1} \cdot \frac{1 \ \cancel{\text{hr}}}{60 \ \cancel{\text{min}}} \cdot \frac{1 \text{ da}}{24 \ \cancel{\text{hr}}} \\ &= \frac{52,560 \cdot 1 \text{ da}}{60 \cdot 24} && \text{Simplify} \\ &= 36.5 \text{ wk} \end{aligned}$$

Thus, $52{,}560 \text{ min} = 36.5 \text{ da}$

The average American spends 36.5 days a year on social media.

## Summary

Temperature is often measured in one of two scales: the Celsius scale and the Fahrenheit scale. A Celsius thermometer will measure the boiling point of water at 100° and its freezing point at 0°; a Fahrenheit thermometer will measure the same events at 212° for the boiling point of water and 32° as its freezing point. You can use conversion formulas to convert a measurement made in one scale to the other scale. Time is universal and is measured in seconds, minutes, hours, days, weeks and years.

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# 1.4: Conversion Between the Metric and US Customary Systems of Measurement

**Learning Objectives**

1. Estimate equivalence between metric units and US customary units.
2. Convert metric units to US customary units.
3. Convert US customary units to metric units.

## Estimate equivalence between the Metric Units and US Customary Units

Before converting metric units to US customary units and vice versa, you need to familiarize the approximate metric equivalents of common customary units. The table below summarizes the commonly used metric and US customary units and their approximate equivalence. The notations enclosed are the abbreviations of each unit.

| **Metric Units** | **US Customary Units** |
|---|---|
| **Lengths** | |
| 1 meter (m) | 3.28 feet (ft or ’) |
| 1 meter (m) | 39.4 inches (in) |
| 1 kilometer (km) | 0.62 mile (mi) |
| 2.54 centimeter (cm) | 1 inch (in) |
| 0.9144 meter (m) | 1 yard (yd) |
| **Weight** | |
| 1 kilogram (kg) | 2.2 pounds (lb) |
| 28.3 grams (g) | 1 ounce (oz) |
| **Fluid Capacity** | |
| 1 milliliter (ml) | 0.203 teaspoon (tsp) |
| 29.57 milliliters (ml) | 1 fluid ounce (fl oz) |
| 1 liter (l) | 4.23 cups (c) |
| 3.79 liters (l) | 1 gallon (gal) |
| 473.176 milliliters (ml) | 1 pint (pt) |
| 0.95 liters (l) | 1 quart (qt) |

### Additional Common Conversions

| **Metric Units** | **US Customary Units** |
|---|---|
| **Area** | |
| 6.45 square centimeters ($cm^2$) | 1 square inch ($in^2$) |
| 1 square meter ($m^2$) | 1.196 square yard ($yd^2$) |
| 4046.873 square meters ($m^2$) | 1 acre (ac) |
| **Volume and Capacity** | |
| 16.39 milliliter (ml) | 1 cubic inch ($in^3$) |
| 28.32 liters (l) | 1 cubic foot ($ft^3$) |
| 764.55 liters (l) | 1 cubic yard ($yd^3$) |

## Convert Metric Units to US Customary Units

You have familiarized the commonly used equivalence between metric and US customary units. At this point, you will learn to convert metric units to US customary units.

To convert metric units to US customary units, use their estimate equivalence as the conversion factor in dimensional analysis. Write the equivalence as a fraction, and multiply it to the given measurement (in metric) to convert. Be sure that the units of measure to convert cancels out. For instance, to convert liters to gallons, be sure to cancel out liters and retain gallons as the final unit.

Now, let us consider some examples.

### ✓ Example 1.4.1

The official regulators of a shipment company classify a parcel weighing more than 51 pounds (lb) as a heavy shipment. Andrea packed 24 terracotta plant containers to be shipped to her customers. If each plant container weighs 1.3 kilograms (kg), will her package be classified as a heavy shipment?

**Solution**

To determine whether the parcel is a heavy shipment or not, you need to convert the total weight of the plant containers to pounds.

Compute for the total weight.

$$\text{Total Weight } = 24 \times 1.3 \text{ kg} = 31.2 \text{ kg}$$

Convert the total weight to pounds. Use the metric to US customary units' equivalence: 1 kg = 2.2 lbs.

$$31.2 \ \cancel{\text{kg}} \cdot \frac{2.2 \text{ lb}}{1 \ \cancel{\text{kg}}} = 68.64 \text{ lb}$$

Since the total weight is more than 51 lb, the shipment is considered heavy.

### Try It 1.4.1

Angelica who lives in Las Vegas, Nevada, visited her friend in Canada. She knows that Canada uses the metric system, so their speed limits are posted in kilometers per hour. On her way, she sees a road sign that says, “max 60 km/h”. She knows the sign means “a maximum speed of 60 kilometers per hour” but she is unsure if this is higher than the speed limit in Las Vegas, Nevada, which is 65 miles per hour (mph). How can Angelica convert 60 km/h to miles per hour? Use the metric to US customary units' equivalence: 1 kilometer (km) = 0.62 mile.

**Answer**

$$\frac{60 \ \cancel{\text{km}}}{1 \text{ hr}} \cdot \frac{0.62 \text{ mi}}{1 \ \cancel{\text{km}}} = 37.2 \text{ mi / hr}$$

In Las Vegas, the limit is 65 miles per hour (mph). Since 37.2 mph is below this limit, 60km/h is not over the speed limit.

### ✓ Example 1.4.2

Half of a meter of ribbon is required for wrapping presents enclosed in a rectangular box. How many presents using the same box size can be wrapped with 10 yards of ribbon?

**Solution**

Convert half a meter (0.5 m) to yards. Use the metric to US customary units' equivalence: 1 yard = 0.9144 meter.

$$0.5 \ \cancel{\text{m}} \cdot \frac{1 \text{ yd}}{0.9144 \ \cancel{\text{m}}} = 0.5468 \text{ yd}$$

The result means that one present requires 0.5468 yards of ribbon. Calculate the number of presents by dividing 10 yards by 0.5468 yards.

$$10\text{ yd} \div \frac{0.5468\text{ yd}}{1\text{ present}} = 10\ \cancel{\text{yd}} \times \frac{1\text{ present}}{0.5468\ \cancel{\text{yd}}} = 18.29\text{ presents}$$

Approximately, 18 presents can be wrapped with 10 yards of ribbons.

### ✎ Try It 1.4.2

Crystal and her friends are planning for a road trip, which they have determined to be 2000 kilometers (km) long. They want to rent a van that can travel 100 km using 9 liters of gas. Suppose the gas price is $3.74 a gallon (gal), how many gallons of gas they will need? How much money will the group of friends need to save for gas? Use the metric to US customary units' equivalence: 3.79 liters (l)= 1 gallon (gal).

**Answer**

They will need $\frac{2000\ \cancel{\text{km}}}{1} \cdot \frac{9\ l}{100\ \cancel{\text{km}}} = 180$ liters of gas.

Converting to gallons: $\frac{180\ \cancel{l}}{1} \cdot \frac{1\text{ gal}}{3.79\ \cancel{l}} = 47.5$ gallons of gas

Calculating the price: $47.5 \times \$3.74 = \$177.63$

The team needs 47.5 gallons of gas for the trip, which will cost them $177.63.

### ✎ Try It 1.4.3

Christine plans to put up a short-course swimming pool at her house. A short-course swimming pool usually measures 22.86 meters in length, at least 18.29 meters in width, and 2.5 meters in depth. Convert the dimensions of the pool to inches. Use the metric to US customary units' equivalence: 1 centimeter (cm) = 0.39 inch (in.) and 1 centimeter (cm) = 0.01 meter (m).

**Answer**

Length: $\frac{22.86\ \cancel{\text{m}}}{1} \cdot \frac{1\ \cancel{\text{cm}}}{0.01\ \cancel{\text{m}}} \cdot \frac{0.39\text{ in}}{1\ \cancel{\text{cm}}} = 891.54\ \text{ inches}$

Width: $\frac{18.29\ \cancel{\text{m}}}{1} \cdot \frac{1\ \cancel{\text{cm}}}{0.01\ \cancel{\text{m}}} \cdot \frac{0.39\text{ in}}{1\ \cancel{\text{cm}}} = 713.31\ \text{ inches}$

Depth: $\frac{2.5\ \cancel{\text{m}}}{1} \cdot \frac{1\ \cancel{\text{cm}}}{0.01\ \cancel{\text{m}}} \cdot \frac{0.39\text{ in}}{1\ \cancel{\text{cm}}} = 97.5\ \text{ inches}$

Approximately, the pool measures 891.54 in by 713.31 in by 97.5 in.

## Convert US Customary Units to Metric Units

You have learned how to convert metric units to US customary units. Now, you will learn to convert US customary units to metric units.

### ✓ Example 1.4.3

Daniel drove 40 miles in two hours. What distance (in meters) did he cover in one minute? Use the US customary to metric units' equivalence: 0.62 mile (mi) = 1 kilometer (km), 1 km is equal to 1000 meters (m), and 1 hour = 60 minutes.

## Solution

Convert 40 miles in two hours to kilometers per hour.

$$\frac{40\ \cancel{\text{mi}}}{2\text{ hrs}} \cdot \frac{1\text{ km}}{0.62\ \cancel{\text{mi}}} = 32.2581\text{ km/hr}$$

Convert 32.2581 kilometers per hour to meters per hour.

$$\frac{32.2581\ \cancel{\text{km}}}{1\ \text{hr}} \cdot \frac{1000\ \text{m}}{1\ \cancel{\text{km}}} = 32,258.1\ \text{m/hr}$$

Convert 32,258.1 meters per hour to meters per minute.

$$\frac{32,258.1\ \text{m}}{1\ \cancel{\text{hr}}} \cdot \frac{1\ \cancel{\text{hr}}}{60\ \text{min}} = 537.635\ \text{m/min}$$

Daniel has covered 537.635 meters in one minute.

### ✎ Try It 1.4.4

A supermarket has a car parking spot that is 9 feet wide and 18 feet long. Its ground-level parking has 50 parallel parking spots (side by side) of the same sizes. What is the area of the parking lot in square centimeters? Use the US customary units' equivalence: 1 foot (ft) = 12 inches (in) and 1 square inch (in$^2$) = 6.45 square centimeters (cm$^2$). (Hint: You will need to use the formula to find the area of a rectangle)

**Answer**

Convert 9 feet to inches: $\frac{9\ \cancel{\text{ft}}}{1} \cdot \frac{12\ \text{in}}{1\ \cancel{\text{ft}}} = 108\ \text{in}$

Convert 18 feet to inches: $\frac{18\ \cancel{\text{ft}}}{1} \cdot \frac{12\ \text{in}}{1\ \cancel{\text{ft}}} = 216\ \text{in}$

Find the area of one parking spot: 108 in x 216 in = 23,328 in$^2$

Find the area of the whole parking lot: 23,328 in$^2$ x 50 = 1,166,400 in$^2$

Convert to cm$^2$: $\frac{1,166,400\ \cancel{\text{in}^2}}{1} \cdot \frac{6.45\ \text{cm}^2}{1\ \cancel{\text{in}^2}} = 7,523,280\ \text{cm}^2$

The area of the parking lot is 7,523,280 square centimeters.

### ✓ Example 1.4.4

Melody's garage is 45 feet long while David's garage is 10 yards and 10 feet long. Which house has a longer garage? How much is the difference (in meters)? Use the US customary to metric equivalence: 3.28 feet (ft) = 1 meter (m) and 1 foot (ft) = 0.3333 yard (yd)

**Solution**

Convert 10 yards to feet: $\frac{10\ \cancel{\text{yd}}}{1} \cdot \frac{1\ \text{ft}}{0.3333\ \cancel{\text{yd}}} = 30\ \text{ft}$

David's garage is 30 ft + 10 ft = 40 ft long. His garage is 5 feet shorter than Melody's garage.

Convert 5 feet to meters: $\frac{5\ \cancel{\text{ft}}}{1} \cdot \frac{1\ \text{m}}{3.28\ \cancel{\text{ft}}} = 1.52\ \text{m}$

Melody's garage is longer by 1.52 meters.

### ✎ Try It 1.4.5

How many liters of water are needed to fill an Olympic swimming pool with a total volume of 3300 cubic yards? Assume that the water level is as deep as the pool. Use the US customary to metric units' equivalence: 1 cubic yard (yd$^3$) = 746.56 liters (l).

**Answer**

Convert cubic yards to liters: $\frac{3300\ \cancel{yd^3}}{1}\cdot\frac{746.56\ l}{1\ \cancel{yd^3}}=2,463,648\,\text{liters}$

At most 2,463,648 liters of water are needed to fill the swimming pool.

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# 1.5: Chapter Review and Glossary

## Chapter Review

### 1.1: Length

The four basic units of measurement that are used for length in the U.S. customary measurement system are: inch, foot, yard, and mile. Typically, people use yards, miles, and sometimes feet to describe long distances. Measurement in inches is common for shorter objects or lengths. You need to convert from one unit of measure to another if you are solving problems that include measurements involving more than one type of measurement. Each of the units can be converted to one of the other units using the table of equivalents, the conversion factors, and/or the factor label method shown in this topic.

### 1.1: Weight

In the U.S. customary system of measurement, weight is measured in three units: ounces, pounds, and tons. A pound is equivalent to 16 ounces, and a ton is equivalent to 2,000 pounds. While an object’s weight can be described using any of these units, it is typical to describe very heavy objects using tons and very light objects using an ounce. Pounds are used to describe the weight of many objects and people. Often, in order to compare the weights of two objects or people or to solve problems involving weight, you must convert from one unit of measurement to another unit of measurement. Using conversion factors with the factor label method is an effective strategy for converting units and solving problems.

### 2.1: Capacity

There are five basic units for measuring capacity in the U.S. customary measurement system. These are the fluid ounce, cup, pint, quart, and gallon. These measurement units are related to one another, and capacity can be described using any of the units. Typically, people use gallons to describe larger quantities and fluid ounces, cups, pints, or quarts to describe smaller quantities. Often, in order to compare or to solve problems involving the amount of liquid in a container, you need to convert from one unit of measurement to another.

### 1.2: The Metric System

The metric system is an alternative system of measurement used in most countries, as well as in the United States. The metric system is based on joining one of a series of prefixes, including kilo- , hecto-, deka-, deci-, centi-, and milli-, with a base unit of measurement, such as meter, liter, or gram. Units in the metric system are all related by a power of 10, which means that each successive unit is 10 times larger than the previous one. This makes converting one metric measurement to another a straightforward process, and is often as simple as moving a decimal point. It is always important, though, to consider the direction of the conversion. If you are converting a smaller unit to a larger unit, then the decimal point has to move to the left (making your number smaller); if you are converting a larger unit to a smaller unit, then the decimal point has to move to the right (making your number larger).

### 2.2: Converting within the Metric System

To convert among units in the metric system, identify the unit that you have, the unit that you want to convert to, and then count the number of units to move between them. If you are going from a larger unit to a smaller unit, you multiply by 10 successively. If you are going from a smaller unit to a larger unit, you divide by 10 successively. You can also move the decimal point of your number the same direction and same number of spots that you need to move from the original unit to the new unit. The factor label method can also be applied to conversions within the metric system. To use the factor label method, you multiply the original measurement by unit fractions; this allows you to represent the original measurement in a different measurement unit.

### 1.2: Using Metric Conversion to Solve Problems

Understanding the context of real-life application problems is important. Look for words within the problem that help you identify what operations are needed, and then apply the correct unit conversions. Checking your final answer by using another conversion method (such as the “move the decimal” method, if you have used the factor label method to solve the problem) can cut down on errors in your calculations.

### 1.3: Temperature and time

Temperature is often measured in one of two scales: the Celsius scale and the Fahrenheit scale. A Celsius thermometer will measure the boiling point of water at 100° and its freezing point at 0°; a Fahrenheit thermometer will measure the same events at

212° for the boiling point of water and 32° as its freezing point. You can use conversion formulas to convert a measurement made in one scale to the other scale. Time is universal and is measured in seconds, minutes, hours, days, weeks and years.

### 1.4: Conversion Between the Metric and US Customary Systems of Measurement

To convert metric units to US customary units or vice versa, use their estimate equivalence as the conversion factor in dimensional analysis. Write the equivalence as a fraction, and multiply it to the given measurement to convert. Be sure that the units of measure to convert cancels out.

## Glossary:

| | |
|---|---|
| **capacity** | The amount of liquid (or other pourable substance) that an object can hold when it's full. |
| **Celsius** | A measure of temperature commonly used in countries that use the metric system. On the Celsius scale, water freezes at 0° and boils at 100°. |
| **cup** | A unit of capacity equal to 8 fluid ounces. |
| **factor label method** | One method of converting a measurement from one unit of measurement to another unit of measurement. In this method, you multiply the original measurement by unit fractions containing different units of measurement to obtain the new unit of measurement. |
| **Fahrenheit** | A measure of temperature commonly used in the United States. On the Fahrenheit scale, water freezes at 32° F and boils at 212° F. |
| **fluid ounce** | A unit of capacity equal to $\frac{1}{8}$ of a cup. One fluid ounce of water at 62°F weighs about one ounce. |
| **foot** | A unit for measuring length in the U.S. customary measurement system. 1 foot = 12 inches |
| **gallon** | A unit equal to 4 quarts, or 128 fluid ounces. |
| **gram** | The base unit of mass in the metric system. |
| **inch** | A unit for measuring length in the U.S. customary measurement system. 1 foot = 12 inches |
| **length** | The distance from one end to the other or the distance from one point to another. |
| **liter** | The base unit of capacity in the metric system. |
| **measurement** | The use of standard units to find out the size or quantity of items such as length, width, height, mass, weight, volume, temperature or time. |
| **meter** | The base unit of length in the metric system. |
| **metric system** | A widely-used system of measurement that is based on the decimal system and multiples of 10. |
| **mile** | A unit for measuring length in the U.S. customary measurement system. 1 mile = 5280 feet or 1760 yards. |
| **ounce** | A unit for measuring weight in the U.S. customary measurement system. 16 ounces = 1 pound. |
| **pint** | A unit of capacity equal to 16 fluid ounces, or 2 cups. |
| **pound** | A unit for measuring weight in the U.S. customary measurement system. 16 ounces = 1 pound. |
| **prefix** | A short set of letters that denote the size of measurement units in the metric system. Metric prefixes include kilo-, hecto-, deka-, deci-, centi-, and milli- |
| **quart** | A unit of capacity equal to 32 fluid ounces, or 4 cups. |
| **ton** | A unit for measuring the weight of heavier items in the U.S. customary measurement system. 1 ton = 2000 pounds. |
| **U.S. customary measurement system** | The most common system of measurement used in the United States. It is based on English measurement systems of the 18th century. |
| **unit equivalents** | Statements of equivalence between measurement units within a system or in comparison to another system of units. For example, 1 foot = 12 inches or 1 inch = 2.54 centimeters are both examples of unit equivalents. |

| | |
|---|---|
| **unit fractions** | A fraction where the numerator and denominator are equal amounts, as in $\frac{1\text{ kg}}{1000\text{ g}}$ or $\frac{12\text{ inches}}{1\text{ foot}}$. Unit fractions serve to help with conversions in the factor label method (dimensional analysis). |
| **unit of measurement** | A standard amount or quantity. For example, an inch is a unit of measurement. |
| **weight** | A mathematical description of how heavy an object is. |
| **yard** | A unit for measuring length in the U.S. customary measurement system. 1 yard = 3 feet or 36 inches. |

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# 1.6: Exercises

For 1 - 26, convert the measurements. *Round your answer up to 3 decimal places, if needed.*

## Exploration

1. 5 yards = _____ feet
2. 6 yards = _____ inches
3. 108 inches = _____ yards
4. 257 miles = _____ feet
5. 253 miles = _____ feet
6. 8 pounds = _____ ounces
7. 370 ounces = _____ ton
8. 417 ounces = _____ ton
9. 112 cups = _____ gallons
10. $8\frac{1}{2}$ gallons = _____ cups
11. 7 quarts = _____ fluid ounces
12. 9 fluid ounces = _____ quarts
13. 8 meters = _____ centimeters
14. 4580 milliliters = _____ liters
15. 3520 milliliters = _____ liters
16. 2 millimeters = _____ meters
17. 171 centimeters = _____ meters
18. 5180 milliliters = _____ liters
19. 2610 grams = _____ kilograms
20. 2 kilograms = _____ grams
21. 58.3 millimeters = _____ centimeters
22. 0.3429 kilograms = _____ milligrams
23. $14^\circ C$ = ______ $^\circ F$
24. $41^\circ F$ = ______ $^\circ C$
25. $12^\circ C$ = ______ $^\circ F$
26. $99^\circ F$ = ______ $^\circ C$
27. Convert the following between U.S. and Metric measurement systems.
    a. 52 inches to centimeters
    b. 2 miles to meters
    c. 11 feet to decimeters
    d. 47 ounces to grams
    e. 4580 milliliters to cups
    f. 22 millimeters to inches
    g. 2.3 tons to kilograms
    h. 2,610 grams to pounds
    i. 7 gallons to liters
    j. 10 cups to quarts
    k. 3.6 quarts to cups
    l. 29 cups to gallons
    m. 3 pints to milliliters
    n. 8 meters to yards
28. What unit of measurement do these abbreviations belong to?
    a. da
    b. wk
    c. F
    d. C

   e. yr
   f. sec
   g. min
   h. hr
29. Convert the following:
   a. $21^\circ\mathrm{C}$ to $^\circ\mathrm{F}$
   b. $81^\circ\mathrm{F}$ to $^\circ\mathrm{C}$
   c. 235,000 sec to yr
   d. 67 hr to da
   e. 19 wk to hr
   f. 4 da to min
30. Australia uses Celsius to measure temperature. Your travel agent said it will be between $23^\circ$ C and $27^\circ$ C during your visit. What is the temperature range in Fahrenheit?
31. You want to binge watch a TV show that has 11 episodes that are 45 minutes each. How many hours do you need to set aside this weekend to watch the entire series?
32. A friend from Switzerland is coming to visit the U.S. in December. You tell them it will be about $35^\circ$ when they're here. You forgot to tell them that the measurement was in degrees Fahrenheit. They are used to Celsius and assume that is the temperature you told them. Are they planning for cold or hot weather? What Celsius temperature, should you have told them?
33. Your aunt always tells you her new child's age in weeks. The last time you saw them she said her child was 93 weeks old. How much is that in years?
34. To tell your colleague in Europe the temperature you are currently experiencing, you need to tell it to them in Celsius. If it is currently $64^\circ\mathrm{F}$, what would you tell them?
35. If you drove 100 km/hour on the freeway, were you exceeding the speed limit?
36. Would you expect to pay more for 10 gallons (40 quarts) of gasoline or for 40 liters of gasoline?
37. Which is the better deal: apples priced at $1.69 per pound or the same apples priced $3.72 per kilogram?
38. If you ran 100 yards in the same time as I ran 100 meters, who ran faster?
39. Apollo Spas services 281 hot tubs. If each hot tub needs 105 milliliters of muriatic acid, how many liters of acid are needed for all of the hot tubs?
40. The photo sharing site Flickr had 2.7 billion photos in June 2012. Create a comparison to understand this number by assuming each picture is about 2 megabytes in size, and comparing to the data stored on other media like DVDs, iPods, or flash drives.
41. In June 2012, Twitter was reporting 400 million tweets per day. Each tweet can consist of up to 140 characters (letter, numbers, etc.). Create a comparison to help understand the amount of tweets in a year by imagining each character was a drop of water and comparing to filling something up.
42. During the landing of the Mars Science Laboratory *Curiosity*, it was reported that the signal from the rover would take 14 minutes to reach earth. Radio signals travel at the speed of light, about 186,000 miles per second. How far was Mars from Earth when *Curiosity* landed?
43. It is estimated that a driver takes, on average, 1.5 seconds from seeing an obstacle to reacting by applying the brake or swerving. How far will a car traveling at 60 miles per hour travel (in feet) before the driver reacts to an obstacle?

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# CHAPTER OVERVIEW

## 2: Geometry

You use geometric terms in everyday language, often without thinking about it. For example, any time you say “walk along this line” or “watch out, this road quickly angles to the left” you are using geometric terms to make sense of the environment around you. You use these terms flexibly, and people generally know what you are talking about.

In the world of mathematics, each of these geometric terms has a specific definition. It is important to know these definitions, as well as how different figures are constructed, to become familiar with the language of geometry.

The material in this chapter is from Developmental Math by NROC.

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# 2.1: Basic Geometric Concepts

**Learning Objectives**

- Identify and define points, lines, line segments, rays, and planes.
- Classify angles as acute, right, obtuse, or straight.

## Introduction

You use geometric terms in everyday language, often without thinking about it. For example, any time you say “walk along this line” or “watch out, this road quickly angles to the left” you are using geometric terms to make sense of the environment around you. You use these terms flexibly, and people generally know what you are talking about.

In the world of mathematics, each of these geometric terms has a specific definition. It is important to know these definitions, as well as how different figures are constructed, to become familiar with the language of geometry. Let’s start with a basic geometric figure: the plane.

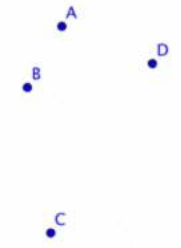
## Figures on a Plane

A **plane** is a flat surface that continues forever (or, in mathematical terms, infinitely) in every direction. It has two dimensions: length and width.

You can visualize a plane by placing a piece of paper on a table. Now imagine that the piece of paper stays perfectly flat and extends as far as you can see in two directions, left-to-right and front-to-back. This gigantic piece of paper gives you a sense of what a geometric plane is like: it continues infinitely in two directions. (Unlike the piece of paper example, though, a geometric plane has no height.)

A plane can contain a number of geometric figures. The most basic geometric idea is a **point**, which has no dimensions. A point is simply a location on the plane. It is represented by a dot. Three points that don’t lie in a straight line will determine a plane.

The image below shows four points, labeled *A*, *B*, *C*, and *D*.



Two points on a plane determine a line. A **line** is a one-dimensional figure that is made up of an infinite number of individual points placed side by side. In geometry, all lines are assumed to be straight; if they bend, they are called a curve. A line continues infinitely in two directions.

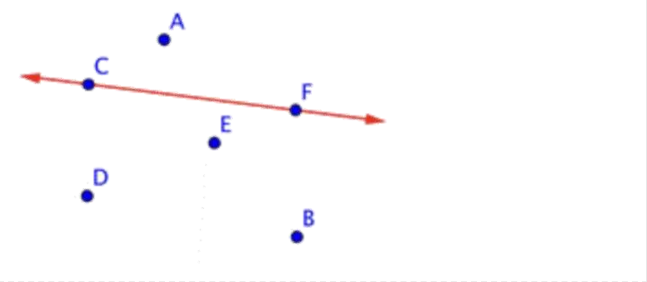
Below is $\text{line } AB$ or, in geometric notation, $\overleftrightarrow{AB}$. The arrows indicate that the line keeps going forever in the two directions. This line could also be called $\text{line } BA$. While the order of the points does not matter for a line, it is customary to name the two points in alphabetical order.

The image below shows the points $A$ and $B$ and the line $\overleftrightarrow{AB}$.

## ✓ Example 2.1.1

Name the line shown in red.

**Solution**

| | |
|---|---|
|  | The red line goes through the points $C$ and $F$, so the line is $\overleftrightarrow{CF}$. |

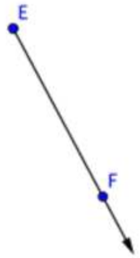
Line: $\overleftrightarrow{CF}$

There are two more figures to consider. The section between any two points on a line is called a **line segment**. A line segment can be very long, very short, or somewhere in between. The difference between a line and a line segment is that the line segment has two endpoints and a line goes on forever. A line segment is denoted by its two endpoints, as in $\overline{CD}$.

A **ray** has one endpoint and goes on forever in one direction. Mathematicians name a ray with notation like $\overrightarrow{EF}$, where point $E$ is the endpoint and $F$ is another point on the ray. When naming a ray, we always say the endpoint first. Note that $\overrightarrow{FE}$ would have the endpoint at $F$, and continue through $E$, which is a different ray than $\overrightarrow{EF}$, which would have a endpoint at $E$, and continue through $F$.

The term “ray” may be familiar because it is a common word in English. “Ray” is often used when talking about light. While a ray of light resembles the geometric term “ray,” it does not go on forever, and it has some width. A geometric ray has no width; only length.

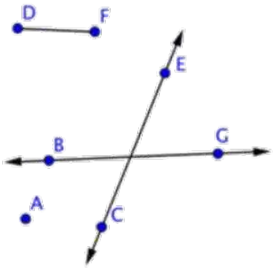
Below is an image of ray $EF$ or $\overrightarrow{EF}$. Notice that the endpoint is $E$.



**Example 2.1.2**

Identify each line and line segment in the picture below.

**Solution**



Two points define a line, and a line is denoted with arrows. There are two lines in this picture: $\overleftrightarrow{CE}$ and $\overleftrightarrow{BG}$.

A line segment is a section between two points. $\overline{DF}$ is a line segment. But there are also two more line segments on the lines themselves: $\overline{CE}$ and $\overline{BG}$.

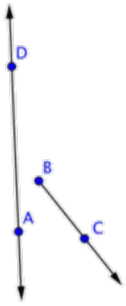
Lines: $\overleftrightarrow{CE}, \overleftrightarrow{BG}$

Line segments: $\overline{DF}, \overline{CE}, \overline{BG}$

**Example 2.1.3**

Identify each point and ray in the picture below.

**Solution**



There are four points: $A, B, C$ and $D$.

There are also three rays, though only one may be obvious.

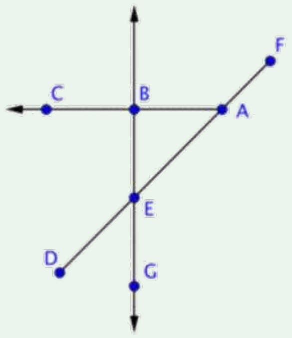
Ray $\overrightarrow{BC}$ begins at point $B$ and goes through $C$. Two more rays exist on line $\overleftrightarrow{AD}$: they are $\overrightarrow{DA}$ and $\overrightarrow{AD}$.

Points: $A, B, C, D$

Rays: $\overrightarrow{BC}, \overrightarrow{AD}, \overrightarrow{DA}$

**Try It 2.1.1**

Which of the following is *not* represented in the image below?



A. $\overleftrightarrow{BG}$
B. $\overrightarrow{BA}$
C. $\overline{DF}$
D. $\overrightarrow{AC}$

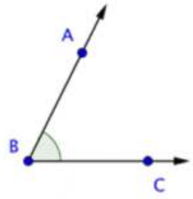
**Answer**

B. This image does not show any ray that begins at point $B$ and goes through point $A$.

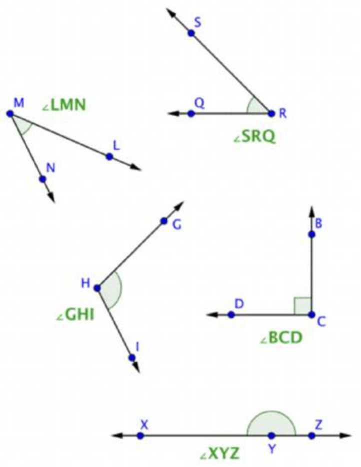
## Angles

Lines, line segments, points, and rays are the building blocks of other figures. For example, two rays with a common endpoint make up an **angle**. The common endpoint of the angle is called the **vertex**.

The angle $ABC$ is shown below. This angle can also be called $\angle ABC$, $\angle CBA$, or simply $\angle B$. When you are naming angles, be careful to include the vertex (here, point $B$) as the middle letter.



The image below shows a few angles on a plane. Notice that the label of each angle is written "point-vertex-point," and the geometric notation is in the form $\angle ABC$.



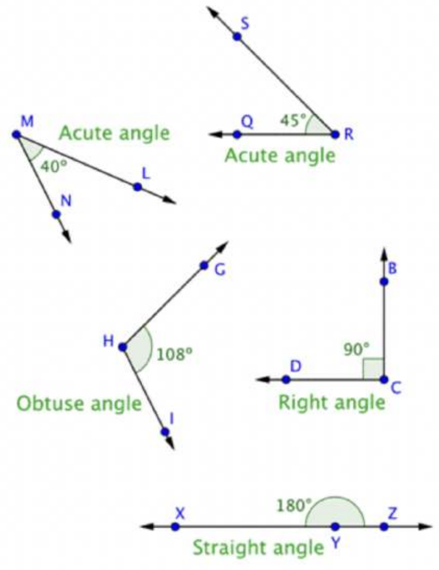
Sometimes angles are very narrow; sometimes they are very wide. When people talk about the "size" of an angle, they are referring to the arc between the two rays. The length of the rays has nothing to do with the size of the angle itself. Drawings of angles will often include an arc (as shown above) to help the reader identify the correct ‘side’ of the angle.

Think about an analog clock face. The minute and hour hands are both fixed at a point in the middle of the clock. As time passes, the hands rotate around the fixed point, making larger and smaller angles as they go. The length of the hands does not impact the angle that is made by the hands.

An angle is measured in degrees, represented by the degree symbol, which is a small circle at the upper right of a number. For example, a circle is defined as having 360°. (In skateboarding and basketball, “doing a 360" refers to jumping and doing one complete body rotation.)

A **right angle** is any degree that measures exactly 90°. This represents exactly one-quarter of the way around a circle. Rectangles contain exactly four right angles. A corner mark is often used to denote a right angle, as shown in right angle $DCB$ below.

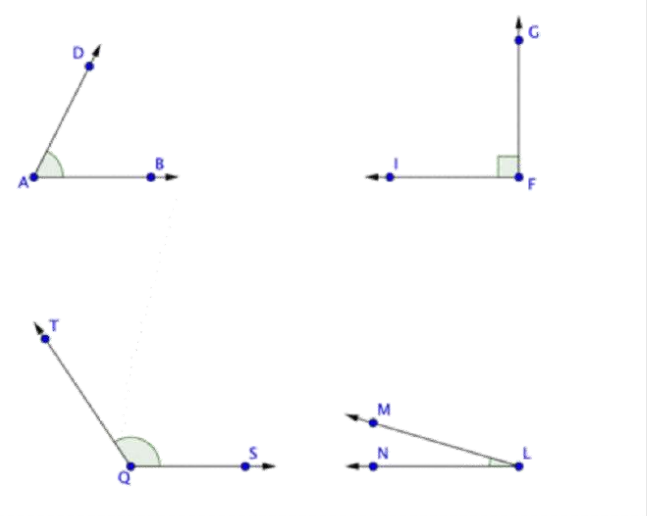
Angles that are between 0° and 90° (smaller than right angles) are called **acute angles**. Angles that are between 90° and 180° (larger than right angles and less than 180°) are called **obtuse angles**. And an angle that measures exactly 180° is called a **straight angle** because it forms a straight line!



**Example 2.1.4**

Label each angle below as acute, right, or obtuse.

**Solution**

| | |
|---|---|
|  | You can start by identifying any right angles.<br>$\angle GFI$ is a right angle, as indicated by the corner mark at vertex $F$.<br>Acute angles will be smaller than $\angle GFI$ (or less than 90$^{o}$). This means that $\angle DAB$ and $\angle MLN$ are acute.<br>$\angle TQS$ is larger than $\angle GFI$, so it is an obtuse angle. |

$\angle DAB$ and $\angle MLN$ are acute angles.

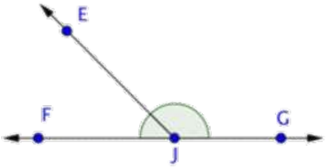
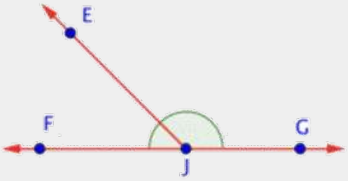
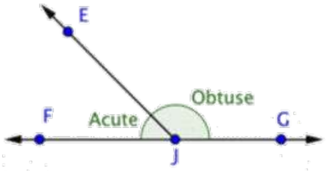
$\angle GFI$ is a right angle.

$\angle TQS$ is an obtuse angle.

**Example 2.1.5**

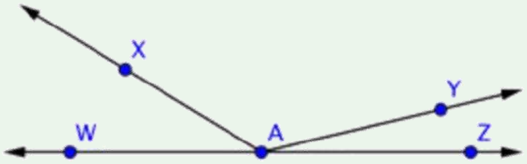
Identify each point, ray, and angle in the figure below.

**Solution**

| | |
|---|---|
|  | Begin by identifying each point in the figure. There are 4: $E, F, G,$ and $J$. |
|  | Now find rays. A ray begins at one point, and then continues through another point towards infinity (indicated by an arrow).<br>Three rays start at point $J$: $\overrightarrow{JE}$, $\overrightarrow{JF}$, and $\overrightarrow{JG}$. But also notice that a ray could start at point $F$ and go through $J$ and $G$, and another could start at point $G$ and go through $J$ and $F$. These rays can be represented by $\overrightarrow{GF}$ and $\overrightarrow{FG}$ |
|  | Finally, look for angles.<br>$\angle EJG$ is obtuse, $\angle EJF$ is acute, and $\angle FJG$ is straight. (Don't forget those straight angles!) |
| Points: $E, F, G, J$<br>Rays: $\overrightarrow{JE}, \overrightarrow{JG}, \overrightarrow{JF}, \overrightarrow{GF}, \overrightarrow{FG}$<br>Angles: $\angle EJG, \angle EJF, \angle FJG$ | |

**Try It 2.1.2**

Identify the acute angles in the image below.



**Answer**

Both $\angle WAX$ and $\angle YAZ$ are acute angles.

$\angle WAZ$ is a straight angle, and $\angle WAY$, $\angle XAY$ and $\angle XAZ$ are obtuse angles.

## Measuring Angles with a Protractor

Learning how to measure angles can help you become more comfortable identifying the difference between angle measurements. For instance, how is a 135° angle different from a 45° angle?

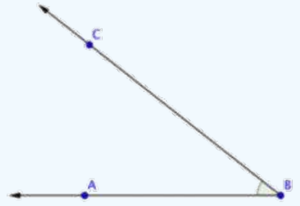
Measuring angles requires a **protractor**, which is a semi-circular tool containing 180 individual hash marks. Each hash mark represents 1°. (Think of it like this: a circle is 360°, so a semi-circle is 180°.) To use the protractor, do the following three steps:

1. line up the vertex of the angle with the dot in the middle of the flat side (bottom) of the protractor,
2. align one side of the angle with the line on the protractor that is at the zero degree mark, and
3. look at the curved section of the protractor to read the measurement.

The example below shows you how to use a protractor to measure the size of an angle.

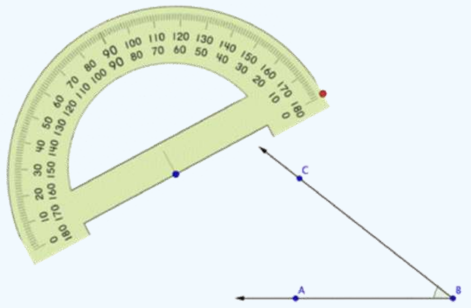
**Example 2.1.6**

Use a protractor to measure the angle shown below.

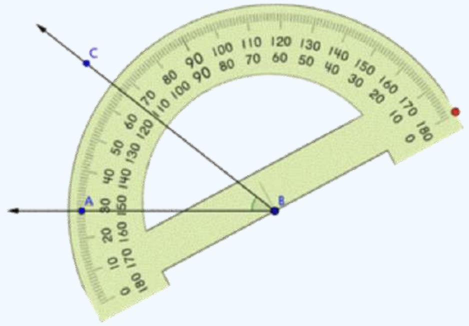


**Solution**

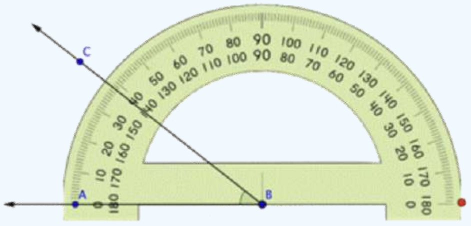
Use a protractor to measure the angle:



Align the blue dot on the protractor with the vertex of the angle you want to measure:

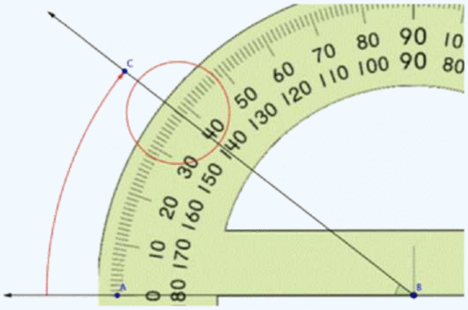


Rotate the protractor around the vertex of the angle until the side of the angle is aligned with the 0 degree mark of the protractor:



Read the measurement, in degrees, of the angle. Begin with the side of the angle that is aligned with the 0° mark of the protractor

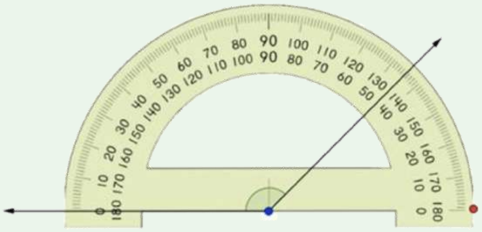
and count up from 0º :



The angle measures 38°.

### Try It 2.1.3

What is the measurement of the angle shown below?



**Answer**

135°

## Summary

Geometric shapes and figures are all around us. A point is a zero-dimensional object that defines a specific location on a plane. A line is made up of an infinite number of points, all arranged next to each other in a straight pattern, and going on forever. A ray begins at one point and goes on towards infinity in one direction only. A plane can be described as a two-dimensional canvas that goes on forever.

When two rays share an endpoint, an angle is formed. Angles can be described as acute, right, obtuse, or straight, and are measured in degrees. You can use a protractor (a special math tool) to closely measure the size of any angle.

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# 2.2: Properties of Angles

**Learning Objectives**

- Identify parallel and perpendicular lines.
- Find measures of angles.
- Identify complementary and supplementary angles.

## Introduction

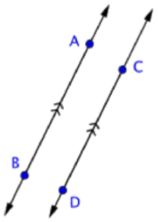
Imagine two separate and distinct lines on a plane. There are two possibilities for these lines: they will either intersect at one point, or they will never intersect. When two lines intersect, four angles are formed. Understanding how these angles relate to each other can help you figure out how to measure them, even if you only have information about the size of one angle.

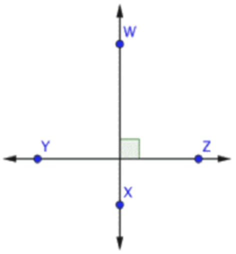
## Parallel and Perpendicular

**Parallel lines** are two or more lines that never intersect. Likewise, parallel line segments are two line segments that never intersect even if the line segments were turned into lines that continued forever. Examples of parallel line segments are all around you, in the two sides of this page and in the shelves of a bookcase. When you see lines or structures that seem to run in the same direction, never cross one another, and are always the same distance apart, there’s a good chance that they are parallel.

**Perpendicular lines** are two lines that intersect at a 90° (right) angle. And perpendicular line segments also intersect at a 90° (right) angle. You can see examples of perpendicular lines everywhere as well: on graph paper, in the crossing pattern of roads at an intersection, to the colored lines of a plaid shirt. In our daily lives, you may be happy to call two lines perpendicular if they merely seem to be at right angles to one another. When studying geometry, however, you need to make sure that two lines intersect at a 90° angle before declaring them to be perpendicular.

The image below shows some parallel and perpendicular lines. The geometric symbol for parallel is $||$, so you can show that $\overleftrightarrow{AB}||\overleftrightarrow{CD}$. Parallel lines are also often indicated by the marking >> on each line (or just a single > on each line). Perpendicular lines are indicated by the symbol $\perp$, so you can write $\overleftrightarrow{WX} \perp \overleftrightarrow{YZ}$.

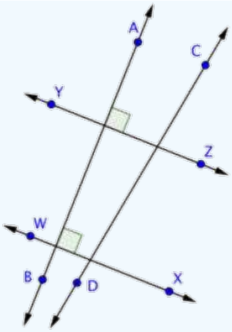




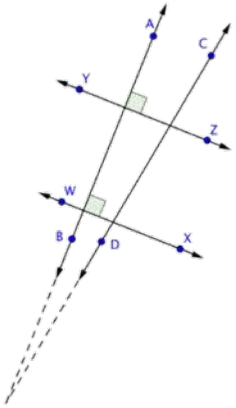
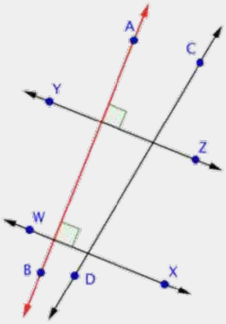
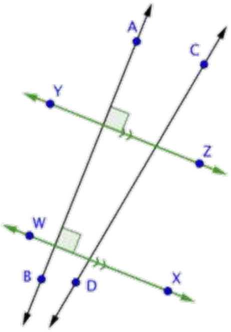
If two lines are parallel, then any line that is perpendicular to one line will also be perpendicular to the other line. Similarly, if two lines are both perpendicular to the same line, then those two lines are parallel to each other. Let’s take a look at one example and identify some of these types of lines.

**Example 2.2.1**

Identify a set of parallel lines and a set of perpendicular lines in the image below.



**Solution**

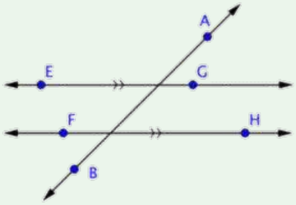
| | |
|---|---|
|  | Parallel lines never meet, and perpendicular lines intersect at a right angle.<br>$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ do not intersect in this image, but if you imagine extending both lines, they will intersect soon. So, they are neither parallel nor perpendicular. |
|  | $\overleftrightarrow{AB}$ is perpendicular to both $\overleftrightarrow{WX}$ and $\overleftrightarrow{YZ}$, as indicated by the right-angle marks at the intersection of those lines. |
|  | Since $\overleftrightarrow{AB}$ is perpendicular to both lines, then $\overleftrightarrow{WX}$ and $\overleftrightarrow{YZ}$ are parallel. |

Parallel lines: $\overleftrightarrow{WX} \| \overleftrightarrow{YZ}$

Perpendicular lines: $\overleftrightarrow{AB} \perp \overleftrightarrow{WX}, \overleftrightarrow{AB} \perp \overleftrightarrow{YZ}$

**Try It 2.2.1**

Which statement most accurately represents the image below?



A. $\overleftrightarrow{EF} \| \overleftrightarrow{GH}$
B. $\overleftrightarrow{AB} \perp \overleftrightarrow{EG}$
C. $\overleftrightarrow{FH} \| \overleftrightarrow{EG}$
D. $\overleftrightarrow{AB} \| \overleftrightarrow{FH}$

**Answer**

A. Incorrect. This image shows the lines $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$, not $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$.

B. Incorrect. $\overleftrightarrow{AB}$ does intersect $\overleftrightarrow{EG}$, but the intersection does not form a right angle. This means that they cannot be perpendicular.

C. Correct. Both $\overleftrightarrow{EG}$ and $\overleftrightarrow{FH}$ are marked with >> on each line, and those markings mean they are parallel.

D. Incorrect. $\overleftrightarrow{AB}$ and $\overleftrightarrow{FH}$ intersect, so they cannot be parallel.

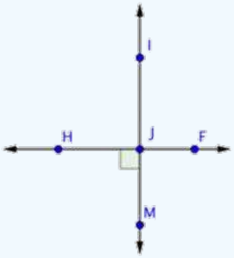
## Finding Angle Measurements

Understanding how parallel and perpendicular lines relate can help you figure out the measurements of some unknown angles. To start, all you need to remember is that perpendicular lines intersect at a 90° angle, and that a straight angle measures 180°.

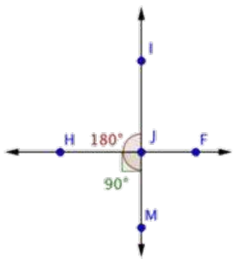
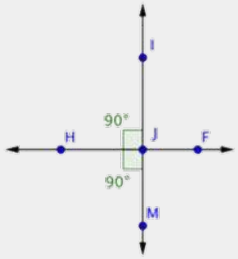
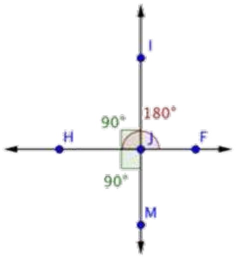
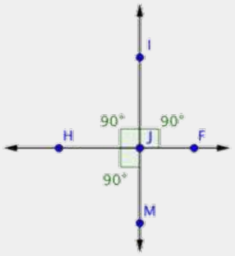
The measure of an angle such as $\angle A$ is written as $m\angle A$. Look at the example below. How can you find the measurements of the unmarked angles?

**Example 2.2.2**

Find the measurement of $\angle IJF$.



**Solution**

| | |
|---|---|
|  | Only one angle, $\angle HJM$, is marked in the image. Notice that it is a right angle, so it measures 90°.<br>$\angle HJM$ is formed by the intersection of lines $\overleftrightarrow{IM}$ and $\overleftrightarrow{HF}$. Since $\overleftrightarrow{IM}$ is a line, $\angle IJM$ is a straight angle measuring 180°. |
|  | You can use this information to find the measurement of $\angle HJI$:<br>$m\angle HJM + m\angle HJI = m\angle IJM$<br>$90^\circ + m\angle HJI = 180^\circ$<br>$m\angle HJI = 90^\circ$ |
|  | Now use the same logic to find the measurement of $\angle IJF$.<br>$\angle IJF$ is formed by the intersection of lines $\overleftrightarrow{IM}$ and $\overleftrightarrow{HF}$.<br>Since $\overleftrightarrow{HF}$ is a line, $\angle HJF$ will be a straight angle measuring 180º. |
|  | You know that $\angle HJI$ measures 90°. Use this information to find the measurement of $\angle IJF$:<br>$m\angle HJI + m\angle IJF = m\angle HJF$<br>$90^\circ + m\angle IJF = 180^\circ$<br>$m\angle IJF = 90^\circ$ |

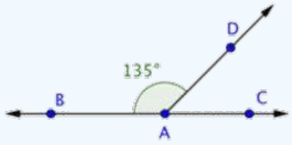
Thus, $m\angle IJF = 90^\circ$.

In this example, you may have noticed that angles $\angle HJI, \angle IJF$, and $\angle HJM$ are all right angles. (If you were asked to find the measurement of $\angle FJM$, you would find that angle to be 90°, too.) This is what happens when two lines are perpendicular: the four angles created by the intersection are all right angles.

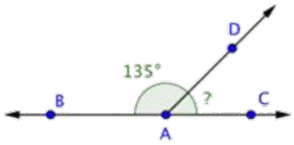
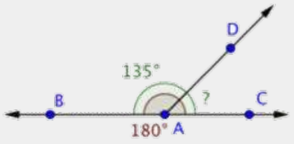
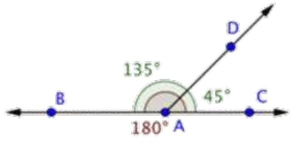
Not all intersections happen at right angles, though. In the example below, notice how you can use the same technique as shown above (using straight angles) to find the measurement of a missing angle.

✓ Example 2.2.3

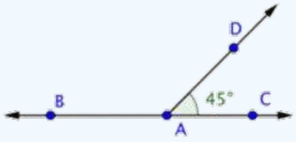
Find the measurement of $\angle DAC$.



**Solution**

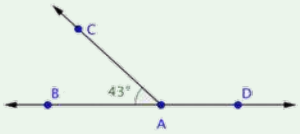
| | |
|---|---|
|  | This image shows the line $\overleftrightarrow{BC}$ and the ray $\overrightarrow{AD}$ intersecting at point $A$. The measurement of $\angle BAD$ is 135°. You can use straight angles to find the measurement of $\angle DAC$. |
|  | $\angle BAC$ is a straight angle, so it measures 180°. |
|  | Use this information to find the measurement of $\angle DAC$.<br>$m\angle BAD + m\angle DAC = m\angle BAC$<br>$135^\circ + m\angle DAC = 180^\circ$<br>$m\angle DAC = 45^\circ$ |

Thus, $m\angle DAC = 45^\circ$.



**Try It 2.2.2**

Find the measurement of $\angle CAD$.



**Answer**

$m\ \angle CAD$ = 137° since 180° - 43° = 137°.

## Supplementary and Complementary

In the example above, $m\angle BAC$ and $m\angle DAC$ add up to 180°. Two angles whose measures add up to 180° are called **supplementary angles**. There’s also a term for two angles whose measurements add up to 90°; they are called **complementary angles**.

One way to remember the difference between the two terms is that “corner” and “complementary” each begin with c (a 90° angle looks like a corner), while straight and “supplementary” each begin with s (a straight angle measures 180°).

If you can identify supplementary or complementary angles within a problem, finding missing angle measurements is often simply a matter of adding or subtracting.

**Example 2.2.4**

Two angles are supplementary. If one of the angles measures 48°, what is the measurement of the other angle?

**Solution**

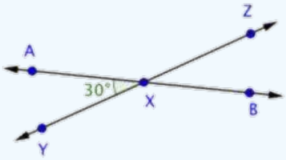
| | |
|---|---|
| $m\angle A + m\angle B = 180^\circ$ | Two supplementary angles make up a straight angle, so the sum of the measurements of the two angles will be 180°. |

| | |
|---|---|
| $48^\circ + m\angle B = 180^\circ$<br>$m\angle B = 180^\circ - 48^\circ$<br>$m\angle B = 132^\circ$ | You know the measurement of one angle. To find the measurement of the second angle, subtract 48° from 180°. |

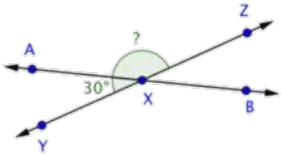
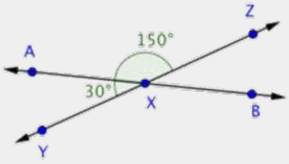
The measurement of the other angle is 132°.

**Example 2.2.5**

Find the measurement of $\angle AXZ$.



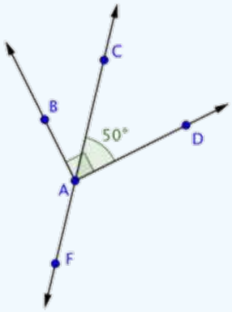
**Solution**

| | |
|---|---|
|  | This image shows two intersecting lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{YZ}$. They intersect at point $X$, forming four angles.<br>Angles $\angle AXY$ and $\angle AXZ$ are supplementary because together they make up the straight angle $\angle YXZ$. |
|  | Use this information to find the measurement of $\angle AXZ$.<br>$m\angle AXY + m\angle AXZ = m\angle YXZ$<br>$30^\circ + m\angle AXZ = 180^\circ$<br>$m\angle AXZ = 150^\circ$ |

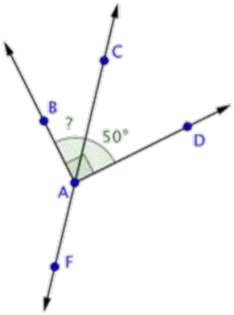
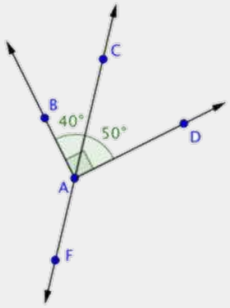
Thus, $m\angle AXZ = 150^\circ$.

**Example 2.2.6**

Find the measurement of $\angle BAC$.



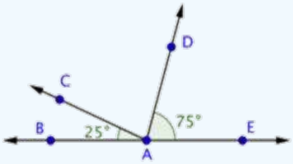
**Solution**

| | |
|---|---|
|  | This image shows the line $\overleftrightarrow{CF}$ and the rays $\overrightarrow{AB}$ and $\overrightarrow{AD}$, all intersecting at point $A$. Angle $\angle BAD$ is a right angle. Angles $\angle BAC$ and $\angle CAD$ are complementary, because together they create $\angle BAD$. |
|  | Use this information to find the measurement of $\angle BAC$.<br>$m\angle BAC + m\angle CAD = m\angle BAD$<br>$m\angle BAC + 50^\circ = 90^\circ$<br>$m\angle BAC = 40^\circ$ |

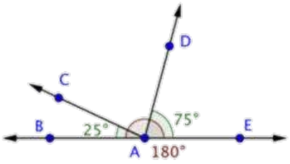
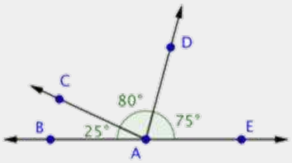
Thus, $m\angle BAC = 40^\circ$.

## Example 2.2.7

Find the measurement of $\angle CAD$.



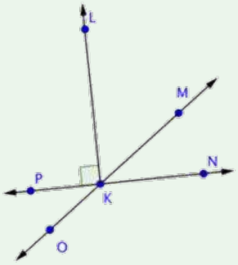
**Solution**

| | |
|---|---|
|  | You know the measurements of two angles here: $\angle CAB$ and $\angle DAE$. You also know that $m\angle BAE = 180^\circ$. |
|  | Use this information to find the measurement of $\angle CAD$.<br>$m\angle BAC + m\angle CAD + m\angle DAE = m\angle BAE$<br>$25^\circ + m\angle CAD + 75^\circ = 180^\circ$<br>$m\angle CAD + 100^\circ = 180^\circ$<br>$m\angle CAD = 80^\circ$ |

Thus, $m\angle CAD = 80^\circ$.

## Try It 2.2.3

Which pair of angles is complementary?



**Answer**

$\angle LKM$ and $\angle MKN$ . The measurements of two complementary angles will add up to 90°. $\angle LKP$ is a right angle, so $\angle LKN$ must be a right angle as well. $\angle LKM + \angle MKN = \angle LKN$ , so $\angle LKM$ and $\angle MKN$ are complementary.

## Summary

Parallel lines do not intersect, while perpendicular lines cross at a 90°. angle. Two angles whose measurements add up to 180° are said to be supplementary, and two angles whose measurements add up to 90° are said to be complementary. For most pairs of intersecting lines, all you need is the measurement of one angle to find the measurements of all other angles formed by the intersection.

---

# 2.3: Triangles

## Learning Objectives

- Identify equilateral, isosceles, scalene, acute, right, and obtuse triangles.
- Identify whether triangles are similar, congruent, or neither.
- Identify corresponding sides of congruent and similar triangles.
- Find the missing measurements in a pair of similar triangles.
- Solve application problems involving similar triangles.

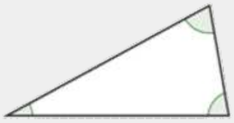
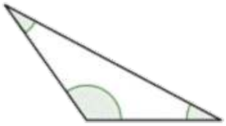
## Introduction

Geometric shapes, also called figures, are an important part of the study of geometry. The **triangle** is one of the basic shapes in geometry. It is the simplest shape within a classification of shapes called **polygons**. All triangles have three sides and three angles, but they come in many different shapes and sizes. Within the group of all triangles, the characteristics of a triangle's sides and angles are used to classify it even further. Triangles have some important characteristics, and understanding these characteristics allows you to apply the ideas in real-world problems.

## Classifying and Naming Triangles

A **polygon** is a closed plane figure with three or more straight sides. Polygons each have a special name based on the number of sides they have. For example, the polygon with three sides is called a triangle because "tri" is a prefix that means "three." Its name also indicates that this polygon has three angles. The prefix "poly" means many.

The table below shows and describes three classifications of triangles. Notice how the types of angles in the triangle are used to classify the triangle.

| **Name of Triangle** | **Picture of Triangle** | **Description** |
|---|---|---|
| **Acute Triangle** |  | A triangle with 3 acute angles (3 angles measuring between 0° and 90°). |
| **Obtuse Triangle** |  | A triangle with 1 obtuse angle (1 angle measuring between 90° and 180°) |
| **Right Triangle** |  | A triangle containing one right angle (1 angle that measures 90°). Note that the right angle is shown with a corner mark and does not need to be labeled 90°. |

**The sum of the measures of the three interior angles of a triangle is always 180°**. This fact can be applied to find the measure of the third angle of a triangle, if you are given the other two. Consider the examples below.

## Example 2.3.1

A triangle has two angles that measure 35° and 75°. Find the measure of the third angle.

### Solution

| | |
|---|---|
| $35^\circ + 75^\circ + x = 180^\circ$ | The sum of the three interior angles of a triangle is 180°. |

| | |
|---|---|
| $110^\circ + x = 180^\circ$<br>$x = 180^\circ - 110^\circ$<br>$x = 70^\circ$ | Find the value of $x$. |

The third angle of the triangle measures 70°.

### Example 2.3.2

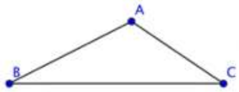
One of the angles in a right triangle measures 57°. Find the measurement of the third angle.

**Solution**

| | |
|---|---|
| $57^\circ + 90^\circ + x = 180^\circ$ | The sum of the three angles of a triangle is always 180°. One of the angles has a measure of 90° as it is a right triangle. |
| $147^\circ + x = 180^\circ$<br>$x = 180^\circ - 147^\circ$<br>$x = 33^\circ$ | Simplify.<br>Find the value of $x$. |

The third angle of the right triangle measures 33°.

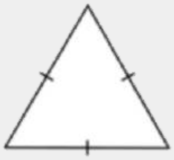
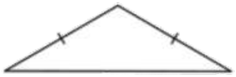
There is an established convention for naming triangles. The labels of the vertices of the triangle, which are generally capital letters, are used to name a triangle.



You can call this triangle $ABC$ or $\triangle ABC$ since $A, B,$ and $C$ are vertices of the triangle. When naming the triangle, you can begin with any vertex. Then keep the letters in order as you go around the polygon. The triangle above could be named in a variety of ways: $\triangle ABC$, or $\triangle CBA$. The sides of the triangle are line segments $\overline{AB}$, $\overline{AC}$, and $\overline{CB}$.

Just as triangles can be classified as acute, obtuse, or right based on their angles, they can also be classified by the length of their sides. Sides of equal length are called **congruent** sides. While we designate a segment joining points $A$ and $B$ by the notation $\overline{AB}$, we designate the length of a segment joining points $A$ and $B$ by the notation $AB$ without a segment bar over it. The length $AB$ is a number, and the segment $\overline{AB}$ is the collection of points that make up the segment.

Mathematicians show congruency by putting a hash mark symbol through the middle of sides of equal length. If the hash mark is the same on one or more sides, then those sides are congruent. If the sides have different hash marks, they are *not* congruent. The table below shows the classification of triangles by their side lengths.

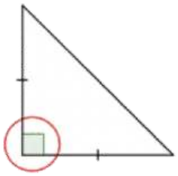
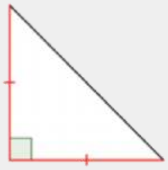
| **Name of Triangle** | **Picture of Triangle** | **Description** |
|---|---|---|
| **Equilateral Triangle** |  | A triangle whose three sides have the same length. These sides of equal length are called congruent sides. |
| **Isosceles Triangle** |  | A triangle with exactly two congruent sides. |
| **Scalene Triangle** |  | A triangle in which all three sides are a different length. |

To describe a triangle even more specifically, you can use information about both its sides and its angles. Consider this example.

**Example 2.3.3**

Classify the triangle below.

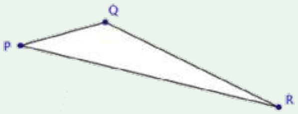
**Solution**

| | |
|---|---|
|  | Notice what kind of angles the triangle has. Since one angle is a right angle, this is a right triangle. |
|  | Notice the lengths of the sides. Are there congruence marks or other labels?<br>The congruence marks tell us there are two sides of equal length. So, this is an isosceles triangle. |

This is an isosceles right triangle.

**Try It 2.3.1**

Classify the triangle shown below.

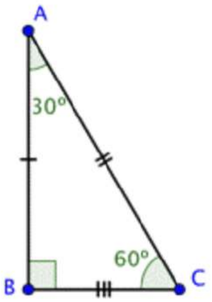


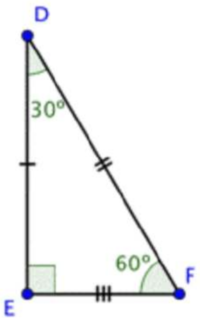
**Answer**

$\triangle PQR$ is an obtuse scalene triangle. The triangle has vertices $P$, $Q$, and $R$, one angle (angle $Q$) that is between 90° and 180°, and sides of three different lengths.

## Identifying Congruent and Similar Triangles

Two triangles are **congruent** if they are exactly the same size and shape. In congruent triangles, the measures of **corresponding angles** and the lengths of **corresponding sides** are equal. Consider the two triangles shown below:





Since both $\angle B$ and $\angle E$ are right angles, these triangles are right triangles. Let's call these two triangles $\triangle ABC$ and $\triangle DEF$. These triangles are congruent if every pair of corresponding sides has equal lengths and every pair of corresponding angles has the same measure.

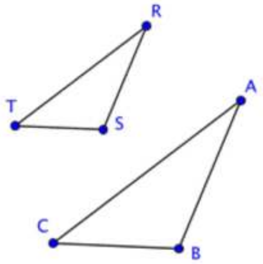
The corresponding sides are opposite the corresponding angles.

A double headed arrow means "corresponds to"

$\angle B \leftrightarrow \angle E$
$\angle A \leftrightarrow \angle D$
$\angle C \leftrightarrow \angle F$
$\overline{AB} \leftrightarrow \overline{DE}$
$\overline{AC} \leftrightarrow \overline{DF}$
$\overline{BC} \leftrightarrow \overline{EF}$

$\triangle ABC$ and $\triangle DEF$ are congruent triangles as the corresponding sides and corresponding angles are equal.

Let's take a look at another pair of triangles. Below are the triangles $\triangle ABC$ and $\triangle RST$.



These two triangles are surely not congruent because $\triangle RST$ is clearly smaller in size than $\triangle ABC$. But, even though they are not the same size, they do resemble one another. They are the same shape. The corresponding angles of these triangles look like they might have the same exact measurement, and if they did they would be congruent angles and we would call the triangles **similar triangles**.

Congruent angles are marked with hash marks, just as congruent sides are.

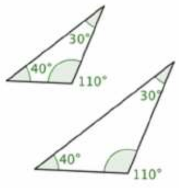


Image showing angle measurements of both triangles.

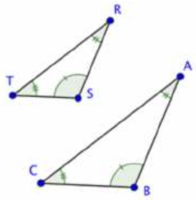


Image showing triangles $ABC$ and $RST$ using hash marks to show angle congruency.

We can also show congruent angles by using multiple bands within the angle, rather than multiple hash marks on one band. Below is an image using multiple bands within the angle.

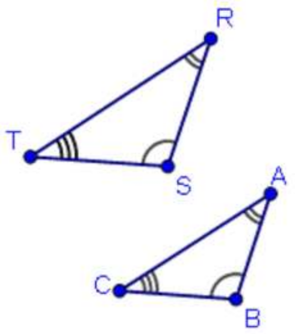


Image showing triangles $ABC$ and $RST$ using bands to show angle congruency.

If the corresponding angles of two triangles have the same measurements, they are similar triangles. This name makes sense because they have the same shape, but not necessarily the same size. When a pair of triangles is similar, the corresponding sides are proportional to one another. That means that there is a consistent scale factor that can be used to compare the corresponding sides. In the previous example, the side lengths of the larger triangle are all 1.4 times the length of the smaller. So, similar triangles are proportional to one another.

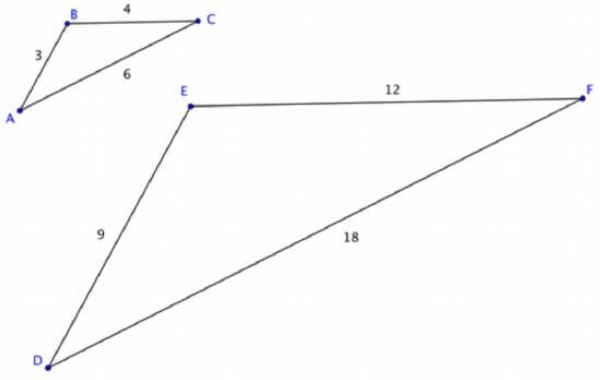
Just because two triangles *look* similar does not mean they *are* similar triangles in the mathematical sense of the word. Checking that the corresponding angles have equal measure is one way of being sure the triangles are similar.

## Corresponding Sides of Similar Triangles

There is another method for determining similarity of triangles that involves comparing the ratios of the lengths of the corresponding sides.

If the ratios of the pairs of corresponding sides are equal, the triangles are similar.

Consider the two triangles below.



$\triangle ABC$ is *not* congruent to $\triangle DEF$ because the side lengths of $\triangle DEF$ are longer than those of $\triangle ABC$. So, are these triangles similar? If they are, the corresponding sides should be proportional.

Since these triangles are oriented in the same way, you can pair the left, right, and bottom sides: $\overline{AB}$ and $\overline{DE}$, $\overline{BC}$ and $\overline{EF}$, $\overline{AC}$ and $\overline{DF}$. (You might have called these the two shortest sides, the two longest sides, and the two leftover sides and arrived at the same ratios). Now we will look at the ratios of their lengths.

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$$

Substituting the side length values into the proportion, you see that it is true:

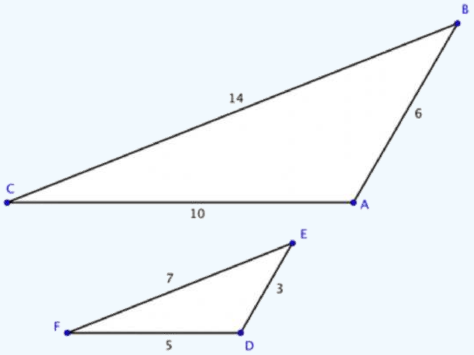
$$\frac{3}{9}=\frac{4}{12}=\frac{6}{18}$$

If the corresponding sides are proportional, then the triangles are similar. Triangles $ABC$ and $DEF$ are similar, but not congruent.

Let's use this idea of proportional corresponding sides to determine whether two more triangles are similar.

**Example 2.3.4**

Determine if the triangles below are similar by checking if their corresponding sides are proportional.



**Solution**

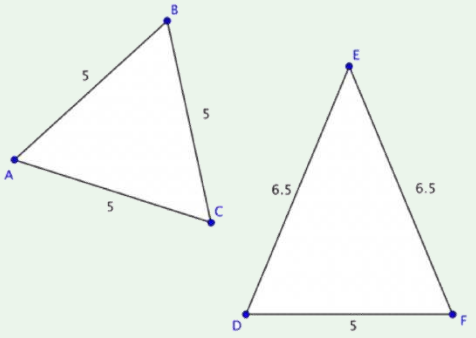
| | |
|---|---|
| $\overline{CA} \leftrightarrow \overline{FD}$<br>$\overline{AB} \leftrightarrow \overline{DE}$<br>$\overline{BC} \leftrightarrow \overline{EF}$ | First determine the corresponding sides, which are opposite corresponding angles. |
| $\frac{CA}{FD} = \frac{AB}{DE} = \frac{BC}{EF}$ | Write the corresponding side lengths as ratios. |
| $\frac{10}{5} = \frac{6}{3} = \frac{14}{7}$<br>$2 = 2 = 2$ | Substitute the side lengths into the ratios, and determine if the ratios of the corresponding sides are equivalent. They are, so the triangles are similar. |

$\triangle ABC$ and $\triangle DEF$ are similar.

The mathematical symbol ~ means "is similar to." So, you can write $\triangle ABC$ is similar to $\triangle DEF$ as $\triangle ABC \sim \triangle DEF$.

**Try It 2.3.2**

Determine whether the two triangles are similar, congruent, or neither.



**Answer**

$\triangle ABC$ and $\triangle DEF$ are neither similar nor congruent. The corresponding angle measures are not known to be equal as shown by the absence of congruence marks on the angles. Also, the ratios of the corresponding sides are not equal:

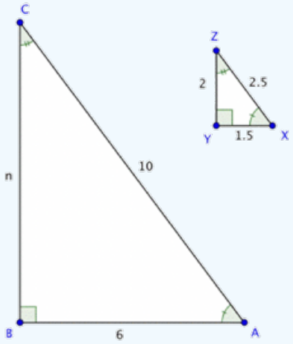
$$\frac{6.5}{5} = \frac{6.5}{5} \neq \frac{5}{5} \ .$$

## Finding Missing Measurements in Similar Triangles

You can find the missing measurements in a triangle if you know some measurements of a similar triangle. Let’s look at an example.

**Example 2.3.5**

$\triangle ABC$ and $\triangle XYZ$ are similar triangles. What is the length of side $BC$?



**Solution**

| | |
|---|---|
| $\frac{BC}{YZ} = \frac{AB}{XY}$ | In similar triangles, the ratios of corresponding sides are proportional. Set up a proportion of two ratios, one that includes the missing side. |
| $\frac{n}{2} = \frac{6}{1.5}$ | Substitute in the known side lengths for the side names in the ratio. Let the unknown side length be $n$. |
| $1.5 \cdot n = 2 \cdot 6$<br>$1.5n = 12$<br>$n = 8$ | Solve for $n$ using cross multiplication. |

The missing length of side $BC$ is 8 units.

This process is fairly straightforward, but be careful that your ratios represent corresponding sides, recalling that corresponding sides are opposite corresponding angles.

## Solving Application Problems Involving Similar Triangles

Applying knowledge of triangles, similarity, and congruence can be very useful for solving problems in real life. Just as you can solve for missing lengths of a triangle drawn on a page, you can use triangles to find unknown distances between locations or objects.

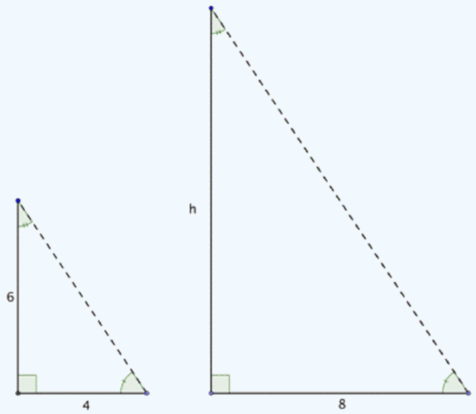
Let’s consider the example of two trees and their shadows. Suppose the sun is shining down on two trees, one that is 6 feet tall and the other whose height is unknown. By measuring the length of each shadow on the ground, you can use triangle similarity to find the unknown height of the second tree.

First, let’s figure out where the triangles are in this situation! The trees themselves create one pair of corresponding sides. The shadows cast on the ground are another pair of corresponding sides. The third side of these imaginary similar triangles runs from the top of each tree to the tip of its shadow on the ground. This is the hypotenuse of the triangle.

If you know that the trees and their shadows form similar triangles, you can set up a proportion to find the height of the tree.

**Example 2.3.6**

When the sun is at a certain angle in the sky, a 6-foot tree will cast a 4-foot shadow. How tall is a tree that casts an 8-foot shadow?



**Solution**

| | |
|---|---|
| | The angle measurements are the same, so the triangles are similar triangles. Since they are similar triangles, you can use proportions to find the size of the missing side. |
| $\frac{\text{Tree 1}}{\text{Tree 2}}=\frac{\text{Shadow 1}}{\text{Shadow 2}}$ | Set up a proportion comparing the heights of the trees and the lengths of their shadows. |
| $\frac{6}{h}=\frac{4}{8}$ | Substitute in the known lengths. Call the missing tree height $h$. |
| $6\cdot 8=4h$<br>$48=4h$<br>$12=h$ | Solve for $h$ using cross-multiplication. |

The tree is 12 feet tall.

## Summary

Triangles are one of the basic shapes in the real world. Triangles can be classified by the characteristics of their angles and sides, and triangles can be compared based on these characteristics. The sum of the measures of the interior angles of any triangle is 180°. Congruent triangles are triangles of the same size and shape. They have corresponding sides of equal length and corresponding angles of the same measurement. Similar triangles have the same shape, but not necessarily the same size. The lengths of their sides are proportional. Knowledge of triangles can be a helpful in solving real-world problems.

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# 2.4: The Pythagorean Theorem

**Learning Objectives**

- Use the Pythagorean Theorem to find the unknown side of a right triangle.
- Solve application problems involving the Pythagorean Theorem.

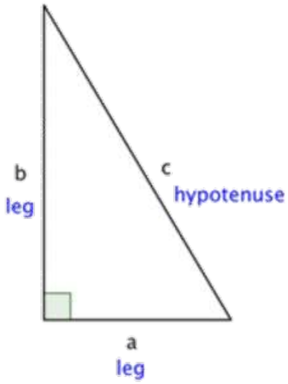
## Introduction

A long time ago, a Greek mathematician named **Pythagoras** discovered an interesting property about **right triangles**: the sum of the squares of the lengths of each of the triangle's **legs** is the same as the square of the length of the triangle's **hypotenuse**. This property, which has many applications in science, art, engineering, and architecture, is now called the **Pythagorean Theorem**.

Let's take a look at how this theorem can help you learn more about the construction of triangles. And the best part is you don't even have to speak Greek to apply Pythagoras' discovery.

## The Pythagorean Theorem

Pythagoras studied right triangles, and the relationships between the legs and the hypotenuse of a right triangle, before deriving his theory. The legs of a right triangle are the sides attached to the right angle and the hypotenuse is the side opposite the right angle. By convention the legs are refrred to with the letters a and b, and the hypotenuse is labeled c.



**The Pythagorean Theorem**

If $a$ and $b$ are the lengths of the legs of a right triangle and $c$ is the length of the hypotenuse, then the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

This relationship is represented by the formula:

$$a^2+b^2=c^2 \tag{2.4.1}$$

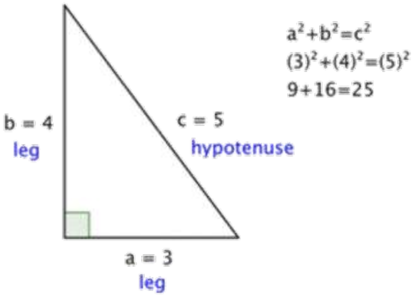
In the box above, you may have noticed the word "square," as well as the small 2s to the top right of the letters in $a^2+b^2=c^2$ . To **square** a number means to multiply it by itself. So, for example, to square the number 5 you multiply $5\cdot 5$, and to square the number 12, you multiply $12\cdot 12$. Some common squares are shown in the table below.

| Number | Number Times Itself | Square |
|---|---|---|
| 1 | $1^2=1\cdot 1$ | 1 |
| 2 | $2^2=2\cdot 2$ | 4 |
| 3 | $3^2=3\cdot 3$ | 9 |
| 4 | $4^2=4\cdot 4$ | 16 |
| 5 | $5^2=5\cdot 5$ | 25 |

| | | |
|---|---|---|
| 10 | $10^2 = 10 \cdot 10$ | 100 |

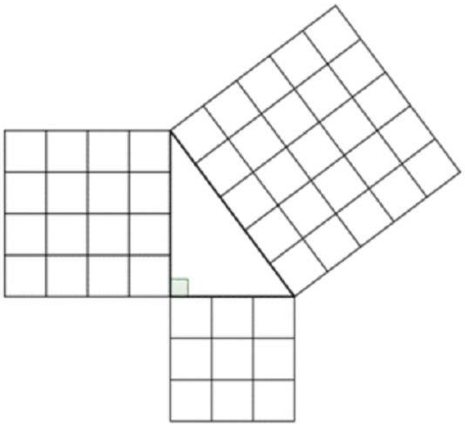
When you see the equation $a^2 + b^2 = c^2$ , you can think of this as “the length of side $a$ times itself, plus the length of side $b$ times itself is the same as the length of side $c$ times itself.”

Let’s try out the Pythagorean Theorem with an actual right triangle.



This theorem holds true for this right triangle: the sum of the squares of the lengths of both legs is the same as the square of the length of the hypotenuse. And, in fact, it holds true for all right triangles.

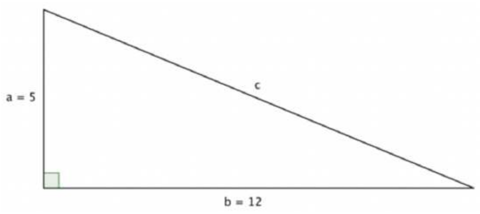
The Pythagorean Theorem can also be represented in terms of area. In any right triangle, the area of the square drawn from the hypotenuse is equal to the sum of the areas of the squares that are drawn from the two legs. You can see this illustrated below in the same 3-4-5 right triangle.



Note that the Pythagorean Theorem **only works with *right* triangles**.

## Finding the Length of the Hypotenuse

You can use the Pythagorean Theorem to find the length of the hypotenuse of a right triangle if you know the length of the triangle’s other two sides, called the legs. Put another way, if you know the lengths of $a$ and $b$, you can find $c$.



In the triangle above, you are given measures for legs $a$ and $b$: 5 and 12, respectively. You can use the Pythagorean Theorem to find a value for the length of $c$, the hypotenuse.

| | |
|---|---|
| $a^2 + b^2 = c^2$ | The Pythagorean Theorem |

| | |
|---|---|
| $(5)^2+(12)^2=c^2$ | Substitute known values for $a$ and $b$. |
| $25+144=c^2$ | Evaluate. |
| $169=c^2$ | Simplify. To find the value of $c$, think about a number that, when multiplied by itself, equals 169. Does 10 work? How about 11? Does 12? 13? (You can use a calculator to multiply if the numbers are unfamiliar.) |
| $13=c$ | The square root of 169 is 13. |

Using the formula, you find that the length of $c$, the hypotenuse, is 13.

In this case, you did not know the value of $c$, you were given the square of the length of the hypotenuse, and had to figure it out from there. When you are given an equation like $169=c^2$ and are asked to find the value of $c$, this is called finding the **square root** of a number. (Notice you found a number, c, whose square was 169.)

Finding a square root takes some practice, but it also takes knowledge of multiplication, division, and a little bit of trial and error. Look at the table below.

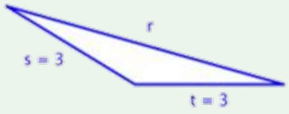
| **Number $x$** | **Number $y$ which, when multiplied by itself, equals number $x$** | **Square root $y$** |
|---|---|---|
| 1 | $1\cdot 1$ | 1 |
| 4 | $2\cdot 2$ | 2 |
| 9 | $3\cdot 3$ | 3 |
| 16 | $4\cdot 4$ | 4 |
| 25 | $5\cdot 5$ | 5 |
| 100 | $10\cdot 10$ | 10 |

It is a good habit to become familiar with the squares of the numbers from 0 through 10, as these arise frequently in mathematics. If you can remember those square numbers, or if you can use a calculator to find them, then finding many common square roots will be just a matter of recall.

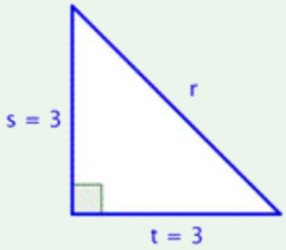
**Try It 2.4.1**

For which of these triangles is $(3)^2+(3)^2=r^2$ ?

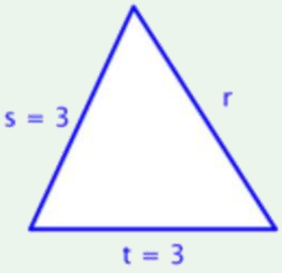
A.



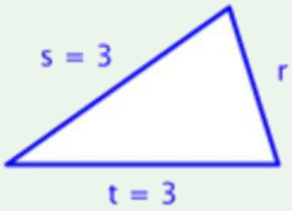
B.



C.



D.



**Answer**

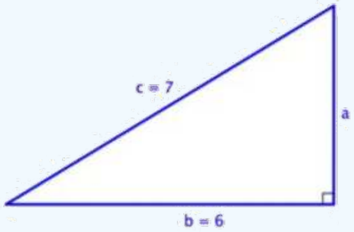
B, since it is the only right triangle.

## Finding the Length of a Leg

You can use the same formula to find the length of a right triangle's leg if you are given measurements for the lengths of the hypotenuse and the other leg. Consider the example below.

**Example 2.4.1**

Find the length of side *a* in the triangle below. Use a calculator to estimate the square root to one decimal place.



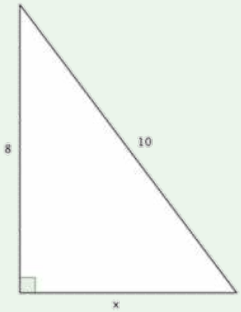
**Solution**

| | |
|---|---|
| $a = ?$<br>$b = 6$<br>$c = 7$ | In this right triangle, you are given the measurements for the hypotenuse, $c$, and one leg, $b$. The hypotenuse is always opposite the right angle and it is always the longest side of the triangle. |
| $a^2 + b^2 = c^2$<br>$a^2 + 6^2 = 7^2$ | To find the length of leg $a$, substitute the known values into the Pythagorean Theorem. |
| $a^2 + 36 = 49$ | Solve for $a^2$ by subtracting 36 from both side of the equation |
| $a^2 = 13$ | Take a square root on both sides |
| $a = \sqrt{13} \approx 3.6$ | Use a calculator to find the square root of 13.<br>The calculator gives an answer of 3.6055..., which you can round to 3.6. (Since you are approximating, you use the symbol $\approx$.) |

Thus, $a \approx 3.6$.

### Try It 2.4.2

Which of the following correctly uses the Pythagorean Theorem to find the missing side, $x$?



A. $8^2+10^2=x^2$
B. $x+8=10$
C. $x^2+8^2=10^2$
D. $x^2+10^2=8^2$

**Answer**

A. Incorrect. In this triangle, you know the hypotenuse (the side opposite the right angle) has a length of 10. The lengths of the legs are 8 and $x$.
B. Incorrect. The Pythagorean Theorem is a relationship between the lengths of the sides squared.
C. Correct. In this triangle, the hypotenuse has length 10, and the legs have length 8 and $x$. Substituting into the Pythagorean Theorem you have: $x^2+8^2=10^2$ ; this equation is the same as $x^2+64=100$, or $x^2=36$. What number, times itself, equals 36? That would make $x=6$.
D. Incorrect. In this triangle, the hypotenuse has length 10 (always the longest side of the triangle and the side opposite the right angle) not 8.

## Using the Theorem to Solve Real World Problems

The Pythagorean Theorem is perhaps one of the most useful formulas you will learn in mathematics because there are so many applications of it in real-world settings. Architects and engineers use this formula extensively when building ramps, bridges, and buildings. Look at the following examples.

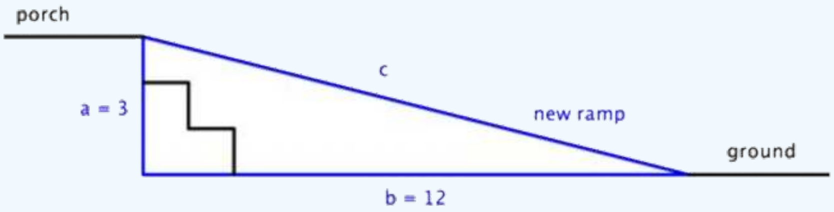
### Example 2.4.2

The owners of a house want to convert a stairway leading from the ground to their back porch into a ramp. The porch is 3 feet off the ground and, due to building regulations, the ramp must start 12 feet away from the base of the porch. How long will the ramp be?

Use a calculator to find the square root, and round the answer to the nearest tenth.

**Solution**

To solve a problem like this one, it often makes sense to draw a simple diagram showing where the legs and hypotenuse of the triangle lie.



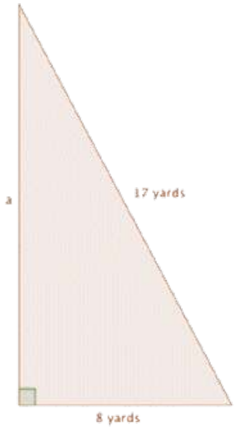
| | |
|---|---|
| $a=3$<br>$b=12$<br>$c=?$ | Identify the legs and the hypotenuse of the triangle. You know that the triangle is a *right* triangle since the ground and the raised portion of the porch are perpendicular.<br>This means you can use the Pythagorean Theorem to solve this problem. Identify $a$, $b$, and $c$. |
| $a^2+b^2=c^2$<br>$3^2+12^2=c^2$<br>$9+144=c^2$<br>$153=c^2$ | Use the Pythagorean Theorem to find the length of $c$. |
| $c=\sqrt{153}\approx 12.4$ | Use your calculator to find $c$. The square root of 153 is 12.369..., so you can round that to 12.4. |

The ramp will be 12.4 feet long.

### ✓ Example 2.4.3

A sailboat has a large sail in the shape of a right triangle. The longest edge of the sail measures 17 yards and the bottom edge of the sail is 8 yards. How tall is the sail?

**Solution**

| | |
|---|---|
|  | Draw an image to help you visualize the problem. In a right triangle, the hypotenuse will always be the longest side, so here it must be 17 yards. The problem also tells you that the bottom edge of the triangle is 8 yards. |
| $a^2+b^2=c^2$<br>$a^2+8^2=17^2$<br>$a^2+64=289$<br>$a^2=225$ | Set up the Pythagorean Theorem. |
| $a=15$ | $15\cdot 15=225,\ \text{so } a=15$ |

The height of the sail is 15 yards.

## Summary

The Pythagorean Theorem states that in any right triangle, the sum of the squares of the lengths of the triangle's legs is the same as the square of the length of the triangle's hypotenuse. This theorem is represented by the formula $a^2+b^2=c^2$ . Put simply, if you know the lengths of two sides of a right triangle, you can apply the Pythagorean Theorem to find the length of the third side. Remember, this theorem only works for right triangles.

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- **7.1.4: The Pythagorean Theorem** by The NROC Project is licensed CC BY-NC-SA 4.0. Original source: https://content.nroc.org/DevelopmentalMath.HTML5/Common/toc/toc_en.html.

# 2.5: Quadrilaterals

**Learning Objectives**

- Identify properties, including angle measurements, of quadrilaterals.

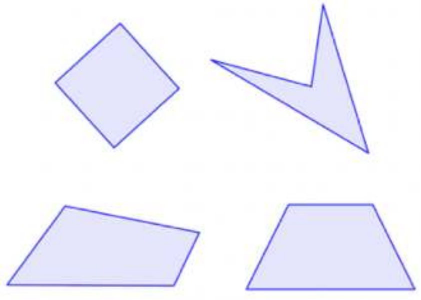
## Introduction

**Quadrilaterals** are a special type of polygon. As with triangles and other polygons, quadrilaterals have special properties and can be classified by characteristics of their angles and sides. Understanding the properties of different quadrilaterals can help you in solving problems that involve this type of polygon.

## Defining a Quadrilateral

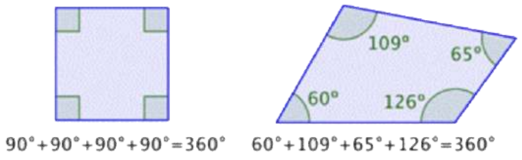
Picking apart the name “quadrilateral” helps you understand what it refers to. The prefix “quad-” means “four,” and “lateral” is derived from the Latin word for “side.” So a quadrilateral is a four-sided polygon.

Since it is a **polygon**, you know that it is a two-dimensional figure made up of straight sides. A quadrilateral also has four angles formed by its four sides. Below are some examples of quadrilaterals. Notice that each figure has four straight sides and four angles.



## Interior Angles of a Quadrilateral

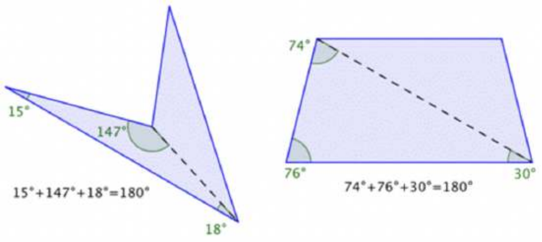
**The sum of the interior angles of any quadrilateral is 360°**. Consider the two examples below.



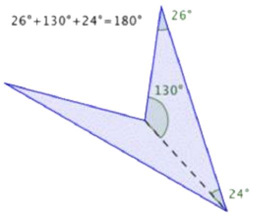
You could draw many quadrilaterals such as these and carefully measure the four angles. You would find that for every quadrilateral, the sum of the interior angles will always be 360°.

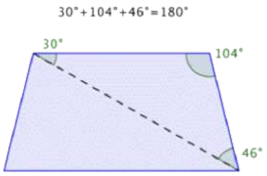
You can also use your knowledge of triangles as a way to understand why the sum of the interior angles of any quadrilateral is 360°. Any quadrilateral can be divided into two triangles as shown in the images below.

In the first image, the quadrilaterals have each been divided into two triangles. The angle measurements of one triangle are shown for each.



These measurements add up to 180°. Now look at the measurements for the other triangles. They also add up to 180°!

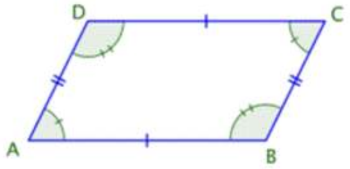




Since the sum of the interior angles of any triangle is 180° and there are two triangles in a quadrilateral, the sum of the angles for each quadrilateral is 360°.

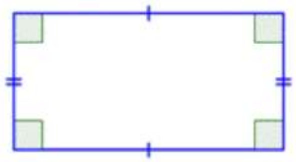
## Specific Types of Quadrilaterals

Let's start by examining the group of quadrilaterals that have two pairs of parallel sides. These quadrilaterals are called **parallelograms** They take a variety of shapes, but one classic example is shown below.

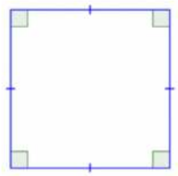


Imagine extending the pairs of opposite sides. They would never intersect because they are parallel. Notice, also, that the opposite angles of a parallelogram are congruent, as are the opposite sides. (Remember that "congruent" means "the same size.") The geometric symbol for congruent is $\cong$, so you can write $\angle A \cong \angle C$ and $\angle B \cong \angle D$. The parallel sides are also the same length: $\overline{AB} \cong \overline{DC}$ and $\overline{BC} \cong \overline{AD}$. These relationships are true for all parallelograms.

There are two special cases of parallelograms that will be familiar to you from your earliest experiences with geometric shapes. The first special case is called a **rectangle**. By definition, a rectangle is a parallelogram because its pairs of opposite sides are parallel. A rectangle also has the special characteristic that all of its angles are right angles; all four of its angles are congruent.

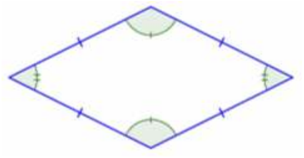


The other special case of a parallelogram is a special type of rectangle, a **square**. A square is one of the most basic geometric shapes. It is a special case of a parallelogram that has four congruent sides and four right angles.



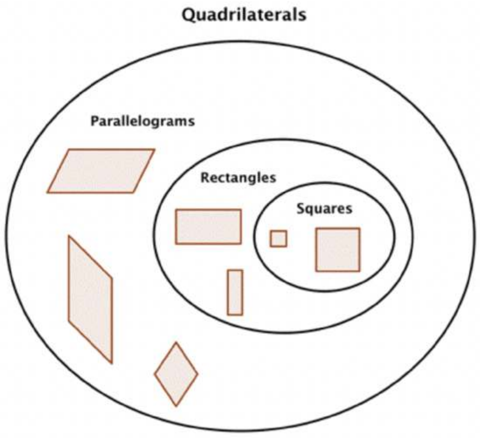
A square is also a rectangle because it has two sets of parallel sides and four right angles. A square is also a parallelogram because its opposite sides are parallel. So, a square can be classified in any of these three ways, with "parallelogram" being the least specific description and "square," the most descriptive.

Another quadrilateral that you might see is called a **rhombus**. All four sides of a rhombus are congruent. Its properties include that each pair of opposite sides is parallel, also making it a parallelogram.



In summary, all squares are rectangles, but not all rectangles are squares. All rectangles are parallelograms, but not all parallelograms are rectangles. And *all* of these shapes are quadrilaterals.

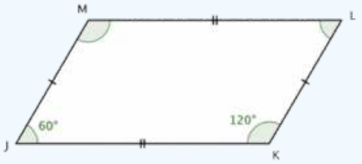
The diagram below illustrates the relationship between the different types of quadrilaterals.



You can use the properties of parallelograms to solve problems. Consider the example that follows.

**Example 2.5.1**

Determine the measures of $\angle M$ and $\angle L$.



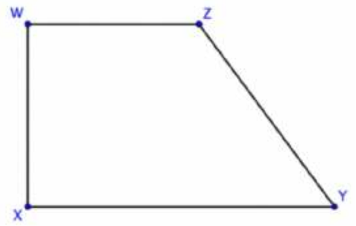
**Solution**

| | |
|---|---|
| $\angle L$ is opposite $\angle J$<br>$\angle M$ is opposite $\angle K$ | Identify opposite angles. |
| $\angle L \cong \angle J$<br>$\angle M \cong \angle K$ | A property of parallelograms is that opposite angles are congruent. |
| $m\angle J = 60^\circ$, so $m\angle L = 60^\circ$<br>$m\angle K = 120^\circ$, so $m\angle M = 120^\circ$ | Use the given angle measurements to determine measures of opposite angles. |

Thus, $m\angle L = 60^\circ$ and $m\angle M = 120^\circ$ .

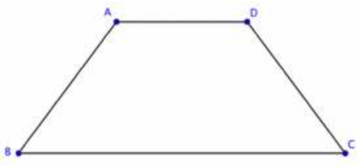
## Trapezoids

There is another special type of quadrilateral. This quadrilateral has the property of having *only one* pair of opposite sides that are parallel. Here is one example of a **trapezoid**.



Notice that $\overline{XY} \| \overline{WZ}$, and that $\overline{WX}$ and $\overline{ZY}$ are not parallel. You can easily imagine that if you extended sides $\overline{WX}$ and $\overline{ZY}$, they would intersect above the figure.

If the non-parallel sides of a trapezoid are congruent, the trapezoid is called an **isosceles trapezoid**. Like the similarly named triangle that has two sides of equal length, the isosceles trapezoid has a pair of opposite sides of equal length. The other pair of opposite sides is parallel. Below is an example of an isosceles trapezoid.



In this trapezoid $ABCD, \overline{BC} \| \overline{AD}$ and $\overline{AB} \cong \overline{CD}$.

### Try It 2.5.1

Which of the following statements is true?

A. Some trapezoids are parallelograms.
B. All trapezoids are quadrilaterals.
C. All rectangles are squares.
D. A shape cannot be a parallelogram and a quadrilateral.

**Answer**

A. Incorrect. Trapezoids have only one pair of parallel sides; parallelograms have two pairs of parallel sides. A trapezoid can never be a parallelogram.
B. Correct. Trapezoids are four-sided polygons, so they are all quadrilaterals.
C. Incorrect. Some rectangles may be squares, but not *all* rectangles have four congruent sides. All squares are rectangles however.
D. Incorrect. All parallelograms are quadrilaterals, so if it is a parallelogram, it is also a quadrilateral.

You can use the properties of quadrilaterals to solve problems involving trapezoids. Consider the example below.

### Example 2.5.2

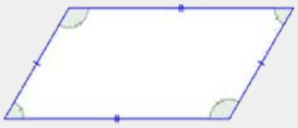
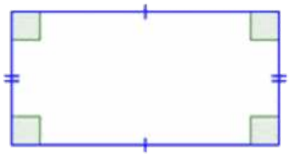
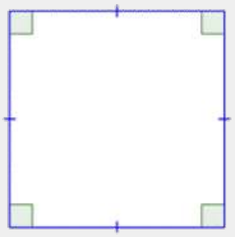
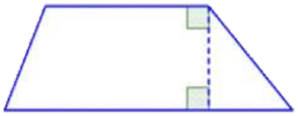
Find the measure of $\angle Q$.

**Solution**

| | |
|---|---|
| $m\angle P + m\angle Q + m\angle R + m\angle S = 360^\circ$ | The sum of the measures of the interior angles of a quadrilateral is 360º. |
| $m\angle R = 90^\circ$<br>$m\angle S = 90^\circ$ | The square symbol indicates a right angle. |
| $60^\circ + m\angle Q + 90^\circ + 90^\circ = 360^\circ$ | Since three of the four angle measures are given, you can find the fourth angle measurement. |
| $m\angle Q + 240^\circ = 360^\circ$<br>$m\angle Q = 120^\circ$ | Now, for $\angle Q$, calculate the measurement. From the image, you can see that it is an obtuse angle, so its measure must be greater than 90°. |

Thus, $m\angle Q = 120^\circ$.

The table below summarizes the special types of quadrilaterals and some of their properties.

| Name of Quadrilateral | Quadrilateral | Description |
|---|---|---|
| Parallelogram |  | 2 pairs of parallel sides.<br>Opposite sides and opposite angles are congruent. |
| Rectangle |  | 2 pairs of parallel sides.<br>4 right angles (90°).<br>Opposite sides are parallel and congruent.<br>All angles are congruent. |
| Square |  | 4 congruent sides.<br>4 right angles (90°).<br>Opposite sides are parallel.<br>All angles are congruent. |
| Trapezoid |  | Only one pair of opposite sides is parallel. |

## Summary

A quadrilateral is a mathematical name for a four-sided polygon. Parallelograms, squares, rectangles, and trapezoids are all examples of quadrilaterals. These quadrilaterals earn their distinction based on their properties, including the number of pairs of parallel sides they have and their angle and side measurements.

---

# 2.6: Perimeter and Area

**Learning Objectives**

- Find the perimeter of a polygon.
- Find the area of a polygon.
- Find the area and perimeter of non-standard polygons.

## Introduction

**Perimeter** and **area** are two important and fundamental mathematical topics. They help you to quantify physical space and also provide a foundation for more advanced mathematics found in algebra, trigonometry, and calculus. Perimeter is a measurement of the distance around a shape and area gives us an idea of how much surface the shape covers.

Knowledge of area and perimeter is applied practically by people on a daily basis, such as architects, engineers, and graphic designers, and is math that is very much needed by people in general. Understanding how much space you have and learning how to fit shapes together exactly will help you when you paint a room, buy a home, remodel a kitchen, or build a deck.

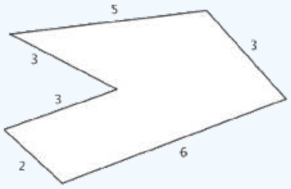
## Perimeter

The perimeter of a two-dimensional shape is the distance around the shape. You can think of wrapping a string around a triangle. The length of this string would be the perimeter of the triangle. Or walking around the outside of a park, you walk the distance of the park's perimeter. Some people find it useful to think "peRIMeter" because the edge of an object is its rim and peRIMeter has the word "rim" in it.

If the shape is a **polygon**, then you can add up all the lengths of the sides to find the perimeter. Be careful to make sure that all the lengths are measured in the same units. You measure perimeter in linear units, which is one dimensional. Examples of units of measure for length are inches, centimeters, or feet.

**Example 2.6.1**

Find the perimeter of the given figure. All measurements indicated are inches.



**Solution**

| | |
|---|---|
| $P = 5+3+6+2+3+3$ | Since all the sides are measured in inches, just add the lengths of all six sides to get the perimeter. |
| $P = 22 \text{ inches}$ | Remember to include units. |

This means that a tightly wrapped string running the entire distance around the polygon would measure 22 inches long.

**Example 2.6.2**

Find the perimeter of a triangle with sides measuring 6 m, 8 m, and 12 m.

**Solution**

| | |
|---|---|
| $P = 6+8+12$ | Since all the sides are measured in meters, just add the lengths of all three sides to get the perimeter. |

Thus, $P = 26 \text{ m}$.

Sometimes, you need to use what you know about a polygon in order to find the perimeter. Let’s look at the rectangle in the next example.

### ✓ Example 2.6.3

A rectangle has a length of 8 centimeters and a width of 3 centimeters. Find the perimeter.

**Solution**

| | |
|---|---|
| $P=3+3+8+8$ | Since this is a rectangle, the opposite sides have the same lengths, 3 cm. and 8 cm. Add up the lengths of all four sides to find the perimeter. |

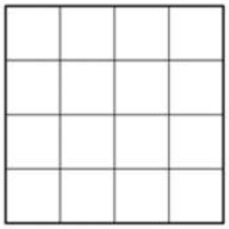
Thus, $P=22\text{ centimeters}.$

Notice that the perimeter of a rectangle always has two pairs of equal length sides. In the above example you could have also written $P=2(3)+2(8)=6+16=22\text{ cm}$ . The formula for the perimeter of a rectangle is often written as $P=2l+2w$, where $l$ is the length of the rectangle and $w$ is the width of the rectangle.

## Area of Parallelograms

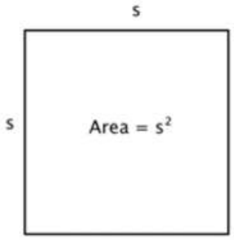
The area of a two-dimensional figure describes the amount of surface the shape covers. You measure area in square units of a fixed size. Examples of square units of measure are square inches, square centimeters, or square miles. When finding the area of a polygon, you count how many squares of a certain size will cover the region inside the polygon.

Let’s look at a 4 by 4 square.



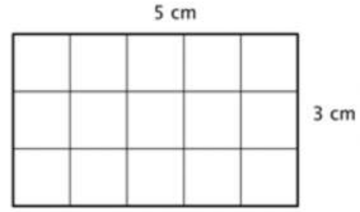
You can count that there are 16 squares, so the area is 16 square units. Counting out 16 squares doesn’t take too long, but what about finding the area if this is a larger square or the units are smaller? It could take a long time to count.

Fortunately, you can use multiplication. Since there are 4 rows of 4 squares, you can multiply $4\cdot 4$ to get 16 squares! And this can be generalized to a formula for finding the area of a square with any length, $s$: $\text{Area }=s\cdot s=s^2$ .

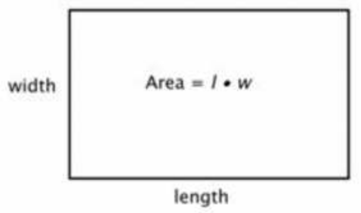


You can write “ $\text{in}^2$ ” for square inches and “ $\text{ft}^2$ ” for square feet.

To help you find the area of the many different categories of polygons, mathematicians have developed formulas. These formulas help you find the measurement more quickly than by simply counting. The formulas you are going to look at are all developed from the understanding that you are counting the number of square units *inside* the polygon. Let’s look at a rectangle.



You can count the squares individually, but it is much easier to multiply 3 times 5 to find the number more quickly. And, more generally, the area of any rectangle can be found by multiplying *length* times *width*.



### Example 2.6.4

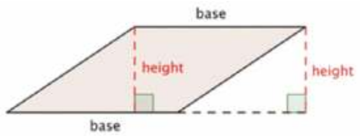
A rectangle has a length of 8 centimeters and a width of 3 centimeters. Find the area.

**Solution**

| | |
|---|---|
| $A = l \cdot w$ | Start with the formula for the area of a rectangle, which multiplies the length times the width. |
| $A = 8 \cdot 3$ | Substitute 8 for the length and 3 for the width. |
| $A = 24\text{ cm}^2$ | Be sure to include the units, in this case square centimeters. |

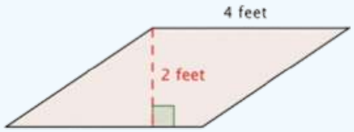
It would take 24 squares, each measuring 1 centimeter on a side, to cover this rectangle.

The formula for the area of any parallelogram (remember, a rectangle is a type of parallelogram) is the same as that of a rectangle: $\text{Area } = l \cdot w$ . Notice in a rectangle, the length and the width are perpendicular. This should also be true for all parallelograms. *Base* ( $b$ for the length (of the base), and *height* ( $h$) for the width of the line perpendicular to the base is often used. So the formula for a parallelogram is generally written, $A = b \cdot h$ .



### Example 2.6.5

Find the area of the parallelogram.



**Solution**

| | |
|---|---|
| $A = b \cdot h$ | Start with the formula for the area of a parallelogram: |
| $A = 4 \cdot 2$ | $\text{Area} = \text{base} \cdot \text{height}$ .<br>Substitute the values into the formula. |
| $A = 8$ | Multiply. |

The area of the parallelogram is 8 $\text{ft}^2$.

### Try It 2.6.1

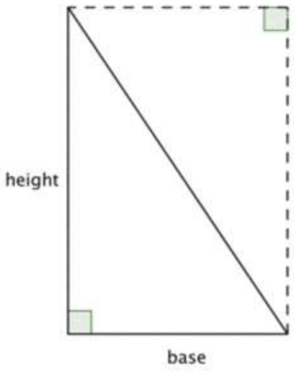
Find the area of a parallelogram with a height of 12 feet and a base of 9 feet.

**Answer**

The height of the parallelogram is 12 and the base of the parallelogram is 9; the area is 12 times 9, or $108 \text{ ft}^2$.

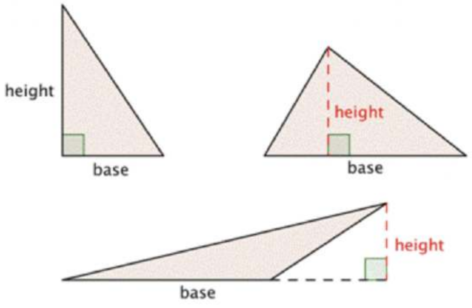
## Area of Triangles and Trapezoids

The formula for the area of a triangle can be explained by looking at a right triangle. In the image below is a rectangle with the same height and base as the original triangle. The area of the triangle is one half of the rectangle!



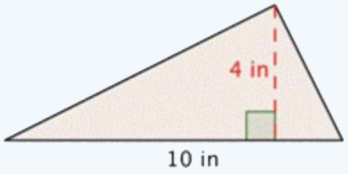
Since the area of two congruent triangles is the same as the area of a rectangle, you can come up with the formula $\text{Area} = \frac{1}{2} b \cdot h$ to find the area of a triangle.

When you use the formula for a triangle to find its area, it is important to identify a base and its corresponding height, which is perpendicular to the base.



### Example 2.6.6

A triangle has a height of 4 inches and a base of 10 inches. Find the area.



**Solution**

| | |
|---|---|
| $A = \frac{1}{2}bh$ | Start with the formula for the area of a triangle. |
| $A = \frac{1}{2} \cdot 10 \cdot 4$ | Substitute 10 for the base and 4 for the height. |
| $A = \frac{1}{2} \cdot 40$<br>$A = 20$ | Multiply. |

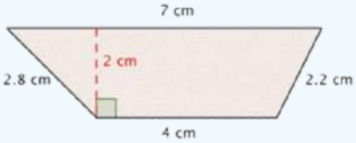
Thus, the area is $A = 20 \text{ in}^2$.

Now let's look at the trapezoid. To find the area of a trapezoid, take the average length of the two parallel bases and multiply that length by the height: $A = \frac{1}{2}(b_1 + b_2)\,h$.

An example is provided below. Notice that the height of a trapezoid will always be perpendicular to the bases (just like when you find the height of a parallelogram).

### ✓ Example 2.6.7

Find the area of the trapezoid.



**Solution**

| | |
|---|---|
| $A = \frac{(b_1 + b_2)}{2}h$ | Start with the formula for the area of a trapezoid. |
| $A = \frac{(4+7)}{2} \cdot 2$<br>$A = \frac{11}{2} \cdot 2$<br>$A = 11$ | Substitute 4 and 7 for the bases and 2 for the height, and find $A$. |

The area of the trapezoid is $11 \text{ cm}^2$.

### Area Formulas

Use the following formulas to find the areas of different shapes.

square: $A = s^2$

rectangle: $A = l \cdot w$

parallelogram: $A = b \cdot h$

triangle: $A = \frac{1}{2}b \cdot h$

trapezoid: $A=\frac{(b_1+b_2)}{2}h$

## Working with Perimeter and Area

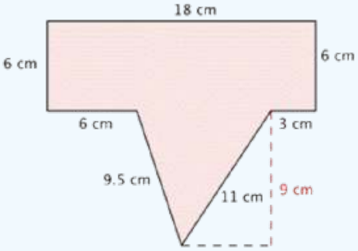
Often you need to find the area or perimeter of a shape that is not a standard polygon. Artists and architects, for example, usually deal with complex shapes. However, even complex shapes can be thought of as being composed of smaller, less complicated shapes, like rectangles, trapezoids, and triangles.

To find the perimeter of non-standard shapes, you still find the distance around the shape by adding together the length of each side.

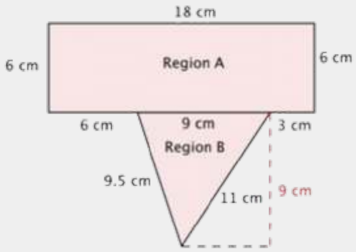
Finding the area of non-standard shapes is a bit different. You need to create regions *within* the shape for which you can find the area, and add these areas together. Have a look at how this is done below.

### ✓ Example 2.6.8

Find the area and perimeter of the polygon.



**Solution**

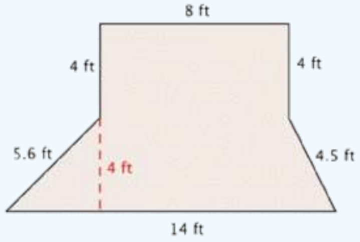
| | |
|---|---|
| $P=18+6+3+11+9.5+6+6$<br>$P=59.5\text{ cm}$ | To find the perimeter, add together the lengths of the sides. To stay organized, start at the top and work clockwise around the shape. |
|  <br>*Area of Polygon* = (*Area of A*) + (*Area of B*) | To find the area, divide the polygon into two separate, simpler regions. The area of the entire polygon will equal the sum of the areas of the two regions. |
| $\begin{aligned}\text{Area of Region }A&=l\cdot w\\&=18\cdot 6\\&=108\end{aligned}$ | Region A is a rectangle. To find the area, multiply the length (18) by the width (6).<br>The area of Region A is 108 cm$^2$. |
| $\begin{aligned}\text{Area of Region B}&=\frac{1}{2}b\cdot h\\&=\frac{1}{2}\cdot 9\cdot 9\\&=\frac{1}{2}\cdot 81\\&=40.5\end{aligned}$ | Region B is a triangle. To find the area, use the formula $\frac{1}{2}bh$, where the base is 9 and the height is 9.<br>The area of Region B is 40.5 cm$^2$. |
| $108\text{ cm}^2+40.5\text{ cm}^2=148.5\text{ cm}^2$ | Add the regions together. |

$$\begin{aligned}\text{Perimeter }&=59.5\text{ cm}\\\text{Area }&=148.5\text{ cm}^2\end{aligned}$$

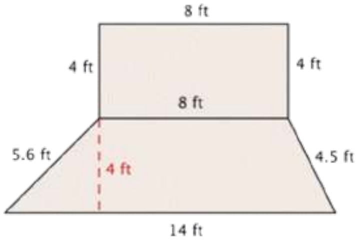
You also can use what you know about perimeter and area to help solve problems about situations like buying fencing or paint, or determining how big a rug is needed in the living room. Here's a fencing example.

**✓ Example 2.6.9**

Rosie is planting a garden with the dimensions shown below. She wants to put a thin, even layer of mulch over the entire surface of the garden. The mulch costs \$3 per square foot. How much money will she have to spend on mulch?



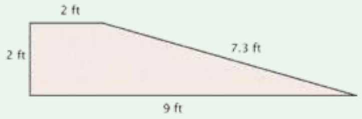
**Solution**

| | |
|---|---|
|  | This shape is a combination of two simpler shapes: a rectangle and a trapezoid. Find the area of each. |
| $A = l \cdot w$<br>$A = 8 \cdot 4$<br>$A = 32\text{ ft}^2$ | Find the area of the rectangle. |
| $A = \frac{(b_1 + b_2)}{2} h$<br>$A = \frac{(14+8)}{2} \cdot 4$<br>$A = \frac{22}{2} \cdot 4$<br>$A = 11 \cdot 4$<br>$A = 44\text{ ft}^2$ | Find the area of the trapezoid. |
| $32\text{ ft}^2 + 44\text{ ft}^2 = 76\text{ ft}^2$ | Add the measurements. |
| $76\text{ ft}^2 \cdot \$3 = \$228$ | Multiply by \$3 to find out how much Rosie will have to spend. |

Rosie will spend \$228 to cover her garden with mulch.

**✎ Try It 2.6.2**

Find the area of the shape shown below.



**Answer**

This shape is a trapezoid, so you can use the formula $A = \frac{(b_1 + b_2)}{2} h$ to find the area: $A = \frac{(2+9)}{2} \cdot 2 = 11\text{ ft}^2$ .

## Summary

The perimeter of a two-dimensional shape is the distance around the shape. For a polygon this is found by adding up all the sides (as long as they are all the same unit). The area of a two-dimensional shape is found by counting the number of squares that cover the shape. Many formulas have been developed to quickly find the area of standard polygons, like triangles and parallelograms.

# 2.7: Circles

**Learning Objectives**

- Identify properties of circles.
- Find the circumference of a circle.
- Find the area of a circle.
- Find the area and perimeter of composite geometric figures.

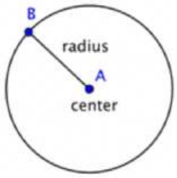
## Introduction

Circles are a common shape. You see them all over: wheels on a car, Frisbees passing through the air, DVDs delivering data. These are all circles.

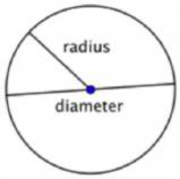
A circle is a two-dimensional figure just like polygons and quadrilaterals. However, circles are measured differently than these other shapes. You even have to use some different terms to describe them. Let's take a look at this interesting shape.

## Properties of Circles

A **circle** represents a set of points, all of which are the same distance away from a fixed, middle point. This fixed point is called the **center**. The distance from the center of the circle (point A) to any point on the circle (point B) is called the **radius**. A circle is named by the point at its center, so this circle would be called Circle A.



When two radii (the plural of radius) are put together to form a line segment across the circle, you have a **diameter**. The diameter of a circle passes through the center of the circle and has its endpoints on the circle itself.



The diameter of any circle is two times the length of that circle's radius. It can be represented by the expression $2r$, or "two times the radius." So if you know a circle's radius, you can multiply it by 2 to find the diameter; this also means that if you know a circle's diameter, you can divide by 2 to find the radius.

**Example 2.7.1**

Find the diameter of the circle.

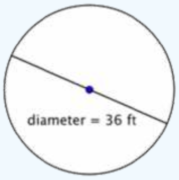
### Solution

| | |
|---|---|
| $d=2r$<br>$d=2(7)$<br>$d=14$ | The diameter is two times the radius, or $2r$. The radius of this circle is 7 inches, so the diameter is $2(7)=14$ inches. |

The diameter is 14 inches.

**✓ Example 2.7.2**

Find the radius of the circle.



**Solution**

| | |
|---|---|
| $r=\frac{1}{2}d$<br>$r=\frac{1}{2}(36)$<br>$r=18$ | The radius is half the diameter, or $\frac{1}{2}d$. The diameter of this circle is 36 feet, so the radius is $\frac{1}{2}(36)=18$ feet. |

The radius is 18 feet.

## Circumference

The distance around a circle is called the **circumference**. (Recall, the distance around a polygon is the perimeter.)

One interesting property about circles is that the ratio of a circle's circumference and its diameter is the same for all circles. No matter the size of the circle, the ratio of the circumference and diameter will be the same.

Some actual measurements of different items are provided below. The measurements are accurate to the nearest millimeter or quarter inch (depending on the unit of measurement used). Look at the ratio of the circumference to the diameter for each one. Although the items are different, the ratio for each is approximately the same.

| **Item** | **Circumference ( $C$) (rounded to nearest hundredth)** | **Diameter ( $d$)** | **Ratio $\frac{C}{d}$** |
|---|---|---|---|
| Cup | 253 mm | 79 mm | $\frac{253}{79}=3.2025\ldots$ |
| Quarter | 84 mm | 27 mm | $\frac{84}{27}=3.1111\ldots$ |
| Bowl | 37.25 inches | 11.75 inches | $\frac{37.25}{11.75}=3.1702\ldots$ |

The circumference and the diameter are approximate measurements, since there is no precise way to measure these dimensions exactly. If you were able to measure them more precisely, however, you would find that the ratio $\frac{C}{d}$ would move towards 3.14 for each of the items given. The mathematical name for the ratio $\frac{C}{d}$ is **pi**, and is represented by the Greek letter $\pi$.

$\pi$ is a non-terminating, non-repeating decimal, so it is impossible to write it out completely. The first 10 digits of $\pi$ are 3.141592653; it is often rounded to 3.14 or estimated as the fraction $\frac{22}{7}$. Note that both 3.14 and $\frac{22}{7}$ are *approximations* of $\pi$, and are used in calculations where it is not important to be precise.

Since you know that the ratio of circumference to diameter (or $\pi$) is consistent for all circles, you can use this number to find the circumference of a circle if you know its diameter.

$$\frac{C}{d}=\pi,\text{ so }C=\pi d$$

$$\text{Also, since }d=2r,\text{ then }C=\pi d=\pi(2r)=2\pi r\ .$$

### Circumference of a Circle

To find the circumference ( $C$) of a circle, use one of the following formulas:

If you know the diameter ( $d$) of a circle: $C=\pi d$

If you know the radius ( $r$) of a circle: $C=2\pi r$

### Example 2.7.3

Find the circumference of the circle.

**Solution**

| | |
|---|---|
| $C=\pi d$<br>$C=\pi\cdot 9$<br>$C\approx 3.14\cdot 9$<br>$C\approx 28.26$ | To calculate the circumference given a diameter of 9 inches, use the formula $C=\pi d$. Use 3.14 as an approximation for $\pi$. |
| | Since you are using an approximation for $\pi$, you cannot give an exact measurement of the circumference. Instead, you use the symbol $\approx$ to indicate “approximately equal to.” |

The circumference is $9\pi$ or approximately 28.26 inches.

### Example 2.7.4

Find the circumference of a circle with a radius of 2.5 yards.

**Solution**

| | |
|---|---|
| $C=2\pi r$<br>$C=2\pi\cdot 2.5$<br>$C=\pi\cdot 5$<br>$C\approx 3.14\cdot 5$<br>$C\approx 15.7$ | To calculate the circumference of a circle given a radius of 2.5 yards, use the formula $C=2\pi r$. Use 3.14 as an approximation for $\pi$. |

The circumference is $5\pi$ or approximately 15.7 yards.

### Try It 2.7.1

A circle has a radius of 8 inches. What is its circumference, rounded to the nearest inch?

**Answer**

The formula for circumference when the radius is given is $C=2\pi r$. Thus, $C=2\pi\cdot 8\approx$ 50 inches.

## Area

$\pi$ is an important number in geometry. You have already used it to calculate the circumference of a circle. You use $\pi$ when you are figuring out the *area* of a circle, too.

**Area of a Circle**

To find the area ( $A$) of a circle, use the formula: $A=\pi r^2$

**Example 2.7.5**

Find the area of the circle.

**Solution**

| | |
|---|---|
| $A=\pi r^2$<br>$A=\pi\cdot 3^2$<br>$A=\pi\cdot 9$<br>$A\approx 3.14\cdot 9$<br>$A\approx 28.26$ | To find the area of this circle, use the formula $A=\pi r^2$ . Remember to write the answer in terms of square units, since you are finding the area. |

The area is $9\pi$ $\text{feet}^2$ or approximately 28.26 $\text{feet}^2$.

**Try It 2.7.2**

A button has a diameter of 20 millimeters. What is the area of the button? Use 3.14 as an approximation of $\pi$.

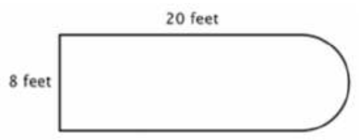
**Answer**

To find the area, use the formula $A=\pi r^2$ . The radius is half of the diameter, which is 10 mm. Thus, $A=\pi\cdot 10^2=100\pi$ . The answer is 314 $\text{mm}^2$.

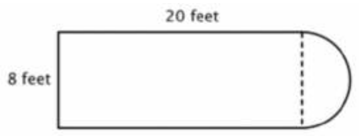
## Composite Figures

Now that you know how to calculate the circumference and area of a circle, you can use this knowledge to find the perimeter and area of composite figures. The trick to figuring out these types of problems is to identify shapes (and parts of shapes) within the composite figure, calculate their individual dimensions, and then add them together.

For example, look at the image below. Is it possible to find the perimeter?

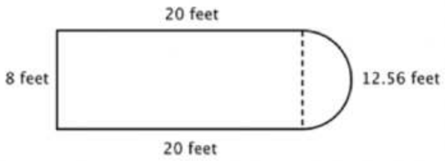


The first step is to identify simpler figures within this composite figure. You can break it down into a rectangle and a semicircle, as shown below.



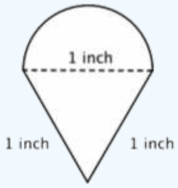
You know how to find the perimeter of a rectangle, and you know how to find the circumference of a circle. Here, the perimeter of the three solid sides of the rectangle is $8+20+20=48\text{ feet}$ . (Note that only three sides of the rectangle will add into the perimeter of the composite figure because the other side is not at an edge; it is covered by the semicircle!)

To find the circumference of the semicircle, use the formula $C = \pi d$ with a diameter of 8 feet, then take half of the result. The circumference of the semicircle is 4 $\pi$, or approximately 12.56 feet, so the total perimeter is about 60.56 feet.

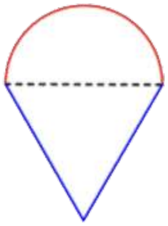
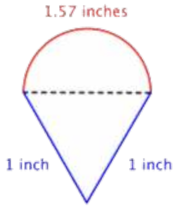


## Example 2.7.6

Find the perimeter (to the nearest hundredth) of the composite figure made up of a semicircle and a triangle.



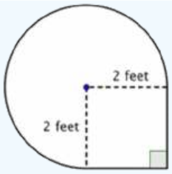
**Solution**

| | |
|---|---|
|  | Identify smaller shapes within the composite figure. This figure contains a semicircle and a triangle. |
| Diameter $(d) = 1$<br>$C = \pi d$<br>$C = \pi(1)$<br>$C = \pi$ | Find the circumference of the circle. Then divide by 2 to find the circumference of the semicircle. |
| Circumference of semicircle $= \frac{1}{2}\pi$ or approximately 1.57 inches<br>$1+1+\frac{1}{2}\pi \approx 3.57$ inches<br><br> | Find the total perimeter by adding the circumference of the semicircle and the lengths of the two legs. Since our measurement of the semicircle's circumference is approximate, the perimeter will be an approximation also. |

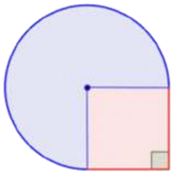
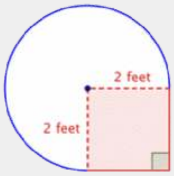
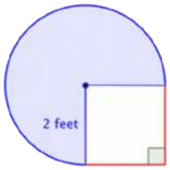
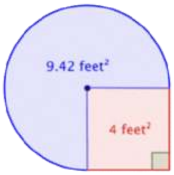
Thus, the perimeter is approximately 3.57 inches.

## Example 2.7.7

Find the area of the composite figure, made up of three-quarters of a circle and a square, to the nearest hundredth.



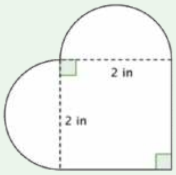
**Solution**

| | |
|---|---|
|  | Identify smaller shapes within the composite figure. This figure contains a circular region and a square. If you find the area of each, you can find the area of the entire figure. |
|  <br>$\text{Area of square} = s^2$<br>$= (2)^2$<br>$= 4\text{ ft}^2$ | Find the area of the square. |
|  <br>$\text{Area of full circle} = \pi r^2$<br>$= \pi(2)^2$<br>$= 4\pi\text{ ft}^2$ | Find the area of the circular region. The radius is 2 feet. |
| $\text{Area of region} = \frac{3}{4}\cdot 4\pi$<br>$= 3\pi$<br>$\approx 3\cdot 3.14\text{ ft}^2$<br>This is approximately $9.42\text{ ft}^2$ | Note that the region is $\frac{3}{4}$ of a whole circle, so you need to multiply the area of the circle by $\frac{3}{4}$.<br>Use 3.14 as an approximation for $\pi$. |
|  <br>$4\text{ ft}^2 + 3\pi\text{ ft}^2 \approx 13.42\text{ ft}^2$ | Add the two regions together. Since your measurement of the circular's area is approximate, the area of the figure will be an approximation also. |

The area is approximately $13.42\text{ ft}^2$.

**Try It 2.7.3**

What is the area (to the nearest hundredth) of the figure shown below? (Both rounded regions are semicircles.)



**Answer**

Imagine the two semicircles being put together to create one circle. The radius of the circle is 1 inch; this means the area of the circle is $\pi r^2 = \pi \cdot 1^2 = \pi$. The area of the square is 2 · 2 = 4. Adding those together yields 7.14 in$^2$.

## Summary

Circles are an important geometric shape. The distance around a circle is called the circumference, and the interior space of a circle is called the area. Calculating the circumference and area of a circle requires a number called pi ( $\pi$), which is a non-terminating, non-repeating decimal. Pi is often approximated by the values 3.14 and $\frac{22}{7}$. You can find the perimeter or area of composite shapes, including shapes that contain circular sections, by applying the circumference and area formulas where appropriate.

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# 2.8: Solids

**Learning Objectives**

- Identify geometric solids.
- Find the volume of geometric solids.
- Find the volume of a composite geometric solid.

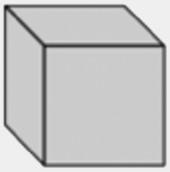
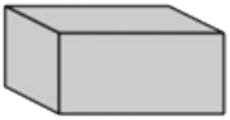
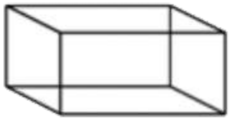
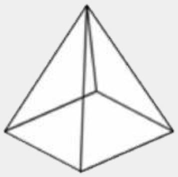
## Introduction

Living in a two-dimensional world would be pretty boring. Thankfully, all of the physical objects that you see and use every day, such as computers, phones, cars, and shoes, exist in three dimensions. They all have length, width, and height. (Even very thin objects like a piece of paper are three-dimensional. The thickness of a piece of paper may be a fraction of a millimeter, but it does exist.)

In the world of geometry, it is common to see three-dimensional figures. In mathematics, a flat side of a three-dimensional figure is called a **face**. **Polyhedrons** are shapes that have four or more faces, each one being a polygon. These include cubes, prisms, and pyramids. Sometimes you may even see single figures that are composites of two of these figures. Let’s take a look at some common polyhedrons.

## Identifying Solids

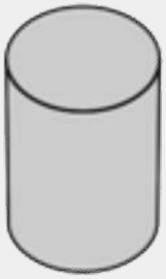
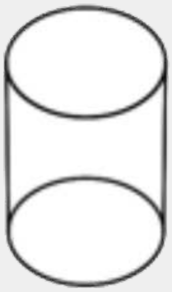
The first set of solids contains rectangular bases. Have a look at the table below, which shows each figure in both solid and transparent form.

| Name | Definition | Solid Form | Transparent Form |
|---|---|---|---|
| Cube | A six-sided polyhedron that has congruent squares as faces. |  |  |
| Rectangular prism | A polyhedron that has three pairs of congruent, rectangular, parallel faces. |  |  |
| Pyramid | A polyhedron with a polygonal base and a collection of triangular faces that meet at a point. |  |  |

Notice the different names that are used for these figures. A **cube** is different than a square, although they are sometimes confused with each other; a cube has three dimensions, while a square only has two. Likewise, you would describe a shoebox as a **rectangular prism** (not simply a rectangle), and the ancient **pyramids** of Egypt as…well, as pyramids (not triangles)! A rectangular prism is casually referred to as a box.

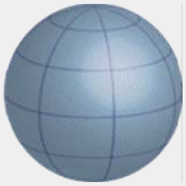
In this next set of solids, each figure has a circular base.

| Name | Definition | Solid Form | Transparent Form |
|---|---|---|---|

|  |  |  |  |
|---|---|---|---|
| Cylinder | A solid figure with a pair of circular, parallel bases and a round, smooth face between them. |  |  |
| Cone | A solid figure with a single circular base and a round, smooth face that diminishes to a single point. |  |  |

Take a moment to compare a pyramid and a **cone**. Notice that a pyramid has a rectangular base and flat, triangular faces; a cone has a circular base and a smooth, rounded body.

Finally, let’s look at a shape that is unique: a **sphere**.

| **Name** | **Definition** | **Solid Form** | **Transparent Form** |
|---|---|---|---|
| Sphere | A solid, round figure where every point on the surface is the same distance from the center. |  |  |

There are many spherical objects all around you: soccer balls, tennis balls, and baseballs being three common items. While they may not be perfectly spherical, they are generally referred to as spheres.

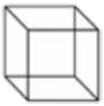
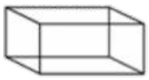
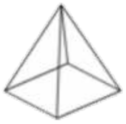
✓ Example 2.8.1

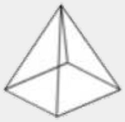
A three-dimensional figure has the following properties:

- It has a rectangular base.
- It has four triangular faces.

What kind of a solid is it?

**Solution**

|  |  |
|---|---|
|    | A rectangular base indicates that it must be a cube, rectangular prism, or pyramid. |

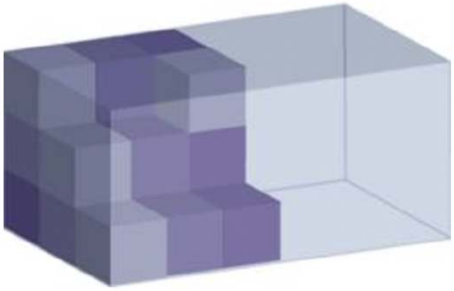
| | |
|---|---|
|  | Since the faces are triangular, it must be a pyramid. |

Thus, the solid is a pyramid.

## Volume

Recall that perimeter measures one dimension (length), and area measures two dimensions (length and width). To measure the amount of space a three-dimensional figure takes up, you use another measurement called **volume**.

To visualize what “volume” measures, look back at the transparent image of the rectangular prism mentioned earlier (or just think of an empty shoebox). Imagine stacking identical cubes inside that box so that there are no gaps between any of the cubes. Imagine filling up the entire box in this manner. If you counted the number of cubes that fit inside that rectangular prism, you would have its volume.

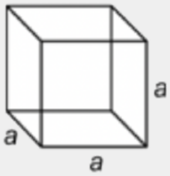
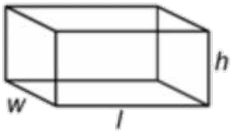


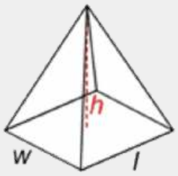
Volume is measured in cubic units. The shoebox illustrated above may be measured in cubic inches (usually represented as $\text{in}^3$ or $\text{inches}^3$), while the Great Pyramid of Egypt would be more appropriately measured in cubic meters ($\text{m}^3$ or $\text{meters}^3$).

To find the volume of a geometric solid, you could create a transparent version of the solid, create a bunch of 1 by 1 by 1 cubes, and then stack them carefully inside. However, that would take a long time! A much easier way to find the volume is to become familiar with some geometric formulas, and to use those instead.

Let’s go through the geometric solids once more and list the volume formula for each.

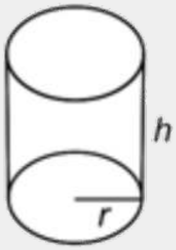
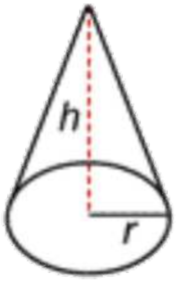
As you look through the list below, you may notice that some of the volume formulas look similar to their area formulas. To find the volume of a rectangular prism, you find the area of the base and then multiply that by the height.

| Name | Transparent Form | Volume Formula |
|---|---|---|
| Cube |  | $V = a \cdot a \cdot a = a^3$<br>$a =$ the length of one side |
| Rectangular prism |  | $V = l \cdot w \cdot h$<br>$l =$ length<br>$w =$ width<br>$h =$ height |

| | | |
|---|---|---|
| Pyramid |  | $V = \frac{l \cdot w \cdot h}{3}$<br>$l =$ length<br>$w =$ width<br>$h =$ height |

Remember that all cubes are rectangular prisms, so the formula for finding the volume of a cube is the area of the base of the cube times the height.

Now let’s look at solids that have a circular base.

| **Name** | **Transparent Form** | **Volume Formula** |
|---|---|---|
| Cylinder |  | $V = \pi \cdot r^2 \cdot h$<br>$r =$ radius<br>$h =$ height |
| Cone |  | $V = \frac{\pi \cdot r^2 \cdot h}{3}$<br>$r =$ radius<br>$h =$ height |

Here you see the number $\pi$ again.

The volume of a **cylinder** is the area of its base, $\pi r^2$, times its height, $h$.

Compare the formula for the volume of a cone ( $V = \frac{\pi \cdot r^2 \cdot h}{3}$ ) with the formula for the volume of a pyramid ( $V = \frac{l \cdot w \cdot h}{3}$ ). The numerator of the cone formula is the volume formula for a cylinder, and the numerator of the pyramid formula is the volume formula for a rectangular prism. Then divide each by 3 to find the volume of the cone and the pyramid. Looking for patterns and similarities in the formulas can help you remember which formula refers to a given solid.

Finally, the formula for a sphere is provided below. Notice that the radius is cubed, not squared and that the quantity $\pi r^3$ is multiplied by $\frac{4}{3}$.

| **Name** | **Transparent Form** | **Volume Formula** |
|---|---|---|
| Sphere |  | $V = \frac{4}{3}\pi r^3$<br>$r =$ radius |

## Applying the Formulas

You know how to identify the solids, and you also know the volume formulas for these solids. To calculate the actual volume of a given shape, all you need to do is substitute the solid's dimensions into the formula and calculate.

In the examples below, notice that cubic units (meters$^3$, inches$^3$, feet$^3$) are used.

### ✓ Example 2.8.2

Find the volume of a cube with side lengths of 6 meters.

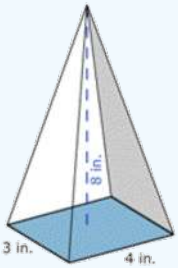
**Solution**

| | |
|---|---|
| $\begin{aligned} V &= a \cdot a \cdot a = a^3 \\ a &= \text{side length} \end{aligned}$ | Identify the proper formula to use. |
| $V = 6 \cdot 6 \cdot 6 = 6^3$ | Substitute $a = 6$ into the formula. |
| $V = 6 \cdot 6 \cdot 6 = 216$ | Calculate the volume. |

Thus, the volume of the cube is $216$ meters$^3$.

### ✓ Example 2.8.3

Find the volume of the solid figure shown below.



**Solution**

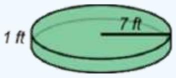
| | |
|---|---|
| Pyramid | Identify the shape. It has a rectangular base and rises to a point, so it is a pyramid. |
| $V = \frac{l \cdot w \cdot h}{3}$ | Identify the proper formula to use. |
| $\begin{aligned} l &= \text{length} \\ w &= \text{width} \\ h &= \text{height} \end{aligned}$<br>$\begin{aligned} 4 &= \text{length} \\ 3 &= \text{width} \\ 8 &= \text{height} \end{aligned}$ | Use the image to identify the dimensions. Then substitute $l = 4, w = 3, \text{and } h = 8$ into the formula. |
| $V = \frac{4 \cdot 3 \cdot 8}{3}$<br>$V = \frac{96}{3}$<br>$V = 32$ | Calculate the volume and choose the appropriate units. |

The volume of the pyramid is $32$ inches$^3$.

### ✓ Example 2.8.4

Find the volume of the solid shown below.

Use 3.14 for $\pi$, and round the answer to the nearest hundredth.



**Solution**

| | |
|---|---|
| Cylinder | Identify the shape. It has a circular base and has uniform thickness (or height), so it is a cylinder. |
| $V = \pi \cdot r^2 \cdot h$ | Identify the proper formula to use. |
| $V = \pi \cdot 7^2 \cdot 1$ | Use the image to identify the dimensions. Then substitute $r = 7$ and $h = 1$ into the formula. |
| $V = \pi \cdot 49 \cdot 1$<br>$V = 49\pi$<br>$V \approx 153.86$ | Calculate the volume, using 3.14 as an approximation for $\pi$. |

The volume of the cylinder is $49\pi$ or approximately $153.86 \text{ feet}^3$.

### ✎ Try It 2.8.1

Find the volume of a rectangular prism that is 8 inches long, 3 inches wide, and 10 inches tall.

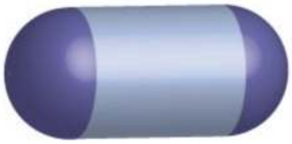
**Answer**

To find the volume of the rectangular prism, use the formula $V = l \cdot w \cdot h$, and then substitute in the values for the length, width, and height. $V = 8 \text{ inches } \cdot 3 \text{ inches } \cdot 10 \text{ inches } = 240 \text{ inches}^3$ .

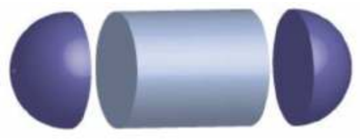
## Composite Solids

Composite geometric solids are made from two or more geometric solids. You can find the volume of these solids as well, as long as you are able to figure out the individual solids that make up the composite shape.

Look at the image of a capsule below. Each end is a half-sphere. You can find the volume of the solid by taking it apart. What solids can you break this shape into?



You can break it into a cylinder and two half-spheres.

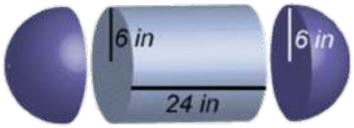


Two half-spheres form a whole one, so if you know the volume formulas for a cylinder and a sphere, you can find the volume of this capsule.

### Example 2.8.5

If the radius of the spherical ends is 6 inches, find the volume of the solid below. Use 3.14 for $\pi$. Round your final answer to the nearest whole number.

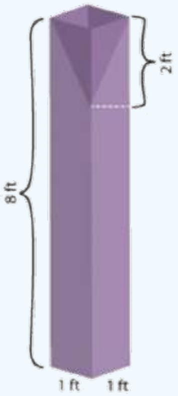
**Solution**

| | |
|---|---|
|  | Identify the composite solids. This capsule can be thought of as a cylinder with a half-sphere on each end. |
| Volume of a cylinder: $\pi \cdot r^2 \cdot h$<br>Volume of a sphere: $\frac{4}{3}\pi r^3$ | Identify the proper formulas to use. |
| Volume of a cylinder: $\pi \cdot 6^2 \cdot 24$<br>Volume of a sphere: $\frac{4}{3}\pi \cdot 6^3$ | Substitute the dimensions into the formulas.<br>The height of a cylinder refers to the section between the two circular bases. This dimension is given as 24 inches, so $h = 24$. |
| Volume of the cylinder:<br>$V = \pi \cdot 36 \cdot 24$<br>$= 864 \cdot \pi$<br>$\approx 2712.96$<br>Volume of the sphere:<br>$V = \frac{4}{3}\pi \cdot 216$<br>$= 288 \cdot \pi$<br>$\approx 904.32$ | The radius of the sphere is 6 inches.<br>You can use $r = 6$ in both formulas.<br>Calculate the volume of the cylinder and the sphere. |
| Volume of capsule:<br>$2712.96 + 904.32 \approx 3617.28$ | Add the volumes. |

The volume of the capsule is $1152\pi$ or approximately $3617$ inches$^3$.

### Example 2.8.6

A sculptor carves a rectangular prism out of a solid piece of wood. Then, at the top, she hollows out an inverted pyramid. The solid, and its dimensions, are shown at right. What is the volume of the finished piece?



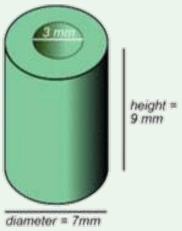
**Solution**

| | |
|---|---|
| | Identify the composite solids. This sculpture can be thought of as a rectangular prism with a pyramid removed. Hence the volume of the pyramid will be subtracted (taken away) from the volume of the box. |
| Volume of rectangular prism: $l \cdot w \cdot h$<br>Volume of pyramid: $\frac{l \cdot w \cdot h}{3}$ | Identify the proper formulas to use. |
| Volume of rectangular prism: $1 \cdot 1 \cdot 8 = 8$<br>Volume of pyramid: $\frac{1 \cdot 1 \cdot 2}{3} = \frac{2}{3}$ | Substitute the dimensions into the formulas, and calculate. |
| Volume of sculpture: $V = 8 - \frac{2}{3} = 7\frac{1}{3}$ | Subtract the volume of the pyramid from the volume of the rectangular prism. |

The volume of the sculpture is $7\frac{1}{3}$ feet$^3$.

## Try It 2.8.2

A machine takes a solid cylinder with a height of 9 mm and a diameter of 7 mm, and bores a hole all the way through it. The hole that it creates has a diameter of 3 mm. Which of the following expressions would correctly find the volume of the solid?



A. $\left(\pi \cdot 7^2 \cdot 9\right) - \left(\pi \cdot 3^2 \cdot 9\right)$
B. $\left(\pi \cdot 3.5^2 \cdot 9\right) - \left(\pi \cdot 1.5^2 \cdot 9\right)$
C. $\left(\pi \cdot 7^2 \cdot 9\right) + \left(\pi \cdot 3^2 \cdot 9\right)$
D. $\left(\pi \cdot 3.5^2 \cdot 9\right) + \left(\pi \cdot 1.5^2 \cdot 9\right)$

**Answer**

B. You find the volume of the entire cylinder by multiplying $\pi \cdot 3.5^2 \cdot 9$, then subtract the empty cylinder in the middle, which is found by multiplying $\pi \cdot 1.5^2 \cdot 9$.

## Summary

Three-dimensional solids have length, width, and height. You use a measurement called volume to figure out the amount of space that these solids take up. To find the volume of a specific geometric solid, you can use a volume formula that is specific to that solid. Sometimes, you will encounter composite geometric solids. These are solids that combine two or more basic solids. To find the volume of these, identify the simpler solids that make up the composite figure, find the volumes of those solids, and combine them as needed.

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# 2.9: Chapter Review and Glossary

## Chapter Review

### Geometric Concepts

Geometric shapes and figures are all around us. A point is a zero-dimensional object that defines a specific location on a plane. A line is made up of an infinite number of points, all arranged next to each other in a straight pattern, and going on forever. A ray begins at one point and goes on towards infinity in one direction only. A plane can be described as a two-dimensional canvas that goes on forever.

When two rays share an endpoint, an angle is formed. Angles can be described as acute, right, obtuse, or straight, and are measured in degrees. You can use a protractor (a special math tool) to closely measure the size of any angle.

### Properties of Angles

Parallel lines do not intersect, while perpendicular lines cross at a 90°. angle. Two angles whose measurements add up to 180° are said to be supplementary, and two angles whose measurements add up to 90° are said to be complementary. For most pairs of intersecting lines, all you need is the measurement of one angle to find the measurements of all other angles formed by the intersection.

### Triangles

Triangles are one of the basic shapes in the real world. Triangles can be classified by the characteristics of their angles and sides, and triangles can be compared based on these characteristics. The sum of the measures of the interior angles of any triangle is 180°. Congruent triangles are triangles of the same size and shape. They have corresponding sides of equal length and corresponding angles of the same measurement. Similar triangles have the same shape, but not necessarily the same size. The lengths of their sides are proportional. Knowledge of triangles can be a helpful in solving real-world problems.

### The Pythagorean Theorem

The Pythagorean Theorem states that in any right triangle, the sum of the squares of the lengths of the triangle’s legs is the same as the square of the length of the triangle’s hypotenuse. This theorem is represented by the formula $a^2+b^2=c^2$ . Put simply, if you know the lengths of two sides of a right triangle, you can apply the Pythagorean Theorem to find the length of the third side. Remember, this theorem only works for right triangles.

### Quadrilaterals

A quadrilateral is a mathematical name for a four-sided polygon. Parallelograms, squares, rectangles, and trapezoids are all examples of quadrilaterals. These quadrilaterals earn their distinction based on their properties, including the number of pairs of parallel sides they have and their angle and side measurements.

### Perimeter and Area

The perimeter of a two-dimensional shape is the distance around the shape. For a polygon this is found by adding up all the sides (as long as they are all the same unit). The area of a two-dimensional shape is found by counting the number of squares that cover the shape. Many formulas have been developed to quickly find the area of standard polygons, like triangles and parallelograms.

### Circles

Circles are an important geometric shape. The distance around a circle is called the circumference, and the interior space of a circle is called the area. Calculating the circumference and area of a circle requires a number called pi ( $\pi$), which is a non-terminating, non-repeating decimal. Pi is often approximated by the values 3.14 and $\frac{22}{7}$. You can find the perimeter or area of composite shapes, including shapes that contain circular sections, by applying the circumference and area formulas where appropriate.

### Solids

Three-dimensional solids have length, width, and height. You use a measurement called volume to figure out the amount of space that these solids take up. To find the volume of a specific geometric solid, you can use a volume formula that is specific to that solid. Sometimes, you will encounter composite geometric solids. These are solids that combine two or more basic solids. To find the volume of these, identify the simpler solids that make up the composite figure, find the volumes of those solids, and combine them as needed.

## Glossary:

| | |
|---|---|
| **acute angle** | Angles that are between 0° and 90° |
| **acute triangle** | triangle with three acute angles |
| **angle** | two rays with a common endpoint |
| **area** | amount of surface the shape covers |
| **circle** | set of points, all of which are the same distance away from a center |
| **circumference** | perimeter of a circle |
| **complementary angles** | two angles whose measurements add up to 90° are called complementary angles |
| **Cone** | A solid figure with a single circular base and a round, smooth face that diminishes to a single point |
| **congruent sides** | Sides of equal length |
| **congruent triangles** | triangles are congruent if they are exactly the same size and shape |
| **cube** | A six-sided polyhedron that has congruent squares as faces |
| **Cylinder** | A solid figure with a pair of circular, parallel bases and a round, smooth face between them |
| **diameter** | distance across the circle through the center |
| **Equilateral Triangle** | triangle whose three sides have the same length |
| **face** | flat side of a 3-dimensional figure |
| **hypotenuse** | side of a right triangle opposite of the right angle |
| **isosceles trapezoid** | trapezoid with congruent non-parallel sides |
| **Isosceles Triangle** | triangle with exactly two congruent sides |
| **leg** | side of a right triangle attached to the right angle |
| **line** | one-dimensional figure that is made up of an infinite number of individual points |
| **line segment** | section between any two points on a line |
| **obtuse angle** | Angles that are between 90° and 180° |
| **obtuse triangle** | triangle with one obtuse angle |
| **parallel** | two lines are parallel if the never intersect |
| **parallelogram** | quadrilaterals that have two pairs of parallel sides |
| **perimeter** | distance around a 2-dimensional shape |
| **perpendicular** | two lines are perpendicular if they intersect at a right angle |
| **plane** | triangle whose three sides have the same length |
| **point** | location on the plane that has no dimensions |
| **polygon** | closed plane figure with three or more straight sides |
| **polyhedron** | shapes that have four or more faces, each one being a polygon |
| **protractor** | tool to measure angles |
| **Pyramid** | A polyhedron with a polygonal base and a collection of triangular faces that meet at a point |

| | |
|---|---|
| **quadrilateral** | a quadrilateral is a four-sided polygon |
| **radius** | distance from the center of the circle to any point on the circle |
| **ray** | one endpoint and goes on forever in one direction |
| **rectangle** | quadrilateral with four right angles |
| **Rectangular prism** | A polyhedron that has three pairs of congruent, rectangular, parallel faces |
| **rhombus** | quadrilateral with all sides congruent |
| **right angle** | measures exactly 90° |
| **right triangle** | triangle with one right angle |
| **Scalene Triangle** | triangle in which all three sides are a different length |
| **similar triangles** | triangles have the same angle measurements |
| **sphere** | A solid, round figure where every point on the surface is the same distance from the center |
| **square** | rectangle with all sides congruent |
| **straight angle** | measures exactly 180° |
| **supplementary angles** | Two angles whose measures add up to 180° are called supplementary angles |
| **trapzoid** | quadrilateral with only one pair of opposite sides that are parallel |
| **triangle** | polygon with three sides |
| **vertex** | common endpoint of two rays that form an angle |
| **volume** | amount of space a three-dimensional figure takes up |

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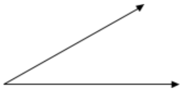
## 2.10: Exercises

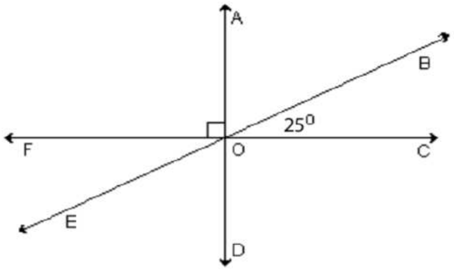
1. Classify the angle below as acute, obtuse, or right.

2. Classify the angle below as acute, obtuse, or right.



3. Classify the angle shown as acute, obtuse, or right.

4. Use the picture below to answer the following questions. *Note, ∠AOF is a right angle.*



a) Which angle is supplementary to $\angle BOC$?

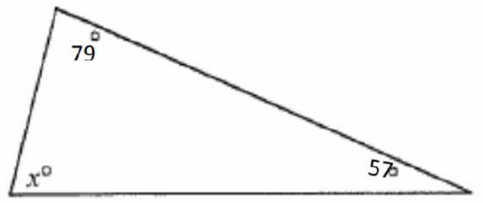
b) Which angle is complementary to $\angle BOC$?

c) What is the measure of $\angle EOF$?

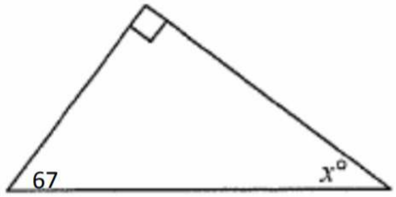
d) What is the measure of $\angle AOE$?

e) What is the measure of $\angle BOF$?

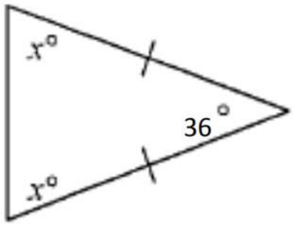
5. Find the unknown angle measure.



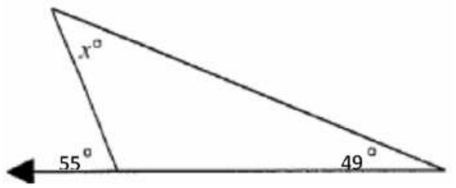
6. Find the unknown angle measure.



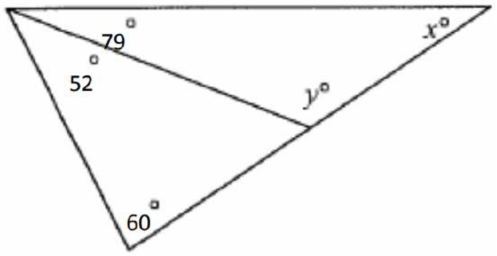
7. Find the unknown angle measure.



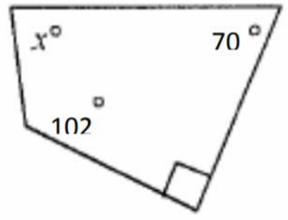
8. Find the unknown angle measure.



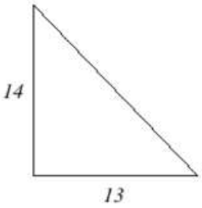
9. Find the unknown angle measures.



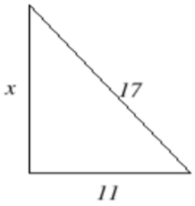
10. Find the unknown angle measure.



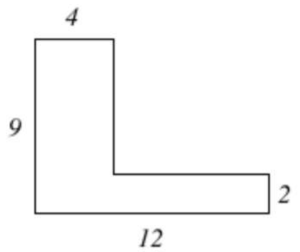
11. Find the length of the hypotenuse of the given right triangle pictured below. *Round to two decimal places.*



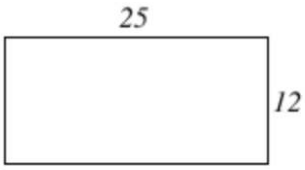
12. Find the length of the leg $x$. *Enter the exact value, not a decimal approximation.*



13. Find the perimeter of the figure pictured below.

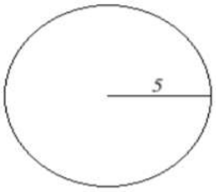


14. Find the perimeter of the rectangle pictured below.

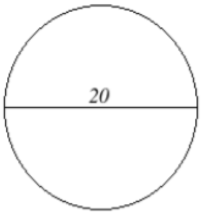


15. Find the perimeter of the parallelogram shown below.

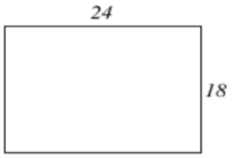
16. Find the circumference of the circle pictured below. *Round your answer to the nearest hundredth.*



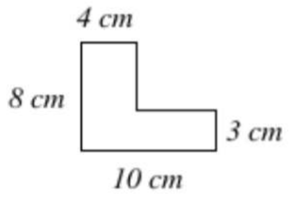
17. Find the circumference of the circle pictured below. Round your answer to the nearest hundredth.



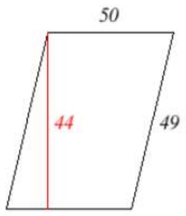
18. Find the area of the rectangle pictured below.



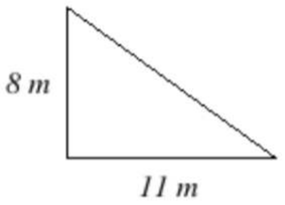
19. Find the area of the figure pictured below and state the correct units.



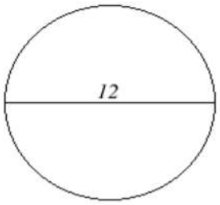
20. Find the area of the parallelogram shown below.



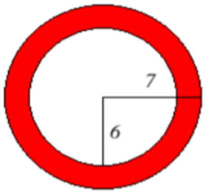
21. The area of a triangle can be found using the formula: $\text{Area } = \frac{1}{2} \cdot \text{base} \cdot \text{height}$ . Find the area of the triangle pictured below, where the measurements are given in meters (m).



22. Find the area of the circle pictured below. *Round your answer to the nearest hundredth.*



23. Find the area of the shaded area. *Round your answer to the nearest tenth.*



24. Match the formula for each volume to the figure to which it applies.

| **Figure** | **Volume** |
|---|---|
| Right Circular Cylinder | A. $V = \frac{4}{3}\pi r^3$ |
| Rectangular Solid | B. $V = \pi r^2 h$ |
| Sphere | C. $V = l \cdot w \cdot h$ |

25. The volume of a cylinder with height $h$ and radius $r$ can be found using the formula $V = \pi r^2 h$. Sketch a cylinder with radius $7$ feet and height $4$ feet, then find the volume and select the correct units. *Round your answer to the nearest tenth.*

26. The volume of a cone with height $h$ and radius $r$ can be found using the formula $V = \frac{1}{3}\pi r^2 h$. Sketch a cone with radius $9$ feet and height $3$ feet, then find the volume and select the correct units. *Round your answer to the nearest tenth.*

27. A sports ball has a diameter of $26 \text{ cm}$. Find the volume of the ball and select the correct units. *Round your answer to 2 decimal places.*

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# CHAPTER OVERVIEW

## 3: Finance

We have to work with money every day. While balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more mathematics.

The material in this chapter is from Math In Society by David Lippman.

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# 3.1: Percents

## Learning Objectives

- Convert between percent, decimal, and fraction form.
- Solve problems involving percents.
- Find percent increase or decrease.

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties [1]. Who is correct? How can we make sense of these numbers?

**Percent** literally means "per 100," or "parts per hundred." When we write 40%, this is equivalent to the fraction $\frac{40}{100}$ or the decimal 0.40. Notice that 80 out of 200 and 10 out of 25 are also 40%, since $\frac{80}{200} = \frac{10}{25} = \frac{40}{100}$.

## Example 3.1.1

243 people out of 400 state that they like dogs. What percent is this?

### Solution

$\frac{243}{400} = 0.6075 = \frac{60.75}{100}$. This is 60.75%.

Notice that the percent can be found from the equivalent decimal by moving the decimal point two places to the right.

## Converting between Decimals and Percents

- To change a decimal to a percent, move the decimal point two places to the right and add the percent sign
- To change a percent to a decimal, drop the percent sign and move the decimal point two places to the left

## Converting between Fractions and Percents

- To change a fraction to a percent, first change the fraction to a decimal by dividing, then change the decimal to a percent
- To change a percent to a fraction, write a fraction with the percent number (without the percent sign) in the numerator and 100 in the denominator, then simplify if necessary.

## Example 3.1.2

Write each as a percent:

a. $\frac{1}{4}$
b. 0.02
c. 2.35

### Solution

a. $\frac{1}{4} = 0.25 = 25\%$
b. $0.02 = 2\%$
c. $2.35 = 235\%$

### Percents

If we have a *part* that is some *percent* of a *whole*, then

$\text{percent} = \dfrac{\text{part}}{\text{whole}}$ , or equivalently, $\text{part} = \text{percent} \cdot \text{whole}$

To do the calculations, we write the percent as a decimal.

When writing a percentage statement in symbols, the word "of" becomes multiplication, and the word "is" becomes an equal sign. Also, we must change the percent into decimal or fractional form. For example, the statement $2 is 5% of $40 would read as $2 = 0.05 * 40$ as a mathematical equation.

### Example 3.1.3

The sales tax in a town is 9.4%. How much tax will you pay on a $140 purchase?

**Solution**

Here, $140 is the whole, and we want to find 9.4% *of* $140. We start by writing the percent as a decimal by moving the decimal point two places to the left (which is equivalent to dividing by 100). We can then compute:

$\text{tax} = 0.094(140) = \$13.16$in tax.

### Example 3.1.4

In the news, you hear "tuition is expected to increase by 7% next year." If tuition this year was $1200 per semester, what will it be next year?

**Solution**

The tuition next year will be the current tuition plus an additional 7%, so it will be 107% of this year's tuition:

$\$1200(1.07) = \$1284$

Alternatively, we could have first calculated 7% of $1200: $\$1200(0.07) = \$84$

Notice this is *not* the expected tuition for next year (we could only wish). Instead, this is the expected *increase*, so to calculate the expected tuition, we'll need to add this change to the previous year's tuition:

$\$1200 + \$84 = \$1284$.

### Try It 3.1.1

A TV originally priced at $799 is on sale for 30% off. There is a 9.2% sales tax. Find the final price including the discount and sales tax.

**Answer**

The sale price is $\$799(0.70) = \$559.30$After tax, the price is $\$559.30(1.092) = \$610.76$

### Calculating a Percent Increase or Decrease

Procedure to find **percent increase (decrease):**

- Find the amount of increase (decrease)
- Calculate the percent increase (decrease) by using a fraction $\dfrac{\text{amount of increase (decrease)}}{\text{original amount}}$
- Convert the fraction to a percent

### ✓ Example 3.1.5

The value of a car dropped from $7400 to $6800 over the last year. What percent decrease is this?

**Solution**

To compute the percent change, we first need to find the dollar value change: $\$6800 - \$7400 = -\$600$ Often we will take the absolute value of this amount, which is called the **absolute change**: $|-600| = 600$.

Since we are computing the decrease relative to the starting value, we compute this percent out of $\$7400$:

$\dfrac{600}{7400} = 0.081 = 8.1\%$ decrease. This is called a **relative change**.

### Absolute and Relative Change

Given two quantities,

Absolute change = $|\text{ending quantity } - \text{ starting quantity}|$

Relative change: $\dfrac{\text{absolute change}}{\text{starting quantity}}$

Absolute change has the same units as the original quantity.

Relative change gives a percent change.

The starting quantity is called the **base** of the percent change.

The base of a percent is very important. For example, while Nixon was president, it was argued that marijuana was a “gateway” drug, claiming that 80% of marijuana smokers went on to use harder drugs like cocaine. The problem is, this isn’t true. The true claim is that 80% of harder drug users first smoked marijuana. The difference is one of base: 80% of marijuana smokers using hard drugs, vs. 80% of hard drug users having smoked marijuana. These numbers are not equivalent. As it turns out, only one in 2,400 marijuana users actually go on to use harder drugs [2].

### ✓ Example 3.1.6

There are about 75 QFC supermarkets in the U.S. Albertsons has about 215 stores. Compare the size of the two companies.

**Solution**

When we make comparisons, we must ask first whether an absolute or relative comparison. The absolute difference is $215 - 75 = 140$. From this, we could say “Albertsons has 140 more stores than QFC.” However, if you wrote this in an article or paper, that number does not mean much. The relative difference may be more meaningful. There are two different relative changes we could calculate, depending on which store we use as the base:

Using QFC as the base, $\dfrac{140}{75} = 1.867$.

This tells us Albertsons is 186.7% larger than QFC.

Using Albertsons as the base, $\dfrac{140}{215} = 0.651$.

This tells us QFC is 65.1% smaller than Albertsons.

Notice both of these are showing percent *differences*. We could also calculate the size of Albertsons relative to QFC: , which tells us Albertsons is 2.867 times the size of QFC. Likewise, we could calculate the size of QFC relative to Albertsons: , which tells us that QFC is 34.9% of the size of Albertsons.

### ✓ Example 3.1.7

Suppose a stock drops in value by 60% one week, then increases in value the next week by 75%. Is the value higher or lower than where it started?

**Solution**

To answer this question, suppose the value started at $100. After one week, the value dropped by 60%:

$$\$100 - \$100(0.60) = \$100 - \$60 = \$40.$$

In the next week, notice that base of the percent has changed to the new value, $40. Computing the 75% increase:

$$\$40 + \$40(0.75) = \$40 + \$30 = \$70.$$

In the end, the stock is still $30 lower, or $\dfrac{\$30}{\$100} = 30\%$ lower, valued than it started.

### ✎ Try It 3.1.2

The U.S. federal debt at the end of 2001 was $5.77 trillion, and grew to $6.20 trillion by the end of 2002. At the end of 2005 it was $7.91 trillion, and grew to $8.45 trillion by the end of 2006 [3]. Calculate the absolute and relative increase for 2001-2002 and 2005-2006. Which year saw a larger increase in federal debt?

**Answer**

2001-2002: Absolute change: $0.43 trillion. Relative change: 7.45%

2005-2006: Absolute change: $0.54 trillion. Relative change: 6.83%

2005-2006 saw a larger absolute increase, but a smaller relative increase.

### ✓ Example 3.1.8

A Seattle Times article on high school graduation rates reported "The number of schools graduating 60 percent or fewer students in four years – sometimes referred to as "dropout factories" – decreased by 17 during that time period. The number of kids attending schools with such low graduation rates was cut in half."

a. Is the "decrease by 17" number a useful comparison?
b. Considering the last sentence, can we conclude that the number of "dropout factories" was originally 34?

**Solution**

a. This number is hard to evaluate, since we have no basis for judging whether this is a larger or small change. If the number of "dropout factories" dropped from 20 to 3, that'd be a very significant change, but if the number dropped from 217 to 200, that'd be less of an improvement.
b. The last sentence provides relative change which helps put the first sentence in perspective. We can estimate that the number of "dropout factories" was probably previously around 34. However, it's possible that students simply moved schools rather than the school improving, so that estimate might not be fully accurate.

### ✓ Example 3.1.9

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties. Who is correct?

**Solution**

Without more information, it is hard for us to judge who is correct, but we can easily conclude that these two percents are talking about different things, so one does not necessarily contradict the other. Edward's claim was a percent with coalition forces as the base of the percent, while Cheney's claim was a percent with both coalition and Iraqi security forces as the base of the percent. It turns out both statistics are in fact fairly accurate.

## Try It 3.1.3

In the 2012 presidential elections, one candidate argued that "the president's plan will cut $716 billion from Medicare, leading to fewer services for seniors," while the other candidate rebuts that "our plan does not cut current spending and actually expands benefits for seniors, while implementing cost saving measures." Are these claims in conflict, in agreement, or not comparable because they're talking about different things?

**Answer**

> Without more information, it is hard to judge these arguments. This is compounded by the complexity of Medicare. As it turns out, the $716 billion is not a cut in current spending, but a cut in future increases in spending, largely reducing future growth in health care payments. In this case, at least the numerical claims in both statements could be considered at least partially true. Here is one source of more information if you're interested: http://factcheck.org/2012/08/a-campaign-full-of-mediscare/

We'll wrap up our review of percents with a couple cautions. First, when talking about a change of quantities that are already measured in percents, we have to be careful in how we describe the change.

## Example 3.1.10

A politician's support increases from 40% of voters to 50% of voters. Describe the change.

### Solution

We could describe this using an absolute change: $|50\% - 40\%| = 10\%$. Notice that since the original quantities were percents, this change also has the units of percent. In this case, it is best to describe this as an increase of 10 **percentage points**.

In contrast, we could compute the percent change: $\frac{10\%}{40\%} = 0.25 = 25\%$ increase. This is the relative change, and we'd say the politician's support has increased by 25%.

Lastly, a caution against averaging percents.

## Example 3.1.11

A basketball player scores on 40% of 2-point field goal attempts, and on 30% of 3-point of field goal attempts. Find the player's overall field goal percentage.

### Solution

It is very tempting to average these values, and claim the overall average is 35%, but this is likely not correct, since most players make many more 2-point attempts than 3-point attempts. We don't actually have enough information to answer the question. Suppose the player attempted 200 2-point field goals and 100 3-point field goals. Then they made 200(0.40) = 80 2-point shots and $100(0.30) = 30$ 3-point shots. Overall, they made 110 shots out of 300, for a $\frac{110}{300} = 0.367 = 36.7\%$ overall field goal percentage.

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[1] www.factcheck.org/cheney_edwards_mangle_facts.html

[2] http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics

[3] www.whitehouse.gov/sites/defa...s/hist07z1.xls

# 3.2: Simple Interest

**Learning Objectives**

- Calculate simple interest
- Find the annual percentage rate of an account

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed \$100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $\$100(0.05)=\$5$ The total amount you would repay would be \$105, the original principal plus the interest.

**Simple One-time Interest**

$$I=Pr$$

$$A=P+I=P+Pr=P(1+r)$$

where

- $I$ is the interest
- $A$ is the accumulated amount: principal plus interest
- $P$ is the principal (starting amount)
- $r$ is the interest rate (in decimal form. Example: $5\%=0.05$)

**Example 3.2.1**

A friend asks to borrow \$300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

**Solution**

$$\begin{array}{ll} P=\$300 & \text{the principal} \\ r=0.03 & 3\%\text{ rate} \\ I=\$300(0.03)=\$9. & \text{You will earn \$9 interest.} \end{array}$$

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

**Example 3.2.2**

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a \$1000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

**Solution**

Each year, you would earn 5% interest: $\$1000(0.05)=\$50$in interest. So over the course of five years, you would earn a total of \$250 in interest. When the bond matures, you would receive back the \$1000 you originally paid, leaving you with a total of \$1250.

We can generalize this idea of simple interest over time.

**Simple Interest over Time**

$$I=Prt$$

$$A=P+I=P+Prt=P(1+rt)$$

where

- $I$ is the interest
- $A$ is the accumulated amount: principal plus interest (also known as the future value)
- $P$ is the principal (starting amount)
- $r$ is the interest rate in decimal form
- $t$ is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

### ✓ Example 3.2.3

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

**Solution**

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$$\begin{array}{ll} P=\$1000 & \text{the principal} \\ r=0.02 & 2\% \text{ rate per half-year} \\ t=8 & 4 \text{ years} = 8 \text{ half-years} \\ I=\$1000(0.02)(8)=\$160. & \text{You will earn } \$160 \text{ interest total over the four years.} \end{array}$$

### APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a $6\%$ APR paid monthly would be divided into twelve $0.5\%$ payments.
A $4\%$ annual rate paid quarterly would be divided into four $1\%$ payments.

### Try it 3.2.1

A loan company charges $30 interest for a one month loan of $500. Find the annual interest rate they are charging.

**Answer**

We want to find the interest rate. Solving $I=Pr$ for $r$, we get $r=\dfrac{I}{P}$.

$I=\$30$ interest

$P=\$500$ principal

$r=\dfrac{I}{P}=\dfrac{30}{500}=0.06$ = 6% for one month

$(0.06)(12)=0.72$= 72% for one year

They are charging an annual interest rate of 72%.

# 3.3: Compound Interest

## Compound Interest

### Learning Objectives

- Calculate compound interest, accumulated amount and present value on an account
- Find the effective rate of an account
- Use the Rule of 70s to estimate doubling time

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit $1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3\%}{12} = 0.25\%$ per month.

In the first month,

$P = \$1000$

$r = 0.25\% = 0.0025$

$I = \$1000(0.0025) = \$2.50$

$A = \$1000 + \$2.50 = \$1002.50$

In the first month, we will earn $2.50 in interest, raising our account balance to $1002.50.

In the second month,

$P = \$1002.50$

$I = \$1002.50(0.0025) = \$2.51 \text{(rounded)}$

$A = \$1002.50 + \$2.51 = \$1005.01$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original $1000 we deposited, but we also earned interest on the $2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:

| Month | Starting balance | Interest earned | Ending Balance |
|---|---|---|---|
| 1 | $1000.00 | $2.50 | $1002.50 |
| 2 | $1002.50 | $2.51 | $1005.01 |
| 3 | $1005.01 | $2.51 | $1007.52 |
| 4 | $1007.52 | $2.52 | $1010.04 |
| 5 | $1010.04 | $2.53 | $1012.57 |
| 6 | $1012.57 | $2.53 | $1015.10 |
| 7 | $1015.10 | $2.54 | $1017.64 |
| 8 | $1017.64 | $2.54 | $1020.18 |
| 9 | $1020.18 | $2.55 | $1022.73 |
| 10 | $1022.73 | $2.56 | $1025.29 |
| 11 | $1025.29 | $2.56 | $1027.85 |
| 12 | $1027.85 | $2.57 | $1030.42 |

To find an equation to represent this, if $P_m$ represents the amount of money after $m$ months, then we could write the recursive equation:

$P_0 = \$1000$

$P_m = (1+0.0025)P_{m-1}$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$P_0 = \$1000$

$P_1 = 1.0025P_0 = 1.0025(1000)$

$P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = 1.0025^2(1000)$

$P_3 = 1.0025P_2 = 1.0025\left(1.0025^2(1000)\right) = 1.0025^3(1000)$

$P_4 = 1.0025P_3 = 1.0025\left(1.0025^3(1000)\right) = 1.0025^4(1000)$

Observing a pattern, we could conclude

$P_m = (1.0025)^m(\$1000)$

Notice that the \$1000 in the equation was $P_0$, the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

$P_m = P_0\left(1+\frac{r}{n}\right)^m$

In this formula:

$m$ is the number of compounding periods (months in our example)

$r$ is the annual interest rate

$n$ is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If $t$ is the number of years, then $m = nt$. Making this change gives us the standard formula for compound interest.

**Compound Interest**

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

- $A$ is the balance in the account (accumulated amount) after $t$ years (also called the future value)
- $P$ is the starting balance of the account (also called the initial deposit, principal or present value)
- $r$ is the annual interest rate in decimal form
- $n$ is the number of compounding periods in one year

If the compounding is done annually (once a year), $n = 1$.

If the compounding is done quarterly, $n = 4$.

If the compounding is done monthly, $n = 12$.

If the compounding is done daily, $n = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

### Example 3.3.1

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit $3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

**Solution**

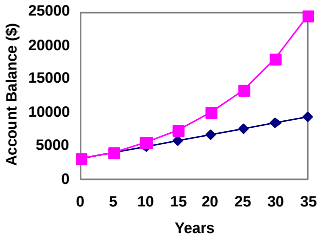
In this example,

$P = \$3000$ the initial deposit
$r = 0.06$ 6% annual rate
$n = 12$ 12 months in 1 year
$t = 20$ since we're looking for how much we'll have after 20 years

So $A = 3000\left(1+\frac{0.06}{12}\right)^{12\times 20} = \$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest.

| Years | Simple Interest ($15 per month) | 6% compounded monthly = 0.5% each month |
|---|---|---|
| 5 | $3900 | $4046.55 |
| 10 | $4800 | $5458.19 |
| 15 | $5700 | $7362.28 |
| 20 | $6600 | $9930.61 |
| 25 | $7500 | $13,394.91 |
| 30 | $8400 | $18,067.73 |
| 35 | $9300 | $24,370.65 |



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

### Evaluating exponents on the calculator

When we need to calculate something like $5^3$ it is easy enough to just multiply $5\cdot5\cdot5=125$. But when we need to calculate something like $1.005^{240}$, it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

$[\wedge]$, $[y^x]$, or $[x^y]$

To evaluate $1.005^{240}$ we'd type 1.005 $[\wedge]$ 240, or 1.005 $[y^x]$ 240. Try it out - you should get something around 3.3102044758.

### Example 3.3.2

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

**Solution**

We're looking for $P$, the present value.

$r = 0.04$ 4%
$n = 4$ 4 quarters in 1 year
$t = 18$ Since we know the balance in 18 years
$A = \$40,000$ The accumulated amount we will have in 18 years

In this case, we're going to have to set up the equation, and solve for $P$.

$$40,000 = P\left(1 + \frac{0.04}{4}\right)^{4\times18}$$

$$40,000 = P(2.0471)$$

$$P = \frac{40,000}{2.0471} = \$19,539.84$$

So you would need to deposit $19,539.84 now to have $40,000 in 18 years.

Note: this value is also known as the *present value* of the investment--the amount that must be invested now to have a specific amount later with interest.

**Rounding**

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better.

**Example 3.3.3**

To see why not over-rounding is so important, suppose you were investing $1000 at 5% interest compounded monthly for 30 years.

**Solution**

$P = \$1000$ the initial deposit
$r = 0.05$ 5%
$n = 12$ 12 months in 1 year
$t = 30$ since we're looking for the amount after 30 years

If we first compute $\frac{r}{n}$, we find $\frac{0.05}{12} = 0.00416666666667$

Here is the effect of rounding this to different values:

| $\frac{r}{n}$ rounded to: | Gives $A$ to be: | Error |
|---|---|---|
| 0.004 | \$4208.59 | \$259.15 |
| 0.0042 | \$4521.45 | \$53.71 |
| 0.00417 | \$4473.09 | \$5.35 |
| 0.004167 | \$4468.28 | \$0.54 |
| 0.0041667 | \$4467.80 | \$0.06 |
| no rounding | \$4467.74 | |

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough - $5 off of $4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

### Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$$A=1000\left(1+\frac{0.05}{12}\right)^{12\times30}$$

We can quickly calculate $12\times30=360$, giving $A=1000\left(1+\frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.

| **Type this** | **Calculator shows** |
|---|---|
| $0.05[\div]12[=]$ | $0.00416666666667$ |
| $[+]1[=]$ | $1.00416666666667$ |
| $[y^x]360[=]$ | $4.46774431400613$ |
| $[\times]1000[=]$ | $4467.74431400613$ |

### Using your calculator continued

The previous steps were assuming you have a “one operation at a time” calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 $[\times]$ ( 1 $[+]$ 0.05 $[\div]$ 12 ) $[y^x]$ 360 $[=]$

## Effective Interest Rate

Banks are required to state their interest rate in terms of an **“effective yield”** ” or **“effective interest rate”**, for comparison purposes. The effective rate is also called the Annual Percentage Yield (APY) or Annual Percentage Rate (APR).

### Effective rate

The **effective rate** is the annual simple interest rate that would be equivalent to the annual compound interest accrued for the stated rate and number of compounding periods.

Formula:

$$r_{EFF}=\left(1+\frac{r}{n}\right)^{n}-1$$

- $r_{EFF}$ is the effective rate, in decimal form
- $r$ is the stated annual interest rate, in decimal form
- $n$ is the number of compounding periods in one year

Be sure to change the answer back to a percent from decimal form.

The formula is derived from the compound interest formula by depositing $P$ = \$1 in an account and calculating how much interest it will accrue in a year ($t$ = 1). The original \$1 deposit must be subtracted out to find the total interest. To examine several investments to see which has the best rate, we find and compare the effective rate for each investment.

### Example 3.3.4

If Bank A pays 7.2% interest compounded monthly, what is the effective interest rate?
If Bank B pays 7.25% interest compounded semiannually, what is the effective interest rate?
Which bank pays more interest?

**Solution**

Bank A: Using the formula with $n$ = 12, we will get:

$$\mathbf{r}_{\text{EFF}} = \left(1+\frac{0.072}{12}\right)^{12} - 1 = 1.0744 - 1 = 0.0744$$

We earned interest of $.0744 on an investment of $1. The effective interest rate is 7.44%, often referred to as the APY or APR.

Bank B: The effective rate is calculated with $n$ = 2:

$$\mathbf{r}_{\text{EFF}} = \left(1+\frac{0.072}{2}\right)^{2} - 1 = .0738$$

The effective interest rate is 7.38%.

Bank A pays slightly higher interest, with an effective rate of 7.44%, compared to Bank B with effective rate 7.38%.

## Rule of 70

The rule of 70 is a useful tool for estimating the time needed for an investment to double in value. It is an approximation and is not exact, but is a quick-and-easy way to estimate how long it will take for an investment to double.

### Rule of 70

**Rule of 70:** The approximate number of years required to double an investment with compound interest

Formula:

$$t \approx \frac{70}{r}$$

- $t$ is the approximate number of years to double the investment
- $r$ is the annual interest rate as a percent (not in decimal form)

Note this formula does not take into account the number of compounding periods per year. This is because it is based on the idea of interest that is continuously compounded, a concept that makes calculations easier. In the real world, however, there is no bank or financial institution that offers interest that is continuously compounded.

Here are some approximate doubling times for various interest rates using the Rule of 70.

Approximate Doubling Time in Years By Interest Rate

| Annual interest rate | 1% | 2% | 3% | 4% | 5% | 6% | 7% | 8% | 9% | 10% |
|---|---|---|---|---|---|---|---|---|---|---|
| Number of years to double money | 70 | 35 | 23.3 | 17.5 | 14 | 11.7 | 10 | 8.75 | 7.8 | 7 |

The rule of 70 can be useful to help quickly estimate many “doubling time” problems mentally, which can be useful in compound interest applications as well as other applications involving exponential growth.

### Example 3.3.5

As of 2015, the world population’s annual growth rate was approximately 1.14%. Based on that rate, find the approximate doubling time.

**Solution**

According to the rule of 70,

$$t \approx \frac{70}{1.14} \approx 61.4 \text{ years}$$

> If the world population were to continue to grow at the annual growth rate of 1.14% , it would take approximately 61.4 years for the population to double.

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# 3.4: Annuities

**Learning Objectives**

- Understand the difference between an annuity and a compound interest account
- Calculate the accumulated amount of a savings or payout annuity
- Calcluate the deposit or withdrawal amount for a savings or payout annuity

In the last two sections of this chapter, we examined problems where an amount of money was deposited as a lump sum (one-time deposit) in an account and was left there for the entire time period. Now we will do problems where timely payments are made in an account. When a sequence of payments of some fixed amount are made in an account at equal intervals of time, we call that an **annuity**.

## Savings Annuity

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

**Annuity**

A **savings annuity** is a savings account in which equal deposits are made at equal intervals of time, and earns compound interest.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m=\left(1+\frac{r}{n}\right)P_{m-1}$$

For a savings annuity, we simply need to add a deposit, $d$, to the account with each compounding period:

$$P_m=\left(1+\frac{r}{n}\right)P_{m-1}+d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit \$100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$r=0.06$ (6%)

$n=12$ (12 compounds/deposits per year)

$d=\$100$ (our deposit per month)

Writing out the recursive equation gives

$$P_m=\left(1+\frac{0.06}{12}\right)P_{m-1}+100=(1.005)P_{m-1}+100$$

Assuming we start with an empty account, we can begin using this relationship:

$P_0=0$

$P_1=(1.005)P_0+100=100$

$P_2=(1.005)P_1+100=(1.005)(100)+100=100(1.005)+100$

$P_3=(1.005)P_2+100=(1.005)(100(1.005)+100)+100=100(1.005)^2+100(1.005)+100$

Continuing this pattern, after $m$ deposits, we'd have saved:

$P_m=100(1.005)^{m-1}+100(1.005)^{m-2}+\cdots+100(1.005)+100$

In other words, after $m$ months, the first deposit will have earned compound interest for $m-1$ months. The second deposit will have earned interest for $m-2$ months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005P_m = 1.005\left(100(1.005)^{m-1} + 100(1.005)^{m-2} + \cdots + 100(1.005) + 100\right)$$

Distributing on the right side of the equation gives

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \cdots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$\begin{aligned} 1.005P_m &= 100(1.005)^m + &100(1.005)^{m-1} + \cdots + &\quad 100(1.005) \\ P_m &= &100(1.005)^{m-1} + \cdots + &\quad 100(1.005) + 100 \end{aligned}$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005P_m - P_m = 100(1.005)^m - 100$$

Solving for $P_m$

$$0.005P_m = 100\left((1.005)^m - 1\right)$$

$$P_m = \frac{100\left((1.005)^m - 1\right)}{0.005}$$

Replacing $m$ months with $12t$, where $t$ is measured in years, gives

$$P_t = \frac{100\left((1.005)^{12\mathrm{t}} - 1\right)}{0.005}$$

Recall 0.005 was $\frac{r}{n}$ and 100 was the deposit $d$. 12 was $n$, the number of deposits each year. Generalizing this result, we get the saving annuity formula.

**Annuity Formula**

$$A = \frac{d\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}{\left(\frac{r}{n}\right)}$$

- $A$ is the balance in the account (accumulated amount) after $t$ years
- $d$ is the regular deposit (the amount you deposit each year, each month, etc.)
- $r$ is the annual interest rate in decimal form
- $n$ is the number of compounding periods in one year

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, $n=12$.

If you make your deposits every year, use yearly compounding, $n=1$.

If you make your deposits every quarter, use quarterly compounding, $n=4$.

**When do you use this**

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

### ✓ Example 3.4.1

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

**Solution**

In this example,

$d = \$100$ the monthly deposit
$r = 0.06$ 6% annual rate
$n = 12$ since we're doing monthly deposits, we'll compound monthly
$t = 20$ we want the amount after 20 years

Putting this into the equation:

$$A = \frac{100\left[\left(1+\frac{0.06}{12}\right)^{12\times 20}-1\right]}{\left(\frac{0.06}{12}\right)}$$

$$A = \frac{100\left((1.005)^{240}-1\right)}{(0.005)}$$

$$A = \frac{100(3.310-1)}{(0.005)}$$

$$A = \frac{100(2.310)}{(0.005)} = \$46,200$$

The account will grow to $46,200 after 20 years.

Notice that you deposited into the account a total of $24,000 ($100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $\$46,200 - \$24,000 = \$22,200$

### ✎ Try It 3.4.1

A more conservative investment account pays 3% interest. If you deposit $5 a day into this account, how much will you have after 10 years? How much is from interest?

**Answer**

$d = \$5$ the daily deposit
$r = 0.03$ 3% annual rate
$n = 365$ since we're doing daily deposits, we'll compound daily
$t = 10$ we want the amount after 10 years

$$A = \frac{5\left[\left(1+\frac{0.03}{365}\right)^{365\times 10}-1\right]}{\frac{0.03}{365}} = \$21,282.07$$

We would have deposited a total of $\$5 \cdot 365 \cdot 10 = \$18,250$ so $3,032.07 is from interest.

We can solve the annuity formula for $d$ in order to calculate the amount of money that must be deposited regularly in order to reach a savings goal in a specific amount of time.

**Deposit Formula**

$$d=\frac{A\left(\frac{r}{n}\right)}{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}$$

- $d$ is the regular deposit (the amount you deposit each year, each month, etc.)
- $A$ is the desired balance in the account (accumulated amount) after $t$ years
- $r$ is the annual interest rate in decimal form
- $n$ is the number of compounding periods in one year

**Example 3.4.2**

You want to have $200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

### Solution

In this example, we're looking for $d$.

$$\begin{array}{ll} r=0.08 & 8\% \text{ annual rate} \\ n=12 & \text{since we're doing monthly deposits, we'll compound monthly} \\ t=30 & 30 \text{ years} \\ A=\$200,000 & \text{The amount we want to have in } 30 \text{ years} \end{array}$$

In this case, we're going to have to set up the equation, and solve for $d$.

$$d=\frac{200,000\left(\frac{0.08}{12}\right)}{\left[\left(1+\frac{0.08}{12}\right)^{12\times 30}-1\right]}$$

$$d=\frac{200,000(0.00667)}{[(1.00667)^{360}-1]}$$

$$d=\frac{1333.3333}{9.9357}=\$134.20$$

So you would need to deposit $\$134.20$each month to have $\$200,000$in 30 years if your account earns 8% interest.

**Try It 3.4.2**

A business needs $450,000 in five years. How much should be deposited each quarter in an account that earns 9% compounded quarterly to have this amount in five years?

**Answer**

$$\begin{array}{ll} r=0.09 & 9\% \text{ annual rate} \\ n=4 & \text{compounded quarterly} \\ t=5 & 5 \text{ years} \\ A=\$450,000 & \text{The amount we want to have in } 5 \text{ years} \end{array}$$

$$d=\frac{450,000\left(\frac{0.09}{4}\right)}{\left[\left(1+\frac{0.09}{4}\right)^{4\times5}-1\right]}$$

$$d=\frac{450,000(0.0225)}{[(1.0225)^{20}-1]}$$

$$d=\frac{10,125}{0.5605092}=\$18,063.93$$

The business needs to deposit $18,063.93 at the end of each quarter for 5 years into an sinking fund earning interest of 9% compounded quarterly in order to have $450,000 at the end of 5 years.

This type of investment is called a *sinking fund,* when a business deposits money at regular intervals into an account in order to save for a future purchase of equipment.

## Payout Annuities

There are other types of annuities besides savings annuities. In a savings annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

Another common type of annuity is a **payout annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved $500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

### Payout Annuity Formula

$$A=\frac{d\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}{\left(\frac{r}{n}\right)}$$

- $A$ is the amount in the account at the beginning (starting amount, or principal)
- $d$ is the regular withdrawal (the amount you take out each year, each month, etc.)
- $r$ is the annual interest rate in decimal form
- $n$ is the number of compounding periods in one year
- $t$ is the number of years we plan to take withdrawals

Like with savings annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

### When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

### Example 3.4.3

After retiring, you want to be able to take \$1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

**Solution**

In this example,

$d = \$1000$ the monthly withdrawal
$r = 0.06$ $6\%$ annual rate
$n = 12$ since we're doing monthly withdrawals, we'll compound monthly
$t = 20$ since were taking withdrawals for 20 years

We're looking for $A$; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$A = \frac{1000\left[1-\left(1+\frac{0.06}{12}\right)^{-12\times 20)}\right]}{\left(\frac{0.06}{12}\right)}$$

$$A = \frac{1000\left[1-(1.005)^{-240}\right]}{0.005}$$

$$A = \frac{1000\,[1-0.302]}{0.005} = \$139,600$$

You will need to have \$139,600 in your account when you retire.

Notice that you withdrew a total of \$240,000 (\$1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is $\$240,000 - \$139,600 = \$100,400$in interest.

### Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this [(-)], some with [+/-] . The button is often near the = key or the decimal point.

If your calculator displays operations on it (typically a calculator with multi-line display), to calculate $1.005^{-240}$you'd type something like: $1.005[\wedge][(-)]240$

If your calculator only shows one value at a time, then usually you hit the (-) key after a number to negate it, so you'd hit: $1.005[y^x]240[(-)] =$

Give it a try - you should get $1.005^{-240} = 0.302096$

We can solve the payout annuity formula for $d$, which will tell us how much can be withdrawn at regular intervals over a set number of years if we have a specific amount in the annuity account.

### Withdrawal Amount Formula

$$d = \frac{A\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$$

- $d$ is the regular withdrawal (the amount you take out each year, each month, etc.)
- $A$ is the amount in the account at the beginning (starting amount, or principal)
- $r$ is the annual interest rate in decimal form

- $n$ is the number of compounding periods in one year
- $t$ is the number of years we plan to take withdrawals

### ✓ Example 3.4.4

You know you will have \$500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

**Solution**

In this example, we're looking for $d$.

$$\begin{array}{ll} r=0.08 & 8\% \text{ annual rate} \\ n=12 & \text{since we're doing monthly withdrawals} \\ t=30 & \text{ since were taking withdrawals for 30 years} \\ A=\$500,000 & \text{we are beginning with } \$500,000 \end{array}$$

Using the formula:

$$d=\frac{500,000\left(\frac{0.08}{12}\right)}{\left[1-\left(1+\frac{0.08}{12}\right)^{-12\times 30}\right]}$$

$$d=\frac{500,000(0.00667)}{\left[1-(1.00667)^{-360}\right]}$$

$$d=\frac{3335}{0.90856}$$

$$d=\frac{500,000}{0.9085566}=\$3670.64$$

You would be able to withdraw \$3670.64 each month for 30 years.

Notice that you withdrew a total of \$3670.64 x 12 months x 30 years = \$1,321, 430.40. This amounts to \$821,430.40 in interest over 30 years when the interest rate is 8%.

### Try It 3.4.3

A donor gives \$100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

**Answer**

$$\begin{array}{ll} d= & \text{unknown} \\ r=0.04 & 4\% \text{ annual rate} \\ n=1 & \text{since we're doing annual scholarships} \\ t=20 & \text{ since were taking withdrawals for 20 years} \\ A=\$100,000 & \text{we are starting with } \$100,000 \end{array}$$

$$d=\frac{100,000\left(\frac{0.04}{1}\right)}{\left[1-\left(1+\frac{0.04}{1}\right)^{-1\times 20}\right]}$$

Solving for $d$ gives \$7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, $\$100,000(0.04) = \$4,000$ a year would have been available.

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# 3.5: Loans

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto and student loans, and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

## Loans

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had \$10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests \$10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

**Loan Formula**

$$A=\frac{d\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}{\left(\frac{r}{n}\right)}$$

- $A$ is the amount in the account at the beginning (the principal, or amount of the loan)
- $d$ is your loan payment (your monthly payment, annual payment, etc)
- $r$ is the annual interest rate in decimal form
- $n$ is the number of compounding periods in one year
- $t$ is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

**When do you use this**

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Loans: Many payments

**Example 3.5.1**

You can afford \$200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with \$200 per month?

**Solution**

In this example,

$d=\$200$ the monthly loan payment
$r=0.03$ 3% annual rate
$n=12$ since we're doing monthly payments, we'll compound monthly
$t=5$ since we're making monthly payments for 5 years

We're looking for $A$, the starting amount of the loan.

$$A = \frac{200\left[1-\left(1+\frac{0.03}{12}\right)^{-12\times 5}\right]}{\left(\frac{0.03}{12}\right)}$$

$$A = \frac{200\left[1-(1.0025)^{-60}\right]}{0.0025}$$

$$A = \frac{200(1-0.861)}{0.0025} = \$11,120$$

You can afford a $\$11,120$loan.

You will pay a total of $12,000 ($200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying $\$12,000-\$11,120=\$880$interest total.

Similar to the previous section, the amount of payments is the same as the withdrawal formula.

**Loan Payment Formula**

$$d = \frac{A\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$$

- $d$ is the loan payment
- $A$ is the loan amount
- $r$ is the annual interest rate in decimal form
- $n$ is the number of compounding periods in one year
- $t$ is the length of the loan, in years

**Example 3.5.2**

You want to take out a $400,000 mortgage (home loan). The interest rate on the loan is 5%, and the loan is for 30 years. How much will your monthly payments be?

### Solution

In this example, we're looking for $d$.

| | |
|---|---|
| $r=0.05$ | $5\%$ annual rate |
| $n=12$ | since we're doing monthly payments, we'll compound monthly |
| $t=30$ | since we're making monthly payments for 30 years |
| $A=\$400,000$ | the starting loan amount |

Using the formula,

$$d = \frac{400,000\left(\frac{0.05}{12}\right)}{\left[1-\left(1+\frac{0.05}{12}\right)^{-12\times 30}\right]}$$

$$d = \frac{400,000(0.0041667)}{\left[1-(1.0041667)^{-360}\right]}$$

$$d = \frac{1666.68}{0.776176} = \$2147.30$$

You will make payments of $2147.30 per month for 30 years.

You're paying a total of $\$773,028$to the loan company: $\$2147.30$per month for 360 months. You are paying a total of $\$773,028-\$400,000=\$373,028$in interest over the life of the loan.

### Try It 3.5.1

Janine bought $3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

**Answer**

$$\begin{array}{ll} d= \text{ unknown} & \\ r=0.16 & 16\% \text{ annual rate} \\ n=12 & \text{since we're doing monthly payments, we'll compound monthly} \\ t=2 & 2 \text{ year to repay} \\ A=3,000 & \text{the starting loan amount } \$3,000 \text{ loan} \end{array}$$

$$d=\frac{3000\left(\frac{0.16}{12}\right)}{\left[1-\left(1+\frac{0.16}{12}\right)^{-12\times 2}\right]}$$

Solving for $d$ gives $\$146.89$as monthly payments.

In total, she will pay $\$3,525.36$to the store, meaning she will pay $\$525.36$in interest over the two years.

## Amortization for a loan

### Amortization

**Amortization** is the process of paying off a loan with equal payments over time. Part of each payment will go toward paying off the principal or loan amount, and part of the payment will go toward interest.

An amortization schedule is a table that lists all payments on a loan, splits them into the portion devoted to interest and the portion that is applied to repay principal, and calculates the outstanding balance on the loan after each payment is made.

### Example 3.5.3

An amount of $500 is borrowed for 6 months at a rate of 12%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment contributing toward reducing the debt, and the outstanding balance.

**Solution**

Use the loan payment formula to verify that the monthly payment is $86.27.

The first month, the outstanding balance is $500, and therefore, the monthly interest on the outstanding balance is the simple interest calculated by $I=Pr$ where $P$ = $500 and the monthly interest rate is $\frac{r}{n}=\frac{0.12}{12}=0.01$:

$$I=Pr= \text{(outstanding balance)(the monthly interest rate) = \$500(0.01) = \$5}$$

This means, the first month, out of the $86.27 payment, $5 goes toward the interest and the remaining $81.27 toward the balance leaving a new balance of $500 - $81.27 = $418.73.

Similarly, the second month, the outstanding balance is $418.73, and the monthly interest on the outstanding balance is ($418.73)(.01) = $4.19. Again, out of the $86.27 payment, $4.19 goes toward the interest and the remaining $82.08 toward the balance leaving a new balance of $418.73 - $82.08 = $336.65. The process continues in the table below.

| Payment # | Payment | Interest | Principal Payment | Balance |
|---|---|---|---|---|

| Payment # | Payment | Interest | Principal Payment | Balance |
|---|---|---|---|---|
| 1 | \$86.27 | \$5 | \$81.27 | \$418.73 |
| 2 | \$86.27 | \$4.19 | \$82.08 | \$336.65 |
| 3 | \$86.27 | \$3.37 | \$82.90 | \$253.75 |
| 4 | \$86.27 | \$2.54 | \$83.73 | \$170.02 |
| 5 | \$86.27 | \$1.70 | \$84.57 | \$85.45 |
| 6 | \$86.27 | \$0.85 | \$85.42 | \$0.03 |

Note that the last balance of 3 cents is due to error in rounding off.

The total interest paid on the loan is \$17.65. Notice that the interest payments decrease over the course of the loan payments, while the principal payment increases over the course of the loan. You will pay more interest in the beginning of the loan, and pay almost no interest by the time that you're done paying off the loan (if you make payments for the entire life of the loan).

An amortization schedule is usually lengthy and tedious to calculate by hand. For example, an amortization schedule for a 30-year mortgage loan with monthly payments would have $12 \times 30 = 360$ rows of calculations in the amortization schedule table. A car loan with 5 years of monthly payments would have $12 \times 5 = 60$ rows of calculations in the amortization schedule table. However it would be straightforward to use a spreadsheet application on a computer to do these repetitive calculations by inputting and copying formulas for the calculations into the cells.

## Remaining Loan Balance

One of the most common loan problems deals with finding the balance owed at a given time during the life of a loan. Suppose a person buys a house and amortizes the loan over 30 years, but decides to sell the house a few years later. At the time of the sale, he is obligated to pay off his lender, therefore, he needs to know the balance he owes. Since most long-term loans are paid off prematurely, we are often confronted with this problem.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments goes towards paying down the loan balance; a portion is going to go towards interest. For example, if your payments were \$1,000 a month, after a year you will *not* have paid off \$12,000 of the loan balance.

To determine the remaining loan balance, we can think "how much loan will these loan payments be able to pay off in the *remaining* time on the loan?" Use the loan amount formula, but in this case, $t$ does not represent the entire term of the loan. Instead:

- $t$ represents the time that still remains on the loan
- $nt$ represents the total number of future payments

### ✓ Example 3.5.4

A couple purchases a home with a $\$180,000$mortgage at $4\%$ for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

**Solution**

First we will calculate their monthly payments.

We're looking for $d$, the loan payment amount.

$$\begin{array}{ll} r=0.04 & 4\% \text{ annual rate} \\ n=12 & \text{since they're paying monthly} \\ t=30 & 30 \text{ years} \\ A=\$180,000 & \text{the starting loan amount} \end{array}$$

Using the loan payment formula to calculate $d$:

$$d = \frac{180,000\left(\frac{0.04}{12}\right)}{\left[1-\left(1+\frac{0.04}{12}\right)^{-12\times 30}\right]}$$

$$d = \frac{180,000(0.00333)}{[1-(1.00333)^{-360}]}$$

$$d = \frac{600}{0.698204}$$

$$= \$859.35$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

$d = \$859.35$ the monthly loan payment we calculated above
$r = 0.04$ $4\%$ annual rate
$n = 12$ since they're paying monthly
$t = 25$ since they'll be making monthly payments for 25 more years

$$A = \frac{859.35\left[1-\left(1+\frac{0.04}{12}\right)^{-12\times 25}\right]}{\left(\frac{0.04}{12}\right)}$$

$$= \frac{859.35\left[1-(1.0033333)^{-300}\right]}{0.0033333}$$

$$= \frac{859.35(1-0.3684954)}{0.0033333}$$

$$\approx \$162,807$$

The loan balance after 5 years, with 25 years remaining on the loan, will be $\$162,807$

Over that 5 years, the couple has paid off $\$180,000 - \$162,807 = \$17,193$ of the loan balance. They have paid a total of $\$859.35$ a month for 5 years (60 months), for a total of $\$51,561$, so $\$51,561 - \$17,193 = \$34,368$ of what they have paid so far has been interest.

**Try It 3.5.2**

If a mortgage at a 6% interest rate has payments of $1000 a month, how much will the loan balance be 10 years from the end of the loan?

**Answer**

To determine this, we are looking for the amount of a loan that can be paid off by $1000 monthly payments for 10 years. In other words, we're looking for $A$ when

$d = \$1000$ the monthly loan payment
$r = 0.06$ $6\%$ annual rate
$n = 12$ since we're doing monthly payments, we'll compound monthly
$t = 10$ since we're making monthly payments for 10 more years

$$A=\frac{1000\left[1-\left(1+\frac{0.06}{12}\right)^{-12\times 10}\right]}{\left(\frac{0.06}{12}\right)}$$

$$=\frac{1000\left[1-(1.005)^{-120}\right]}{0.005}$$

$$=\frac{1000(1-0.5496)}{0.005}$$

$$=\$90,073.45$$

The loan balance with 10 years left of the loan will be $\$90,073.45$

---

# 3.6: Which equation to use?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan", "amortize", "finance (a car)", or "mortgage" (a home loan). Look for these words. If they are part of the problem, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annual/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you’re looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a savings annuity problem. Savings annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a payout annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

### Try It 3.6.1

For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loan problem. Then solve each problem.

a. Marcy received an inheritance of $20,000, and invested it at 6% interest. She is going to use it for college, withdrawing money for tuition and expenses each semester. How much can she take out each semester if she has 3 years of school left?
b. Paul wants to buy a new car. Rather than take out a loan, he decides to save $200 a month in an account earning 3% interest compounded monthly. How much will he have saved up after 3 years?
c. Keisha is managing investments for a non-profit company. They want to invest some money in an account earning 5% interest compounded annually with the goal to have $30,000 in the account in 6 years. How much should Keisha deposit into the account?
d. Miao is going to finance new office equipment at a 2% rate over a 4 year term. If he can afford monthly payments of $100, how much new equipment can he buy?
e. How much would you need to save every month in an account earning 4% interest to have $5000 saved up in two years?

**Answer**

a. This is a payout annuity problem. She can pull out $3691.95 a semester.
b. This is a savings annuity problem. He will have saved up $7,524.11.
c. This is compound interest problem. She would need to deposit $22,386.46.
d. This is a loan problem. He can buy $4,609.33 of new equipment.
e. This is a savings annuity problem. You would need to save $200.46 each month.

---

# 3.7: Solving for Time

Often we are interested in how long it will take to accumulate money or how long we'd need to extend a loan to bring payments down to a reasonable level.

Note: This section assumes you've covered solving exponential equations using logarithms in prior classes.

✓ Example 3.7.1

If you invest \$2000 at 6% compounded monthly, how long will it take the account to double in value?

**Solution**

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

$$\begin{array}{ll} P=\$2000 & \text{the initial deposit} \\ r=0.06 & 6\% \text{ annual rate} \\ n=12 & 12 \text{ months in } 1 \text{ year} \end{array}$$

So our general equation is $A=2000\left(1+\frac{0.06}{12}\right)^{12t}$. We also know that we want our ending amount to be double of $\$2000$, which is $\$4000,$so we're looking for $t$ so that $A=4000$. To solve this, we set our equation for $A$ equal to 4000.

$$\begin{array}{ll} 4000=2000\left(1+\frac{0.06}{12}\right)^{12t} & \text{Divide both sides by } 2000 \\ 2=(1.005)^{12t} & \text{To solve for the exponent, take the log of both sides} \\ \log(2)=\log\left((1.005)^{12t}\right) & \text{Use the exponent property of logs on the right side} \\ \log(2)=12t\log(1.005) & \text{Now we can divide both sides by } 12\log 1.005 \\ \frac{\log(2)}{12\log(1.005)}=t & \text{Approximating this to a decimal} \\ t=11.581 & \end{array}$$

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded log(2) to 0.301 and log(1.005) to 0.00217, then your final answer would have been about 11.577 years.

This answer is close to the answer you would get from the rule of 70 ($t=\frac{70}{6}\approx 11.67$), which is just a quick approximation of doubling time. The rule of 70 is based on this method of solving for time using logarithms.

✓ Example 3.7.2

If you invest \$100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to \$10,000?

**Solution**

This is a savings annuity problem since we are making regular deposits into the account.

$$\begin{array}{ll} d=\$1000 & \text{the monthly deposit} \\ r=0.03 & 3\% \text{ annual rate} \\ n=12 & \text{since we're doing monthly deposits, we'll compound monthly} \end{array}$$

We don't know $t$, but we want $A$ to be $\$10,000$

Putting this into the equation:

$$10,000=\frac{100\left[\left(1+\frac{0.03}{12}\right)^{12t}-1\right]}{\left(\frac{0.03}{12}\right)} \quad \text{Simplifying the fractions a bit}$$

$$10,000=\frac{100\left[(1.0025)^{12t}-1\right]}{0.0025}$$

We want to isolate the exponential term, $1.0025^{12t}$, so multiply both sides by 0.0025

$$\begin{array}{ll}
25=100\left[(1.0025)^{12t}-1\right] & \text{Divide both sides by 100} \\
0.25=(1.0025)^{12t}-1 & \text{Add 1 to both sides} \\
1.25=(1.0025)^{12t} & \text{Now take the log of both sides} \\
\log(1.25)=\log\left((1.0025)^{12t}\right) & \text{Use the exponent property of logs} \\
\log(1.25)=12t\log(1.0025) & \text{Divide by 12log(1.0025)} \\
\frac{\log(1.25)}{12\log(1.0025)}=t & \text{Approximating to a decimal} \\
t=7.447 \text{ years} &
\end{array}$$

It will take about 7.447 years to grow the account to $\$10,000$.

## Try It 3.7.1

Joel is considering putting a $1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of $30 a month?

**Answer**

$$\begin{array}{ll}
d=\$30 & \text{The monthly payments} \\
r=0.12 & 12\% \text{ annual rate} \\
n=12 & \text{since we're doing monthly deposits} \\
A=\$1000 & \text{we're starting with a \$1,000 loan}
\end{array}$$

We are solving for $t$, the time to pay off the loan.

$$1000=\frac{30\left[1-\left(1+\frac{0.12}{12}\right)^{-12t}\right]}{\frac{0.12}{12}}$$

Solving for $t$ gives $3.396$.It will take about 3.4 years to pay off the purchase.

---

# 3.8: Credit Cards

## Understanding Credit Cards

**Revolving credit** is an installment loan that remains open to debt increases and credits. These loans require a regular payment schedule but do not have a fixed amount that is being paid off. The balance of the loan changes as the credit is used for products and services as decided by the account holder.

**Credit cards** are used by a financial institution to give users access to a loan based on revolving credit. When a credit card is used, the account holder is borrowing money from the institution to make a purchase. The maximum a user can borrow is set by the financial institution based on the credit worthiness of the account holder. Each billing period (typically 25-30 days) the account accrues interest for any remaining balance based on the **average daily balance**. The interest is compounded each billing period, which is why it can be easy to get into debt when using credit cards.

## Credit Card Myths

**Myth 1: Carrying a credit card balance can increase your credit score.**

> FALSE! Having a high balance on your cards can actually decrease your credit score since financial institutions see that you have a lot of debt. It is best to pay off the balance of the card every billing period if possible. If not, then keeping your total balance for all credit cards below 15% of the maximum you can borrow is the second-best idea.

**Myth 2: Your income is a factor in your credit score.**

> FALSE! No institution knows your income except you, your employer, and the IRS. The only way an institution knows your income is if you disclose it to them.

**Myth 3: Credit repair companies can fix your credit.**

> FALSE! Credit repair companies can only remove information from a credit report that is incorrect, which any user can do for free. Credit repair companies often will create a new loan to consolidate your debt into one payment. This does not necessarily improve your credit score.

**Myth 4: Closing old credit cards can hurt your credit score.**

> FALSE/TRUE This one is a bit of both.
>
> FALSE: If you have old credit cards that have high maximums that you do not use or excessive number of credit cards, it can be beneficial to close those since financial institutions look at how much debt you COULD get into if you maxed out all your cards.
>
> TRUE: If you have old cards that are not being used, they will show on the credit report as an account that is in good standing which can help your credit score. Some financial institutions will close accounts that have not been used for an extended period of time.

**Myth 5: Checking your credit score can hurt your credit score.**

> FALSE! When you check your personal credit score it is considered a soft inquiry. These inquiries do not affect your credit score. Everyone can request a free credit report from the three main credit reporting agencies once a year.
>
> Applying for a loan or a new service that requires a credit check is considered a hard inquiry. These can negatively affect your score if there are several hard inquiries within a short period of time for different types of loans. For example: if you are applying to purchase a car at several financial institutions to find the best rate that may not affect your score negatively. If you apply for several different credit cards in a short period of time that will negatively affect your score.

## Average Daily Balance Method

### Average Daily Balance

Since the balance of a revolving loan changes often, they typical way of determining interest cannot be used. A financial institution cannot charge interest for the entire billing period on money that was not borrowed at the beginning of that period. For this reason, financial institutions use the average balance over the length of the billing period.

$$\text{average daily balance} = \frac{\text{sum of unpaid balances for each day in the billing period}}{\text{number of days in the billing period}}$$

## Example 3.8.1

Victor got a new credit card. He made the following charges in the first billing cycle. Determine the average daily balance for his first billing period.

| | |
|---|---|
| March 1 | The first billing cycle begins. |
| March 3 | He bought a jacket for \$55. |
| March 8 | He went to the movies and spent \$42. |
| March 15 | He put gas in his car and spent \$67. |
| March 21 | He treated his family to dinner and spent \$73. |
| March 25 | The billing cycle ended. |

**Solution**

To determine the average daily balance we first need to find the sum of unpaid balances for each day in the billing period. The calendar below lists the balances for each day of the billing cycle.

| | | | | |
|---|---|---|---|---|
| March 1 - \$ 0.00 | March 2 - \$ 0.00 | March 3 - \$ 55.00 | March 4 - \$ 55.00 | March 5 - \$ 55.00 |
| March 6 - \$ 55.00 | March 7 - \$ 55.00 | March 8 - \$ 97.00 | March 9 - \$ 97.00 | March 10 - \$ 97.00 |
| March 11 - \$ 97.00 | March 12 - \$ 97.00 | March 13 - \$ 97.00 | March 14 - \$ 97.00 | March 15 - \$ 164.00 |
| March 16 - \$ 164.00 | March 17 - \$ 164.00 | March 18 - \$ 164.00 | March 19 - \$ 164.00 | March 20 - \$ 164.00 |
| March 21 - \$ 237.00 | March 22 - \$ 237.00 | March 23 - \$ 237.00 | March 24 - \$ 237.00 | March 25 - \$ 237.00 |

Begin on March $3^{\text{rd}}$ since that is the first date with a balance to find the sum. The sum will be the numerator of the average daily balance fraction.

$$55+55+55+55+55+97+97+97+97+97+97+97+164+164+164+164+164+164+237+237 \\ +237+237+237=3123$$

The denominator will be the number of days in the billing cycle which, in this case, is 25 days.

$$\text{average daily balance} = \frac{3123}{25} = 125.92$$

Interest will be calculated on the amount \$125.92 since that is the average daily balance for this billing cycle.

## Example 3.8.2

Jasmine has had her credit card for several months. She only pays the minimum payment, which is 2% of the balance or \$10, whichever is more. Review the transactions of her latest billing cycle to determine the average daily balance. Then find the interest that will be charged to her account with a 7.99% APR and the minimum payment required.

| | |
|---|---|
| June 1 | The billing cycle begins with a balance of \$2750 |
| June 5 | She makes a payment of \$55. |
| June 6 | She got a coffee and snacks \$27. |
| June 12 | She put gas in her car and spent \$33. |
| June 17 | She bought tickets to a concert \$119. |
| June 25 | The billing cycle ended. |

**Solution**

In this example, there is an existing balance and a payment is made. The payment will decrease the balance while purchases will increase the balance. Instead of using a calendar to determine the sum, we will use multiplication.

First determine when the balance changed and what the change was. Then determine how many days the account was at the balance.

| | |
|---|---|
| June 1 | $2750.00 |
| June 5 | $\$2750-\$55=\$2695$ |
| June 6 | $\$2695+\$27=\$2722$ |
| June 12 | $\$2722+\$33=\$2755$ |
| June 17 | $\$2755+\$119=\$2874$ |
| June 30 | The billing cycle ended. |

| | |
|---|---|
| 4 days | $2750 |
| 1 day | $2695 |
| 6 days | $2722 |
| 5 days | $2755 |
| 14 days | $2874 |

This billing cycle has 30 days. The total number of days should add to 30. Now we can use multiplication to determine the sum of the daily balances.

$$\text{average daily balance}=\frac{4\times 2750+1\times 2695+6\times 2722+5\times 2755+14\times 2874}{30}=\frac{84,038}{30}=2801.27$$

To find the interest, use the $I=Prt$ formula, where $P$ is the average daily balance, $r$ is the APR, and $t$ is the time in years. Since this is a monthly account $t=\frac{1}{12}$.

$$I=2801.27\times 0.0799\times\frac{1}{12}=18.65$$

The interest is added to the balance before the minimum payment is determined. Recall that the minimum payment is either 2% or $10, whichever is greater. The balance is the last value before the billing cycle ended. Do not use the average daily balance to calculate the minimum payment. Note: If the balance is paid in full by the end of the cycle, no interest is charged.

$$2874+18.65=2892.65\times 0.02=57.85$$

The minimum payment due for this billing period is $57.85.

### Try It 3.8.1

Refer to the following credit card transactions. Determine the average daily balance and the interest charged for this billing cycle for an account with a 24.99% APR. Find the final balance. Assume the account holder wanted to pay the credit card off in one year. Use the loan payment formula from the previous sections to determine how much should be paid each month if no additional charges are made to meet that goal.

| | |
|---|---|
| May 1 | The billing cycle begins with a balance of $4212 |
| May 4 | A payment of $25 is made. |
| May 12 | A purchase of $148 is made. |
| May 16 | A purchase of $16 is made. |
| May 17 | A purchase of $96 is made. |
| May 30 | The billing cycle ended. |

**Answer**

Find the sum of the daily balances and divide by the number of days in the billing cycle.

$$\text{average daily balance}=\frac{3\times 4212+8\times 4187+4\times 4335+1\times 4351+13\times 4447}{30}=\frac{125,634}{30}=4187.80$$

Determine the interest for the current billing cycle.

$$I=4187.80\times 0.2499\times\frac{1}{12}=87.21$$

Use the loan payment formula to determine how much should be paid each month to pay off the credit card in one year.

The final balance would be $4447 + 87.21 = 4534.21$ which will be used as $P$, $r = 24.99\% = 0.2499$, $t = 1$, and $n = 12$.

$$d = \frac{4534.24\left(\frac{0.2499}{12}\right)}{\left[1 - \left(1 + \frac{0.2499}{12}\right)^{-12\times 1}\right]} = 430.93$$

It would require a monthly payment of \$430.93 to pay off the balance of the given credit card in one year if no additional charges are made.

## Buy Now, Pay Later: An Alternative to Credit Cards?

The payment method known as "Buy Now, Pay Later," or BNPL, has been around for many years, but has become popular recently due to the pandemic. As more consumers buy goods online, merchants have started offering BNPL options for making purchases. You can determine whether a company offers BNPL on the checkout page - you might be given the option to pay off the whole bill immediately or to break it up into regular installments, such as one payment per month. These payments are calculated using the **Add-On Interest Method**.

### Add-On Interest

Companies that charge interest for the use of a BNPL plan typically calculate interest on the entire amount borrowed, then add it to the principal. This total is then divided into equal-sized payments.

$$\text{size of payment} = \frac{\text{sum of amount borrowed and interest}}{\text{number of payments}}$$

More specifically, if the company uses a monthly payment plan, we can use the following formula.

$$\text{monthly payment} = \frac{P + I}{n}$$

where

- $P$ is the principal
- $I = Prt$ is the interest
- $r$ is the annual interest rate as a decimal
- $t$ is the length of time (in years) before the bill is fully paid off
- $n$ is the number of monthly payments

### Example 3.8.3

Jacques is looking to buy a \$2,000 couch online. The merchant selling the couch offers a BNPL option for purchasing, where the customer would pay 15% simple interest, and the bill would be spread out in four equal monthly payments. Calculate the amount of Jacques's monthly payment.

#### Solution

Since the couch costs \$2,000, the principal $P$ is 2,000. The interest rate is $r = 0.15$. Since it will take 4 months (or $\frac{1}{3}$ of a year) to pay off the bill, $n = 4$ and $t = \frac{1}{3}$.

$$I = 2000 \times 0.15 \times \frac{1}{3} = 100$$

and

$$\text{monthly payment} = \frac{2,000 + 100}{4} = 525$$

So Jacques would pay \$525 in each of his monthly payments.

When would it be preferable to pay using Buy Now, Pay Later? In general, it is a good rule of thumb to avoid as much debt as possible, especially when for a non-essential item. But there are some situations when it might be better to use BNPL instead of, say, a credit card. You typically do not need to have a credit history to use BNPL, and many payment options charge zero interest. But be careful! These payment plans frequently charge late fees if you do not complete a payment on time, and it is easy to overspend.

# 3.9: Basic Budgeting

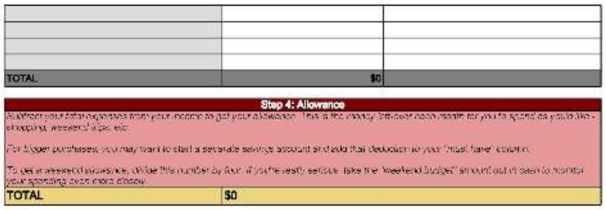
Budgeting is an important step in managing your money and spending habits. A **budget** is a plan to balance income and expenses, which may also include saving money for future use. To create a budget you need to identify how much money you are spending. Some expenses to keep in mind when creating a budget are rent, car payment, fuel, auto insurance, utilities, groceries, cell phone, personal, gym membership, entertainment, gifts, dining out, medical expenses, etc.

There are several apps out there that can help you budget your money. Just a few examples are Mint, Manilla, and Goodbudget. These are all free apps that help you keep track of bills and your accounts. Your bank also keeps track of your spending and what categories each item falls under. Log into your bank account online and look for “Track Spending” or a similar item. Many banks give you a pie chart showing you how much you spent in each category in the last month. You can edit your categories, change the number of months, and sometimes even set a budget goal.

**Table $3.9.1$: Example of Budget in Excel**

When you are creating a monthly budget, many experts say if you want to have control of your money, you should know where every dollar is going. In order to keep track of this, a written budget is essential. Below is one example of a budget in Excel. This was a free template from the “Life After College” blog. There are hundreds of free templates out there so you should find the template that suits you the best – or create your own Excel budget!

| | | |
|---|---|---|
| **Four-Step Budget Template** | | **Last Updated:** |
| Brought to you by Jenny Blake | | {Insert date here} |
| Life After College Blog: | http://lifeaftercollege.org | |
| Life After College Book: | http://amzn.to/jennyblake | |
| Related post: | http://bit.ly/VuSJB | |
| | | |
| ***Note: Enter amounts in Column B and the totals will automatically calculate.*** | | |
| | | |
| **Step 1: Income** | | |
| *This includes: paychecks, side jobs, anything that brings money into your bank account* | | ***Notes:*** |
| **Income Source: {Fill In Name}** | | |
| **Income Source: {Fill In Name}** | | |
| **Income Source: {Fill In Name}** | | |
| TOTAL | $0 | |
| | | |
| **Step 2: Must-Have Expenses** | | |
| *This includes: Rent, utilities, cell phone bills, anything that will incur late fees. Includes other essentials like groceries and automatic savings account deductions. Saving is a must!* | | **Notes:** |
| **Rent or Mortgage** | | |
| **Utilities** | | |
| **Cell Phone Bill** | | |
| **Savings 1** | | |
| **Savings 2** | | |
| **Other (add rows as needed)** | | |
| | | |
| TOTAL | $0 | |
| | | |
| **Step 3: Nice-to-Have Expenses** | | |
| *This includes: things that you KNOW you spend money on every month like going out to eat. This does not include: one-off purchases (like a TV), major shopping trips, major travel (unless you take frequent weekend trips)* | | ***Notes:*** |
| **Going out to eat (estimate)** | | |
| **Fill in...** | | |
| | | |



| | | |
|---|---|---|
| | | |
| | | |
| | | |
| | | |
| TOTAL | $0 | |
| **Step 4: Allowance** | | |
| TOTAL | $0 | |

(“Four-Step Budget Template,” n.d., https://docs.google.com/spreadsheets...mplate/preview)

**✓ Example 3.9.1**

You make $32,000 a year and want to save 10% of your income every year. How much should you put into savings every month?

**Solution**

$$\$32,000 \cdot 0.10 = \$3200$$

You want to save $3200 a year.

$$\frac{\$3200}{12} = \$266.67$$

You should be saving $266.67 a month or $133.33 a paycheck if you are paid biweekly.

---

# 3.10: Extension- Taxes

Governments collect taxes to pay for the services they provide. In the United States, federal income taxes help fund the military, the Environmental Protection Agency, and thousands of other programs. Property taxes help fund schools. Gasoline taxes help pay for road improvements. While very few people enjoy paying taxes, they are necessary to pay for the services we all depend upon.

Taxes can be computed in a variety of ways, but are typically computed as a percentage of a sale, of one's income, or of one's assets.

### ✓ Example 3.10.1

The sales tax rate in a city is 9.3%. How much sales tax will you pay on a $140 purchase?

**Solution**

The sales tax will be 9.3% of $140. To compute this, we multiply $140 by the percent written as a decimal: $140(0.093) = $13.02.

When taxes are not given as a fixed percentage rate, sometimes it is necessary to calculate the **effective rate**.

### Definition: Effective Rate

The **effective tax rate** is the equivalent percent rate of the tax paid out of the dollar amount the tax is based on.

### ✓ Example 3.10.2

Joan paid $3,200 in property taxes on her house valued at $215,000 last year. What is the effective tax rate?

**Solution**

We can compute the equivalent percentage: $\frac{3200}{215000}$ = 0.01488, or about 1.49% effective rate.

Taxes are often referred to as progressive, regressive, or flat.

### Definition: Tax Categories

A **flat tax**, or proportional tax, charges a constant percentage rate.

A **progressive tax** increases the percent rate as the base amount increases.

A **regressive tax** decreases the percent rate as the base amount increases.

### ✓ Example 3.10.3

The United States federal income tax on earned wages is an example of a progressive tax. People with a higher wage income pay a higher percent tax on their income.

For a single person in 2011, adjusted gross income (income after deductions) under $8,500 was taxed at 10%. Income over $8,500 but under $34,500 was taxed at 15%.

**Solution**

A person earning $10,000 would pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

8500(0.10) = 850 (10% of $8500)

1500(0.15) = 225 (15% of the remaining $1500 of income)

Total tax: = $1075

The effective tax rate paid is $\frac{1075}{10000}$ = 10.75%

A person earning $30,000 would also pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

8500(0.10) = 850 (10% of $8500)

21500(0.15) = 3225 (15% of the remaining $21500 of income)

Total tax: = $4075

The effective tax rate paid is $\frac{4075}{30000}=13.58\%$

Notice that the effective rate has increased with income, showing this is a progressive tax.

### ✓ Example 3.10.4

A gasoline tax is a flat tax when considered in terms of consumption, a tax of, say, $0.30 per gallon is proportional to the amount of gasoline purchased. Someone buying 10 gallons of gas at $4 a gallon would pay $3 in tax, which is $\frac{\$3}{\$40}=7.5\%$. Someone buying 30 gallons of gas at $4 a gallon would pay $9 in tax, which is $\frac{\$9}{\$120}=7.5\%$, the same effective rate.

**Solution**

However, in terms of income, a gasoline tax is often considered a regressive tax. It is likely that someone earning $30,000 a year and someone earning $60,000 a year will drive about the same amount. If both pay $60 in gasoline taxes over a year, the person earning $30,000 has paid 0.2% of their income, while the person earning $60,000 has paid 0.1% of their income in gas taxes.

### Try It 3.10.1

A sales tax is a fixed percentage tax on a person's purchases. Is this a flat, progressive, or regressive tax?

**Answer**

While sales tax is a flat percentage rate, it is often considered a regressive tax for the same reasons as the gasoline tax.

---

# 3.11: Income Taxation

Many people have proposed various revisions to the income tax collection in the United States. Some, for example, have claimed that a flat tax would be fairer. Others call for revisions to how different types of income are taxed since currently investment income is taxed at a different rate than wage income.

The following two projects will allow you to explore some of these ideas and draw your own conclusions.

## Project 1: Flat Tax, Modified Flat Tax, and Progressive Tax.

Imagine the country is made up of 100 households. The federal government needs to collect $800,000 in income taxes to be able to function. The population consists of 6 groups:

- Group A: 20 households that earn $12,000 each
- Group B: 20 households that earn $29,000 each
- Group C: 20 households that earn $50,000 each
- Group D: 20 households that earn $79,000 each
- Group E: 15 households that earn $129,000 each
- Group F: 5 households that earn $295,000 each

This scenario is roughly proportional to the actual United States population and tax needs. We are going to determine new income tax rates.

The first proposal we'll consider is a flat tax – one where every income group is taxed at the same percentage tax rate.

1) Determine the total income for the population (all 100 people together)

2) Determine what flat tax rate would be necessary to collect enough money.

The second proposal we'll consider is a modified flat-tax plan, where everyone only pays taxes on any income over $20,000. So, everyone in group A will pay no taxes. Everyone in group B will pay taxes only on $9,000.

3) Determine the total *taxable* income for the whole population

4) Determine what flat tax rate would be necessary to collect enough money in this modified system

5) Complete this table for both the plans

| | | Flat Tax Plan | | Modified Flat Tax Plan | |
|---|---|---|---|---|---|
| **Group** | **Income per Household** | **Income Tax per Household** | **Income after Taxes** | **Income Tax per Household** | **Income after Taxes** |
| A | $12,000 | | | | |
| B | $29,000 | | | | |
| C | $50,000 | | | | |
| D | $79,000 | | | | |
| E | $129,000 | | | | |
| F | $295,000 | | | | |

The third proposal we'll consider is a progressive tax, where lower income groups are taxed at a lower percent rate, and higher income groups are taxed at a higher percent rate. For simplicity, we're going to assume that a household is taxed at the same rate on all their income.

6) Set progressive tax rates for each income group to bring in enough money. There is no one right answer here – just make sure you bring in enough money!

| Group | Income per Household | Tax Rate (%) | Income Tax per Household | Total Tax Collected for All Households | Income after Taxes per Household |
|---|---|---|---|---|---|
| A | $12,000 | | | | |
| B | $29,000 | | | | |
| C | $50,000 | | | | |
| D | $79,000 | | | | |
| E | $129,000 | | | | |
| F | $295,000 | | | | |
| | | | | This better total to $800,000 | |

7) Discretionary income is the income people have left over after paying for necessities like rent, food, transportation, etc. The cost of basic expenses does increase with income, since housing and car costs are higher, however usually not proportionally. For each income group, estimate their essential expenses, and calculate their discretionary income. Then compute the effective tax rate for each plan relative to discretionary income rather than income.

| Group | Income per Household | Discretionary Income (Estimated) | Effective Rate, Flat | Effective Rate, Modified | Effective Rate, Progressive |
|---|---|---|---|---|---|
| A | $12,000 | | | | |
| B | $29,000 | | | | |
| C | $50,000 | | | | |
| D | $79,000 | | | | |
| E | $129,000 | | | | |
| F | $295,000 | | | | |

8) Which plan seems the fairest to you? Which plan seems the least fair to you? Why?

## Project 2: Calculating Taxes

Visit www.irs.gov, and download the most recent version of forms 1040, and schedules A, B, C, and D.

**Scenario 1**: Calculate the taxes for someone who earned $60,000 in standard wage income (W-2 income), has no dependents, and takes the standard deduction.

**Scenario 2**: Calculate the taxes for someone who earned $20,000 in standard wage income, $40,000 in qualified dividends, has no dependents, and takes the standard deduction. (Qualified dividends are earnings on certain investments such as stocks.)

**Scenario 3**: Calculate the taxes for someone who earned $60,000 in small business income, has no dependents, and takes the standard deduction.

Based on these three scenarios, what are your impressions of how the income tax system treats these different forms of income (wage, dividends, and business income)?

**Scenario 4**: To get a more realistic sense for calculating taxes, you'll need to consider itemized deductions. Calculate the income taxes for someone with the income and expenses listed below.

Married with 2 children, filing jointly

Wage income: $50,000 combined

Paid sales tax in Washington State

Property taxes paid: $3200

Home mortgage interest paid: $4800

Charitable gifts: $1200

# 3.12: Chapter Review and Glossary

## Chapter Review

### Percents

A percentage is a number that tells us how much a part is out of 100 and can also be written as a decimal or a fraction. Percentages describe a relative portion of a whole.

Simple Interest

Compound Interest

Annuities

Loans

Credit Cards

Basic Budgeting

Leases

Taxes

# 3.13: Exercises

## Skills

1. A friend lends you \$200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?
2. Suppose you obtain a \$3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?
3. A T-bill is a type of bond that is sold at a discount over the face value. For example, suppose you buy a 13-week T-bill with a face value of \$10,000 for \$9,800. This means that in 13 weeks, the government will give you the face value, earning you \$200. What annual interest rate have you earned?
4. Suppose you are looking to buy a \$5000 face value 26-week T-bill. If you want to earn at least 1% annual interest, what is the most you should pay for the T-bill?
5. You deposit \$300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?
6. How much will \$1000 deposited in an account earning 7% interest compounded annually be worth in 20 years?
7. You deposit \$2000 in an account earning 3% interest compounded monthly.
   a. How much will you have in the account in 20 years?
   b. How much interest will you earn?
8. You deposit \$10,000 in an account earning 4% interest compounded monthly.
   a. How much will you have in the account in 25 years?
   b. How much interest will you earn?
9. How much would you need to deposit in an account now in order to have \$6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.
10. How much would you need to deposit in an account now in order to have \$20,000 in the account in 4 years? Assume the account earns 5% interest.
11. You deposit \$200 each month into an account earning 3% interest compounded monthly.
    a. How much will you have in the account in 30 years?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
12. You deposit \$1000 each year into an account earning 8% compounded annually.
    a. How much will you have in the account in 10 years?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
13. Jose has determined he needs to have \$800,000 for retirement in 30 years. His account earns 6% interest.
    a. How much would he need to deposit in the account each month?
    b. How much total money will he put into the account?
    c. How much total interest will he earn?
14. You wish to have \$3000 in 2 years to buy a fancy new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?
15. You want to be able to withdraw \$30,000 each year for 25 years. Your account earns 8% interest.
    a. How much do you need in your account at the beginning
    b. How much total money will you pull out of the account?
    c. How much of that money is interest?
16. How much money will I need to have at retirement so I can withdraw \$60,000 a year for 20 years from an account earning 8% compounded annually?
    a. How much do you need in your account at the beginning
    b. How much total money will you pull out of the account?

   c. How much of that money is interest?

17. You have $500,000 saved for retirement. Your account earns 6% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 20 years?
18. Loren already knows that he will have $500,000 when he retires. If he sets up a payout annuity for 30 years in an account paying 10% interest, how much could the annuity provide each month?
19. You can afford a $700 per month mortgage payment. You’ve found a 30 year loan at 5% interest.
    a. How big of a loan can you afford?
    b. How much total money will you pay the loan company?
    c. How much of that money is interest?
20. Marie can afford a $250 per month car payment. She’s found a 5 year loan at 7% interest.
    a. How expensive of a car can she afford?
    b. How much total money will she pay the loan company?
    c. How much of that money is interest?
21. You want to buy a $25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?
22. You decide finance a $12,000 car at 3% compounded monthly for 4 years. What will your monthly payments be? How much interest will you pay over the life of the loan?
23. You want to buy a $200,000 home. You plan to pay 10% as a down payment, and take out a 30 year loan for the rest.
    a. How much is the loan amount going to be?
    b. What will your monthly payments be if the interest rate is 5%?
    c. What will your monthly payments be if the interest rate is 6%?
24. Lynn bought a $300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.
    a. Find her monthly payments.
    b. How much interest will she pay over the life of the loan?
25. Emile bought a car for $24,000 three years ago. The loan had a 5 year term at 3% interest rate, making monthly payments. How much does he still owe on the car?
26. A friend bought a house 15 years ago, taking out a $120,000 mortgage at 6% for 30 years, making monthly payments. How much does she still owe on the mortgage?
27. Pat deposits $6,000 into an account earning 4% compounded monthly. How long will it take the account to grow to $10,000?
28. Kay is saving $200 a month into an account earning 5% interest. How long will it take her to save $20,000?
29. James has $3,000 in credit card debt, which charges 14% interest. How long will it take to pay off the card if he makes the minimum payment of $60 a month?
30. Chris has saved $200,000 for retirement, and it is in an account earning 6% interest. If she withdraws $3,000 a month, how long will the money last?

## Concepts

31. Suppose you invest $50 a month for 5 years into an account earning 8% compounded monthly. After 5 years, you leave the money, without making additional deposits, in the account for another 25 years. How much will you have in the end?
32. Suppose you put off making investments for the first 5 years, and instead made deposits of $50 a month for 25 years into an account earning 8% compounded monthly. How much will you have in the end?
33. Mike plans to make contributions to his retirement account for 15 years. After the last contribution, he will start withdrawing $10,000 a quarter for 10 years. Assuming Mike's account earns 8% compounded quarterly, how large must his quarterly contributions be during the first 15 years, in order to accomplish his goal?
34. Kendra wants to be able to make withdrawals of $60,000 a year for 30 years after retiring in 35 years. How much will she have to save each year up until retirement if her account earns 7% interest?

35. You have $2,000 to invest, and want it to grow to $3,000 in two years. What interest rate would you need to find to make this possible?
36. You have $5,000 to invest, and want it to grow to $20,000 in ten years. What interest rate would you need to find to make this possible?
37. You plan to save $600 a month for the next 30 years for retirement. What interest rate would you need to have $1,000,000 at retirement?
38. You really want to buy a used car for $11,000, but can only afford $200 a month. What interest rate would you need to find to be able to afford the car, assuming the loan is for 60 months?

## Exploration

39. Pay day loans are short term loans that you take out against future paychecks: The company advances you money against a future paycheck. Either visit a pay day loan company, or look one up online. Be forewarned that many companies do not make their fees obvious, so you might need to do some digging or look at several companies.
    a. Explain the general method by which the loan works.
    b. We will assume that we need to borrow $500 and that we will pay back the loan in 14 days. Determine the total amount that you would need to pay back and the effective loan rate. The effective loan rate is the percentage of the original loan amount that you pay back. It is not the same as the APR (annual rate) that is probably published.
    c. If you cannot pay back the loan after 14 days, you will need to get an extension for another 14 days. Determine the fees for an extension, determine the total amount you will be paying for the now 28 day loan, and compute the effective loan rate.
40. Suppose that 10 years ago you bought a home for $110,000, paying 10% as a down payment, and financing the rest at 9% interest for 30 years.
    a. Let's consider your existing mortgage:
        i. How much money did you pay as your down payment?
        ii. How much money was your mortgage (loan) for?
        iii. What is your current monthly payment?
        iv. How much total interest will you pay over the life of the loan?
40. b. This year, you check your loan balance. Only part of your payments have been going to pay down the loan; the rest has been going towards interest. You see that you still have $88,536 left to pay on your loan. Your house is now valued at $150,000.

How much of the loan have you paid off? (i.e., how much have you reduced the loan balance by? Keep in mind that interest is charged each month - it's not part of the loan balance.)

    i. How much money have you paid to the loan company so far?
    ii. How much interest have you paid so far?
    iii. How much equity do you have in your home (equity is value minus remaining debt)

40. c. Since interest rates have dropped, you consider refinancing your mortgage at a lower 6% rate.
        i. If you took out a new 30 year mortgage at 6% for your remaining loan balance, what would your new monthly payments be?
        ii. How much interest will you pay over the life of the new loan?
40. d. Notice that if you refinance, you are going to be making payments on your home for another 30 years. In addition to the 10 years you've already been paying, that's 40 years total.
        i. How much will you save each month because of the lower monthly payment?
        ii. How much total interest will you be paying (you need to consider the amount from 2c and 3b)
        iii. Does it make sense to refinance? (there isn't a correct answer to this question. Just give your opinion and your reason)

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# CHAPTER OVERVIEW

## 4: Sets

The material in this chapter is from Math In Society by David Lippman.

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# 4.1: Basics of Sets

**Learning Objectives**

- Define a set using a description, roster form or set-builder notation
- Understand notations used in sets
- Identify a subset of a set

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

**Set**

A **set** is a collection of distinct objects, called **elements** of the set.

A set can be defined by:

- describing the contents in words
- by listing the elements of the set enclosed in curly brackets (known as the *roster method*)
- by using mathematical symbols to state a rule to determine the elements of the set (known as *set-builder notation*)

**Notation**

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later. We usually use capital letters to represent the set, and lower case letters for elements of a set.

Symbols used with sets:

- $\in$ means “is an element of”
- | means "such that"
- $\mathbb{N}$ means the set of natural numbers, which is {1, 2, 3, 4, ... }
- $\varnothing$ means the **empty set** or { }, which is a set that contains no elements
  - Note: do not put curly brackets around $\varnothing$. It is a symbol used to represent the empty set. $\varnothing$ = { }.

**Example 4.1.1**

Some examples of sets defined by describing the contents in words:

a. The set of all even numbers
b. The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:

a. *A* = {1, 3, 9, 12}
b. *B* = {red, orange, yellow, green, blue, indigo, violet}

Some examples of sets defined by set-builder notation:

a. $C=\{x|x\in\mathbb{N} \text{ and } x<10\}$ (which is *C* = {1, 2, 3, 4, 5, 6, 7, 8, 9} in roster form)
b. $D=\{x|x \text{ is a day of the week that starts with S}\}$ (which is *D* = {Saturday, Sunday} in roster form)

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

**Example 4.1.2**

Let $A=\{1,2,3,4\}$

To notate that 2 is element of the set, we'd write $2\in A$.

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

### Subset

A **subset** of a set $A$ is another set that contains only elements from the set $A$, but may not contain all the elements of $A$.

If $B$ is a subset of $A$, we write $B \subseteq A$.

A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.

If $B$ is a proper subset of $A$, we write $B \subset A$.

Note that the empty set is a subset of every set. Also, every set is a subset of itself.

### Example 4.1.3

Consider these three sets:

$A =$ the set of all even numbers $\quad B = \{2, 4, 6\} \quad C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of $B$ is an even number, so every element in $B$ is an element of $A$.

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.

It is also true that $B \subset C$.

$C$ is not a subset of $A$, since $C$ contains an element, 3, that is not contained in $A$. It is written $C \not\subseteq A$.

### Example 4.1.4

Suppose a set contains the plays "Much Ado About Nothing", "MacBeth", and "A Midsummer's Night Dream". What is a larger set this might be a subset of?

#### Solution

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

### Try it 4.1.1

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

**Answer**

> There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

---

# 4.2: Union, Intersection, and Complement

**Learning Objectives**

- Find the union of sets
- Find the intersection of sets
- Find the complement of a set

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

**Universal Set**

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context. It is often denoted by $U$.

**Example $4.2.1$**

a. If we were discussing searching for books, the universal set might be all the books in the library.
b. If we were grouping your Discord friends, the universal set would be all your Discord friends.
c. If you were working with sets of numbers, the universal set might be all natural numbers, all integers, or all real numbers.

**Set Operations: Union, Intersection, and Complement**

The **union** of two sets contains all the elements in either set (or both sets).

The union is notated $A \cup B$.

More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both).

The **intersection** of two sets contains only the elements that are in both sets.

The intersection is notated $A \cap B$.

More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$.

The **complement** of a set *A* contains everything in the universal set that is *not* in the set *A*.

The complement is notated $A'$, or $A^C$, or sometimes $\sim A$.

**Example $4.2.2$**

Suppose the universal set is $U =$ all natural numbers from 1 to 9 inclusive. If $A = \{1, 2, 4\}$, then $A' = \{3, 5, 6, 7, 8, 9\}$.

**Example $4.2.3$**

Consider the sets:

$$A = \{\text{red, green, blue}\} \quad B = \{\text{red, yellow, orange}\} \quad C = \{\text{red, orange, yellow, green, blue, violet}\}$$

a. Find $A \cup B$.
b. Find $A \cap B$.
c. Find $A' \cap C$.

**Solution**

a. The union contains all the elements in either set: $A \cup B = \{\text{red, green, blue, yellow, orange}\}$. Notice we only list red once.
b. The intersection contains all the elements in both sets: $A \cap B = \{\text{red}\}$
c. Here we're looking for all of the elements that are not in set $A$ and are also in $C$. $A' \cap C = \{\text{orange, yellow, violet}\}$

**Try It 4.2.1**

Using the sets from the previous example, find $A \cup C$ and $B' \cap A$.

**Answer**

$A \cup C = \{\text{red, orange, yellow, green, blue, violet}\}$

$B' \cap A = \{\text{green, blue}\}$

As we saw earlier with the expression $A' \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic - to force an order of operations.

**Example 4.2.4**

Suppose

$H = \{\text{cat, dog, rabbit, mouse}\}, F = \{\text{dog, cow, duck, pig, rabbit}\} \quad W = \{\text{duck, rabbit, deer, frog, mouse}\}$

a. Find $(H \cap F) \cup W$.
b. Find $H \cap (F \cup W)$.
c. Find $(H \cap F)' \cap W$.

**Solution**

a. We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$.
Now we union that result with $W$ : $(H \cap F) \cup W = \{\text{dog, duck, rabbit, deer, frog, mouse}\}$ .
b. We start with the union: $F \cup W = \{\text{dog, cow, rabbit, duck, pig, deer, frog, mouse}\}$.
Now we intersect that result with $H$ : $H \cap (F \cup W) = \{\text{dog, rabbit, mouse}\}$ .
c. We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$.
Now we want to find the elements of $W$ that are not in $H \cap F$. $(H \cap F)' \cap W = \{\text{duck, deer, frog, mouse}\}$ .

---

# 4.3: Venn Diagrams

## Learning Objectives

- Use a Venn diagram to illustrate the relationship between 2 and 3 sets
- Create an expression relating 2 and 3 sets from a Venn diagram

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

## Venn Diagram

A **Venn diagram** represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

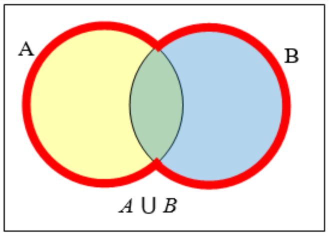
Basic Venn diagrams can illustrate the interaction of two or three sets.

## Example 4.3.1

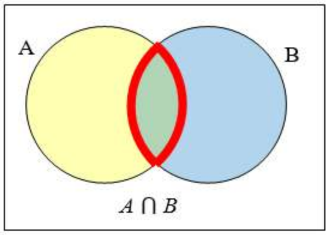
Create Venn diagrams to illustrate $A \cup B, A \cap B,$ and $A' \cap B$.

### Solution

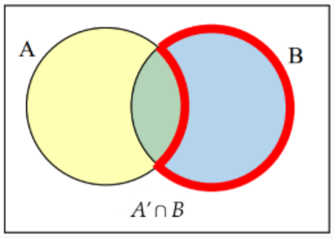
$A \cup B$ contains all elements in either set.



$A \cap B$ contains only those elements in both sets - in the overlap of the circles.



$A' \cap B$ contains the elements of $B$ that are not in $A$.

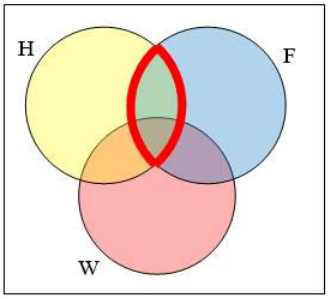


### ✓ Example 4.3.2

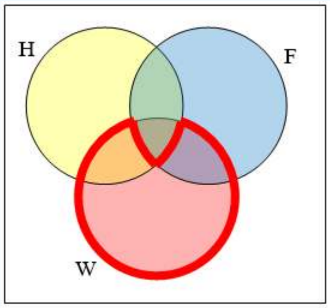
Use a Venn diagram to illustrate $(H \cap F)' \cap W$ .

**Solution**

We'll start by identifying everything in the set $\mathrm{H} \cap F$ .

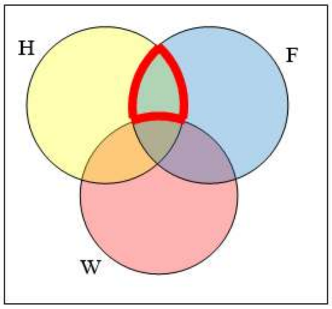


Now, $(H \cap F)' \cap W$ will contain everything not in the set identified above that is also in set $W$.



### ✓ Example 4.3.3

Create an expression to represent the outlined part of the Venn diagram shown.

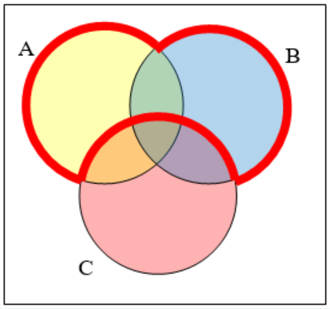


**Solution**

The elements in the outlined set are in sets H and $F$, but are not in set $W$. So we could represent this set as $H \cap F \cap W'$ .

### ✎ Try It 4.3.1

Create an expression to represent the outlined portion of the Venn diagram shown



**Answer**

$(A \cup B) \cap C'$

---

# 4.4: Cardinality

**Learning Objectives**

- Find the cardinality of a set
- Use cardinality properties to solve survey problems using a Venn diagram

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

**Cardinality**

The number of elements in a set is the **cardinality** of that set.

The cardinality of the set $A$ is often notated as $|A|$ or $n(A)$.

Note: a set can be finite or infinite. A finite set will have a cardinality of 0 or a natural number. An infinite set has a cardinality of the form $\aleph_0$ (aleph null), which represents the cardinality of the natural numbers.

**Example 4.4.1**

Let $A=\{1,2,3,4,5,6\}$and $B=\{2,4,6,8\}$.

Find the cardinality of:

a. $B$
b. $A\cup B$
c. $A\cap B$

**Solution**

a. The cardinality of $B$ is $4$, since there are 4 elements in the set.
b. The cardinality of $A\cup B$ is $7$, since $A\cup B=\{1,2,3,4,5,6,8\}$, which contains 7 elements.
c. The cardinality of $A\cap B$ is 3 , since $A\cap B=\{2,4,6\}$, which contains 3 elements.

**Example 4.4.2**

What is the cardinality of $P=$ the set of English names for the months of the year?

**Solution**

The cardinality of this set is $12$, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

**Example 4.4.3**

A survey asks 200 people “What beverage do you drink in the morning?”, and offers choices:

- Tea only
- Coffee only
- Both coffee and tea

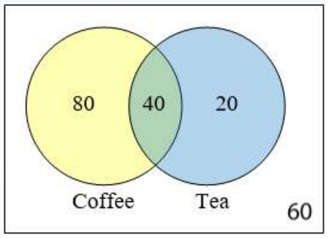
Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

**Solution**

This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

$200 - 20 - 80 - 40 = 60$ people who drink neither.



### ✓ Example 4.4.4

A survey asks: Which social media have you used in the last month?

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter nor Facebook?

**Solution**

Let $T$ be the set of all people who have used Twitter, and $F$ be the set of all people who have used Facebook. Notice that while the cardinality of $F$ is $70\%$ and the cardinality of $T$ is $40\%$, the cardinality of $F \cup T$ is not simply $70\% + 40\%$, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of $F$ and the cardinality of $T$, then subtract those in intersection that we've counted twice. In symbols,

$$n(F \cup T) = n(F) + n(T) - n(F \cap T)$$

$$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)'$. Since the universal set contains $100\%$ of people and the cardinality of $F \cup T = 90\%$, the cardinality of $(F \cup T)'$ must be the other $10\%$.

The previous example illustrated two important properties.

### Cardinality properties

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A') = n(U) - n(A)$$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

### ✓ Example 4.4.5

Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next semester.

| | |
|---|---|
| 21 were taking a SS course | 26 were taking a HM course |
| 19 were taking a NS course | 9 were taking SS and HM |
| 7 were taking SS and NS | 10 were taking HM and NS |
| 3 were taking all three | 7 were taking none |

How many students are only taking a SS course?

**Solution**

It might help to look at a Venn diagram.

From the given data, we know that there are 3 students in region $e$ and 7 students in region $h$.

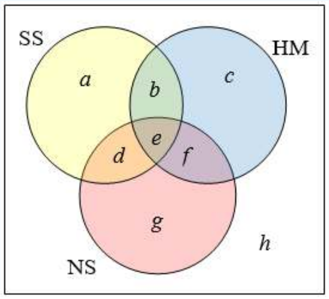
Since 7 students were taking a SS and NS course, we know that $n(d) + n(e) = 7$ . Since we know there are 3 students in region $e$, there must be $7 - 3 = 4$ students in region $d$.

Similarly, since there are 10 students taking HM and NS, which includes regions $e$ and $f$, there must be $10-3=7$ students in region $f$.

Since 9 students were taking SS and HM, there must be $9-3=6$ students in region $b$.

Now, we know that 21 students were taking a SS course. This includes students from regions $a, b, d$, and $e$. Since we know the number of students in all but region $a$, we can determine that $21-6-4-3=8$ students are in region $a$.

Thus, 8 students are taking only a SS course.



### Try It 4.4.1

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

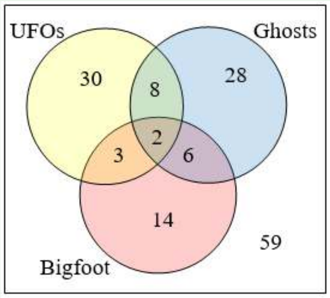
| | |
|---|---|
| 43 believed in UFOs | 44 believed in ghosts |
| 25 believed in Bigfoot | 10 believed in UFOs and ghosts |
| 8 believed in ghosts and Bigfoot | 5 believed in UFOs and Bigfoot |
| 2 believed in all three | |

How many people surveyed believed in at least one of these things?

**Answer**

Starting with the intersection of all three circles, we work our way out. Since 10 people believe in UFOs and ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and ghosts. We work our way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves $150-91=59$ who believe in none.

Thus, 91 people believed in at least one of these things.



# 4.5: Chapter Review and Glossary

## Chapter Review

### Basics of Sets

A set is a collection of elements. Sets can be written in roster form (listing the elements) or set-builder notation (using mathematical symbols to describe the properties of elements in the set).

### Union, Intersection, and Complement

Sets can interact with each other. The union of two sets is the collection of all elements that are in either set. The intersection of two sets is the collection of all elemnts that are in both sets.

### Venn Diagrams

A **Venn diagram** represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

### Cardinality

Cardinality indicates the number of elements in a set. There is a relationship between the cardinality of two sets, their union and their intersection which can be used to compute nummbers not directly given in a situation.

## Glossary:

| | |
|---|---|
| **cardinality** | number of items in a set |
| **complement** | The complement of a set A contains everything in the universal set that is not in the set A |
| **intersection** | The intersection of two sets contains only the elements that are in both sets |
| **proper subset** | a subset that is not identical to the original set – it contains fewer elements |
| **set** | collection of distinct objects, called elements of the set |
| **subset** | A subset of a set $A$A is another set that contains only elements from the set $A$A, but may not contain all the elements of $A$ |
| **union** | The union of two sets contains all the elements in either set (or both sets) |
| **universal set** | set that contains all the elements we are interested in |
| **Venn diagram** | A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set |

# 4.6: Exercises

1. List out the elements of the set “The letters of the word Mississippi.”

2. List out the elements of the set “Months of the year.”

3. Write a word description of the set $\{3, 6, 9\}$.

4. Write a word description of the set $\{a, e, i, o, u\}$.

5. Is $\{1, 3, 5\}$ a subset of the set of odd integers?

6. Is $\{A, B, C\}$a subset of the set of letters of the alphabet?

For problems 7-12, consider the sets below, and indicate if each statement is true or false.

$A = \{1, 2, 3, 4, 5\} \quad B = \{1, 3, 5\} \quad C = \{4, 6\} \quad U = \text{ whole numbers from 0 to 10}$

7. $3 \in B$

8. $5 \in C$

9. $B \subset A$

10. $C \subset A$

11. $C \subset B$

12. $C \subset U$

Using the sets from above, and treating $U$ as the Universal set, find each of the following:

13. $A \cup B$

14. $A \cup C$

15. $A \cap C$

16. $B \cap C$

17. $A'$

18. $B'$

Let $D = \{b, a, c, k\}, \quad E = \{t, a, s, k\}, \quad F = \{b, a, t, h\}$ . Using these sets, find the following:

19. $D' \cap E$

20. $F' \cap D$

21. $(D \cap E) \cup F$

22. $D \cap (E \cup P)$

23. $(F \cap E)' \cap D$

24. $(D \cup E)' \cap F$

Create a Venn diagram to illustrate each of the following:

25. $(F \cap E) \cup D$

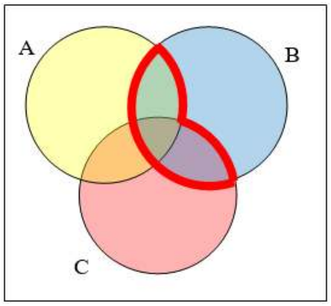
26. $(D \cup E)' \cap F$

27. $(F' \cap E') \cap D$

28. $(D \cup E) \cup F$

Write an expression for the shaded region.

29.



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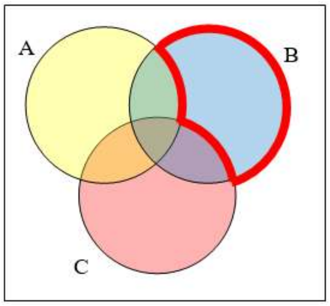
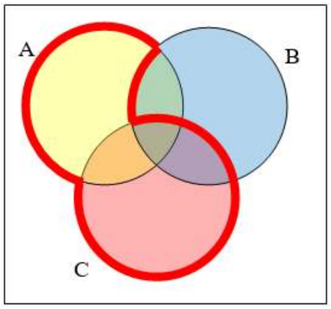
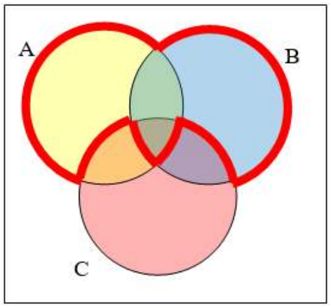


31.



32.



Let $A = \{1, 2, 3, 4, 5\}$ $\quad B = \{1, 3, 5\}$ $\quad C = \{4, 6\}$ Find the cardinality of the given set.

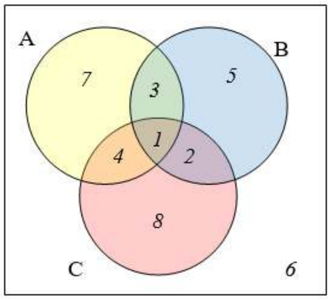
33. $n(A)$

34. $n(B)$

35. $n(A \cup C)$

36. $n(A \cap C)$

The Venn diagram here shows the cardinality of each set. Use this in 37-40 to find the cardinality of given set.



37. $n(A \cap C)$

38. $n(B \cup C)$

39. $n(A \cap B \cap C')$

40. $n(A \cap B' \cap C)$

41. If $n(G) = 20, n(H) = 30, n(G \cap H) = 5,\ $ find $n(G \cup H)$.

42. If $n(G) = 5, n(H) = 8, n(G \cap H) = 4,\ $ find $n(G \cup H)$.

43. A survey was given asking whether respondents if they watch movies at home from Netflix, Redbox, or Hulu. Use the results to determine how many people use Redbox.

52 only use Netflix
24 only use Hulu
48 use only Netflix and Redbox
10 use all three
62 only use Redbox
16 use only Hulu and Redbox
30 use only Hulu and Netflix
25 use none of these

44. A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?

5 only said color
16 only said brand
42 said only color and brand
102 said all three
8 only said size
20 said only color and size
53 said only size and brand
20 said none of these

45. Use the given information to complete a Venn diagram, then determine: a) how many students have seen exactly one of these movies, and b) how many had seen only *Star Wars*.

18 had seen The Matrix (M)
20 had seen Lord of the Rings (LotR)
14 had seen LotR and SW
6 had seen all three
24 had seen Star Wars (SW)
10 had seen M and SW
12 had seen M and LotR

46. A survey asked people what alternative transportation modes they use. Use the data to complete a Venn diagram, then determine: a) what percent of people only ride the bus, and b) how many people don't use any alternate transportation.

30% use the bus
25% walk
10% ride a bicycle and walk
2% use all three
20% ride a bicycle
5% use the bus and ride a bicycle
12% use the bus and walk

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# CHAPTER OVERVIEW

## 5: Logic

Logic is, basically, the study of valid reasoning. When searching the internet, we use Boolean logic – terms like “and” and “or” – to help us find specific web pages that fit in the sets we are interested in. After exploring this form of logic, we will look at logical arguments and how we can determine the validity of a claim.

The material in this chapter is from Math In Society by David Lippman.

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# 5.1: Logic Statements

**Learning Objectives**

- Identify a logical statement
- Construct the negation of a statement, including the use of quantifiers
- Construct a compound statement using conjunctions and disjunctions

Logic is the study of the methods and principles of reasoning. In logic, a **statement** is a declarative sentence that is either true or false, but not both. The key to constructing a good logical statement is that there must be no ambiguity. To be a statement, a sentence must be true or false. It cannot be both. In logic, the truth of a statement is established beyond ANY doubt by a well-reasoned argument.

So, a sentence such as "The house is beautiful" is not a statement, since whether the sentence is true or not is a matter of opinion.

A question such as "Is it snowing?" is not a statement, because it is a question and is not declaring that something is true.

Some sentences that are mathematical in nature often are not logical statements because we may not know precisely what a variable represents. For example, the equation $3x+5=10$ is not a logical statement, since we do not know what $x$ represents. If we substitute a specific value for $x$ (such as $x=4$), then the resulting equation, $3x+5=10$ is a logical statement (which is a false statement).

**Statement**

A **statement** is a sentence that is either true or false.

In logic, lower case letters are often used to represent statements such as $p$, $q$ or $r$.

**Example 5.1.1**

The following are statements:

a. Zero times any real number is zero.
b. $1+1=2$.
c. All birds can fly. (This is a false statement. How can you establish that?)

The following are not statements:

a. Come here.
b. Who are you?
c. I am lying right now. (This is a paradox. If I'm lying I'm telling the truth and if I'm telling the truth I'm lying.)

**Try It 5.1.1**

Which of the following are statements?

a. I like sports cars.
b. $2+3=6$ .
c. Where are you?

**Answer**

Only b is correct since you cannot determine if a is true, and c is a question that is neither true nor false.

**Negation**

The **negation** of a statement is a statement that has the opposite truth value of the original statement.

Notation: $\sim p$ (read: "the negation of $p$" or "not $p$")

The negation of a true statement must be a false statement and vice-versa. A simple statement can often be negated by adding or removing the word "not." A statement can also be negated by adding "It is not true that ... (statement)" or "It is not the case that ... (statement)."

✓ **Example 5.1.2**

Find the negation of the following statements:

a. $p$: The sky is blue.
b. $q$: Homework is not due today.

**Solution**

a. $\sim p$: The sky is not blue. Note that $p$ is true and $\sim p$ is false.
b. $\sim q$: Homework is due today. Note that $q$ and $\sim q$ have opposite truth values. If $q$ is true, then $\sim q$ is false, or if $q$ is false, then $\sim q$ is true.

Logical statements are related to sets and set operations. Words that describe an entire set, such as “all”, “every”, or “none”, are called **universal quantifiers** because that set could be considered a universal set. In contrast, words or phrases such as “some”, “one”, or “at least one” are called **existential quantifiers** because they describe the existence of at least one element in a set.

**Quantifiers**

A **universal quantifier** states that an entire set of things share a characteristic.

An **existential quantifier** states that a set contains at least one element.

Something interesting happens when we negate a quantified statement. When we negate a statement with a universal quantifier, we get a statement with an existential quantifier, and vice-versa.

**Negating a quantified statement**

| **Statement** | **Negation** |
|---|---|
| All $A$ are $B$. | At least one $A$ is not $B$.<br>Some $A$ are not $B$. |
| No $A$ are $B$. | At least one $A$ is $B$.<br>Some $A$ are $B$. |
| At least one $A$ is $B$.<br>Some $A$ are $B$. | No $A$ are $B$. |
| At least one $A$ is not $B$.<br>Some $A$ are not $B$. | All $A$ are $B$. |

In logic, when you have a statement and a negation, one must be negative, meaning it contains "no" or "not", and the other must be positive. For example, for the statement "All students love math," the negation cannot be "Some students love math" since neither statement is negative, even though they have opposite truth values. The correct negation is "Some students do not love math."

✓ **Example 5.1.3**

Write the negation of “Somebody brought a flashlight.”

**Solution**

Since the statement is of the form "Some $A$ are $B$," the negation will be of the form "No $A$ are $B$." The negation is “Nobody brought a flashlight.”

## ✓ Example 5.1.4

Write the negation of “There are no prime numbers that are even.”

**Solution**

Since the statement is of the form "No *A* are *B*," the negation will be of the form "At least one *A* is *B*." The negation is “At least one prime number is even.”

## Try It 5.1.2

Write the negation of “All Icelandic children learn English in school.”

**Answer**

Some Icelandic children do not learn English in school.

We can make a new statement from other statements; we call these **compound statements**. Compound statements are formed by connecting 2 or more simple statements with operators such as *and* and *or*.

## Symbols

The symbol $\wedge$ is a conjunction and is used for "and": $p$ and $q$ is notated $p \wedge q$

The symbol $\vee$ is a disjunction and is used for "or" (where "or" is not exclusive): $p$ or $q$ is notated $p \vee q$

You can remember the first two symbols by relating them to the shapes for the union and intersection. $p \wedge q$ would be the elements that exist in both sets, in $p \cap q$. Likewise, $p \vee q$ would be the elements that exist in either set, in $p \cup q$. When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

## ✓ Example 5.1.5

Translate each statement into symbolic notation. Let $p$ represent "I like Pepsi" and let $c$ represent "I like Coke".

a. I like Pepsi or I like Coke.
b. I like Pepsi and I like Coke.
c. I do not like Pepsi.
d. It is not the case that I like Pepsi or Coke.
e. I like Pepsi but I do not like Coke.

**Solution**

a. $p \vee c$
b. $p \wedge c$
c. $\sim p$
d. $\sim (p \vee c)$
e. $p \wedge \sim c$

As you can see, we can use parentheses to organize more complicated statements.

## Try It 5.1.3

Translate “We have carrots or we will not make soup” into symbols. Let $c$ represent “we have carrots” and let $s$ represent “we will make soup”.

**Answer**

$c \vee \sim s$

# 5.2: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

**Learning Objectives**

- Construct a truth table for the negation of a statement
- Construct a truth table for the conjunction and disjunction of statements
- Determine if a statement is a tautology or contradiction

Because compound statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the compound statement true and false.

**Truth Table**

A **truth table** is a table showing what the resulting truth value of a compound statement is for all the possible truth values for the simple statements.

**Example 5.2.1**

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise".

This is a compound statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use $p$ to designate "is a sectional", and $q$ to designate "has a chaise".

A truth table for this situation would look like this:

| $p$ | $q$ | $p$ or $q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

In the table, T is used for true, and F for false. In the first row, if $p$ is true and $q$ is also true, then the compound statement "$p$ or $q$" is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that *or* in logic is not exclusive; if the couch has both features, it meets the condition.) If you get a couch that is a sectional but not a chaise (row 2), you have still satisfied your partner's wish. Similarly, if you get a couch that is not a sectional but has a chaise (row 3), you have still satisfied your partner's wish. The only time you have not met your partner's wish is when you pick a couch that is neither a sectional nor a chaise (row 4).

In the previous example about the couch, the truth table was really just summarizing what we already know about how the *or* statement works. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

**Basic Truth Tables**

**Negation** - Expresses "not" which means the opposite truth value.

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

**Conjunction** - Expresses "and" which means both $p$ and $q$ must be true. It is only true when both $p$ and $q$ are true.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Disjunction** - Expresses "or" which means either $p$ or $q$ can be true, or both are true. It is only false when both $p$ and $q$ are false.

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Truth tables have different numbers of rows, depending on how many variables (or simple statements) that we have. When there is only one simple statement (such as our truth table for negation), we have two rows, one for true and another for false. When we have two simple statements, $p$ and $q$, there are four rows in the truth table. For three simple statements, $p$, $q$ and $r$, there are eight rows in the truth table. So for each new simple statement, the number of rows doubles. This pattern continues, so we can generalize it as follows.

### Definition: The Number of Lines in a Truth Table

A statement with $k$ variables - or simple statements - will have a truth table with $2^k$ rows.

Truth tables really become useful when we analyze more complex compound statements.

### Example $5.2.2$

Create a truth table for the statement $p \vee \sim q$.

**Solution**

When we create the truth table, we need to list all the possible truth value combinations for $p$ and $q$. Notice how the first column contains 2 Ts followed by 2 Fs, and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.

| $p$ | $q$ |
|---|---|
| T | T |
| T | F |
| F | T |
| F | F |

After creating columns with those initial values, we create a third column for the expression $\sim q$. Now we will temporarily ignore the column for $p$ and write the truth values for $\sim q$.

| $p$ | $q$ | $\sim q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | F |
| F | F | T |

Next we can find the truth values of $p \vee \sim q$, using the first and third columns.

| $p$ | $q$ | $\sim q$ | $p \vee \sim q$ |
|---|---|---|---|
| T | T | F | T |
| T | F | T | T |
| F | T | F | F |
| F | F | T | T |

The truth table shows that $p \vee \sim q$ is true in three cases and false in one case. If you're wondering what the point of this is, suppose it is the last day of the baseball season and two teams, who are not playing each other, are competing for the final playoff spot. Anaheim will make the playoffs if it wins its game <u>or</u> if Boston does not win its game. (Anaheim owns the tiebreaker; if both teams win, or if both teams lose, then Anaheim gets the playoff spot.) If $p =$ Anaheim wins its game and $q =$ Boston wins its game, then $p \vee \sim q$ represents the situation "Anaheim wins its game or Boston does not win its game". The truth table shows us the different scenarios related to Anaheim making the playoffs. In the first row, Anaheim wins its game and Boston wins its game, so it is true that Anaheim makes the playoffs. In the second row, Anaheim wins and Boston does not win, so it is true that Anaheim makes the playoffs. In the third row, Anaheim does not win its game and Boston wins its game, so it is false that Anaheim makes the playoffs. In the fourth row, Anaheim does not win and Boston does not win, so it is true that Anaheim makes the playoffs.

### Try It 5.2.1

Create a truth table for this statement: $\sim p \wedge q$

**Answer**

| $p$ | $q$ | $\sim p$ | $\sim p \wedge q$ |
|---|---|---|---|
| T | T | F | F |
| T | F | F | F |
| F | T | T | T |
| F | F | T | F |

### Example 5.2.3

Create a truth table for the statement $p \wedge \sim (q \vee r)$ .

**Solution**

It helps to work from the inside out when creating a truth table, and to create columns in the table for intermediate operations. We start by listing all the possible truth value combinations for $p, q,$ and $r$. Notice how the first column contains 4 Ts followed by 4 Fs, the second column contains 2 Ts, 2 Fs, then repeats, and the last column alternates T, F, T, F, ... This pattern ensures that all 8 combinations are considered. After creating columns with those initial values, we create a fourth column for the innermost expression, $q \vee r$. Now we will temporarily ignore the column for $p$ and focus on $q$ and $r$, writing the truth values for $q \vee r$.

| $p$ | $q$ | $r$ |
|---|---|---|
| T | T | T |
| T | T | F |
| T | F | T |
| T | F | F |
| F | T | T |
| F | T | F |
| F | F | T |
| F | F | F |

| $p$ | $q$ | $r$ | $q \vee r$ |
|---|---|---|---|
| T | T | T | T |
| T | T | F | T |
| T | F | T | T |
| T | F | F | F |
| F | T | T | T |
| F | T | F | T |
| F | F | T | T |
| F | F | F | F |

Next we can find the negation of $q \vee r$, working off the $q \vee r$ column we just created. (Ignore the first three columns and simply negate the values in the $q \vee r$ column.)

| $p$ | $q$ | $r$ | $q \vee r$ | $\sim(q \vee r)$ |
|---|---|---|---|---|
| T | T | T | T | F |
| T | T | F | T | F |
| T | F | T | T | F |
| T | F | F | F | T |
| F | T | T | T | F |
| F | T | F | T | F |
| F | F | T | T | F |
| F | F | F | F | T |

Finally, we find the values of $p$ and $\sim(q \vee r)$. (Ignore the second, third, and fourth columns.)

| $p$ | $q$ | $r$ | $q \vee r$ | $\sim(q \vee r)$ | $p \wedge \sim(q \vee r)$ |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| T | T | F | T | F | F |
| T | F | T | T | F | F |
| T | F | F | F | T | T |
| F | T | T | T | F | F |
| F | T | F | T | F | F |
| F | F | T | T | F | F |
| F | F | F | F | T | F |

It turns out that this complex expression is true in only one case: when $p$ is true, $q$ is false, and $r$ is false. To illustrate this situation, suppose that Anaheim will make the playoffs if: (1) Anaheim wins, and (2) neither Boston nor Cleveland wins. TFF is the only scenario in which Anaheim will make the playoffs.

### Try It 5.2.2

Create a truth table for this statement: $(\sim p \wedge q) \vee \sim q$

**Answer**

| $p$ | $q$ | $\sim p$ | $\sim p \wedge q$ | $\sim q$ | $(\sim p \wedge q) \vee \sim q$ |
|---|---|---|---|---|---|
| T | T | F | F | F | F |
| T | F | F | F | T | T |
| F | T | T | T | F | T |
| F | F | T | F | T | T |

### Tautology and Contradiction

A **tautology** is a compound statement that is true for all possible truth values of its variables.

A **contradiction** is a compound statement that is false for all possible truth values of its variables.

### Example 5.2.4

The compound statement "Either it is raining or it is not raining" is a tautology. This can be demonstrated with a truth table. First, let $p$ be the statement "it is raining." The symbolic form of the compound statement is $p \vee \sim p$. From the truth table below, we can see that the compound statement is always true.

| $p$ | $\sim p$ | $p \vee \sim p$ |
|---|---|---|
| T | F | T |
| F | T | T |

**Try It 5.2.3**

Is the statement "It is both raining and not raining" a tautology, contradiction or neither?

**Answer**

First, let $p$ be the statement "it is raining." The symbolic form of the compound statement is $p \wedge \sim p$. From the truth table below, we can see that the compound statement is always false, making the statement a contradiction.

| $p$ | $\sim p$ | $p \wedge \sim p$ |
|---|---|---|
| T | F | F |
| F | T | F |

---

# 5.3: Truth Tables- Conditional, Biconditional

**Learning Objectives**

- Construct a truth table for a conditional statement
- Construct a truth table for a biconditional statement

Conditional statements are those in which we take an action based on the value of the condition. We are now going to look at a version of a conditional statement, sometimes called an implication, which states that the second part must logically follow from the first.

**Conditional Statement**

A **conditional statement** is a logical compound statement in which a statement $p$, called the antecedent, implies a statement $q$, called the consequent.

A conditional statement is written as $p \rightarrow q$ and is translated as "if $p$, then $q$".

**Example $5.3.1$**

The English statement “If it is raining, then there are clouds in the sky” is a conditional statement. It makes sense because if the antecedent “it is raining” is true, then the consequent “there are clouds in the sky” must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consequent becomes irrelevant.

**Example $5.3.2$**

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday’s game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

There are four possible outcomes:

1. You pay for expedited shipping and receive the jersey by Friday.
2. You pay for expedited shipping and don’t receive the jersey by Friday.
3. You don’t pay for expedited shipping and receive the jersey by Friday.
4. You don’t pay for expedited shipping and don’t receive the jersey by Friday.

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don’t receive the jersey by Friday. The first outcome is exactly what was promised, so there’s no problem with that. The third outcome is not a lie because the website never said what would happen if you didn’t pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn’t make any promises about when the jersey would arrive if you didn’t pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the antecedent is false, we cannot make any judgment about the consequent. The website never said that paying for expedited shipping was the *only* way to receive the jersey by Friday.

**Truth Table for the Conditional**

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

> Again, if the antecedent $p$ is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true. The conditional statement is only false when the antecedent $p$ is true and the consequent $q$ is false.

✓ Example $5.3.3$

Construct a truth table for the statement $(m \wedge \sim p) \rightarrow r$

**Solution**

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the antecedent, $m \wedge \sim p$.

| $m$ | $p$ | $r$ |
|---|---|---|
| T | T | T |
| T | T | F |
| T | F | T |
| T | F | F |
| F | T | T |
| F | T | F |
| F | F | T |
| F | F | F |

| $m$ | $p$ | $r$ | $\sim p$ |
|---|---|---|---|
| T | T | T | F |
| T | T | F | F |
| T | F | T | T |
| T | F | F | T |
| F | T | T | F |
| F | T | F | F |
| F | F | T | T |
| F | F | F | T |

| $m$ | $p$ | $r$ | $\sim p$ | $m \wedge \sim p$ |
|---|---|---|---|---|
| T | T | T | F | F |
| T | T | F | F | F |
| T | F | T | T | T |
| T | F | F | T | T |
| F | T | T | F | F |
| F | T | F | F | F |
| F | F | T | T | F |
| F | F | F | T | F |

Now we can create a column for the conditional. Because it can be confusing to keep track of all the Ts and $\mathrm{F}$s, why don't we copy the column for $r$ to the right of the column for $m \wedge \sim p$ ? This makes it a lot easier to read the conditional from left to right.

| $m$ | $p$ | $r$ | $\sim p$ | $m\wedge\sim p$ | $r$ | $(m\wedge\sim p)\rightarrow r$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T |
| T | T | F | F | F | F | T |
| T | F | T | T | T | T | T |
| T | F | F | T | T | F | F |
| F | T | T | F | F | T | T |
| F | T | F | F | F | F | T |
| F | F | T | T | F | T | T |
| F | F | F | T | F | F | T |

When $m$ is true, $p$ is false, and $r$ is false- -the fourth row of the table-then the antecedent $m\wedge\sim p$ will be true but the consequent false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by $(m\wedge\sim p)\rightarrow r$, consider this: let $m =$ we order meatballs, $p =$ we order pasta, and $r =$ Rob is happy. The statement $(m\wedge\sim p)\rightarrow r$ is "if we order meatballs and don't order pasta, then Rob is happy". If $m$ is true (we order meatballs), $p$ is false (we don't order pasta), and $r$ is false (Rob is not happy), then the statement is false, because we satisfied the antecedent but Rob did not satisfy the consequent.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider "If you submit your hours today, then you will be paid next Friday." What the payroll rep really means is "If you submit your hours today, then you will be paid next Friday, and if you don't submit your hours today, then you won't be paid next Friday." The conditional statement if *t*, then *p* also includes the inverse of the statement: if not *t*, then not *p*. A more compact way to express this statement is "You will be paid next Friday *if and only if* you submit your timesheet today." A statement of this form is called a **biconditional**.

### Biconditional

A **biconditional statement** is a logical conditional statement in which the antecedent and consequent are interchangeable.

A biconditional is written as $p\leftrightarrow q$ and is translated as " $p$ if and only if $q$."

Because a biconditional statement $p\leftrightarrow q$ actually means $(p\rightarrow q)\wedge(q\rightarrow p)$, we may think of it as a compound conditional statement: if *p*, then *q* and if *q*, then *p*. The double-headed arrow shows that the conditional statement goes from left to right and from right to left. A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

### Truth Table for the Biconditional

| $p$ | $q$ | $p\leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

A biconditional is true as long as the antecedent and the consequent have the same truth value. If they are either both true or both false, the biconditional is true.

Notice that the fourth row, where both components are false, is true; if you don't submit your timesheet and you don't get paid, the person from payroll told you the truth.

### ✓ Example 5.3.4

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?

a. It is noon on Thursday and the garbage truck did not come down my street this morning.

b. It is Monday and the garbage truck is coming down my street.
c. It is Wednesday at 11:59PM and the garbage truck did not come down my street today.

### Solution

Let $p$ be "it is Thursday morning," and let $q$ be "the garbage truck comes down my street."

a. This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.
b. This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.
c. This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

## Try It 5.3.1

Suppose this statement is true: "I wear my running shoes if and only if I am exercising." Determine whether each of the following statements must be true or false.

a. I am exercising and I am not wearing my running shoes.
b. I am wearing my running shoes and I am not exercising.
c. I am not exercising and I am not wearing my running shoes.

**Answer**

Choices a & b are false; c is true.

## Example 5.3.5

Create a truth table for the statement $(A \vee B) \leftrightarrow \sim C$

### Solution

Whenever we have three component statements, we start by listing all the possible truth value combinations for $A$, $B$, and $C$.

| $A$ | $B$ | $C$ |
|---|---|---|
| T | T | T |
| T | T | F |
| T | F | T |
| T | F | F |
| F | T | T |
| F | T | F |
| F | F | T |
| F | F | F |

After creating those three columns, we can create a fourth column for the antecedent, $A \vee B$. Now we will temporarily ignore the column for $C$ and focus on $A$ and $B$, writing the truth values for $A \vee B$.

| $A$ | $B$ | $C$ | $A \vee B$ |
|---|---|---|---|
| T | T | T | T |
| T | T | F | T |
| T | F | T | T |
| T | F | F | T |
| F | T | T | T |
| F | T | F | T |
| F | F | T | F |
| F | F | F | F |

Next we can create a column for the negation of $C$. (Ignore the $A \vee B$ column and simply negate the values in the $C$ column.)

| $A$ | $B$ | $C$ | $A \vee B$ | $\sim C$ |
|---|---|---|---|---|
| T | T | T | T | F |
| T | T | F | T | T |
| T | F | T | T | F |
| T | F | F | T | T |
| F | T | T | T | F |
| F | T | F | T | T |
| F | F | T | F | F |
| F | F | F | F | T |

Finally, we find the truth values of $(A \vee B) \leftrightarrow \sim C$ . Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.

| $A$ | $B$ | $C$ | $A \vee B$ | $\sim C$ | $(A \vee B) \leftrightarrow \sim C$ |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| T | T | F | T | T | T |
| T | F | T | T | F | F |
| T | F | F | T | T | T |
| F | T | T | T | F | F |
| F | T | F | T | T | T |
| F | F | T | F | F | T |
| F | F | F | F | T | F |

To illustrate this situation, suppose your boss needs you to do either project $A$ or project $B$ (or both, if you have the time). If you do one of the projects, you will not get a crummy review ( $C$ is for crummy). So $(A \vee B) \leftrightarrow \sim C$ means "You will not get a crummy review if and only if you do project $A$ or project $B$." Looking at a few of the rows of the truth table, we can see how this works out. In the first row, $A$, $B$, and $C$ are all true: you did both projects and got a crummy review, which is not what your boss told you would happen! That is why the final result of the first row is false. In the fourth row, $A$ is true, $B$ is false, and $C$ is false: you did project $A$ and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row, $A$, $B$, and $C$ are all false: you didn't do either project and did not get a crummy review. This is not what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)

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# 5.4: Equivalent Statements and Variations of the Conditional

**Learning Objectives**

- Determine if 2 statements are logically equivalent
- Apply De Morgan's Laws
- Determine the converse, inverse and contrapositive of a conditional statement
- Determine the negation of a conditional statement

It is often of interest to determine if logic statements have the same truth values. For example, "It is not raining or there are clouds in the sky" has the same truth values as "If it is raining, then there are clouds in the sky." See the example below to verify the values in their truth tables are the same.

**Logically Equivalent**

Two statements are **logically equivalent** if they have the same simple statements and when their truth tables are computed, the final columns in the tables are identical. The symbol for equivalent statements is $\equiv$.

**Example 5.4.1**

Show that "It is not raining or there are clouds in the sky" is equivalent to "If it is raining, then there are clouds in the sky."

### Solution

Let $p$ be the statement "it is raining" and $q$ be the statement "there are clouds in the sky." "It is not raining or there are clouds in the sky" is written symbolically as $\sim p \vee q$ and "If it is raining, then there are clouds in the sky" is written symbolically as $p \rightarrow q$. The truth tables are as follows:

| $p$ | $q$ | $\sim p$ | $\sim p \vee q$ |
|---|---|---|---|
| T | T | F | T |
| T | F | F | F |
| F | T | T | T |
| F | F | T | T |

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Since the final answer columns of the truth tables are the same, the statements are equivalent. $\sim p \vee q \equiv p \rightarrow q$

Augustus De Morgan formalized two rules of logic that had previously been known informally. They allow us to rewrite the negation of a conjunction as a disjunction, and vice-versa.

For example, suppose you want to schedule a meeting with two colleagues at 4:30 PM on Friday, and you need both of them to be available at that time. What situation would make it impossible to have the meeting? It is NOT the case that colleague $A$ is available AND colleague $B$ is available: $\sim(A \wedge B)$ . This situation is equivalent to either colleague $A$ NOT being available OR colleague $B$ NOT being available: $\sim A \vee \sim B$ .

**De Morgan's Laws**

The negation of a conjunction is logically equivalent to the disjunction of the negation of the statements making up the conjunction. To negate an “and” statement, negate each part and change the “and” to “or”.

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

The negation of a disjunction is logically equivalent to the conjunction of the negation of the statements making up the disjunction. To negate an “or” statement, negate each part and change the “or” to “and”.

$$\sim (p \vee q) \equiv \ \sim p \wedge \sim q$$

### ✓ Example 5.4.2

For Valentine’s Day, you did not get your sweetie flowers or candy. Which of the following statements is logically equivalent?

a. You did not get them flowers or did not get them candy.
b. You did not get them flowers and did not get them candy.
c. You got them flowers or got them candy.

**Solution**

Let $f$ be the statement "You got your sweetie flowers" and let $c$ be the statement "You got your sweetie candy." The compound statement "You did not get your sweetie flowers or candy" can be written symbolically as $\sim (f \vee c)$. By De Morgan's Laws,

$$\sim (f \vee c) \equiv \ \sim f \wedge \sim c$$

This is answer b, you did not get your sweetie flowers and you did not get your sweetie candy.

### Try It 5.4.1

To serve as the President of the US, a person must have been born in the US, must be at least 35 years old, and must have lived in the US for at least 14 years. What minimum set of conditions would disqualify someone from serving as President?

**Answer**

Failing to meet just one of the three conditions is all it takes to be disqualified since the compound statement is made up of conjunctions. A person is disqualified if they were not born in the US, or are not at least 35 years old, or have not lived in the US for at least 14 years. The key word here is “or” instead of “and”.

For any conditional, there are three related statements, the converse, the inverse, and the contrapositive.

### Derived Forms of a Conditional

The original **conditional** is "if $p$, then $q$" $p \rightarrow q$

The **converse** is "if $q$, then $p$" $q \rightarrow p$

The **inverse** is "if not $p$, then not $q$" $\sim p \rightarrow \sim q$

The **contrapositive** is "if not $q$, then not $p$" $\sim q \rightarrow \sim p$

### ✓ Example 5.4.3

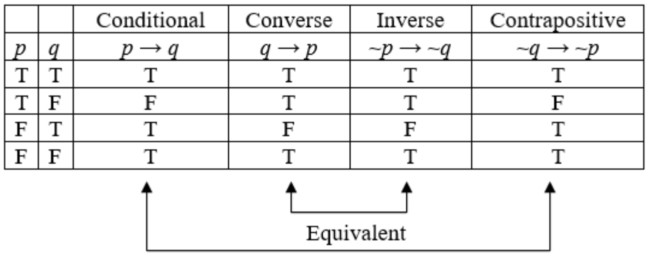
Consider again the conditional “If it is raining, then there are clouds in the sky.” It seems reasonable to assume that this is true.

The converse would be “If there are clouds in the sky, then it is raining.” This is not always true.

The inverse would be “If it is not raining, then there are not clouds in the sky.” Likewise, this is not always true.

The contrapositive would be “If there are not clouds in the sky, then it is not raining.” This statement is true, and is equivalent to the original conditional.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.



|  |  | Conditional | Converse | Inverse | Contrapositive |
|---|---|---|---|---|---|
| $p$ | $q$ | $p \to q$ | $q \to p$ | $\sim p \to \sim q$ | $\sim q \to \sim p$ |
| T | T | T | T | T | T |
| T | F | F | T | T | F |
| F | T | T | F | F | T |
| F | F | T | T | T | T |

### Equivalence

A conditional statement and its contrapositive are logically equivalent: $p \to q \equiv \ \sim q \to \sim p$ .

The converse and inverse of a conditional statement are logically equivalent: $q \to p \equiv \ \sim p \to \sim q$ .

A conditional statement is logically equivalent to the disjunction of the negation of its antecedent and its consequent: $p \to q \equiv \ \sim p \vee q$ .

### Negation of a conditional

The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

$$\sim(p \to q) \equiv p \wedge \sim q$$

### Example 5.4.4

Write the negation of the statement "If you study, then you will pass this class."

**Solution**

Let $p$ be "you study," and $q$ be "you will pass this class." The symbolic form of the statement is $p \to q$. The negation is $p \wedge \sim q$. Translating back to English, the negated statement is "You study and you will not pass this class." Note that this statement is only true when you do study but do not pass the class.

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## 5.5: Arguments with Truth Tables

### Learning Objectives

- Define a logical argument
- Determine if an argument is valid using a truth table

Logic is the study of the methods and principles of reasoning. An **argument** is a set of facts or assumptions, called **premises**, used to support a conclusion. For a logical argument to be **valid**, it is the case that, if the premises are true then the conclusion **must** be true.

### Argument

An **argument** is a set of statements called **premises** together with a **conclusion**. An argument consisting of two premises and a conclusion is called **a syllogism**.

There are two general types of arguments: inductive and deductive arguments.

### Argument types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

### Example $5.5.1$

The argument “When I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store” is an inductive argument.

The premises are:

I forgot my purse last week

I forgot my purse today

The conclusion is:

I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

### Example $5.5.2$

The argument “Every day for the past year, a plane flies over my house at 2:00 P.M. A plane will fly over my house every day at 2:00 P.M.” is a stronger inductive argument, since it is based on a larger set of evidence. While it is not necessarily true—the airline may have cancelled its afternoon flight—it is probably a safe bet.

### Evaluating inductive arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest that it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. Common scientific theories, like Newton’s theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence, such as when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes it easier to evaluate.

### Evaluating deductive arguments

A deductive argument is considered valid if, assuming that all the premises are true, the conclusion follows logically from those premises. In other words, when the premises are all true, the conclusion *must* be true.

One way to determine if an argument is valid is by using truth tables.

### Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically.
2. Create a conditional statement, joining all the premises with a conjunction to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for the conditional statement. If it is always true, then the argument is valid.

### Example 5.5.3

Consider the argument

| | |
|---|---|
| Premise: | If you bought bread, then you went to the store. |
| Premise: | You bought bread. |
| Conclusion: | You went to the store. |

**Solution**

While this example is fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then form a conditional statement showing that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll let $b$ represent "you bought bread" and $s$ represent "you went to the store". Then the argument becomes:

| | |
|---|---|
| Premise: | $b \rightarrow s$ |
| Premise: | $b$ |
| Conclusion: | $s$ |

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(b \rightarrow s) \wedge b] \rightarrow s$?

| $b$ | $s$ | $b \rightarrow s$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

| $b$ | $s$ | $b \rightarrow s$ | $(b \rightarrow s) \wedge b$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | F |
| F | T | T | F |
| F | F | T | F |

| $b$ | $s$ | $b \rightarrow s$ | $(b \rightarrow s) \wedge b$ | $[(b \rightarrow s) \wedge b] \rightarrow s$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | T |

Since the truth table for $[(b \rightarrow s) \wedge b] \rightarrow s$ is always true, this is a valid argument.

### Try It 5.5.1

Determine whether the argument is valid:

Premise: If I have a shovel, I can dig a hole.
Premise: I dug a hole.
Conclusion: Therefore, I had a shovel.

**Answer**

Let $S =$ have a shovel, $D =$ dig a hole. The first premise is equivalent to $S \rightarrow D$. The second premise is $D$. The conclusion is $S$. We are testing $[(S \rightarrow D) \wedge D] \rightarrow S$

| $S$ | $D$ | $S \rightarrow D$ | $(S \rightarrow D) \wedge D$ | $[(S \rightarrow D) \wedge D] \rightarrow S$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | T | F |
| F | F | T | F | T |

This is not a tautology, so this is an invalid argument.

### Example 5.5.4

Premise: If I go to the mall, then I'll buy new jeans.
Premise: If I buy new jeans, I'll buy a shirt to go with it.
Conclusion: If I go to the mall, I'll buy a shirt.

**Solution**

Let $m =$ I go to the mall, $j =$ I buy jeans, and $s =$ I buy a shirt.

The premises and conclusion can be stated as:

Premise: $m \rightarrow j$
Premise: $j \rightarrow s$
Conclusion: $m \rightarrow s$

We can construct a truth table for $[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$. Try to recreate each step and see how the truth table was constructed.

| $m$ | $j$ | $s$ | $m \rightarrow j$ | $j \rightarrow s$ | $(m \rightarrow j) \wedge (j \rightarrow s)$ | $m \rightarrow s$ | $[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | T | F | T | F | T | T |
| T | F | F | F | T | F | F | T |
| F | T | T | T | T | T | T | T |
| F | T | F | T | F | F | T | T |
| F | F | T | T | T | T | T | T |
| F | F | F | T | T | T | T | T |

From the final column of the truth table, we can see this is a valid argument.

---

# 5.6: Forms of Valid and Invalid Arguments

## Learning Objectives

- Recognize common valid and invalid arguments
- Draw a valid conclusion from given premises

Rather than making a truth table for every argument, we may be able to recognize certain common forms of arguments that are valid (or invalid). If we can determine that an argument fits one of the common forms, we can immediately state whether it is valid or invalid.

## The Law of Detachment (*Modus Ponens*)

The **law of detachment** applies when a conditional and its antecedent are given as premises, and the consequent is the conclusion. The general form is:

$$\begin{array}{ll}\text{Premise:} & p \rightarrow q \\ \text{Premise:} & p \\ \text{Conclusion:} & q\end{array}$$

The Latin name, *modus ponens*, translates to “mode that affirms”.

## Example 5.6.1

Recall this argument from an earlier example:

$$\begin{array}{ll}\text{Premise:} & \text{If you bought bread, then you went to the store.} \\ \text{Premise:} & \text{You bought bread.} \\ \text{Conclusion:} & \text{You went to the store.}\end{array}$$

In symbolic form:

$$\begin{array}{ll}\text{Premise:} & b \rightarrow s \\ \text{Premise:} & b \\ \text{Conclusion:} & s\end{array}$$

This argument has the structure described by the law of detachment. (The second premise and the conclusion are simply the two parts of the first premise detached from each other.) Instead of making a truth table, we can say that this argument is valid by stating that it satisfies the law of detachment.

## The Law of Contraposition (*Modus Tollens*)

The **law of contraposition** applies when a conditional and the negation of its consequent are given as premises, and the negation of its antecedent is the conclusion. The general form is:

$$\begin{array}{ll}\text{Premise:} & p \rightarrow q \\ \text{Premise:} & \sim q \\ \text{Conclusion:} & \sim p\end{array}$$

The Latin name, *modus tollens*, translates to “mode that denies”.

Notice that the second premise and the conclusion look like the contrapositive of the first premise, $\sim q \rightarrow \sim p$, but they have been detached. You can think of the law of contraposition as a combination of the law of detachment and the fact that the contrapositive is logically equivalent to the original statement.

### Example 5.6.2

Premise: If I drop my phone into the swimming pool, my phone will be ruined.
Premise: My phone isn't ruined.
Conclusion: I didn't drop my phone into the swimming pool.

If we let $d =$ I drop the phone in the pool and $r =$ the phone is ruined, then we can represent the argument this way:

Premise: $d \rightarrow r$
Premise: $\sim r$
Conclusion: $\sim d$

The form of this argument matches what we need to invoke the law of contraposition, so it is a valid argument.

### Try It 5.6.1

Is this argument valid?

Premise: If you brushed your teeth before bed, then your toothbrush will be wet.
Premise: Your toothbrush is dry.
Conclusion: You didn't brush your teeth before bed.

**Answer**

Let $b =$ brushed teeth and $w =$ toothbrush is wet.

Premise: $b \rightarrow w$
Premise: $\sim w$
Conclusion: $\sim b$

This argument is valid by the law of contraposition.

### The Transitive Property (Hypothetical Syllogism)

The **transitive property** has as its premises a series of conditionals, where the consequent of one is the antecedent of the next. The conclusion is a conditional with the same antecedent as the first premise and the same consequent as the final premise. The general form is:

Premise: $p \rightarrow q$
Premise: $q \rightarrow r$
Conclusion: $p \rightarrow r$

The earlier example about buying a shirt at the mall is an example illustrating the transitive property. It describes a chain reaction: if the first thing happens, then the second thing happens, and if the second thing happens, then the third thing happens. Therefore, if we want to ignore the second thing, we can say that if the first thing happens, then we know the third thing will happen. We don't have to mention the part about buying jeans; we can simply say that the first event leads to the final event. We could even have more than two premises; as long as they form a chain reaction, the transitive property will give us a valid argument.

### Example 5.6.3

Premise: If a soccer player commits a reckless foul, she will receive a yellow card.
Premise: If Hayley receives a yellow card, she will be suspended for the next match.
Conclusion: If Hayley commits a reckless foul, she will be suspended for the next match.

If we let $r =$ committing a reckless foul, $y =$ receiving a yellow card, and $s =$ being suspended, then our argument looks like this:

Premise: $r \rightarrow y$
Premise: $y \rightarrow s$
Conclusion: $r \rightarrow s$

This argument has the exact structure required to use the transitive property, so it is a valid argument.

### Try It 5.6.2

Is this argument valid?

Premise: If the old lady swallows a fly, she will swallow a spider.
Premise: If the old lady swallows a spider, she will swallow a bird.
Premise: If the old lady swallows a bird, she will swallow a cat.
Premise: If the old lady swallows a cat, she will swallow a dog.
Premise: If the old lady swallows a dog, she will swallow a goat.
Premise: If the old lady swallows a goat, she will swallow a cow.
Premise: If the old lady swallows a cow, she will swallow a horse.
Premise: If the old lady swallows a horse, she will die, of course.
Conclusion: If the old lady swallows a fly, she will die, of course.

**Answer**

This argument is valid by the transitive property, which can involve more than two premises, as long as they continue the chain reaction. The premises $f \rightarrow s, s \rightarrow b, b \rightarrow c, c \rightarrow d \quad d \rightarrow g, g \rightarrow w, w \rightarrow h, h \rightarrow x$ can be reduced to $f \rightarrow x$. (Because we had already used $c$ and $d$ we decided to use $w$ for cow and $x$ for death. If the old lady swallows the fly, she will eventually eat a horse and die.

### Disjunctive Syllogism

In a **disjunctive syllogism**, the premises consist of an *or* statement and the negation of one of the options. The conclusion is the other option. The general form is:

Premise: $p \vee q$
Premise: $\sim p$
Conclusion: $q$

The order of the two parts of the disjunction isn't important. In other words, we could have the premises $p \vee q$ and $\sim q$, and the conclusion $p$

### Example 5.6.4

Premise: I can either drive or take the train.
Premise: I refuse to drive.
Conclusion: I will take the train.

If we let $d =$ I drive and $t =$ I take the train, then the symbolic representation of the argument is:

Premise: $d \vee t$
Premise: $\sim d$
Conclusion: $t$

This argument is valid because it has the form of a disjunctive syllogism. I have two choices, and one of them is not going to happen, so the other one must happen.

### Try It 5.6.3

Is this argument valid?

| | |
|---|---|
| Premise: | Alison was required to write a 10-page paper or give a 5-minute speech. |
| Premise: | Alison did not give a 5-minute speech. |
| Conclusion: | Alison wrote a 10-page paper. |

**Answer**

Let $p =$ wrote a paper and $s =$ gave a speech.

| | |
|---|---|
| Premise: | $p \vee s$ |
| Premise: | $\sim s$ |
| Conclusion: | $p$ |

This argument is valid by disjunctive syllogism. Alison had to do one or the other; she didn't choose the speech, so she must have chosen the paper.

Keep in mind that, when you are determining the validity of an argument, you must assume that the premises are true. If you don't agree with one of the premises, you need to keep your personal opinion out of it. Your job is to pretend that the premises are true and then determine whether they force you to accept the conclusion. You may attack the premises in a court of law or a political discussion, of course, but here we are focusing on the structure of the arguments, not the truth of what they actually say.

We have just looked at four forms of valid arguments; there are two common forms that represent *invalid* arguments, which are also called *fallacies*.

### The Fallacy of the Converse

The **fallacy of the converse** arises when a conditional and its consequent are given as premises, and the antecedent is the conclusion. The general form is:

| | |
|---|---|
| Premise: | $p \rightarrow q$ |
| Premise: | $q$ |
| Conclusion: | $p$ |

Notice that the second premise and the conclusion look like the converse of the first premise, $q \rightarrow p$, but they have been detached. The fallacy of the converse incorrectly tries to assert that the converse of a statement is equivalent to that statement.

### Example 5.6.5

| | |
|---|---|
| Premise: | If I drink coffee after noon, then I have a hard time falling asleep that night. |
| Premise: | I had a hard time falling asleep last night. |
| Conclusion: | I drank coffee after noon yesterday. |

If we let $c =$ I drink coffee after noon and $h =$ I have a hard time falling asleep, then our argument looks like this:

| | |
|---|---|
| Premise: | $c \rightarrow h$ |
| Premise: | $h$ |
| Conclusion: | $c$ |

This argument uses converse reasoning, so it is an invalid argument. There could be plenty of other reasons why I couldn't fall asleep: I could be worried about money, my neighbors might have been setting off fireworks, …

### Try It 5.6.4

Is this argument valid?

Premise: If you pull that fire alarm, you will get in big trouble.
Premise: You got in big trouble.
Conclusion: You must have pulled the fire alarm.

**Answer**

Let $f =$ pulled fire alarm and $t =$ got in big trouble.

Premise: $f \rightarrow t$
Premise: $t$
Conclusion: $f$

This is the fallacy of the converse and is an invalid argument.

### The Fallacy of the Inverse

The **fallacy of the inverse** occurs when a conditional and the negation of its antecedent are given as premises, and the negation of the consequent is the conclusion. The general form is:

Premise: $p \rightarrow q$
Premise: $\sim p$
Conclusion: $\sim q$

Again, notice that the second premise and the conclusion look like the inverse of the first premise, $\sim p \rightarrow \sim q$, but they have been detached. The fallacy of the inverse incorrectly tries to assert that the inverse of a statement is equivalent to that statement.

### Example 5.6.6

Premise: If you listen to the Grateful Dead, then you are a hippie.
Premise: Sky doesn't listen to the Grateful Dead.
Conclusion: Sky is not a hippie.

If we let $g =$ listen to the Grateful Dead and $h =$ is a hippie, then this is the argument:

Premise: $g \rightarrow h$
Premise: $\sim g$
Conclusion: $\sim h$

This argument is invalid because it uses inverse reasoning. The first premise does not imply that all hippies listen to the Grateful Dead; there could be some hippies who listen to Phish instead.

### Try It 5.6.5

Is this argument valid?

Premise: If a hockey player trips an opponent, he will be assessed a 2-minute penalty.
Premise: Alexei did not trip an opponent.
Conclusion: Alexei will not be assessed a 2-minute penalty.

**Answer**

Let $t =$ tripped an opponent and $p =$ got a penalty.

Premise: $t \rightarrow p$
Premise: $\sim t$
Conclusion: $\sim p$

This argument is invalid because it has the form of the fallacy of the inverse. Alexei may have gotten a penalty for an infraction other than tripping.

Of course, arguments are not limited to these six basic forms; some arguments have more premises, or premises that need to be rearranged before you can see what is really happening. There are plenty of other forms of arguments that are invalid. If an argument doesn't seem to fit the pattern of any of these common forms, though, you may want to use a Venn diagram or a truth table instead.

Lewis Carroll, author of *Alice's Adventures in Wonderland*, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected using syllogisms. The following example is one such puzzle.

### ✓ Example 5.6.7

Solve the puzzle. In other words, find a logical conclusion from these premises.

All babies are illogical.

Nobody is despised who can manage a crocodile.

Illogical persons are despised.

Let $b =$ is a baby, $d =$ is despised, $i =$ is illogical, and $m =$ can manage a crocodile.

Then we can write the premises as:

$b \rightarrow i$

$m \rightarrow \sim d$

$i \rightarrow d$

Writing the second premise correctly can be a challenge; it can be rephrased as "If you can manage a crocodile, then you are not despised."

Using the transitive property with the first and third premises, we can conclude that $b \rightarrow d$, that all babies are despised. Using the contrapositive of the second premise, $d \rightarrow \sim m$, we can then use the transitive property with $b \rightarrow d$ to conclude that $b \rightarrow \sim m$, that babies cannot manage crocodiles. While it is silly, this is a logical conclusion from the given premises.

### ✓ Example 5.6.8

| | |
|---|---|
| Premise: | If I work hard, I'll get a raise. |
| Premise: | If I get a raise, I'll buy a boat. |
| Conclusion: | If I don't buy a boat, I must not have worked hard. |

If we let $h =$ working hard, $r =$ getting a raise, and $b =$ buying a boat, then we can represent our argument symbolically:

| | |
|---|---|
| Premise: | $h \rightarrow r$ |
| Premise: | $r \rightarrow b$ |
| Conclusion: | $\sim b \rightarrow \sim h$ |

Using the transitive property with the two premises, we can conclude that $h \rightarrow b$, if I work hard, then I will buy a boat. When we learned about the contrapositive, we saw that the conditional statement $h \rightarrow b$ is equivalent to $\sim b \rightarrow \sim h$. Therefore, the conclusion is indeed a logical syllogism derived from the premises.

### ✎ Try It 5.6.6

Is this argument valid?

| | |
|---|---|
| Premise: | If I go to the party, I'll be really tired tomorrow. |
| Premise: | If I go to the party, I'll get to see friends. |
| Conclusion: | If I don't see friends, I won't be tired tomorrow. |

**Answer**

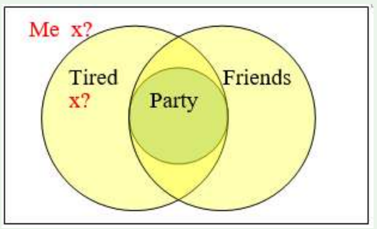
Let $p =$ go to party, $t =$ be tired, and $f =$ see friends.

Premise: $p \rightarrow t$

Premise: $p \rightarrow f$

Conclusion: $\sim f \rightarrow \sim t$

We could try to rewrite the second premise using the contrapositive to state $\sim f \rightarrow \sim p$, but that does not allow us to form a syllogism. If I don't see friends, then I didn't go the party, but that is not sufficient to claim I won't be tired tomorrow. Maybe I stayed up all night watching movies.



A Venn diagram can help, if we set it up correctly. The "party" circle must be completely contained within the intersection of the other circles. We know that I am somewhere outside the "friends" circle, but we cannot determine whether I am in the "tired" circle. All we really know for sure is that I didn't go to the party.

---

# 5.7: Arguments with Euler Diagrams

**Learning Objectives**

- Determine the validity of an argument using an Euler diagram

We can interpret a deductive argument visually with an Euler diagram, which is essentially the same thing as a Venn diagram, where we use circles to represent statements. This can make it easier to determine whether the argument is valid or invalid.

**Analyzing arguments with Euler diagrams**

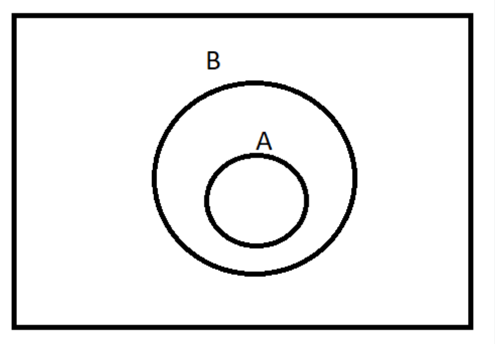
To analyze an argument with an Euler diagram:

1. Draw an Euler diagram based on the premises of the argument.
2. The argument is invalid if there is a way to draw the diagram that makes the conclusion false.
3. The argument is valid if the diagram cannot be drawn to make the conclusion false.
4. If the premises are insufficient to determine the location of an element or a set mentioned in the conclusion, then the argument is invalid.

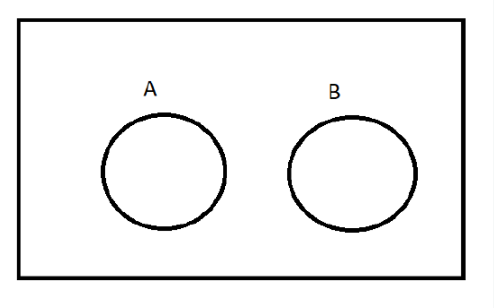
Remember, it only takes one counter-example to show the conclusion is false and make the argument invalid.

**Standard premise structures for Euler diagrams**

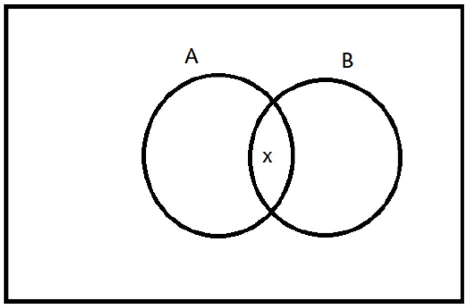
Universal Affirmative: All A are B.



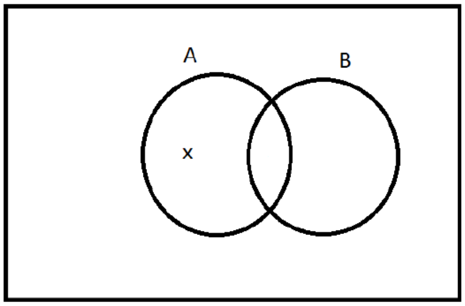
Universal Negative: No A are B.



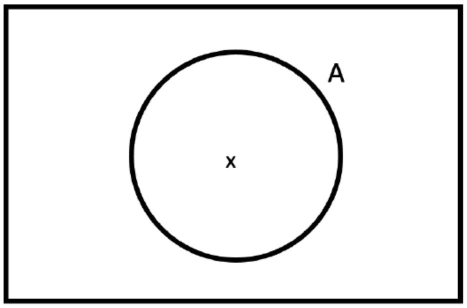
Particular Affirmative: Some A are B. Note that the "x" signifies the region where at least one A is B.



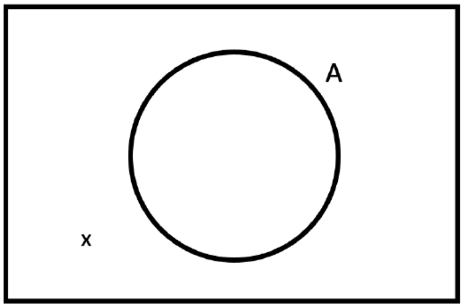
Particular Negative: Some A are not B. The "x" signifies the region where at least one A is not B.



Specific Affirmative: x is A.



Specific Negative: x is not A.



**✓ Example 5.7.1**

Consider the deductive argument “All cats are mammals and a tiger is a cat, so a tiger is a mammal.” Use an Euler diagram to determine if this argument is valid.

**Solution**

The premises are:

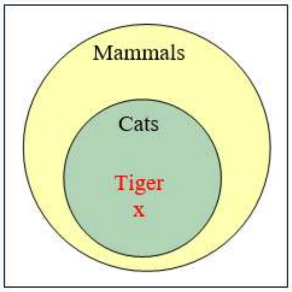
All cats are mammals.

A tiger is a cat.

The conclusion is:

A tiger is a mammal.

Both the premises are true. From the first premise, we draw the set of cats as a subset of the set of mammals. From the second premise, we are told that a tiger is contained within the set of cats. From that, we can see in the Euler diagram that the tiger must also be inside the set of mammals, so the conclusion is valid.



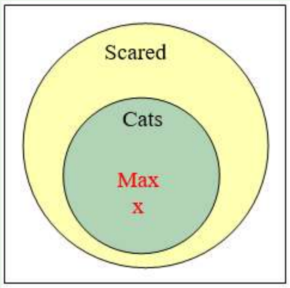
## Try It 5.7.1

Determine the validity of this argument:

| | |
|---|---|
| Premise: | All cats are scared of vacuum cleaners. |
| Premise: | Max is a cat. |
| Conclusion: | Max is scared of vacuum cleaners. |

**Answer**

Valid. Cats are a subset of creatures that are scared by vacuum cleaners. Max is in the set of cats, so he must also be in the set of creatures that are scared by vacuum cleaners.



## Example 5.7.2

Determine the validity of this argument:

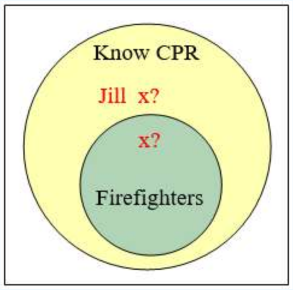
| | |
|---|---|
| Premise: | All firefighters know CPR. |
| Premise: | Jill knows CPR. |
| Conclusion: | Jill is a firefighter. |

### Solution

From the first premise, we know that firefighters all lie inside the set of those who know CPR. (Firefighters are a subset of people who know CPR.) From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know whether she also is a member of the smaller subset that is firefighters.

Since the conclusion does not necessarily follow from the premises, this is an invalid argument. It's possible that Jill is a firefighter, but the structure of the argument doesn't allow us to conclude that she definitely is.

It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are concerned with whether the premises are enough to prove the conclusion.



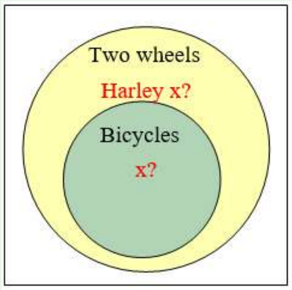
### Try It 5.7.2

Determine the validity of this argument:

| | |
|---|---|
| Premise: | All bicycles have two wheels. |
| Premise: | This Harley-Davidson has two wheels. |
| Conclusion: | This Harley-Davidson is a bicycle. |

**Answer**

Invalid. The set of bicycles is a subset of the set of vehicles with two wheels; the Harley-Davidson is in the set of two-wheeled vehicles but not necessarily in the smaller circle.



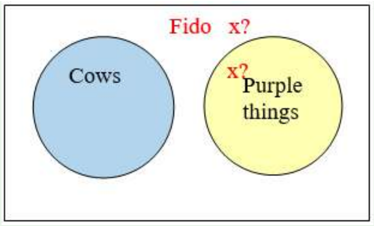
### Try It 5.7.3

Determine the validity of this argument:

| | |
|---|---|
| Premise: | No cows are purple. |
| Premise: | Fido is not a cow. |
| Conclusion: | Fido is purple. |

**Answer**

Invalid. Since no cows are purple, we know there is no overlap between the set of cows and the set of purple things. We know Fido is not in the cow set, but that is not enough to conclude that Fido is in the purple things set.



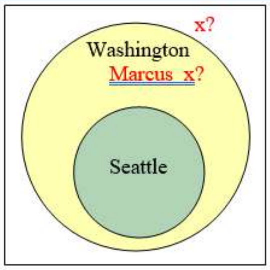
In addition to these categorical style premises of the form “all ___”, “some ____”, and “no ____”, it is also common to see premises that are conditionals.

## Example 5.7.3

| | |
|---|---|
| Premise: | If you live in Seattle, you live in Washington. |
| Premise: | Marcus does not live in Seattle. |
| Conclusion: | Marcus does not live in Washington. |

**Solution**

From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether Marcus lives in Washington or not. This is an invalid argument.



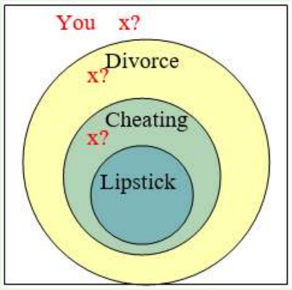
## Try It 5.7.4

Determine the validity of this argument:

| | |
|---|---|
| Premise: | If you have lipstick on your collar, then you are cheating on me. |
| Premise: | If you are cheating on me, then I will divorce you. |
| Premise: | You do not have lipstick on your collar. |
| Conclusion: | I will not divorce you. |

**Answer**

Invalid. Lipstick on your collar is a subset of scenarios in which you are cheating, and cheating is a subset of the scenarios in which I will divorce you. Although it is wonderful that you don’t have lipstick on your collar, you could still be cheating on me, and I will divorce you. In fact, even if you aren’t cheating on me, I might divorce you for another reason. You’d better shape up.



# 5.8: Logical Fallacies in Common Language

In a previous section, we saw that logical arguments can be invalid when the premises are not true, when the premises are not sufficient to guarantee the conclusion, or when there are invalid chains in logic. There are a number of other ways in which arguments can be invalid, a sampling of which are given here.

**Ad hominem**

An ad hominem argument attacks the person making the argument, ignoring the argument itself.

**✓ Example 5.8.1**

“Jane says that whales aren’t fish, but she’s only in the second grade, so she can’t be right.”

Here the argument is attacking Jane, not the validity of her claim, so this is an ad hominem argument.

**✓ Example 5.8.2**

“Jane says that whales aren’t fish, but everyone knows that they’re really mammals. She’s so stupid.”

This certainly isn’t very nice, but it is *not* ad hominem since a valid counterargument is made along with the personal insult.

**Appeal to ignorance**

This type of argument assumes something it true because it hasn’t been proven false.

**✓ Example 5.8.3**

“Nobody has proven that photo isn’t of Bigfoot, so it must be Bigfoot.”

**Appeal to authority**

These arguments attempt to use the authority of a person to prove a claim. While often authority can provide strength to an argument, problems can occur when the person’s opinion is not shared by other experts, or when the authority is irrelevant to the claim.

**✓ Example 5.8.4**

“A diet high in bacon can be healthy; Doctor Atkins said so.”

Here, an appeal to the authority of a doctor is used for the argument. This generally would provide strength to the argument, except that the opinion that eating a diet high in saturated fat runs counter to general medical opinion. More supporting evidence would be needed to justify this claim.

**✓ Example 5.8.5**

“Jennifer Hudson lost weight with Weight Watchers, so their program must work.”

Here, there is an appeal to the authority of a celebrity. While her experience does provide evidence, it provides no more than any other person’s experience would.

**Appeal to consequence**

An appeal to consequence concludes that a premise is true or false based on whether the consequences are desirable or not.

**✓ Example 5.8.6**

“Humans will travel faster than light: faster-than-light travel would be beneficial for space travel.”

**False dilemma**

A false dilemma argument falsely frames an argument as an “either or” choice, without allowing for additional options.

**✓ Example 5.8.7**

“Either those lights in the sky were an airplane or aliens. There are no airplanes scheduled for tonight, so it must be aliens.”

This argument ignores the possibility that the lights could be something other than an airplane or aliens.

**Circular reasoning**

Circular reasoning is an argument that relies on the conclusion being true for the premise to be true.

**✓ Example 5.8.8**

“I shouldn’t have gotten a C in that class; I’m an A student!”

In this argument, the student is claiming that because they’re an A student, though shouldn’t have gotten a C. But because they got a C, they’re not an A student.

**Post hoc (post hoc ergo propter hoc)**

A post hoc argument claims that because two things happened sequentially, then the first must have caused the second.

**✓ Example 5.8.9**

“Today I wore a red shirt, and my football team won! I need to wear a red shirt every time they play to make sure they keep winning.”

**Straw man**

A straw man argument involves misrepresenting the argument in a less favorable way to make it easier to attack.

**✓ Example 5.8.10**

“Senator Jones has proposed reducing military funding by 10%. Apparently he wants to leave us defenseless against attacks by terrorists.”

Here the arguer has represented a 10% funding cut as equivalent to leaving us defenseless, making it easier to attack Senator Jones’ position.

**Correlation implies causation**

Similar to post hoc, but without the requirement of sequence, this fallacy assumes that just because two things are related one must have caused the other. Often there is a third variable not considered.

**Example 5.8.11**

“Months with high ice cream sales also have a high rate of deaths by drowning. Therefore, ice cream must be causing people to drown.”

This argument is implying a causal relation, when really both are more likely dependent on the weather; that ice cream and drowning are both more likely during warm summer months.

**Try It 5.8.1**

Identify the logical fallacy in each of the arguments

a. Only an untrustworthy person would run for office. The fact that politicians are untrustworthy is proof of this.
b. Since the 1950s, both the atmospheric carbon dioxide level and obesity levels have increased sharply. Hence, atmospheric carbon dioxide causes obesity.
c. The oven was working fine until you started using it, so you must have broken it.
d. You can’t give me a D in the class because I can’t afford to retake it.
e. The senator wants to increase support for food stamps. He wants to take the taxpayers’ hard-earned money and give it away to lazy people. This isn’t fair, so we shouldn’t do it.

**Answer**

a. Circular
b. Correlation does not imply causation
c. Post hoc
d. Appeal to consequence
e. Straw man

It may be difficult to identify one particular fallacy for an argument. Consider this argument: “Emma Watson says she’s a feminist, but she posed for these racy pictures. I’m a feminist, and no self-respecting feminist would do that.” Could this be ad hominem, saying that Emma Watson has no self-respect? Could it be appealing to authority because the person making the argument claims to be a feminist? Could it be a false dilemma because the argument assumes that a woman is either a feminist or not, with no gray area in between?

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# 5.9: Chapter Review

## Summary:

| Operation | Notation | Summary of truth values |
|---|---|---|
| Negation | $\sim p$ | The opposite truth value of $p$ |
| Conjunction | $p \wedge q$ | True only when both $p$ and $q$ are true |
| Disjunction | $p \vee q$ | False only when both $p$ and $q$ are false |
| Conditional | $p \rightarrow q$ | False only when $p$ is true and $q$ is false |
| Biconditional | $p \leftrightarrow q$ | True only when both $p$ and $q$ are true or both are false |

## Notations & Definitions:

- Negation: $\sim$ or "**not**"
- Conjunction: $\wedge$ or "**and**"
- Disjunction: $\vee$ or "**or**"
- Conditional: $\rightarrow$ or "**implies**" or "**if/then**"
- Biconditional: $\leftrightarrow$ or "**if and only if**" or "**iff**"
- Counter-example: An example that disproves a mathematical proposition or statement.
- Logically Equivalent: $\equiv$ Two propositions that have the same truth table result.
- Tautology: A statement that is always true, and a truth table yields only true results.
- Contradiction: A statement which is always false, and a truth table yields only false results.

# 5.10: Exercises

**Boolean Logic**

For questions 1-2, list the set of integers that satisfy the given conditions.

1. A positive multiple of 5 and not a multiple of 2

2. Greater than 12 and less than or equal to 18

**Quantified Statements**

For questions 3-4, write the negation of each quantified statement.

3. Everyone failed the quiz today.

4. Someone in the car needs to use the restroom.

**Truth Tables**

5. Translate each statement from symbolic notation into English sentences. Let $A$ represent “Elvis is alive” and let $G$ represent “Elvis gained weight”.

a. $A \vee G$
b. $\sim(A \wedge G)$
c. $G \rightarrow \sim A$
d. $A \leftrightarrow \sim G$

For questions 6-9, create a truth table for each statement.

6. $A \wedge \sim B$

7. $\sim(\sim A \vee B)$

8. $(A \wedge B) \rightarrow C$

9. $(A \vee B) \rightarrow \sim C$

Questions 10-13: In this lesson, we have been studying the inclusive or, which allows both $A$ and $B$ to be true. The exclusive or does not allow both to be true; it translates to "either $A$ or $B$, but not both."

10. For each situation, decide whether the “or” is most likely exclusive or inclusive.

a. An entrée at a restaurant includes soup or a salad.
b. You should bring an umbrella or a raincoat with you.
c. We can keep driving on I-5 or get on I-405 at the next exit.
d. You should save this document on your computer or a flash drive.

11. Complete the truth table for the exclusive or.

| $A$ | $B$ | $A \vee B$ |
|---|---|---|
| T | T | |
| T | F | |
| F | T | |
| F | F | |

12. Complete the truth table for $(A \vee B) \wedge \sim(A \wedge B)$ .

| $A$ | $B$ | $A \vee B$ | $A \wedge B$ | $\sim(A \wedge B)$ | $(A \vee B) \wedge \sim(A \wedge B)$ |
|---|---|---|---|---|---|
| T | T | | | | |
| T | F | | | | |
| F | T | | | | |
| F | F | | | | |

13. Compare your answers for questions 11 and 12. Can you explain the similarities?

**Conditional Statements**

14. Consider the statement “If you are under age 17, then you cannot attend this movie.”

a. Write the converse.
b. Write the inverse.
c. Write the contrapositive.

15. Assume that the statement “If you swear, then you will get your mouth washed out with soap” is true. Which of the following statements must also be true?

a. If you don’t swear, then you won’t get your mouth washed out with soap.
b. If you don’t get your mouth washed out with soap, then you didn’t swear.
c. If you get your mouth washed out with soap, then you swore.

For questions 16-18, write the negation of each conditional statement.

16. If you don’t look both ways before crossing the street, then you will get hit by a car.

17. If Luke faces Vader, then Obi-Wan cannot interfere.

18. If you weren’t talking, then you wouldn’t have missed the instructions.

19. Assume that the biconditional statement “You will play in the game if and only if you attend all practices this week” is true. Which of the following situations could happen?

a. You attended all practices this week and didn’t play in the game.
b. You didn’t attend all practices this week and played in the game.
c. You didn’t attend all practices this week and didn’t play in the game.

**De Morgan’s Laws**

For questions 20-21, use De Morgan’s Laws to rewrite each conjunction as a disjunction, or each disjunction as a conjunction.

20. It is not true that Tina likes Sprite or 7-Up.

21. It is not the case that you need a dated receipt and your credit card to return this item.

22. Go back and look at the truth tables in Exercises 6 & 7. Explain why the results are identical.

**Deductive Arguments**

For questions 23-28, use a Venn diagram or truth table or common form of an argument to decide whether each argument is valid or invalid.

23. If a person is on this reality show, they must be self-absorbed. Laura is not self-absorbed. Therefore, Laura cannot be on this reality show.

24. If you are a triathlete, then you have outstanding endurance. LeBron James is not a triathlete. Therefore, LeBron does not have outstanding endurance.

25. Jamie must scrub the toilets or hose down the garbage cans. Jamie refuses to scrub the toilets. Therefore, Jamie will hose down the garbage cans.

26. Some of these kids are rude. Jimmy is one of these kids. Therefore, Jimmy is rude!

27. Every student brought a pencil or a pen. Marcie brought a pencil. Therefore, Marcie did not bring a pen.

28. If a creature is a chimpanzee, then it is a primate. If a creature is a primate, then it is a mammal. Bobo is a mammal. Therefore, Bobo is a chimpanzee.

**Logical Fallacies**

For questions 28-31, name the type of logical fallacy being used.

29. If you don’t want to drive from Boston to New York, then you will have to take the train.

30. New England Patriots quarterback Tom Brady likes his footballs slightly underinflated. The “Cheatriots” have a history of bending or breaking the rules, so Brady must have told the equipment manager to make sure that the footballs were underinflated.

31. Whenever our smoke detector beeps, my kids eat cereal for dinner. The loud beeping sound must make them want to eat cereal for some reason.

# CHAPTER OVERVIEW

## 6: Probability

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# 6.1: Basic Probability Concepts

## Learning Objectives

- Find the sample space and an event of a probability experiment
- Find the theoretical probability of an event

## Introduction

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says “there is a 10% chance of rain tomorrow,” she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose there is a situation with $n$ equally likely possible outcomes and that $m$ of those $n$ outcomes correspond to a particular event; then the **probability** of that event is defined as $\frac{m}{n}$.

### Types of Probability

An **empirical probability** is based on an experiment or observation and is the relative frequency of the event occurring.

A **subjective probability** is an estimate (a guess) based on experience or intuition.

A **theoretical probability** is based on a mathematical model where all outcomes are equally likely to occur.

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

### Events and Outcomes

A probability **experiment** is an activity or an observation whose result cannot be predicted ahead of time.

The result of an experiment is called an **outcome**.

The **sample space** is the set of all possible outcomes for a probability experiment. It is usually denoted by $S$.

An **event** is a subset of the sample space. It is a collection of outcomes that are grouped together. It is usually denoted with capital letters such as $A$.

A **simple event** is an event that consists of only one outcome.

### Example 6.1.1

If we roll a standard 6-sided die, describe the sample space and some simple events.

**Solution**

The sample space is the set of all possible outcomes: $S = \{1, 2, 3, 4, 5, 6\}$.

Some examples of simple events:

- We roll a 1: $A=\{1\}$
- We roll a 5: $A=\{5\}$

Some compound events (more than one outcome):

- We roll a number bigger than 4: $A=\{5,6\}$
- We roll an even number: $A=\{2,4,6\}$

## Basic Probability

### Probability

The **probability** of an event is the measure of how likely it is to happen. The notation for the probability of event $A$ is $P(A)$.

- $P(A)$ can be expressed as a number between 0 and 1, or as a percentage between 0% and 100%.
- The sum of the probabilities of all of the outcomes in the sample space is 1 (or 100%).
- The probability of an **impossible event** is $P(A)$ = 0 (or 0%).
- The probability of a **certain event** is $P(A)$ = 1 (or 100%).

In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

### Probability Formula

An experiment has **equally likely outcomes** if every outcome has the same probability of occurring.

Given that all outcomes are equally likely, we can compute the theoretical probability of event $A$ using this formula:

$$P(A)=\frac{\text{Number of ways for } A \text{ to occur}}{\text{Total number of outcomes}}$$

### Example 6.1.2

If we roll a 6-sided die, find:

a. $P$(rolling a 1)
b. $P$(rolling a number bigger than 4)

#### Solution

Recall that the sample space is $S=\{1,2,3,4,5,6\}$. The outcomes are equally likely, and there are 6 total outcomes on the die.

a. There is one outcome corresponding to “rolling a 1”, so the probability is
$P$(rolling a 1) = $\frac{\text{number of ways to roll a 1}}{\text{total number of ways to roll a die}}=\frac{1}{6}$.
b. There are two outcomes bigger than a 4, so the probability is
$P$(rolling a number bigger than 4) = $\frac{\text{number of ways to roll a number bigger than 4}}{\text{total number of ways to roll a die}}=\frac{2}{6}=\frac{1}{3}$.

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

### Try It 6.1.1

If we roll a 6-sided die, find:

a. $P$(rolling an odd number)
b. $P$(rolling a number less than 5)

**Answer**

a. The event rolling an odd number is $A = \{1, 3, 5\}$.

$P(\text{rolling an odd number}) = \dfrac{\text{number of ways to roll an odd number}}{\text{total number of ways to roll a die}} = \dfrac{3}{6} = \dfrac{1}{2}$.

The probability of rolling an odd number is $\dfrac{1}{2}$.

b. The event rolling a number less than five is $A = \{1, 2, 3, 4\}$.

$P(\text{rolling a number less than five}) = \dfrac{\text{number of ways to roll number less than five}}{\text{total number of ways to roll a die}} = \dfrac{4}{6} = \dfrac{2}{3}$.

The probability of rolling a number less than five is $\dfrac{2}{3}$.

### ✓ Example 6.1.3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

**Solution**

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\dfrac{14}{20} = \dfrac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

### Try It 6.1.2

If you flip a coin twice, find the probability of getting exactly two heads.

**Answer**

There are 4 outcomes in the sample space, $S = \{HH, HT, TH, TT\}$. The event of getting exactly two heads is $A = \{HH\}$. The number of ways $A$ can occur is 1. Thus $P(A) = \dfrac{1}{4}$.

### Cards

A standard deck of 52 playing cards consists of four **suits** (clubs, spades, hearts, and diamonds). Clubs and spades are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King. The Jack, Queen and King cards are also called **face** cards.

The image below gives an example of a complete deck of 52 cards.

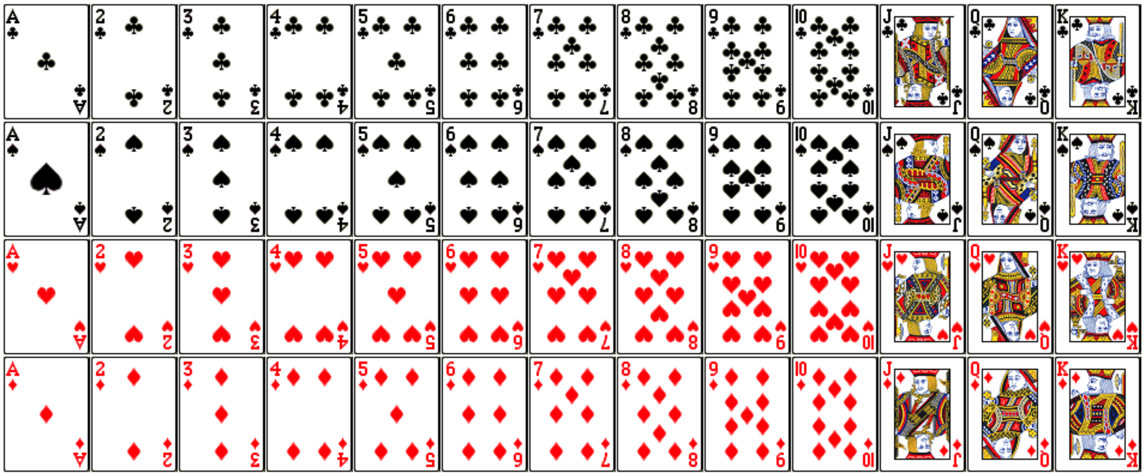

image credit: http://www.milefoot.com/math/discrete/counting/images/cards.png

**Example 6.1.4**

Compute the probability of randomly drawing one card from a deck and getting an Ace.

**Solution**

There are 52 cards in the deck and 4 Aces.

Thus, $P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$.

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

**Try It 6.1.3**

Draw a single card from a well shuffled deck of 52 cards. Find the following probabilities:

a. $P(\text{red})$
b. $P(\text{heart})$
c. $P(\text{red } 5)$

**Answer**

a. $P(\text{red}) = \frac{\text{number of red cards}}{\text{total number of cards}} = \frac{26}{52} = \frac{1}{2}$ . The probability that the card is red is $\frac{1}{2}$.
b. $P(\text{heart}) = \frac{\text{number of hearts}}{\text{total number of cards}} = \frac{13}{52} = \frac{1}{4}$ . The probability that the card is a heart is $\frac{1}{4}$.
c. $P(\text{red } 5) = \frac{\text{number of red fives}}{\text{total number of cards}} = \frac{2}{52} = \frac{1}{26}$ . The probability that the card is a red five is $\frac{1}{26}$.

## Law of Large Numbers

When studying probabilities, many times the law of large numbers will apply. If you want to observe what the probability is of getting tails when flipping a coin, you could do an experiment. Suppose you flip a coin 20 times and the coin comes up tails 9 times. Then, using an empirical probability, the probability of getting tails is $\frac{9}{20}$ = 45%. However, we know that the theoretical probability for getting tails should be $\frac{1}{2}$ = 50%. Why is this different? It is because there is error inherent to sampling methods. However, if you flip the coin 100 times or 1000 times, and use the information to calculate an empirical probability for getting tails, then the probabilities you will observe will become closer to the theoretical probability of 50%. This is the law of large numbers.

**Law of Large Numbers**

The **law of large numbers** says that as the number of times an experiment is repeated increases, the observed empirical probability of an event will approach the calculated theoretical probability of the same event.

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# 6.2: Probability Rules with "Not," "Or" and "And"

**Learning Objectives**

- Find the probability of the complement of an event
- Determine if 2 events are mutually exclusive
- Find the probability of the intersection of events
- Use the Addition Rule to find the probability of "or" events

## Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a 6-sided die and first compute the probability of rolling a 6: the answer is $P(6)=\frac{1}{6}$. Now consider the probability that we do *not* roll a 6: there are 5 outcomes that are not a 6, so the answer is $P(\text{not a } 6)=\frac{5}{6}$. Notice that

$$P(6)+P(\text{not a } 6)=\frac{1}{6}+\frac{5}{6}=\frac{6}{6}=1$$

This is not a coincidence. Consider a generic situation with $n$ possible outcomes and an event $A$ that corresponds to $m$ of these outcomes. Then the remaining $n-m$ outcomes correspond to $A$ not happening, thus

$$P(\text{not } A)=\frac{n-m}{n}=\frac{n}{n}-\frac{m}{n}=1-\frac{m}{n}=1-P(A)$$

**Complement of an Event**

The **complement** of event $A$ is the event "$A$ does not happen." It is the set of outcomes in the sample space $S$ that are not in event $A$.

- The notation $A'$ is used for the complement of event $A$.
- The probability of the complement of event $A$ is $P(A')=1-P(A)$ .
- Notice also that $P(A)=1-P(A')$ since event $A$ and its complement make up the entire sample space, $P(A)+P(A')=1$ .

Note that the complement of an event in probability is essentially the same as the complement of a set, since an event is a subset of the sample space.

**Example 6.2.1**

If you pick a random card from a deck of playing cards, what is the probability it is not a heart?

### Solution

There are 13 hearts in the deck, so

$$P(\text{heart})=\frac{13}{52}=\frac{1}{4}.$$

The probability of *not* drawing a heart is the complement:

$$P(\text{not heart})=1-P(\text{heart})=1-\frac{1}{4}=\frac{3}{4}.$$

**Try It 6.2.1**

A 6-sided die is rolled. Find the probability that the number rolled is not a multiple of 3.

**Answer**

Let $A$ = rolling a multiple of 3 = $\{3,6\}$. The probability of $A$ is $P(A)=\frac{2}{6}=\frac{1}{3}$ . The probability of not rolling a multiple of 3 is $P(A')=1-P(A)=1-\frac{1}{3}=\frac{2}{3}$ .

Many probabilities in real life involve more than one event. If we draw a single card from a deck we might want to know the probability that it is either red or a jack. If we look at a group of students, we might want to know the probability that a single student has brown hair and blue eyes. When we combine two events we make a single event called a **compound event**. To create a compound event, we can use the word “and” or the word “or” to combine events. It is very important in probability to pay attention to the words “and” and “or” if they appear in a problem. The word “and” restricts the field of possible outcomes to only those outcomes that simultaneously describe all events. The word “or” broadens the field of possible outcomes to those that describe one or more events.

✓ Example 6.2.2

Suppose a teacher wants to know the probability that a student in her class of 30 students is taking either Art or English. She asks the class to raise their hands if they are taking Art and counts 13 hands. Then she asks the class to raise their hands if they are taking English and counts 21 hands. The teacher then calculates

$$P(\text{Art or English})=\frac{13+21}{30}=\frac{33}{30}$$

The teacher knows that this is wrong because probabilities must be between 0 and 1, inclusive. After thinking about it she remembers that nine students are taking both Art and English. These students raised their hands each time she counted, so the teacher counted them twice. When we calculate probabilities, we have to be careful to count each outcome only once.

## Mutually Exclusive Events

An experiment consists of drawing one card from a well shuffled deck of 52 cards. Consider the events $E$ = the card is red, $F$ = the card is a 5, and $G$ = the card is a spade. It is possible for a card to be both red and a 5 at the same time but it is not possible for a card to be both red and a spade at the same time. It would be easy to accidentally count a red 5 twice by mistake. It is not possible to count a red spade twice.

If two events do have events in common or they can happen at the same time, the overlap is called the **intersection** of the events. The intersection of events is denoted as $A$ "and" $B$, and is the same as the intersection of two sets $A$ and $B$.

Mutually Exclusive

Two events are **mutually exclusive** if they have no outcomes in common. They cannot occur at the same time.

The probability of 2 mutually exclusive events $A$ and $B$ is $P(A \text{ and } B)=0$ .

✓ Example 6.2.3

Two fair dice are tossed and different events are recorded. Let the events $A$, $B$ and $C$ be as follows:

- $A=\text{the sum is 5}=\{(1,4),(2,3),(3,2),(4,1)\}$
- $B=\text{both numbers are even}=\{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$
- $C=\text{both numbers are less than 5}=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

a. Are events $A$ and $B$ mutually exclusive?
b. Are events $A$ and $C$ mutually exclusive?
c. Are events $B$ and $C$ mutually exclusive?

**Solution**

a. Yes. $A$ and $B$ are mutually exclusive because they have no outcomes in common. It is not possible to add 2 even numbers to get a sum of 5.

b. No. $A$ and $C$ are not mutually exclusive because they have some outcomes in common. The pairs (1, 4), (2, 3), (3, 2) and (4, 1) all have sums of 5 and both numbers are less than 5.

c. No. $B$ and $C$ are not mutually exclusive because they have some outcomes in common. The pairs (2, 2), (2, 4), (4, 2) and (4, 4) all have 2 even numbers that are less than 5.

## Addition Rule for “Or” Probabilities

The addition rule for probabilities is used when the events are connected by the word “or”. Remember our teacher in Example $6.2.2$? She wanted to know the probability that her students were taking either Art or English. Her problem was that she counted some students twice. She needed to add the number of students taking Art to the number of students taking English and then subtract the number of students she counted twice. After dividing the result by the total number of students she will find the desired probability. The calculation is as follows:

$$\begin{aligned} P(\text{Art or English}) &= \frac{\#\text{ taking Art} + \#\text{ taking English - }\#\text{ taking both}}{\text{total number of students}} \\ &= \frac{13+21-9}{30} \\ &= \frac{25}{30} \approx 0.833 \end{aligned}$$

The probability that a student is taking Art or English is 0.833 or 83.3%.

When we calculate the probability for compound events connected by the word “or” we need to be careful not to count the same thing twice. If we want the probability of drawing a red card or a 5 we cannot count the red 5s twice. If we want the probability a person is blonde-haired or blue-eyed we cannot count the blue-eyed blondes twice. The addition rule for probabilities adds the number of blonde-haired people to the number of blue-eyed people then subtracts the number of people we counted twice.

### Addition Rule for “Or” Probabilities

If $A$ and $B$ are any events, then the probability of either $A$ or $B$ occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

If $A$ and $B$ are mutually exclusive events then $P(A \text{ and } B) = 0$, so then

$$P(A \text{ or } B) = P(A) + P(B).$$

### Example 6.2.4

A single card is drawn from a well shuffled deck of 52 cards. Find the probability that the card is a club or a face card.

**Solution**

There are 13 cards that are clubs, 12 face cards (J, Q, K in each suit) and 3 face cards that are clubs.

$$\begin{aligned} P(\text{club or face card}) &= P(\text{club}) + P(\text{face card}) - P(\text{club and face card}) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} = \frac{11}{26} \approx 0.423 \end{aligned}$$

The probability that the card is a club or a face card is approximately 0.423 or 42.3%.

A simple way to check this answer is to take the 52 card deck and count the number of physical cards that are either clubs or face cards. If you were to set aside all of the clubs and face cards in the deck, you would end up with the following:

{2 Clubs, 3 Clubs, 4 Clubs, 5 Clubs, 6 Clubs, 7 Clubs, 8 Clubs, 9 Clubs, 10 Clubs, J Clubs, Q Clubs, K Clubs, A Clubs, J Hearts, Q Hearts, K Hearts, J Spades, Q Spades, K Spades, J Diamonds, Q Diamonds, K Diamonds}

That is 22 cards out of the 52 card deck, which gives us a probably of:

$$\frac{22}{52}=\frac{11}{26}\approx 0.423$$

This confirms our earlier answer using the formal Addition Rule.

### Try It 6.2.2

An experiment consists of tossing a coin then rolling a die. Find the probability that the coin lands on heads or the number is 5 is rolled.

**Answer**

Let $H$ represent heads and $T$ represent tails. The sample space for this experiment is $S=\{H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6\}$

There are 6 ways the coin can land on heads, $\{H1,H2,H3,H4,H5,H6\}$There are 2 ways the die can land on 5, $\{H5,T5\}$. There is 1 way for the coin to land on heads and the die to land on 5, $\{H5\}$.

$$\begin{aligned} P(\text{heads or }5) &= P(\text{heads})+P(5)-P(\text{both heads and }5)\\ &=\frac{6}{12}+\frac{2}{12}-\frac{1}{12}\\ &=\frac{7}{12}=\approx 0.583 \end{aligned}$$

The probability that the coin lands on heads or the number is 5 is approximately 0.583 or 58.3%.

### Example 6.2.5

250 people who recently purchased a car were questioned and the results are summarized in the following table.

Satisfaction of Car Buyers

| | Satisfied | Not Satisfied | Total |
|---|---|---|---|
| New Car | 92 | 28 | 120 |
| Used Car | 83 | 47 | 130 |
| Total | 175 | 75 | 250 |

Find the probability that a person bought a new car or was not satisfied.

**Solution**

$$\begin{aligned} P(\text{new car or not satisfied}) &= P(\text{new car})+P(\text{not satisfied})-P(\text{new car and not satisfied})\\ &=\frac{120}{250}+\frac{75}{250}-\frac{28}{250}=\frac{167}{250}\approx 0.668 \end{aligned}$$

The probability that a person bought a new car or was not satisfied is approximately 0.668 or 66.8%.

### Example 6.2.6

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

**Solution**

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{ King or Queen})=\frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen}) = 0$ . Using the addition rule, we could have said:

$$P(\text{ King or Queen}) = P(\text{ King }) + P(\text{ Queen}) - P(\text{ King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52} = \frac{2}{13}$$

In the last example, the events were **mutually exclusive**, so $P(A \text{ or } B) = P(A) + P(B)$ .

### Try It 6.2.3

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

**Answer**

- Half the cards are red, so $P(\text{red}) = \frac{26}{52}$
- There are 4 kings, so $P(\text{King}) = \frac{4}{52}$
- There are 2 red kings, so $P(\text{red and King}) = \frac{2}{52}$

We can then calculate

$$P(\text{red or King}) = P(\text{red}) + P(\text{King}) - P(\text{red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$$

### Example 6.2.7

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a. Has a red car *and* got a speeding ticket
b. Has a red car *or* got a speeding ticket.

| | Speeding ticket | No speeding ticket | Total |
|---|---|---|---|
| Red car | 15 | 135 | 150 |
| Not red car | 45 | 470 | 515 |
| Total | 60 | 605 | 665 |

**Solution**

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $P(\text{red car and ticket}) = \frac{15}{665} \approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665} \approx 0.2932$.

We also could have found this probability by:

$$\text{P(had a red car)} + \text{P(got a speeding ticket)} - \text{P(had a red car and got a speeding ticket)} = \frac{150}{665} + \frac{60}{665} - \frac{15}{665}$$
$$= \frac{195}{665}$$

# 6.3: Independent Events and Conditional Probabilities

## Learning Objectives

- Understand the difference between independent and dependent events
- Use the Multiplication Rule to find the probability of independent and dependent events
- Find conditional probabilities

## Independent events

Sometimes we need to calculate probabilities for compound events that are connected by the word "and." Tossing a coin multiple times or rolling dice are independent events. Each time you toss a fair coin the probability of getting heads is $\frac{1}{2}$. It does not matter what happened the last time you tossed the coin. It's similar for dice. If you rolled double sixes last time that does not change the probability that you will roll double sixes this time. Drawing two cards without replacement is not an independent event. When you draw the first card and set it aside, the probability for the second card is now out of 51 cards not 52 cards.

### Independent Events

Events $A$ and $B$ are **independent events** if the occurrence of $A$ has no effect on the probability of the occurrence of $B$. In other words, the probability of event $B$ occurring is the same whether or not event $A$ occurs.

If 2 events are not independent, then they are **dependent** events.

### Example 6.3.1

Are these events independent?

a. A fair coin is tossed two times. The two events are (1) first toss is heads and (2) second toss is heads.
b. The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).
c. You draw a card from a deck, then draw a second card without replacing the first.

**Solution**

a. The probability that heads comes up on the second toss is $\frac{1}{2}$ regardless of whether or not heads came up on the first toss, so these events are independent.
b. These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.
c. The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

### Multiplication Rule for Independent Events

If events $A$ and $B$ are independent, then the probability of both $A$ and $B$ occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

where $P(A \text{ and } B)$ is the probability of events $A$ and $B$ both occurring, $P(A)$ is the probability of event $A$ occurring, and $P(B)$ is the probability of event $B$ occurring.

### Example 6.3.2

Suppose we flipped a coin and rolled a die. What is the probability of getting heads on the coin and a 6 on the die?

**Solution**

Let $A$ = getting heads and $B$ = rolling 6. We know that $P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{6}$. $A$ and $B$ are independent since the probability of rolling a 6 does not depend on the outcome of the coin toss. Using the multiplication rule, we get:

$$P(A \text{ and } B)=P(A)\cdot P(B)=\frac{1}{2}\cdot\frac{1}{6}=\frac{1}{12}.$$

To confirm this answer, we could list all possible outcomes: $S=\{H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6\}$ Notice there are $2\cdot 6=12$ total outcomes. Out of these, only 1 is the desired outcome $\{H6\}$, so the probability is $\frac{1}{12}$.

### ✓ Example 6.3.3

A bag contains 5 red and 4 white marbles. A marble is drawn from the bag, its color recorded and the marble is returned to the bag. A second marble is then drawn. What is the probability that the first marble is red and the second marble is white?

**Solution**

Since the first marble is put back in the bag before the second marble is drawn these are independent events.

$$\begin{aligned} P(\text{1st red and 2nd white}) &= P(\text{1st red})\cdot P(\text{2nd white}) \\ &= \frac{5}{9}\cdot\frac{4}{9}=\frac{20}{81} \end{aligned}$$

The probability that the first marble is red and the second marble is white is $\frac{20}{81}$.

### Try It 6.3.1

A card is drawn from a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is drawn and noted. What is the probability that both cards are Aces?

**Answer**

Since the second draw is made after replacing the first card, these events are independent. The probability of an Ace on each draw is $P(\text{Ace})=\frac{4}{52}=\frac{1}{13}$, so the probability of an Ace on both draws is $P(\text{Ace and Ace})=\frac{1}{13}\cdot\frac{1}{13}=\frac{1}{169}$ .

## Conditional Probability

What do you think the probability is that a man is over six feet tall? If you knew that both his parents were tall would you change your estimate of the probability? A conditional probability is a probability that is based on some prior knowledge.

### Conditional Probability

A **conditional probability** is the probability that event $B$ will occur if event $A$ has already occurred. This is denoted by $P(B|A)$, which is read “the probability of $B$ given $A$.”

### ✓ Example 6.3.4

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a. has a speeding ticket *given* they have a red car
b. has a red car *given* they have a speeding ticket

| | Speeding ticket | No speeding ticket | Total |
|---|---|---|---|
| Red car | 15 | 135 | 150 |
| Not red car | 45 | 470 | 515 |
| Total | 60 | 605 | 665 |

### Solution

a. Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so $P(\text{ticket} \mid \text{red car}) = \frac{15}{150} = \frac{1}{10} = 0.1$

b. Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so $P(\text{red car} \mid \text{ticket}) = \frac{15}{60} = \frac{1}{4} = 0.25$.

Notice from the last example that $P(B|A)$ is **not** equal to $P(A|B)$.

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

### Multiplication Rule for Dependent Events

For events dependent events $A$ and $B$, $P(A \text{ and } B) = P(A) \cdot P(B|A)$ .

### ✓ Example 6.3.5

What is the probability that two cards drawn at random from a deck of playing cards will both be Aces?

### Solution

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$ . This would be incorrect, however, because the two events are not independent. If the first card drawn is an Ace, then the probability that the second card is also an Ace would be lower because there would only be 3 Aces left in the deck out of 51 total cards left. This means that the conditional probability of drawing an Ace after one Ace has already been drawn is $P(\text{Ace on the second draw} \mid \text{Ace on the first draw}) = \frac{3}{51} = \frac{1}{17}$ .

Thus, the probability of both cards being aces is $P(\text{Ace and Ace}) = \frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$ .

### Try It 6.3.2

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

**Answer**

$$P(\text{both white}) = \frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$$

### Conditional Probability Rule

For events $A$ and $B$, $P(B|A) = \dfrac{P(A \text{ and } B)}{P(A)}$

This formula basically reduces the sample space to $A$ instead of all events in $S$.

### Example 6.3.6

One card is drawn from a deck of cards. What is the probability that it is a face card, if we know the card is black?

**Solution**

We need to find the conditional probability of drawing a face card, given it is black, $P(\text{face card} \mid \text{black})$. The formula is:

$$P(\text{face card} \mid \text{black}) = \frac{P(\text{face card and black})}{P(\text{black})}$$

There are 6 cards that are black face cards (the Jack, Queen and King of spades and clubs). So the probability of face card and black is $P(\text{face card and black}) = \frac{6}{52}$. There are 26 black cards, so the probability of drawing a black card is $P(\text{black}) = \frac{26}{52}$. Plugging these probabilities in the formula we get:

$$P(\text{face card} \mid \text{black}) = \frac{\frac{6}{52}}{\frac{26}{52}} = \frac{6}{\cancel{52}} \cdot \frac{\cancel{52}}{26} = \frac{6}{26} = \frac{3}{13}$$

This answer should make sense: 3 out of every 13 black cards are face cards.

### Try It 6.3.3

One card is drawn from a deck of cards. What is the probability that it is a black card, if we know the card is a face card?

**Answer**

We need to find the conditional probability of drawing a black card, given it is a face card, $P(\text{black} \mid \text{face})$. The formula is:

$$P(\text{black} \mid \text{face}) = \frac{P(\text{black and face})}{P(\text{face})}$$

There are 6 cards that are black face cards (the Jack, Queen and King of spades and clubs). So the probability of face card and black is $P(\text{black and face}) = \frac{6}{52}$. There are 12 face cards, so the probability of drawing a face card is $P(\text{face}) = \frac{12}{52}$. Plugging these probabilities in the formula we get:

$$P(\text{black} \mid \text{face}) = \frac{\frac{6}{52}}{\frac{12}{52}} = \frac{6}{\cancel{52}} \cdot \frac{\cancel{52}}{12} = \frac{6}{12} = \frac{1}{2}$$

This answer should make sense: half of the face cards are black.

### Example 6.3.7

Two hundred fifty people who recently purchased a car were questioned and the results are summarized in the following table.

Satisfaction of Car Buyers

| | Satisfied | Not Satisfied | Total |
|---|---|---|---|
| New Car | 92 | 28 | 120 |
| Used Car | 83 | 47 | 130 |

| | Satisfied | Not Satisfied | Total |
|---|---|---|---|
| Total | 175 | 75 | 250 |

What is the probability that a person is satisfied if it is known that the person bought a used car?

### Solution

This is a conditional probability because we already know that the person bought a used car.

$$P(\text{satisfied} \mid \text{used car}) = \frac{P(\text{satisfied and used})}{P(\text{used})}$$

$$= \frac{\frac{83}{250}}{\frac{130}{250}} = \frac{83}{\cancel{250}} \cdot \frac{\cancel{250}}{130} = \frac{83}{130} \approx 0.638$$

The probability that a person is satisfied if it is known that the person bought a used car is approximately 0.638 or 63.8%

Note: it is faster to do a table problem like this using the method from Example $6.3.4$. There are 83 people who bought a used car and are satisfied out of the 130 people who bought a used car.

### Try It 6.3.4

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

a. $P(\text{not pregnant} \mid \text{positive test result})$
b. $P(\text{positive test result} \mid \text{not pregnant})$

| | Positive test | Negative test | Total |
|---|---|---|---|
| Pregnant | 70 | 4 | 74 |
| Not Pregnant | 5 | 14 | 19 |
| Total | 75 | 18 | 93 |

**Answer**

a. Since we know the test result was positive, we’re limited to the 75 women in the first column, of which 5 were not pregnant.

$P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} \approx 0.067.$

b. Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test.

$P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263.$

This result is what is usually called a false positive: a positive result when the woman is not actually pregnant.

---

- **3.3: Conditional Probabilities** by Maxie Inigo, Jennifer Jameson, Kathryn Kozak, Maya Lanzetta, & Kim Sonier is licensed CC BY-SA 4.0. Original source: https://www.coconino.edu/open-source-textbooks#college-mathematics-for-everyday-life-by-inigo-jameson-kozak-lanzetta-and-sonier.

# 6.4: Bayes Theorem

In this section we concentrate on the more complex conditional probability problems we began looking at in the last section.

✓ Example 6.4.1

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

**Solution**

There are two ways to approach the solution to this problem. One involves an important result in probability theory called Bayes' theorem. We will discuss this theorem a bit later, but for now we will use an alternative and, we hope, much more intuitive approach.

Let's break down the information in the problem piece by piece.

*Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population).* The percentage 0.1% can be converted to a decimal number by moving the decimal place two places to the left, to get 0.001. In turn, 0.001 can be rewritten as a fraction: $\frac{1}{1000}$. This tells us that about 1 in every 1000 people has the disease. (If we wanted we could write $P(\text{disease})=0.001$.)

*A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it).* This part is fairly straightforward: everyone who has the disease will test positive, or alternatively everyone who tests negative does not have the disease. (We could also say $P(\text{positive} \mid \text{disease})=1$.)

*The false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease).* This is even more straightforward. Another way of looking at it is that of every 100 people who are tested and do not have the disease, 5 will test positive even though they do not have the disease. (We could also say that $P(\text{positive} \mid \text{no disease})=0.05$.)

*Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?* Here we want to compute $P(\text{disease|positive})$. We already know that $P(\text{positive} \mid \text{disease})=1$, but remember that conditional probabilities are not equal if the conditions are switched.

Rather than thinking in terms of all these probabilities we have developed, let's create a hypothetical situation and apply the facts as set out above. First, suppose we randomly select 1000 people and administer the test. How many do we expect to have the disease? Since about $\frac{1}{1000}$ of all people are afflicted with the disease, $\frac{1}{1000}$ of 1000 people is 1. (Now you know why we chose 1000.) Only 1 of 1000 test subjects actually has the disease; the other 999 do not.

We also know that 5% of all people who do not have the disease will test positive. There are 999 disease-free people, so we would expect $(0.05)(999)=49.95$(so, about 50) people to test positive who do not have the disease.

Now back to the original question, computing $P(\text{disease} \mid \text{positive})$. There are 51 people who test positive in our example (the one unfortunate person who actually has the disease, plus the 50 people who tested positive but don't). Only one of these people has the disease, so

$$P(\text{disease} \mid \text{positive}) \approx \frac{1}{51} \approx 0.0196$$

or less than 2%. Does this surprise you? This means that of all people who test positive, over 98% *do not have the disease*.

The answer we got was slightly approximate, since we rounded 49.95 to 50. We could redo the problem with 100,000 test subjects, 100 of whom would have the disease and $(0.05)(99,900)=4995$test positive but do not have the disease, so the exact probability of having the disease if you test positive is

$$P(\text{disease} \mid \text{positive}) \approx \frac{100}{5095} \approx 0.0196$$

which is pretty much the same answer.

But back to the surprising result. *Of all people who test positive, over 98% do not have the disease.* If your guess for the probability a person who tests positive has the disease was wildly different from the right answer (2%), don't feel bad. The exact same problem was posed to doctors and medical students at the Harvard Medical School 25 years ago and the results revealed in a 1978 *New England Journal of Medicine* article. Only about 18% of the participants got the right answer. Most of the rest thought the answer was closer to 95% (perhaps they were misled by the false positive rate of 5%).

So at least you should feel a little better that a bunch of doctors didn't get the right answer either (assuming you thought the answer was much higher). But the significance of this finding and similar results from other studies in the intervening years lies not in making math students feel better but in the possibly catastrophic consequences it might have for patient care. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. Or worse, as in the early days of the AIDS crisis when being HIV-positive was often equated with a death sentence, the patient might take a drastic action and commit suicide.

As we have seen in this hypothetical example, the most responsible course of action for treating a patient who tests positive would be to counsel the patient that they most likely do *not* have the disease and to order further, more reliable, tests to verify the diagnosis.

One of the reasons that the doctors and medical students in the study did so poorly is that such problems, when presented in the types of statistics courses that medical students often take, are solved by use of Bayes' theorem, which is stated as follows:

Bayes' Theorem

$$P(A|B)=\frac{P(A)\cdot P(B|A)}{P(A)\cdot P(B|A)+P(A')\cdot P(B|A')}$$

In our earlier example, this translates to

$$P(\text{ disease }|\text{ positive })=\frac{P(\text{ disease })\cdot P(\text{ positive }|\text{ disease })}{P(\text{ disease })\cdot P(\text{ positive }|\text{ disease })+P(\text{ no disease })\cdot P(\text{ positive }|\text{ no disease })}$$

Plugging in the numbers gives

$$P(\text{ disease }|\text{ positive })=\frac{(0.001)(1)}{(0.001)(1)+(0.999)(0.05)}\approx 0.0196$$

which is exactly the same answer as our original solution.

The problem is that you (or the typical medical student, or even the typical math professor) are much more likely to be able to remember the original solution than to remember Bayes' theorem. Psychologists, such as Gerd Gigerenzer, author of *Calculated Risks: How to Know When Numbers Deceive You*, have advocated that the method involved in the original solution (which Gigerenzer calls the method of "natural frequencies") be employed in place of Bayes' Theorem. Gigerenzer performed a study and found that those educated in the natural frequency method were able to recall it far longer than those who were taught Bayes' theorem. When one considers the possible life-and-death consequences associated with such calculations it seems wise to heed his advice.

✓ Example 6.4.2

A certain disease has an incidence rate of 2%. If the false negative rate is 10% and the false positive rate is 1%, compute the probability that a person who tests positive actually has the disease.

**Solution**

Imagine 10,000 people who are tested. Of these 10,000, 200 will have the disease; 10% of them, or 20, will test negative and the remaining 180 will test positive. Of the 9800 who do not have the disease, 98 will test positive. So of the 278 total people who test positive, 180 will have the disease. Thus

$$P(\text{ disease }|\text{ positive })=\frac{180}{278}\approx 0.647$$

so about 65% of the people who test positive will have the disease.

Using Bayes theorem directly would give the same result:

$$P(\text{ disease } \mid \text{ positive }) = \frac{(0.02)(0.90)}{(0.02)(0.90)+(0.98)(0.01)} = \frac{0.018}{0.0278} \approx 0.647$$

**Try It 6.4.1**

A certain disease has an incidence rate of 0.5%. If there are no false negatives and if the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

**Answer**

Out of 100,000 people, 500 would have the disease. Of those, all 500 would test positive. Of the 99,500 without the disease, 2,985 would falsely test positive and the other 96,515 would test negative.

$$P(\text{ disease } \mid \text{ positive }) = \frac{500}{500+2985} = \frac{500}{3485} \approx 14.3\%$$

---

# 6.5: Counting Methods

## Learning Objectives

- Use the Fundamental Counting Rule
- Compute factorials, permutations and combinations
- Find probabilities using counting methods

Recall that

$$P(A)=\frac{\text{number of ways for } A \text{ to occur}}{\text{total number of outcomes}} \tag{6.5.1}$$

for theoretical probabilities. So far the problems we have looked at had rather small total number of outcomes. We could easily count the number of elements in the sample space. If there are a large number of elements in the sample space we can use counting techniques such as permutations or combinations to count them.

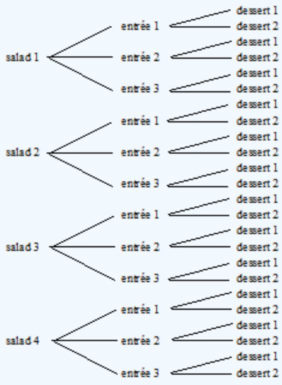
## Fundamental Counting Rule and Tree Diagrams

The simplest of the counting techniques is the Fundamental Counting Rule. A **tree diagram** is a useful tool for visualizing the counting rule. It uses branches to represent the outcomes in a multiple-step experiment.

### Example 6.5.1

Let's say that a person walks into a restaurant for a three course dinner. There are 4 different salads, 3 different entrées, and 2 different desserts to choose from. Assuming the person wants to eat a salad, an entrée and a dessert, how many different meals are possible?

**Solution**



Looking at the tree diagram we can see that the total number of meals is $4 \cdot 3 \cdot 2 = 24$ meals.

### Fundamental Counting Rule

If event 1 can be done in $m_1$ ways, event 2 can be done in $m_2$ ways, and so forth to event $n$ being done in $m_n$ ways, then the number of ways to do event 1, followed by event 2, ... followed by event $n$ together is:

$$m_1 \cdot m_2 \cdot m_3 \cdot \ldots \cdot m_n. \tag{6.5.2}$$

### Example 6.5.2

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

**Solution**

There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in $2 \cdot 2 \cdot 2 = 8$ different ways. Recall that another way to write $2 \cdot 2 \cdot 2$ is $2^3$ which is much more compact.

The Fundamental Counting Rule may seem like a very simple idea but it is very powerful. Many complex counting problems can be solved using the Fundamental Counting Rule.

### ✓ Example 6.5.3

Some license plates in Arizona consist of 3 digits followed by 3 letters. How many license plates of this type are possible if:

a. there are no restrictions on digits or letters?
b. letters can be repeated but digits cannot?
c. the first digit cannot be zero and both digits and letters can be repeated?
d. neither digits nor numbers can be repeated.

**Solution**

a. There are 10 digits (0, 1, 2, 3, …, 9) and 26 letters.

$$\underbrace{(\underline{10} \cdot \underline{10} \cdot \underline{10})}_{\text{digits}} \cdot \underbrace{(\underline{26} \cdot \underline{26} \cdot \underline{26})}_{\text{letters}} = 17,576,000 \text{ license plates}$$

b. There are 26 letters but the number of choices for digits will decrease by 1 after each chosen digit.

$$\underbrace{(\underline{10} \cdot \underline{9} \cdot \underline{8})}_{\text{digits}} \cdot \underbrace{(\underline{26} \cdot \underline{26} \cdot \underline{26})}_{\text{letters}} = 12,654,720 \text{ license plates}$$

c. There are 9 choices for the first digit, and 10 choices for the second and third digits. There are 26 choices for letters.

$$\underbrace{(\underline{9} \cdot \underline{10} \cdot \underline{10})}_{\text{digits}} \cdot \underbrace{(\underline{26} \cdot \underline{26} \cdot \underline{26})}_{\text{letters}} = 15,818,400 \text{ license plates}$$

d. The number of choices for digits and letters will decrease by 1 after each choice.

$$\underbrace{(\underline{10} \cdot \underline{9} \cdot \underline{8})}_{\text{digits}} \cdot \underbrace{(\underline{26} \cdot \underline{25} \cdot \underline{24})}_{\text{letters}} = 11,232,000 \text{ license plates}$$

### ✎ Try It 6.5.1

How many different ways can the letters of the word MATH be rearranged to form a 4-letter code word?

**Answer**

We can choose any of the 4 letters to be first. Then there are 3 choices for the second letter and 2 choices for the third. At this point there is only 1 letter left to be last. Using the Fundamental Counting Rule there are

$$4 \cdot 3 \cdot 2 \cdot 1 = 24 \text{ ways}$$

for to form a 4-letter code word from the letters of MATH.

This type of calculation occurs frequently in counting problems so we have some notation to simplify the problem.

### Factorial

The **factorial** of $n$, read “n factorial” is

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

By this definition, $0! = 1$.

Some examples of calculations of factorials:

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40,320$

Factorials get very large very fast.

$20! = 2.43 \cdot 10^{18}$ and $40! = 8.16 \cdot 10^{47}$.

70! is larger than most calculators can handle.

**Try It 6.5.2**

How many ways can five different door prizes be distributed among five people?

**Answer**

There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

## Permutations

Consider the following counting problems:

1. In how many ways can three runners finish a race?
2. In how many ways can a group of three people be chosen to work on a project?

What is the difference between these two problems? In the first problem the order that the runners finish the race matters. In the second problem the order in which the three people are chosen is not important, only which three people are chosen matters.

**Permutation**

A **permutation** is an arrangement of a set of items in which repeats are not allowed and order matters. The number of permutations of $n$ total items taken $r$ at a time is given by:

$${}_nP_r = P(n,r) = \frac{n!}{(n-r)!} \tag{6.5.3}$$

*Note: Many calculators can calculate permutations directly. Look for a function that looks like* ${}_nP_r$ *or* $P(n,r)$

**Example 6.5.4**

Assume that 10 cars are in a race. In how many ways can three cars finish in first, second and third place?

**Solution**

Repeats are not allowed and the order in which the cars finish is important. This is a permutation of 10 items taken 3 at a time.

Using the permutation formula (Equation 6.5.3):

$$P(10,3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 = 720$$

Note that all of the numbers from 7 down to 1 will cancel in the numerator and denominator, which leaves us with just $10 \cdot 9 \cdot 8$. The calculation starts at $n$ (which is 10) and you multiply $r$ (which is 3) descending numbers, starting from $n$.

You could have also used the Fundamental Counting Rule (Equation 6.5.2):

$$\underline{10} \cdot \underline{9} \cdot \underline{8} = 720$$

There are 720 different ways for cars to finish in the top three places.

**✓ Example 6.5.5**

The Circle K International Club has 18 members. An election is held to choose a president, vice-president and secretary. In how many ways can the 3 officers be chosen?

**Solution**

Repeats are not allowed and the order in which the officers are chosen matters, so this is a permutation.

Using the permutation formula (Equation 6.5.3):

$$P(18,3)=\frac{18!}{(18-3)!}=\frac{18!}{15!}=18\cdot 17\cdot 16=4896$$

There are 4896 different ways the three officers can be chosen.

**✎ Try It 6.5.3**

I have 9 paintings and have room to display only 4 of them at a time on my wall. How many different ways could I do this?

**Answer**

Since we are choosing 4 paintings out of 9 where repeats are not allowed and the order matters, there are ${}_9P_4=9\cdot 8\cdot 7\cdot 6=3,024$ ways to display 4 out of 9 paintings on a wall.

## Combinations

Choose a committee of two people from persons A, B, C, D and E. By the Fundamental Counting Rule there are $5\cdot 4=20$ ways to arrange the two people.

AB AC AD AE BA BC BD BE CA CB

CD CE DA DB DC DE EA EB EC ED

Committees AB and BA are the same committee. Similarly for committees CD and DC. Every committee is counted twice.

$$\frac{20}{2}=10$$

so there are 10 possible different committees.

Now choose a committee of three people from persons A, B, C, D and E. There are $5\cdot 4\cdot 3=60$ ways to pick three people in order. Think about the committees with persons A, B and C. There are $3!=6$ of them.

ABC ACB BAC BCA CAB CBA

Each of these is counted as one of the 60 possibilities but they are the same committee. Each committee is counted six times so there are

$$\frac{60}{6}=10\text{ different committees.}$$

In both cases we divided the number of permutations by the number of ways to rearrange the people chosen.

The number of permutations of $n$ people taken $r$ at a time is $P(n,r)$ and the number of ways to rearrange the people chosen is $r!$. Putting these together we get

$$\begin{aligned}\frac{n!}{\#\text{ ways to arrange r items}} &= \frac{P(n,r)}{r!} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{1}} \\ &= \frac{n!}{(n-r)!}\cdot\frac{1}{r!} \\ &= \frac{n!}{(n-r)!r!}\end{aligned}$$

**Combination**

A **combination** is an arrangement of a set of items in which repeats are not allowed and the order does not matter. The number of combinations of $n$ items taken $r$ at a time is is given by:

$$_nC_r = C(n,r) = \frac{n!}{r!(n-r)!} \tag{6.5.4}$$

*Note: Many calculators can calculate combinations directly. Look for a function that looks like* $_nC_r$ *or* $C(n,r)$.

**Example 6.5.6**

A student has a summer reading list of 8 books. The student must read 5 of the books before the end of the summer. In how many ways can the student read 5 of the 8 books?

**Solution**

Repeats are not allowed, and the order of the books is not important, only which books are read. This is a combination of 8 items taken 5 at a time.

$$C(8,5) = \frac{8!}{5!(8-5)!} = \frac{8!}{5!3!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = \frac{8\cdot7\cdot6}{3\cdot2\cdot1} = 8\cdot7 = 56 \tag{6.5.5}$$

There are 56 ways to choose 5 of the books to read.

**Try It 6.5.4**

A child wants to pick three pieces of Halloween candy to take in her school lunch box. If there are 13 pieces of candy to choose from, how many ways can she pick the three pieces?

**Answer**

This is a combination because it does not matter in what order the candy is chosen.

$$\begin{aligned}C(13,3) &= \frac{13!}{3!(13-3)!} = \frac{13!}{3!10!} \\ &= \frac{13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1)} \\ &= \frac{13\cdot12\cdot11}{3\cdot2\cdot1} \\ &= \frac{1716}{6} = 286\end{aligned}$$

There are 286 ways to choose the three pieces of candy to pack in her lunch.

**Example 6.5.7**

A class consists of 15 men and 12 women. In how many ways can two men and two women be chosen to participate in an in-class activity?

### Solution

This is a combination since repeats are not allowed and the order in which the people is chosen is not important.

Choose two men:

$$C(15,2)=\frac{15!}{2!(15-2)!}=\frac{15!}{2!13!}=105$$

Choose two women:

$$C(12,2)=\frac{12!}{2!(12-2)!}=\frac{12!}{2!10!}=66$$

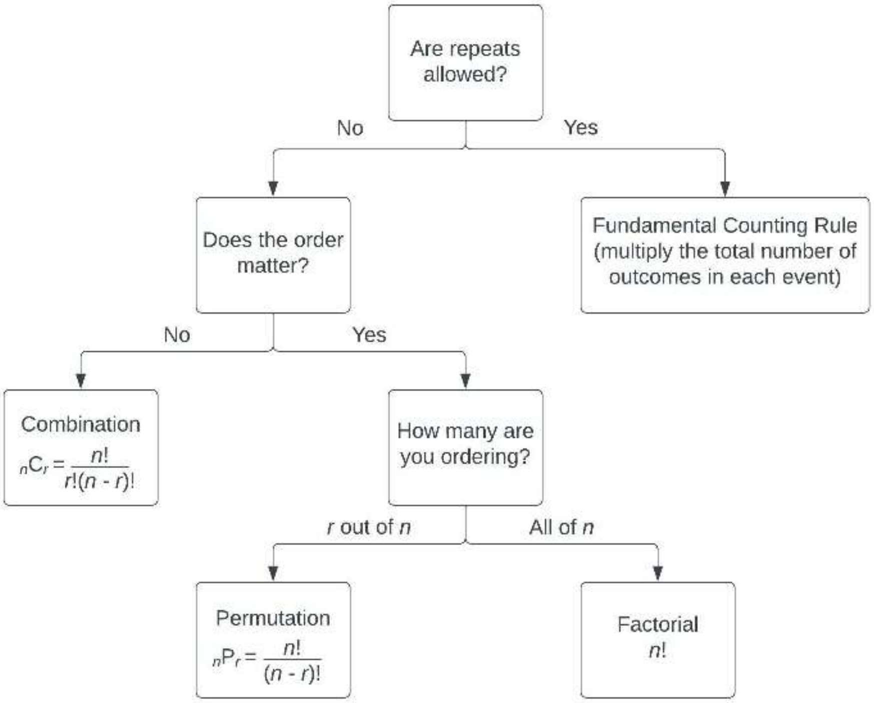
We want 2 men and 2 women so by the Fundamental Counting Rule, we multiply these results.

$$105\cdot 66=6930$$

There are 6930 ways to choose two men and two women to participate.

Note: The difference between a combination and a permutation is whether order matters or not. If the order of the items is important, use a permutation. If the order of the items is not important, use a combination.

The following flow chart below may help with deciding which counting rule to use. Start at the top: ask yourself if the same item can be repeated. For instance, a person on a committee cannot be counted as two distinct people; however, a number on a car license plate may be used twice. If repeats are not allowed, then ask, does the order in which the item is chosen matter? If it does not, then use a combination. If it does, then ask if you are ordering the entire group. If so, use a factorial. If only ordering some from the total, use a permutation.



## Probabilities Involving Permutations and Combinations

Now that we can calculate the number of permutations or combinations, we can use that information to calculate probabilities.

**✓ Example 6.5.8**

There are 20 students in a class. 12 of the students are women. The names of the students are put into a hat and 5 names are drawn. What is the probability that all of the chosen students are women?

**Solution**

This is a combination because repeats are not allowed and the order of the students is not important.

$$P(\text{ all females }) = \frac{\#\text{ ways to pick 5 women}}{\#\text{ ways to pick 5 students}} \tag{6.5.6}$$

The number of way to choose 5 women is

$$C(12,5) = 792$$

The number of ways to choose 5 students is

$$C(20,5) = 15,504$$

$$P(\text{ all females }) = \frac{\#\text{ ways to pick 5 women}}{\#\text{ ways to pick 5 students}} = \frac{792}{15,504} = 0.051$$

The probability that all the chosen students are women is 0.051 or 5.1%.

**✓ Example 6.5.9**

The local Boys and Girls Club holds a duck race to raise money. Community members buy a rubber duck marked with a numeral between 1 and 30, inclusive. The box of 30 ducks is emptied into a creek and allowed to float downstream to the finish line. What is the probability that ducks numbered 5, 18 and 21 finish in first, second and third, respectively?

**Solution**

This is a permutation since repeats are not allowed and the order of finish is important.

$$P(5,18\ \&\ 21\text{ finish 1st, 2nd }\&\text{ 3 rd}) = \frac{\#\text{ ways 5, 18 }\&\text{ 21 finish 1st, 2nd }\&\text{ 3rd}}{\#\text{ ways any ducks can finish 1st, 2nd }\&\text{ 3rd}}$$

There is only one way that the ducks can finish with #5 in first, #18 in second and #21 in third.

The number of ways any ducks can finish in first, second and third is

$$P(30,3) = 24,360$$

$$\begin{aligned} P(5,18\ \&\ 21\text{ finish 1st, 2nd }\&\text{ 3rd}) &= \frac{\#\text{ ways 5, 18 }\&\text{ 21 finish 1st, 2nd }\&\text{ 3rd}}{\#\text{ ways any ducks can finish 1st, 2nd }\&\text{ 3rd}} \\ &= \frac{1}{24,360} \approx 4.10 \times 10^{-5} \end{aligned}$$

The probability that ducks numbered 5, 18 and 21 finish in first, second and third, respectively, is approximately 0.000041 or 0.0041%.

**✎ Try It 6.5.5**

A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

**Answer**

There are 10 possible values for each digit of the PIN, namely: 0,1,2,3,4,5,6,7,8,9, so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10,000$ total possible PIN numbers.

To have no repeated digits, all 4 digits would have to be different, which is selecting without replacement (no repeats). We could either compute $10 \cdot 9 \cdot 8 \cdot 7$, or notice that this is the same as the permutation ${}_{10}P_4 = 5040$.

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is $P(\text{no repeated digits in 4 digit PIN}) = \frac{{}_{10}P_4}{10^4} = \frac{5040}{10,000} = 0.504$ .

### Example 6.5.10

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and 6 of them are drawn at random. If the 6 numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

**Solution**

In order to compute the probability, we need to count the total number of ways 6 numbers can be drawn, and the number of ways the 6 numbers on the player's ticket could match the 6 numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. Of these possible outcomes, only one would match all 6 numbers on the player's ticket, so the probability of winning the grand prize is:

$$P(\text{win grand prize}) = \frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12,271,512} \approx 0.0000000815$$

### Try It 6.5.6

In the state lottery from the previous example, if 5 of the 6 numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

**Answer**

As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$ In order to win the second prize, 5 of the 6 numbers on the ticket must match 5 of the 6 winning numbers; in other words, we must have chosen 5 of the 6 winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by ${}_6C_5 = 6$ and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1 = 42$. Thus the number of favorable outcomes is then given by the Fundamental Counting Rule: ${}_6C_5 \cdot {}_{42}C_1 = 6 \cdot 42 = 252$ . So the probability of winning the second prize is:

$$P(\text{win second prize}) = \frac{({}_6C_5)\,({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12,271,512} \approx 0.0000205$$

### Example 6.5.11

Compute the probability of randomly drawing 5 cards from a deck and getting exactly one Ace.

**Solution**

In many card games (such as poker) repeats are not allowed and the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses). Thus we use combinations to compute the possible number of 5-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and 4 other cards (none of them Aces) from the deck. Since there are 4 Aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be ${}_{48}C_4$\) ways to select the four non-Aces. Now we use the multiplication principle to calculate that there will be ${}_4C_1 \cdot {}_{48}C_4$ ways to choose one Ace and 4 non-Aces.

Putting this all together, we have

$$P(\text{one Ace}) = \frac{\left({}_4C_1\right)\left({}_{48}C_4\right)}{{}_{52}C_5} = \frac{778,320}{2,598,960} \approx 0.29947$$

The probability of getting exactly one Ace in a 5-card hand is approximately 0.29947 or 29.95%.

### ✎ Try It 6.5.7

Compute the probability of randomly drawing 5 cards from a deck of cards and getting 3 Aces and 2 Kings.

**Answer**

$$P(3 \text{ Aces and } 2 \text{ Kings}) = \frac{\left({}_4C_3\right)\left({}_4C_2\right)}{{}_{52}C_5} = \frac{24}{2,598,960} \approx 0.0000092$$

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# 6.6: Odds and Expected Value

**Learning Objectives**

- Find the odds for and against an event
- Find the probability of an event given the odds of the event
- Find the expected value of an event

## Odds

Probabilities are always numbers between 0 and 1. Many people are not comfortable working with such small values. Another way of describing the likelihood of an event happening is to use the ratio of how often it happens to how often it does not happen. The ratio is called the odds of the event happening. There are two types of odds, odds for and odds against. Casinos, race tracks and other types of gambling usually state the odds against an event happening.

**Odds**

If the probability of an event $A$ is $P(A)$, then the **odds for event $A$**, $O(A)$, are given by:

$$O(A)=\frac{P(A)}{P(A')} \text{ or } O(A)=\frac{\text{number of ways for } A \text{ to occur}}{\text{number of ways for } A \text{ to not occur}}$$

Also, the **odds against event $A$**, $O(A')$, are given by:

$$O(A')=\frac{P(A')}{P(A)} \text{ or } O(A')=\frac{\text{number of ways for } A \text{ to not occur}}{\text{number of ways for } A \text{ to occur}}$$

**Example 6.6.1**

A single card is drawn from a well shuffled deck of 52 cards. Find the odds that the card is a red 8.

There are two red 8s in the deck.

$$P(\text{red } 8)=\frac{2}{52}=\frac{1}{26}\ .$$

$$P(\text{not a red } 8)=\frac{50}{52}=\frac{25}{26}\ .$$

$$O(\text{red } 8)=\frac{P(\text{red } 8)}{P(\text{not a red } 8)}=\frac{\frac{1}{26}}{\frac{25}{26}}=\frac{1}{\cancel{26}}\cdot\frac{\cancel{26}}{25}=\frac{1}{25}$$

The odds of drawing a red 8 are 1 to 25. This can also be written as 1:25.

*Note: Do not write odds as a decimal or a percent.*

**Example 6.6.2**

Many roulette wheels have slots numbered 0, 00, and 1 through 36. The slots numbered 0 and 00 are green. Half of the remaining slots are red and the other half are black. The game is played by spinning the wheel one direction and rolling a marble around the outer edge the other direction. Players bet on which slot the marble will fall into. What are the odds the marble will land in a red slot?

### Solution

There are 38 slots in all. The slots 2, 4, 6, …, 36 are red so there are 18 red slots. The other 20 slots are not red.

$$P(\text{red})=\frac{18}{38}=\frac{9}{19}\ .$$

$$P(\text{not red}) = 1 - \frac{9}{19} = \frac{19}{19} - \frac{9}{19} = \frac{10}{19} \quad .$$

$$O(\text{red}) = \frac{P(\text{red})}{P(\text{not red})} = \frac{\frac{9}{19}}{\frac{10}{19}} = \frac{9}{\cancel{19}} \cdot \frac{\cancel{19}}{10} = \frac{9}{10}$$

The odds of the marble landing in a red slot are 9 to 10. This can also be written as 9:10.

**Try It 6.6.1**

Two fair dice are tossed and the sum is recorded. Find the odds against rolling a sum of 9.

**Answer**

The event $A$, roll a sum of 9 is: $A$ = {(3, 6), (4, 5), (5, 4), (6, 3)}

There are 36 ways to roll 2 dice and 4 ways to roll a sum of 9. That means there are 32 ways to roll a sum that is not 9.

$$P(A) = \frac{4}{36} = \frac{1}{9} \quad .$$

$$P(A') = \frac{32}{36} = \frac{8}{9} \quad .$$

$$O(\text{against sum is } 9) = \frac{P(\text{sum is not } 9)}{P(\text{sum is } 9)} = \frac{P(A')}{P(A)} = \frac{\frac{8}{9}}{\frac{1}{9}} = \frac{8}{\cancel{9}} \cdot \frac{\cancel{9}}{1} = \frac{8}{1}$$

The odds against rolling a sum of 9 are 8 to 1 or 8:1.

We can also find the probability of an event happening based on the odds for the event. Saying that the odds of an event are 3 to 5 means that the event happens 3 times for every 5 times it does not happen. If we add up the possibilities of both we get a sum of 8. So the event happens about 3 out of every 8 times. We would say the probability is $\frac{3}{8}$.

**Odds to Probability**

If the odds favoring event $A$ are $a$ to $b$, then:

$$P(A) = \frac{a}{a+b} \text{ and } P(A') = \frac{b}{a+b}.$$

**Example 6.6.3**

A local little league baseball team is going to a tournament. The odds of the team winning the tournament are 3 to 7. Find the probability of the team winning the tournament.

**Solution**

$$P(\text{winning}) = \frac{3}{3+7} = \frac{3}{10} = 0.3$$

## Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

## Example 6.6.4

In the casino game roulette [1], a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

### Solution

Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won" $\frac{-\$2}{\$38} \approx -\$0.053$. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

| Outcome | Probability of outcome |
|---|---|
| $\$35$ | $\frac{1}{38}$ |
| $-\$1$ | $\frac{37}{38}$ |

Notice that if we multiply each outcome by its corresponding probability we get $\$35 \cdot \frac{1}{38} = 0.9211$ and $-\$1 \cdot \frac{37}{38} = -0.9737$, and if we add these numbers we get $0.9211 + (-0.9737) \approx -0.053$, which is the expected value we computed above.

## Expected Value

**Expected value** is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products. We will use $E$ as the notation for the expected value.

## Try It 6.6.2

You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

**Answer**

| Outcome | Probability of outcome |
|---|---|
| $\$3995$ | $\frac{1}{2000}$ |
| $-\$5$ | $\frac{1999}{2000}$ |

$$E = (\$3995) \cdot \frac{1}{2000} + (-\$5) \cdot \frac{1999}{2000} \approx -\$3.00$$

In the long run, you should expect to lose \$3.00 on average each time you buy a \$5 raffle ticket.

**Example 6.6.5**

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. If they match 5 numbers, then win $1,000. It costs $1 to buy a ticket. Find the expected value.

**Solution**

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$$P(\text{match all 6 numbers}) = \frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12,271,512} \approx 0.0000000815$$

$$P(\text{match 5 numbers}) = \frac{\left({}_6C_5\right)\left({}_{42}C_1\right)}{{}_{48}C_6} = \frac{252}{12,271,512} \approx 0.0000205$$

Our probabilities and outcome values are:

| Outcome | Probability of outcome |
|---|---|
| $\$999,999$ | $\frac{1}{12,271,512}$ |
| $\$999$ | $\frac{252}{12,271,512}$ |
| $-\$1$ | $1-\frac{253}{12,271,512}=\frac{12,271,259}{12,271,512}$ |

The expected value, then is:

$$E = (\$999,999)\cdot\frac{1}{12,271,512} + (\$999)\cdot\frac{252}{12,271,512} + (-\$1)\cdot\frac{12,271,259}{12,271,512} \approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose $1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

**Try It 6.6.3**

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him $1. If any 2 dice match values, you get $2. What is the expected value of this game? Would you play?

**Answer**

Suppose you roll the first die. The probability the second will be different is $\frac{5}{6}$. The probability that the third roll is different than the previous 2 is $\frac{4}{6}$, so the probability that the 3 dice are different is $\frac{5}{6}\cdot\frac{4}{6}=\frac{20}{36}$. The probability that 2 dice will match is the complement, $1-\frac{20}{36}=\frac{16}{36}$.

Our probabilities and outcome values are:

| Outcome | Probability of outcome |
|---|---|
| $\$2$ | $\frac{16}{36}$ |
| $-\$1$ | $\frac{20}{36}$ |

The expected value is: $E=(\$2)\cdot\frac{16}{36}+(-\$1)\cdot\frac{20}{36}=\frac{12}{36}\approx\$0.33$ . Yes, it is in your advantage to play. On average, you'd win \$0.33 per play.

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

**Example 6.6.6**

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year [2]. An insurance company charges \$275 for a life-insurance policy that pays a \$100,000 death benefit. What is the expected value for the person buying the insurance?

### Solution

The probabilities and outcomes are

| Outcome | Probability of outcome |
|---|---|
| $\$100,000-\$275=\$99,725$ | $0.00242$ |
| $-\$275$ | $1-0.00242=0.99758$ |

The expected value is $E=(\$99,725)(0.00242)+(-\$275)(0.99758)=-\$33$

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

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[1] Photo CC-BY-SA http://www.flickr.com/photos/stoneflower/

[2] According to the estimator at www.numericalexample.com/inde...=article&id=91

# 6.7: Chapter Review and Glossary

# 6.8: Exercises

1. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of the given event.

a. A red ball is drawn
b. A white ball is drawn

2. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. What is the probability of drawing one slip of paper from the hat at random and getting:

a. A consonant
b. A vowel

3. A group of people were asked if they had run a red light in the last year. 150 responded "yes", and 185 responded "no". Find the probability that if a person is chosen at random, they have run a red light in the last year.

4. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dots, and 40 indicated they don't enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.

5. Compute the probability of tossing a six-sided die (with sides numbered 1 through 6) and getting a 5.

6. Compute the probability of tossing a six-sided die and getting a 7.

7. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female.

| | A | B | C | Total |
|---|---|---|---|---|
| Male | 8 | 18 | 13 | 39 |
| Female | 10 | 4 | 12 | 26 |
| Total | 18 | 22 | 25 | 65 |

8. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person had no credit cards.

| | Zero | One | Two or more | Total |
|---|---|---|---|---|
| Male | 9 | 5 | 19 | 33 |
| Female | 18 | 10 | 20 | 48 |
| Total | 27 | 15 | 39 | 81 |

9. Compute the probability of tossing a six-sided die and getting an even number.

10. Compute the probability of tossing a six-sided die and getting a number less than 3.

11. If you pick one card at random from a standard deck of cards, what is the probability it will be a King?

12. If you pick one card at random from a standard deck of cards, what is the probability it will be a Diamond?

13. Compute the probability of rolling a 12-sided die and getting a number other than 8.

14. If you pick one card at random from a standard deck of cards, what is the probability it is not the Ace of Spades?

15. Referring to the grade table from question #7, what is the probability that a student chosen at random did NOT earn a C?

16. Referring to the credit card table from question #8, what is the probability that a person chosen at random has at least one credit card?

17. A six-sided die is rolled twice. What is the probability of showing a 6 on both rolls?

18. A fair coin is flipped twice. What is the probability of showing heads on both flips?

19. A die is rolled twice. What is the probability of showing a 5 on the first roll and an even number on the second roll?

20. Suppose that 21% of people own dogs. If you pick two people at random, what is the probability that they both own a dog?

21. Suppose a jar contains 17 red marbles and 32 blue marbles. If you reach in the jar and pull out 2 marbles at random, find the probability that both are red.

22. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. If you pull out two slips at random, find the probability that both are vowels.

23. Bert and Ernie each have a well-shuffled standard deck of 52 cards. They each draw one card from their own deck. Compute the probability that:

a. Bert and Ernie both draw an Ace.
b. Bert draws an Ace but Ernie does not.
c. neither Bert nor Ernie draws an Ace.
d. Bert and Ernie both draw a heart.
e. Bert gets a card that is not a Jack and Ernie draws a card that is not a heart.

24. Bert has a well-shuffled standard deck of 52 cards, from which he draws one card; Ernie has a 12-sided die, which he rolls at the same time Bert draws a card. Compute the probability that:

a. Bert gets a Jack and Ernie rolls a five.
b. Bert gets a heart and Ernie rolls a number less than six.
c. Bert gets a face card (Jack, Queen or King) and Ernie rolls an even number.
d. Bert gets a red card and Ernie rolls a fifteen.
e. Bert gets a card that is not a Jack and Ernie rolls a number that is not twelve.

25. Compute the probability of drawing a King from a deck of cards and then drawing a Queen.

26. Compute the probability of drawing two spades from a deck of cards.

27. A math class consists of 25 students, 14 female and 11 male. Two students are selected at random to participate in a probability experiment. Compute the probability that

a. a male is selected, then a female.
b. a female is selected, then a male.
c. two males are selected.
d. two females are selected.
e. no males are selected.

28. A math class consists of 25 students, 14 female and 11 male. Three students are selected at random to participate in a probability experiment. Compute the probability that

a. a male is selected, then two females.
b. a female is selected, then two males.
c. two females are selected, then one male.
d. three males are selected.
e. three females are selected.

29. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female and earned an A.

| | A | B | C | Total |
|---|---|---|---|---|
| Male | 8 | 18 | 13 | 39 |
| Female | 10 | 4 | 12 | 26 |
| Total | 18 | 22 | 25 | 65 |

30. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person was male and had two or more credit cards.

| | Zero | One | Two or more | Total |
|---|---|---|---|---|
| Male | 9 | 5 | 19 | 33 |
| Female | 18 | 10 | 20 | 48 |
| Total | 27 | 15 | 39 | 81 |

31.A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd-numbered.

32. A jar contains 4 red marbles numbered 1 to 4 and 10 blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability the marble is blue or even-numbered.

33. Referring to the table from #29, find the probability that a student chosen at random is female or earned a B.

34. Referring to the table from #30, find the probability that a person chosen at random is male or has no credit cards.

35. Compute the probability of drawing the King of hearts or a Queen from a deck of cards.

36. Compute the probability of drawing a King or a heart from a deck of cards.

37. A jar contains 5 red marbles numbered 1 to 5 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is

a. Even-numbered given that the marble is red.
b. Red given that the marble is even-numbered.

38. A jar contains 4 red marbles numbered 1 to 4 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is

a. Odd-numbered given that the marble is blue.
b. Blue given that the marble is odd-numbered.

39. Compute the probability of flipping a coin and getting heads, given that the previous flip was tails.

40. Find the probability of rolling a “1” on a fair die, given that the last 3 rolls were all ones.

41. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student speaks French, given that the student is female.

42. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student is male, given that the student speaks French.

43. A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 90% of the time if the person has the virus and 10% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".

a. Find the probability that a person has the virus given that they have tested positive, i.e. find $P(A|B)$.
b. Find the probability that a person does not have the virus given that they test negative, i.e. find $P(\text{ not } A| \text{ not } B)$ .

44. A certain virus infects one in every 2000 people. A test used to detect the virus in a person is positive 96% of the time if the person has the virus and 4% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".

a. Find the probability that a person has the virus given that they have tested positive, i.e. find $P(A|B)$.
b. Find the probability that a person does not have the virus given that they test negative, i.e. find $P(\text{ not } A| \text{ not } B)$ .

45. A certain disease has an incidence rate of 0.3%. If the false negative rate is 6% and the false positive rate is 4%, compute the probability that a person who tests positive actually has the disease.

46. A certain disease has an incidence rate of 0.1%. If the false negative rate is 8% and the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

47. A certain group of symptom-free women between the ages of 40 and 50 are randomly selected to participate in mammography screening. The incidence rate of breast cancer among such women is 0.8%. The false negative rate for the mammogram is 10%. The false positive rate is 7%. If a the mammogram results for a particular woman are positive (indicating that she has breast cancer), what is the probability that she actually has breast cancer?

48. About 0.01% of men with no known risk behavior are infected with HIV. The false negative rate for the standard HIV test 0.01% and the false positive rate is also 0.01%. If a randomly selected man with no known risk behavior tests positive for HIV, what is the probability that he is actually infected with HIV?

49. A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?

50. At a restaurant you can choose from 3 appetizers, 8 entrees, and 2 desserts. How many different three-course meals can you have?

51. How many three-letter "words" can be made from 4 letters "FGHI" if

a. repetition of letters is allowed
b. repetition of letters is not allowed

52. How many four-letter "words" can be made from 6 letters "AEBWDP" if

a. repetition of letters is allowed
b. repetition of letters is not allowed

53. All of the license plates in a particular state feature three letters followed by three digits (e.g. ABC 123). How many different license plate numbers are available to the state's Department of Motor Vehicles?

54. A computer password must be eight characters long. How many passwords are possible if only the 26 letters of the alphabet are allowed?

55. A pianist plans to play 4 pieces at a recital. In how many ways can she arrange these pieces in the program?

56. In how many ways can first, second, and third prizes be awarded in a contest with 210 contestants?

57. Seven Olympic sprinters are eligible to compete in the 4 x 100 m relay race for the USA Olympic team. How many four-person relay teams can be selected from among the seven athletes?

58. A computer user has downloaded 25 songs using an online file-sharing program and wants to create a CD-R with ten songs to use in his portable CD player. If the order that the songs are placed on the CD-R is important to him, how many different CD-Rs could he make from the 25 songs available to him?

59. In western music, an octave is divided into 12 pitches. For the film *Close Encounters of the Third Kind*, director Steven Spielberg asked composer John Williams to write a five-note theme, which aliens would use to communicate with people on Earth. Disregarding rhythm and octave changes, how many five-note themes are possible if no note is repeated?

60. In the early twentieth century, proponents of the Second Viennese School of musical composition (including Arnold Schönberg, Anton Webern and Alban Berg) devised the twelve-tone technique, which utilized a tone row consisting of all 12 pitches from the chromatic scale in any order, but with not pitches repeated in the row. Disregarding rhythm and octave changes, how many tone rows are possible?

61. In how many ways can 4 pizza toppings be chosen from 12 available toppings?

62. At a baby shower 17 guests are in attendance and 5 of them are randomly selected to receive a door prize. If all 5 prizes are identical, in how many ways can the prizes be awarded?

63. In the 6/50 lottery game, a player picks six numbers from 1 to 50. How many different choices does the player have if order doesn’t matter?

64. In a lottery daily game, a player picks three numbers from 0 to 9. How many different choices does the player have if order doesn’t matter?

65. A jury pool consists of 27 people. How many different ways can 11 people be chosen to serve on a jury and one additional person be chosen to serve as the jury foreman?

66. The United States Senate Committee on Commerce, Science, and Transportation consists of 23 members, 12 Republicans and 11 Democrats. The Surface Transportation and Merchant Marine Subcommittee consists of 8 Republicans and 7 Democrats. How many ways can members of the Subcommittee be chosen from the Committee?

67. You own 16 CDs. You want to randomly arrange 5 of them in a CD rack. What is the probability that the rack ends up in alphabetical order?

68. A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.

69. In a lottery game, a player picks six numbers from 1 to 48. If 5 of the 6 numbers match those drawn, they player wins second prize. What is the probability of winning this prize?

70. In a lottery game, a player picks six numbers from 1 to 48. If 4 of the 6 numbers match those drawn, they player wins third prize. What is the probability of winning this prize?

71. Compute the probability that a 5-card poker hand is dealt to you that contains all hearts.

72. Compute the probability that a 5-card poker hand is dealt to you that contains four Aces.

73. A bag contains 3 gold marbles, 6 silver marbles, and 28 black marbles. Someone offers to play this game: You randomly select on marble from the bag. If it is gold, you win $3. If it is silver, you win $2. If it is black, you lose $1. What is your expected value if you play this game?

74. A friend devises a game that is played by rolling a single six-sided die once. If you roll a 6, he pays you $3; if you roll a 5, he pays you nothing; if you roll a number less than 5, you pay him $1. Compute the expected value for this game. Should you play this game?

75. In a lottery game, a player picks six numbers from 1 to 23. If the player matches all six numbers, they win 30,000 dollars. Otherwise, they lose $1. Find the expected value of this game.

76. A game is played by picking two cards from a deck. If they are the same value, then you win $5, otherwise you lose $1. What is the expected value of this game?

77. A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of $350. If they offer a 2 year extended warranty for $48, what is the company's expected value of each warranty sold?

78. An insurance company estimates the probability of an earthquake in the next year to be 0.0013. The average damage done by an earthquake it estimates to be $60,000. If the company offers earthquake insurance for $100, what is their expected value of the policy?

## Exploration

Some of these questions were adapted from puzzles at mindyourdecisions.com.

79. A small college has been accused of gender bias in its admissions to graduate programs.

a. Out of 500 men who applied, 255 were accepted. Out of 700 women who applied, 240 were accepted. Find the acceptance rate for each gender. Does this suggest bias?
b. The college then looked at each of the two departments with graduate programs, and found the data below. Compute the acceptance rate within each department by gender. Does this suggest bias?
c. Looking at our results from Parts *a* and *b*, what can you conclude? Is there gender bias in this college’s admissions? If so, in which direction?

80. A bet on “black” in Roulette has a probability of 18/38 of winning. If you win, you double your money. You can bet anywhere from $1 to $100 on each spin.

a. Suppose you have $10, and are going to play until you go broke or have $20. What is your best strategy for playing?
b. Suppose you have $10, and are going to play until you go broke or have $30. What is your best strategy for playing?

81. Your friend proposes a game: You flip a coin. If it’s heads, you win $1. If it’s tails, you lose $1. However, you are worried the coin might not be fair coin. How could you change the game to make the game fair, without replacing the coin?

82. Fifty people are in a line. The first person in the line to have a birthday matching someone in front of them will win a prize. Of course, this means the first person in the line has no chance of winning. Which person has the highest likelihood of winning?

83. Three people put their names in a hat, then each draw a name, as part of a randomized gift exchange. What is the probability that no one draws their own name? What about with four people?

84. How many different “words” can be formed by using all the letters of each of the following words exactly once?

a. “ALICE”
b. “APPLE”

85. How many different “words” can be formed by using all the letters of each of the following words exactly once?

a. “TRUMPS”
b. “TEETER”

86. The *Monty Hall problem* is named for the host of the game show *Let’s make a Deal*. In this game, there would be three doors, behind one of which there was a prize. The contestant was asked to choose one of the doors. Monty Hall would then open one of the other doors to show there was no prize there. The contestant was then asked if they wanted to stay with their original door, or switch to the other unopened door. Is it better to stay or switch, or does it matter?

87. Suppose you have two coins, where one is a fair coin, and the other coin comes up heads 70% of the time. What is the probability you have the fair coin given each of the following outcomes from a series of flips?

a. 5 Heads and 0 Tails
b. 8 Heads and 3 Tails
c. 10 Heads and 10 Tails
d. 3 Heads and 8 Tails

88. Suppose you have six coins, where five are fair coins, and one coin comes up heads 80% of the time. What is the probability you have a fair coin given each of the following outcomes from a series of flips?

a. 5 Heads and 0 Tails
b. 8 Heads and 3 Tails
c. 10 Heads and 10 Tails
d. 3 Heads and 8 Tails

89. In this problem, we will explore probabilities from a series of events.

a. If you flip 20 coins, how many would you *expect* to come up “heads”, on average? Would you expect *every* flip of 20 coins to come up with exactly that many heads?
b. If you were to flip 20 coins, what would you consider a “usual” result? An “unusual” result?
c. Flip 20 coins (or one coin 20 times) and record how many come up “heads”. Repeat this experiment 9 more times. Collect the data from the entire class.
d. When flipping 20 coins, what is the theoretic probability of flipping 20 heads?
e. Based on the class’s experimental data, what appears to be the probability of flipping 10 heads out of 20 coins?
f. The formula ${}_nC_xp^x(1-p)^{n-x}$ will compute the probability of an event with probability $p$ occurring $x$ times out of $n$, such as flipping $x$ heads out of $n$ coins where the probability of heads is $p=\frac{1}{2}$. Use this to compute the theoretic probability of flipping 10 heads out of 20 coins.
g. If you were to flip 20 coins, based on the class’s experimental data, what range of values would you consider a “usual” result? What is the combined probability of these results? What would you consider an “unusual” result? What is the combined probability of these results?
h. We’ll now consider a simplification of a case from the 1960s. In the area, about 26% of the jury eligible population was black. In the court case, there were 100 men on the juror panel, of which 8 were black. Does this provide evidence of racial bias in jury selection?

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## CHAPTER OVERVIEW

# 7: Statistics

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading, and push you into decisions that you might find cause to regret. These chapters will help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

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# 7.1: Basic Concepts of Statistics

**Learning Objectives**

- Understand the basic terminology used in statistics
- Understand the difference between populations and samples
- Classify data as categorical or quantitative

## Introduction

Like most people, you probably feel that it is important to "take control of your life." But what does this mean? Partly it means being able to properly evaluate the data and claims that bombard you every day. If you cannot distinguish good from faulty reasoning, then you are vulnerable to manipulation and to decisions that are not in your best interest. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, Statistics is one of the most important things that you can study.

To be more specific, here are some claims that we have heard on several occasions. (We are *not* saying that each one of these claims is true!)

- 4 out of 5 dentists recommend Dentyne.
- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- Condoms are effective 94% of the time.
- Native Americans are significantly more likely to be hit crossing the streets than are people of other ethnicities.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one's life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 80% chance that in a room full of 30 people that at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

All of these claims are statistical in character. We suspect that some of them sound familiar; if not, we bet that you have heard other claims like them. Notice how diverse the examples are; they come from psychology, health, law, sports, business, etc. Indeed, data and data-interpretation show up in discourse from virtually every facet of contemporary life.

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading, and push you into decisions that you might find cause to regret. For these reasons, learning about statistics is a long step towards taking control of your life. (It is not, of course, the only step needed for this purpose.) These chapters will help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

You can take the first step right away. To be an intelligent consumer of statistics, your first reflex must be to question the statistics that you encounter. The British Prime Minister Benjamin Disraeli famously said, "There are three kinds of lies -- lies, damned lies, and statistics." This quote reminds us why it is so important to understand statistics. So let us invite you to reform your statistical habits from now on. No longer will you blindly accept numbers or findings. Instead, you will begin to think about the numbers, their sources, and most importantly, the procedures used to generate them.

We have put the emphasis on defending ourselves against fraudulent claims wrapped up as statistics. Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must also learn to recognize statistical evidence that supports a stated conclusion. When a research team is testing a new treatment for a disease, statistics allows them to conclude based on a relatively small trial that there is good evidence their drug is effective. Statistics allowed prosecutors in the 1950’s and 60’s to demonstrate racial bias existed in jury panels. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases.

## Basic Terms

In order to study and understand statistics, you must first be acquainted with the basic terminology.

**Data**

**Data** are the individual items of information such as measurements or survey responses that have been collected for a study or analysis.

**Statistics**

**Statistics** is a collection of methods for collecting, displaying, analyzing, and drawing conclusions from data.

There are 2 branches of statistics: descriptive and inferential.

**Descriptive Statistics**

**Descriptive statistics** is the branch of statistics that involves collecting, organizing, displaying, and describing data.

**Inferential Statistics**

**Inferential statistics** is the branch of statistics that uses probability to analyze, make predictions and draw conclusions based on the data.

We will mainly be exploring descriptive statistics in this class. To learn more about the methods of inferential statistics, you should take a course in introductory statistics.

Before we begin gathering and analyzing data we need to characterize the **population** we are studying. If we want to study the amount of money spent on textbooks by a typical first-year college student, our population might be all first-year students at your college. Or it might be:

- All first-year community college students in the state of California.
- All first-year students at public colleges and universities in the state of California.
- All first-year students at all colleges and universities in the state of California.
- All first-year students at all colleges and universities in the entire United States.
- And so on.

**Population**

The **population** of a study is the group the collected data is intended to describe.

Sometimes the intended population is called the **target population**, since if we design our study badly, the collected data might not actually be representative of the intended population.

Why is it important to specify the population? We might get different answers to our question as we vary the population we are studying. First-year students at Cal State Fullerton might take slightly more diverse courses than those at your college, and some of these courses may require less popular textbooks that cost more; or, on the other hand, the University Bookstore might have a larger pool of used textbooks, reducing the cost of these books to the students. Whichever the case (and it is likely that some combination of these and other factors are in play), the data we gather from your college will probably not be the same as that from Cal State Fullerton. Particularly when conveying our results to others, we want to be clear about the population we are describing with our data.

**✓ Example 7.1.1**

A newspaper website contains a poll asking people their opinion on a recent news article. What is the population?

**Solution**

While the target (intended) population may have been all people, the real population of the survey is readers of the website.

If we were able to gather data on every member of our population, say the average (we will define "average" more carefully in a subsequent section) amount of money spent on textbooks by each first-year student at your college during the 2019-2020 academic year, the resulting number would be called a **parameter**.

**Parameter**

A **parameter** is a value (average, percentage, etc.) calculated using all the data from a population.

We seldom see parameters, however, since surveying an entire population is usually very time-consuming and expensive, unless the population is very small or we already have the data collected.

**Census**

A survey of an entire population is called a **census**.

You are probably familiar with two common censuses: the official government Census that attempts to count the population of the U.S. every ten years, and voting, which asks the opinion of all eligible voters in a district. The first of these demonstrates one additional problem with a census: the difficulty in finding and getting participation from everyone in a large population, which can bias, or skew, the results.

There are occasionally times when a census is appropriate, usually when the population is fairly small. For example, if the manager of Starbucks wanted to know the average number of hours her employees worked last week, she should be able to pull up payroll records or ask each employee directly.

Since surveying an entire population is often impractical, we usually select a **sample** to study.

**Sample**

A **sample** is a smaller subset of the entire population, ideally one that is fairly representative of the whole population.

We will discuss sampling methods in greater detail in a later section. For now, let us assume that samples are chosen in an appropriate manner. If we survey a sample, say 100 first-year students at your college, and find the average amount of money spent by these students on textbooks, the resulting number is called a **statistic**.

**Statistic**

A **statistic** is a value (average, percentage, etc.) calculated using the data from a sample.

**✓ Example 7.1.2**

A researcher wanted to know how citizens of Brea felt about a voter initiative. To study this, she goes to the Brea Mall and randomly selects 200 shoppers and asks them their opinion. 60% indicate they are supportive of the initiative. What is the sample and population? Is the 60% value a parameter or a statistic?

**Solution**

The sample is the 200 shoppers questioned. The population is less clear. While the intended population of this survey was Brea citizens, the effective population was mall shoppers. There is no reason to assume that the 200 shoppers questioned would be representative of all Brea citizens.

The 60% value was based on the sample, so it is a statistic.

**Try It 7.1.1**

To determine the average length of trout in a lake, researchers catch 20 fish and measure them. What is the sample and population in this study?

**Answer**

The sample is the 20 fish caught. The population is all fish in the lake. The sample may be somewhat unrepresentative of the population since not all fish may be large enough to catch the bait.

### Try It 7.1.2

A college reports that the average age of their students is 28 years old. Is this a statistic or a parameter?

**Answer**

This is a parameter, since the college would have access to data on all students (the population).

## Classifying Data

Once we have gathered data, we might wish to classify it. Roughly speaking, data can be classified as **categorical data** or **quantitative data**.

### Categorical and Quantitative Data

- **Categorical (qualitative) data** are pieces of information that allow us to classify the objects under investigation into various categories. They are measurements for which there is no natural numerical scale, but which consist of attributes, labels, or other non-numerical characteristics.
- **Quantitative data** are responses that are numerical in nature and with which we can perform meaningful arithmetic calculations.

### Example 7.1.3

We might conduct a survey to determine the name of the favorite movie that each person in a math class saw in a movie theater. Is the data collected categorical or quantitative?

**Solution**

When we conduct such a survey, the responses would look like: *Top Gun: Maverick*, *Doctor Strange in the Multiverse of Madness*, or *Turning Red*. We might count the number of people who give each answer, but the answers themselves do not have any numerical values: we cannot perform computations with an answer like "*Turning Red*." This would be categorical data.

### Example 7.1.4

A survey could ask the number of movies you have seen in a movie theater in the past 12 months (0, 1, 2, 3, 4, ...). Is the data collected categorical or quantitative?

**Solution**

This would be quantitative data since the responses are numerical. We could perform meaningful arithmetic calculations on the data such as finding the average number of movies that people saw in a movie theater in the last year.

Other examples of quantitative data would be the running time of the movie you saw most recently (131 minutes, 126 minutes, 100 minutes, ...) or the amount of money you paid for a movie ticket the last time you went to a movie theater ($10.50, $13.75, $16, ...).

Sometimes, determining whether or not data is categorical or quantitative can be a bit trickier.

### Example 7.1.5

Suppose we gather respondents' ZIP codes in a survey to track their geographical location. Is the data collected categorical or quantitative?

**Solution**

ZIP codes are numbers, but we can't do any meaningful mathematical calculations with them (it doesn't make sense to say that 92806 is "twice" 46403 — that's like saying that Anaheim, CA is "twice" Gary, IN, which doesn't make sense at all), so ZIP codes are really categorical data.

### ✓ Example 7.1.6

A survey about the movie you most recently attended includes the question "How would you rate the movie you just saw?" with these possible answers:

1 - It was awful
2 - It was just OK
3 - I liked it
4 - It was great
5 - Best movie ever!

Is the data collected categorical or quantitative?

**Solution**

Again, there are numbers associated with the responses, but we can't really do any calculations with them: a movie that rates a 4 is not necessarily twice as good as a movie that rates a 2, whatever that means; if two people see the movie and one of them thinks it stinks and the other thinks it's the best ever it doesn't necessarily make sense to say that "on average they liked it."

As we study movie-going habits and preferences, we shouldn't forget to specify the population under consideration. If we survey 3-7 year-olds the runaway favorite might be *Turning Red*. 13-17 year-olds might prefer *Doctor Strange*. And 33-37 year-olds might prefer *Top Gun*.

### ✎ Try It 7.1.3

Classify each measurement as categorical or quantitative:

a. Eye color of a group of people
b. Daily high temperature of a city over several weeks
c. Annual income

**Answer**

a. Categorical
b. Quantitative
c. Quantitative

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# 7.2: Sampling Methods

**Learning Objectives**

- Understand different methods of sampling
- Understand error from sampling

As we mentioned in a previous section, the first thing we should do before conducting a survey is to identify the population that we want to study. Suppose we are hired by a politician to determine the amount of support he has among the electorate should he decide to run for another term. What population should we study? Every person in the district? Not every person is eligible to vote, and regardless of how strongly someone likes or dislikes the candidate, they don't have much to do with him being re-elected if they are not able to vote.

What about eligible voters in the district? That might be better, but if someone is eligible to vote but does not register by the deadline, they won't have any say in the election either. What about registered voters? Many people are registered but choose not to vote. What about "likely voters?"

This is the criteria used in much political polling, but it is sometimes difficult to define a "likely voter." Is it someone who voted in the last election? In the last general election? In the last presidential election? Should we consider someone who just turned 18 a "likely voter?" They weren't eligible to vote in the past, so how do we judge the likelihood that they will vote in the next election?

In November 1998, former professional wrestler Jesse "The Body" Ventura was elected governor of Minnesota. Up until right before the election, most polls showed he had little chance of winning. There were several contributing factors to the polls not reflecting the actual intent of the electorate:

- Ventura was running on a third-party ticket and most polling methods are better suited to a two-candidate race.
- Many respondents to polls may have been embarrassed to tell pollsters that they were planning to vote for a professional wrestler.
- The mere fact that the polls showed Ventura had little chance of winning might have prompted some people to vote for him in protest to send a message to the major-party candidates.

But one of the major contributing factors was that Ventura recruited a substantial amount of support from young people, particularly college students, who had never voted before and who registered specifically to vote in the gubernatorial election. The polls did not deem these young people likely voters (since in most cases young people have a lower rate of voter registration and a turnout rate for elections) and so the polling samples were subject to **sampling bias**: they omitted a portion of the electorate that was weighted in favor of the winning candidate.

**Sampling bias**

**Sampling bias** occurs if every member of the population doesn't have equal likelihood of being in the sample.

So even identifying the population can be a difficult job, but once we have identified the population, how do we choose an appropriate sample? Remember, although we would prefer to survey all members of the population, this is usually impractical unless the population is very small, so we choose a sample. There are many ways to sample a population, but there is one goal we need to keep in mind: we would like the sample to be *representative of the population*.

Returning to our hypothetical job as a political pollster, we would not anticipate very accurate results if we drew all of our samples from among the customers at a Starbucks, nor would we expect that a sample drawn entirely from the membership list of the local Kiwanis club would provide a useful picture of district-wide support for our candidate.

One way to ensure that the sample has a reasonable chance of mirroring the population is to employ *randomness*. The most basic random method is simple random sampling.

**Simple random sample**

A **random sample** is one in which each member of the population has an equal probability of being chosen. A **simple random sample** is one in which every member of the population and any group of members has an equal probability of being chosen.

✓ Example 7.2.1

How could you obtain a simple random sample of 1000 likely voters in the state?

**Solution**

If we could somehow identify all likely voters in the state, put each of their names on a piece of paper, toss the slips into a (very large) hat and draw 1000 slips out of the hat, we would have a simple random sample.

In practice, computers are better suited for this sort of endeavor than millions of slips of paper and extremely large headgear.

It is always possible, however, that even a random sample might end up not being totally representative of the population. If we repeatedly take samples of 1000 people from among the population of likely voters in the state of California, some of these samples might tend to have a slightly higher percentage of Democrats (or Republicans) than does the general population; some samples might include more older people and some samples might include more younger people; etc. In most cases, this **sampling variability** is not significant.

Sampling variability

The natural variation of samples is called **sampling variability**.

This is unavoidable and expected in random sampling, and in most cases is not an issue.

To help account for variability, pollsters might instead use a **stratified sample**.

Stratified sampling

In **stratified sampling**, a population is divided into a number of subgroups (or strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

✓ Example 7.2.2

Suppose in a particular state that previous data indicated that the electorate was comprised of 39% Democrats, 37% Republicans and 24% independents. How would you obtain a stratified sample of 1000 voters?

**Solution**

In a sample of 1000 people, you would then expect to get about 390 Democrats, 370 Republicans and 240 independents. To accomplish this, you could randomly select 390 people from among those voters known to be Democrats, 370 from those known to be Republicans, and 240 from those with no party affiliation.

Stratified sampling can also be used to select a sample with people in desired age groups, a specified mix ratio of males and females, etc. A variation on this technique is called **quota sampling**.

Quota Sampling

**Quota sampling** is a variation on stratified sampling, wherein samples are collected in each subgroup until the desired quota is met.

✓ Example 7.2.3

How would you implement quota sampling to obtain the stratified sample from the previous example?

**Solution**

Suppose the pollsters call people at random, but once they have met their quota of 390 Democrats, they only gather people who do not identify themselves as a Democrat.

You may have had the experience of being called by a telephone pollster who started by asking you your age, income, etc. and then thanked you for your time and hung up before asking any "real" questions. Most likely, they already had contacted enough people in your demographic group and were looking for people who were older or younger, richer or poorer, etc. Quota sampling is usually a bit easier than stratified sampling, but also does not ensure the same level of randomness.

Another sampling method is **cluster sampling**, in which the population is divided into groups, and one or more groups are randomly selected to be in the sample.

### Cluster sampling

In **cluster sampling**, the population is divided into subgroups (clusters), and a set of subgroups are selected to be in the sample.

In both stratified and cluster sampling, the population is divided into non-overlapping subgroups. The difference between them is that in stratified sampling, subjects from each subgroup are randomly selected, but in cluster sampling, some subgroups are selected and everyone in the subgroup is included in the sample.

### ✓ Example $7.2.4$

How could you obtain a cluster sample of faculty from Fullerton College?

**Solution**

If the college wanted to do a cluster sample of faculty, since faculty are already divided into departments, they could randomly select 10 departments and give the survey to all the faculty in those departments. Each faculty member will be listed in only one department (their primary discipline), even if they are members of multiple departments.

Other sampling methods include **systematic sampling**.

### Systematic sampling

In **systematic sampling**, every $n^{th}$ member of the population is selected to be in the sample.

### ✓ Example $7.2.5$

How could a pollster select a sample using systematic sampling?

**Solution**

The pollster calls every $100^{th}$ name in the phone book. Systematic sampling is not as random as a simple random sample (if your name is Albert Aardvark and your sister Alexis Aardvark is right after you in the phone book, there is no way you could both end up in the sample) but it can yield acceptable samples.

Perhaps the worst types of sampling methods are **convenience samples** and **voluntary response samples**.

### Convenience sampling and voluntary response sampling

In **convenience sampling,** the sample is chosen by selecting whoever is convenient.

In **voluntary response sampling,** the sample is chosen by allowing the respondents to volunteer.

### ✓ Example $7.2.6$

How could a pollster select a sample using convenience sampling?

**Solution**

The pollster stands on a street corner and interviews the first 100 people who agree to speak to him. This is convenient for the pollster but does not generate a sample that is representative of the entire population.

### ✓ Example 7.2.7

How could a researcher collect data on how voters feel about a tax proposal that will be on the ballot using voluntary response sampling?

**Solution**

A website has a survey asking readers to give their opinion on a tax proposal. This is a self-selected sample, or voluntary response sample, in which respondents volunteer to participate.

Usually voluntary response samples are skewed towards people who have a particularly strong opinion about the subject of the survey or who just have way too much time on their hands and enjoy taking surveys.

### Try It 7.2.1

In each case, indicate what sampling method was used:

a. Every 4th person in the class was selected.
b. A sample was selected to contain 25 men and 35 women.
c. Viewers of a new show are asked to vote on the show’s website.
d. A website randomly selects 50 of their customers to send a satisfaction survey.
e. To survey voters in a town, a polling company randomly selects 10 city blocks, and interviews everyone who lives on those blocks.

**Answer**

a. Systematic
b. Stratified or Quota
c. Voluntary response
d. Simple random
e. Cluster

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# 7.3: Sources of Bias

**Learning Objectives**

- Identify the different sources of bias in sampling

There are number of ways that a study can be ruined before you even start collecting data. The first we have already explored – **sampling** or **selection bias**, which is when the sample is not representative of the population. One example of this is **voluntary response bias**, which is bias introduced by only collecting data from those who volunteer to participate. This is not the only potential source of bias.

**Sources of bias**

- **Sampling bias** – when the sample is not representative of the population
- **Voluntary response bias** – the sampling bias that often occurs when the respondents in the sample volunteered to participate
- **Self-interest study** – bias that can occur when the researchers have an interest in the outcome
- **Response bias** – when the respondent gives inaccurate responses for any reason
- **Perceived lack of anonymity** – when the respondent fears giving an honest answer might negatively affect them
- **Loaded questions** – when the question wording influences the responses
- **Non-response bias** – when people refusing to participate in the study can influence the validity of the outcome

**Example 7.3.1**

A recent study which found that chewing gum may raise math grades in teenagers [1]. This study was conducted by the Wrigley Science Institute, a branch of the Wrigley chewing gum company. Identify a potential source of bias for the study.

**Solution**

This is an example of a **self-interest study**; one in which the researchers have a vested interest in the outcome of the study. While this does not necessarily ensure that the study was biased, it certainly suggests that we should subject the study to extra scrutiny.

**Example 7.3.2**

A survey asks people “when was the last time you visited your doctor?” Identify a potential source of bias in this survey.

**Solution**

This survey might suffer from **response bias**, since many people might not remember exactly when they last saw a doctor and give inaccurate responses.

Sources of response bias may be innocent, such as bad memory, or as intentional as pressuring by the pollster. Consider, for example, how many voting initiative petitions people sign without even reading them.

**Example 7.3.3**

A survey asks participants a question about their interactions with members of other races. Identify a potential source of bias in this survey.

**Solution**

Here, a **perceived lack of anonymity** could influence the outcome. The respondent might not want to be perceived as racist even if they are, and give an untruthful answer.

### Try It 7.3.1

An employer puts out a survey asking their employees if they have a drug abuse problem and need treatment help. Identify a potential source of bias in this survey.

**Answer**

Here, both response bias and perceived lack of anonymity may be sources of bias. Answering truthfully might have consequences; responses might not be accurate if the employees do not feel their responses are anonymous or fear retribution from their employer.

### Example 7.3.4

A survey asks “do you support funding research of alternative energy sources to reduce our reliance on high-polluting fossil fuels?” Identify a potential source of bias in this survey.

**Solution**

This is an example of a **loaded** or **leading question** – questions whose wording leads the respondent towards an answer.

Loaded questions can occur intentionally by pollsters with an agenda, or accidentally through poor question wording. Also a concern is **question order**, where the order of questions changes the results. A psychology researcher provides an example [2]:

“My favorite finding is this: we did a study where we asked students, 'How satisfied are you with your life? How often do you have a date?' The two answers were not statistically related - you would conclude that there is no relationship between dating frequency and life satisfaction. But when we reversed the order and asked, 'How often do you have a date? How satisfied are you with your life?' the statistical relationship was a strong one. You would now conclude that there is nothing as important in a student's life as dating frequency.”

### Example 7.3.5

A telephone poll asks the question “Do you often have time to relax and read a book?”, and 50% of the people called refused to answer the survey. Identify a potential source of bias in this poll.

**Solution**

It is unlikely that the results will be representative of the entire population. This is an example of **non-response bias**, introduced by people refusing to participate in a study or dropping out of an experiment. When people refuse to participate, we can no longer be so certain that our sample is representative of the population.

### Try It 7.3.2

In each situation, identify a potential source of bias:

a. A survey asks how many sexual partners a person has had in the last year.
b. A radio station asks readers to phone in their choice in a daily poll.
c. A substitute teacher wants to know how students in the class did on their last test. The teacher asks the 10 students sitting in the front row to state their latest test score.
d. High school students are asked if they have consumed alcohol in the last two weeks.
e. The Beef Council releases a study stating that consuming red meat poses little cardiovascular risk.
f. A poll asks “Do you support a new transportation tax, or would you prefer to see our public transportation system fall apart?”

**Answer**

a. response bias: historically, men are likely to over-report and women are likely to under-report
b. voluntary response bias: self-selection of the sample may not be representative of the entire population
c. sampling bias: the sample may not be representative of the entire class
d. perceived lack of anonymity or response bias: student may fear getting in trouble

e. self-interest study: researchers have an interest in the outcome
f. loaded question: wording leads the respondent toward an answer

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[1] Reuters. news.yahoo.com/s/nm/20090423/...k_gum_learning. Retrieved 4/27/09

[2] Swartz, Norbert. www.umich.edu/~newsinfo/MT/01...01/mt6f01.html. Retrieved 3/31/2009

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# 7.4: Experiments

**Learning Objectives**

- Understand the difference between an observational study and an experiment
- Plan a well-designed experiment
- Minimize confounding in an experiment

So far, we have primarily discussed **observational studies** – studies in which conclusions would be drawn from observations of a sample or the population. In some cases these observations might be unsolicited, such as studying the percentage of cars that turn right at a red light even when there is a "no turn on red" sign. In other cases the observations are solicited, like in a survey or a poll.

In contrast, it is common to use **experiments** when exploring how subjects react to an outside influence. In an experiment, some kind of **treatment** is applied to the subjects and the results are measured and recorded.

**Observational Studies and Experiments**

An **observational study** is a study based on observations or measurements. An **experiment** is a study in which the effects of a **treatment** are measured.

Here are some examples of experiments:

**Example 7.4.1**

a. A pharmaceutical company tests a new medicine for treating Alzheimer's disease by administering the drug to 50 elderly patients with recent diagnoses. The treatment is the new drug.
b. A gym tests out a new weight loss program by enlisting 30 volunteers to try out the program. The treatment is the new program.
c. You test a new kitchen cleaner by buying a bottle and cleaning your kitchen. The new cleaner is the treatment.
d. A psychology researcher explores the effect of music on temperament by measuring people's temperament while listening to different types of music. The music is the treatment.

**Try It 7.4.1**

Is each scenario describing an observational study or an experiment?

a. The weights of 30 randomly selected people are measured.
b. Subjects are asked to do 20 jumping jacks, and then their heart rates are measured.
c. 20 coffee drinkers and 20 tea drinkers are given a concentration test.

**Answer**

a. Observational study; there is no treatment
b. Experiment; the treatment is the jumping jacks
c. Experiment; the treatments are coffee and tea

When conducting experiments, it is essential to isolate the treatment being tested.

**Example 7.4.2**

Suppose a middle school (junior high) finds that their students are not scoring well on the state's standardized math test. How could the school conduct an experiment to improve math test scores?

**Solution**

The school decides to run an experiment to see if an alternate curriculum would improve scores. To run the experiment, they hire a math specialist to come in and teach a class using the new curriculum. To their delight, they see an improvement in test scores. The treatment is the new curriculum.

The difficulty with this scenario is that it is not clear whether the curriculum is responsible for the improvement, or whether the improvement is due to a math specialist teaching the class. This is called **confounding** – when it is not clear which factor or factors caused the observed effect. Confounding is the downfall of many experiments, though sometimes it is hidden.

### Confounding

**Confounding** occurs when there are two or more potential variables that could have caused the outcome and it is not possible to determine which actually caused the result.

### ✓ Example 7.4.3

A drug company study about a weight loss pill might report that people lost an average of 8 pounds while using their new drug. How could confounding be a problem with their weight loss claim?

**Solution**

In the fine print you find a statement saying that participants were encouraged to also diet and exercise. It is not clear in this case whether the weight loss is due to the pill, to diet and exercise, or a combination of both.

### Try It 7.4.2

Researchers conduct an experiment to determine whether students will perform better on an arithmetic test if they listen to music during the test. They first give the student a test without music, then give a similar test while the student listens to music. How could confounding be a problem with this experiment?

**Answer**

> In this case, the student might perform better on the second test, regardless of the music, simply because it was the second test and they were warmed up.

There are a number of measures that can be introduced to help reduce the likelihood of confounding. The primary measure is to use a **control group**.

### Control Group

When conducting an experiment, the participants are divided into two or more groups, typically a **control group** and a treatment group. The treatment group receives the treatment being tested; the control group does not receive the treatment.

Ideally, the groups are otherwise as similar as possible, isolating the treatment as the only potential source of difference between the groups. For this reason, the method of dividing groups is important. Some researchers attempt to ensure that the groups have similar characteristics (same number of females, same number of people over 50, etc.), but it is nearly impossible to control for every characteristic. Because of this, random assignment is very commonly used.

### ✓ Example 7.4.4

How could a control group be used in an experiment to determine if a 2-day prep course would help high school students improve their scores on the SAT test?

**Solution**

A group of students takes the SAT and their scores are recorded. The group is then randomly divided into 2 subgroups. The first group, the treatment group, is given a 2-day prep course. The second group, the control group, is not given the prep course. Afterwards, both groups take the SAT again and their before-and-after scores are compared.

**Try It 7.4.3**

How could a control group be used in an experiment conducted by a company testing a new plant food?

**Answer**

They grow 2 crops of plants in adjacent fields, the treatment group receiving the new plant food and the control group not. The crop yield would then be compared. By growing them at the same time in adjacent fields, they are controlling for weather and other confounding factors.

Sometimes not giving the control group anything does not completely control for confounding variables. For example, suppose a medicine study is testing a new headache pill by giving the treatment group the pill and the control group nothing. If the treatment group showed improvement, we would not know whether it was due to the medicine in the pill, or a response to having taken any pill. This is called a **placebo effect**.

**Placebo Effect**

The **placebo effect** is when the effectiveness of a treatment is influenced by the patient's perception of how effective they think the treatment will be, so a result might be seen even if the treatment is ineffectual.

**Example 7.4.5**

A study found that when doing painful dental tooth extractions, patients told they were receiving a strong painkiller while actually receiving a saltwater injection found as much pain relief as patients receiving a dose of morphine. [1]

To control for the placebo effect, a **placebo**, or dummy treatment, is often given to the control group. This way, both groups are truly identical except for the specific treatment given.

**Placebo and Placebo-controlled experiments**

A **placebo** is a dummy treatment given to control for the placebo effect. An experiment that gives the control group a placebo is called a **placebo-controlled experiment**.

Here are some examples of placebo-controlled experiments:

**Example 7.4.6**

a. In a study for a new medicine that is dispensed in a pill form, a sugar pill could be used as a placebo.
b. In a study on the effect of alcohol on memory, a non-alcoholic beer might be given to the control group as a placebo.
c. In a study of a frozen meal diet plan, the treatment group would receive the diet food, and the control could be given standard frozen meals stripped of their original packaging.

In some cases, it is more appropriate to compare to a conventional treatment than a placebo. For example, in a cancer research study, it would not be ethical to deny any treatment to the control group or to give a placebo treatment. In this case, the currently acceptable medicine would be given to the control group, sometimes called a **comparison group** in this case. In our SAT test example, the non-treatment group would most likely be encouraged to study on their own, rather than be asked to not study at all, to provide a meaningful comparison.

When using a placebo, it would defeat the purpose if the participant knew they were receiving the placebo.

**Blind studies**

A **blind study** is one in which the participant does not know whether or not they are receiving the treatment or a placebo. A **double-blind study** is one in which those interacting with the participants don't know who is in the treatment group and who is in the control group.

## ✓ Example 7.4.7

In a study about anti-depression medicine, you would not want the psychological evaluator to know whether the patient is in the treatment or control group either, as it might influence their evaluation, so the experiment should be conducted as a double-blind study.

It should be noted that not every experiment needs a control group.

## ✓ Example 7.4.8

If a researcher is testing whether a new fabric can withstand fire, she simply needs to torch multiple samples of the fabric – there is no need for a control group.

## Try It 7.4.4

To test a new lie detector, two groups of subjects are given the new test. One group is asked to answer all the questions truthfully, and the second group is asked to lie on one set of questions. The person administering the lie detector test does not know what group each subject is in.

Does this experiment have a control group? Is it blind, double-blind, or neither?

**Answer**

> The truth-telling group could be considered the control group, but really both groups are treatment groups here, since it is important for the lie detector to be able to correctly identify lies, and also not identify truth telling as lying. This study is blind, since the person running the test does not know what group each subject is in.

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[1] Levine JD, Gordon NC, Smith R, Fields HL. (1981) Analgesic responses to morphine and placebo in individuals with postoperative pain. Pain. 10:379-89.

# 7.5: Chapter Review and Glossary

# 7.6: Exercises

## Skills

1. A political scientist surveys 28 of the current 106 representatives in a state's congress. Of them, 14 said they were supporting a new education bill, 12 said there were not supporting the bill, and 2 were undecided.
   a. What is the population of this survey?
   b. What is the size of the population?
   c. What is the size of the sample?
   d. Give the sample statistic for the proportion of voters surveyed who said they were supporting the education bill.
   e. Based on this sample, we might expect how many of the representatives to support the education bill?
2. The city of Raleigh has 9500 registered voters. There are two candidates for city council in an upcoming election: Brown and Feliz. The day before the election, a telephone poll of 350 randomly selected registered voters was conducted. 112 said they'd vote for Brown, 207 said they'd vote for Feliz, and 31 were undecided.
   a. What is the population of this survey?
   b. What is the size of the population?
   c. What is the size of the sample?
   d. Give the sample statistic for the proportion of voters surveyed who said they'd vote for Brown.
   e. Based on this sample, we might expect how many of the 9500 voters to vote for Brown?
3. Identify the most relevant source of bias in this situation: A survey asks the following: Should the mall prohibit loud and annoying rock music in clothing stores catering to teenagers?
4. Identify the most relevant source of bias in this situation: To determine opinions on voter support for a downtown renovation project, a surveyor randomly questions people working in downtown businesses.
5. Identify the most relevant source of bias in this situation: A survey asks people to report their actual income and the income they reported on their IRS tax form.
6. Identify the most relevant source of bias in this situation: A survey randomly calls people from the phone book and asks them to answer a long series of questions.
7. Identify the most relevant source of bias in this situation: A survey asks the following: Should the death penalty be permitted if innocent people might die?
8. Identify the most relevant source of bias in this situation: A study seeks to investigate whether a new pain medication is safe to market to the public. They test by randomly selecting 300 men from a set of volunteers.
9. In a study, you ask the subjects their age in years. Is this data qualitative or quantitative?
10. In a study, you ask the subjects their gender. Is this data qualitative or quantitative?
11. Does this describe an observational study or an experiment: The temperature on randomly selected days throughout the year was measured.
12. Does this describe an observational study or an experiment? A group of students are told to listen to music while taking a test and their results are compared to a group not listening to music.
13. In a study, the sample is chosen by separating all cars by size, and selecting 10 of each size grouping. What is the sampling method?
14. In a study, the sample is chosen by writing everyone’s name on a playing card, shuffling the deck, then choosing the top 20 cards. What is the sampling method?
15. A team of researchers is testing the effectiveness of a new HPV vaccine. They randomly divide the subjects into two groups. Group 1 receives new HPV vaccine, and Group 2 receives the existing HPV vaccine. The patients in the study do not know which group they are in.
   a. Which is the treatment group?
   b. Which is the control group (if there is one)?
   c. Is this study blind, double-blind, or neither?

d. Is this best described as an experiment, a controlled experiment, or a placebo controlled experiment?

16. For the clinical trials of a weight loss drug containing *Garcinia cambogia t*he subjects were randomly divided into two groups. The first received an inert pill along with an exercise and diet plan, while the second received the test medicine along with the same exercise and diet plan. The patients do not know which group they are in, nor do the fitness and nutrition advisors.
    a. Which is the treatment group?
    b. Which is the control group (if there is one)?
    c. Is this study blind, double-blind, or neither?
    d. Is this best described as an experiment, a controlled experiment, or a placebo controlled experiment?

## Concepts

17. A teacher wishes to know whether the males in his/her class have more conservative attitudes than the females. A questionnaire is distributed assessing attitudes.
    a. Is this a sampling or a census?
    b. Is this an observational study or an experiment?
    c. Are there any possible sources of bias in this study?
18. A study is conducted to determine whether people learn better with spaced or massed practice. Subjects volunteer from an introductory psychology class. At the beginning of the semester 12 subjects volunteer and are assigned to the massed-practice group. At the end of the semester 12 subjects volunteer and are assigned to the spaced-practice condition.
    a. Is this a sampling or a census?
    b. Is this an observational study or an experiment?
    c. This study involves two kinds of non-random sampling: (1) Subjects are not randomly sampled from some specified population and (2) Subjects are not randomly assigned to groups. Which problem is more serious? What affect on the results does each have?
19. A farmer believes that playing Barry Manilow songs to his peas will increase their yield. Describe a controlled experiment the farmer could use to test his theory.
20. A sports psychologist believes that people are more likely to be extroverted as adults if they played team sports as children. Describe two possible studies to test this theory. Design one as an observational study and the other as an experiment. Which is more practical?

## Exploration

21. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 50 AIDS patients from the start of treatment until their deaths.
    a. What is the population of this study?
    b. List two reasons why the data may differ.
    c. Can you tell if one researcher is correct and the other one is incorrect? Why?
    d. Would you expect the data to be identical? Why or why not?
    e. If the first researcher collected her data by randomly selecting 40 states, then selecting 1 person from each of those states. What sampling method is that?
    f. If the second researcher collected his data by choosing 40 patients he knew. What sampling method would that researcher have used? What concerns would you have about this data set, based upon the data collection method?
22. Find a newspaper or magazine article, or the online equivalent, describing the results of a recent study (the results of a poll are not sufficient). Give a summary of the study’s findings, then analyze whether the article provided enough information to determine the validity of the conclusions. If not, produce a list of things that are missing from the article that would help you determine the validity of the study. Look for the things discussed in the text: population, sample, randomness, blind, control, placebos, etc.

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# CHAPTER OVERVIEW

## 8: Describing Data

Once we have collected data from surveys or experiments, we need to summarize and present the data in a way that will be meaningful to the reader. We will begin with graphical presentations of data then explore numerical summaries of data.

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# 8.1: Presenting Categorical Data Graphically

**Learning Objectives**

- Summarize categorical data using a frequency table
- Construct a bar graph, Pareto chart and pie chart
- Recognize misleading graphs

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table.**

## Frequency Table

**Frequency Table**

A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

**Example 8.1.1**

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. The data is summarized in the frequency table below.

**Vehicle color involved in total-loss collision**

| Color | Frequency |
|---|---|
| Blue | 25 |
| Green | 52 |
| Red | 41 |
| White | 36 |
| Black | 39 |
| Grey | 23 |

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. In this section we will work with bar graphs that display categorical data; the next section will be devoted to bar graphs that display quantitative data.

## Bar Graph

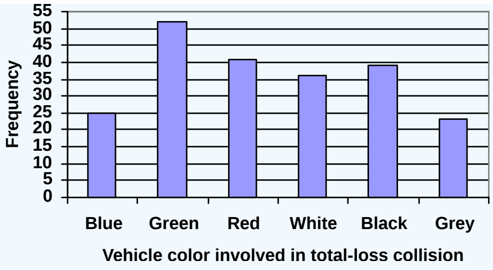
**Bar graph**

A **bar graph** is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

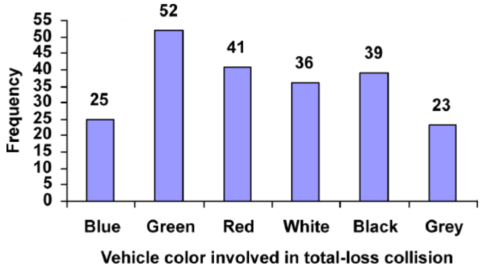
To construct a bar graph, we need to draw a vertical axis and a horizontal axis. The vertical direction will have a scale and measure the frequency of each category; the horizontal axis has no scale in this instance. **Be sure to always give any graph a title and label the axes.** The construction of a bar chart is most easily described by use of an example.

**Example 8.1.2**

Using our car data from above, note the highest frequency is 52, so our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units:



Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal gridlines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the gridlines will already be in place, or using technology. Instead of gridlines, we might also list the frequencies at the top of each bar, like this:



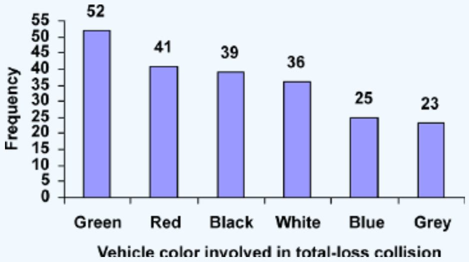
In this case, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart, even without gridlines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart**.

### Pareto chart

A **Pareto chart** is a bar graph ordered from highest to lowest frequency.

### Example 8.1.3

Transforming our bar graph from earlier into a Pareto chart, we get:



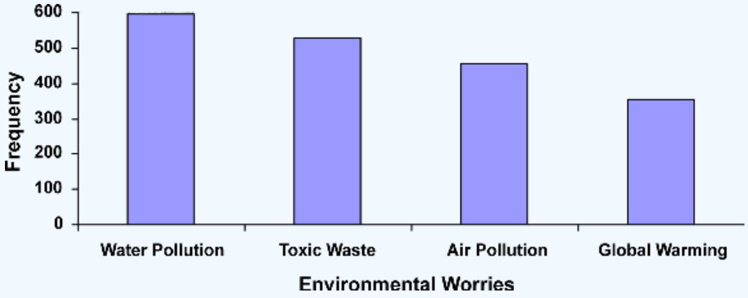
### Example 8.1.4

In a survey [1], adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of 1012 surveyed) who indicated that they worried "a great deal" about some selected concerns are summarized below. Construct a bar graph of the survey data.

| Environmental Issue | Frequency |
|---|---|
| Pollution of drinking water | 597 |
| Contamination of soil and water by toxic waste | 526 |
| Air pollution | 455 |
| Global warming | 354 |

**Solution**

This data could be shown graphically in a bar graph:



## Pie Chart

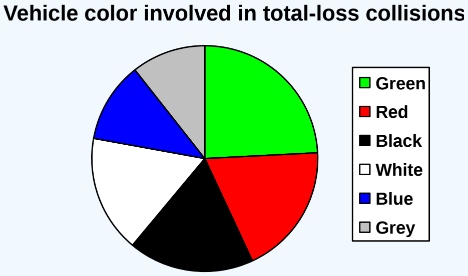
To show relative sizes, it is common to use a pie chart.

### Pie Chart

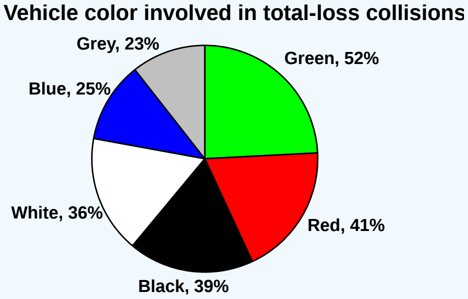
A **pie chart** is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies (or percents) of the categories.

### Example 8.1.5

For our vehicle color data, a pie chart might look like this:

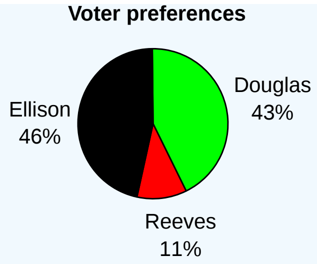


Pie charts can often benefit from including frequencies or relative frequencies (percents) in the chart next to the pie slices. Often having the category names next to the pie slices also makes the chart clearer.



### Example 8.1.6

The pie chart to the below shows the percentage of voters supporting each candidate running for a local senate seat.



If there are 20,000 voters in the district, the pie chart shows that about 11% of those, about 2,200 voters, support Reeves.

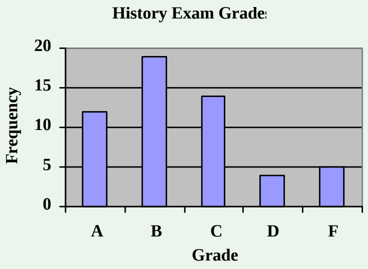
Pie charts look nice, but are harder to draw by hand than bar charts since to draw them accurately we would need to compute the angle each wedge cuts out of the circle, then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org Write or Calc, or Google Docs are able to create bar graphs, pie charts, and other graph types. There are also numerous online tools that can create graphs [2].

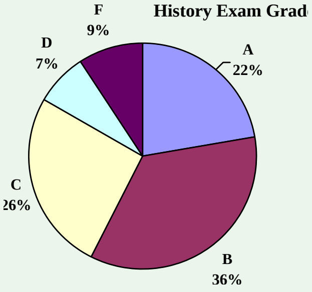
### ✎ Try It 8.1.1

Create a bar graph and a pie chart to illustrate the grades on a history exam below.

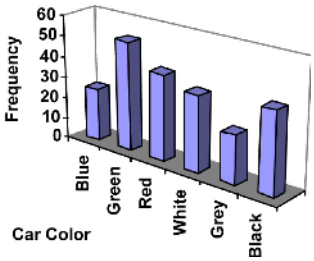
A: 12 students, B: 19 students, C: 14 students, D: 4 students, F: 5 students

**Answer**





Don’t get fancy with graphs! People sometimes add features to graphs that don’t help to convey their information. For example, 3-dimensional bar charts like the one shown below are usually not as effective as their two-dimensional counterparts.



## Misleading Graphs

Here is another way that fanciness can lead to trouble. Instead of plain bars, it is tempting to substitute meaningful images. This type of graph is called a **pictogram**.

### Pictogram

A **pictogram** is a statistical graphic in which the size of the picture is intended to represent the frequencies or size of the values being represented.

### ✓ Example 8.1.7

A labor union might produce the graph below to show the difference between the average manager salary and the average worker salary. What is deceptive about the graph?

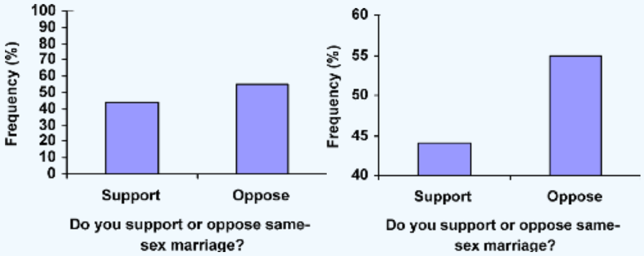
**Solution**

Looking at the picture, it would be reasonable to guess that the manager salaries is 4 times as large as the worker salaries – the area of the bag looks about 4 times as large. However, the manager salaries are in fact only twice as large as worker salaries, which were reflected in the picture by making the manager bag twice as tall.

Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

### ✓ Example 8.1.8

Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December 2008 [3]. What is deceptive about the second graph?

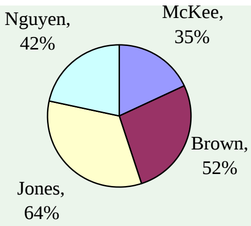


**Solution**

The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages; the second graph makes it look like twice as many people oppose marriage rights as support it.

### Try It 8.1.2

A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart present a good representation of this data? Explain.



**Answer**

While the pie chart accurately depicts the relative size of the people agreeing with each candidate, the chart is confusing, since usually percents on a pie chart represent the percentage of the pie the slice represents.

---

[1] Gallup Poll. March 5-8, 2009. http://www.pollingreport.com/enviro.htm

[2] For example: http://nces.ed.gov/nceskids/createAgraph/ or http://docs.google.com

[3]CNN/Opinion Research Corporation Poll. Dec 19-21, 2008, from http://www.pollingreport.com/civil.htm

# 8.2: Presenting Quantitative Data Graphically

**Learning Objectives**

- Summarize quantitative data using a frequency distribution
- Construct a histogram, frequency polygon and stem plot

## Frequency Distributions

Quantitative, or numerical, data can also be summarized into frequency tables, also known as frequency distributions.

**Example 8.2.1**

A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

Construct a frequency table for the data.

### Solution

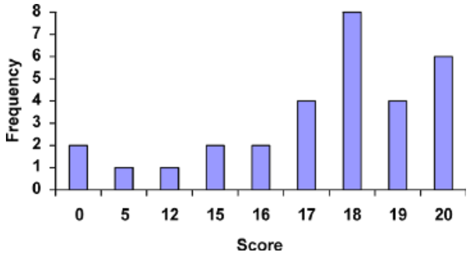
These scores could be summarized into a frequency table by grouping like values:

| Score | Frequency |
|---|---|
| 0 | 2 |
| 5 | 1 |
| 12 | 1 |
| 15 | 2 |
| 16 | 2 |
| 17 | 4 |
| 18 | 8 |
| 19 | 4 |
| 20 | 6 |

## Histograms

Using the table above, it would be possible to create a standard bar chart from this summary, like we did for categorical data:

**Class Quiz Scores**



However, since the scores are numerical values, this chart doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be more correct to treat the horizontal axis as a number line. This type of graph is called a **histogram**.

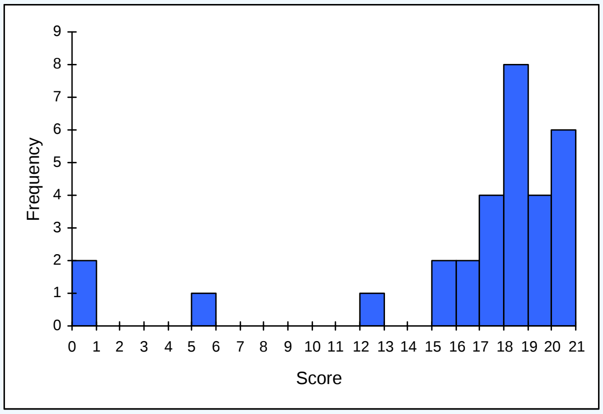
**Histogram**

A **histogram** is a graphical representation of quantitative data, similar to a bar graph. The horizontal axis is a number line and the bars are touching.

### ✓ Example 8.2.2

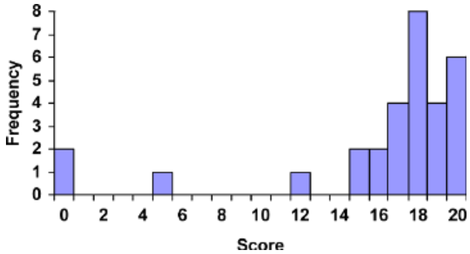
For the values above, a histogram would look like:

**Class Quiz Scores**



Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. Some people choose to have bars start at $\frac{1}{2}$ values to avoid this ambiguity.

**Class Quiz Scores**



Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table, and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals**.

### Class Intervals

**Class intervals** are groupings of the data. In general, we define class intervals so that:

- Each interval is equal in size. For example, if the first class contains values from 120-129, the second class should include values from 130-139.
- Each interval has a **lower limit** and an **upper limit**. For example, for the class interval of 120-129, the lower limit is 120 and the upper limit is 129.
- The size of the interval is called the **class width**. It is the difference between 2 consecutive lower limits. For example, the class width for a class interval of 120-129 is 10 since the next class interval starts at 130 (and 130 - 120 = 10).
- We have between 5 and 20 classes, typically, depending upon the number of data we’re working with.

### ✓ Example 8.2.3

Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total range of $263 - 121 = 142$. Construct a frequency distribution for the data and a histogram.

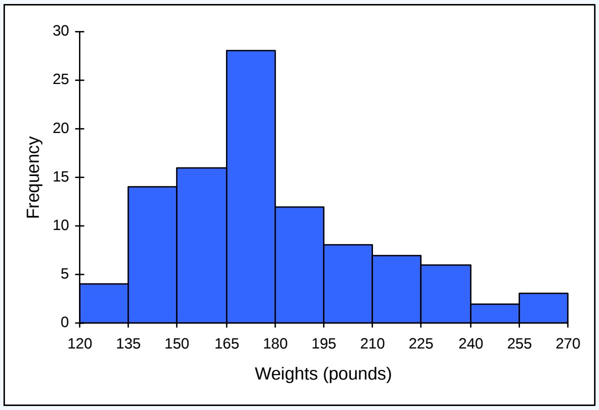
**Solution**

We could create 7 intervals with a width of around 20, 14 intervals with a width of around 10, or somewhere in between. Often times we have to experiment with a few possibilities to find something that represents the data well. Let us try using a class width of 15. We could start at 121, or at 120 since it is a nice round number. The second class interval will start at 120 + 15 = 135.

| Interval | Frequency |
|---|---|
| 120 – 134 | 4 |
| 135 – 149 | 14 |
| 150 – 164 | 16 |
| 165 – 179 | 28 |
| 180 – 194 | 12 |
| 195 – 209 | 8 |
| 210 – 224 | 7 |
| 225 – 239 | 6 |
| 240 – 254 | 2 |
| 255 – 269 | 3 |

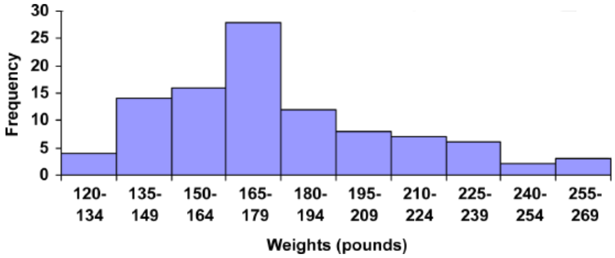
A histogram of this data would look like:

**Weights of Subjects in Nutrition Study**

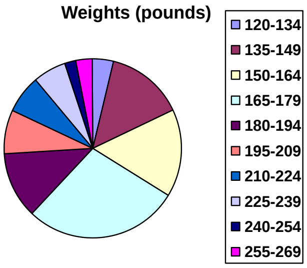


In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.

**Weights of Subjects in Nutrition Study**



Other graph types such as pie charts are possible for quantitative data. The usefulness of different graph types will vary depending upon the number of intervals and the type of data being represented. For example, a pie chart of our weight data is difficult to read because of the quantity of intervals we used.



## Try It 8.2.1

The total cost of textbooks for the term was collected from 36 students. Create a histogram for this data.

\$140 \$160 \$160 \$165 \$180 \$220 \$235 \$240 \$250 \$260 \$280 \$285

\$285 \$285 \$290 \$300 \$300 \$305 \$310 \$310 \$315 \$315 \$320 \$320

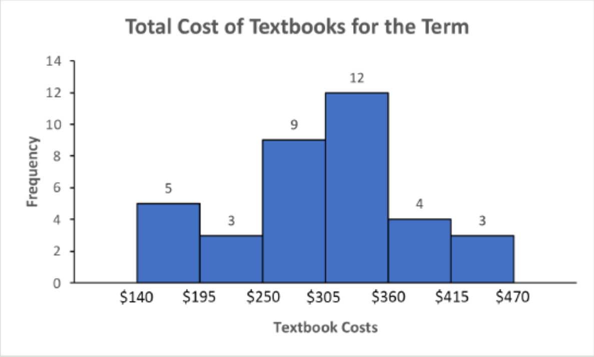
\$330 \$340 \$345 \$350 \$355 \$360 \$360 \$380 \$395 \$420 \$460 \$460

**Answer**

Using a class intervals of size 55, we can group our data into 6 intervals:

| cost interval | Frequency |
|---|---|
| \$140 – 194 | 5 |
| \$195 – 249 | 3 |
| \$250 – 304 | 9 |
| \$305 – 359 | 12 |
| \$360 – 414 | 4 |
| \$415 – 469 | 3 |

We can use the frequency distribution to generate the histogram.



When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

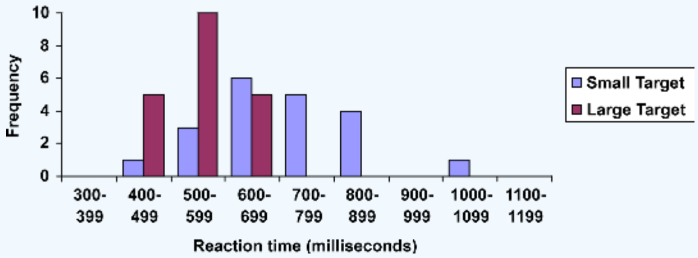
## Example 8.2.4

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

| Interval (milliseconds) | Frequency small target | Frequency large target |
|---|---|---|
| $300-399$ | 0 | 0 |
| $400-499$ | 1 | 5 |
| $500-599$ | 3 | 10 |
| $600-699$ | 6 | 5 |
| $700-799$ | 5 | 0 |
| $800-899$ | 4 | 0 |
| $900-999$ | 0 | 0 |
| $1000-1099$ | 1 | 0 |
| $1100-1199$ | 0 | 0 |

One option to represent this data would be a comparative histogram or side-by-side bar chart, in which bars for the small target group and large target group are placed next to each other.

**Reaction Time for Small and Large Targets**



## Frequency Polygons

An alternative representation is a **frequency polygon**.

### Frequency polygon

A **frequency polygon** is a line graph of a frequency distribution.

It starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval with height equal to the frequency. The midpoint of a class interval is

$$\text{class midpoint} = \frac{\text{lower limit}_1 + \text{lower limit}_2}{2}$$

The points are connected with straight lines to emphasize the distribution of the data. By definition, a polygon is a closed figure, so the graph is "closed" on both ends by connecting the first and last points back to 0 (the *x*-axis) at the appropriate interval midpoint before the first and last class intervals.

### Example 8.2.5

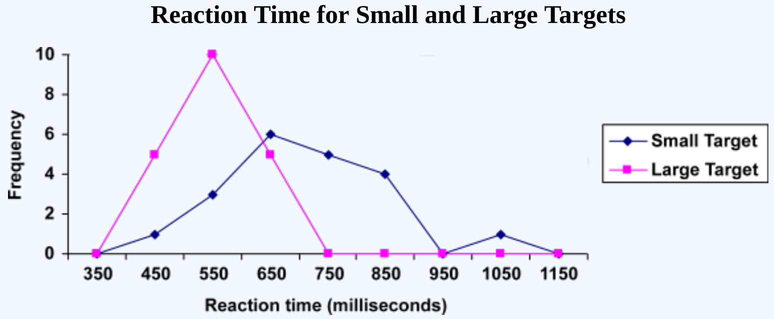
Construct a frequency polygon for both small and large targets on the same graph using the data on reaction time from the previous example.

**Solution**

Find the midpoint of the first class interval: $\text{class midpoint} = \frac{400+500}{2} = \frac{900}{2} = 450$. Since the class width is $500-400=100$, add 100 to find the second midpoint: 450 + 100 = 550. Find the rest of the midpoints, including the midpoint of the class before the first class interval $(450-100=350)$ and the midpoint of the class after the last class interval $(1050+100=1150)$ where the polygons will connect back to the $x$-axis. Plot the midpoints as $x$-coordinates and frequencies as $y$-coordinates and connect the points with straight lines. The table below shows the midpoints and frequencies.

| Midpoint (milliseconds) | Frequency small target | Frequency large target |
|---|---|---|
| 350 | 0 | 0 |
| 450 | 1 | 5 |
| 550 | 3 | 10 |
| 650 | 6 | 5 |
| 750 | 5 | 0 |
| 850 | 4 | 0 |
| 950 | 0 | 0 |
| 1050 | 1 | 0 |
| 1150 | 0 | 0 |

The completed graph is shown below.



This graph makes it easier to see that reaction times were generally shorter for the larger target, and that the reaction times for the smaller target were more spread out.

## Stem Plots

Stem-and-leaf plots, or stem plots, are a quick and easy way to look at small samples of numerical data. You can look for any patterns or any strange data values. It is easy to compare two samples using stem plots.

The first step is to divide each number into 2 parts, the stem (such as the leftmost digit) and the leaf (such as the rightmost digit). There are no set rules, you just have to look at the data and see what makes sense.

### ✓ Example 8.2.6

The following are the percentage grades of 25 students from a statistics course. Draw a stem plot of the data.

Table 8.2.1: Data of Test Grades

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 62 | 87 | 81 | 69 | 87 | 62 | 45 | 95 | 76 | 76 |
| 62 | 71 | 65 | 67 | 72 | 80 | 40 | 77 | 87 | 58 |
| 84 | 73 | 93 | 64 | 89 | | | | | |

**Solution**

Divide each number so that the tens digit is the stem and the ones digit is the leaf. 62 becomes 6|2.

Make a vertical chart with the stems on the left of a vertical bar. Be sure to fill in any missing stems. In other words, the stems should have equal spacing (for example, count by ones or count by tens). Here is what the stems for our data look like:

| | |
|---|---|
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |

Now go through the list of data and add the leaves. Put each leaf next to its corresponding stem. Don’t worry about order yet, just get all the leaves down.

When the data value 62 is placed on the plot it looks like the plot below.

| | |
|---|---|
| 4 | |
| 5 | |
| 6 | 2 |
| 7 | |
| 8 | |
| 9 | |

When the data value 87 is placed on the plot it looks like the plot below.

| | |
|---|---|
| 4 | |
| 5 | |
| 6 | 2 |
| 7 | |
| 8 | 7 |
| 9 | |

Filling in the rest of the leaves to obtain the plot below.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 5 | 0 | | | | | |
| 5 | 8 | | | | | | |
| 6 | 2 | 9 | 2 | 2 | 5 | 7 | 4 |
| 7 | 6 | 6 | 1 | 2 | 7 | 3 | |
| 8 | 7 | 1 | 7 | 0 | 7 | 4 | 9 |
| 9 | 5 | 3 | | | | | |

Now you have to add labels and make the graph look pretty. You need to add a title and sort the leaves into increasing order. You also need to tell people what the stems and leaves mean by inserting a key. **Be careful to line the leaves up in columns**. You need to be able to compare the lengths of the rows when you interpret the graph. The final stem plot for the test grade data is shown below.

Test Grades

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 0 | 5 | | | | | |
| 5 | 8 | | | | | | |
| 6 | 2 | 2 | 2 | 4 | 5 | 7 | 9 |
| 7 | 1 | 2 | 3 | 6 | 6 | 7 | |
| 8 | 0 | 1 | 4 | 7 | 7 | 7 | 9 |
| 9 | 3 | 5 | | | | | |

key: 4|0 = 40%

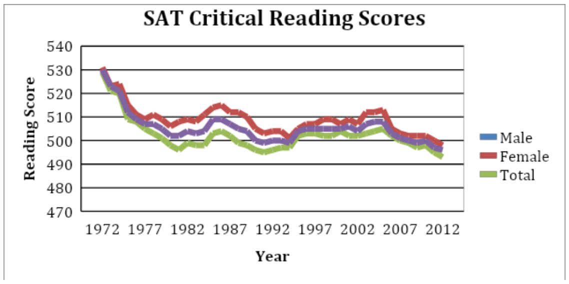
# 8.3: Graphics in the Media

There are many other types of graphs you will encounter in the media.

## Multiple Line Graphs

A line graph is useful for seeing trends over time. Multiple line graphs are useful for seeing trends over time, and also comparing two or more data sets. As an example, suppose you want to examine the average SAT critical reading score over time for Arizona students, but you further want to compare the averages overall and between genders. So, a multiple line graph like the following may be used. As you can see, the average score for SAT critical reading has been going down over the years. You can also see that the average score for male students is higher than the average score for female students for all of the years. The other interesting aspect that you can see is that the average scores for male and female used to be closer to each other, then they separated fairly far, and look to be getting closer to each other again. One last comment is that even though there is a difference between male and female average score, the average scores seem to follow each other. In other words, when the male score was going down, so was the female score. Do be careful. Do not try to make up a reason for the scores to go down. You cannot say why the scores have decreased, since you did not run an experiment. The scores could have decreased because our education system is not teaching as well, funding has decreased for education, or the intelligence of Arizona students has decreased. Or it could be that the percentage of the overall student body that takes the SAT has increased over the years, meaning that more than just the highest ranked students have been taking the SAT in later years, which could lower the averages. Any one of these reasons, or other reasons, could be the right one, and you cannot determine which it is. Do not make unsubstantiated claims. *Note: A next step in analyzing this data could be to compare the AZ trends to national trends.*

**Graph 1.6.1: Multiple Line Graph for SAT Critical Reading Scores in AZ**



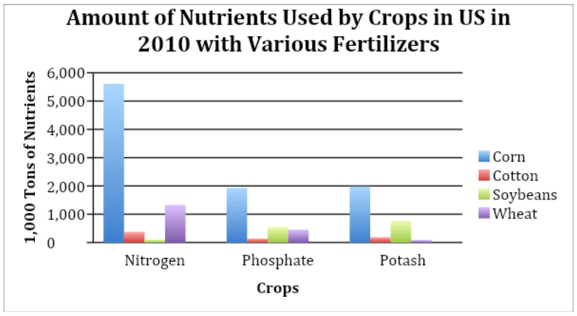
(College Board: Arizona, 2012)

*Note: In this case the vertical axis did not start at zero, because if it did start at zero, the different lines would be very close together and difficult to see. When at all possible, the scale on the vertical axis should start at zero. Always carefully consider that if a vertical scale of a graph does not start at zero, the researchers could be attempting to exaggerate insignificant differences in the data.*

**Multiple Bar Graphs:**

Sometimes you have information for multiple variables and instead of putting the information on different bar graphs, you can put them all on one so that you can compare the variables. The following is an example of where you might use this. The data is the amount of nutrients used by crops with various fertilizers. By creating a graph that has all of the fertilizers and all of the crops, you can see that corn with nitrogen uses the most nutrients, and soybeans with nitrogen uses the least amount of nutrients. You can also see that phosphate seems to use low amounts of nutrients for all of its crops. So, a multiple bar graph is useful to make all of these observations.

**Graph 1.6.2: Multiple Bar Graph for the Amount of Nutrients Used by Crops: 2010**

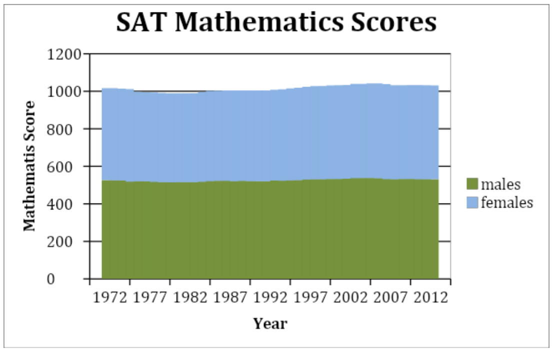


(United States Department of Agriculture [USDA], 2010)

**Stack Plots:**

A stack plot is basically a multiple line graph, but with the lines separated (or stacked) on top of each other instead of overlapping. This can be useful when it is difficult to interpret a multiple line graph since the lines are so close to one another. To read a given line on a stack plot, you must subtract that line from the line below it. In the example of a stack plot below, you can see that the SAT Mathematics score for males in 1972 is about 530 while the SAT Mathematics score for females in 1972 is about 1010 – 530 = 480.

**Graph 1.6.3: Stack Plot of SAT Mathematics Scores**



(College Board: Arizona, 2012)

**Geographical Graphs:**

Weather maps, topographic maps, population distribution maps, gravity maps, and vegetation maps are examples of geographical graphs. They allow you to see a trend of information over a geographic area. The following is an example of a weather map showing temperatures. As you can see, the different colors represent certain temperature ranges. From this graph, you can see that on this date, the red in the south means that the temperature was in the 80s and 90s there, and the blue in the Rockies area means that the temperature was in the 40s there.

**Graph 1.6.4: Geographical Graph**

(Weather Channel, 2013)

**Three-Dimensional Graphics**

Some people like to show a bar graph in three-dimensions. Occasionally, a three-dimensional graph is used to graph three variables together on three axes, but this type of graph may be difficult to read. The following graph just represents two variables and so it is basically the same as a standard bar graph, but the three-dimensional look may add a bit more style.

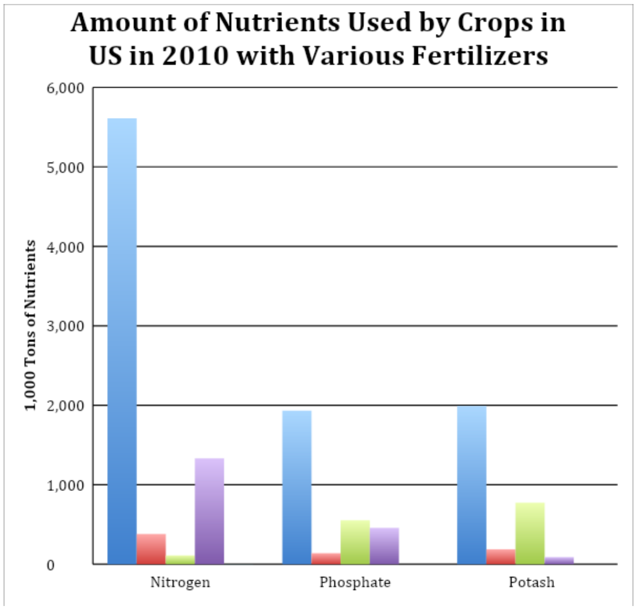
**Graph 1.6.5: Three-Dimensional Graph**

(USDA, 2010)

### Combination Graphics

Some graphs are created so that they combine the data table and a graph or combine two types of graphs in one. The advantage is that you can see a graphical representation of the data, and still have the data to find exact values. The disadvantage is that they are busier, and usually people show graphical representation of data because people do not like looking at the data.

**Graph 1.6.6: Combination Graph**



(USDA, 2010)

These are just a few of the different types of graphs that exist in the world. There are many other ones. A quick Google search on statistical graphs will show you many more. Just open up a newspaper, magazine, or website and you are likely to see others. The most important thing to remember is that you need to look at the graph objectively, and interpret for yourself what it says. Also, do not read any cause and effect into what you see.

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# 8.4: Measures of Central Tendency

**Learning Objectives**

- Find the mean of a data set
- Find the median of a data set
- Find the mode of a data set

It is often desirable to use a few numbers to summarize a distribution. One important aspect of a distribution is where its center is located. Numbers that describe a distribution's center are called measures of central tendency.

Let's begin by trying to find the most "typical" value of a data set.

## Mean

Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average".

**Mean**

The **mean** of a set of data is the sum of the data values divided by the number of values.

We will use $n$ to represent the number of values in a data set.

**Example 8.4.1**

Marci's exam scores for her last math class were: 79, 86, 82, 94. Find the mean of her exam scores.

**Solution**

$$\frac{79+86+82+94}{4}=\frac{341}{4}=85.25.$$

Typically we round the mean to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3. Marci's mean exam score was 85.3.

**Example 8.4.2**

The number of touchdown (TD) passes thrown by each of the 32 teams in the National Football League (NFL) in the 2021 season are shown below [1].

40 43 38 36 37 39 36 27 41 27 24 20 26 22 21 12

30 34 21 29 21 21 21 20 23 20 23 23 16 20 14 15

Find the mean number of touchdown passes thrown in the NFL in the 2021 season.

**Solution**

Adding these values, we get 840 total TDs. Dividing by $n$ = 32, the number of data values, we get $\frac{840}{32}=26.25$. It would be appropriate to round this to 26.3.

It would be most correct for us to report that "The mean number of touchdown passes thrown per team in the NFL in the 2021 season was 26.3 passes," but it is not uncommon to see the more casual word "average" used in place of "mean.

### Try It 8.4.1

The price of a jar of peanut butter at 5 stores was: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the mean price.

**Answer**

$$\frac{\$3.29+\$3.59+\$3.79+\$3.75+\$3.99}{5}=\frac{\$18.41}{5}=\$3.682$$

Let's look at an example for calculating the mean given a frequency table.

### Example 8.4.3

The 100 families in a particular neighborhood are asked their annual household income, to the nearest \$5 thousand dollars. The results are summarized in a frequency table below.

| Income (thousands of dollars) | Frequency |
|---|---|
| 15 | 6 |
| 20 | 8 |
| 25 | 11 |
| 30 | 17 |
| 35 | 19 |
| 40 | 20 |
| 45 | 12 |
| 50 | 7 |

Find the mean annual household income.

**Solution**

Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\frac{15+15+15+15+15+15+20+20+\ldots+50+50+50+50+50+50+50}{100}$$

We could calculate this more easily by noticing that adding 15 to itself 6 times is the same as $15\cdot 6=90$. Using this simplification, we get

$$\frac{15\cdot 6+20\cdot 8+25\cdot 11+30\cdot 17+35\cdot 19+40\cdot 20+45\cdot 12+50\cdot 7}{100}=\frac{3390}{100}=33.9$$

The mean household income of our sample is 33.9 thousand dollars (\$33,900).

### Example 8.4.4

Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of \$5 million (\$5000 thousand). What is the new mean household income?

**Solution**

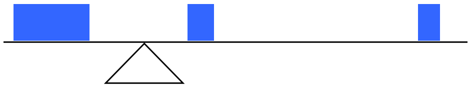
Adding this to our sample, our mean is now:

$$\frac{15\cdot 6+20\cdot 8+25\cdot 11+30\cdot 17+35\cdot 19+40\cdot 20+45\cdot 12+50\cdot 7+5000\cdot 1}{101}=\frac{8390}{101}=83.069$$

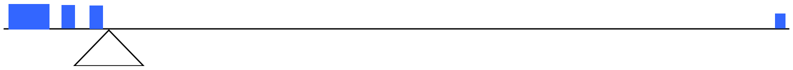
While 83.1 thousand dollars (\$83,069) is the correct mean household income, it no longer represents a "typical" value.

## Median

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



If we graph our household data, the \$5 million data value is so far out to the right that the mean has to adjust up to keep things in balance



For this reason, when working with data that have **outliers** – values far outside the primary grouping – it is common to use a different measure of center, the **median**.

**Median**

The **median** of a set of data is the value in the middle when the data is in order.

To find the median:

- List the data in order from smallest to largest, or largest to smallest
- If the number of data values, $n$, is odd, then the median is the middle data value. This is the data value in the $\frac{n+1}{2}$ position.
- If the number of data values is even, there is no one middle value, so we find the mean of the 2 middle values (values in the positions of $\frac{n}{2}$ and $\frac{n}{2}+1$).

We can interpret the median as "half of the data is less than or equal to the median and the other half is more than or equal to the median." Of course, we can rewrite this in context of the problem.

**Example 8.4.5**

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7 9

### Solution

We start by listing the data in order: 2 4 5 5 6 7 7 8 8 9 10

Since there are $n$ = 11 data values, an odd number, the middle data value will be in the $\frac{11+1}{2}=\frac{12}{2}=6^{\text{th}}$ position. So the median quiz score is 7. Half of the quiz scores are 7 or less, and half of the quiz scores are 7 or more.

**Example 8.4.6**

Returning to the football touchdown data, find the median number of touchdown passes thrown per team in the NFL in the 2021 season.

### Solution

We would start by listing the data in order.

12 14 15 16 20 20 20 20 21 21 21 21 21 22 23 23

23 24 26 27 27 29 30 34 36 36 37 38 39 40 41 43

Since there are $n$ = 32 data values, an even number, the median will be the mean of the 2 middle values, the $16^{\text{th}}$ and $17^{\text{th}}$ data values. The $16^{\text{th}}$ and $17^{\text{th}}$ data values are both 23, so the mean of them is also 23 (since $\frac{23+23}{2}=23$). The median number of touchdown passes per team in the 2021 season was 23 passes. Half of the teams had 23 touchdown passes thrown or less for the season, and half of the teams had 23 or more.

### Try It 8.4.2

The price of a jar of peanut butter at 5 stores were: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the median price.

**Answer**

First we put the data in order: \$3.29, \$3.59, \$3.75, \$3.79, \$3.99. Since there are an odd number of data, the median will be the middle value, the 3rd data value, \$3.75.

### Example 8.4.7

Let us return now to our original household income data. Find the median household income.

| Income (thousands of dollars) | Frequency |
|---|---|
| 15 | 6 |
| 20 | 8 |
| 25 | 11 |
| 30 | 17 |
| 35 | 19 |
| 40 | 20 |
| 45 | 12 |
| 50 | 7 |

**Solution**

Here we have $n$ = 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values - the 50th and 51st data values. To find these, we start counting up from the bottom:

$$\begin{array}{ll}\text{There are 6 data values of } 15\text{, so} & \text{values 1 to 6 are \$15 thousand} \\ \text{The next 8 data values are } 20\text{, so} & \text{values } 7 \text{ to } (6+8)=14 \text{ are \$20 thousand} \\ \text{The next 11 data values are } 25\text{, so} & \text{values } 15 \text{ to } (14+11)=25 \text{ are \$25 thousand} \\ \text{The next 17 data values are } 30\text{, so} & \text{values } 26 \text{ to } (25+17)=42 \text{ are \$30 thousand} \\ \text{The next 19 data values are } 35\text{, so} & \text{values } 43 \text{ to } (42+19)=61 \text{ are \$35 thousand}\end{array}$$

From this we can tell that values 50 and 51 will be \$35 thousand, and the mean of these two values is \$35 thousand. The median income in this neighborhood is \$35 thousand. Thus, half of the households' annual income is \$35,000 or less and the other half is \$35,000 or more.

### Example 8.4.8

If we add in the new neighbor with a \$5 million household income, what is the median household income?

**Solution**

There will be $n$ = 101 data values, and the 51st value will be the median. As we discovered in the last example, the 51st value is \$35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

Let's think about the previous example. When we added the 101st family's income, the mean was \$81,069 from \$31,900. That's a big difference in the average household income. We see that the mean is influenced by the values of the data, i.e., the mean could get larger or smaller depending on the values of the data. However, when calculating the median including the 101st family's income, the median wasn't influenced at all. In fact, in general, the median is known as a better statistic for household income since there is a wide spread of income among families. Thus, the values of the data influence the mean, but not the median.

## Mode

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode**.

**Mode**

The **mode** is the value of the data set that occurs most frequently.

The mode is most commonly used for categorical data, for which the median and mean cannot be computed. Also, the mode is the only measure of central tendency that is used for both categorical and quantitative data. The mean and median are only used with quantitative data.

**Example 8.4.8**

In our vehicle color survey, we collected the data below. Find the mode.

| Color | Frequency |
|---|---|
| Blue | 3 |
| Green | 5 |
| Red | 4 |
| White | 3 |
| Black | 2 |
| Grey | 3 |

**Solution**

For this data, Green is the mode, since it is the data value that occurred the most frequently. This is often called the *modal class* when referring to categorical data.

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

**Try It 8.4.3**

Reviewers were asked to rate a product on a scale of 1 to 5. The results of the survey are below. Find:

a. The mean rating
b. The median rating
c. The mode rating

| Rating | Frequency |
|---|---|
| 1 | 4 |
| 2 | 8 |
| 3 | 7 |
| 4 | 3 |
| 5 | 1 |

**Answer**

a. The mean rating is $\frac{1\cdot 4+2\cdot 8+3\cdot 7+4\cdot 3+5\cdot 1}{23}=\frac{58}{23}\approx 2.5$ . Note that this is actually categorical data, so it is not appropriate to calculate the mean for this data. This number is meaningless since the rating values are based on a customer's opinion. Think about restaurant ratings or ratings for products on Amazon: some customers will rate the product itself, while others will rate the customer service. There is no clear criteria for what makes a product a "1" rating versus a "5" rating. This erroneous type of "mean rating" is used quite extensively when shopping online.

b. There are 23 data values, so the median will be the 12th data value. Ratings of 1 are the first 4 values, while a rating of 2 are the next 8 values, so the 12th value will be a rating of 2. The median rating is 2. Half of the ratings are 2 or less, and the other half of the ratings are 2 or more.
c. The mode is the most frequent rating. The mode rating is 2.

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[1] https://www.statmuse.com/nfl/ask/mos...-nfl-2021-team

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# 8.5: Measures of Variation and Location

**Learning Objectives**

- Find the range of a data set
- Find and interpret the standard deviation of a data set
- Interpret percentiles and calculate quartiles of a data set

A second aspect of a distribution is how spread out it is. In other words, how much the data in the distribution vary from one another. Numbers that describe a distribution's spread or amount of variability are called measures of variation.

Consider these three sets of quiz scores:

- section A: 5 5 5 5 5 5 5 5 5 5
- section B: 0 0 0 0 0 10 10 10 10 10
- section C: 4 4 4 5 5 5 5 6 6 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the "typical" or "middle" value, we also need a measure of how "spread out" or varied each data set is.

## Range

There are several ways to measure this "spread" of the data. The first is the simplest and is called the **range**.

**Range**

The range is the difference between the maximum value and the minimum value of the data set.

**Example 8.5.1**

Using the quiz scores from above, find the range of each section.

**Solution**

For section A, the range is 0 since both maximum and minimum are 5 and $5-5=0$.

For section B, the range is 10 since $10-0=10$.

For section C, the range is 2 since $6-4=2$.

In the last example, the range seems to be revealing how spread out the data is. However, suppose we add a fourth section, section D, with scores 0 5 5 5 5 5 5 5 5 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different than section B. To better illuminate the differences, we'll have to turn to more sophisticated measures of variation.

## Standard Deviation

**Standard deviation**

The **standard deviation** is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is always positive. Standard deviation will be zero if all the data values are equal, and will get larger as the data spreads out.
- Standard deviation has the same units as the original data.

- Standard deviation, like the mean, can be highly influenced by outliers.

Using the data from section D, we could compute for each data value the difference between the data value and the mean:

| data value | deviation: data value - mean |
|---|---|
| 0 | $0-5=-5$ |
| 5 | $5-5=0$ |
| 5 | $5-5=0$ |
| 5 | $5-5=0$ |
| 5 | $5-5=0$ |
| 5 | $5-5=0$ |
| 5 | $5-5=0$ |
| 5 | $5-5=0$ |
| 5 | $5-5=0$ |
| 10 | $10-5=5$ |

We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column the negative and positive values cancel each other out (this will always happen), so to prevent this we square every value in the second column:

We then add the squared deviations up to get $25+0+0+0+0+0+0+0+0+25=50$ . Ordinarily we would then divide by the number of scores, $n$, (in this case, 10) to find the mean of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by $n-1$ (in this case, $10-1=9$ ). [1]

So in our example, we would have $\frac{50}{10}=5$ if section D represents a population and $\frac{50}{9}\approx 5.56$ if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

Variance can be a useful statistical concept, but note that the units of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

$$\text{population standard deviation}=\sqrt{\frac{50}{10}}=\sqrt{5}\approx 2.2$$

or

$$\text{sample standard deviation}=\sqrt{\frac{50}{9}}\approx 2.4$$

What does this say about section D? We can say that the average score was 5 give or take 2.4. The “give or take” part is the prefix for standard deviation. Or we can say that the data values in section D are, on average, 2.4 points away from the mean 5. One data value is below the mean, some data values are the same as the mean, and one value is above the mean. But if you average out how far the data values are away from the mean, you will get about 2.4 points.

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data, as we have done above.

### To compute standard deviation:

1. Find the deviation of each data from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by $n$, the number of data values, if the data represents a whole population; divide by $n-1$ if the data is from a sample. The value of this answer is the variance.
5. Compute the square root of the result from the previous step. The value of this answer is the standard deviation.

### Example 8.5.2

Compute the standard deviation for section B above.

**Solution**

We first calculate the mean, which is 5. Using a table can help keep track of your computations for the standard deviation:

| data value | deviation: data value - mean | deviation squared |
|---|---|---|
| 0 | $0-5=-5$ | $(-5)^2=25$ |
| 0 | $0-5=-5$ | $(-5)^2=25$ |
| 0 | $0-5=-5$ | $(-5)^2=25$ |
| 0 | $0-5=-5$ | $(-5)^2=25$ |
| 0 | $0-5=-5$ | $(-5)^2=25$ |
| 10 | $10-5=5$ | $(5)^2=25$ |
| 10 | $10-5=5$ | $(5)^2=25$ |
| 10 | $10-5=5$ | $(5)^2=25$ |
| 10 | $10-5=5$ | $(5)^2=25$ |
| 10 | $10-5=5$ | $(5)^2=25$ |

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$$\sqrt{\frac{25+25+25+25+25+25+25+25+25+25}{10}}=\sqrt{\frac{250}{10}}=5$$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out. On average, the data values in section B are 5 points away from the mean of 5. This is exactly what you would expect with half of the data values equal to 0 and the other half of the data values equal to 10.

For comparison, the (population) standard deviations of all four sections are:

| | |
|---|---|
| section A: 5 5 5 5 5 5 5 5 5 5 | standard deviation: 0 |
| section B: 0 0 0 0 0 10 10 10 10 10 | standard deviation: 5 |
| section C: 4 4 4 5 5 5 5 6 6 6 | standard deviation: 0.8 |
| section D: 0 5 5 5 5 5 5 5 5 10 | standard deviation: 2.2 |

Note: most scientific and graphing calculators have functions for calculating the mean and standard deviation of a data set. Check your calculator's user manual to find out how yours works.

### Try It 8.5.1

The price of a jar of peanut butter at 5 stores were: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the standard deviation of the prices.

**Answer**

Earlier we found the mean of the data was \$3.682.

| data value | deviation: data value− mean | deviation squared |
|---|---|---|
| 3.29 | $3.29-3.682=-0.391$ | 0.153664 |
| 3.59 | $3.59-3.682=-0.092$ | 0.008464 |
| 3.79 | $3.79-3.682=0.108$ | 0.011664 |
| 3.75 | $3.75-3.682=0.068$ | 0.004624 |
| 3.99 | $3.99-3.682=0.308$ | 0.094864 |

This data is from a sample, so we will add the squared deviations, divide by $n-1$ = 4, the number of data values minus 1, and compute the square root:

$$\sqrt{\frac{0.153664+0.008464+0.011664+0.004624+0.094864}{4}} \approx \$0.261$$

On average, the prices of peanut butter from the 5 stores are \$0.261 away from the mean of \$3.682.

## Percentiles and Quartiles

There are other calculations that we can do to look at spread. One of those is called the **percentile**. This looks at what data value has a certain percent of the data at or below it. It is also known as a measure of location since it gives the location or position of a data value in the data set relative to the other data values.

### Percentile

The $k^{\text{th}}$ **percentile** is the value with $k$ % of the data at or below this value.

For example, if a data value is in the 80th percentile, then 80% of the data values fall at or below this value, and 20% of the data values fall at or above it.

We see percentiles in many places in our lives. If you take any standardized tests, your score is given as a percentile. If you take your child to the doctor, their height and weight are given as percentiles. If your child is tested for gifted or behavior problems, the score is given as a percentile. If your child has a score on a gifted test that is in the 92nd percentile, then that means that 92% of all of the children who took the same gifted test scored the same or lower than your child. That also means that 8% scored the same or higher than your child. This may mean that your child is gifted.

### Example 8.5.3

Suppose you took the SAT mathematics test and received your score as a percentile.

a. What does a score in the 90th percentile mean?
b. What does a score in the 70th percentile mean?
c. If the test was out of 800 points and you scored in the 80th percentile, what was your score on the test?
d. If your score was in the 95th percentile, does that mean you passed the test?

**Solution**

a. 90% of the scores were at or below your score and 10% of the scores were at or above your score. (You did the same as or better than 90% of the test takers.)
b. 70% of the scores were at or below your score and 30% of the scores were at or above your score.
c. You do not know! All you know is that you scored the same as or better than 80% of the people who took the test. If all the scores were really low, you could have still failed the test. On the other hand, if many of the scores were high you could have gotten a 95% on the test.
d. No, it just means you did the same as or better than 95% of the other people who took the test. You could have failed the test, but still did the same as or better than 95% of the rest of the people.

While standard deviation is a measure of variation based on the mean, **quartiles** are a measure of location based on the median. Quartiles are a type of percentile.

### Quartiles

**Quartiles** are values that divide the data in quarters.

The **first quartile** ($Q_1$) is the value so that 25% of the data values are less than or equal to it; the **third quartile** ($Q_3$) is the value so that 75% of the data values are less than or equal to it. You may have guessed that the **second quartile** is the same as the median, since the median is the value so that 50% of the data values are less than or equal to it.

This divides the data into quarters; 25% of the data is between the minimum and $Q_1$, 25% is between $Q_1$ and the median, 25% is between the median and $Q_3$, and 25% is between $Q_3$ and the maximum value.

To find the first quartile, we need to find the median of the first half of the data set. Similarly, to find the third quartile, we need to find the median of the second half of the data set. It is easiest to find the median first to divide the data set into the first half and second half.

### Finding the Quartiles

1. Begin by ordering the data from smallest to largest.
2. Find the median. This is the second quartile.
3. Separate the data set into 2 data sets: the half before the median and the half after the median. Do not include the median in either of the half sets.
4. Find the median of the first half of the data set. This is the first quartile.
5. Find the median of the second half of the data set. This is the third quartile.

Examples should help make this clearer.

### ✓ Example $8.5.4$

Suppose we have measured 9 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70 72

Find the first and third quartiles.

**Solution**

Since there are $n$ = 9 data values, the median will be the 5th data value, 66. (This is the second quartile.)

To find the first quartile we find the median of the first half of the data set: 59 60 62 64. Since there are now $n$ = 4 data values, the median will be the average of the 2nd and 3rd data values: $\frac{60+62}{2} = 61$. The first quartile is 61 inches. We can say that 25% of females are 61 inches or shorter and 75% of females are 61 inches or taller.

To find the third quartile, we find the median of the second half of the data set: 67 69 70 72. The median will again be the average of the 2nd and 3rd data values: $\frac{69+70}{2} = 69.5$. The third quartile is 69.5 inches. We can say that 75% of females are 69.5 inches or shorter and 25% of females are 69.5 inches or taller.

Note the locations of the quartiles denoted by the red lines and the median shaded in red:

59 60 | 62 64 66 67 69 | 70 72

The median separates the data set into 2 halves. The first quartile is the median of the first half of the data set and the third quartile is the median of the second half of the data set.

It is also worth noting that if there were only 8 females with the following data: 59 60 62 64 66 67 69 70, the median would be 65 inches (the average of the 4th and 5th data values) but the first quartile would be the same since the first half of the data set would be the same. The third quartile would be 68 inches. It would look like:

59 60 | 62 64 | 66 67 | 69 70

### Try It $8.5.2$

The prices of a jar of peanut butter at 5 stores were: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the first and third quartiles.

**Answer**

The data in order are: \$3.29, \$3.59, \$3.75, \$3.79, \$3.99.

The median is the middle value, \$3.75.

The first quartile is the median of the first half of the data, \$3.29 and \$3.59. Since there are only 2 values, the first quartile will be the average of the data values: $\frac{3.29+3.59}{2}=3.44$. The first quartile is \$3.44. 25% of peanut butter costs \$3.44 or less and 75% of peanut butter costs \$3.44 or more.

The third quartile is the median of the second half of the data, \$3.79 and \$3.99. It will be the average of the 2 data values: $\frac{3.79+3.99}{2}=3.89$. The third quartile is \$3.89. 75% of peanut butter costs \$3.89 or less and 25% of peanut butter costs \$3.89 or more.

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[1] The reason we do this is highly technical, but we can see how it might be useful by considering the case of a small sample from a population that contains an outlier, which would increase the average deviation: the outlier very likely won't be included in the sample, so the mean deviation of the sample would underestimate the mean deviation of the population; thus we divide by a slightly smaller number to get a slightly bigger average deviation.

# 8.6: Box Plots

**Learning Objectives**

- Find the 5-number summary of a data set
- Construct and interpret a box plot for a data set

## 5-Number Summary

While quartiles are not a 1-number summary of spread like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a **5-number summary** of the data.

**5-Number Summary**

The **5-number summary** takes the form:

Minimum, $Q_1$, Median, $Q_3$, Maximum

The 5-number summary is written as a list of numbers. It is understood by context what each value in the list represents.

**Example $8.6.1$**

Find the 5-number summaries for the 9 female sample and the 8 female sample from the previous section.

59 60 62 64 66 67 69 70 72

59 60 62 64 66 67 69 70

### Solution

For the 9 female sample, the median is 66, the minimum is 59, and the maximum is 72. The 5-number summary is: 59, 61, 66, 69.5, 72.

For the 8 female sample, the median is 65, the minimum is 59, and the maximum is 70, so the 5-number summary would be: 59, 61, 65, 68, 70.

**Example $8.6.2$**

Find the 5-number summary of the quiz score data from the previous section.

- section A: 5 5 5 5 5 5 5 5 5 5
- section B: 0 0 0 0 0 10 10 10 10 10
- section C: 4 4 4 5 5 5 5 6 6 6
- section D: 0 5 5 5 5 5 5 5 5 10

### Solution

In each section, the median is the average of 5th and 6th data values. The first quartile is the 3rd data value, and the third quartile is the 8th data value. Creating the 5-number summaries:

| section and data | 5-number summary |
|---|---|
| section A: 5 5 5 5 5 5 5 5 5 5 | 5, 5, 5, 5, 5 |
| section B: 0 0 0 0 0 10 10 10 10 10 | 0, 0, 5, 10, 10 |
| section C: 4 4 4 5 5 5 5 6 6 6 | 4, 4, 5, 6, 6 |
| section D: 0 5 5 5 5 5 5 5 5 10 | 0, 5, 5, 5, 10 |

Of course, with a relatively small data set, finding a 5-number summary is a bit silly, since the summary contains almost as many values as the original data.

### Try It 8.6.1

The total cost of textbooks for the term was collected from 36 students. Find the 5-number summary of this data.

\$140 \$160 \$160 \$165 \$180 \$220 \$235 \$240 \$250 \$260 \$280 \$285

\$285 \$285 \$290 \$300 \$300 \$305 \$310 \$310 \$315 \$315 \$320 \$320

\$330 \$340 \$345 \$350 \$355 \$360 \$360 \$380 \$395 \$420 \$460 \$460

**Answer**

The data is already in order, so we don't need to sort it first.

The minimum value is \$140 and the maximum is \$460.

There are 36 data values so $n = 36$. 36 is an even number, so the median is the average of the $18^{\text{th}}$ and $19^{\text{th}}$ data values, $\$305 \text{and} \$310$ The median is $\$307.50$

To find the first quartile, the number of data in the first half is $n = 18$. Since 18 is an even number, we know $\mathrm{Q}_1$ is the average of the $9^{\text{th}}$ and $10^{\text{th}}$ data values, $\$250 \text{and} \$260$. $\mathrm{Q}_1 = \$255$

To find the third quartile, the number of data in the second half is also $n = 18$. Since 18 is an even number, we know $\mathrm{Q}_3$ is the average of the $9^{\text{th}}$ and $10^{\text{th}}$ data values of the second half of the data set (or the 18 + 9 = 27th and 28th data values in the original data set), $\$345 \text{and} \$350$ $\mathrm{Q}_3 = \$347.50$

The 5-number summary of this data is: \$140, \$255, \$307.50, \$347.50, \$460

### Example 8.6.3

Returning to the household income data from earlier, find the 5-number summary.

| Income (thousands of dollars) | Frequency |
|---|---|
| 15 | 6 |
| 20 | 8 |
| 25 | 11 |
| 30 | 17 |
| 35 | 19 |
| 40 | 20 |
| 45 | 12 |
| 50 | 7 |

**Solution**

By adding the frequencies, we can see there are $n$ = 100 data values represented in the table. In a previous example, we found the median was the mean of the 50th and 51st data values, \$35 thousand. We can see in the table that the minimum income is \$15 thousand, and the maximum is \$50 thousand.

To find $Q_1$, we find the median of the first 50 data values. It will be the mean of the 25th and 26th data values.

Counting up in the data as we did before,

$$\begin{array}{ll} \text{There are 6 data values of \$15, so} & \text{values 1 to 6 are \$15 thousand} \\ \text{The next 8 data values are \$20, so} & \text{values 7 to } (6+8)=14 \text{ are \$20 thousand} \\ \text{The next 11 data values are \$25, so} & \text{values 15 to } (14+11)=25 \text{ are \$25 thousand} \\ \text{The next 17 data values are \$30, so} & \text{values 26 to } (25+17)=42 \text{ are \$30 thousand} \end{array}$$

The 25th data value is \$25 thousand, and the 26th data value is \$30 thousand, so $Q_1$ will be the mean of these: $\dfrac{(25+30)}{2} = \$27.5$ thousand.

To find $Q_3$, we find the median of the second 50 data values. It will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

$$\begin{array}{ll}\text{The next 19 data values are \$35, so} & \text{values 43 to } (42+19)=61 \text{ are \$35 thousand} \\ \text{The next 20 data values are \$40, so} & \text{values 61 to } (61+20)=81 \text{ are \$40 thousand}\end{array}$$

Both the 75th and 76th data values lie in this group, so $Q_3$ will be $40 thousand.

Putting these values together into a 5-number summary, we get: 15, 27.5, 35, 40, 50 (in thousands of dollars).

Note that the 5 number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between $40 thousand and $50 thousand.

## Box Plots

For visualizing data, there is a graphical representation of a 5-number summary called a **box plot**, or box-and-whisker graph.

### Box plot

A **box plot** is a graphical representation of a 5-number summary.

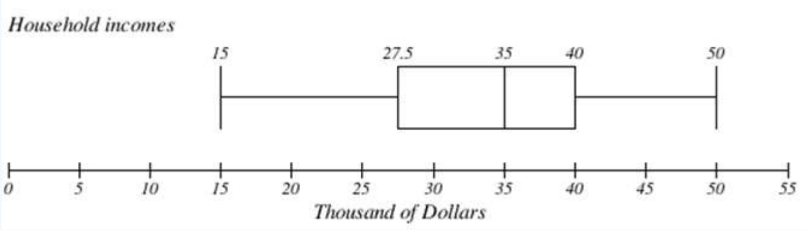
To create a box plot, a scaled number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. “Whiskers” are extended out to the minimum and maximum values. Be sure to give the graph a title and label the number line.

### Example 8.6.4

Construct a box plot for the household income data.

### Solution

The box plot below is based on the household income data with 5-number summary: 15, 27.5, 35, 40, 50

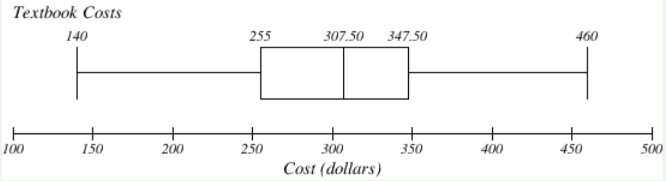


The box plot illustrates how spread or concentrated the data is. Since each part of the box plot represents a quarter of the data set, we can see that the data in the first quarter is more spread out, but the data between the second and third quartiles is more concentrated. There are the same number of data values in each part: 25 values between $15 to $27.5, 25 values between $27.5 to $35 thousand, 25 values between $35 to $40 thousand and 25 values between $40 to $50 thousand.

### Try It 8.6.2

Create a box plot based on the textbook price data from the last Try It.

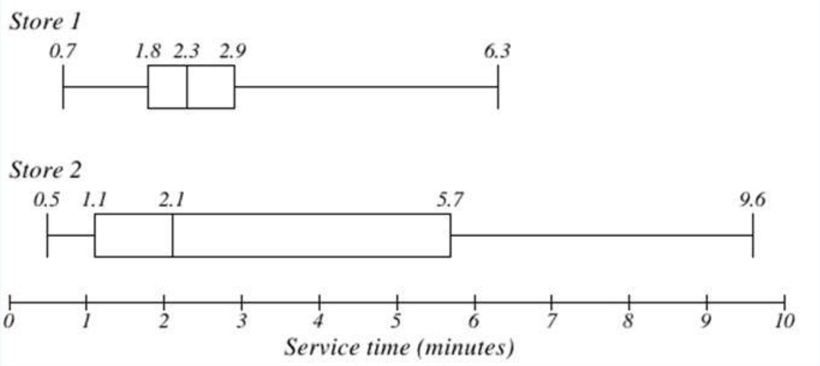
**Answer**



Box plots are particularly useful for comparing data from two populations.

**✓ Example 8.6.5**

The box plots of service times for two fast-food restaurants are shown below. Discuss the differences between the stores based on the box plots.



**Solution**

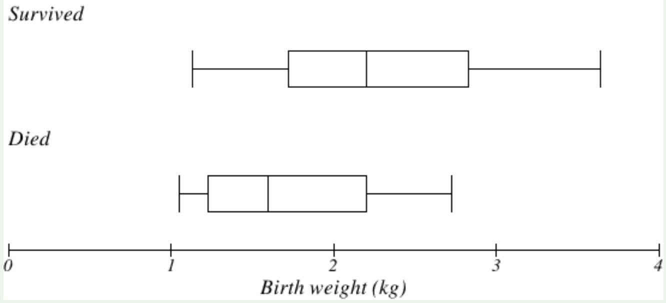
While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry? That depends upon your opinions about luck: 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes!

**Try It 8.6.3**

The box plot below is based on the birth weights in kilograms of infants with severe idiopathic respiratory distress syndrome (SIRDS) [1]. There are separate box plots to show the birth weights of infants who survived and those that did not. Compare and contrast the groups based on the box plots.



**Answer**

Comparing the two groups, the box plots reveal that the birth weights of the infants that died appear to be, overall, less than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

Similarly, we can see that the first quartile of the survivors is greater than the median weight of those that died, meaning that over 75% of the survivors had a birth weight greater than the median birth weight of those that died.

Looking at the maximum value for those that died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights greater than the heaviest infant that died.

> The box plots give us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.

[1] van Vliet, P.K. and Gupta, J.M. (1973) Sodium bicarbonate in idiopathic respiratory distress syndrome. *Arch. Disease in Childhood*, **48**, 249–255. As quoted on http://openlearn.open.ac.uk/mod/ouco...&section=1.1

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# 8.7: Correlation and Causation, Scatter Plots

The label on a can of Planters Cocktail Peanuts says, “Scientific evidence suggest but does not prove that eating 1.5 ounces per day of most nuts, such as peanuts, as part of a diet low in saturated fat and cholesterol & not resulting in increased caloric intake may reduce the risk of heart disease. See nutritional information for fat content (1.5 oz. is about 53 pieces).” Why is it written this way and what does this statement mean? There are many studies that exist that show that two variables are related to one another. The strength of a relationship between two variables is called **correlation**. Variables that are strongly related to each other have strong correlation. However, if two variables are correlated it does not mean that one variable caused the other variable to occur. The above example from the Planters Cocktail Peanuts label is an example of this. There is a strong correlation between eating a diet that is low in saturated fat and cholesterol and heart disease. But that correlation does not mean that eating a diet that is low in saturated fat and cholesterol will cause your risk of heart disease to go down. There could be many different variables that could cause both variables in question to go down or up. One example is that a person’s genetic makeup could make them not want to eat fatty food and also not develop heart disease. No matter how strong a correlation is between two variables, you can never know for sure if one variable causes the other variable to occur without conducting experimentation. The only way to find out if eating a diet low in saturated fat and cholesterol actually lowers the risk of heart disease is to do an experiment. This is where you tell one group of people that they have to eat a diet low in saturated fat and cholesterol and another group of people that they have to eat a diet high in saturated fat and cholesterol, and then observe what happens to both groups over the years. You cannot morally do this experiment, so there is no way to prove the statement. That is why the word “may” is in the statement. We see many correlations like this one. Always be sure not to make a correlation statement into a causation statement.

**Example $8.7.1$: Correlation vs Causation**

For each of the following scenarios answer the question and give an example of another variable that could explain the correlation.

1. There is a negative correlation between number of children a woman has and her life expectancy. Does that mean that having children causes a woman to die earlier?

A correlation between two variables does not mean that one causes the other. A possible cause for both variables could be better health care. If there is better health care, then life expectancy goes up, and also with better health care birth control is more readily available.

2. There is a positive correlation between ice cream sales and the number of drownings at the beach. Does that mean that eating ice cream can cause a person to drown?

A correlation between two variables does not mean that one causes the other. The cause for both could be that the temperature is going up. The higher the temperature, the more likely someone will buy ice cream and the more people at the beach.

3. There is a correlation between waist measures and wrist measures. Does this mean that your waist measurement causes your wrist measurement to change?

A correlation between two variables does not mean that one causes the other. The cause of both could be a person’s genetics, eating habits, exercise habits, etc.

How do we tell if there is a correlation between two variables? The easiest way is to graph the two variables together as ordered pairs on a graph called a **scatter plot**. To create a scatter plot, consider that one variable is the independent variable and the other is the dependent variable. This means that the dependent variable depends on the independent variable. We usually set up these two variables as ordered pairs where the independent variable is first and the dependent variable is second. Thus, when graphed, the independent variable is graphed along the horizontal axis and the dependent variable is graphed along the vertical axis. You do not connect the dots after plotting these ordered pairs. Instead look to see if there is a pattern, such as a line, that fits the data well. Here are some examples of scatter plots and how strong the linear correlation is between the two variables.

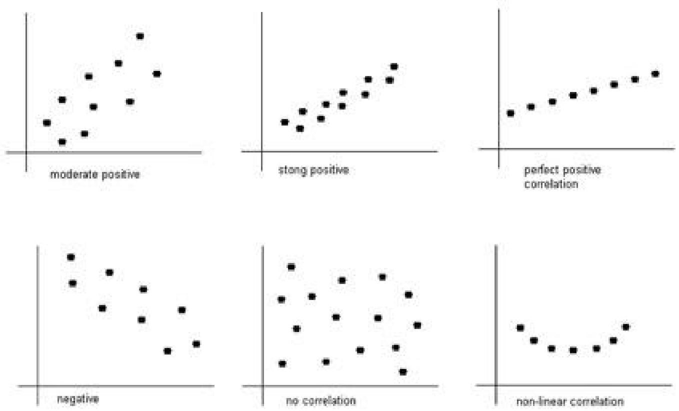


Figure 8.7.1: Scatter Plots Showing Types of Linear Correlation

Creating a scatter plot is not difficult. Just make sure that you set up your axes with scaling before you start to plot the ordered pairs.

**Example 8.7.2: Creating a Scatter Plot**

Data has been collected on the life expectancy and the fertility rate in different countries ("World health rankings," 2013). A random sample of 10 countries was taken, and the data is:

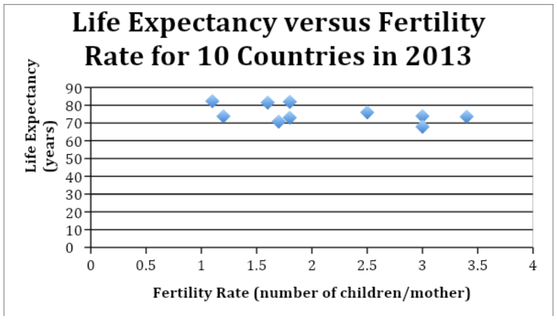
Table 8.7.2: Life Expectancy and Fertility Rate in 2013

| Country | Life Expectancy (years) | Fertility Rate (number of children per mother) |
|---|---|---|
| SINGAPORE | 82.3 | 1.1 |
| MONACO | 81.9 | 1.8 |
| CANADA | 81.5 | 1.6 |
| ECUADOR | 76 | 2.5 |
| MALAYSIA | 73.9 | 3 |
| LITHUANIA | 73.8 | 1.2 |
| BELIZE | 73.6 | 3.4 |
| ALGERIA | 73 | 1.8 |
| TRINIDAD/TOB. | 70.8 | 1.7 |
| TAJIKISTAN | 67.9 | 3 |

To make the scatter plot, you have to decide which variable is the independent variable and which one is the dependent variable. Sometimes it is obvious which variable is which, and in some case it does not seem to be obvious. In this case, it seems to make more sense to predict what the life expectancy is doing based on fertility rate, so choose life expectancy to be the dependent variable and fertility rate to be the independent variable. The horizontal axis needs to encompass 1.1 to 3.4, so have it range from zero to four, with tick marks every one unit. The vertical axis needs to encompass the numbers 70.8 to 81.9, so have it range from zero to 90, and have tick marks every 10 units.

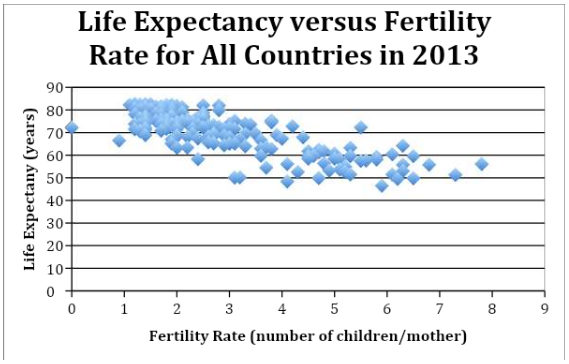
*Note: Always start the vertical axis at zero to avoid exaggeration of the data.*

**Graph 8.7.3: Scatter Plot of Life Expectancy versus Fertility Rate**



From the graph, you can see that there is somewhat of a downward trend, but it is not prominent. What this says is that as fertility rate increases, life expectancy decreases. The trend is not strong which could be due to not having enough data or this could represent the actual relationship between these two variables. Let's see what the scatter plot looks like with data from all countries in 2013 ("World health rankings," 2013).

**Graph $8.7.4$: Scatter Plot of Life Expectancy versus Fertility Rate for All Countries in 2013**



Again, there is a downward trend. It looks a little stronger than the previous scatter plot and the trend looks more obvious. This correlation would probably be considered moderate negative correlation. It appears that there is a trend that the higher the fertility rate, the lower the life expectancy. Caution: just because there is a correlation between higher fertility rate and lower life expectancy, do not assume that having fewer children will mean that a person lives longer. The fertility rate does not necessarily cause the life expectancy to change. There are many other factors that could influence both, such as medical care and education. Remember a correlation does not imply causation.

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# 8.8: Chapter Review and Glossary

# 8.9: Exercises

## Skills

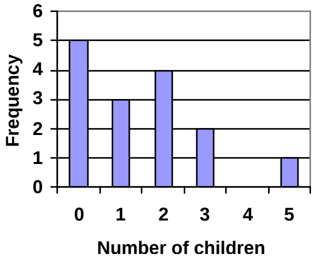
1. The table below shows scores on a Math test.
   a. Complete the frequency table for the Math test scores
   b. Construct a histogram of the data
   c. Construct a pie chart of the data

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 80 | 50 | 50 | 90 | 70 | 70 | 100 | 60 | 70 | 80 | 70 | 50 |
| 90 | 100 | 80 | 70 | 30 | 80 | 80 | 70 | 100 | 60 | 60 | 50 |

2. A group of adults where asked how many cars they had in their household
   a. Complete the frequency table for the car number data
   b. Construct a histogram of the data
   c. Construct a pie chart of the data

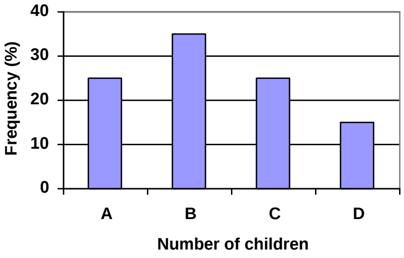
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 2 | 2 | 1 | 2 | 3 | 3 | 1 | 4 | 2 | 2 |
| 1 | 2 | 1 | 3 | 2 | 2 | 1 | 2 | 1 | 1 | 1 | 2 |

3. A group of adults were asked how many children they have in their families. The bar graph to the right shows the number of adults who indicated each number of children.
   a. How many adults where questioned?
   b. What percentage of the adults questioned had 0 children?

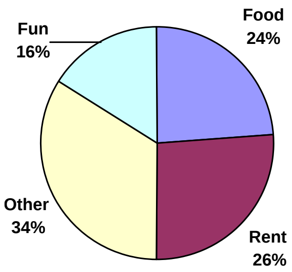


4. Jasmine was interested in how many days it would take an order from Netflix to arrive at her door. The graph below shows the data she collected.
   a. How many movies did she order?
   b. What percentage of the movies arrived in one day?

5. The bar graph below shows the *percentage* of students who received each letter grade on their last English paper. The class contains 20 students. What number of students earned an A on their paper?



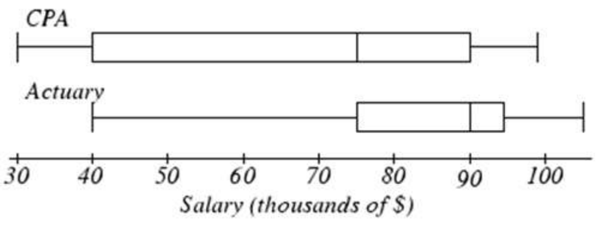
6. Kori categorized her spending for this month into four categories: Rent, Food, Fun, and Other. The percents she spent in each category are pictured here. If she spent a total of $2600 this month, how much did she spend on rent?



7. A group of diners were asked how much they would pay for a meal. Their responses were: $7.50, $8.25, $9.00, $8.00, $7.25, $7.50, $8.00, $7.00.
    a. Find the mean
    b. Find the median
    c. Write the 5-number summary for this data
    d. Find the standard deviation of this data
8. You recorded the time in seconds it took for 8 participants to solve a puzzle. The times were: 15.2, 18.8, 19.3, 19.7, 20.2, 21.8, 22.1, 29.4.
    a. Find the mean
    b. Find the median
    c. Write the 5-number summary for this data
    d. Find the standard deviation of this data
9. Refer back to the histogram from question #3.
    a. Compute the mean number of children for the group surveyed
    b. Compute the median number of children for the group surveyed
    c. Write the 5-number summary for this data.
    d. Create box plot.
10. Refer back to the histogram from question #4.
    a. Compute the mean number of shipping days
    b. Compute the median number of shipping days
    c. Write the 5-number summary for this data.
    d. Create box plot.

## Concepts

11. The box plot below shows salaries for Actuaries and CPAs. Kendra makes the median salary for an Actuary. Kelsey makes the first quartile salary for a CPA. Who makes more money? How much more?



12. Referring to the boxplot above, what percentage of actuaries makes more than the median salary of a CPA?

## Exploration

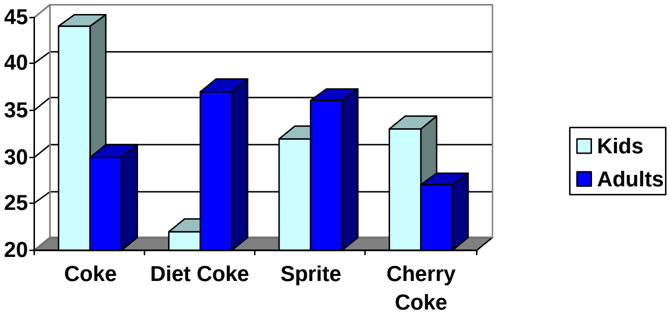
13. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 40 AIDS patients from the start of treatment until their deaths. The following data (in months) are collected.

Researcher 1: 3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34

Researcher 2: 3; 14; 11; 5; 16; 17; 28; 41; 31; 18; 14; 14; 26; 25; 21; 22; 31; 2; 35; 44; 23; 21; 21; 16; 12; 18; 41; 22; 16; 25; 33; 34; 29; 13; 18; 24; 23; 42; 33; 29

a. Create comparative histograms of the data
b. Create comparative boxplots of the data

14. A graph appears below showing the number of adults and children who prefer each type of soda. There were 130 adults and kids surveyed. Discuss some ways in which the graph below could be improved



15. Make up three data sets with 5 numbers each that have:
    a. the same mean but different standard deviations.
    b. the same mean but different medians.
    c. the same median but different means.
16. A sample of 30 distance scores measured in yards has a mean of 7, a variance of 16, and a standard deviation of 4.
    a. You want to convert all your distances from yards to feet, so you multiply each score in the sample by 3. What are the new mean, median, variance, and standard deviation?
    b. You then decide that you only want to look at the distance past a certain point. Thus, after multiplying the original scores by 3, you decide to subtract 4 feet from each of the scores. Now what are the new mean, median, variance, and standard deviation?
17. In your class, design a poll on a topic of interest to you and give it to the class.
    a. Summarize the data, computing the mean and five-number summary.
    b. Create a graphical representation of the data.
    c. Write several sentences about the topic, using your computed statistics as evidence in your writing.

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# CHAPTER OVERVIEW

## 9: Normal Distribution

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# 9.1: The Normal Distribution

**Learning Objectives**

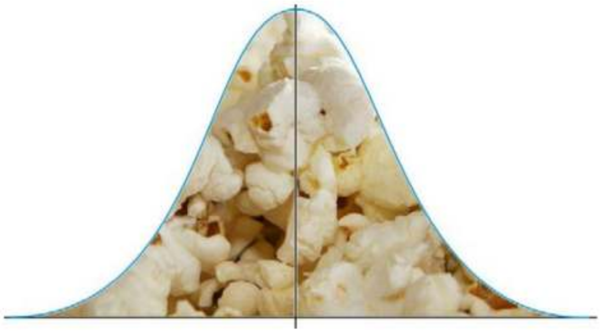
- Identify the characteristics of a normal distribution
- Apply the Empirical Rule (68-95-99.7 Rule) for normal distributions
- Calculate and interpret z-scores

## Introduction

Most high schools have a set amount of time in between classes during which students must get to their next class. If you were to stand at the door of your math class and watch the students coming in, think about how the students would enter. Usually, one or two students enter early, then more students come in, then a large group of students enter, and finally, the number of students entering decreases again, with one or two students barely making it on time, or perhaps even coming in late!

Now consider this. Have you ever popped popcorn in a microwave? Think about what happens in terms of the rate at which the kernels pop. For the first few minutes, nothing happens, and then, after a while, a few kernels start popping. This rate increases to the point at which you hear most of the kernels popping, and then it gradually decreases again until just a kernel or two pop.

Here's something else to think about. Try measuring the height, shoe size, or the width of the hands of the students in your class. In most situations, you will probably find that there are a couple of students with very low measurements and a couple with very high measurements, with the majority of students centered on a particular value.



All of these examples show a typical pattern that seems to be a part of many real-life phenomena. In statistics, because this pattern is so pervasive, it seems to fit to call it normal, or more formally, the normal distribution. The normal distribution is an extremely important concept, because it occurs so often in the data we collect from the natural world, as well as in many of the more theoretical ideas that are the foundation of statistics. This chapter explores the details of the normal distribution.

## The Characteristics of a Normal Distribution

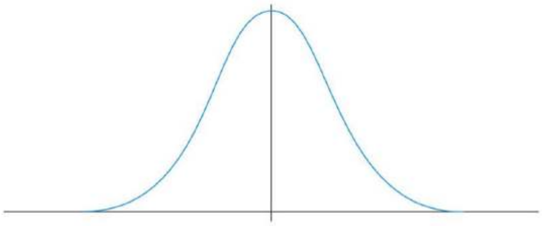
**Normal Distribution**

The **normal distribution** is a commonly occurring statistical distribution whose bell shape is determined by the mean and standard deviation of the distribution.

The normal distribution is a continuous probability distribution. The distribution is represented by a smoothed-out histogram. If you recall from the probability chapter, the sum of the probabilities of all outcomes in the sample space is 1. This is also true for the normal distribution: the total area under the normal distribution curve, which represents the probabilities of all outcomes in the sample space, is always 1.

### Shape

When graphing the data from each of the examples in the introduction, the distributions from each of these situations would be mound or bell-shaped and mostly symmetric. A normal distribution is a perfectly symmetric, bell-shaped distribution. It is commonly referred to the as a normal curve, or bell curve.

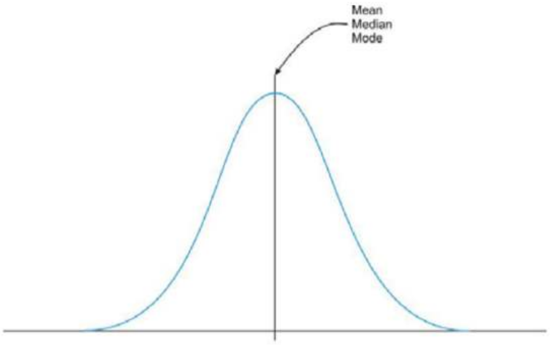


The Normal Distribution

Because so many real data sets closely approximate a normal distribution, we can use the idealized normal curve to learn a great deal about such data. With a practical data collection, the distribution will never be exactly symmetric, so just like situations involving probability, a true normal distribution only results from an infinite collection of data.

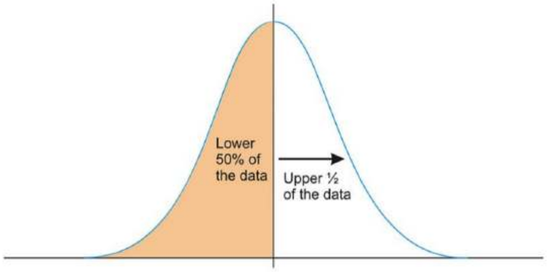
## Center

Due to the exact symmetry of a normal curve, the center of a normal distribution, or a data set that approximates a normal distribution, is located at the highest point of the distribution, and all the statistical measures of center we have already studied (the mean, median, and mode) are equal. The normal distribution is symmetric about the mean, median and mode. In statistics, the mean is represented by the Greek letter mu $\mu$.



Normal Distribution Center

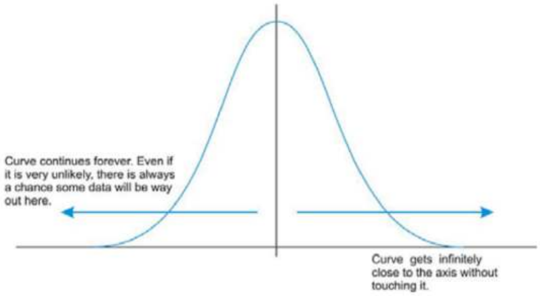
It is also important to realize that this center peak divides the data into two equal parts since it represents the median.



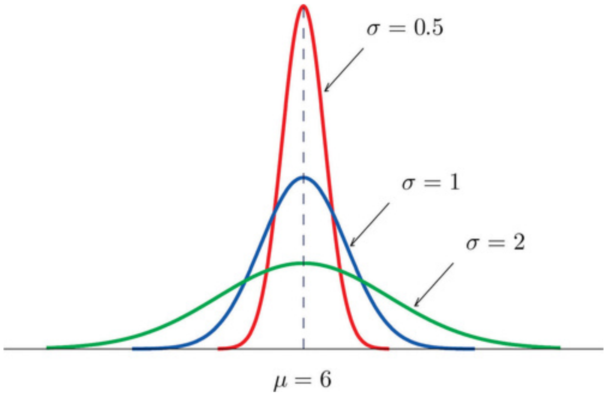
Normal Distribution divided into equal halves by the center

## Spread

Let’s go back to our popcorn example. The bag advertises a certain time, beyond which you risk burning the popcorn. From experience, the manufacturers know when most of the popcorn will stop popping, but there is still a chance that there are those rare kernels that will require more (or less) time to pop than the time advertised by the manufacturer. The directions usually tell you to stop when the time between popping is a few seconds, but aren’t you tempted to keep going so you don’t end up with a bag full of un-popped kernels? Because this is a real, and not theoretical, situation, there will be a time when the popcorn will stop popping and start burning, but there is always a chance, no matter how small, that one more kernel will pop if you keep the microwave going. In an idealized normal distribution, the distribution continues infinitely in both directions. This means that the curve never actually touches the $x$-axis. The curve approaches the $x$-axis, but technically never touches it.



Because of this infinite spread, the range would not be a useful statistical measure of spread. The most common way to measure the spread of a normal distribution is with the standard deviation, or the typical distance away from the mean. Because of the symmetry of a normal distribution, the standard deviation, which is denoted by the Greek letter sigma $\sigma$ in statistics, indicates how far away from the maximum peak the data will be, on average. The value of $\sigma$ determines whether the bell curve is tall and thin or short and squat, subject always to the condition that the total area under the curve be equal to $1$. This is shown below, where we have arbitrarily chosen to center the curves at $\mu = 6$.



The distribution with $\sigma = 0.5$ pictured above has the smallest standard deviation, and more of the data are heavily concentrated around the mean than in the distribution with $\sigma = 1$ or $\sigma = 2$. Also, in the distribution with $\sigma = 0.5$, there are fewer data values at the extremes than in the other 2 distributions. Because the distribution with $\sigma = 2$ has the largest standard deviation, the data are spread farther from the mean value, with more of the data appearing in the tails.

## The Empirical Rule for Normal Distributions

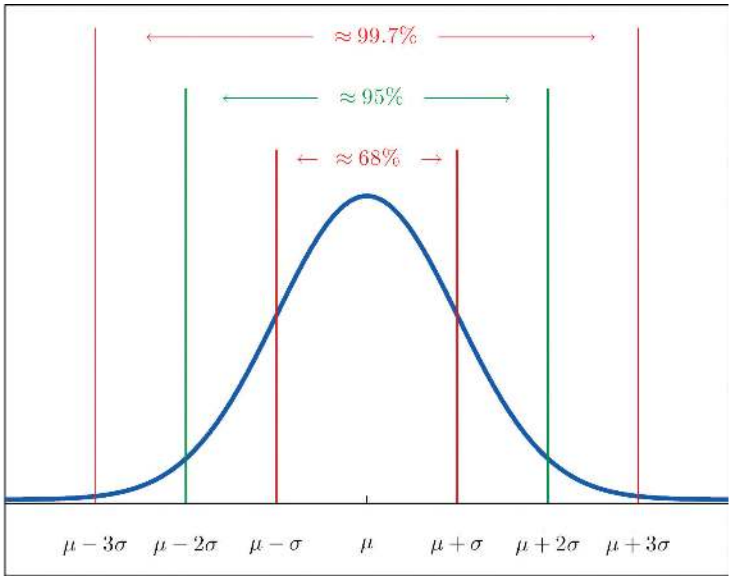
Since all normal distributions have the same shape and the total area under the curve is 1 or 100%, it turns out that the amount of data between the mean and the standard deviations for any approximately normal distribution is consistent.

### Empirical Rule

For data sets that are approximately bell-shaped:

- approximately $68\%$ of the data lie within one standard deviation of the mean, that is, in the interval with endpoints $\mu \pm \sigma$
- approximately $95\%$ of the data lie within two standard deviations of the mean, that is, in the interval with endpoints $\mu \pm 2\sigma$
- approximately $99.7\%$ of the data lie within three standard deviations of the mean, that is, in the interval with endpoints $\mu \pm 3\sigma$

The Empirical Rule is also referred to as the 68-95-99.7 Rule. The figure below illustrates the Empirical Rule.



Two key points in regard to the Empirical Rule are that the data distribution must be approximately bell-shaped and that the percentages are only approximately true. The Empirical Rule does not apply to data sets that are not bell-shaped, especially severely asymmetric distributions, and the actual percentage of observations in any of the intervals specified by the rule could be either greater or less than those given in the rule.

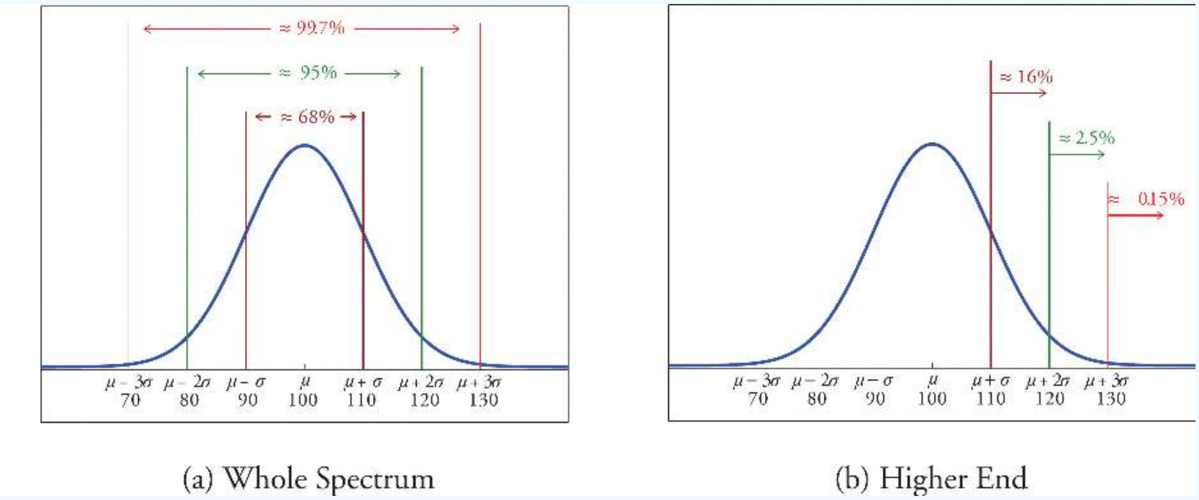
**Example 9.1.1**

Scores on IQ tests have a bell-shaped distribution with mean $\mu=100$ and standard deviation $\sigma=10$. Discuss what the Empirical Rule implies concerning individuals with IQ scores of 110, 120, and 130.

**Solution**:

A sketch of the IQ distribution is shown below. The Empirical Rule states that

- approximately $68\%$ of the IQ scores in the population lie between 90 and 110
- approximately $95\%$ of the IQ scores in the population lie between 80 and 120
- approximately $99.7\%$ of the IQ scores in the population lie between 70 and 130.



(a) Whole Spectrum

(b) Higher End

1. Since $68\%$ of the IQ scores lie *within* the interval from 90 to 110, it must be the case that $100\%-68\%=32\%$ lie *outside* that interval. By symmetry approximately half of that $32\%$, or $16\%$ of all IQ scores, will lie above 110. If $16\%$ lie above 110, then $100\%-16\%=84\%$ lie below. We conclude that the IQ score 110 is the $84^{th}$ percentile.
2. The same analysis applies to the score 120. Since approximately $95\%$ of all IQ scores lie within the interval form 80 to 120, only $5\%$ lie outside it, and half of them, or $2.5\%$ of all scores, are above 120. The IQ score 120 is thus higher than $97.5\%$ of all

IQ scores, and is quite a high score.

3. By a similar argument, only $\frac{15}{100}$ of $1\%$ of all adults, or about one or two in every thousand, would have an IQ score above $130$. This fact makes the score $130$ extremely high.

**Try It 9.1.1**

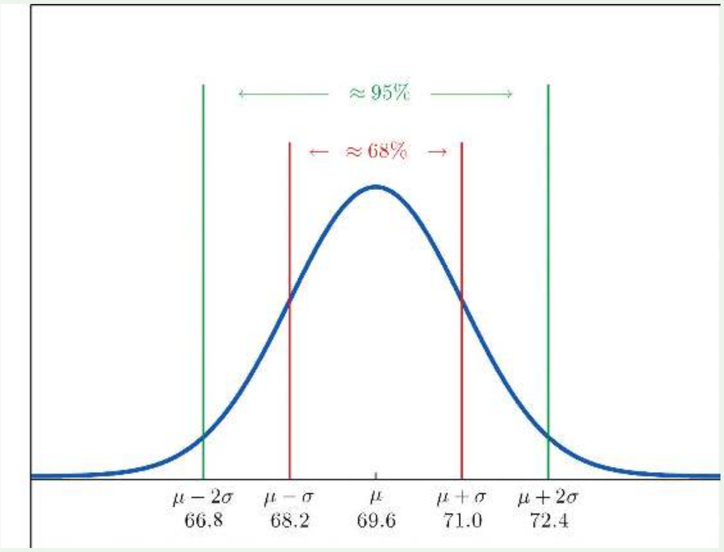
Heights of 18-year-old males have a bell-shaped distribution with mean $69.6$ inches and standard deviation $1.4$ inches.

a. About what proportion of all such men are between $68.2$ and $71$ inches tall?
b. What interval centered on the mean should contain about $95\%$ of all such men?

**Answer**

A sketch of the distribution of heights is shown below.

a. Since the interval from $68.2$ to $71.0$ has endpoints $\mu-\sigma$ and $\mu+\sigma$, by the Empirical Rule about $68\%$ of all 18-year-old males should have heights in this range.
b. By the Empirical Rule, the shortest such interval has endpoints $\mu-2\sigma$ and $\mu+2\sigma$. Since $\mu-2\sigma=69.6-2(1.4)=66.8$ and $\mu+2\sigma=69.6+2(1.4)=72.4$ the interval in question is the interval from $66.8$ inches to $72.4$ inches.



## Calculating and Interpreting z-Scores

**z-score**

A **z-score** is a measure of the number of standard deviations a particular data point is away from the mean.

For example, let's say the mean score on a test for your math class was an 82, with a standard deviation of 7 points. If your score was an 89, it is exactly one standard deviation to the right of the mean; therefore, your z-score would be 1. If, on the other hand, you scored a 75, your score would be exactly one standard deviation below the mean, and your z-score would be −1. All values that are below the mean have negative z-scores, while all values that are above the mean have positive z-scores. A z-score of −2 would represent a value that is exactly 2 standard deviations below the mean, so in this case, the value would be $82-2(7)=82-14=68$.

To calculate a z-score for which the numbers are not so obvious, you take the deviation and divide it by the standard deviation.

$$z=\frac{\text{deviation}}{\text{standard deviation}}$$

You may recall that deviation is the mean value of the variable, $\mu$, subtracted from the observed value, $x$, so in symbolic terms, the z-score would be:

$$z = \frac{x-\mu}{\sigma}$$

As previously stated, since $\sigma$ is always positive, $z$ will be positive when $x$ is greater than $\mu$ and negative when $x$ is less than $\mu$. A z-score of zero means that the term has the same value as the mean. The value of $z$ represents the number of standard deviations the given value of $x$ is above or below the mean and is usually rounded to 2 decimal places.

### ✓ Example $9.1.2$

What is the z-score for an A on the test described above, which has a mean score of 82? (Assume that an A is a 93.)

**Solution**

For this problem, $x = 93$, $\mu = 82$ and $\sigma = 7$. The z-score can be calculated as follows:

$$z = \frac{x-\mu}{\sigma}$$

$$z = \frac{93-82}{7}$$

$$z \approx 1.57$$

A score of 93 on the test is about 1.57 standard deviations above the average test score.

If we know that the test scores from the last example are distributed normally, then a z-score can tell us something about how our test score relates to the rest of the class. From the Empirical Rule, we know that about 68% of the students would have scored between a z-score of $-1$ and 1, or between 75 and 89, on the test. 95% of the students scored between a z-score of -2 and 2, or between 68 and 96.

### Try It $9.1.2$

On a nationwide math test, the mean was 65 and the standard deviation was 10. If Robert scored 81, what was his z-score?

**Answer**

For this problem, $x = 81$, $\mu = 65$ and $\sigma = 10$. The z-score can be calculated as follows:

$$z = \frac{x-\mu}{\sigma}$$

$$z = \frac{81-65}{10}$$

$$z = 1.60$$

Robert scored 1.6 standard deviations above the mean.

z-scores are useful when comparing data values that come from different data sets.

### ✓ Example $9.1.3$

Two students, John and Ali, from different high schools, wanted to find out who had the highest GPA when compared to his school. Which student had the highest GPA when compared to his school?

| Student | GPA | School Mean GPA | School Standard Deviation |
|---|---|---|---|
| John | 2.85 | 3.0 | 0.7 |
| Ali | 77 | 80 | 10 |

**Answer**

For each student, determine how many standard deviations his GPA is away from the average, for his school. This is the z-score. Pay careful attention to signs when comparing and interpreting the answer.

$$z=\frac{x-\mu}{\sigma}$$

For John,

$$z=\frac{2.85-3.0}{0.7}=-0.21$$

For Ali,

$$z=\frac{77-80}{10}=-0.3$$

John has the better GPA when compared to his school because his GPA is 0.21 standard deviations **below** his school's mean while Ali's GPA is 0.3 standard deviations **below** his school's mean.

John's *z*-score of –0.21 is higher than Ali's *z*-score of –0.3. For GPA, higher values are better, so we conclude that John has the better GPA when compared to his school.

### ✓ Example 9.1.4

On a college entrance exam, the mean was 70, and the standard deviation was 8. If Helen's z-score was −1.5, what was her exam score?

**Solution**

Since $z=\frac{x-\mu}{\sigma}$, then we can rewrite this formula solving for $x$:

$$x=\mu+z\cdot\sigma$$

Now, we can obtain Helen's exam score with the given parameters $\mu=70, \sigma=8$ and $z=-1.5$:

$$x=\mu+z\cdot\sigma$$

$$x=70+(-1.5)(8)$$

$$x=58$$

Thus, Helen's exam score was 58. Notice a score of 58 is below the mean and this makes sense since her z-score was negative.

## Notes:

1. Data available on the College Board Website: http://professionals.collegeboard.com/data-reportsresearch/ap/archi ved/2007

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# 9.2: The Standard Normal Distribution

**Learning Objectives**

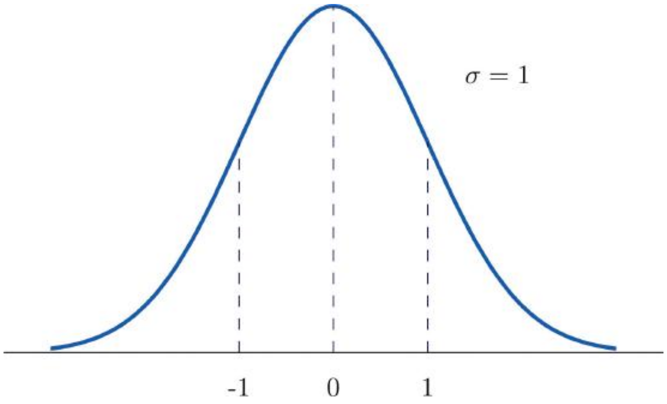
- Identify the characteristics of a standard normal distribution.
- Compute probabilities using a standard normal distribution.

While it is convenient to estimate areas under a normal curve using the Empirical Rule, we often need more precise methods to calculate these areas. Luckily, we can use formulas or technology to help us with the calculations. All normal distributions have the same basic shape, and therefore, rescaling and re-centering can be implemented to change any normal distribution to one with a mean of 0 and a standard deviation of 1. This configuration is referred to as a **standard normal distribution**. In a standard normal distribution, the variable along the horizontal axis is the z-score.

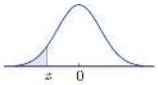
**Standard Normal Distribution**

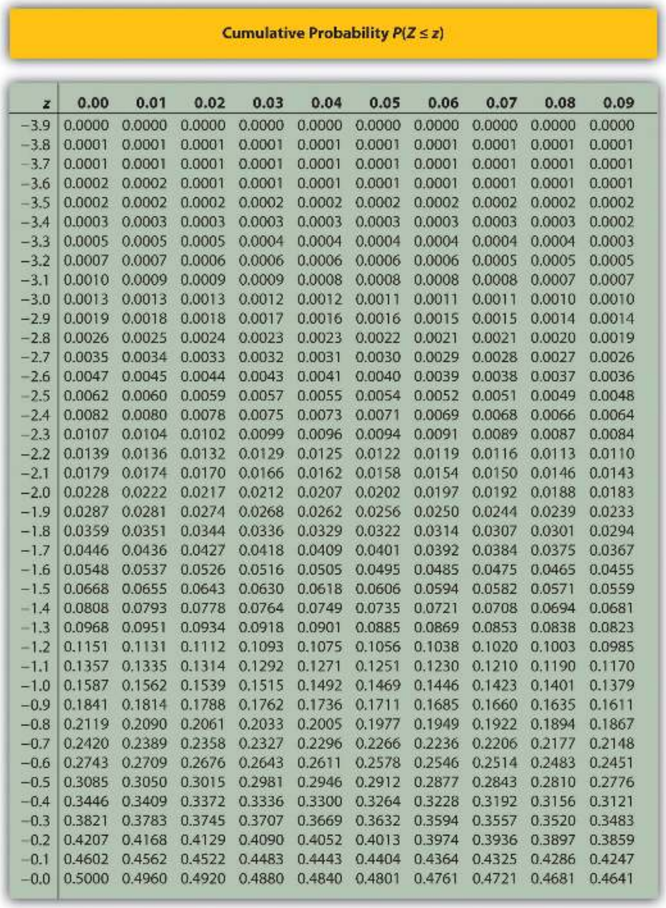
A **standard normal distribution** is a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$. It is denoted by the letter $Z$.

The curve for a standard normal distribution is shown below. Note the mean is 0 and the standard deviation is 1.



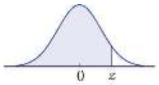
Since areas under normal curves correspond to the probability of an event occurring, a special normal distribution table is used to calculate the probabilities. To find probabilities for $Z$, we will read probabilities out of the tables below. The tables contain *cumulative* probabilities; their entries are probabilities of the form $P(Z \leq z)$. This means the probability in the table is the probability or area to the left of the value of $z$. Probabilities for negative values of $z$ are found in the first page of the table while probabilities for positive values of $z$ are found on the second page. The table is also available online here in section 6. The use of the tables will be explained by the following series of examples.

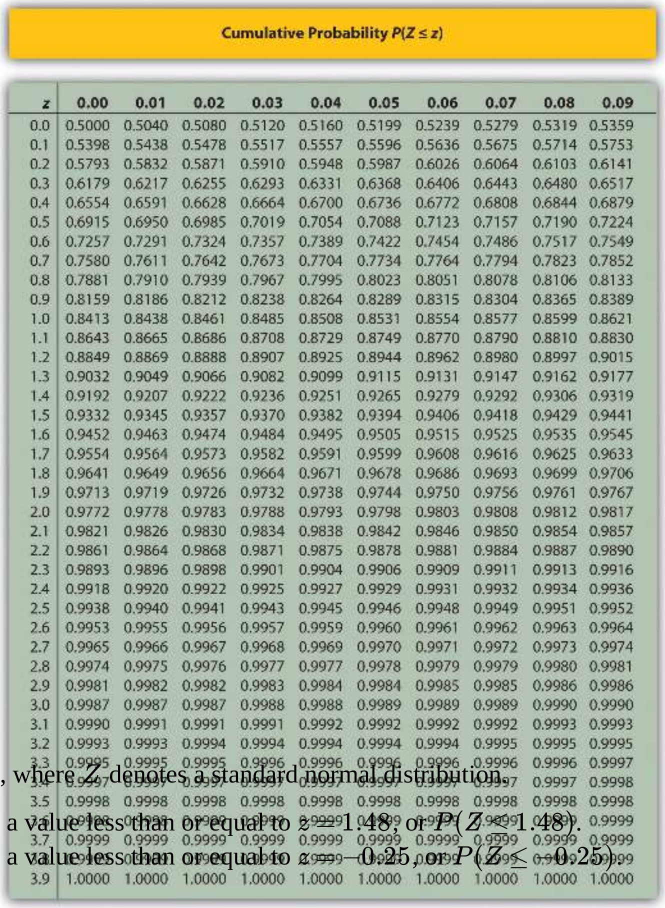




Cumulative Probability $P(Z \le z)$

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| −3.9 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| −3.8 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| −3.7 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| −3.6 | 0.0002 | 0.0002 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| −3.5 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 |
| −3.4 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| −3.3 | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| −3.2 | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| −3.1 | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| −3.0 | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| −2.9 | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| −2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| −2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| −2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| −2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| −2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| −2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| −2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| −2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| −2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| −1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| −1.8 | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| −1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| −1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| −1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| −1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| −1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| −1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| −1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| −1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| −0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| −0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| −0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| −0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| −0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| −0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| −0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| −0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| −0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| −0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |





Cumulative Probability $P(Z \leq z)$

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8304 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.3 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.4 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |
| 3.5 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 |
| 3.6 | 0.9998 | 0.9998 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.7 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.8 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.9 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

### ✓ ✓ Example 9.2.1

Find the probabilities indicated, where $Z$ denotes a standard normal distribution.

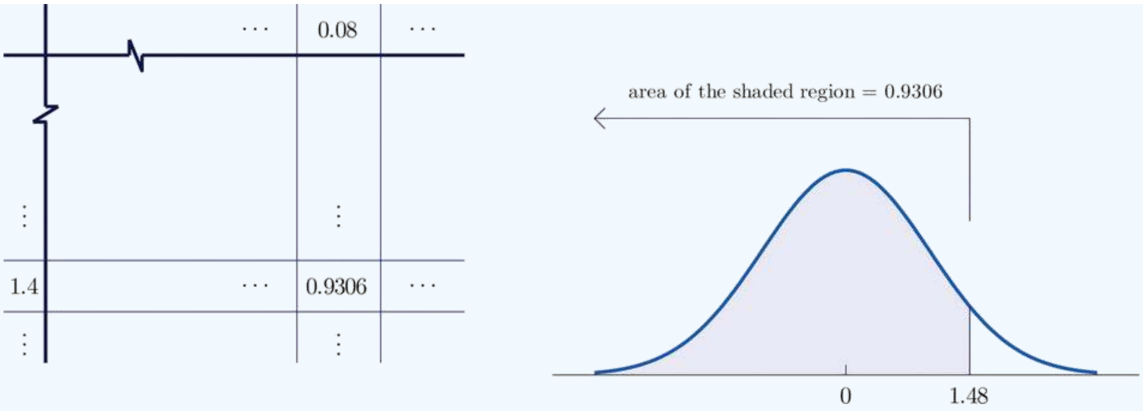
a. The probability of selecting a value less than or equal to $z = 1.48$, or $P(Z \leq 1.48)$.
b. The probability of selecting a value less than or equal to $z = -0.25$, or $P(Z \leq -0.25)$.

**Solution**:

a. It is first helpful to draw the standard normal curve, label the mean, label the given $z$-value, and shade the area of the probability desired. For this problem, we would draw the standard normal curve and label 0 on the $x$-axis at the peak. Then we would estimate where $z = 1.48$ would be to the right of the mean at 0. Since we are finding the probability of selecting a value less than or equal to $z$, we would shade the area to the left.

The figure below shows how this probability is read directly from the table without any computation required. We need to look at the second page of the table since the $z$-value is positive. The digits in the ones and tenths places of $1.48$, namely $1.4$, are used to select the appropriate row of the table; the hundredths part of $1.48$, namely $0.08$, is used to select the appropriate column of the table. The 4-decimal place number in the interior of the table that lies in the intersection of the row and column selected, $0.9306$, is the probability sought since we are looking for the probability to the left of $z = 1.48$:

$$P(Z \leq 1.48) = 0.9306$$



b. The minus sign in $z = -0.25$ makes no difference in the procedure; the table is used in exactly the same way as in part (a): the probability sought is the number that is in the intersection of the row with heading $-0.2$ and the column with heading $0.05$, the number $0.4013$. Thus $P(Z \leq -0.25) = 0.4013$.

## Example 9.2.2

Find the probabilities indicated.

a. The probability of selecting a value greater than $z = 1.60$, or $P(Z > 1.60)$.
b. The probability of selecting a value greater than $z = -1.02$, or $P(Z > -1.02)$.

**Solution**:

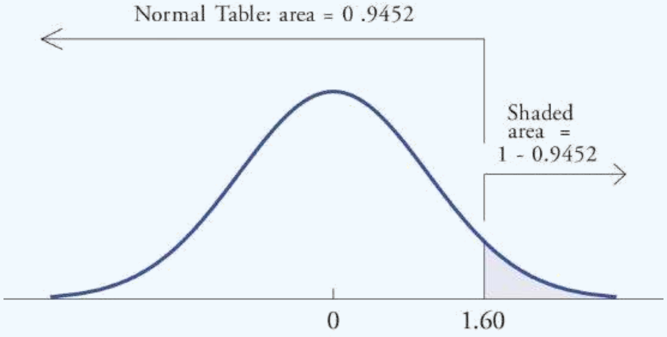
a. Because the events $Z > 1.60$ and $Z \leq 1.60$ are complements, the probability rule for the complement of an event implies that

$$P(Z > 1.60) = 1 - P(Z \leq 1.60) \tag{9.2.1}$$

Since inclusion of the endpoint makes no difference for the standard normal curve $Z$, $P(Z \leq 1.60) = P(Z < 1.60)$, which we know how to find from the table. The number in the row with heading $1.6$ and in the column with heading $0.00$ is $0.9452$. Thus $P(Z < 1.60) = 0.9452$ so

$$P(Z > 1.60) = 1 - P(Z \leq 1.60) = 1 - 0.9452 = 0.0548 \tag{9.2.2}$$

The figure below illustrates the sketch of the probability desired. Since the total area under the curve is $1$ and the area of the region to the left of $1.60$ is (from the table) $0.9452$, the area of the region to the right of $1.60$ must be $1 - 0.9452 = 0.0548$. Thus, (P(Z> 1.60)= 0.0548\).



b. The minus sign in $-1.02$ makes no difference in the procedure; the table is used in exactly the same way as in part (a). The number in the intersection of the row with heading $-1.0$ and the column with heading $0.02$ is $0.1539$. This means that $P(Z < -1.02) = P(Z \leq -1.02) = 0.1539$. Hence $P(Z > -1.02) = 1 - P(Z \leq -1.02) = 1 - 0.1539 = 0.8461$ .

### ✓ ✓ Example 9.2.3

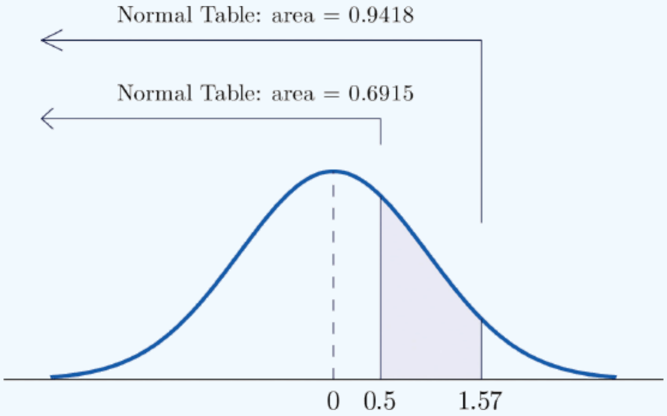
Find the probabilities indicated.

a. The probability of selecting a value between $z = 0.5$ and $z = 1.57$ or $P(0.5 < Z < 1.57)$.
b. The probability of selecting a value between $z = -2.55$ and $z = 0.09$ or $P(-2.55 < Z < 0.09)$.

**Solution**:

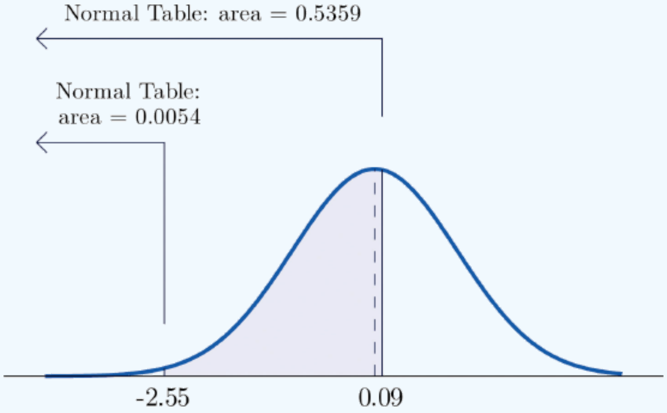
a. The figure below illustrates the ideas involved for intervals of this type. First look up the areas in the table that correspond to the numbers $z = 0.5$ (which we think of as $z = 0.50$ to use the table) and $z = 1.57$. We obtain $0.6915$ and $0.9418$, respectively. From the figure it is apparent that we must take the difference of these two numbers to obtain the probability desired. In symbols,

$$P(0.5 < Z < 1.57) = P(Z < 1.57) - P(Z < 0.50) = 0.9418 - 0.6915 = 0.2503$$



b. The procedure for finding the probability that $Z$ takes a value in an interval whose endpoints have opposite signs is exactly the same procedure used in part (a), and is illustrated in the figure below. In symbols, the computation is

$$P(-2.55 < Z < 0.09) = P(Z < 0.09) - P(Z < -2.55) = 0.5359 - 0.0054 = 0.5305$$



The next example shows what to do if the value of $Z$ that we want to look up in the table is not present there.

### ✓ ✓ Example 9.2.4

Find the probabilities indicated.

1. The probability of selecting a value between $z = 1.13$ and $z = 4.16$ or $P(1.13 < Z < 4.16)$.
2. The probability of selecting a value between $z = -5.22$ and $z = 2.15$ or $P(-5.22 < Z < 2.15)$.

**Solution**:

1. We attempt to compute the probability exactly as in Example $9.2.3$ by looking up the numbers $z = 1.13$ and $z = 4.16$ in the table. We obtain the value $0.8708$ for the area of the region under the density curve to left of $z = 1.13$ without any problem, but when we go to look up the number $z = 4.16$ in the table, it is not there. We can see from the last row of numbers in the table that the area to the left of $z = 4.16$ must be so close to 1 that to 4 decimal places, it rounds to $1.0000$. Therefore

$$P(1.13 < Z < 4.16) = 1.0000 - 0.8708 = 0.1292$$

2. Similarly, here we can read directly from the table that the area under the curve and to the left of $z = 2.15$ is $0.9842$, but $z = -5.22$ is too far to the left on the number line to be in the table. We can see from the first line of the table that the area to the left of $z = -5.22$ must be so close to $0$ that to 4 decimal places, it rounds to $0.0000$. Therefore

$$P(-5.22 < Z < 2.15) = 0.9842 - 0.0000 = 0.9842$$

The Try It below shows the origin of the percents given in the Empirical Rule.

**Try It $9.2.1$**

Find the probabilities indicated.

a. $P(-1 < Z < 1)$.
b. $P(-2 < Z < 2)$.
c. $P(-3 < Z < 3)$.

**Answer**

1. Using the table as was done in Example $9.2.3$ we obtain

$$P(-1 < Z < 1) = 0.8413 - 0.1587 = 0.6826$$

Since $Z$ has mean $0$ and standard deviation $1$, for $Z$ to take a value between $-1$ and $1$ means that $Z$ takes a value that is within one standard deviation of the mean. Our computation shows that the probability that this happens is about $0.68$ or 68%, the percent given by the Empirical Rule for distributions that are bell-shaped and symmetrical.

2. Using the table in the same way,

$$P(-2 < Z < 2) = 0.9772 - 0.0228 = 0.9544$$

This corresponds to about 0.95 or 95%, which is the percent of data within two standard deviations of the mean.

3. Similarly,

$$P(-3 < Z < 3) = 0.9987 - 0.0013 = 0.9974$$

which corresponds to about 0.997 or 99.7%, the percent of data within three standard deviations of the mean.

## Contributor

- Anonymous

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# 9.3: Probability Computations for General Normal Distributions

**Learning Objectives**

- Compute probabilities for any normal distribution using the *Z* table

In most practical problems involving normal distributions, the curve will not be as we have seen so far, with $\mu = 0$ and $\sigma = 1$. When using a *Z* table, you will first have to standardize the distribution by calculating the *z*-score(s).

**Standardizing**

**Standardizing** a data value, $x$, is the process of converting it to a *z*-score using the formula $z = \dfrac{x-\mu}{\sigma}$ .

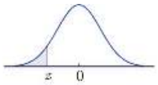
To compute a probability of the form $P(a < X < b)$ we use the following process.

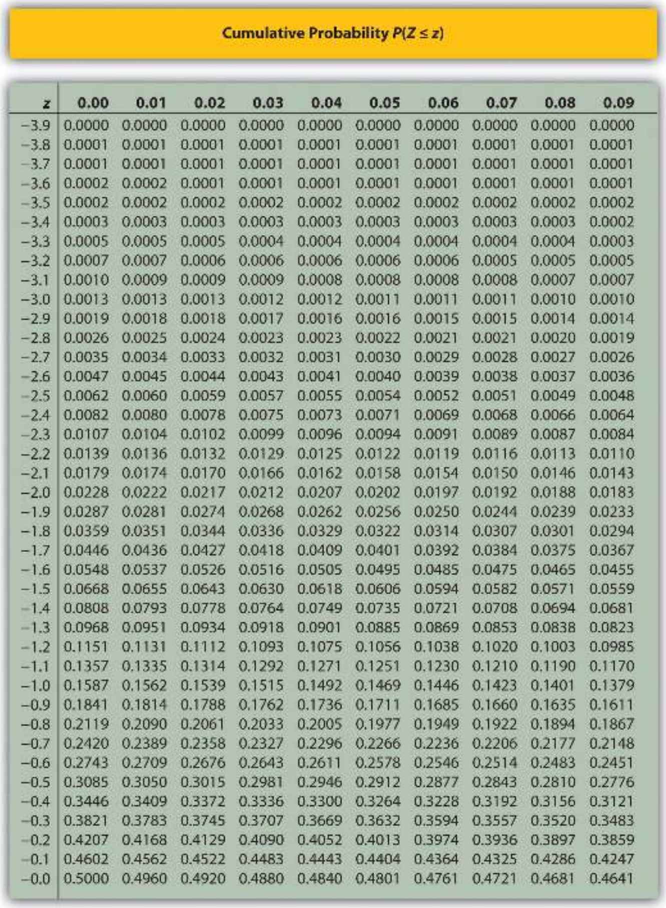
**Probability for General Normal Distributions**

If $X$ is normally distributed with mean $\mu$ and standard deviation $\sigma$, then

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

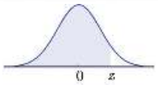
where $Z$ denotes a standard normal distribution. $a$ can be any decimal number or $-\infty$; $b$ can be any decimal number or $\infty$.

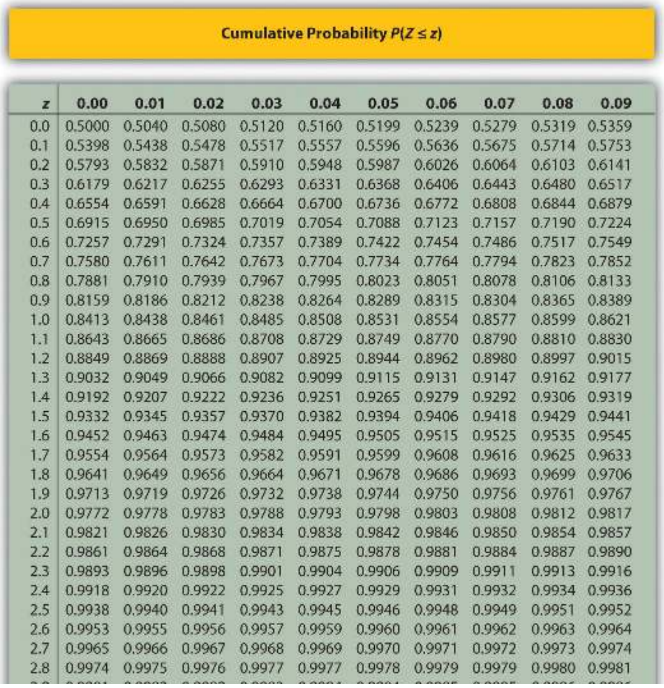




Cumulative Probability $P(Z \le z)$

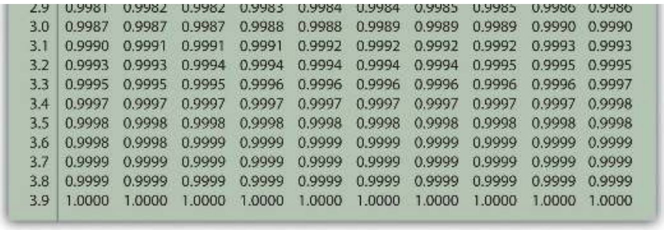
| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| −3.9 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| −3.8 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| −3.7 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| −3.6 | 0.0002 | 0.0002 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| −3.5 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 |
| −3.4 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| −3.3 | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| −3.2 | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| −3.1 | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| −3.0 | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| −2.9 | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| −2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| −2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| −2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| −2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| −2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| −2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| −2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| −2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| −2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| −1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| −1.8 | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| −1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| −1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| −1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| −1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| −1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| −1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| −1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| −1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| −0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| −0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| −0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| −0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| −0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| −0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| −0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| −0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| −0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| −0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |





Cumulative Probability $P(Z \le z)$

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8304 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |

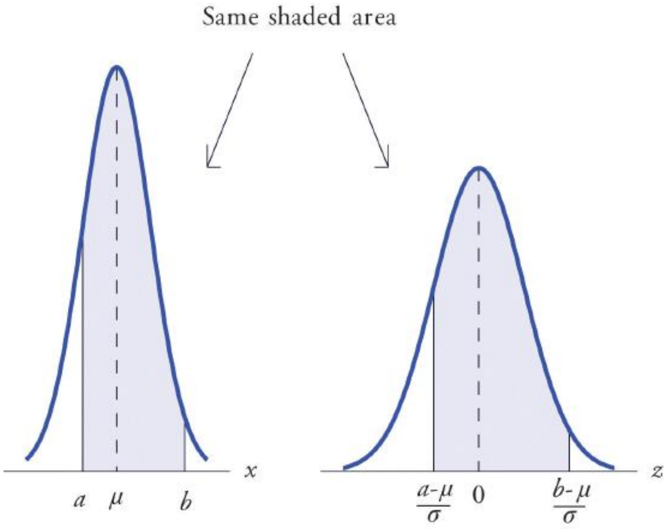


| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.3 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.4 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |
| 3.5 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 |
| 3.6 | 0.9998 | 0.9998 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.7 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.8 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.9 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

The table can also be found online here in section 6.

The new endpoints $\frac{(a-\mu)}{\sigma}$ and $\frac{(b-\mu)}{\sigma}$ are the $z$-scores of $a$ and $b$.

The figure below illustrates the meaning of the process geometrically: the two shaded regions, one under the curve for $X$ and the other under the curve for $Z$, have the same area. Instead of drawing both bell curves, though, we will draw a single generic bell-shaped curve with both an $x$-axis and a $z$-axis below it.



## Example 9.3.1

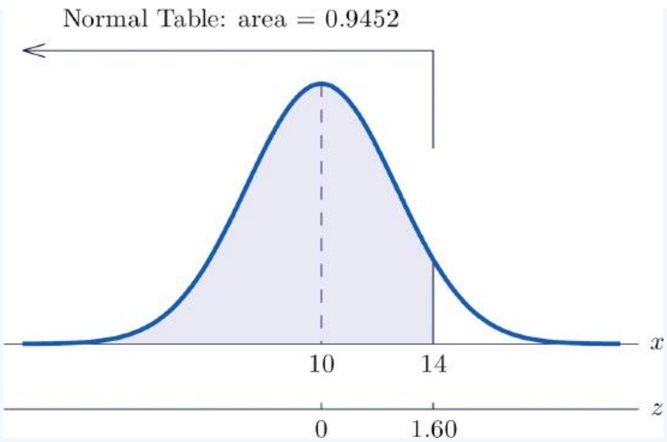
Let $X$ be normally distributed with mean $\mu = 10$ and standard deviation $\sigma = 2.5$. Compute the following probabilities.

a. $P(X < 14)$.
b. $P(8 < X < 14)$.

**Solution**:

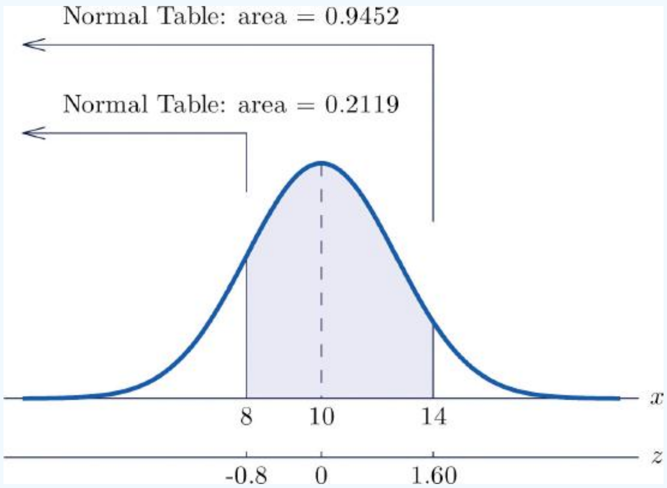
a. See the figure below.

$$\begin{aligned} P(X<14) &= P\left(Z<\frac{14-\mu}{\sigma}\right) \\ &= P\left(Z<\frac{14-10}{2.5}\right) \\ &= P(Z<1.60) \\ &= 0.9452 \end{aligned}$$



b. See the figure below.

$$\begin{aligned} P(8 < X < 14) &= P\left(\frac{8-10}{2.5} < Z < \frac{14-10}{2.5}\right) \\ &= P(-0.80 < Z < 1.60) \\ &= 0.9452 - 0.2119 \\ &= 0.7333 \end{aligned}$$



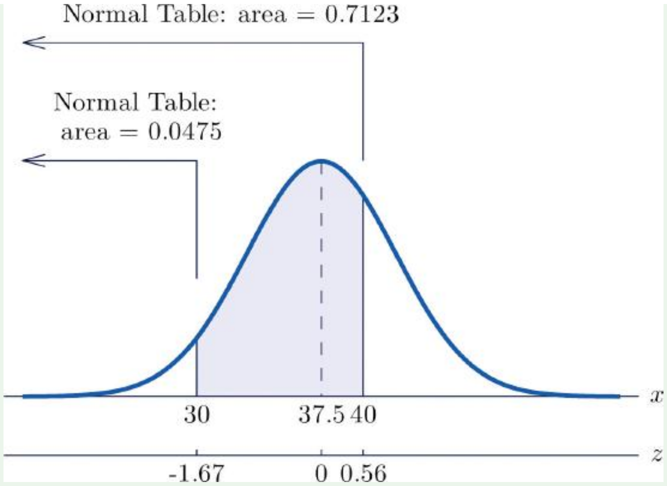
### Try It 9.3.1

Let $X$ be normally distributed with mean $37.5$ and standard deviation $4.5$. Find the probability that $X$ will be between $30$ and $40$.

**Answer**

For $X$, $\mu = 37.5$ and $\sigma = 4.5$. The problem is to compute $P(30 < X < 40)$. The figure below illustrates the following computation:

$$\begin{aligned} P(30 < X < 40) &= P\left(\frac{30-\mu}{\sigma} < Z < \frac{40-\mu}{\sigma}\right) \\ &= P\left(\frac{30-37.5}{4.5} < Z < \frac{40-37.5}{4.5}\right) \\ &= P(-1.67 < Z < 0.56) \\ &= 0.7123 - 0.0475 \\ &= 0.6648 \end{aligned}$$



Note that the two $z$-scores were rounded to two decimal places in order to use the table.

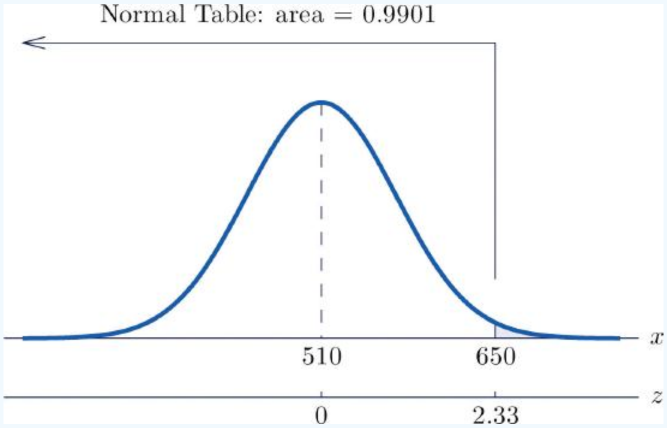
✓ ✓ Example $9.3.2$

Let $X$ be normally distributed with mean $510$ and standard deviation $60$. Find the percentage of data values that exceed 650.

**Solution**:

$X$ is normally distributed with mean $\mu = 510$ and standard deviation $\sigma = 60$. The probability that $X$ lies in a particular interval is the same as the proportion or percent of all data values that lie in that interval. Thus the solution to the problem is $P(X > 650)$, expressed as a percentage. The figure below illustrates the following computation:

$$\begin{aligned} P(X > 650) &= P\left(Z > \frac{650 - \mu}{\sigma}\right) \\ &= P\left(Z > \frac{650 - 510}{60}\right) \\ &= P(Z > 2.33) \\ &= 1 - 0.9901 \\ &= 0.0099 \end{aligned}$$



The percent of all data values that exceed $650$ is $0.0099$, hence $0.99\%$ or about $1\%$.

## Contributor

- Anonymous

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# 9.4: Applications of Normal Distributions

**Learning Objectives**

- Apply the characteristics of a normal distribution to solve application problems

## Introduction

This section will cover some of the types of questions that can be answered using the properties of a normal distribution. The first examples deal with more theoretical questions that will help you master basic understandings and computational skills, while the later problems will provide examples with real data, or at least a real context.

## Normal Distributions with Real Data

The foundation of performing experiments by collecting surveys and samples is most often based on the normal distribution. Here are two examples to get you started.

**Example 9.4.1**

The Information Centre of the National Health Service in Britain collects and publishes a great deal of information and statistics on health issues affecting the population. One such comprehensive data set tracks information about the health of children [1]. According to its statistics, in 2006, the mean height of 12-year-old boys was 152.9 cm, with a standard deviation of 8.5 cm. If 12-year-old Cecil is 158 cm, approximately what percentage of all 12-year-old boys in Britain is he taller than?

**Solution**

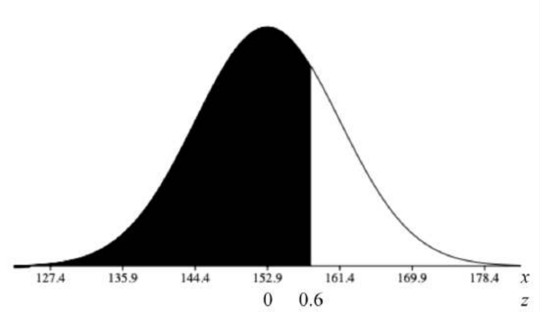
We first must assume that the height of 12-year-old boys in Britain is normally distributed, and this seems like a reasonable assumption to make. We need to find the percentage, or probability, of a 12-year-old boy being shorter than 158 cm, i.e., $P(X<158)$. We let $x=158$ and calculate the z-score:

$$z=\frac{x-\mu}{\sigma}$$

$$z=\frac{158-152.9}{8.5}$$

$$z=0.6$$

As always, draw a sketch and estimate a reasonable answer prior to calculating the percentage.



From the table, the probability is $P(Z<0.6)=0.7257$. We would estimate that Cecil is taller than about 73% of 12-year-old boys. We could also phrase our assumption this way: the probability of a randomly selected British 12- year-old boy being shorter than Cecil is about 0.73. Often with data like this, we use percentiles. We would say that Cecil is in the 73rd percentile for height among 12-year-old boys in Britain.

### ✓ Example 9.4.2

Suppose that the distribution of the mass of female marine iguanas in Puerto Villamil in the Galapagos Islands is approximately normal, with a mean mass of 950 g and a standard deviation of 325 g. There are very few young marine iguanas in the populated areas of the islands because feral cats tend to kill them. How rare is it that we would find a female marine iguana with a mass less than 400 g in this area?

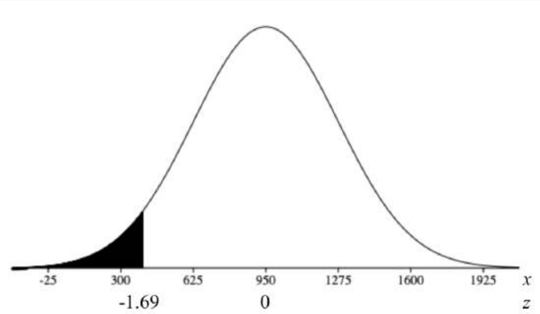
**Solution**

We need to find the probability of a female marine iguana being less than 400 grams, i.e., $P(X<400)$. We let $x=400$ and calculate the z-score:

$$z=\frac{x-\mu}{\sigma}$$

$$z=\frac{400-950}{325}$$

$$z\approx -1.69$$



We need to find $P(Z<-1.69)$. Using the table, we obtain a probability of $0.0455$. With a probability of approximately 0.0455, or only about 5%, we could say it is rather unlikely that we would find an iguana this small.

### ✎ Try It 9.4.1

A candy company sells small bags of candy and attempts to keep the number of pieces in each bag the same, though small differences due to random variation in the packaging process lead to different amounts in individual packages. A quality control expert from the company has determined that the mean number of pieces in each bag is normally distributed, with a mean of 57.3 and a standard deviation of 1.2. Endy opened a bag of candy and felt he was cheated. His bag contained only 55 candies. Does Endy have reason to complain?

**Answer**

To determine if Endy was cheated, we need to find the probability of selecting a bag of candy with 55 or fewer candies, i.e., \(P(X \leq 55)\

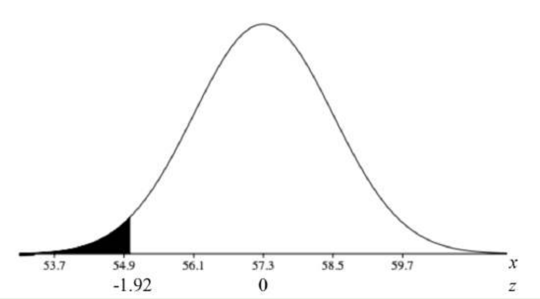
). We let $x=55$ and calculate the z-score:

$$z=\frac{x-\mu}{\sigma}$$

$$z=\frac{55-57.3}{1.2}$$

$$z\approx -1.92$$

Next, we can draw a figure to see the shaded region:



We need to find $P(Z < -1.92)$. Using the table, we obtain a probability of $0.0274$ Hence, there is about a 3% chance that he would get a bag of candy with 55 or fewer pieces, so Endy should feel cheated because the chances of getting a bag with 55 or fewer candies is so low.

## ✓ Example 9.4.3

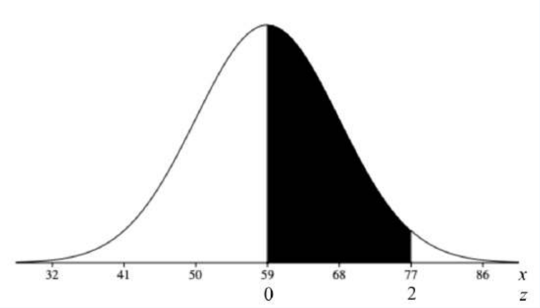
The physical plant at the main campus of a large state university receives daily requests to replace florescent light bulbs. The distribution of the number of daily requests is bell-shaped and has a mean of 59 and a standard deviation of 9. What is the percentage of light bulb replacement requests numbering between 59 and 77?

**Solution**

We need to calculate $P(59 < X < 77)$ given $\mu = 59$ and $\sigma = 9$. Re-writing the probability with z-scores:

$$P(59 < X < 77) = P\left(\frac{59-59}{9} < Z < \frac{77-59}{9}\right) = P(0 < Z < 2)$$

Using the table, the probability is $0.9772 - 0.5000 = 0.4772$ Thus, $47.72\%$ of daily light bulb replacement requests number between 59 and 77.



Note: this problem could have been approximated using the Empirical Rule.

## ✎ Try It 9.4.2

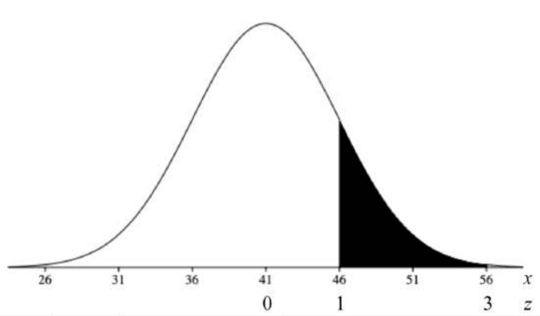
A company has a policy of retiring company cars; this policy looks at number of miles driven, purpose of trips, style of car and other features. The distribution of the number of months in service for the fleet of cars is bell-shaped and has a mean of 41 months and a standard deviation of 5 months. What is the percentage of cars that remain in service between 46 and 56 months?

**Answer**

We need to calculate $P(46 < X < 56)$ given $\mu = 41$ and $\sigma = 5$. Re-writing the probability with z-scores:

$$P(46 < X < 56) = P\left(\frac{46-41}{5} < Z < \frac{56-41}{5}\right) = P(1 < Z < 3)$$

Using the table, the probability is $0.9987 - 0.8413 = 0.1574$Thus, $15.74\%$of cars remain in service between 46 and 56 months.



Note: this problem could also have been approximated using the Empirical Rule.

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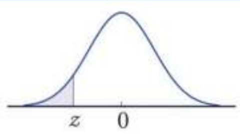
# 9.5: Chapter Review and Glossary

# 9.6: Cumulative Standard Normal Distribution Table

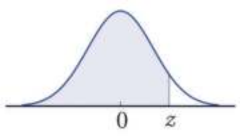
## Cumulative Standard Normal Distribution Table



Cumulative Probability $P(Z \leq z)$

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **-3.9** | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **-3.8** | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| **-3.7** | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| **-3.6** | 0.0002 | 0.0002 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| **-3.5** | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 |
| **-3.4** | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| **-3.3** | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| **-3.2** | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| **-3.1** | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| **-3.0** | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| **-2.9** | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| **-2.8** | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| **-2.7** | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| **-2.6** | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| **-2.5** | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| **-2.4** | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| **-2.3** | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| **-2.2** | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| **-2.1** | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| **-2.0** | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| **-1.9** | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| **-1.8** | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| **-1.7** | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| **-1.6** | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| **-1.5** | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| **-1.4** | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| **-1.3** | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| **-1.2** | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **-1.1** | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| **-1.0** | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| **-0.9** | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| **-0.8** | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| **-0.7** | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| **-0.6** | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| **-0.5** | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| **-0.4** | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| **-0.3** | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| **-0.2** | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| **-0.1** | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| **-0.0** | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |



Cumulative Probability $P(Z \le z)$

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| **0.1** | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| **0.2** | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| **0.3** | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| **0.4** | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| **0.5** | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| **0.6** | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| **0.7** | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| **0.8** | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| **0.9** | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8304 | 0.8365 | 0.8389 |
| **1.0** | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| **1.1** | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| **1.2** | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| **1.3** | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| **1.4** | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| **1.5** | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| **1.6** | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| **1.7** | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **1.8** | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| **1.9** | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| **2.0** | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| **2.1** | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| **2.2** | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| **2.3** | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| **2.4** | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| **2.5** | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| **2.6** | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| **2.7** | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| **2.8** | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| **2.9** | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| **3.0** | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| **3.1** | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| **3.2** | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| **3.3** | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| **3.4** | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |
| **3.5** | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 |
| **3.6** | 0.9998 | 0.9998 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| **3.7** | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| **3.8** | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| **3.9** | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

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# 9.7: Exercises

## Exercises

1. Which of the following data sets is most likely to be normally distributed? For the other choices, explain why you believe they would not follow a normal distribution.

a) The hand span (measured from the tip of the thumb to the tip of the extended 5th finger) of a random sample of high school seniors

b) The annual salaries of all employees of a large shipping company

c) The annual salaries of a random sample of 50 CEOs of major companies, 25 women and 25 men

d) The dates of 100 pennies taken from a cash drawer in a convenience store

2. The grades on a statistics mid-term for a high school are normally distributed, with $\mu = 81$ and $\sigma = 6.3$. Calculate the z-scores for each of the following exam grades. Draw and label a sketch for each example. 65, 83, 93, 100

3. Assume that the mean weight of 1-year-old girls in the USA is normally distributed, with a mean of about 9.5 kilograms and a standard deviation of approximately 1.1 kilograms. Without using a calculator, estimate the percentage of 1-year-old girls who meet the following conditions. Draw a sketch and shade the proper region for each problem.

a) Less than 8.4 kg

b) Between 7.3 kg and 11.7 kg

c) More than 12.8 kg

4. For a standard normal distribution, place the following in order from smallest to largest.

a) The percentage of data below 1

b) The percentage of data below $-1$

c) The mean

d) The standard deviation

e) The percentage of data above 2

5. The 2007 AP Statistics examination scores were not normally distributed, with $\mu = 2.8$ and $\sigma = 1.34$ [1]. What is the approximate z-score that corresponds to an exam score of 5? (The scores range from 1 to 5.)

a) 0.786

b) 1.46

c) 1.64

d) 2.20

e) A z-score cannot be calculated because the distribution is not normal.

6. The heights of 5th-grade boys in the USA is approximately normally distributed, with a mean height of 143.5 cm and a standard deviation of about 7.1 cm. What is the probability that a randomly chosen 5th-grade boy would be taller than 157.7 cm?

7. A statistics class bought some sprinkle (or jimmies) doughnuts for a treat and noticed that the number of sprinkles seemed to vary from doughnut to doughnut, so they counted the sprinkles on each doughnut.

Here are the results: 241, 282, 258, 223, 133, 335, 322, 323, 354, 194, 332, 274, 233, 147, 213, 262, 227, and 366.

Create a histogram, dot plot, or box plot for this data. Comment on the shape, center, and spread of the distribution.

1. The physical plant at the main campus of a large state university receives daily requests to replace fluorescent light bulbs. The distribution of the number of daily requests is bell-shaped and has a mean of 56 and a standard deviation of 4. Using the Empirical Rule, what is the approximate percentage of light bulb replacement requests numbering between 56 and 68?

2. A company has a policy of retiring company cars; this policy looks at the number of miles driven, the purpose of trips, style of car and other features. The distribution of the number of months in service for the fleet of cars is bell-shaped and has a mean of 65 months and a standard deviation of 4 months. Using the Empirical Rule, what is the approximate percentage of cars that remain in service between 57 and 61 months?

3. The Acme Company manufactures widgets. The distribution of widget weights is bell-shaped. The widget weights have a mean of 48 ounces and a standard deviation of 11 ounces. Suggestion: sketch the distribution in order to answer these questions.

a. 99.7% of the widget weights lie between ________ and ________.

b. What percentage of the widget weights lie between 26 and 81 ounces?

c. What percentage of the widget weights lie above 37?

4. For a standard normal distribution, find the following probabilities:

a. $P(z<1.42)$

b. $P(z>-2.52)$

c. $P(-2.06<z<2.81)$

5. For a standard normal distribution, given $P(z<c)=0.7055$, find $c$.

6. For a standard normal distribution, given $P(z>c)=0.7109$, find $c$.

7. On a nationwide math test, the mean was 72 and the standard deviation was 10. If Roberto scored 70, what was his z-score?

8. On a nationwide math test, the mean was 66 and the standard deviation was 4. If Roberto scored 75, what was his z-score?

9. On a nationwide math test, the mean was 57 and the standard deviation was 4. If Roberto scored 85, what was his z-score?

10. A quick survey of peanut butter prices had a standard deviation and mean of \$0.26 and \$3.68, respectively. Compute the area for a peanut butter jar costing less than \$3.50.

11. A quick survey of peanut butter prices had a standard deviation and mean of \$0.26 and \$3.68, respectively. Compute the area for a peanut butter jar costing more than \$4.25.

12. A quick survey of peanut butter prices had a standard deviation and mean of \$0.26 and \$3.68, respectively. Compute the area for a peanut butter jar costing between \$3.50 and \$4.25.

13. A quick survey of peanut butter prices had a standard deviation and mean of \$0.81 and \$3.22, respectively. Compute the price for a peanut butter jar costing given the area from the mean is 0.48422.

14. A quick survey of peanut butter prices had a standard deviation and mean of \$1.53 and \$2.22, respectively. Compute the price for a peanut butter jar costing given the area from the mean is 0.13683.

1. Which of the following intervals contains the middle 95% of the data in a standard normal distribution?

a) $z<2$

b) $z\leq 1.645$

c) $z\leq 1.96$

d) $-1.645\leq z\leq 1.645$

e) $-1.96\leq z\leq 1.96$

2. The manufacturing process at a metal-parts factory produces some slight variation in the diameter of metal ball bearings. The quality control experts claim that the bearings produced have a mean diameter of 1.4 cm. If the diameter is more than 0.0035 cm too wide or too narrow, they will not work properly. In order to maintain its reliable reputation, the company wishes to ensure that no more than one-tenth of 1% of the bearings that are made are ineffective. What would the standard deviation of the manufactured bearings need to be in order to meet this goal?

3. Suppose that the wrapper of a certain candy bar lists its weight as 2.13 ounces. Naturally, the weights of individual bars vary somewhat. Suppose that the weights of these candy bars vary according to a normal distribution, with $\mu=2.2$ ounces and $\sigma=0.04$ ounces.

a) What proportion of the candy bars weigh less than the advertised weight?

b) What proportion of the candy bars weight between 2.2 and 2.3 ounces?

c) A candy bar of what weight would be heavier than all but 1% of the candy bars out there?

d) If the manufacturer wants to adjust the production process so that no more than 1 candy bar in 1000 weighs less than the advertised weight, what would the mean of the actual weights need to be? (Assume the standard deviation remains the same.)

e) If the manufacturer wants to adjust the production process so that the mean remains at 2.2 ounces and no more than 1 candy bar in 1000 weighs less than the advertised weight, how small does the standard deviation of the weights need to be?

4. The Acme Company manufactures widgets. The distribution of widget weights is bell-shaped. The widget weights have a mean of 51 ounces and a standard deviation of 4 ounces. Use the Empirical Rule to answer the following questions.

a) 99.7% of the widget weights lie between what two weights?

b) What percentage of the widget weights lie between 43 and 63 ounces?

c) What percentage of the widget weights lie above 47?

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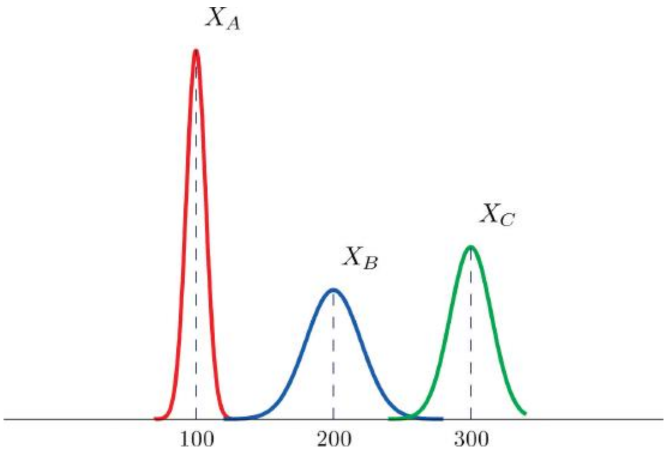
# 9.E: Continuous Random Variables (Exercises)

These are homework exercises to accompany the Textmap created for "Introductory Statistics" by Shafer and Zhang.

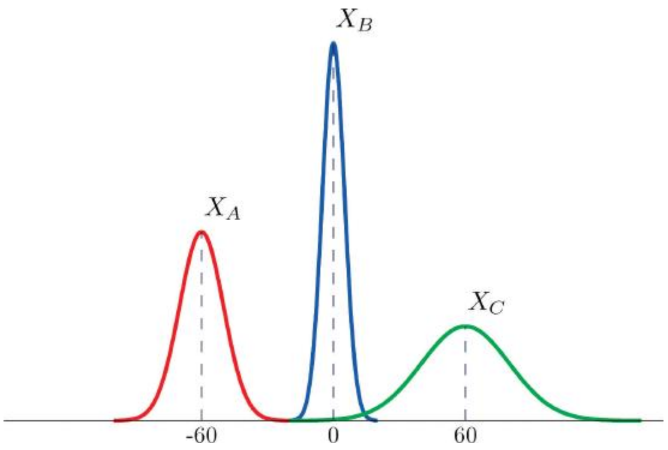
## 5.1: Continuous Random Variables

### Basic

1. A continuous random variable $X$ has a uniform distribution on the interval $[5, 12]$. Sketch the graph of its density function.
2. A continuous random variable $X$ has a uniform distribution on the interval $[-3, 3]$. Sketch the graph of its density function.
3. A continuous random variable $X$ has a normal distribution with mean $100$ and standard deviation $10$. Sketch a qualitatively accurate graph of its density function.
4. A continuous random variable $X$ has a normal distribution with mean $73$ and standard deviation $2.5$. Sketch a qualitatively accurate graph of its density function.
5. A continuous random variable $X$ has a normal distribution with mean $73$. The probability that $X$ takes a value greater than $80$ is $0.212$. Use this information and the symmetry of the density function to find the probability that $X$ takes a value less than $66$. Sketch the density curve with relevant regions shaded to illustrate the computation.
6. A continuous random variable $X$ has a normal distribution with mean $169$. The probability that $X$ takes a value greater than $180$ is $0.17$. Use this information and the symmetry of the density function to find the probability that $X$ takes a value less than $158$. Sketch the density curve with relevant regions shaded to illustrate the computation.
7. A continuous random variable $X$ has a normal distribution with mean $50.5$. The probability that $X$ takes a value less than $54$ is $0.76$. Use this information and the symmetry of the density function to find the probability that $X$ takes a value greater than $47$. Sketch the density curve with relevant regions shaded to illustrate the computation.
8. A continuous random variable $X$ has a normal distribution with mean $12.25$. The probability that $X$ takes a value less than $13$ is $0.82$. Use this information and the symmetry of the density function to find the probability that $X$ takes a value greater than $11.50$. Sketch the density curve with relevant regions shaded to illustrate the computation.
9. The figure provided shows the density curves of three normally distributed random variables $X_A,\ X_B$ and $X_C$. Their standard deviations (in no particular order) are $15$, $7$, and $20$. Use the figure to identify the values of the means $\mu_A,\ \mu_B,$ and $\mu_C$ and standard deviations $\sigma_A,\ \sigma_B,$ and $\sigma_C$ of the three random variables.



10. The figure provided shows the density curves of three normally distributed random variables $X_A,\ X_B$ and $X_C$. Their standard deviations (in no particular order) are $20$, $5$, and $10$. Use the figure to identify the values of the means $\mu_A,\ \mu_B,$ and $\mu_C$ and standard deviations $\sigma_A,\ \sigma_B,$ and $\sigma_C$ of the three random variables.



## Applications

11. Dogberry's alarm clock is battery operated. The battery could fail with equal probability at any time of the day or night. Every day Dogberry sets his alarm for $6:30\ a.m.$ and goes to bed at $10:00\ p.m.$. Find the probability that when the clock battery finally dies, it will do so at the most inconvenient time, between $10:00\ p.m.$ and $6:30\ a.m.$.
12. Buses running a bus line near Desdemona's house run every $15$ minutes. Without paying attention to the schedule she walks to the nearest stop to take the bus to town. Find the probability that she waits more than $10$ minutes.
13. The amount $X$ of orange juice in a randomly selected half-gallon container varies according to a normal distribution with mean $64$ ounces and standard deviation $0.25$ ounce.
    a. Sketch the graph of the density function for $X$.
    b. What proportion of all containers contain less than a half gallon ($64$ ounces)? Explain.
    c. What is the median amount of orange juice in such containers? Explain.
14. The weight $X$ of grass seed in bags marked $50$ lb varies according to a normal distribution with mean $50$ lb and standard deviation $1$ ounce ($0.0625$ lb).
    a. Sketch the graph of the density function for $X$.
    b. What proportion of all bags weigh less than $50$ pounds? Explain.
    c. What is the median weight of such bags? Explain.

## Answers

1. The graph is a horizontal line with height $1/7$ from $x=5$ to $x=12$
2. 
3. The graph is a bell-shaped curve centered at $100$ and extending from about $70$ to $130$.
4. 
5. $0.212$
6. 
7. $0.76$
8. 
9. $\mu_A=100,\ \mu_B=200,\ \mu_C=300,\ \sigma_A=7,\ \sigma_B=20,\ \sigma_C=15$
10. 
11. $0.3542$
12. 
13. a. The graph is a bell-shaped curve centered at $64$ and extending from about $63.25$ to $64.75$.
    b. $0.5$
    c. $64$

## 5.2: The Standard Normal Distribution

## Basic

1. Use Figure 7.1.5: Cumulative Normal Probability to find the probability indicated.$/* <![CDATA[*/9.E.5/*]] > */$
   1. $P(Z < -1.72)$
   2. $P(Z < 2.05)$
   3. $P(Z < 0)$
   4. $P(Z > -2.11)$
   5. $P(Z > 1.63)$
   6. $P(Z > 2.36)$
2. Use Figure 7.1.5: Cumulative Normal Probability to find the probability indicated.$/* <![CDATA[*/9.E.5/*]] > */$
   1. $P(Z < -1.17)$
   2. $P(Z < -0.05)$
   3. $P(Z < 0.66)$
   4. $P(Z > -2.43)$
   5. $P(Z > -1.00)$
   6. $P(Z > 2.19)$
3. Use Figure 7.1.5: Cumulative Normal Probability to find the probability indicated.
   1. $P(-2.15 < Z < -1.09)$
   2. $P(-0.93 < Z < 0.55)$
   3. $P(0.68 < Z < 2.11)$
4. Use Figure 7.1.5: Cumulative Normal Probability to find the probability indicated.
   1. $P(-1.99 < Z < -1.03)$
   2. $P(-0.87 < Z < 1.58)$
   3. $P(0.33 < Z < 0.96)$
5. Use Figure 7.1.5: Cumulative Normal Probability to find the probability indicated.
   1. $P(-4.22 < Z < -1.39)$
   2. $P(-1.37 < Z < 5.11)$
   3. $P(Z < -4.31)$
   4. $P(Z < 5.02)$
6. Use Figure 7.1.5: Cumulative Normal Probability to find the probability indicated.
   1. $P(Z > -5.31)$
   2. $P(-4.08 < Z < 0.58)$
   3. $P(Z < -6.16)$
   4. $P(-0.51 < Z < 5.63)$
7. Use Figure 7.1.5: Cumulative Normal Probability to find the probability listed. Find the second probability without referring to the table, but using the symmetry of the standard normal density curve instead. Sketch the density curve with relevant regions shaded to illustrate the computation.
   1. $P(Z < -1.08),\ P(Z > 1.08)$
   2. $P(Z < -0.36),\ P(Z > 0.36)$
   3. $P(Z < 1.25),\ P(Z > -1.25)$
   4. $P(Z < 2.03),\ P(Z > -2.03)$
8. Use Figure 7.1.5: Cumulative Normal Probability to find the probability listed. Find the second probability without referring to the table, but using the symmetry of the standard normal density curve instead. Sketch the density curve with relevant regions shaded to illustrate the computation.
   1. $P(Z < -2.11),\ P(Z > 2.11)$
   2. $P(Z < -0.88),\ P(Z > 0.88)$
   3. $P(Z < 2.44),\ P(Z > -2.44)$
   4. $P(Z < 3.07),\ P(Z > -3.07)$
9. The probability that a standard normal random variable $Z$ takes a value in the union of intervals $(-\infty, -\alpha] \cup [\alpha, \infty)$, which arises in applications, will be denoted $P(Z \leq -a \ or \ Z \geq a)$ . Use Figure 7.1.5: Cumulative Normal Probability to find the

following probabilities of this type. Sketch the density curve with relevant regions shaded to illustrate the computation. Because of the symmetry of the standard normal density curve you need to use Figure 7.1.5: Cumulative Normal Probability only one time for each part.

   a. $P(Z < -1.29 \; or \; Z > 1.29)$
   b. $P(Z < -2.33 \; or \; Z > 2.33)$
   c. $P(Z < -1.96 \; or \; Z > 1.96)$
   d. $P(Z < -3.09 \; or \; Z > 3.09)$

10. The probability that a standard normal random variable $Z$ takes a value in the union of intervals $(-\infty, -\alpha] \cup [\alpha, \infty)$, which arises in applications, will be denoted $P(Z \leq -a \; or \; Z \geq a)$ . Use Figure 7.1.5: Cumulative Normal Probability to find the following probabilities of this type. Sketch the density curve with relevant regions shaded to illustrate the computation. Because of the symmetry of the standard normal density curve you need to use Figure 7.1.5: Cumulative Normal Probability only one time for each part.
    a. $P(Z < -2.58 \; or \; Z > 2.58)$
    b. $P(Z < -2.81 \; or \; Z > 2.81)$
    c. $P(Z < -1.65 \; or \; Z > 1.65)$
    d. $P(Z < -2.43 \; or \; Z > 2.43)$

## Answers

1. a. $0.0427$
   b. $0.9798$
   c. $0.5$
   d. $0.9826$
   e. $0.0516$
   f. $0.0091$
2. 
3. a. $0.1221$
   b. $0.5326$
   c. $0.2309$
4. 
5. a. $0.0823$
   b. $0.9147$
   c. $0.0000$
   d. $1.0000$
6. 
7. a. $0.1401, \; 0.1401$
   b. $0.3594, \; 0.3594$
   c. $0.8944, \; 0.8944$
   d. $0.9788, \; 0.9788$
8. 
9. a. $0.1970$
   b. $0.01980$
   c. $0.0500$
   d. $0.0020$

## 5.3: Probability Computations for General Normal Random Variables

### Basic

1. $X$ is a normally distributed random variable with mean $57$ and standard deviation $6$. Find the probability indicated.
   a. $P(X < 59.5)$
   b. $P(X < 46.2)$
   c. $P(X > 52.2)$

   d. $P(X>70)$
2. $X$ is a normally distributed random variable with mean $-25$ and standard deviation $4$. Find the probability indicated.
   a. $P(X<-27.2)$
   b. $P(X<-14.8)$
   c. $P(X>-33.1)$
   d. $P(X>-16.5)$
3. $X$ is a normally distributed random variable with mean $112$ and standard deviation $15$. Find the probability indicated.
   a. $P(100<X<125)$
   b. $P(91<X<107)$
   c. $P(118<X<160)$
4. $X$ is a normally distributed random variable with mean $72$ and standard deviation $22$. Find the probability indicated.
   a. $P(78<X<127)$
   b. $P(60<X<90)$
   c. $P(49<X<71)$
5. $X$ is a normally distributed random variable with mean $500$ and standard deviation $25$. Find the probability indicated.
   a. $P(X<400)$
   b. $P(466<X<625)$
6. $X$ is a normally distributed random variable with mean $0$ and standard deviation $0.75$. Find the probability indicated.
   a. $P(-4.02<X<3.82)$
   b. $P(X>4.11)$
7. $X$ is a normally distributed random variable with mean $15$ and standard deviation $1$. Use Figure 7.1.5 $/\ast<![CDATA[\ast/9.E.5/\ast]]>\ast/$ Cumulative Normal Probability to find the first probability listed. Find the second probability using the symmetry of the density curve. Sketch the density curve with relevant regions shaded to illustrate the computation.
   a. $P(X<12),\ P(X>18)$
   b. $P(X<14),\ P(X>16)$
   c. $P(X<11.25),\ P(X>18.75)$
   d. $P(X<12.67),\ P(X>17.33)$
8. $X$ is a normally distributed random variable with mean $100$ and standard deviation $10$. Use Figure 7.1.5 $/\ast<![CDATA[\ast/9.E.5/\ast]]>\ast/$ Cumulative Normal Probability to find the first probability listed. Find the second probability using the symmetry of the density curve. Sketch the density curve with relevant regions shaded to illustrate the computation.
   a. $P(X<80),\ P(X>120)$
   b. $P(X<75),\ P(X>125)$
   c. $P(X<84.55),\ P(X>115.45)$
   d. $P(X<77.42),\ P(X>122.58)$
9. $X$ is a normally distributed random variable with mean $67$ and standard deviation $13$. The probability that $X$ takes a value in the union of intervals $(-\infty,67-a]\cup[67+a,\infty)$ will be denoted $P(X\le 67-a \text{ or } X\ge 67+a)$ . Use Figure 7.1.5 $/\ast<![CDATA[\ast/9.E.5/\ast]]>\ast/$ Cumulative Normal Probability to find the following probabilities of this type. Sketch the density curve with relevant regions shaded to illustrate the computation. Because of the symmetry of the density curve you need to use Figure 7.1.5$/\ast<![CDATA[\ast/9.E.5/\ast]]>\ast/$ Cumulative Normal Probability only one time for each part.
   a. $P(X<57 \text{ or } X>77)$
   b. $P(X<47 \text{ or } X>87)$
   c. $P(X<49 \text{ or } X>85)$
   d. $P(X<37 \text{ or } X>97)$
10. $X$ is a normally distributed random variable with mean $288$ and standard deviation $6$. The probability that $X$ takes a value in the union of intervals $(-\infty,288-a]\cup[288+a,\infty)$ will be denoted $P(X\le 288-a \text{ or } X\ge 288+a)$ . Use Figure 7.1.5 $/\ast<![CDATA[\ast/9.E.5/\ast]]>\ast/$ Cumulative Normal Probability to find the following probabilities of this type. Sketch the

density curve with relevant regions shaded to illustrate the computation. Because of the symmetry of the density curve you need to use Figure $7.1.5/* <![CDATA[*/9.E.5/*]] > */$: Cumulative Normal Probability only one time for each part.

a. $P(X < 278 \text{ or } X > 298)$
b. $P(X < 268 \text{ or } X > 308)$
c. $P(X < 273 \text{ or } X > 303)$
d. $P(X < 280 \text{ or } X > 296)$

## Applications

11. The amount $X$ of beverage in a can labeled $12$ ounces is normally distributed with mean $12.1$ ounces and standard deviation $0.05$ ounce. A can is selected at random.
    a. Find the probability that the can contains at least $12$ ounces.
    b. Find the probability that the can contains between $11.9$ and $12.1$ ounces.
12. The length of gestation for swine is normally distributed with mean $114$ days and standard deviation $0.75$ day. Find the probability that a litter will be born within one day of the mean of $114$.
13. The systolic blood pressure $X$ of adults in a region is normally distributed with mean $112$ mm Hg and standard deviation $15$ mm Hg. A person is considered “prehypertensive” if his systolic blood pressure is between $120$ and $130$ mm Hg. Find the probability that the blood pressure of a randomly selected person is prehypertensive.
14. Heights $X$ of adult women are normally distributed with mean $63.7$ inches and standard deviation $2.71$ inches. Romeo, who is $69.25$ inches tall, wishes to date only women who are shorter than he but within $4$ inches of his height. Find the probability that the next woman he meets will have such a height.
15. Heights $X$ of adult men are normally distributed with mean $69.1$ inches and standard deviation $2.92$ inches. Juliet, who is $63.25$ inches tall, wishes to date only men who are taller than she but within $6$ inches of her height. Find the probability that the next man she meets will have such a height.
16. A regulation hockey puck must weigh between $5.5$ and $6$ ounces. The weights $X$ of pucks made by a particular process are normally distributed with mean $5.75$ ounces and standard deviation $0.11$ ounce. Find the probability that a puck made by this process will meet the weight standard.
17. A regulation golf ball may not weigh more than $1.620$ ounces. The weights $X$ of golf balls made by a particular process are normally distributed with mean $1.361$ ounces and standard deviation $0.09$ ounce. Find the probability that a golf ball made by this process will meet the weight standard.
18. The length of time that the battery in Hippolyta's cell phone will hold enough charge to operate acceptably is normally distributed with mean $25.6$ hours and standard deviation $0.32$ hour. Hippolyta forgot to charge her phone yesterday, so that at the moment she first wishes to use it today it has been $26$ hours $18$ minutes since the phone was last fully charged. Find the probability that the phone will operate properly.
19. The amount of non-mortgage debt per household for households in a particular income bracket in one part of the country is normally distributed with mean $\$28,350$ and standard deviation $\$3,425$. Find the probability that a randomly selected such household has between $\$20,000$ and $\$30,000$ in non-mortgage debt.
20. Birth weights of full-term babies in a certain region are normally distributed with mean $7.125$ lb and standard deviation $1.290$ lb. Find the probability that a randomly selected newborn will weigh less than $5.5$ lb, the historic definition of prematurity.
21. The distance from the seat back to the front of the knees of seated adult males is normally distributed with mean $23.8$ inches and standard deviation $1.22$ inches. The distance from the seat back to the back of the next seat forward in all seats on aircraft flown by a budget airline is $26$ inches. Find the proportion of adult men flying with this airline whose knees will touch the back of the seat in front of them.
22. The distance from the seat to the top of the head of seated adult males is normally distributed with mean $36.5$ inches and standard deviation $1.39$ inches. The distance from the seat to the roof of a particular make and model car is $40.5$ inches. Find the proportion of adult men who when sitting in this car will have at least one inch of headroom (distance from the top of the head to the roof).

## Additional Exercises

23. The useful life of a particular make and type of automotive tire is normally distributed with mean $57,500$ miles and standard deviation $950$ miles.
    a. Find the probability that such a tire will have a useful life of between $57,000$ and $58,000$ miles.

b. Hamlet buys four such tires. Assuming that their lifetimes are independent, find the probability that all four will last between $57,000$ and $58,000$ miles. (If so, the best tire will have no more than $1,000$ miles left on it when the first tire fails.) Hint: There is a binomial random variable here, whose value of $p$ comes from part (a).

24. A machine produces large fasteners whose length must be within $0.5$ inch of $22$ inches. The lengths are normally distributed with mean $22.0$ inches and standard deviation $0.17$ inch.
    a. Find the probability that a randomly selected fastener produced by the machine will have an acceptable length.
    b. The machine produces $20$ fasteners per hour. The length of each one is inspected. Assuming lengths of fasteners are independent, find the probability that all $20$ will have acceptable length. Hint: There is a binomial random variable here, whose value of $p$ comes from part (a).
25. The lengths of time taken by students on an algebra proficiency exam (if not forced to stop before completing it) are normally distributed with mean $28$ minutes and standard deviation $1.5$ minutes.
    a. Find the proportion of students who will finish the exam if a $30$-minute time limit is set.
    b. Six students are taking the exam today. Find the probability that all six will finish the exam within the $30$-minute limit, assuming that times taken by students are independent. Hint: There is a binomial random variable here, whose value of $p$ comes from part (a).
26. Heights of adult men between $18$ and $34$ years of age are normally distributed with mean $69.1$ inches and standard deviation $2.92$ inches. One requirement for enlistment in the military is that men must stand between $60$ and $80$ inches tall.
    a. Find the probability that a randomly elected man meets the height requirement for military service.
    b. Twenty-three men independently contact a recruiter this week. Find the probability that all of them meet the height requirement. Hint: There is a binomial random variable here, whose value of $p$ comes from part (a).
27. A regulation hockey puck must weigh between $5.5$ and $6$ ounces. In an alternative manufacturing process the mean weight of pucks produced is $5.75$ ounce. The weights of pucks have a normal distribution whose standard deviation can be decreased by increasingly stringent (and expensive) controls on the manufacturing process. Find the maximum allowable standard deviation so that at most $0.005$ of all pucks will fail to meet the weight standard. (Hint: The distribution is symmetric and is centered at the middle of the interval of acceptable weights.)
28. The amount of gasoline $X$ delivered by a metered pump when it registers $5$ gallons is a normally distributed random variable. The standard deviation $\sigma$ of $X$measures the precision of the pump; the smaller $\sigma$ is the smaller the variation from delivery to delivery. A typical standard for pumps is that when they show that $5$ gallons of fuel has been delivered the actual amount must be between $4.97$ and $5.03$ gallons (which corresponds to being off by at most about half a cup). Supposing that the mean of $X$ is 5, find the largest that $\sigma$ can be so that $P(4.97 < X < 5.03)$ is $1.0000$to four decimal places when computed using Figure 7.1.5: Cumulative Normal Probability which means that the pump is sufficiently accurate. (Hint: The $z$-score of $5.03$ will be the smallest value of $Z$ so that Figure 7.1.5: Cumulative Normal Probability gives $P(Z < z) = 1.0000$).

## Answers

1. a. $0.6628$
   b. $0.0359$
   c. $0.7881$
   d. $0.0150$
2. 
3. a. $0.5959$
   b. $0.2899$
   c. $0.3439$
4. 
5. a. $0.0000$
   b. $0.9131$
6. 
7. a. $0.0013, \ 0.0013$
   b. $0.1587, \ 0.1587$
   c. $0.0001, \ 0.0001$
   d. $0.0099, \ 0.0099$

8.
9. a. $0.4412$
   b. $0.1236$
   c. $0.1676$
   d. $0.0208$
10.
11. a. $0.9772$
    b. $0.5000$
12.
13. $0.1830$
14.
15. $0.4971$
16.
17. $0.9980$
18.
19. $0.6771$
20.
21. $0.0359$
22.
23. a. $0.4038$
    b. $0.0266$
24.
25. a. $0.9082$
    b. $0.5612$
26.
27. $0.089$

## 5.4: Areas of Tails of Distributions

### Basic

1. Find the value of $z*$ that yields the probability shown.
   a. $P(Z < z*) = 0.0075$
   b. $P(Z < z*) = 0.9850$
   c. $P(Z > z*) = 0.8997$
   d. $P(Z > z*) = 0.0110$
2. Find the value of $z*$ that yields the probability shown.
   a. $P(Z < z*) = 0.3300$
   b. $P(Z < z*) = 0.9901$
   c. $P(Z > z*) = 0.0055$
   d. $P(Z > z*) = 0.7995$
3. Find the value of $z*$ that yields the probability shown.
   a. $P(Z < z*) = 0.1500$
   b. $P(Z < z*) = 0.7500$
   c. $P(Z > z*) = 0.3333$
   d. $P(Z > z*) = 0.8000$
4. Find the value of $z*$ that yields the probability shown.
   a. $P(Z < z*) = 0.2200$
   b. $P(Z < z*) = 0.6000$
   c. $P(Z > z*) = 0.0750$
   d. $P(Z > z*) = 0.8200$

5. Find the indicated value of $Z$. (It is easier to find $-z_c$and negate it.)
   a. $Z_{0.025}$
   b. $Z_{0.20}$
6. Find the indicated value of $Z$. (It is easier to find $-z_c$and negate it.)
   a. $Z_{0.002}$
   b. $Z_{0.02}$
7. Find the value of $x*$ that yields the probability shown, where $X$ is a normally distributed random variable $X$ with mean $83$ and standard deviation $4$.
   a. $P(X < x*) = 0.8700$
   b. $P(X > x*) = 0.0500$
8. Find the value of $x*$ that yields the probability shown, where $X$ is a normally distributed random variable $X$ with mean $54$ and standard deviation $12$.
   a. $P(X < x*) = 0.0900$
   b. $P(X > x*) = 0.6500$
9. $X$ is a normally distributed random variable $X$ with mean $15$ and standard deviation $0.25$. Find the values $X_L$ and $X_R$ of $X$ that are symmetrically located with respect to the mean of $X$ and satisfy $P(X_L < X < X_R) = 0.80$ . (Hint. First solve the corresponding problem for $Z$).
10. $X$ is a normally distributed random variable $X$ with mean $28$ and standard deviation $3.7$. Find the values $X_L$ and $X_R$ of $X$ that are symmetrically located with respect to the mean of $X$ and satisfy $P(X_L < X < X_R) = 0.65$ . (Hint. First solve the corresponding problem for $Z$).

## Applications

11. Scores on a national exam are normally distributed with mean $382$ and standard deviation $26$.
    a. Find the score that is the $50^{th}$ percentile.
    b. Find the score that is the $90^{th}$ percentile.
12. Heights of women are normally distributed with mean $63.7$ inches and standard deviation $2.47$ inches.
    a. Find the height that is the $10^{th}$ percentile.
    b. Find the height that is the $80^{th}$ percentile.
13. The monthly amount of water used per household in a small community is normally distributed with mean $7,069$ gallons and standard deviation $58$ gallons. Find the three quartiles for the amount of water used.
14. The quantity of gasoline purchased in a single sale at a chain of filling stations in a certain region is normally distributed with mean $11.6$ gallons and standard deviation $2.78$ gallons. Find the three quartiles for the quantity of gasoline purchased in a single sale.
15. Scores on the common final exam given in a large enrollment multiple section course were normally distributed with mean $69.35$ and standard deviation $12.93$. The department has the rule that in order to receive an $A$ in the course his score must be in the top $10\%$ of all exam scores. Find the minimum exam score that meets this requirement.
16. The average finishing time among all high school boys in a particular track event in a certain state is $5$ minutes $17$ seconds. Times are normally distributed with standard deviation $12$ seconds.
    a. The qualifying time in this event for participation in the state meet is to be set so that only the fastest $5\%$ of all runners qualify. Find the qualifying time. (Hint: Convert seconds to minutes.)
    b. In the western region of the state the times of all boys running in this event are normally distributed with standard deviation $12$ seconds, but with mean $5$ minutes $22$ seconds. Find the proportion of boys from this region who qualify to run in this event in the state meet.
17. Tests of a new tire developed by a tire manufacturer led to an estimated mean tread life of $67,350$ miles and standard deviation of $1,120$ miles. The manufacturer will advertise the lifetime of the tire (for example, a "$50,000$ mile tire") using the largest value for which it is expected that $98\%$ of the tires will last at least that long. Assuming tire life is normally distributed, find that advertised value.
18. Tests of a new light led to an estimated mean life of $1,321$ hours and standard deviation of $106$ hours. The manufacturer will advertise the lifetime of the bulb using the largest value for which it is expected that $90\%$ of the bulbs will last at least that long. Assuming bulb life is normally distributed, find that advertised value.

19. The weights $X$ of eggs produced at a particular farm are normally distributed with mean $1.72$ ounces and standard deviation $0.12$ ounce. Eggs whose weights lie in the middle $75\%$ of the distribution of weights of all eggs are classified as “medium.” Find the maximum and minimum weights of such eggs. (These weights are endpoints of an interval that is symmetric about the mean and in which the weights of $75\%$ of the eggs produced at this farm lie.)
20. The lengths $X$ of hardwood flooring strips are normally distributed with mean $28.9$ inches and standard deviation $6.12$ inches. Strips whose lengths lie in the middle 80% of the distribution of lengths of all strips are classified as “average-length strips.” Find the maximum and minimum lengths of such strips. (These lengths are endpoints of an interval that is symmetric about the mean and in which the lengths of $80\%$ of the hardwood strips lie.)
21. All students in a large enrollment multiple section course take common in-class exams and a common final, and submit common homework assignments. Course grades are assigned based on students' final overall scores, which are approximately normally distributed. The department assigns a $C$ to students whose scores constitute the middle $2/3$ of all scores. If scores this semester had mean $72.5$ and standard deviation $6.14$, find the interval of scores that will be assigned a $C$.
22. Researchers wish to investigate the overall health of individuals with abnormally high or low levels of glucose in the blood stream. Suppose glucose levels are normally distributed with mean $96$ and standard deviation $8.5\ mg/dl$, and that “normal” is defined as the middle $90\%$ of the population. Find the interval of normal glucose levels, that is, the interval centered at $96$ that contains $90\%$ of all glucose levels in the population.

## Additional Exercises

23. A machine for filling $2$-liter bottles of soft drink delivers an amount to each bottle that varies from bottle to bottle according to a normal distribution with standard deviation $0.002$ liter and mean whatever amount the machine is set to deliver.
    a. If the machine is set to deliver $2$ liters (so the mean amount delivered is $2$ liters) what proportion of the bottles will contain at least $2$ liters of soft drink?
    b. Find the minimum setting of the mean amount delivered by the machine so that at least $99\%$ of all bottles will contain at least $2$ liters.
24. A nursery has observed that the mean number of days it must darken the environment of a species poinsettia plant daily in order to have it ready for market is $71$ days. Suppose the lengths of such periods of darkening are normally distributed with standard deviation $2$ days. Find the number of days in advance of the projected delivery dates of the plants to market that the nursery must begin the daily darkening process in order that at least $95\%$ of the plants will be ready on time. (Poinsettias are so long-lived that once ready for market the plant remains salable indefinitely.)

## Answers

1. a. $-2.43$
   b. $2.17$
   c. $-1.28$
   d. $2.29$
2. 
3. a. $-1.04$
   b. $0.67$
   c. $0.43$
   d. $-0.84$
4. 
5. a. $1.96$
   b. $0.84$
6. 
7. a. $87.52$
   b. $89.58$
8. 
9. $15.32$
10. 
11. a. $382$
    b. $415$

12.
13. $7030.14,\ 7069,\ 7107.86$
14.
15. $85.90$
16.
17. $65,054$
18.
19. $1.58,\ 1.86$
20.
21. $66.5,\ 78.5$
22.
23.
    a. $0.5$
    b. $2.005$

## Contributor

- Anonymous

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## CHAPTER OVERVIEW

# 10: Appendix

# SECTION OVERVIEW

## 10.1: Whole Numbers

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# 10.1.1: Introduction to Whole Numbers

# 10.1.1.1: Place Value and Names for Whole Numbers

### Learning Objectives

- Find the place value of a digit in a whole number.
- Write a whole number in words and in standard form.
- Write a whole number in expanded form.

## Introduction

Mathematics involves solving problems that involve numbers. We will work with **whole numbers**, which are any of the numbers 0, 1, 2, 3 and so on. We first need to have a thorough understanding of the number system we use. Suppose the scientists preparing a lunar command module know it has to travel 382, 564 kilometers to get to the moon. How well would they do if they didn’t understand this number? Do you think it would make more of a difference if the 8 was off by 1 or if the 4 was off by 1?

In this section, you will take a look at digits and place value. You will also learn how to write whole numbers in words, standard form, and expanded form based on the place values of their digits.

## The Number System

A **digit** is one of the symbols is one of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. All numbers are made up of one or more digits. Numbers such as 2 have one digit, whereas numbers such as 89 have two digits. To understand what a number really means, you need to understand what the digits represent in a given number.

The position of each digit in a number tells its value, or **place value**. We can use a **place-value chart** like the one below to easily see the place value for each digit. The place values for the digits in 1, 456 are shown in this chart.

Place-Value Chart

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Trillions | | | Billions | | | Millions | | | Thousands | | | Ones | | |
| | | | | | | | | | | | 1 | 4 | 5 | 6 |
| Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones |

In the number 1, 456 the digit 1 is in the thousands place. The digit 4 is in the hundreds place. The digit 5 is in the tens place, and the digit 6 is in the ones place.

As you see above, you can tell a digit’s value by looking at its position. Look at the number of digits to the right of the digit, or write your number into a place-value chart, with the last digit in the ones column. Both these methods are shown in the example below.

### Example

**The development of a city over the past twenty years cost** $ 962, 234, 532, 274, 312. **What is the value of the digit 6 in this number?**

### Solution

Place-Value Chart

| Trillions | | | Billions | | | Millions | | | Thousands | | | Ones | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 6 | 2 | 2 | 3 | 4 | 5 | 3 | 2 | 2 | 7 | 4 | 3 | 1 | 2 |
| Hundreds | **Tens** | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones |

Write the number in the place-value chart. Read the value of the 6 from the chart.

$ 962, 234, 532, 274, 312

60, 000, 000, 000, 000

Answer: The value of the digit 6 is 60 trillion.

### Exercise

In a far away galaxy, there are 2, 968, 351, 472 stars. What does the digit 3 represent in this problem?

a. three hundred thousands
b. three hundreds
c. three hundred trillions
d. three hundred millions

**Answer**

a. Correct. The digit 3 is in the hundred thousands place.
b. Incorrect. The digit 4 is in the hundreds place. The correct answer is three hundred thousands.
c. Incorrect. The number is less than a trillion, so this digit does not exist here. The correct answer is three hundred thousands.
d. Incorrect. The digit 9 is in the hundred millions place. The correct answer is three hundred thousands.

## Periods and Standard Form

The **standard form** of a number refers to a type of notation in which digits are separated into groups of three by commas. These groups of three digits are known as **periods**. For example, 893, 450, 243 has three periods with three digits in each period, as shown below.

Place-Value Chart

| Trillions | | | Billions | | | Millions | | | Thousands | | | Ones | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 8 | 9 | 3 | 4 | 5 | 0 | 2 | 4 | 3 |
| Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones |

Let’s examine the number of digits and periods in a greater number. The number of body cells in an average adult human is about one hundred trillion. This number is written as 100, 000, 000, 000, 000. Notice that there are 15 digits and 5 periods. Here is how the number would look in a place-value chart.

Place-Value Chart

| Trillions | | | Billions | | | Millions | | | Thousands | | | Ones | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones |

You are now familiar with the place values of greater numbers, so let’s examine a problem that involves converting from standard form to a word name.

## Converting Standard Form to Word Names

We often use word names to write numbers. A word name for 42 is “forty-two.” The total number of weeks in a year, 52, is written as “fifty-two.”

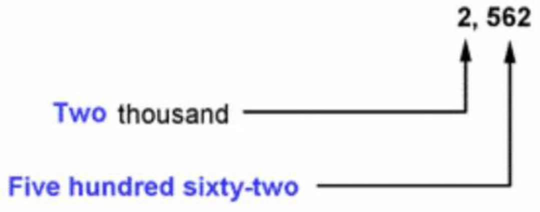
For whole numbers with three digits, use the word “hundred” to describe how many hundreds there are in the number. For example, for the number of days in a normal year, 365, the digit 3 is in the hundreds place. The word name for the number is “three hundred sixty-five.”

For whole numbers with four digits, begin the name with the number of thousands, followed by the period name, as in the example below.

✓ Example

**A man owes $2, 562 on a car. Write the word name for this.**

**Solution**



The word name is two thousand, five hundred sixty-two.

For word names of greater numbers, begin at the left with the greatest period. For each period, write the one- to three-digit number in the period, and then the period name. See the example below.

✓ Example

**The construction of a new athletic center cost $23, 456, 390. Write the word name for this number.**

**Solution**

The word name is twenty-three million, four hundred fifty-six thousand, three hundred ninety.

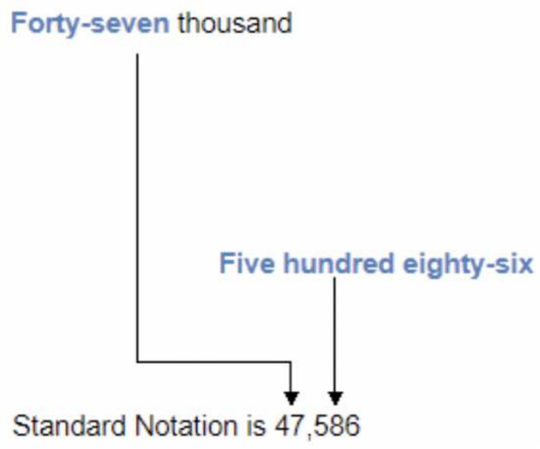
## Converting Word Names to Standard Form

When converting word names to standard form, the word “thousand” tells you which period the digits are in. See the example below.

✓ Example

**Forty-seven thousand, five hundred eighty-six blueberries are produced on a farm over the course of three years. Write this number in standard form.**

**Solution**



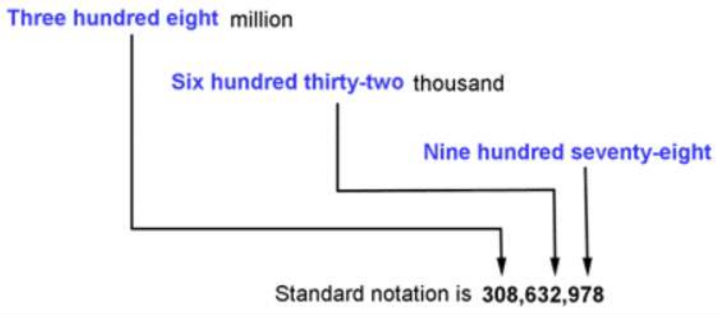
The number in standard form is 47, 586.

Below is an example with a number containing more digits. The words “million” and “thousand” tell you which periods the digits are in. The periods are separated by commas.

✓ Example

**There are three hundred eight million, six hundred thirty-two thousand, nine hundred seventy-eight bacteria in a sample of soil. Write this number in standard form.**

**Solution**



The number in standard form is 308, 632, 978

Some numbers in word form may not mention a specific period. For example, three million, one hundred twelve written in standard form is 3, 000, 112. Because the thousands period is not mentioned, you would write three zeros in the thousands period. You can use a place-value chart to make it easier to see the values of the digits. See the example below.

✓ Example

**A company had a new office building constructed. The final cost was seventy-four million, three hundred sixty-two dollars. Write this number in standard form.**

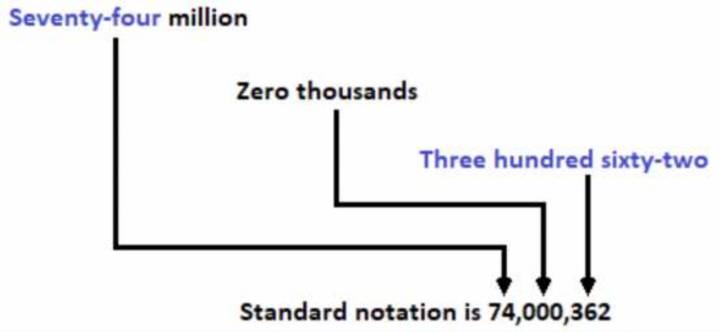
**Solution**

Place-Value Chart

| Trillions | | | Billions | | | Millions | | | Thousands | | | Ones | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 7 | 4 | 0 | 0 | 0 | 3 | 6 | 2 |
| Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones |

Placing this number in a place- value chart shows that the thousands period is zero.

Remember to separate each period with a comma.



The number written in standard form is $74, 000, 362.

## Writing Numbers in Expanded Form

Sometimes it is useful to write numbers in **expanded form**. In expanded form, the number is written as a sum of the value of each digit.

✓ Example

**During the week, Mike drives a total of 264 miles. Write 264 in expanded form.**

### Solution

First, identify the value of each digit.

| | |
|---|---|
| In numerical form: | |
| The 2 in 264 | 200 |
| The 6 in 264 | 60 |
| The 4 in 264 | 4 |
| In word form: | |
| The 2 in 264 | 2 **hundreds** |
| The 6 in 264 | 6 **tens** |
| The 4 in 264 | 4 **ones** |
| Then, write the numbers as a sum. | |

264 written in expanded form is

200 + 60 + 4 or

2 hundreds + 6 tens + 4 or

$(2 \cdot 100)+(6 \cdot 10)+(4 \cdot 1)$

You can also use a place-value chart to help write a number in expanded form. Suppose the number of cars and pick-up trucks in the U.S. at this very moment is 251,834,697. Place this number in a place-value chart.

Place-Value Chart

| Trillions | | | Billions | | | Millions | | | Thousands | | | Ones | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 2 | 5 | 1 | 8 | 3 | 4 | 6 | 9 | 7 |

| Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones | Hundreds | Tens | Ones |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

| | |
|---|---|
| 2 hundred millions | 200,000,000 |
| +5 ten millions | +50,000,000 |
| +1 million | +1,000,000 |
| +8 hundred thousands | +800,000 |
| +3 ten thousands | +30,000 |
| +4 thousands | +4,000 |
| +6 hundreds | +600 |
| +9 tens | +90 |
| +7 ones | +7 |

## Summary

Whole numbers that are greater than 9 consist of multiple digits. Each digit in a given number has a place value. To better understand place value, numbers can be put in a place-value chart so that the value of each digit can be identified. Numbers with more than three digits can be separated into groups of three digits, known as periods. Any whole number can be expressed in standard form, expanded form, or as a word name.

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# 10.1.1.2: Rounding Whole Numbers

**Learning Objectives**

- Learn the rules for rounding.
- Round whole numbers to specific place values, including tens, hundreds, and thousands.

## Introduction

In some situations, you don't need an exact answer. In these cases, **rounding** the number to a specific **place value** is possible. For example, if you travelled 973 miles, you might want to round the distance to 1,000 miles, which is easier to think about. Rounding also comes in handy to see if a calculation is reasonable.

## Rounding Whole Numbers

These are the rules for rounding **whole numbers**:

First, identify the digit with the place value to which you are rounding. You might circle or highlight the digit so you can focus on it better.

Then, determine the possible numbers that you would obtain by rounding. These possible numbers are close to the number that you're rounding to, but have zeros in the digits to the right.

If you are rounding 186 to the nearest ten, then 180 and 190 are the two possible numbers to round to, as 186 is between 180 and 190. But how do you know whether to round to 180 and 190?

> Usually, round a number to the number that is closest to the original number.
>
> When a number is halfway between the two possible numbers, round up to the greater number.

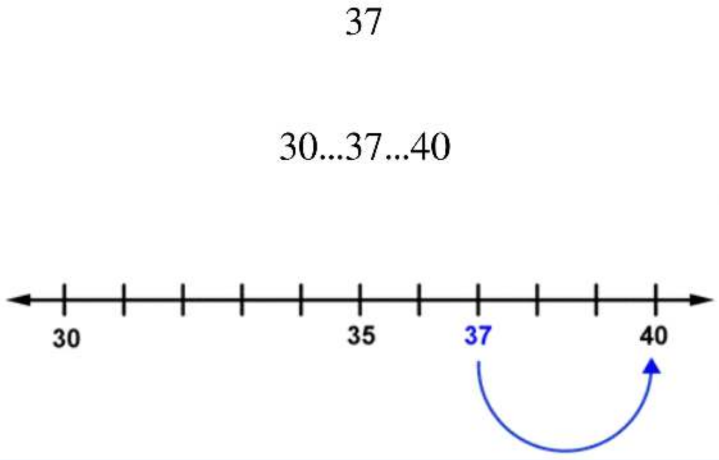
Since 186 is between 180 and 190, and 186 is closer to 190, you round up to 190.

You can use a number line to help you round numbers.

**Example**

**A camera is dropped out of a boat, and sinks to the bottom of a pond that is 37 feet deep. Round 37 to the nearest ten.**

**Solution**



The digit you're rounding to is the tens digit, 3.

37 is between 30 and 40.

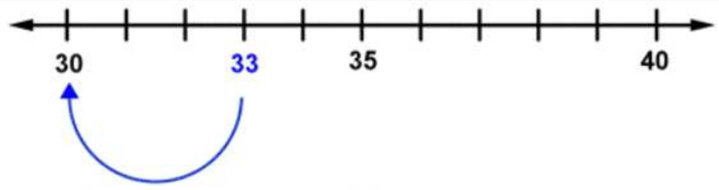
37 is only 3 away from 40, but it's 7 away from 30. So, 37 is closer to 40.

To the nearest ten, 37 rounds to 40.

✓ Example

**Round 33 to the nearest ten.**

**Solution**



33 rounds to 30 because 33 is closer to 30.

To the nearest ten, 33 rounds to 30.

You can determine where to round without using a number line by looking at the digit to the right of the one you're rounding to. If that digit is less than 5, round down. If it's 5 or greater, round up. In the example above, you can see without a number line that 33 is rounded to 30 because the ones digit, 3, is less than 5.

✓ Example

**Round 77 to the nearest ten.**

**Solution**

77 rounds to 80 because the ones digit, 7, is 5 or greater.

77 rounded to the nearest ten is 80.

✓ Example

**There are 576 jellybeans in a jar. Round this number to the nearest ten.**

**Solution**

576 rounds to 580 because the ones digit, 6, is 5 or greater.

576 rounded to the nearest ten is 580.

In the previous examples, you rounded to the tens place. The rounded numbers had a 0 in the ones place. If you round to the nearest hundred, the rounded number will have zeros in the tens and ones places. The rounded number will resemble 100, 500, or 1, 200.

✓ Example

**A runner ran 1,539 meters, but describes the distance he ran with a rounded number. Round 1,539 to the nearest hundred.**

**Solution**

1,539 rounds to 1,500 because the next digit is less than 5.

1,539 rounded to the nearest hundred is 1,500.

If you round to the nearest thousand, the rounded number will have zeros in the hundreds, tens, and ones places. The rounded number will resemble 1,000 , 2,000 , or 14,000.

**✓ Example**

**A plane's altitude increased by 2,721 feet. Round this number to the nearest thousand.**

**Solution**

2,721 rounds to 3,000 because the next digit, 7, is 5 or greater.

2,721 rounded to the nearest thousand is 3,000.

Now that you know how to round to the nearest ten, hundred, and thousand, try rounding to the nearest ten thousand.

**✓ Example**

**Round 326,749 to the nearest ten thousand.**

**Solution**

326,749 rounds to 330,000 because the next digit, 6, is 5 or greater.

326,749 rounded to the nearest ten thousand is 330,000.

**? Exercise**

A record number of 23,386 people voted in a city election. Round this number to the nearest hundred.

A. 23,300
B. 23,400
C. 23,000
D. 23,390

**Answer**

A. Incorrect. The two possible numbers are 23,300 and 23,400, but 23,386 is closer to 23,400. The tens digit, 8, is 5 or greater, so you should round up. The correct answer is 23,400.
B. Correct. The two possible numbers are 23,300 and 23,400, and 23,386 is closer to 23,400. The tens digit, 8, is 5 or greater, so you should round up.
C. Incorrect. This number is rounded to the nearest thousand, not the nearest hundred. The correct answer is 23,400.
D. Incorrect. This number is rounded to the nearest ten, not the nearest hundred. The correct answer is 23,400.

## Summary

In situations when you don't need an exact answer, you can round numbers. When you round numbers, you are always rounding to a particular place value, such as the nearest thousand or the nearest ten. Whether you round up or round down usually depends on which number is closest to your original number. When a number is halfway between the two possible numbers, round up to the larger number.

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# 10.1.1.3: Comparing Whole Numbers

**Learning Objectives**

- Use > or < to compare whole numbers.

## Introduction

There will be times when it’s helpful to compare two numbers and determine which number is greater, and which one is less. This is a useful way to compare quantities such as travel time, income, or expenses. The symbols < and > are used to indicate which number is greater, and which is less than the other.

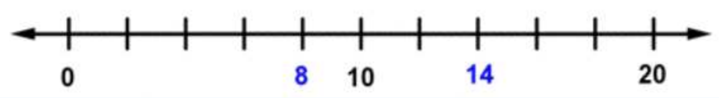
## Comparing Whole Numbers

When comparing the values of two numbers, you can use a number line to determine which number is greater. The number on the right is always greater than the number on the left. In the example below, you can tell that 14 is greater than 8 because 14 is to the right of 8 on the number line.

**Example**

**Which number is greater, 8 or 14?**

### Solution



14 is to the right of 8, so 14 is greater than 8.

14 is greater than 8.

In the example below, you can determine which number is greater by comparing the digits in the ones place value.

**Example**

**Which number is greater, 33 or 38?**

### Solution

In both 33 and 38, the digit in the tens place is 3.

Because they have the same number in the tens place, you can determine which one is greater by comparing the digits in the ones place.

In the number 38, the digit in the ones place is 8.

In the number 33, the digit in the ones place is 3.

Because 8 is greater than 3,38 is greater than 33.

38 is greater than 33. This answer was obtained from comparing their digits in the ones place value, which are 8 and 3, respectively.

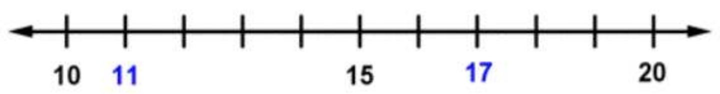
**Exercise**

Which number is greater, 17 or 11?

A. 17
B. 11

**Answer**

A. Correct. The number 17 is 6 units to the right of 11 on the number line.
B. Incorrect. The number 11 is to the left of 17 on the number line, so 17 is greater. The correct answer is 17.



If one number is significantly greater than another number, it may be difficult to compare the numbers effectively with a number line. In general, **whole numbers** with more digits are greater than whole numbers with fewer digits. For example, 542 is greater than 84 because 542 has the digit 5 in the hundreds place. There are no hundreds in 84.

? Exercise

Which number is greater, 71 or 710?

A. 71
B. 710

**Answer**

A. Incorrect. You can see that there is no digit in the hundreds place, which means that 71 is less than 710. The correct answer is 710.
B. Correct. The number 710 has 7 hundreds, but 71 has no hundreds.

## Inequalities

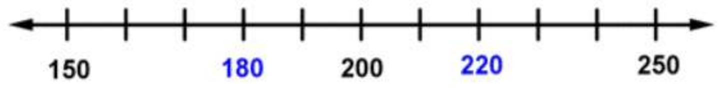
An **inequality** is a mathematical sentence that compares two numbers that aren’t equal. Instead of an equal sign (=), inequalities use greater than (>) or less than (<) symbols. The important thing to remember about these symbols is that the small end points towards the lesser number, and the larger (open) end is always on the side of the greater number.

There are other ways to remember this. For example, the wider part of the symbol represents the jaws of an alligator, which “gobbles up” the greater number. So "35 is greater than 28" can be written as 35 > 28, and "52 is less than 109" can be written as 52 < 109.

✓ Example

**Replace the question mark with < or > to make a true sentence: 180 ? 220.**

**Solution**



180 is to the left of 220, so 180 < 220. The symbol points at 180, which is the lesser number.

180 < 220

? Exercise

Which expression correctly compares the numbers 85 and 19?

A. 85 < 19
B. 19 = 85
C. 85 > 19
D. 19 > 85

**Answer**

A. Incorrect. The symbol should point at the lesser number, 19. On a number line, 85 is to the right of 19, so 85 is greater than 19. The correct answer is 85 > 19

B. Incorrect. This symbol says that 85 is equal to 19, which is false. On a number line, 85 is to the right of 19, so 85 is greater than 19. The correct answer is 85 > 19.
C. Correct. The open part of the symbol faces the larger number, 85, and the symbol points at the smaller number, 19.
D. Incorrect. The symbol should point at the smaller number, 19. On a number line, 85 is to the right of 19, so 85 is greater than 19. The correct answer is 85 > 19.

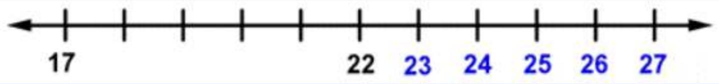
Many times, an answer needs to be a range of values rather than just a single value. For example, you want to make more than $22 an hour. This can be expressed as all numbers greater than 22. See the example below.

✓ Example

**? > 22. What whole number(s) will make this statement true?**

### Solution

The symbol points at 22, so the numbers you want to replace the question mark with are *greater* than 22. There are many numbers that work.



23, 24, 25, 26 and any additional whole numbers that are greater than 26 make this statement true.

? Exercise

A farmer has produced 230 pumpkins for the autumn harvest. Last year, he produced 198. Write an expression that compares these two numbers.

A. 230 > 198
B. 230 < 198
C. 198 = 230
D. 198 > 230

**Answer**

A. Correct. 230 is greater than 198, and this is reflected in the symbol because the open part of the symbol faces 230.
B. Incorrect. 230 is greater than 198, and the symbol is pointing in the wrong direction, with the open part facing the lesser number. The correct answer is 230 > 198.
C. Incorrect. This statement says that 198 is equal to 230, which is incorrect. The correct answer is 230 > 198.
D. Incorrect. 230 is greater than 198, and the symbol is pointing in the wrong direction, with the open part facing the lesser number. The correct answer is 230 > 198.

## Summary

To compare two values that are not the same, you can write an inequality. You can use a number line or place value to determine which number is greater than another number. Inequalities can be expressed using greater than (>) or less than (<) symbols.

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# 10.1.2: Adding and Subtracting Whole Numbers

# 10.1.2.1: Adding Whole Numbers and Applications

**Learning Objectives**

- Add whole numbers without regrouping.
- Add whole numbers with regrouping.
- Find the perimeter of a polygon.
- Solve application problems using addition.

## Introduction

Adding is used to find the total number of two or more quantities. The total is called the **sum**, or the number that results from the addition. You use addition to find the total distance that you travel if the first distance is 1,240 miles and the second distance is 530 miles. The two numbers to be added, 1,240 and 530, are called the **addends**. The total distance, 1,770 miles, is the sum.

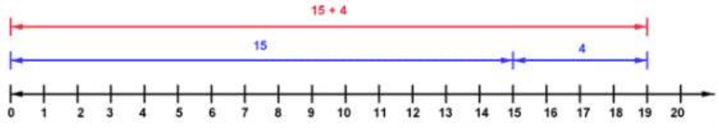
## Adding Whole Numbers without Regrouping

Adding numbers with more than one digit requires an understanding of **place value**. The place value of a digit is the value based on its position within the number. In the number 492, the 4 is in the hundreds place, the 9 is in the tens place, and the 2 is in the ones place. You can use a number line to add. In the example below, the blue lines represent the two quantities, 15 and 4, that are being added together. The red line represents the resulting quantity.

**Example**

15 + 4 = ?

**Solution**



On the number line, the blue line segment stretches across 15 units, representing the number 15. The second blue segment shows that if you add 4 more units, the resulting number is 19.

15 + 4 = 19

You can solve the same problem without a number line, by adding vertically. When adding numbers with more than 1 digit, it is important to line up your numbers by place value, as in the example below. You must add ones to ones, tens to tens, hundreds to hundreds, and so on.

**Example**

15 + 4 = ?

**Solution**

| | |
|---|---|
| $\begin{array}{r} 15 \\ +\quad 4 \\ \hline \end{array}$ | Because 5 and 4 have the same place value, make sure they are aligned when you add. |
| $\begin{array}{r} 15 \\ +\quad 4 \\ \hline 9 \end{array}$ | First, add the ones digits (the numbers on the right). The result goes in the ones place for the answer. |
| $\begin{array}{r} 15 \\ +\quad 4 \\ \hline 19 \end{array}$ | Then, add the tens digits and put the result in the tens place of the answer. In this case, there is no tens digit in the second number, so the result is the same as the tens digit of the first number (1). |

15 + 4 = 19

This strategy of lining up the numbers is effective for adding a series of numbers as well.

**✓ Example**

1 + 2 + 3 + 2 = ?

**Solution**

$$\begin{array}{r} 1 \\ 2 \\ 3 \\ +2 \\ \hline 8 \end{array}$$

1 + 2 + 3 + 2 = 8

## Adding Whole Numbers with Regrouping

When adding whole numbers, a place-value position can have only one digit in it. If the sum of digits in a place value position is more than 10, you have to **regroup** the number of tens to the next greater place value position.

When you add, make sure you line up the digits according to their place values, as in the example below. As you regroup, place the regrouped digit above the appropriate digit in the next higher place value position and add it to the numbers below it.

**✓ Example**

45 + 15 = ?

**Solution**

| | |
|---|---|
| $\begin{array}{r} 1\phantom{5} \\ 45 \\ +15 \\ \hline 0 \end{array}$ | Add the ones. Regroup as needed. The sum of 5 and 5 is 10. This is 1 ten and 0 ones. Write the number of ones (0) in the ones place and the 1 ten in the tens place above the 4. |
| $\begin{array}{rr} & 1\phantom{5} \\ & 45 \\ + & 15 \\ \hline & 60 \end{array}$ | Add the tens, 1 + 4 + 1 is 6 tens. The final sum is 60. |

45 + 15 = 60

You must add digits in the ones place first, the digits in the tens place next, and so on. Go from right to left.

**✓ Example**

4,576 + 698 = ?

**Solution**

| | |
|---|---|
| $\begin{array}{rr} & 4,576 \\ + & 698 \\ \hline \end{array}$ | First, write the problem with one addend on top of the other. Be sure you line up the place values! |

| | |
|---|---|
| $\begin{array}{r} 1 \\ 4,576 \\ +\quad 698 \\ \hline 4 \end{array}$ | Add the numbers, 6 and 8, in the ones place. Since the sum is 14, write the ones value (4) in the ones place of the answer. Write the 1 ten in the tens place above the 7. |
| $\begin{array}{r} 11 \\ 4,576 \\ +\quad 698 \\ \hline 74 \end{array}$ | Add the numbers in the tens place. Since the sum is 17 tens, regroup 17 tens as 1 hundred, 7 in the tens place in the answer and write the 1 hundred in the hundreds place above the 5. |
| $\begin{array}{r} 1\ 11 \\ 4,576 \\ +\quad 698 \\ \hline 274 \end{array}$ | Add the numbers in the hundreds place, including the 1. Again, the sum is more than one digit. Rename 12 hundreds as 2 hundreds and 1 thousand. Write the 2 in the hundreds place and the 1 above the 4 in the thousands place. |
| $\begin{array}{r} 1\ 11 \\ 4,576 \\ +\quad 698 \\ \hline 5,274 \end{array}$ | Add the numbers in the thousands place, including the 1. The final sum is 5,274. |

4,576 + 698 = 5,274

## Adding Numbers Using the Partial Sums Method

Another way to add is the partial sums method. In the example below, the sum of 23 + 46 is found using the partial sums method. In this method, you add together all the numbers with the same place value and record their *values* (not just a single digit). Once you have done this for each place value, add their sums together.

✓ Example

23 + 46 = ?

**Solution**

| | |
|---|---|
| Step 1: Add Tens | |
| $\begin{array}{rr} 23 & 20 \\ 46 & +40 \\ \hline & 60 \end{array}$ | Let’s begin by adding the values in the tens position, the 2 and 4. The values are written as 20 and 40. |
| Step 2: Add Ones | |
| $\begin{array}{rr} 23 & 3 \\ 46 & +6 \\ \hline & 9 \end{array}$ | Add the values in the ones place, the 3 and 6. |
| Step 3: Add Parts | |
| $\begin{array}{r} 60 \\ +\quad 9 \\ \hline 69 \end{array}$ | Finally, add the two sums, 60 and 9, together. |

23 + 46 = 69

The next example adds a series of three numbers. Notice that hundreds is the greatest place value now, so hundreds are added before the tens. (You can add in any order that you prefer.) Also notice that in Step 3, the value in the ones column for 350 is zero, but you still add that in to make sure everything is accounted for.

**Example**

225 + 169 + 350 = ?

### Solution

| | |
|---|---|
| Step 1: Add Hundreds | |
| $\begin{array}{rr} 225 & 200 \\ 169 & 100 \\ 350 & +300 \\ \hline & 600 \end{array}$ | Add the values represented by the digits 2, 1, and 3 in the hundreds place first. This gives a sum of 600. |
| Step 2: Add Tens | |
| $\begin{array}{rr} 225 & 20 \\ 169 & 60 \\ 350 & +50 \\ \hline & 130 \end{array}$ | Next, add the values from the digits in the tens place, the 2, 6, and 9. The sum is 130. |
| Step 3: Add Ones | |
| \\( \begin{array}{r}<br>22\color{blue}5 & \color{blue}5\\\\<br>16\color{blue}9& \color{blue} 9\\\\<br>35\color{blue}0&\color{blue}+0\\\\<br>\hline<br>&\color{blue}14<br>\end{array}\\) | Add the values from the digits in the ones place, the 5, 9, and 0. The sum is 14. |
| Step 4: Add Parts | |
| $\begin{array}{r} 600 \\ 130 \\ +\ \ 14 \\ \hline 744 \end{array}$ | At this point, you have a sum for each place value. Add together these three sums, which gives a final value of 744. |

225 + 169 + 350 = 744

**Exercise**

A local company built a playground at a park. It took the company 124 hours to plan out the playground, 243 hours to prepare the site, and 575 hours to build the playground. Find the total number of hours the company spent on the project.

A. 937 hours
B. 812 hours
C. 742 hours
D. 942 hours

**Answer**

A. Incorrect. You probably did not add the ones correctly. The correct answer is 942 hours.
B. Incorrect. You probably did not add the tens correctly. The correct answer is 942 hours.
C. Incorrect. You probably did not add the hundreds correctly. The correct answer is 942 hours.
D. Correct. You carried out the partial sums process effectively. The parts should be 800 + 130 + 12.

When adding multi-digit numbers, use the partial sums method or any method that works best for you.

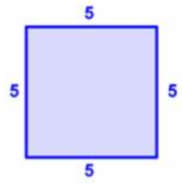
## Finding the Perimeter of a Polygon

A **polygon** is a many-sided closed figure with sides that are straight line segments. Triangles, rectangles, and pentagons (five-sided figures) are polygons, but a circle or semicircle is not. The **perimeter** of a polygon is the distance around the polygon. To find the perimeter of a polygon, add the lengths of its sides, as in the example below.

✓ Example

**One side of a square has a length of 5cm. Find the perimeter.**

**Solution**

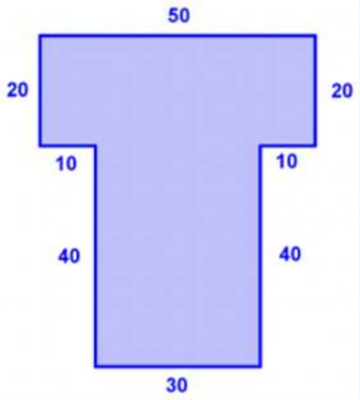
| | |
|---|---|
|  | Draw the polygon and label the lengths of the sides. Since the side lengths of a square are equal, each side is 5cm. |
| $\begin{array}{r} 5 \\ 5 \\ 5 \\ +\ \ 5 \\ \hline 20 \end{array}$ | Add the lengths of each side, 5 + 5 + 5 + 5. |

The perimeter is 20cm.

The key part of completing a polygon problem is correctly identifying the side lengths. Once you know the side lengths, you add them as you would in any other addition problem.

✓ Example

**A company is planning to construct a building. Below is a diagram illustrating the shape of the building's floor plan. The length of each side is given in the diagram. Measurements for each side are in feet. Find the perimeter of the building.**



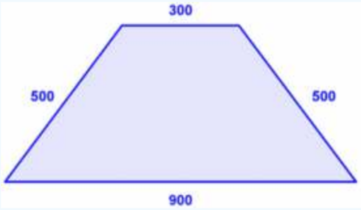
**Solution**

| | |
|---|---|
| $\begin{array}{r} 50 \\ 20 \\ 20 \\ 10 \\ 10 \\ 40 \\ 40 \\ +30 \\ \hline 220 \end{array}$ | Add the lengths of each side, making sure to align all numbers according to place value. |

The perimeter is 220 feet.

**Exercise**

Find the perimeter of the trapezoid in feet.



A. 2,200 feet
B. 1,200 feet
C. 200 feet
D. 3,200 feet

**Answer**

A. Correct. You added the lengths of the sides together successfully.
B. Incorrect. You probably did not add the two diagonal sides of the trapezoid. The correct answer is 2,200 feet.
C. Incorrect. You probably subtracted the total length of the two sides, 1,000 feet, from the total length of the top and bottom, 1,200. The correct answer is 2,200 feet.
D. Incorrect. You may have added some of the sides more than once. The correct answer is 2,200 feet.

## Solving Application Problems

Addition is useful for many kinds of problems. When you see a problem written in words, look for key words that let you know you need to *add* numbers.

**Example**

**A woman preparing an outdoor market is setting up a stand with 321 papayas, 45 peaches, and 213 mangos. How many pieces of fruit in total does the woman have on her stand?**

**Solution**

| | |
|---|---|
| $\begin{array}{r} 321 \\ 45 \\ +213 \\ \hline \end{array}$ | The words “how many… in total” suggest that you need to add the numbers of the different kinds of fruits.<br>Use any method you like to add the numbers. Below, the partial sums method is used. |
| Step 1: Add Hundreds | |

| | |
|---|---|
| $\begin{array}{rr} 321 & 300 \\ 045 & 0 \\ 213 & +200 \\ \hline & 500 \end{array}$ | Add the numbers represented by the digits in the hundreds place first, the 3, 0 and 2. This gives a sum of 500. |
| Step 2: Add Tens | |
| $\begin{array}{rr} 321 & 20 \\ 045 & 40 \\ 213 & +10 \\ \hline & 70 \end{array}$ | Next, add the numbers represented by the digits from the tens place, the 2, 4, and 1. The sum is 70. |
| Step 3: Add Ones | |
| $\begin{array}{rr} 321 & 1 \\ 045 & 5 \\ 213 & +3 \\ \hline & 9 \end{array}$ | Add the numbers from the ones, the 1, 5, and 3. |
| Step 4: Add Parts | |
| $\begin{array}{r} 500 \\ 70 \\ +\quad 9 \\ \hline 579 \end{array}$ | Add together the three previous sums. The final sum is 579. |

The woman has 579 pieces of fruit on her stand.

✓ Example

**Lynn has 23 rock CDs, 14 classical music CDs, 8 country and western CDs, and 6 movie soundtrack CDs. How many CDs does she have in all?**

### Solution

| | |
|---|---|
| $\begin{array}{r} 23 \\ 14 \\ 8 \\ +6 \\ \hline \end{array}$ | The words “how many… in all” suggest that addition is the way to solve this problem.<br>To find how many CDs Lynn has, you need to add the number of CDs she has for each music style. |
| $\begin{array}{r} 2 \\ 23 \\ 14 \\ 8 \\ +\quad 6 \\ \hline 51 \end{array}$ | Use whatever method you prefer to find the sum of the numbers. |

Lynn has 51 CDs.

The following phrases also appear in problem situations that require addition.

| **Phrase** | **Example problem** |
|---|---|

| | |
|---|---|
| Add to | Jonah was planning a trip from Boston to New York City. The distance is 218 miles. His sister wanted him to visit her in Springfield, Massachusetts, on his way. Jonah knew this would **add** 17miles **to** his trip. How long is his trip if he visits his sister? |
| Plus | Carrie rented a DVD and returned it one day late. The store charged $5 for a two-day rental, **plus** a $3 late fee. How much did Carrie pay for the rental? |
| Increased by | One statistic that is important for football players in offensive positions is *rushing*. After four games, one player had rushed 736 yards. After two more games, the number of yards rushed by this player **increased by** 352yards. How many yards had he rushed after the six games? |
| More than | Lavonda posted 38 photos to her social network profile. Chris posted 27 **more** photos to his **than** Lavonda. How many photos did Chris post? |

✓ Example

**Lena was planning a trip from her home in Amherst to the Museum of Science in Boston. The trip is 91 miles. She had to take a detour on the way, which added 13 miles to her trip. What is the total distance she traveled?**

### Solution

The word "added" suggests that addition is the way to solve this problem.

To find the total distance, you need to add the two distances.

$$\begin{array}{r} 91 \\ +\quad 13 \\ \hline 104 \end{array}$$

The total distance is 104 miles.

It can help to seek out words in a problem that imply what operation to use. See if you can find the key word(s) in the following problem that provide you clues on how to solve it.

? Exercise

A city was struck by an outbreak of a new flu strain in December. To prevent another outbreak, 3,462 people were vaccinated against the new strain in January. In February, 1,298 additional people were vaccinated. How many people in total received vaccinations over these two months?

A. 2,164
B. 4,760
C. 4,660
D. 4,750

**Answer**

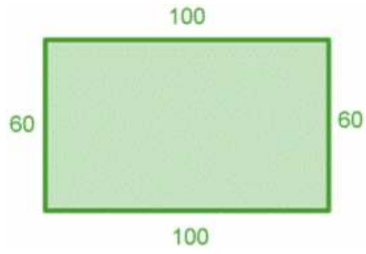
A. Incorrect. You probably subtracted instead of adding. The correct answer is 4,760.
B. Correct. You recognized this as an addition problem and successfully carried out your addition process.
C. Incorrect. You probably did not regroup to the hundreds place, or added the hundreds places incorrectly. The correct answer is 4,760.
D. Incorrect. You probably did not regroup to the tens place, or added the tens place incorrectly. The correct answer is 4,760.

Drawing a diagram to solve problems is very useful in fields such as engineering, sports, and architecture.

✓ Example

**A coach tells her athletes to run one lap around a soccer field. The length of the soccer field is 100 yards, while the width of the field is 60 yards. Find the total distance that each athlete will have run after completing one lap around the perimeter of the field.**

**Solution**

| | |
|---|---|
|  | The words “total distance” and “perimeter” both tell you to add. Draw the soccer field and label the various sides so you can see the numbers you are working with to find the perimeter. |
| $\begin{array}{r} 1\phantom{00} \\ 100 \\ 100 \\ 60 \\ +\quad 60 \\ \hline 20 \end{array}$ | There is a zero in the ones place, and the sum of 6 and 6 in the tens place is 12 tens. Place 2 tens in the tens place in the answer, and regroup 10 tens as 1 hundred. |
| $\begin{array}{r} 1\phantom{00} \\ 100 \\ 100 \\ 60 \\ +\quad 60 \\ \hline 320 \end{array}$ | By adding the 1 hundred to the other digits in the hundreds place, you end up with a 3 in the hundreds place of the answer. |

Each athlete will have run 320 yards.

## Summary

You can add numbers with more than one digit using any method, including the partial sums method. Sometimes when adding, you may need to regroup to the next greater place value position. Regrouping involves grouping ones into groups of tens, grouping tens into groups of hundreds, and so on. The perimeter of a polygon is found by adding the lengths of each of its sides.

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# 10.1.2.2: Subtracting Whole Numbers and Applications

**Learning Objectives**

- Subtract whole numbers without regrouping.
- Subtract whole numbers with regrouping.
- Solve application problems using subtraction.

## Introduction

Subtracting involves finding the difference between two or more numbers. It is a method that can be used for a variety of applications, such as balancing a checkbook, planning a schedule, cooking, or travel. Suppose a government official is out of the U.S. on business for 142 days a year, including travel time. The number of days per year she is in the U.S. is the difference of 365 days and 142 days. Subtraction is one way of calculating the number of days she would be in the U.S. during the year.

When subtracting numbers, it is important to line up your numbers, just as with addition. The **minuend** is the greater number from which the lesser number is subtracted. The **subtrahend** is the number that is subtracted from the minuend. A good way to keep minuend and subtrahend straight is that since subtrahend has “subtra” in its beginning, it goes next to the subtraction sign and is the number being subtracted. The **difference** is the quantity that results from subtracting the subtrahend from the minuend. In 86 - 52 = 34, 86 is the *minuend*, 52 is the *subtrahend*, and 34 is the *difference*.

## Subtracting Whole Numbers

When writing a subtraction problem, the minuend is placed above the subtrahend. This can be seen in the example below, where the minuend is 10 and the subtrahend is 7.

**Example**

10 - 7 = ?

**Solution**

$$\begin{array}{r} 10 \\ -\quad 7 \\ \hline 3 \end{array}$$

10 - 7 = 3

When both numbers have more than one digit, be sure to work with one place value at a time, as in the example below.

**Example**

689 - 353 = ?

**Solution**

| | |
|---|---|
| $\begin{array}{r} 689 \\ -353 \\ \hline \end{array}$ | First, set up the problem and align the numbers by place value. |
| $\begin{array}{r} 689 \\ -353 \\ \hline 6 \end{array}$ | Then, subtract the digits in the ones place, 9 - 3. |
| $\begin{array}{r} 689 \\ -353 \\ \hline 36 \end{array}$ | Next, subtract the digits in the tens place, 8 - 5. |

| | |
|---|---|
| $\begin{array}{r} 689 \\ -353 \\ \hline 336 \end{array}$ | Finally, subtract the digits in the hundreds place, 6 - 3. |

689 - 353 = 336

Lining up numbers by place value becomes especially important when you are working with larger numbers that have more digits, as in the example below.

### Example

9,864 - 743 = ?

**Solution**

| | |
|---|---|
| $\begin{array}{r} 9864 \\ -\;\;743 \\ \hline \end{array}$ | First, set up the problem and align the numbers by place value. |
| $\begin{array}{r} 9864 \\ -\;\;743 \\ \hline 1 \end{array}$ | Then, subtract the digits in the ones place, 4 - 3. |
| $\begin{array}{r} 9864 \\ -\;\;743 \\ \hline 21 \end{array}$ | Next, subtract the digits in the tens place, 6 - 4. |
| $\begin{array}{r} 9864 \\ -\;\;743 \\ \hline 121 \end{array}$ | Now, subtract the digits in the hundreds place, 8 - 7. |
| $\begin{array}{r} 9864 \\ -\;\;743 \\ \hline 9121 \end{array}$ | There is no digit to subtract in the thousands place, so keep the 9. |

9,864 - 743 = 9,121

### Exercise

Subtract: 2,489 - 345

A. 2,144
B. 1,355
C. 2,834
D. 1,134

**Answer**

A. Correct. You successfully subtracted 345 from 2,489.
B. Incorrect. You probably did not line up your numbers correctly by place value. The correct answer is 2,144.
C. Incorrect. You probably added when you should have subtracted. The correct answer is 2,144.
D. Incorrect. You probably did not subtract correctly in the tens and thousands places. The correct answer is 2,144.

## Subtracting Whole Numbers with Regrouping

You may need to regroup when you subtract. When you **regroup**, you rewrite the number so you can subtract a greater digit from a lesser one.

When you're subtracting, just regroup to the next greater place-value position in the minuend and add 10 to the digit you're working with. As you regroup, cross out the regrouped digit in the minuend and place the new digit above it. This method is demonstrated in the example below.

**Example**

3,225 - 476 = ?

**Solution**

| | |
|---|---|
| $\begin{array}{cr} & 3225 \\ - & 476 \\ \hline \end{array}$ | First, set up the problem and align the digits by place value. |
| $\begin{array}{crrrr} & & & 1 & 15 \\ & 3 & 2 & \cancel{2} & \cancel{5} \\ - & & 4 & 7 & 6 \\ \hline & & & & 9 \end{array}$ | Since you can't subtract 6 from 5, regroup, so 2 tens and 5 ones become 1 ten and 15 ones. Now you can subtract 6 from 15 to get 9. |
| $\begin{array}{crrrr} & & 1 & 11 & 15 \\ & 3 & \cancel{2} & \cancel{2} & \cancel{5} \\ - & & 4 & 7 & 6 \\ \hline & & & 4 & 9 \end{array}$ | Next, you need to subtract 7 tens from 1 ten. Regroup 2 hundreds as 1 hundred, 10 tens and add the 10 tens to 1 ten to get 11 tens. Now you can subtract 7 from 11 to get 4. |
| $\begin{array}{crrrr} & 2 & 11 & 11 & 15 \\ & \cancel{3} & \cancel{2} & \cancel{2} & \cancel{5} \\ - & & 4 & 7 & 6 \\ \hline & & 7 & 4 & 9 \end{array}$ | To subtract the digits in the hundreds place, regroup 3 thousands as 2 thousands, 10 hundreds and add the 10 hundreds to the 1 hundred that is already in the hundreds place. Now, subtract 4 from 11 to get 7. |
| $\begin{array}{crrrr} & 2 & 11 & 11 & 15 \\ & \cancel{3} & \cancel{2} & \cancel{2} & \cancel{5} \\ - & & 4 & 7 & 6 \\ \hline & 2 & 7 & 4 & 9 \end{array}$ | Since there is no digit in the thousands place of the subtrahend, bring down the 2 in the thousands place into the answer. |

3,225 - 476 = 2,749

**Exercise**

Subtract: 1,610 - 880

A. 1,522
B. 2,490
C. 730
D. 620

**Answer**

A. Incorrect. You probably did not line up the numbers by place value when you were subtracting. The correct answer is 730.
B. Incorrect. You probably added the numbers instead of subtracting. The correct answer is 730.
C. Correct. You successfully subtracted 880 from 1,610.
D. Incorrect. You probably did not regroup correctly. The correct answer is 730.

## Checking Your Work

You can check subtraction by adding the difference and the subtrahend. The sum should be the same as the minuend.

**✓ Example**

**Check to make sure that 7 subtracted from 12 is equal to 5.**

**Solution**

| | |
|---|---|
| 12 - 7 = 5 | Write out the original equation. The minuend is 12, the subtrahend is 7, and the difference is 5. |
| $\begin{array}{r} 5 \\ +\ \ 7 \\ \hline 12 \end{array}$ | Next, add the difference, 5, to the subtrahend, 7, which results in the number 12. This confirms that your answer is correct. |

The answer of 5 is correct.

Checking your work is very important and should always be performed when time permits.

## Subtracting Numbers Using the Expanded Form

An alternative method to subtract involves writing numbers in expanded form, as shown in the examples below. If you have 4 tens and want to subtract 1 ten, you can just think (4 - 1) tens and get 3 tens. Let's see how that works.

**✓ Example**

45 - 12 = ?

**Solution**

| | |
|---|---|
| $45 = 40 + 5$<br>$12 = 10 + 2$ | Let's write the numbers in expanded form so you can understand what they really mean. |
| $\begin{array}{r} 45 = 40 + 5 \\ 12 = 10 + 2 \\ \hline 30 \ \ \ \ \ \end{array}$ | For the tens, the minuend is 40, or 4 tens. The subtrahend is 10, or 1 ten. Since 4 - 1 = 3, 4 tens -1 ten =3 tens, or 30. |
| $\begin{array}{r} 45 = 40 + 5 \\ 12 = 10 + 2 \\ \hline 30 + 3 \end{array}$ | Now, the ones. 5 - 2 = 3. So, 30 + 3 = 33. |

45 - 12 = 33

Now let's use this method in the example below, which asks for the difference of 467 and 284. In the tens place of this problem, you need to subtract 8 from 6. What can you do?

**✓ Example**

467 - 284 = ?

**Solution**

| | |
|---|---|
| Step 1: Separate by place value | |
| $\begin{array}{r} 4 \text{ hundreds} + 6 \text{ tens} + 7 \text{ ones} \\ -2 \text{ hundreds} + 8 \text{ tens} + 4 \text{ ones} \end{array}$ | Write both the minuend and the subtrahend in expanded form. |
| Step 2: Identify impossible differences | |
| 6 - 8 = [ ] | Identify differences that are not whole numbers. Since 8 is greater than 6, you won't get a whole number difference. |
| Step 3: Regroup | |

| | |
|---|---|
| $\begin{array}{r} 3 \text{ hundreds } + 16 \text{ tens } + 7 \text{ ones} \\ -2 \text{ hundreds } + 8 \text{ tens } + 4 \text{ ones} \\ \hline 1 \text{ hundred } + 8 \text{ tens } + 3 \text{ ones} \end{array}$ | Regroup one of the hundreds from the 4 hundreds into 10 tens and add it to the 6 tens. Now you have 16 tens.<br>Subtracting 8 tens from 16 tens yields a difference of 8 tens. |
| Step 4: Combine the parts | |
| $1 \text{ hundred } + 8 \text{ tens } + 3 \text{ ones } = 183$ | Combining the resulting differences for each place value yields a final answer of 183. |

467 - 284 = 183

**Exercise**

A woman who owns a music store starts her week with 965 CDs. She sells 452 by the end of the week. How many CDs does she have remaining?

A. 313
B. 513
C. 510
D. 1,417

**Answer**

A. Incorrect. You probably made an error when you were subtracting digits in the hundreds place. The correct answer is 513.
B. Correct. You successfully subtracted 452 from 965. In expanded form, 513 is 5 hundreds, 1 ten, and 3 ones.
C. Incorrect. You probably made an error when you were subtracting digits in the ones place. The correct answer is 513.
D. Incorrect. You probably added instead of subtracted. The correct answer is 513.

**Example**

45 - 17 = ?

**Solution**

| | |
|---|---|
| $45 = 40 + 5$<br>$17 = 10 + 7$ | When you try to subtract 17 from 45, you would first try to subtract 7 from 5. But 5 is less than 7.<br>Let’s write the numbers in expanded form so you can see what they really mean. |
| $\begin{array}{l} 45 = 30 + 15 \\ 17 = 10 + 7 \\ \hline \end{array}$ | Now, regroup 4 tens as 3 tens and 10 ones. Add the 10 ones to 5 ones to get 15 ones, which is greater than 7 ones, so you can subtract. |
| $\begin{array}{rrcr} 45 = & 30 & + & 15 \\ -17 = & 10 & + & 7 \\ \hline & 20 & + & 8 \end{array}$ | Finally, subtract 7 from 15, and 10 from 30 and add the results: 20 + 8 = 28. |

45 - 17 = 28

## Solve Application Problems Using Subtraction

You are likely to run into subtraction problems in everyday life, and it helps to identify key phrases in a problem that indicate that subtraction is either used or required. The following phrases appear in problem situations that require subtraction.

| Phrase or word | Example problem |
|---|---|

| | |
|---|---|
| Less than | The cost of gas is 42 cents per gallon **less than** it was last month. The cost last month was 280 cents per gallon. How much is the cost of gas this month? |
| Take away | Howard made 84 cupcakes for a neighborhood picnic. People **took away** 67 cupcakes. How many did Howard have left? |
| Decreased by | The temperature was $84^{\circ}$ Fahrenheit in the early evening. It **decreased by** \15° overnight. What was the temperature in the morning? |
| Subtracted from | Jeannie works in a specialty store on commission. When she sells something for $\$75$, she **subtracts** $\$15$ **from** the $\$75$ and gives the rest to the store. How much of the sale goes to the store? |
| The difference | What is **the difference** between this year's rent of $\$1,530$ and last year's rent of $\$1,450$? |
| Fewer than | The number of pies sold at this year's bake sale was 15 **fewer** than the number sold at the same event last year. Last year, 32 pies were sold. How many pies were sold this year? |

When translating a phrase such as "5 fewer than 39" into a mathematical expression, the order in which the numbers appears is critical. Writing 5 - 39 would not be the correct translation. The correct way to write the expression is 39 - 5. This results in the number 34, which is 5 fewer than 39. The chart below shows how phrases with the key words above can be written as mathematical expressions.

| Phrase | Expression |
|---|---|
| three subtracted from six | 6 - 3 |
| the difference of ten and eight | 10 - 8 |
| Nine fewer than 40 | 40 - 9 |
| Thirty-nine decreased by fourteen | 39 - 14 |
| Eighty-five take away twelve | 85 - 12 |
| Four less than one hundred eight | 108 - 4 |

### ✓ Example

**Each year, John is out of the U.S. on business for 142 days, including travel time. The number of days per year he is in the U.S. is the difference of 365 days and 142 days. How many days during the year is John in the U.S.?**

#### Solution

| | |
|---|---|
| $\begin{array}{r} 365 \\ -142 \\ \hline \end{array}$ | The words "the difference of" suggest that you need to subtract to answer the problem.<br>First, write out the problem based on the information given and align numbers by place value, 365 - 142. |
| $\begin{array}{r} 365 \\ -142 \\ \hline 3 \end{array}$ | Then, subtract numbers in the ones place, 5 - 2. |
| $\begin{array}{r} 365 \\ -142 \\ \hline 23 \end{array}$ | Subtract numbers in the tens place, 6-4. |

| | |
|---|---|
| $\begin{array}{r} 365 \\ -142 \\ \hline 223 \end{array}$ | Finally, subtract numbers in the hundreds place, 3-1. |

John is in the U.S. for 223 days during the year.

### Exercise

To make sure he was paid up for the month on his car insurance, Dave had to pay the difference of the amount on his monthly bill, which was $289, and what he had paid earlier this month, which was $132. Write the difference of $289 and $132 as a mathematical expression.

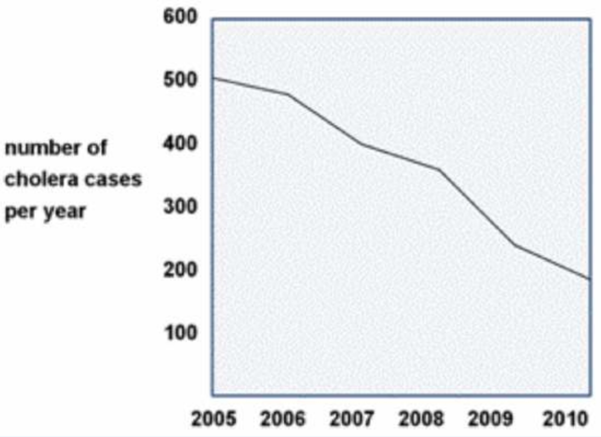
A. 132-289
B. 289+132
C. 132+289
D. 289-132

**Answer**

A. Incorrect. Dave had paid $132, so that $132 can be taken away from the full $289 he owed for the month. The correct answer is 289-132.
B. Incorrect. Dave owes the difference of 132 and 289, not the sum of 289 and 132. The correct answer is 289-132.
C. Incorrect. Dave owes the difference of 132 and 289, not the sum of 132 and 289. The correct answer is 289-132.
D. Correct. The difference of 289 and 132 can be written as 289-132.

### Example

**An African village is now getting cleaner water than it used to get. The number of cholera cases in the village has declined over the past five years. Using the graph below, determine the difference between the number of cholera cases in 2005 and the number of cases in 2010.**



**Solution**

| | |
|---|---|
| $\begin{array}{r} 500 \\ -200 \\ \hline 300 \end{array}$ | The words “the difference” suggest that you need to subtract to answer the problem.<br>First, use the graph to find the number of cholera cases per year for the two years: 500 in 2005 and 200 in 2010.<br>Then write the subtraction problem and align numbers by place value. Subtract the numbers as you usually would. |

500-200=300 cases

## Summary

Subtraction is used in countless areas of life, such as finances, sports, statistics, and travel. You can identify situations that require subtraction by looking for key phrases, such as *difference* and *fewer than*. Some subtraction problems require regrouping to the next greater place value, so that the digit in the minuend becomes greater than the corresponding digit in the subtrahend. Subtraction problems can be solved without regrouping, if each digit in the minuend is greater than the corresponding digit in the subtrahend.

In addition to subtracting using the standard algorithm, subtraction can also can be accomplished by writing the numbers in expanded form so that both the minuend and the subtrahend are written as the sums of their place values.

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# 10.1.2.3: Estimation

**Learning Objectives**

- Use rounding to estimate sums and differences.
- Use rounding to estimate the solutions for application problems.

## Introduction

An **estimate** is an answer to a problem that is close to the solution, but not necessarily exact. Estimating can come in handy in a variety of situations, such as buying a computer. You may have to purchase numerous devices: a computer tower and keyboard for $1,295, a monitor for $679, the printer for $486, the warranty for $196, and software for $374. Estimating can help you know *about* how much you'll spend without actually adding those numbers exactly.

Estimation usually requires **rounding**. When you round a number, you find a new number that's close to the original one. A rounded number uses zeros for some of the place values. If you round to the nearest ten, you will have a zero in the ones place. If you round to the nearest hundred, you will have zeros in the ones and tens places. Because these place values are zero, adding or subtracting is easier, so you can find an estimate to an exact answer quickly.

It is often helpful to estimate answers before calculating them. Then if your answer is not close to your estimate, you know something in your problem-solving process is wrong.

## Using Rounding to Estimate Sums and Differences

Suppose you must add a series of numbers. You can round each addend to the nearest hundred to estimate the sum.

**Example**

**Estimate the sum** 1,472+398+772+164 **by rounding each number to the nearest hundred.**

### Solution

| | |
|---|---|
| $1,472$ rounds to $1,500$<br>$398$ rounds to $400$<br>$772$ rounds to $800$<br>$164$ rounds to $200$ | First, round each number to the nearest hundred. |
| $\begin{array}{r} 1,500 \\ 400 \\ 800 \\ +\quad 200 \\ \hline 2,900 \end{array}$ | Then, add the rounded numbers together. |

The estimate is 2,900

In the example above, the exact sum is 2,806. Note how close this is to the estimate, which is 94 greater.

In the example below, notice that rounding to the nearest ten produces a far more accurate estimate than rounding to the nearest hundred. In general, rounding to the lesser place value is more accurate, but it takes more steps.

**Example**

**Estimate the sum 1,472+398+772+164 by first rounding each number to the nearest ten.**

### Solution

| | |
|---|---|
| $\begin{array}{rlr} 1,472 & \text{rounds to} & 1,470 \\ 398 & \text{rounds to} & 400 \\ 772 & \text{rounds to} & 770 \\ 164 & \text{rounds to} & 160 \end{array}$ | First, round each number to the nearest ten. |
| $\begin{array}{r} 12 \\ 1470 \\ 400 \\ +\ \ 160 \\ \hline 00 \end{array}$ | Next, add the ones and then the tens. Here, the sum of 7, 7, and 6 is 20. Regroup. |
| $\begin{array}{r} 12 \\ 1470 \\ 400 \\ 770 \\ +\ \ 160 \\ \hline 800 \end{array}$ | Now, add the hundreds. The sum of the digits in the hundreds place is 18. Regroup. |
| $\begin{array}{r} 12 \\ 1470 \\ 400 \\ 770 \\ +\ \ 160 \\ \hline 2800 \end{array}$ | Finally, add the thousands. The sum in the thousands place is 2. |

The estimate is 2,800.

Note that the estimate is 2,800, which is only 6 less than the actual sum of 2,806.

**? Exercise**

In three months, a freelance graphic artist earns $1,290 for illustrating comic books, $2,612 for designing logos, and $4,175 for designing web sites. Estimate how much she earned in total by first rounding each number to the nearest hundred.

A. $8,200
B. $7,900
C. $8,000
D. $8,100

**Answer**

A. Incorrect. You probably rounded one or more numbers up when you should have rounded them down. The correct answer is $8,100
B. Incorrect. You probably rounded one or more numbers down when you should have rounded them up. The correct answer is $8,100
C. Incorrect. You probably rounded each number to the nearest thousand instead of to the nearest hundred before adding. The correct answer is $8,100
D. Correct. You probably rounded the numbers to $1,300, $2,600 and $4,200 and added them together successfully.

You can also estimate when you subtract, as in the example below. Because you round, you do not need to subtract in the tens or hundreds places.

**✓ Example**

**Estimate the difference of 5,876 and 4,792 by first rounding each number to the nearest hundred.**

**Solution**

| | |
|---|---|
| $\begin{array}{lll} 5,876 & \text{rounds to} & 5,900 \\ 4,792 & \text{rounds to} & 4,800 \end{array}$ | First, round each number to the nearest hundred. |
| $\begin{array}{r} 5,900 \\ \underline{-4,800} \\ 1,100 \end{array}$ | Subtract. No regrouping is needed since each number in the minuend is greater than or equal to the corresponding number in the subtrahend. |

The estimate is 1,100.

The estimate is 1,100, which is 16 greater than the actual difference of 1,084.

**? Exercise**

Estimate the difference of 474,128 and 262,767 by rounding to the nearest thousand.

A. 212,000
B. 211,000
C. 737,000
D. 447,700

**Answer**

A. Incorrect. You probably rounded 474,128 up to 475,000 when you should have rounded it down to 474,000. The correct answer is 211,000.
B. Correct. You most likely rounded the numbers to 474,000 and 263,000 and subtracted successfully.
C. Incorrect. You probably added instead of subtracted. The correct answer is 211,000.
D. Incorrect. You probably used 26,300 instead of 263,000 as the number that you subtracted from 474,000. The correct answer is 211,000.

## Solving Application Problems by Estimating

Estimating is handy when you want to be sure you have enough money to buy several things.

**✓ Example**

**When buying a new computer, you find that the computer tower and keyboard cost $1,295, the monitor costs $679, the printer costs $486, the 2-year warranty costs $196, and a software package costs $374. Estimate the total cost by first rounding each number to the nearest hundred.**

**Solution**

| | |
|---|---|
| $\begin{array}{rlr} 1,295 & \text{rounds to} & 1,300 \\ 679 & \text{rounds to} & 700 \\ 486 & \text{rounds to} & 500 \\ 196 & \text{rounds to} & 200 \\ 374 & \text{rounds to} & 400 \end{array}$ | First, round each number to the nearest hundred. |

| | |
|---|---|
| $\begin{array}{r} 2 \\ 1300 \\ 700 \\ 500 \\ 200 \\ +\ \ 400 \\ \hline 3,100 \end{array}$ | Add.<br>After adding all of the rounded values, the estimated answer is \$3,100. |

The total cost is approximately \$3,100.

Estimating can also be useful when calculating the total distance one travels over several trips.

✓ Example

**James travels 3,247 meters to the park, then travels 582 meters to the store. He then travels 1,634 meters back to his house. Find the total distance traveled by first rounding each number to the nearest ten.**

**Solution**

| | |
|---|---|
| $\begin{array}{rlr} 3247 & \text{rounds to} & 3250 \\ 582 & \text{rounds to} & 580 \\ 1634 & \text{rounds to} & 1630 \end{array}$ | First, round each number to the nearest ten. |
| $\begin{array}{r} 1\ \ \\ 3250 \\ 580 \\ +\ \ 1630 \\ \hline 60 \end{array}$ | Adding the numbers in the tens place gives 16, so you need to regroup. |
| $\begin{array}{r} 11\ \ \ \\ 3250 \\ 580 \\ +\ \ 1630 \\ \hline 460 \end{array}$ | Adding the numbers in the hundreds place gives 14, so regroup. |
| $\begin{array}{r} 11\ \ \ \\ 3250 \\ 580 \\ +\ \ 1630 \\ \hline 5,460 \end{array}$ | Adding the numbers in the thousands place gives 5. |

The total distance traveled was approximately 5,460

In the example above, the final estimate is 5,460 meters, which is 3 less than the actual sum of 5,463 meters.

Estimating is also effective when you are trying to find the difference between two numbers. Problems dealing with mountains like the example below may be important to a meteorologist, a pilot, or someone who is creating a map of a given region. As in other problems, estimating beforehand can help you find an answer that is close to the exact value, preventing potential errors in your calculations.

✓ Example

**One mountain is 10,496 feet high and another mountain is 7,421 feet high. Find the difference in height by first rounding each number to the nearest 100.**

**Solution**

| | |
|---|---|
| $\begin{array}{rcr} 10,496 & \text{rounds to} & 10,500 \\ 7,421 & \text{rounds to} & 7,400 \end{array}$ | First, round each number to the nearest hundred. |
| $\begin{array}{r} 10,500 \\ -\quad 7,400 \\ \hline 3,100 \end{array}$ | Then, align the numbers and subtract.<br>The final estimate is 3,100, which is 25 greater than the actual value of 3,075. |

The estimated difference in height between the two mountains is 3,100 feet.

### ? Exercise

A space shuttle traveling at 17,581 miles per hour decreases its speed by 7,412 miles per hour. Estimate the speed of the space shuttle after it has slowed down by rounding each number to the nearest hundred.

A. 10,100 miles per hour
B. 10,200 miles per hour
C. 25,000 miles per hour
D. 25,100 miles per hour

**Answer**

A. Incorrect. You probably rounded the subtrahend up or the minuend down. You should have rounded the subtrahend of 7,412 down to 7,400 and the minuend of 17,581 up to 17,600. The correct answer is 10,200.
B. Correct. You correctly rounded both numbers and subtracted them successfully.
C. Incorrect. You probably added when you should have subtracted. The correct answer is 10,200
D. Incorrect. You probably rounded incorrectly and added when you should have subtracted. The correct answer is 10,200

## Summary

Estimation is very useful when an exact answer is not required. You can use estimation for problems related to travel, finances, and data analysis. Estimating is often done before adding or subtracting by rounding to numbers that are easier to think about. Following the rules of rounding is essential to the practice of accurate estimation.

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# 10.1.3: Multiplying and Dividing Whole Numbers

# 10.1.3.1: Multiplying Whole Numbers and Applications

**Learning Objectives**

- Use three different ways to represent multiplication.
- Multiply whole numbers.
- Multiply whole numbers by a power of 10.
- Use rounding to estimate products.
- Find the area of a rectangle.
- Solve application problems using multiplication.

## Introduction

Instead of adding the same number over and over again, an easier way to reach an answer is to use multiplication. Suppose you want to find the value in pennies of 9 nickels. You can use addition to figure this out. Since a nickel is worth 5 pennies, or 5 cents, you can find the value of 9 nickels by adding 5+5+5+5+5+5+5+5+5. This repeated addition shows that 9 nickels have a value of 45 cents.

All of this addition can become very tiring. So, the math operation called *multiplication* can help perform repeated addition of whole numbers much more quickly. To find the value of these nickels, you could write a multiplication equation: $5 \cdot 9 = 45$.

$5 \cdot 9 = 45$ is read “5 times 9 equals 45" or "5 multiplied by 9 is equal to 45." The numbers that are being multiplied are called **factors**. The factors in this example are 5 and 9. The result of the multiplication (or the answer) is called the **product**. The product for $5 \cdot 9$ is 45.

In addition to showing multiplication as $5 \cdot 9 = 45$, you can show multiplication by using the x sign, $5 \times 9 = 45$, and also with parentheses, (5) (9) = 45 or 5(9)=45.

**Three Ways to Write Multiplication**

Using a multiplication or times sign: $2 \times 3 = 6$

Using a dot: $2 \cdot 3 = 6$ (this dot is NOT a decimal point)

Using parentheses: (2)(3)=6 or 2(3)=6

When you are adding the same number over and over again, you can use multiplication. You take the number that you are adding and rewrite it as a multiplication problem, multiplying it by the number of times you are adding it. For example, if you were serving 2 cookies each to 13 children, you could add 2 thirteen times or you could use multiplication to find the answer.

$2+2+2+2+2+2+2+2+2+2+2+2+2 = 2 \cdot 13 = 26$

You could also write this using parentheses: $2(13) = 26$

## What Is Multiplication?

In order to understand what multiplication is, consider three different ways to think about multiplication of whole numbers.

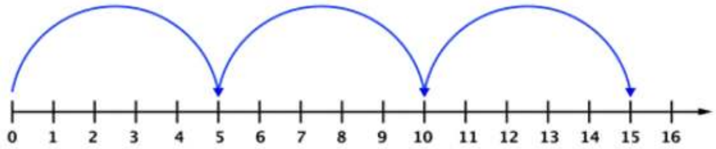
Approach 1: Set Model

Multiplication is a way of writing repeated addition. When you read the problem $3 \cdot 5$ you could think of this as 3 groups of 5 things: 3 plates with 5 cookies on each plate; 3 baskets, each with 5 oranges in it; or 3 piles with 5 coins in each pile. We could show this as a picture:

$3 \cdot 5 = 3$ groups of $5 = 15$

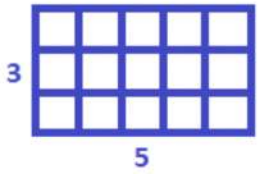
Approach 2: Number Line Model

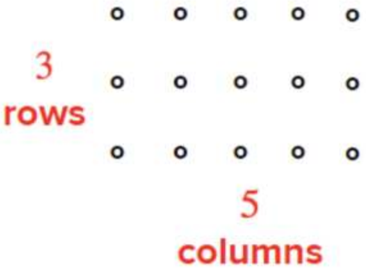
Multiplication can also be shown on a number line. The problem, $3 \cdot 5$ is modeled on the number line below. You can see that the arrows cover a distance of 5 units at a time. After 3 “jumps” on the number line, the arrow ends at 15.



Approach 3: Area Model

Another way of thinking about multiplication is to think about an array or area model to represent multiplication. You could think of $3 \cdot 5$ as 3 rows of 5 things. This might be a box of chocolates that has 3 rows of 5 chocolates, or a meeting room that is set up with 3 rows of 5 chairs. The pictures below show two rectangular arrangements of $3 \cdot 5$.





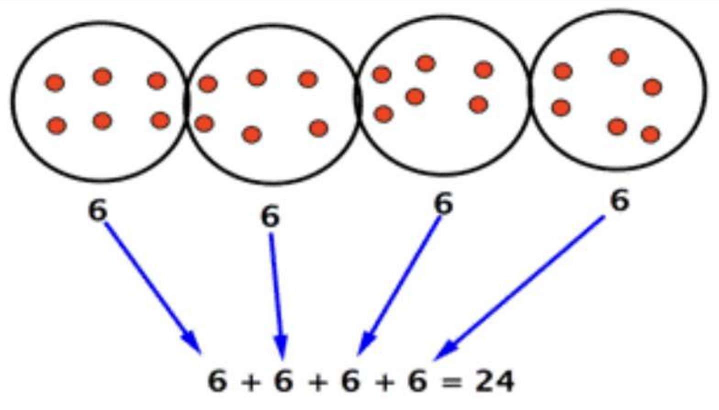
Do you see how both pictures represent the product 15? The picture on the left shows an area of 3 by 5. If you count the small squares that make up the rectangle, they total 15. Similarly, in the picture on the right, you see that 3 rows of 5 circles is equal to 15 circles.

✓ Example

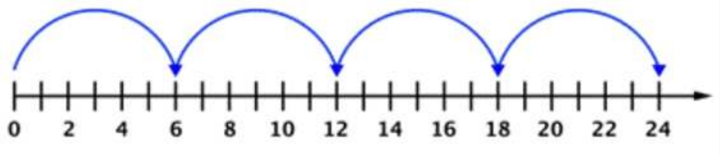
**What is the product of $4 \cdot 6$? Use the set model, number line model and area model to represent the multiplication problem.**

**Solution**

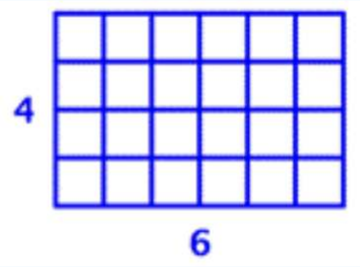
Set Model:



Number Line model:



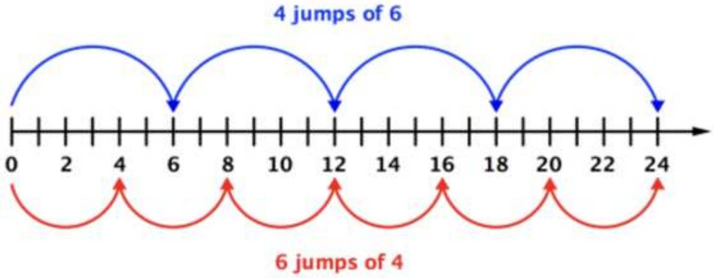
Area Model:



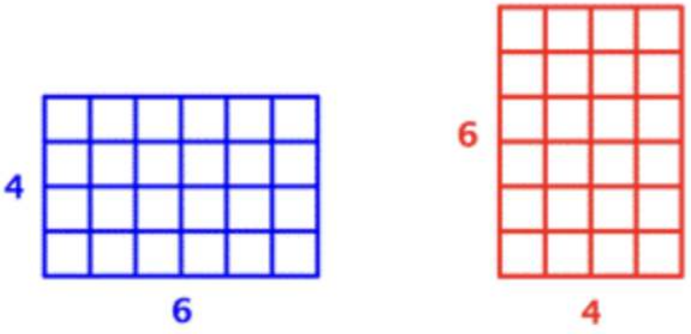
$4 \cdot 6 = 24$

If you switch the order in which you multiply two numbers, the product will not change. This is true for any two numbers you multiply. Think about the problem shown above.

You could make 6 jumps of 4 or 4 jumps of 6 on the number line and end up at 24.



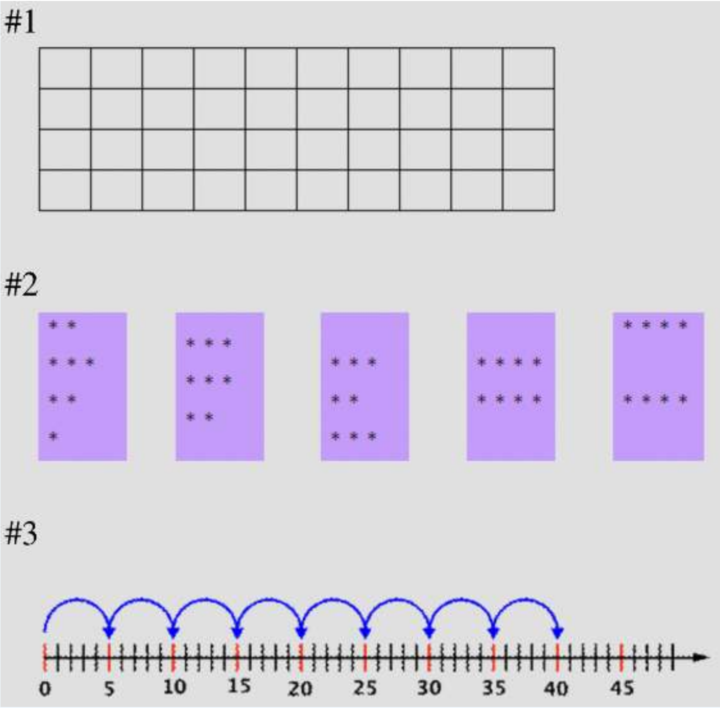
Or, you could make 6 rows of 4 or 4 rows of 6 and still have 24 squares.



$6 \cdot 4 = 24$ and $4 \cdot 6 = 24$.

? Exercise

Tanisha modeled $5 \cdot 8$ using the following models. Which models are accurate representations of the multiplication of these two factors?



A. All three models accurately represent $5 \cdot 8$.
B. Only models #1 and #3 accurately represent $5 \cdot 8$.
C. Only models #2 and #3 accurately represent $5 \cdot 8$.
D. None of the models accurately represents $5 \cdot 8$.

**Answer**

A. All three models accurately represent $5 \cdot 8$. Incorrect. While the first model has the same product, it models the factors $4 \cdot 10$. The correct answer is Only models #2 and #3 accurately represent $5 \cdot 8$.
B. Only models #1 and #3 accurately represent $5 \cdot 8$. Incorrect. Model #1 models $4 \cdot 10$, not $5 \cdot 8$. The correct answer is Only models #2 and #3 accurately represent $5 \cdot 8$.
C. Only models #2 and #3 accurately represent $5 \cdot 8$. Correct. Model #2 shows 5 groups of 8, which equals 40. Model #3 shows skip counting 8 times by 5 to get to 40.
D. None of the models accurately represents $5 \cdot 8$. Incorrect. Models #2 and #3 represent the multiplication $5 \cdot 8$. The correct answer is Only models #2 and #3 accurately represent $5 \cdot 8$.

## Multiplying Greater Numbers

Let’s go back to the question posed at the opening of this topic of study. How can you use multiplication to figure out the total cost of 6 baseball caps that cost $14 each? (You do not have to pay sales tax). You can figure out the cost by multiplying $14 \cdot 6$.

One way to do this computation is to break 14 down into parts and multiply each part by 6.

$$\begin{aligned} 14 &= 10+4 \quad \text{So,} \\ 14 \cdot 6 &= 10 \cdot 6 + 4 \cdot 6 \\ &= 60+24 \\ &= 84 \end{aligned}$$

You may recall the multiplication computed as $14 \cdot 6$:

$$\begin{array}{r} 2 \\ 14 \\ \times \quad 6 \\ \hline 84 \end{array}$$

In this notation, some of the steps are written down with special notation. The product of 6 and 4(24) is written by putting the 4 in the ones place and writing a 2 up above the 1. This 2 actually stands for 20. Then, 6 is multiplied by 1. We are actually multiplying 6 by 1 ten and adding 20 to get the 80 in 84.

An example of multiplying two two-digit numbers is shown below. When performing this multiplication, each part of each number is multiplied by the other number. The numeric notation and an accompanying description are provided.

**Example**

$47 \cdot 52$

**Solution**

| | |
|---|---|
| $\begin{array}{r} 47 \\ \times\ 52 \\ \hline \end{array}$ | Stack the numbers with the place values aligned. |
| $\begin{array}{r} 1\phantom{0} \\ 47 \\ \times\ 52 \\ \hline 4 \end{array}$ | Multiply the ones. $2 \times 7 = 14$ ones. Write 4 ones in the ones place and regroup 10 ones into the tens place. |
| $\begin{array}{r} 1\phantom{0} \\ 47 \\ \times\ 52 \\ \hline 94 \end{array}$ | Multiply 2 ones by 4 tens, and add the regrouped 1 ten. 2 ones times 4 tens + 1 ten = 9 tens |
| $\begin{array}{r} 3\phantom{00} \\ 47 \\ \times\ 52 \\ \hline 94 \\ 50 \end{array}$ | Multiply the tens. $5 \times 7 = 35$ tens. Write 5 tens in the tens place and regroup. |
| $\begin{array}{r} 3\phantom{00} \\ 47 \\ \times 52 \\ \hline 94 \\ 2350 \end{array}$ | Multiply: 5 tens times 4 tens = 20 hundreds.<br>Add the regrouped 3, which is 3 hundred. |
| $\begin{array}{r} 3\phantom{00} \\ 47 \\ \times\ 52 \\ \hline 194 \\ 2350 \\ \hline 2,444 \end{array}$ | Add the two lines. 94+2,350 |

$47 \cdot 52 = 2,444$

Notice that you are multiplying each of the parts of each number by the parts of the other number. You are doing so in a systematic way, from ones places to tens places. You are also using notation to keep track of what you have regrouped. These are shown with raised numbers.

To keep your columns straight and your work organized, consider using grid paper or lined paper turned sideways so the lines form columns. Here is an example of a problem written on grid paper:

| | | | |
|---|---|---|---|
| | | 2 | 3 |
| | x | 1 | 2 |
| | | 4 | 6 |
| + | 2 | 3 | 0 |
| | 2 | 7 | 6 |

When you are multiplying whole numbers, be sure to line up the digits by their place values. In the example above, the digits in the ones place are lined up: the 2 in 12 is directly below the 3 in 23.

## Multiplying Whole Numbers by 10

When you multiply numbers by 10 or powers of 10(100 ; 1,000 ; 10,000 ; 100,000), you'll discover some interesting patterns. These patterns occur because our number system is based on ten: ten *ones* equal ten; ten *tens* equal one hundred; ten *hundreds* equal one thousand. Learning about these patterns can help you compute easily and quickly.

Consider the example of $25 \cdot 100$. First, let's use the standard algorithm method to multiply these numbers.

### ✓ Example

$25 \cdot 100$

**Solution**

$$\begin{array}{r} 100 \\ \underline{\times \quad 25} \\ 500 \\ \underline{2000} \\ 2,500 \end{array}$$

$25 \cdot 100 = 2,500$

Using the standard algorithm, we calculated $25 \cdot 100 = 2,500$.

Look at the table below to find a pattern in the factors and products. See how the number of zeros in the power of 10(10,100, 1,000, etc.) relates to the number of zeros in the product.

| **Factors** | **Product** |
|---:|---|
| $5 \cdot 10$ | = 50 |
| $5 \cdot 100$ | = 500 |
| $5 \cdot 1,000$ | = 5,000 |
| $5 \cdot 10,000$ | = 50,000 |

You can see that the number of zeros in the product matches the number of zeros in the power of 10(10,100,1,000, etc.). Will this always be true or is it true only in certain situations? Look at two more patterns:

| **Factors** | **Product** |
|---:|---|
| $10 \cdot 10$ | = 100 |
| $10 \cdot 100$ | = 1,000 |
| $10 \cdot 1,000$ | = 10,000 |
| $10 \cdot 10,000$ | = 100,000 |

| **Factors** | **Product** |
|---:|---|

| | |
|---|---|
| $120 \cdot 10$ | = 1,200 |
| $120 \cdot 100$ | = 12,000 |
| $120 \cdot 1,000$ | = 120,000 |
| $120 \cdot 10,000$ | = 1,200,000 |

Notice that in these last two examples, both factors had zeros in them. The number of zeros in the product is equal to the sum of the number of zeros at the end of each of the factors.

The example below illustrates how to multiply $140 \cdot 3000$.

## ✓ Example

$140 \cdot 3000$

### Solution

| | |
|---|---|
| $\begin{array}{r} 1 \\ 14 \\ \times\ \ 3 \\ \hline 42 \end{array}$ | Identify the non-zero parts of the factors and multiply these parts.<br>Multiply 3 ones by 4 ones. $4 \cdot 3 = 12$ . |
| 420,000 | Count the number of zeros in each factor.<br>140 has **one** zero; 3,000 has **three** zeros.<br>1+3=4<br>Write another 4 zeros after the 42. |

$140 \cdot 3000 = 420,000$

## Multiplying by Ten

When you multiply a whole number by 10 or a power of 10, first multiply the nonzero parts of the numbers. Then include the number of zeros at the end of the product equal to the total number of zeros at the end of the factors.

$13 \cdot 100 = 1,300$

$180 \cdot 2,000 = 360,000$

## ? Exercise

An apple orchard sold 100 bags of apples. If there are 30 apples in each bag, how many apples did the orchard sell?

A. 130
B. 300
C. 30,000
D. 3,000

**Answer**

A. Incorrect. You need to multiply 100 and 30, not add. The correct answer is 3,000.
B. Incorrect. $30 \cdot 100 = 3,000$ You did not include the correct number of zeros. The correct answer is 3,000.
C. Incorrect. $30 \cdot 100 = 3,000$ You included too many zeros. The correct answer is 3,000.
D. Correct. $30 \cdot 100 = 3,000.3 \cdot 1 = 3$and add three zeros since there is one zero in 30 and there are two zeros in 100.

## Using Rounding to Estimate Products

Sometimes you don’t need an exact product because an estimate is enough. If you’re shopping, stopping to make a calculation with pencil and paper, or even a calculator, is inconvenient. Usually, shoppers will round numbers up so they will be sure that they have enough money for their purchases.

Estimating products is also helpful for checking an answer to a multiplication problem. If your actual calculation is quite different from your estimate, there is a good chance you have made a place value and/or regrouping mistake.

To estimate a product, you often round the numbers first. When you round numbers, you are always rounding to a particular place value, such as the nearest thousand or the nearest ten. If you are rounding a number to the nearest ten, you round it to the ten that is closest to the original number. An example of this is rounding 317 to the nearest ten. In this case, you round 317 to 320. If the number is half way in between (315), generally round up to 320.

Rounding factors can make it easy to multiply in your head. Let’s consider the multiplication problem $145 \cdot 29$. To estimate this product by rounding, you can round to the nearest ten.

### ✓ Example

**Use rounding to estimate the product of $145 \cdot 29$ .**

**Solution**

| | |
|---|---|
| $150 \cdot 30$ | Round numbers to the nearest ten. |
| $15 \cdot 3 = 45$ | Multiply the non-zero numbers. |
| 4,500 | Count the zeros in the factors and include that many zeros after the 45. |

The estimate of $145 \cdot 29$ is $4,500$.

You can use a calculator to see if your estimate seems reasonable. Or you can use estimation to make sure that the answer that you got on a calculator is reasonable. (Have you ever input the wrong numbers?)

| | |
|---|---|
| | Key entries:<br>145<br>x<br>29<br>=<br>Result: 4,205<br>The exact product and the estimate are close enough to give you confidence in your calculations. |

### ? Exercise

A factory produces 58 packages of cookies in one hour. There are 32 cookies in each package. Which is the best estimate of the number of cookies the factory produces in one hour?

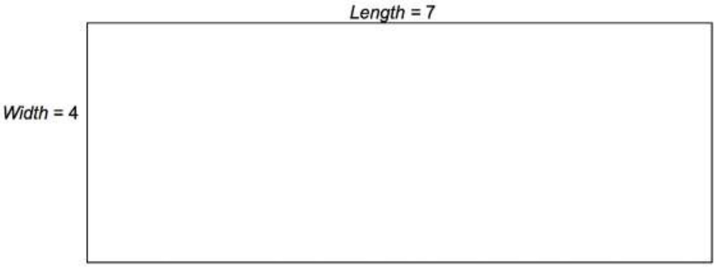
A. 1,800
B. 1,500
C. 18,000
D. 180

**Answer**

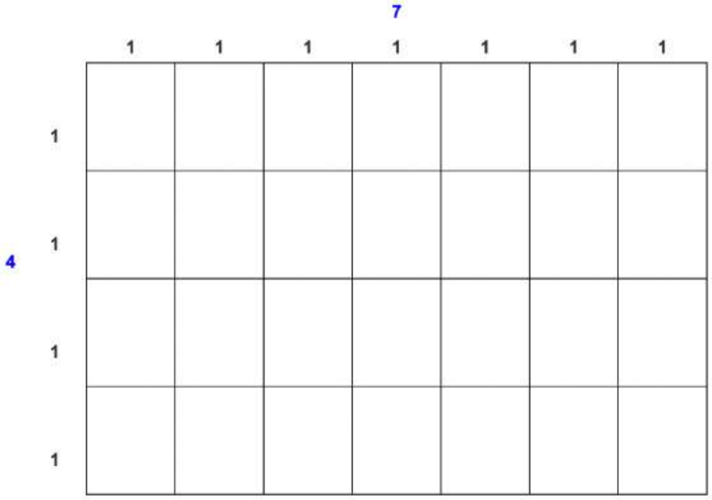
A. Correct. Multiplying $60 \cdot 30$ would give a good estimate. $60 \cdot 30 = 1,800$.
B. Incorrect. $50 \cdot 30 = 1,500$, but 58 rounds to 60, not 50.The correct answer is 1,800.
C. Incorrect. $60 \cdot 30 = 1,800$. The correct answer is 1,800.
D. Incorrect. $60 \cdot 30 = 1,800$. The correct answer is 1,800.

## Finding the Area of a Rectangle

The formula for the area of a rectangle uses multiplication: $\text{length} \cdot \text{width} = \text{area}$ . Applying what you know about multiplication, you can find the area of any rectangle if you know its dimensions (length and width). Consider the rectangle that is 4 by 7 shown below. Its length is 7 and its width is 4.



You can divide the rectangle into units by making 7 columns and 4 rows.



You can see that dividing the rectangle in this way results in 28 squares. You could say that the area of the rectangle is 28 square units. You could also find the area by multiplying $7 \cdot 4$. (Note: Area is always measured in square units: square inches, square centimeters, square feet, etc.)

Consider an example of a larger rectangle, like one that is found on a soccer field. At each end of a soccer field, centered at the goal, is a large rectangle. This rectangle is called the *penalty box* because fouls committed within the lines of this rectangle may result in a penalty kick. On a regulation soccer field, the penalty box is 44 yards by 18 yards. What is the area of a penalty box?

### Example

$44 \text{ yards} \cdot 18 \text{ yards}$

**Solution**

$$\begin{array}{rr} & {}^{3}\phantom{4} \\ & 44 \\ \times & 18 \\ \hline & 352 \\ + & 440 \\ \hline & 792 \end{array}$$

The area of the penalty box is 792 square yards.

**Exercise**

What is the area of a rectangle whose length is 23 feet and whose width is 7 feet?

A. 30 feet
B. 161 feet
C. 161 square feet
D. 1,421 square feet

**Answer**

A. Incorrect. To find the area, you multiply, not add, the length and width. The correct answer is 161 square feet.
B. Incorrect. Area is measured in square units. The correct answer is 161 square feet.
C. Correct. $23 \cdot 7 = 161$
D. Incorrect. There is a place-value mistake. When you multiply $7 \cdot 3$, you need to regroup 20 ones to 2 tens and add it to the product of 2 and 7. The correct answer is 161 square feet.

## Using Multiplication in Problem Solving

Multiplication is used in solving many types of problems. Below are two examples that use multiplication in their solutions to the problem.

**Example**

**A case of cat food has two layers. Each layer has 4 rows of 6 cans. How many cans are in a case of cat food?**

**Solution**

| | |
|---|---|
| $\begin{array}{r} 6 \\ \times \quad 4 \\ \hline 24 \end{array}$ | First find the number of cans in one layer. You can multiply to figure this out. |
| $\begin{array}{r} 24 \\ \times \quad 2 \\ \hline 48 \end{array}$ | Since there are two layers, you multiply the number of cans in one layer by 2. |

There are 48 cans in a case of cat food.

**Example**

**A theater has 45 rows with 40 seats in each row. How many seats are there in the theater?**

**Solution**

| | |
|---|---|
| $\begin{array}{r} 2\phantom{0} \\ 45 \\ \times \qquad 40 \\ \hline 00 \\ + \qquad 1800 \\ \hline 1,800 \end{array}$ | You can solve this problem by adding 40, 45 times, but that would take a lot of work. Multiplication is the way to go. |

There are 1,800 seats in the theater.

**Exercise**

A lawn care company charges $35 to mow a lawn. If the company mows 32 lawns, how much money will it make?

A. $9,760
B. $1,120
C. $130.00
D. $67.00

**Answer**

A. Incorrect. This answer is too large. By estimation, $40 \cdot 30 = 1200$. There must be an error in regrouping or place value. The correct answer is $1,120.
B. Correct. $35 \cdot 32 = 1,200$
C. Incorrect. The answer should be a larger number. By estimation, $40 \cdot 30 = 1200$. There must be an error in regrouping or place value. The correct answer is $1,120.
D. Incorrect. You must multiply, not add, 32 and 35. The correct answer is $1,120.

## Summary

Multiplication can make repeated addition easier to compute in calculations and problem solving. Multiplication can be written using three symbols: parentheses, a times sign, or a multiplication dot. To perform multiplication with two-digit factors or greater, you can use the standard algorithm where you multiply each of the numbers in each factor by the numbers in the other factor. Using strategies such as short cuts for multiplying by powers of 10 and estimation to check your answers can make multiplication easier as well as reduce errors.

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# 10.1.3.2: Dividing Whole Numbers and Applications

**Learning Objectives**

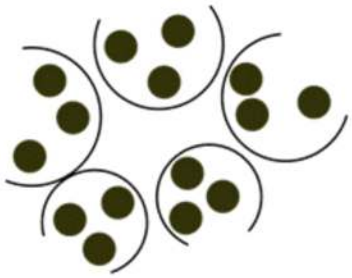
- Use three different ways to represent division.
- Divide whole numbers.
- Perform long division.
- Divide whole numbers by a power of 10.
- Recognize that division by 0 is not defined.
- Solve application problems using division.

## Introduction

Some people think about division as “fair sharing” because when you divide a number you are trying to create equal parts. Division is also the **inverse operation** of multiplication because it “undoes” multiplication. In multiplication, you combine equal sets to create a total. In division, you separate a whole group into sets that have the same amount. For example, you could use division to determine how to share 40 empanadas among 12 guests at a party.

## What is Division?

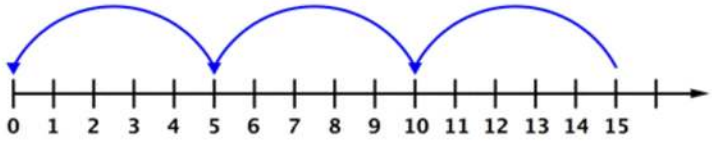
Division is splitting into equal parts or groups. For example, one might use division to determine how to share a plate of cookies evenly among a group. If there are 15 cookies to be shared among five people, you could divide 15 by 5 to find the “fair share” that each person would get. Consider the picture below.



15 cookies split evenly across 5 plates results in 3 cookies on each plate. You could represent this situation with the equation:

$$15 \div 5 = 3$$

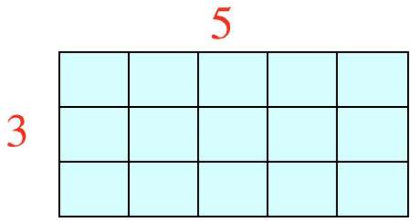
You could also use a number line to model this division. Just as you can think of multiplication as repeated addition, you can think of division as repeated subtraction. Consider how many jumps you take by 5s as you move from 15 back to 0 on the number line.



Notice that there are 3 jumps that you make when you skip count by 5 from 15 back to 0 on the number line. This is like subtracting 5 from 15 three times. This repeated subtraction can be represented by the equation: $15 \div 5 = 3$ .

Finally, consider how an area model can show this division. Ask yourself, if you were to make a rectangle that contained 15 squares with 5 squares in a row, how many rows would there be in the rectangle? Start by making one row of 5 squares:

Then add two more rows of 5 squares so you have 15 squares.



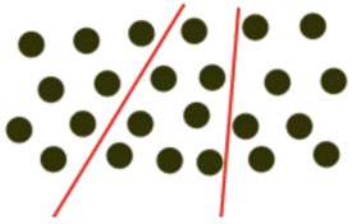
The number of rows is 3. So, 15 divided by 5 is equal to 3.

**Example**

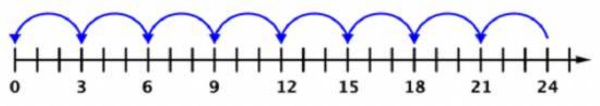
**Find $24 \div 3$ using a set model and a number line model.**

**Solution**

Set Model:



Number line model:



$24 \div 3 = 8$

## Ways to Represent Division

As with multiplication, division can be written using a few different symbols. We showed this division written as $15 \div 5 = 3$, but it can also be written two other ways:

$$\begin{array}{r} 3 \\ 5\backslash\text{longdiv}15 \end{array}$$

$\frac{15}{5} = 3$

Each part of a division problem has a name. The number that is being divided up, that is the total, is called the **dividend**. In the work in this topic, this number will be the larger number, but that is not always true in mathematics. The number that is dividing the dividend is called the **divisor**. The answer to a division problem is called the **quotient**.

The blue box below summarizes the terminology and common ways to represent division.

**Three Ways to Represent Division**

$12 \div 3 = 4$ (**with a division symbol**; this equation is read "12 divided by 3 equals 4."

$\begin{array}{r} 4 \\ 3\backslash\text{longdiv}12 \end{array}$ (**with a division or long division symbol**; this expression is read "12 divided by 3 equals 4." Notice here, though, that you have to start with what is underneath the symbol. This may take some getting used to since you are reading from right to left and bottom to top!)

$\frac{12}{3} = 4$ (**with a fraction bar**; this expression can also be read "12 divided by 3 equals 4." In this format, you read from top to bottom.)

In the examples above, 12 is the **dividend**, 3 is the **divisor** and 4 is the **quotient.**

$$\text{Dividend} \div \text{Divisor} = \text{Quotient}$$

$$\text{Divisor}\backslash\text{longdiv}\overset{\text{Quotient}}{\text{Dividend}}$$

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient}$$

**Exercise**

Which of the following expressions represent dividing $56 equally among 7 people?

#1: $\frac{7}{56}$

#2: $56 \div 7$

#3: $56\backslash\text{longdiv}7$

A. #2 represents the situation.
B. All three expressions represent the situation.
C. #1 represents the situation.
D. #3 represents the situation.

**Answer**

A. Correct. #2 is the only expression that represents 56 divided by 7.
B. Incorrect. #1 and #3 represent 7 divided by 56, not 56 divided by 7. The correct answer is #2 is the only expression that represents the situation.
C. Incorrect. This expression represents 7 divided by 56, not 56 divided by 7. The correct answer is #2 is the only expression that represents the situation.
D. Incorrect. This expression represents 7 divided by 56, not 56 divided by 7. The correct answer is #2 is the only expression that represents the situation.

## Dividing Whole Numbers

Once you understand how division is written, you are on your way to solving simple division problems. You will need your multiplication facts to perform division. If you do not have them memorized, you can guess and check or use a calculator.

Consider the following problems:

$10 \div 5 = ?$

$48 \div 2 = ?$

$30 \div 5 = ?$

In the first problem, $10 \div 5$, you could ask yourself, “how many fives are there in ten?” You can probably answer this easily. Another way to think of this is to consider breaking up 10 into 5 groups and picturing how many would be in each group.

$10 \div 5 = 2$

To solve $48 \div 2$, you might realize that dividing by 2 is like splitting into two groups or splitting the total in half. What number could you double to get 48?

$48 \div 2 = 24$

To figure out $30 \div 5$, you could ask yourself, how many times do you have to skip count by 5 to get from 0 to 30? 5, 10, 15, 20, 25, 30. You have to skip count 6 times to get to 30.

$30 \div 5 = 6$

**Exercise**

Compute $35 \div 5$.

**Answer**

7

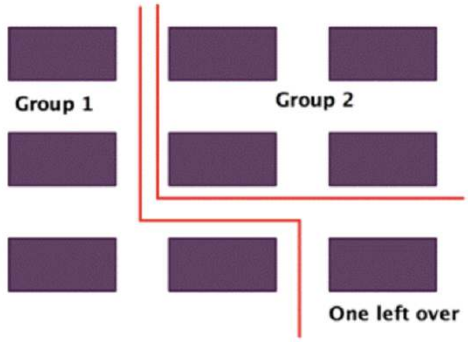
**Exercise**

Compute $32 \div 4$.

**Answer**

8

Sometimes when you are dividing, you cannot easily share the number equally. Think about the division problem $9 \div 2$. You could think of this problem as 9 pieces of chocolate being split between 2 people. You could make two groups of 4 chocolates, and you would have one chocolate left over.



In mathematics, this left over part is called the **remainder**. It is the part that *remains* after performing the division. In the example above, the remainder is 1. We can write this as:

$$9 \div 2 = 4 \quad \text{R1}$$

We read this equation: “Nine divided by two equals four with a remainder of 1."

You might be thinking you could split that extra piece of chocolate in parts to share it. This is great thinking! If you split the chocolate in half, you could give each person another half of a piece of chocolate. They would each get $4\frac{1}{2}$ pieces of chocolate. We are not going to worry about expressing remainders as fractions or decimals right now. We are going to use the remainder notation with the letter R. Here’s an example:

**Example**

$45 \div 6$

**Solution**

| | |
|---|---|
| $6 \cdot 7 = 42$ | How many sixes are there in 45? Try 7. |
| $45 - 42 = 3$ | 3 is not enough for another 6. So, 3 is the remainder. |

$$45 \div 6 = 7 \quad \text{R3}$$

Since multiplication is the inverse of division, you can check your answer to a division problem with multiplication. To check the answer 7 R3, first multiply 6 by 7 and then add 3.

$$6 \cdot 7 = 42$$

$42+3=45$, so the quotient 7 R3 is correct.

**Exercise**

Compute $67 \div 7$.

A. 9
B. 9 R4
C. 60
D. 10

**Answer**

A. Incorrect. $9 \cdot 7=63$. There is a remainder of 4. The correct answer is 9 R4.
B. Correct. $9 \cdot 7=63$ and there are 4 left over.
C. Incorrect. This is a division, not subtraction, problem. The correct answer is 9 R4.
D. Incorrect. $70 \div 7=10$, so the answer to $67 \div 7$ cannot be; $9 \cdot 7=63$ and there are 4 left over. The correct answer is 9 R4.

## Performing Long Division

Long division is a method that is helpful when you are performing division that you cannot do easily in your head, such as division involving larger numbers. Below is an example of a way to write out the division steps.

**Example**

$68 \div 4$

**Solution**

| | |
|---|---|
| 4\longdiv68 | Rewrite the division. |
| 1<br>4\longdiv68<br>−4<br>28 | Divide the tens.<br>What is 6 divided by 4?<br>$4 \cdot 1=4$, so write a 1 above the 6.<br>Subtract 4 from 6 and bring down the next digit of the dividend, 8.<br>What is 28 divided by 4? |
| 17<br>4\longdiv68<br>−4<br>28<br>−28<br>0 | $7 \cdot 4=28$, so write a 7 above the 8.<br>There is no remainder. |
| \(\ \begin{array}{r}<br>17 \cdot 4 \\<br>^217 \\<br>\times 4 \\<br>\hline 68<br>\end{array}\) | Check your answer using multiplication. |

$68 \div 4=17$

**Example**

$6,707 \div 233$

**Solution**

| | |
|---|---|
| 2<br>233\longdiv6707<br>$-466$ | Examine the first 3 digits of the dividend and determine how many 233s are in it.<br>Use guess and check.<br>Try: $2 \cdot 233 = 466$<br>Try: $3 \cdot 233 = 699$ (too large) |
| 2<br>233\longdiv6707<br>$-466$<br>2047 | Subtract 466 from 670 and bring down the next digit of the dividend, 7. |
| 28<br>233\longdiv6707<br>$-466$<br>2047<br>$-1864$<br>183 | How many 233s are in 2,047? It looks like close to 10 because $233 \cdot 10 = 2,330$.<br>Try $9.233 \cdot 9$ equals 2,097.<br>22<br>233<br>$\times$ 9<br>2,097<br>2,097 (Too large)<br>Must be 8!<br>22<br>233<br>$\times$ 8<br>1,864 |
| $233 \cdot 28 = 6,524$<br>$6,524 + 183 = 6,707$ | Check your answer using multiplication.<br>First, multiply $233 \cdot 28$.<br>Then, add the remainder. |

$6,707 \div 233 = 28 \quad \text{R}183$

**Exercise**

Compute $417 \div 34$.

A. 451
B. 12
C. 12 R9
D. 13

**Answer**

A. Incorrect. This is a division problem, not an addition problem. The correct answer is 12 R9.
B. Incorrect. $12 \cdot 34 = 408$. The correct answer is 12 R9.
C. Correct. $12 \cdot 34 = 408$ and $408 + 9 = 417$
D. Incorrect. $13 \cdot 34 = 442$. The correct answer is 12 R9.

## Dividing Whole Numbers by a Power of 10

Just as multiplication by powers of 10 results in a pattern, there is a pattern with division by powers of 10. Consider three quotients: $20 \div 10; 200 \div 10; 2,000 \div 10$.

Think about $20 \div 10$. There are 2 tens in twenty, so $20 \div 10 = 2$. The computations for $200 \div 10$ and $2,000 \div 10$ are shown below.

**Example**

$200 \div 10$

**Solution**

| | |
|---|---|
| 10\longdiv200 | Rewrite the problem. |
| 10\longdiv200 | Divide the first digit of the dividend, 2, by the divisor.<br>Since $2 \div 10$ does not give a whole number, go to the next digit, 0. |
| 2<br>10\longdiv200 | $20 \div 10 = 2$ |
| 2<br>10\longdiv200<br>$-20$<br>0 | $2 \cdot 10 = 20$<br>$20 - 20 = 0$ |
| 2<br>10\longdiv200<br>$-20$<br>00 | Bring down the next digit of the dividend, which is 0. |
| 20<br>10\longdiv200<br>$-20$<br>00<br>0<br>0 | Since 10 still does not go into 00 and we have nothing left to bring down, multiply the 0 by 10.<br>$0 \cdot 10 = 0$<br>$0 - 0 = 0$<br>We have no remainder. |

$200 \div 10 = 20$

**Example**

$2000 \div 10$

**Solution**

| | |
|---|---|
| 10\longdiv2000 | Rewrite the problem. |
| 10\longdiv2000 | Divide the first digit of the dividend, 2, by the divisor.<br>Since $2 \div 10$ does not give a whole number, go to the next digit, 0. |
| 2<br>10\longdiv2000 | $20 \div 10 = 2$ |
| 2<br>10\longdiv2000<br>$-20$<br>0 | $2 \cdot 10 = 20$<br>$20 - 20 = 0$ |

| | |
|---|---|
| $\begin{array}{r} 2 \\ 10\backslash\mathrm{longdiv}2000 \\ -20 \\ \hline 00 \end{array}$ | Bring down the next digit, 0, of the dividend. |
| $\begin{array}{r} 20 \\ 10\backslash\mathrm{longdiv}2000 \\ -20 \\ \hline 000 \end{array}$ | Since 10 does not go into 00, add a 0 to the quotient and bring down the next digit, 0. |
| $\begin{array}{r} 200 \\ 10\backslash\mathrm{longdiv}2000 \\ -20 \\ \hline 000 \\ 0 \\ \hline 0 \end{array}$ | Since 10 still does not go into 000 and we have nothing left to bring down, add a 0 to the quotient, multiply the 0 by 10.<br>$0 \cdot 10 = 0$<br>$0 - 0 = 0$<br>We have no remainder. |

$2,000 \div 10 = 200$

Examine the results of these three problems to try to determine a pattern in division by 10.

$$\begin{array}{rl} 20 \div 10 & = 2 \\ 200 \div 10 & = 20 \\ 2,000 \div 10 & = 200 \end{array}$$

Notice that the number of zeros in the quotient decreases when a dividend is divided by 10: 20 becomes 2; 200 becomes 20 and 2,000 become 200. In each of the examples above, you can see that there is one fewer 0 in the quotient than there was in the dividend.

Continue another example of division by a power of 10.

✓ Example

$2,000 \div 100$

**Solution**

| | |
|---|---|
| $100\backslash\mathrm{longdiv}2000$ | Rewrite the problem. |
| $100\backslash\mathrm{longdiv}2000$ | Divide the first digit of the dividend, 2, by the divisor.<br>Since $2 \div 100$ does not give a whole number, go to the next digit, 0. |
| $100\backslash\mathrm{longdiv}2000$ | Divide the first two digits of the dividend, 20, by the divisor.<br>Since $20 \div 100$ does not give a whole number, go to the next digit, 0. |
| $100\backslash\mathrm{longdiv}2000$ | $200 \div 100 = 2$ |
| $\begin{array}{r} 2 \\ 100\backslash\mathrm{longdiv}2000 \\ -200 \\ \hline 0 \end{array}$ | $2 \cdot 100 = 200$<br>$200 - 200 = 0$ |
| $\begin{array}{r} 2 \\ 100\backslash\mathrm{longdiv}2000 \\ -200 \\ \hline 00 \end{array}$ | Bring down the next digit, 0, of the dividend. |

| | |
|---|---|
| 20<br>100\longdiv2000<br>$-200$<br>00<br>0<br>0 | Since 100 still does not go into 00 and we have nothing left to bring down, add a 0 to the quotient, multiply the 0 by 10.<br>$0\cdot 10=0$<br>$0-0=0$<br>We have no remainder. |

$2,000\div 100=20$

Consider this set of examples of division by powers of 10. What pattern do you see?

$$\begin{array}{rcl} 20 & \div & 10=2 \\ 200 & \div & 10=20 \\ 2,000 & \div & 10=200 \\ 2,000 & \div & 100=20 \\ 2,000 & \div & 1,000=2 \end{array}$$

Notice that when you divide a number by a power of 10, the quotient has fewer zeros. This is because division by a power of 10 has an effect on the place value. For example, when you perform the division $18,000\div 100=180$, the quotient, 180, has two fewer zeros than the dividend, 18,000. This is because the power of 10 divisor, 100, has two zeros.

**? Exercise**

Compute $135,000\div 100$.

A. 13,500
B. 134,900
C. 13,500,000
D. 1,350

**Answer**

A. Incorrect. This answer is too large. $13,500\cdot 100=1,350,000$The correct answer is 1,350.
B. Incorrect. This is a division, not a subtraction, problem. The correct answer is 1,350.
C. Incorrect. This is a division, not a multiplication, problem. The correct answer is 1,350.
D. Correct. 1,350 \cdot 100=135,000.

## Division by Zero

You know what it means to divide by 2 or divide by 10, but what does it mean to divide a quantity by 0? Is this even possible? Can you divide 0 by a number? Consider the two problems written below.

$\frac{0}{8}$ and $\frac{8}{0}$

We can read the first expression, "zero divided by eight" and the second expression, "eight divided by zero." Since multiplication is the inverse of division, we could rewrite these as multiplication problems.

$0\div 8=?$

$?\cdot 8=0$

The quotient must be 0 because $0\cdot 8=0$.

$\frac{0}{8}=0$

Now let's consider $\frac{8}{0}$.

$8\div 0=?$

$?\cdot 0=8$

This is not possible. There is no number that you could multiply by zero and get eight. Any number multiplied by zero is always zero. There is no quotient for $\frac{8}{0}$. There is no quotient for any number when it is divided by zero.

Division by zero is an operation for which you cannot find an answer, so it is not allowed. We say that division by 0 is undefined.

## Using Division in Problem Solving

Division is used in solving many types of problems. Below are three examples from real life that use division in their solutions.

**Example**

**Luana made 40 empanadas for a party. If the empanadas are divided equally among 12 guests, how many will each guest have? Will there be any leftover empanadas?**

**Solution**

| | |
|---|---|
| $40 \div 12$ | Since each guest will have an equal share, we can use division. |
| 3<br>12\longdiv40<br>$-36$<br>4 | Use trial and error. Try 3.<br>$12 \cdot 3 = 36$<br>When 40 empanadas are divided equally among 12 people, there are 4 left over. |

Each guest will have 3 empanadas. There will be 4 empanadas left over.

**Example**

**A case of floor tiles has 12 boxes in it. The case costs $384. How much does one box cost?**

**Solution**

| | |
|---|---|
| $384 \div 12$ | Since the boxes each cost the same amount, you want to divide $384 into 12 equal parts. |
| 12\longdiv384 | Perform the division.<br>Try to divide the first digit in the dividend by the divisor. 12 will not divide into 3, so go to the next digit. |
| 3<br>12\longdiv384<br>$-36$ | Perform $38 \div 12$.<br>Pick a quotient and test it. Try 3.<br>$3 \cdot 12 = 36$. |
| 3<br>12\longdiv384<br>$-36$<br>2 | Subtract 36 from 38. |
| 32<br>12\longdiv384<br>$-36$<br>24<br>$-24$<br>0 | Bring the next digit of the dividend down and perform division.<br>$12 \cdot 2 = 24$<br>$24 - 24 = 0$ |

| | |
|---|---|
| $\begin{array}{r} 32 \\ \times\quad 12 \\ \hline 64 \\ +\quad 320 \\ \hline 384 \end{array}$ | Does $32 \cdot 12$ equal 384?<br>Check your answer by multiplying.<br>Yes! The answer is correct! |

Each box of tiles costs $32

### ✓ Example

**A banana grower is shipping 4,644 bananas. There are 86 crates, each containing the same number of bananas. How many bananas are in each crate?**

#### Solution

| | |
|---|---|
| $4,644 \div 86$ | Since each crate, or box, has the same number of bananas, you can take the total number of bananas and divide by the number of crates.<br>Rewrite the division. |
| $86\backslash\text{longdiv}4644$ | Use trial and error to determine what $464 \div 86$ equals.<br>Try 5:<br>$\begin{array}{r} {}^{3}86 \\ \times\quad 5 \\ \hline 430 \end{array}$ |
| $\begin{array}{r} 5 \\ 86\backslash\text{longdiv}4644 \\ -430 \\ \hline 344 \end{array}$ | $464 - 430 = 34$<br>Then, bring down the next digit of the dividend, 4. |
| $\begin{array}{r} 54 \\ 86\backslash\text{longdiv}4644 \\ -430 \\ \hline 344 \\ -344 \\ \hline 0 \end{array}$ | Use trial and error to determine the quotient of 344 and 86.<br>Try 4:<br>$\begin{array}{r} {}^{2}86 \\ \times 4 \\ \hline 344 \end{array}$ |
| $\begin{array}{r} 86 \\ \times 54 \\ \hline 344 \\ +4300 \\ \hline 4644 \end{array}$ | Check your answer by multiplying.<br>Yes! The answer is correct! |

Each crate contains 54 bananas.

### ? Exercise

A theater has 1,440 seats. The theater has 30 rows of seats. How many seats are in each row?

A. 1,410
B. 48
C. 43,200
D. 480

**Answer**

A. Incorrect. This answer is too large. Use division, $1440 \div 30$, not subtraction for this problem. The correct answer is 48.
B. Correct. $1440 \div 30 = 48$.
C. Incorrect. The answer is too large. Use division, $1440 \div 30$, not multiplication, for this problem. The correct answer is 48.
D. Incorrect. There is a place-value error. The correct answer is 48.

## Summary

Division is the inverse operation of multiplication, and can be used to determine how to evenly share a quantity among a group. Division can be written in three different ways: using a fraction bar, using a division symbol, and using long division. Division can be represented as splitting a total quantity into sets of equal quantities, as skip subtracting on the number line, and as a dimension with an area model. Remainders may result when performing division and they can be represented with the letter R, followed by the number remaining. Since division is the inverse operation of multiplication, you need to know your multiplication facts in order to do division. For larger numbers, you can use long division to find the quotient.

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# 10.1.4: Properties of Whole Numbers

# 10.1.4.1: Properties and Laws of Whole Numbers

**Learning Objectives**

- Simplify by using the addition property of 0.
- Simplify by using the multiplication property of 1.
- Identify and use the commutative law of addition.
- Identify and use the commutative law of multiplication.
- Identify and use the associative law of addition.
- Identify and use the associative law of multiplication.

## Introduction

Mathematics often involves simplifying numerical expressions. When doing so, you can use laws and properties that apply to particular operations. The multiplication property of 1 states that any number multiplied by 1 equals the same number, and the addition property of zero states that any number added to zero is the same number.

Two important laws are the commutative laws, which state that the order in which you add two numbers or multiply two numbers does not affect the answer. You can remember this because if you *commute* to work you go the same distance driving to work and driving home as you do driving home and driving to work. You can move numbers around in addition and multiplication expressions because the order in these expressions does not matter.

You will also learn how to simplify addition and multiplication expressions using the associative laws. As with the commutative laws, there are associative laws for addition and multiplication. Just like people may associate with people in different groups, a number may *associate* with other numbers in one group or another. The associative laws allow you to place numbers in different groups using parentheses.

## Addition and Multiplication Properties of 0 and 1

The **addition property of 0** states that for any number being added to 0, the sum equals that number. Remember that you do not end up with zero as an answer; that only happens when you multiply. Your answer is simply the same as your original number.

**Example**

$62+0=?$

**Solution**

| | |
|---|---|
| $62+0=62$ | Adding zero to 62 does not add any quantity to the sum, so the number remains 62. |

$62+0=62$

**Exercise**

$112+0=?$

A. 112
B. 0
C. 1
D. 1,120

**Answer**

A. Correct. Adding zero to a number does not change a number.
B. Incorrect. Your answer would be zero if you multiply 112 by 0, but not if you add 112 to 0. The correct answer is 112.
C. Incorrect. Adding 112 to 0 does not equal 1. The correct answer is 112.

D. Incorrect. You do not put zero in the ones place and move other digits up by one place value. This only occurs if you are multiplying a number by ten. The correct answer is 112.

According to the **multiplication property of 1**, the product of 1 and any number results in that number. The answer is simply identical to the original number.

**Example**

$2,500 \cdot 1 = ?$

**Solution**

| | |
|---|---|
| $2,500 \cdot 1 = 2,500$ | Multiplying 2,500 by 1 yields the same number. |

$2,500 \cdot 1 = 2,500$

**Exercise**

$72,540 \cdot 1 = ?$

A. 725,401
B. 72,541
C. 72,540
D. 72,539

**Answer**

A. Incorrect. You do not put 1 in the ones place and move other digits up by one place value. The correct answer is 72,540.
B. Incorrect. You do not add one to the number. The correct answer is 72,540.
C. Correct. Multiplying any number by 1 yields the same number, which is in this case 72,540.
D. Incorrect. You do not subtract one from the number. The correct answer is 72,540.

## The Commutative Law of Addition

The **commutative law of addition** states that you can change the position of numbers in an addition expression without changing the sum. For example, 3+2 is the same as 2+3.

3+2=5

2+3=5

You likely encounter daily routines in which the order can be switched. For example, when you get ready for work in the morning, putting on your left glove and right glove is commutative. You could put the right glove on before the left glove, or the left glove on before the right glove. Likewise, brushing your teeth and combing your hair is commutative, because it does not matter which one you do first.

Remember that this law only applies to addition, and not subtraction. For example:

8-2 is not the same as 2-8.

Below, you will find examples of expressions that have been changed with the commutative law. Note that expressions involving subtraction cannot be changed.

| Original Expression | Rewritten Expression |
|---|---|
| 4+5 | 5+4 |
| 6+728 | 728+6 |
| 9+4+1 | 9+1+4 |

|  |  |
|---|---|
| 9-1 | cannot be changed |
| 72-10 | cannot be changed |
| 128-100 | cannot be changed |

You also will likely encounter real life routines that are not commutative. When preparing to go to work, putting on our clothes has to occur before putting on a coat. Likewise, getting in the car has to occur before putting the key in the ignition. In a store, you would need to pick up the items you are buying before proceeding to the cash register for checkout.

✓ Example

**Write the expression 10+25 in a different way, using the commutative law of addition, and show that both expressions result in the same answer.**

**Solution**

|  |  |
|---|---|
| 10+25=35 | Solving the problem yields an answer of 35. |
| 25+10 | Using the commutative property, you can switch the 10 and the 25 so that they are in different positions. |
| 25+10=35 | Adding 25 to 10 in this new order also yields 35. |

10+25=35 and 25+10=35

? Exercise

Rewrite 15+12=27 in a different way, using the commutative law of addition.

A. 15=12+27
B. 12=15+27
C. 15+(12=27)
D. 12+15=27

**Answer**

A. Incorrect. Only the order of the numbers to be added can change. The correct answer is 12+15=27.
B. Incorrect. Only the order of the numbers to be added can change. The correct answer is 12+15=27.
C. Incorrect. You would never put parentheses around an = sign (and parentheses are not used with the commutative laws). The correct answer is 12+15=27.
D. Correct. The commutative law lets you change the order of the numbers being added.

## The Commutative Law of Multiplication

Multiplication also has a commutative law. The **commutative law of multiplication** states that when two or more numbers are being multiplied, their order can be changed without affecting the answer. In the example below, note that 5 multiplied by 4 yields the same result as 4 multiplied by 5. In both cases, the answer is 20.

$5 \cdot 4 = 20$

$4 \cdot 5 = 20$

This example shows how numbers can be switched in a multiplication expression.

✓ Example

**Write the expression $30 \cdot 50$ in a different way, using the commutative law of multiplication, and show that both expressions result in the same answer.**

**Solution**

| | |
|---|---|
| $30 \cdot 50=1,500$ | Solving the problem yields an answer of 1,500. |
| $50 \cdot 30$ | Using the commutative law, you can switch the 30 and the 50 so that they are in different positions. |
| $50 \cdot 30=1,500$ | Multiplying 50 and 30 also yields 1,500. |

$50 \cdot 30$ and $30 \cdot 50=1,500$

Keep in mind that when you are using the commutative law, only the order is affected. The grouping remains unchanged.

**? Exercise**

Rewrite $52 \cdot 46$ in a different way, using the commutative law of multiplication.

A. $42 \cdot 56$
B. $5 \cdot 246$
C. $5 \cdot 24 \cdot 6$
D. $46 \cdot 52$

**Answer**

A. Incorrect. You cannot switch one digit from each number. The correct answer is $46 \cdot 52$.
B. Incorrect. The same two numbers should be multiplied. The numbers themselves should not be changed, only the order in which the numbers appear. The correct answer is $46 \cdot 52$.
C. Incorrect. The same two numbers should be multiplied. The numbers themselves should not be changed, only the order in which the numbers appear. The correct answer is $46 \cdot 52$.
D. Correct. The order of numbers is reversed, and the same two numbers are multiplied.

## The Associative Law of Addition

Below are two ways of simplifying and solving an addition problem. Note that you can add numbers in any order. In the first example, 4 is added to 5 to make 9.

4+5+6=9+6=15

Here, the same problem is solved, but this time, 5 is added to 6 to make 11. Note that solving it this way yields the same answer.

4+5+6=4+11=15

The **associative law of addition** states that numbers in an addition expression can be regrouped using parentheses. You can remember the meaning of the associative law by remembering that when you *associate* with family members, friends, and co-workers, you end up forming groups with them. In the following expression, parentheses are used to group numbers together so that you know what to add first. Note that when parentheses are present, any numbers within parentheses are numbers you will add first. The expression can be re-written with different groups using the associative law.

(4+5)+6=9+6=15

4+(5+6)=4+11=15

Here, it is clear that the parentheses do not affect the final answer. The answer is the same regardless of where the parentheses are.

**✓ Example**

**Rewrite (5+8)+3 using the associative law of addition. Show that the rewritten expression yields the same answer.**

**Solution**

| | |
|---|---|
| (5+8)+3=13+3=16 | The original expression yields an answer of 16. |
| 5+(8+3)=5+11=16 | Grouping 8 and 3 instead of 5 and 8 results in the same answer of 16. |

(5+8)+3=16 and 5+(8+3)=16

When rewriting an expression using the associative law, remember that you are regrouping the numbers and not reversing the order, as in the commutative law.

? Exercise

Rewrite 10+(5+6) using the associative property.

A. (5+6)+10
B. 10+(6+5)
C. (10+5)+6
D. (10+6)+5

**Answer**

A. Incorrect. The *order* of numbers is not changed when you are rewriting the expression using the associative law of addition. How the numbers are *grouped* should change. The correct answer is (10+5)+6.
B. Incorrect. The order of numbers is not changed when you are rewriting the expression using the associative law of addition. How the numbers are *grouped* should change. The correct answer is (10+5)+6.
C. Correct. Here, the numbers are regrouped. Now 10 and 5 are grouped in parentheses instead of 5 and 6.
D. Incorrect. The order of numbers is not changed when you are rewriting the expression using the associative law of addition. Only how the numbers are *grouped* should change. The correct answer is (10+5)+6.

## The Associative Law of Multiplication

Multiplication has an associative law that works exactly the same as the one for addition. The **associative law of multiplication** states that numbers in a multiplication expression can be regrouped using parentheses. The following expression can be rewritten in a different way using the associative law.

$$(2\cdot 3)\cdot 4=2\cdot(3\cdot 4)$$

Here, it is clear that the parentheses do not affect the final answer. The answer is the same regardless of where the parentheses are.

✓ Example

**Rewrite $(10\cdot 200)\cdot 24$ using the associative law of multiplication, and show that the rewritten expression yields the same answer.**

**Solution**

| | |
|---|---|
| $(10\cdot 200)\cdot 24=2000\cdot 24=48,000$ | The original expression yields an answer of 48,000. |
| $10\cdot(200\cdot 24)=10\cdot 4800=48,000$ | Grouping 200 and 24 instead of 10 and 200 results in the same answer of 48,000. |

$$(10\cdot 200)\cdot 24=48,000 \text{ and } 10\cdot(200\cdot 24)=48,000$$

When rewriting an expression using the associative law, remember that you are regrouping the numbers and not changing the order. Changing the order uses the commutative law.

? Exercise

Rewrite $8\cdot(7\cdot 6)$ using the associative property.

A. $(8\cdot 7)\cdot 6$
B. $(7\cdot 6)\cdot 8$
C. $(7\cdot 8)\cdot 6$
D. $(8\cdot 76)$

**Answer**

A. Correct. Here, the numbers are regrouped. Now 8 and 7 are grouped in parentheses instead of 7 and 6.
B. Incorrect. The order of numbers is not changed when you are rewriting the expression using the associative law of multiplication. How they are *grouped* should change. The correct answer is $(8\cdot7)\cdot6$.
C. Incorrect. The order of numbers is not changed when you are rewriting the expression using the associative law of multiplication. Only how they are *grouped* should change. The correct answer is $(8\cdot7)\cdot6$.
D. Incorrect. The digits of different numbers are not combined to create new numbers. How the numbers are *grouped* should change. The correct answer is $(8\cdot7)\cdot6$.

## Commutative or Associative?

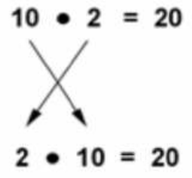
When an expression is being rewritten, you can tell whether it is being rewritten using the commutative or associative laws based on whether the order of the numbers change or the numbers are being regrouped using parentheses.

If an expression is rewritten so that the order of the numbers is changed, the commutative law is being used.

### ✓ Example

**$10\cdot2=20$ is rewritten as $2\cdot10=20$. Was this expression rewritten using the commutative law or the associative law?**

**Solution**



Rewriting the expression involves switching the order of the numbers. Therefore, the commutative law is being used.

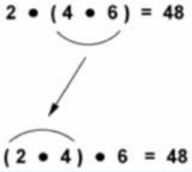
The commutative law is being used to rewrite the expression.

Remember that when you associate with friends and family, typically you are *grouping* yourself with other people. So, if numbers in an expression are regrouped using parentheses and the order of numbers remains the same, then the associative law is being used.

### ✓ Example

**$2\cdot(4\cdot6)=48$ is rewritten as $(2\cdot4)\cdot6=48$. Was this expression rewritten using the commutative law or the associative law?**

**Solution**



Regrouping using parentheses does not change the order of the numbers. Therefore, the associative law is being used.

The associative law is being used to rewrite the expression.

### ? Exercise

$12\cdot(6\cdot2)=144$ is rewritten as $3\cdot17=51$. Was this expression rewritten using the commutative law or associative law?

A. commutative law
B. associative law

**Answer**

A. Correct. The order of numbers is being switched, which shows that the commutative law is being used.
B. Incorrect. The associative law involves regrouping numbers using parentheses, which does not occur in this problem. Rather, the order of numbers is switched. The correct answer is commutative law.

## Using the Associative and Commutative Laws

The associative and commutative laws are useful when you have an expression with only addition. Using the commutative law, the numbers can be reordered so that the numbers that are easiest to add are next to each other, and using the associative law, you can group them in any way.

For example, here are some of the ways we can add 6+5+4 using the associative and commutative laws. Note that the answer is always the same.

(6+5)+4=11+4=15 (grouping 6 and 5 to add first)

(5+6)+4=11+4=15 (reordering 6 and 5)

5+(6+4)=5+10=15 (grouping 6 and 4 to add first)

6+(5+4)=6+9=15 (grouping 5 and 4 to add first)

6+(4+5)=6+9=15 (reordering 4 and 5)

(6+4)+5=10+5=15 (grouping 6 and 4 to add first)

✓ Example

**Write the expression 13+28+7 a different way to make it easier to simplify. Then simplify.**

### Solution

| | |
|---|---|
| 13+28+7<br>13+7+28 | Using the commutative property, reorder the numbers 7 and 28 since 13+7 is easier to add than 13+28. |
| 20+28 | Using the associative property, group the 13 and 7 together and add them first. |
| 48 | Add 20 and 28. |

13+28+7=13+7+28=48

Sometimes the commutative and associative laws can make the problem easy enough to do in your head.

✓ Example

**Jim is buying 8 pears, 7 apples, and 2 oranges. He decided the total number of fruits is 8+7+2. Use the commutative property to write this expression in a different way. Then find the total.**

### Solution

| | |
|---|---|
| 8+7+2<br>8+2+7 | Using the commutative property, reorder 2 and 7. |
| 10+7 | Using the associative property, group the 8 and 2 together and add them first. |
| 17 | Add 10 and 7. |

8+7+2=8+2+7=17

This also works when you are multiplying more than two numbers. You can use the commutative and associative laws freely if the expression involves only multiplication.

✓ Example

**There are 2 trucks in a garage, and each truck holds 60 boxes. There are 5 laptop computers in each box. Find the number of computers in the garage.**

**Solution**

| | |
|---|---|
| $2 \cdot 60 \cdot 5$ | In order to find the answer, you need to multiply the number of trucks times the number of boxes in each truck, and, then by the number of computers in each box. |
| $2 \cdot 5 \cdot 60$ | Using the commutative property, reorder the 5 and the 60. Now you can multiply $2 \cdot 5$ first. |
| $10 \cdot 60$ | Using the associative property, multiply the 2 and the 5, 2 \cdot 5=10. |
| 600 | Now it's easier to multiply 10 and 60 to get 600. |

There are 600 computers in the garage.

## Summary

The addition property of 0 states that for any number being added to zero, the sum is the same number. The multiplication property of 1 states that for any number multiplied by one, that answer is that same number. Zero is called the additive identity, and one is called the multiplicative identity.

When you rewrite an expression by a commutative law, you change the order of the numbers being added or multiplied. When you rewrite an expression using an associative law, you group a different pair of numbers together using parentheses.

You can use the commutative and associative laws to regroup and reorder any number in an expression that involves only addition. You can also use the commutative and associative laws to regroup and reorder any number in an expression that involves only multiplication.

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# 10.1.4.2: The Distributive Property

**Learning Objectives**

- Simplify using the distributive property of multiplication over addition.
- Simplify using the distributive property of multiplication over subtraction.

## Introduction

The distributive property of multiplication is a very useful property that lets you simplify expressions in which you are multiplying a number by a sum or difference. The property states that the product of a sum or difference, such as 6(5-2), is equal to the sum or difference of the products; in this case, 6(5)-6(2).

Remember that there are several ways to write multiplication. $3\times 6=3(6)=3\cdot 6$

$3\cdot(2+4)=3\cdot 6=18$

## Distributive Property of Multiplication over Addition

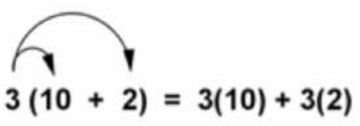
The **distributive property of multiplication over addition** can be used when you multiply a number by a sum. For example, suppose you want to multiply 3 by the sum of 10+2.

3(10+2)=?

According to this property, you can add the numbers and then multiply by 3.

3(10+2)=3(12)=36

Or, you can first multiply each addend by the 3. (This is called **distributing** the 3.) Then, you can add the products.



The multiplication of 3(10) and 3(2) will each be done before you add.

3(10)+3(2)=30+6=36

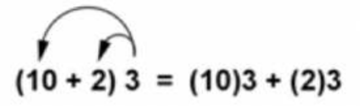
Note that the answer is the same as before.

You probably use this property without knowing that you are using it. When a group (let’s say 5 of you) orders food, and orders the same thing (let’s say you each order a hamburger for $3 each and a soda for $1 each), you can compute the bill (without tax) in two ways. You can figure out how much each of you needs to pay and multiply the sum times the number of you. So, you each pay (3+1) and then multiply times 5. That’s 5(3+1)=5(4)=20. Or, you can figure out how much the 5 hamburgers will cost and the 5 sodas and then find the total. That’s 5(3)+5(1)=15+5=20. Either way, the answer is the same, $20.

The two methods are represented by the equations below. On the left side, we add 10 and 2, and then multiply by 3. The expression is rewritten using the distributive property on the right side, where we distribute the 3, then multiply each by 3 and add the results. Notice that the result is the same in each case.

$$\begin{aligned} 3(10+2) &= 3(10)+3(2) \\ 3(12) &= 30+6 \\ 36 &= 36 \end{aligned}$$

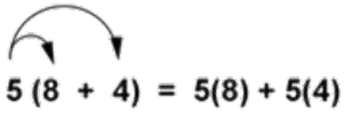
The same process works if the 3 is on the other side of the parentheses, as in the example below.



**✓ Example**

**Rewrite the expression 5(8+4) using the distributive property of multiplication over addition. Then simplify the result.**

**Solution**

| | |
|---|---|
|  | In the original expression, the 8 and the 4 are grouped in parentheses. Using arrows, you can see how the 5 is distributed to each addend. The 8 and 4 are each multiplied by 5. |
| 40+20=60 | The resulting products are added together, resulting in a sum of 60. |

5(8+4)=5(8)+5(4)=60

**? Exercise**

Rewrite the expression 30(2+4) using the distributive property of addition.

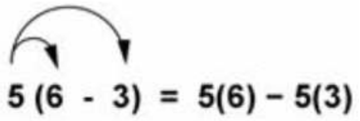
A. 30(2+4)+30(2+4)
B. 30(2)+30(4)
C. 30(6)
D. 30(24)

**Answer**

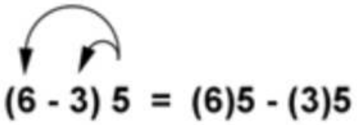
A. Incorrect. This would be doubling your original value. To distribute the 30, multiply the 2 by 30 and the 4 by 30. The correct answer is 30(2)+30(4).
B. Correct. The number 30 is distributed to both the 2 and the 4, so that both 2 and 4 are multiplied by 30.
C. Incorrect. The number 30 is not distributed in this answer. To distribute the 30, multiply the 2 by 30 and the 4 by 30. The correct answer is 30(2)+30(4).
D. Incorrect. The digits 2 and 4 should not be combined to form 24 because the addition process is incorrect. The number 30 is not distributed in this answer. To distribute the 30, multiply the 2 by 30 and the 4 by 30. The correct answer is 30(2)+30(4).

## Distributive Property of Multiplication over Subtraction

The **distributive property of multiplication over subtraction** is like the distributive property of multiplication over addition. You can subtract the numbers and then multiply, or you can multiply and then subtract as shown below. This is called “distributing the multiplier.”



The same number works if the 5 is on the other side of the parentheses, as in the example below.



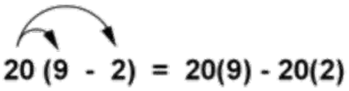
In both cases, you can then simplify the distributed expression to arrive at your answer. The example below, in which 5 is the outside multiplier, demonstrates that this is true. The expression on the right, which is simplified using the distributive property, is shown to be equal to 15, which is the resulting value on the left as well.

$$\begin{aligned} 5(6-3) &= 5(6)-5(3) \\ 5(3) &= 30-15 \\ 15 &= 15 \end{aligned}$$

### ✓ Example

**Rewrite the expression 20(9-2) using the distributive property of multiplication over subtraction. Then simplify.**

**Solution**

| | |
|---|---|
|  20 (9 - 2) = 20(9) - 20(2) | In the original expression, the 9 and the 2 are grouped in parentheses. Using arrows, you can see how the 20 is distributed to each number so that the 9 and 2 are both multiplied by 20 individually. |
| 180-40=140 | Here, the resulting product of 40 is subtracted from the product of 180, resulting in an answer of 140. |

20(9-2)=20(9)-20(2)=140

### ? Exercise

Rewrite the expression 10(15-6) using the distributive property of subtraction.

A. 10(6)-10(15)
B. 10(9)
C. 10(6-15)
D. 10(15)-10(6)

**Answer**

A. Incorrect. Here, a greater number would be subtracted from a lesser number, and the answer would not be a whole number. The correct answer is 10(15)-10(6).
B. Incorrect. The numbers in parentheses were subtracted before the number 10 could be distributed. The correct answer is 10(15)-10(6).
C. Incorrect. You probably used the commutative law instead of the distributive property. The correct answer is 10(15)-10(6).
D. Correct. The 10 is correctly distributed so that it is used to multiply the 15 and the 6 separately.

## Summary

The distributive properties of addition and subtraction can be used to rewrite expressions for a variety of purposes. When you are multiplying a number by a sum, you can add and then multiply. You can also multiply each addend first and then add the products. This can be done with subtraction as well, multiplying each number in the difference before subtracting. In each case, you are distributing the outside multiplier to each number in the parentheses, so that multiplication occurs with each number before addition or subtraction occurs. The distributive property will be useful in future math courses, so understanding it now will help you build a solid math foundation.

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# 10.1.5: Exponents, Square Roots, and the Order of Operations

# 10.1.5.1: Understanding Exponents and Square Roots

**Learning Objectives**

- Evaluate expressions containing exponents.
- Write repeated factors using exponential notation.
- Find a square root of a perfect square.

## Introduction

**Exponents** provide a special way of writing repeated multiplication. Numbers written in this way have a specific form, with each part providing important information about the number. Writing numbers using exponents can save a lot of space, too. The inverse operation of multiplication of a number by itself is called finding the square root of a number. This **operation** is helpful for problems about the area of a square.

## Understanding Exponential Notation

**Exponential notation** is a special way of writing repeated factors, for example $7 \cdot 7$. Exponential notation has two parts. One part of the notation is called the **base**. The base is the number that is being multiplied by itself. The other part of the notation is the **exponent**, or power. This is the small number written up high to the right of the base. The exponent, or power, tells how many times to use the base as a **factor** in the multiplication. In the example, $7 \cdot 7$ can be written as $7^2$ (7 is the base and 2 is the exponent). The exponent 2 means there are two factors.

$$7^2 = 7 \cdot 7 = 49$$

You can read $7^2$ as “seven squared.” This is because multiplying a number by itself is called “**squaring** a number.” Similarly, raising a number to a power of 3 is called “**cubing** the number.” You can read $7^3$ as “seven cubed.”

You can read $2^5$ as “two to the fifth power” or “two to the power of five,” or “two raised to the power of five.” Read $8^4$ as “eight to the fourth power,” or “eight to the power of four,” or “eight raised to the power of four.” This format can be used to read *any* number written in exponential notation. In fact, while $6^3$ is most commonly read “six cubed,” it can also be read “six to the third power,” or “six to the power of three,” or "six raised to the power of three.”

To find the value of a number written in exponential form, rewrite the number as repeated multiplication and perform the multiplication. Two examples are shown below.

**Example**

**Find the value of $4^2$.**

**Solution**

| | |
|---|---|
| 4 is the base.<br>2 is the exponent. | An exponent means repeated multiplication.<br>The base is 4; 4 is the number being multiplied.<br>The exponent is 2; This means to use two factors of 4 in the multiplication. |
| $4^2 = 4 \cdot 4$ | Rewrite as repeated multiplication. |
| $4 \cdot 4 = 16$ | Multiply. |

$$4^2 = 16$$

**Example**

**Find the value of $2^5$.**

**Solution**

| | |
|---|---|
| $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ | Rewrite $2^5$ as repeated multiplication.<br>The base is 2, the number being multiplied.<br>The exponent is 5, the number of times to use 2 in the multiplication. |
| $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$<br>$4 \cdot 2 \cdot 2 \cdot 2$<br>$8 \cdot 2 \cdot 2$<br>$16 \cdot 2$<br>$32$ | Perform multiplication. |

$2^5 = 32$

## Exercise

Find the value of $4^3$.

A. 12
B. 64
C. 256
D. 43

**Answer**

A. Incorrect. Multiply $4 \cdot 4 \cdot 4$, not $4 \cdot 3$. The correct answer is 64.
B. Correct. 4 $\cdot 4 \cdot 4 = 64$
C. Incorrect. Use three factors of 4 in the multiplication. The correct answer is 64.
D. Incorrect. $4^3$ means $4 \cdot 4 \cdot 4$. The correct answer is 64.

## Writing Repeated Multiplication Using Exponents

Writing repeated multiplication in exponential notation can save time and space. Consider the example $5 \cdot 5 \cdot 5 \cdot 5$. We can use exponential notation to write this repeated multiplication as $5^4$. Since 5 is being multiplied, it is written as the base. Since the base is used 4 times in the multiplication, the exponent is 4. The expression $5 \cdot 5 \cdot 5 \cdot 5$ can be rewritten in shorthand exponential notation as $5^4$ and is read, "five to the fourth power" or "five to the power of 4."

To write repeated multiplication of the same number in exponential notation, first write the number being multiplied as the base. Then count how many times that number is used in the multiplication, and write that number as the exponent. Be sure to count the numbers, *not* the multiplication signs, to determine the exponent.

## Example

**Write $7 \cdot 7 \cdot 7$ in exponential notation.**

### Solution

| | |
|---|---|
| 7 is the base. | The base is the number being multiplied, 7. |
| Since 7 is used 3 times, 3 is the exponent. | The exponent tells the number of times the base is multiplied. |

$7 \cdot 7 \cdot 7 = 7^3$ This is read "seven cubed."

## Exercise

Write $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$. in exponential notation.

A. 1,000,000
B. 60

C. $10^5$
D. $10^6$

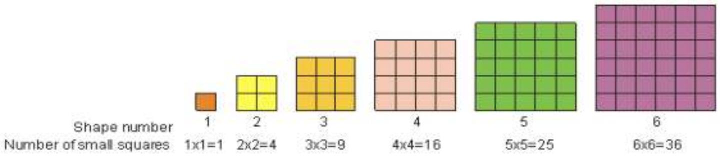
**Answer**

A. Incorrect. 1,000,000 is equivalent to $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$, but it is not written in exponential notation. The correct answer is $10^6$.

B. Incorrect. This answer is way less than the correct value and is not in exponential notation. You may have thought that six 10s should be written $6 \cdot 10$ and then simplified that to 60. The correct answer is $10^6$ because the six 10s are multiplied together.

## Understanding and Computing Square Roots

As you saw earlier, $5^2$ is called “five squared.” “Five squared” means to multiply five by itself. In mathematics, we call multiplying a number by itself “squaring” the number. We call the result of squaring a whole number a square or a **perfect square**. A perfect square is any number that can be written as a whole number **raised to the power** of 2. For example, 9 is a perfect square because $3^2$ is 9. A perfect square number can be represented as a square shape, as shown below. We see that 1, 4, 9, 16, 25, and 36 are examples of perfect squares.



To square a number, multiply the number by itself. 3 squared $= 3^2 = 3 \cdot 3 = 9$ .

Below are some more examples of perfect squares.

| | | | |
|---|---|---|---|
| 1 squared | $1^2$ | $1 \cdot 1$ | 1 |
| 2 squared | $2^2$ | $2 \cdot 2$ | 4 |
| 3 squared | $3^2$ | $3 \cdot 3$ | 9 |
| 4 squared | $4^2$ | $4 \cdot 4$ | 16 |
| 5 squared | $5^2$ | $5 \cdot 5$ | 25 |
| 6 squared | $6^2$ | $6 \cdot 6$ | 36 |
| 7 squared | $7^2$ | $7 \cdot 7$ | 49 |
| 8 squared | $8^2$ | $8 \cdot 8$ | 64 |
| 9 squared | $9^2$ | $9 \cdot 9$ | 81 |
| 10 squared | $10^2$ | $10 \cdot 10$ | 100 |

The **inverse operation** of squaring a number is called finding the **square root** of a number. Finding a square root is like asking, “what number multiplied by itself will give me this number?” The square root of 25 is 5, because 5 multiplied by itself is equal to 25. Square roots are written with the mathematical symbol, called a **radical sign**, which looks like this: $\sqrt{}$. The “square root of 25" is written $\sqrt{25}$.

**✓ Example**

**Find $\sqrt{81}$.**

**Solution**

| | |
|---|---|
| $\sqrt{81} = 9$ | Think, what number times itself gives 81?<br>$9 \cdot 9 = 81$ |

$\sqrt{81}=9$

## Exercise

Find $\sqrt{36}$.

A. 6
B. 18
C. 72
D. 7

**Answer**

A. Correct. Since $6 \cdot 6=36$, $\sqrt{36}=6$.
B. Incorrect. The square root of 36 is the number that you can multiply by itself to get 36. The square root is not the number you multiply by 2 to get 36. The correct answer is 6, because $6 \cdot 6=36$.
C. Incorrect. You may have incorrectly added 36 to itself to get 72. The square root of 36 is the number that you can multiply by itself to get 36. The correct answer is 6, because $6 \cdot 6=36$.
D. Incorrect. $7 \cdot 7=49$, so $\sqrt{49}=7$. The correct answer is 6, because $6 \cdot 6=36$. This means $\sqrt{36}=6$.

## Summary

Exponential notation is a shorthand way of writing repeated multiplication of the same number. A number written in exponential notation has a base and an exponent, and each of these parts provides information for finding the value of the expression. The base tells what number is being repeatedly multiplied, and the exponent tells how many times the base is used in the multiplication. Exponents 2 and 3 have special names. Raising a base to a power of 2 is called "squaring" a number. Raising a base to a power of 3 is called "cubing" a number. The inverse of squaring a number is finding the square root of a number. To find the square root of a number, ask yourself, "What number can I multiply by itself to get this number?"

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# 10.1.5.2: Order of Operations

**Learning Objectives**

- Use the order of operations to simplify expressions, including those with parentheses.
- Use the order of operations to simplify expressions containing exponents and square roots.

## Introduction

People need a common set of rules for performing computation. Many years ago, mathematicians developed a standard **order of operations** that tells you which calculations to make first in an expression with more than one **operation**. Without a standard procedure for making calculations, two people could get two different answers to the same problem. For example, $3+5\cdot 2$ has only one correct answer. Is it 13 or 16?

## The Order of Addition, Subtraction, Multiplication, and Division Operations

First, consider expressions that include one or more of the arithmetic operations: addition, subtraction, multiplication, and division. The order of operations requires that all multiplication and division be performed first, going from left to right in the **expression**. The order in which you compute multiplication and division is determined by which one comes first, reading from left to right.

After multiplication and division has been completed, add or subtract in order from left to right. The order of addition and subtraction is also determined by which one comes first when reading from left to right.

Below are three examples showing the proper order of operations for expressions with addition, subtraction, multiplication, and/or division.

**Example**

**Simplify $3+5\cdot 2$ .**

**Solution**

| | |
|---|---|
| $3+5\cdot 2$ | Order of operations tells you to perform multiplication before addition. |
| 3+10 | Then add. |

$3+5\cdot 2=13$

**Example**

**Simplify $20-16\div 4$ .**

**Solution**

| | |
|---|---|
| $20-16\div 4$ | Order of operations tells you to perform division before subtraction. |
| 20-4<br>16 | Then subtract. |

$20-16\div 4=16$

**Example**

**Simplify $60-30\div 3\cdot 5+7$ .**

**Solution**

| | |
|---|---|
| $60-30\div3\cdot5+7$ | Order of operations tells you to perform multiplication and division first, working from left to right, before doing addition and subtraction. |
| $60-10\cdot5+7$<br>$60-50+7$ | Continue to perform multiplication and division from left to right. |
| 10+7<br>17 | Next, add and subtract from left to right. (Note that addition is not necessarily performed before subtraction.) |

$60-30\div3\cdot5+7=17$

## Grouping Symbols and the Order of Operations

**Grouping symbols** such as parentheses ( ), brackets [ ], braces { }, and fraction bars can be used to further control the order of the four basic arithmetic operations. The rules of the order of operations require computation within grouping symbols to be completed first, even if you are adding or subtracting within the grouping symbols and you have multiplication outside the grouping symbols. After computing within the grouping symbols, divide or multiply from left to right and then subtract or add from left to right.

✓ Example

**Simplify** $\mathbf{900\div(6+3\cdot8)-10}$ .

**Solution**

| | |
|---|---|
| $900\div((6+3\cdot8))-10$ | Order of operations tells you to perform what is inside the parentheses first. |
| $900\div(6+(3\cdot8))-10$<br>$900\div(6+24)-10$ | Simplify the expression in the parentheses. Multiply first. |
| $900\div30-10$ | Then add 6+24. |
| $900\div30-10$<br>$30-10$<br>$20$ | Now perform division; then subtract. |

$900\div(6+3\cdot8)-10=20$

When there are grouping symbols within grouping symbols, compute from the inside to the outside. That is, begin simplifying the innermost grouping symbols first. Two examples are shown.

✓ Example

**Simplify** $4-3[20-3\cdot4-(2+4)]\div2$ .

**Solution**

| | |
|---|---|
| $4-3\div2$ | There are brackets and parentheses in this problem. Compute inside the innermost grouping symbols first. |
| $4-3[20-3\cdot4-((2+4)]]\div2$<br>$4-3\div2$ | Simplify within parentheses. |
| $4-3\div2$<br>$4-3\div2$<br>$4-3\div2$ | Then, simplify within the brackets by multiplying and then subtracting from left to right. |
| $4-3(2)\div2$ | |

| | |
|---|---|
| $4-3((2)\div 2$<br>$4-6\div 2$<br>$4-3$ | Multiply and divide from left to right. |
| $4-3$<br>$1$ | Subtract. |

$4-3[20-3\cdot 4-(2+4)]\div 2=1$

Remember that parentheses can also be used to show multiplication. In the example that follows, the parentheses are not a grouping symbol; they are a multiplication symbol. In this case, since the problem only has multiplication and division, we compute from left to right. Be careful to determine what parentheses mean in any given problem. Are they a grouping symbol or a multiplication sign?

### ✓ Example

**Simplify $6\div(3)(2)$.**

**Solution**

| | |
|---|---|
| $6\div 3\cdot 2$ | This expression has multiplication and division only. The multiplication operation can be shown with a dot. |
| $6\div 3\cdot 2$<br>$2\cdot 2$<br>$4$ | Since this expression has only division and multiplication, compute from left to right. |

$6\div(3)(2)=4$

Consider what happens if braces are added to the problem above: $6\div\{(3)(2)\}$. The parentheses still mean multiplication; the additional braces are a grouping symbol. According to the order of operations, compute what is inside the braces first. This problem is now evaluated as $6\div 6=1$ . Notice that the braces caused the answer to change from 1 to 4.

### ? Exercise

Simplify $40-(4+6)\div 2+3$ .

A. 18
B. 38
C. 24
D. 32

**Answer**

A. Incorrect. Compute the addition in parentheses first. $40-10\div 2+3$ . Then, perform division. $40-5+3$ . Finally, add and subtract from left to right. The correct answer is 38.
B. Correct. Compute the addition in parentheses first. $40-10\div 2+3$ . Then, perform division. $40-5+3$ . Finally, add and subtract from left to right.
C. Incorrect. Compute the addition in parentheses first. $40-10\div 2+3$ . Then, perform division. $40-5+3$ . Finally, add and subtract from left to right. The correct answer is 38.
D. Incorrect. Compute the addition in parentheses first. $40-10\div 2+3$ . Then, perform division. $40-5+3$ . Finally, with only subtraction and addition left, add and subtract from left to right. The correct answer is 38.

The Order of Operations

1. Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), braces { }, brackets [ ], and fraction bars.
2. Multiply and Divide, from left to right.
3. Add and Subtract, from left to right.

## Performing the Order of Operations with Exponents and Square Roots

So far, our rules allow us to simplify expressions that have multiplication, division, addition, subtraction or grouping symbols in them. What happens if a problem has **exponents** or **square roots** in it? We need to expand our order of operation rules to include exponents and square roots.

If the expression has exponents or square roots, they are to be performed after parentheses and other grouping symbols have been simplified and *before* any multiplication, division, subtraction and addition that are outside the parentheses or other grouping symbols.

Note that you compute from more complex operations to more basic operations. Addition and subtraction are the most basic of the operations. You probably learned these first. Multiplication and division, often thought of as repeated addition and subtraction, are more complex and come before addition and subtraction in the order of operations. Exponents and square roots are repeated multiplication and division, and because they're even more complex, they are performed before multiplication and division. Some examples that show the order of operations involving exponents and square roots are shown below.

✓ Example

**Simplify $14+28\div 2^2$ .**

**Solution**

| | |
|---|---|
| $14+28\div 2^2$ | This problem has addition, division, and exponents in it. Use the order of operations. |
| $14+28\div 4$ | Simplify $2^2$ |
| $14+7$ | Perform division before addition. |
| $21$ | Add. |

$14+28\div 2^2=21$

✓ Example

**Simplify $3^2\cdot 2^3$ .**

**Solution**

| | |
|---|---|
| $3^2\cdot 2^3$ | This problem has exponents and multiplication in it. |
| $9\cdot 8$ | Simplify $3^2$ and $2^3$. |
| $72$ | Perform multiplication. |

$3^2\cdot 2^3=72$

✓ Example

**Simplify $(3+4)^2+(8)(4)$ .**

**Solution**

| | |
|---|---|
| $(3+4)^2+(8)(4)$ | This problem has parentheses, exponents, and multiplication in it. The first set of parentheses is a grouping symbol. The second set indicates multiplication.<br>Grouping symbols are handled first. |
| $7^2+(8)(4)$<br>$49+(8)(4)$ | Add the numbers inside the parentheses that are serving as grouping symbols. Simplify $7^2$. |
| $49+32$ | Perform multiplication. |
| $81$ | Add. |

$(3+4)^2+(8)(4)=81$

## Exercise

Simplify $77-(1+4-2)^2$ .

A. 68
B. 28
C. 71
D. 156

**Answer**

A. Correct. $77-(1+4-2)^2=77-(3)^2=77-9=68$
B. Incorrect. Simplify the expression in parentheses first. $77-(1+4-2)^2=77-(3)^2=77-9=68$
C. Incorrect. The exponent of 2 tells you to multiply the number by itself, not by 2; $77-(3)^2=77-9$ , not $77-6$. The correct answer is $68$.
D. Incorrect. Parentheses are a grouping symbol, and numbers inside them should be computed first. The exponent of 2 tells you to multiply the number by itself, not by $2.77-(1+4-2)^2=77-(3)^2=77-9=68$ . The correct answer is 68.

## The Order of Operations

1. Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), braces { }, brackets [ ], and fraction bars.
2. Evaluate exponents and roots of numbers, such as square roots.
3. Multiply and Divide, from left to right.
4. Add and Subtract, from left to right.

Some people use a saying to help them remember the order of operations. This saying is called PEMDAS or “Please Excuse My Dear Aunt Sally.” The first letter of each word begins with the same letter of an arithmetic operation.

The P in Please stands for Parentheses (and other grouping symbols).

The E in Excuse stands for Exponents.

The M and D in My Dear stand for Multiplication and Division (from left to right).

The A and S in Aunt Sally stand for Addition and Subtraction (from left to right).

Note: Even though multiplication comes before division in the saying, division could be performed first. Which is performed first, between multiplication and division, is determined by which comes first when reading from left to right. The same is true of addition and subtraction. Don’t let the saying confuse you about this!

## Summary

The order of operations gives us a consistent sequence to use in computation. Without the order of operations, you could come up with different answers to the same computation problem. (Some of the early calculators, and some inexpensive ones, do NOT use the order of operations. In order to use these calculators, the user has to input the numbers in the correct order.)

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# SECTION OVERVIEW

## 10.2: Fractions and Mixed Numbers

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# 10.2.1: Introduction to Fractions and Mixed Numbers

# 10.2.1.1: Introduction to Fractions and Mixed Numbers

**Learning Objectives**

- Identify the numerator and denominator of a fraction.
- Represent a fraction as part of a whole or part of a set.

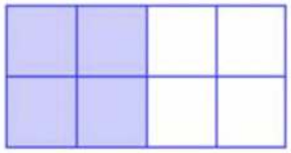
## Introduction

Many problems in mathematics deal with whole numbers, which are used to count whole units of things. For example, you can count students in a classroom and the number of dollar bills. You need other kinds of numbers to describe units that are not whole. For example, an aquarium might be partly full. A group may have a meeting, but only some of the members are present.

**Fractions** are numbers used to refer to a part of a whole. This includes measurements that cannot be written as whole numbers. For example, the width of a piece of notebook paper is more than 8 inches but less than 9 inches. The part longer than 8 inches is written as a fraction. Here, you will investigate how fractions can be written and used to represent quantities that are parts of the whole.

## Identifying Numerators and Denominators

A whole can be divided into parts of equal size. In the example below, a rectangle has been divided into eight equal squares. Four of these eight squares are shaded.



The shaded area can be represented by a fraction. A fraction is written vertically as two numbers with a line between them.

The **denominator** (the bottom number) represents the number of equal parts that make up the whole. The **numerator** (the top number) describes the number of parts that you are describing. So returning to the example above, the rectangle has been divided into 8 equal parts, and 4 of them have been shaded. You can use the fraction $\frac{4}{8}$ to describe the shaded part of the whole.

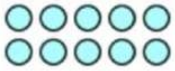
In $\frac{4}{8}$, the 4 is the numerator and tells how many parts are shaded. The 8 is the denominator and tells how many parts are required to make the whole.

## Parts of a Set

The rectangle model above provides a good, basic introduction to fractions. However, what do you do with situations that cannot be as easily modeled by shading part of a figure? For example, think about the following situation:

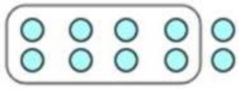
Marc works as a Quality Assurance Manager at an automotive plant. Every hour, he inspects 10 cars; $\frac{4}{5}$ of those pass inspection.

In this case, 10 cars make up the whole group. Each car can be represented as a circle, as shown below.

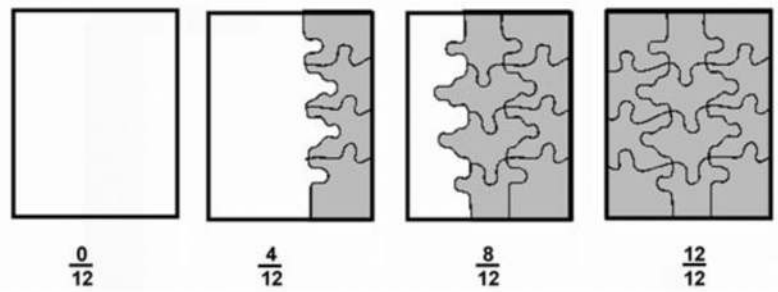


To show $\frac{4}{5}$ of the whole group, you first need to divide the whole group into 5 equal parts. (You know this because the fraction has a denominator of 5.)

To show $\frac{4}{5}$, circle 4 of the equal parts.



Here is another example. Imagine that Aneesh is putting together a puzzle made of 12 pieces. At the beginning, none of the pieces has been put into the puzzle. This means that $\frac{0}{12}$ of the puzzle is complete. Aneesh then puts four pieces together. The puzzle is $\frac{4}{12}$ complete. Soon, he adds four more pieces; 8 out of 12 pieces are now connected. This fraction can be written as $\frac{8}{12}$. Finally, Aneesh adds four more pieces. The puzzle is whole, using all 12 pieces. The fraction can be written as $\frac{12}{12}$.



Note that the number in the denominator cannot be zero. The denominator tells how many parts make up the whole. So if this number is 0, then there are no parts and therefore there can be no whole.

The numerator can be zero, as it tells how many parts you are describing. Notice that in the puzzle example above, you can use the fraction $\frac{0}{12}$ to represent the state of the puzzle when 0 pieces have been placed.

Fractions can also be used to analyze data. In the data table below, 3 out of 5 tosses of a coin came up heads, and 2 out of 5 tosses came up tails. Out of the total number of coin tosses, the portion that was heads can be written as $\frac{3}{5}$. The portion that was tails can be written as $\frac{2}{5}$.

| Coin Toss | Result |
|---|---|
| 1 | Heads |
| 2 | Tails |
| 3 | Heads |
| 4 | Heads |
| 5 | Tails |

**? Exercise**

Sophia, Daphne, and Charlie are all participating in a relay race to raise money for charity. First, Sophia will run 2 miles. Then, Daphne will run 5 miles. Finally, Charlie will end the race by running 3 miles. What fraction of the race will Daphne run?

A. 5 miles
B. $\frac{5}{10}$
C. $\frac{2}{5}$
D. $\frac{5}{3}$

**Answer**

A. Incorrect. Daphne will run 5 miles, but that does not indicate the fractional part of the race that she will run. To find the fraction, first find the whole length of the race by combining the distances the three people will run: 2+5+3=10. Then consider the distance Daphne will run. The correct answer is $\frac{5}{10}$.
B. Correct. The entire race is 10 miles long, and Daphne will run 5 miles. This means she will run $\frac{5}{10}$ of the race.
C. Incorrect. To find the fraction, first find the whole length of the race by combining the distances the three people will run: 2+5+3=10. Then consider the distance Daphne will run. The correct answer is $\frac{5}{10}$.
D. Incorrect. To find the fraction, first find the whole length of the race by combining the distances the three people will run: 2+5+3=10. Then consider the distance Daphne will run. The correct answer is $\frac{5}{10}$.

## Parts of a Whole

The “parts of a whole” concept can be modeled with pizzas and pizza slices. For example, imagine a pizza is cut into 4 pieces, and someone takes 1 piece. Now, $\frac{1}{4}$ of the pizza is gone and $\frac{3}{4}$ remains. Note that both of these fractions have a denominator of 4, which refers to the number of slices the whole pizza has been cut into.

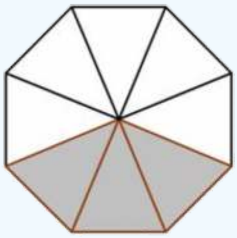
### ✓ Example

**Joaquim bakes a blueberry pie for a potluck dinner. The total pie is cut into 6 equal slices. After everybody eats dessert, only one slice of the pie remains. What fraction of the pie remains?**

**Solution**

| | |
|---|---|
| $\frac{?}{6}$ | The pie was cut into six equal slices, so the denominator of the fraction will be 6. |
| $\frac{1}{6}$ | Only 1 slice remains, so the numerator of the fraction will be 1. |
| | $\frac{1}{6}$ of the pie remains. |

### ✓ Example



**Write a fraction to represent the portion of the octagon that is *not* shaded.**

**Solution**

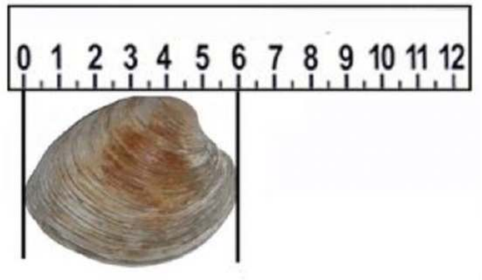
| | |
|---|---|
| $\frac{?}{8}$ | The octagon has eight equal sections, so the denominator of the fraction will be 8. |
| $\frac{5}{8}$ | Five sections are not shaded, so the numerator of the fraction will be 5. |
| | $\frac{5}{8}$ of the octagon is not shaded. |

## Measurement Contexts

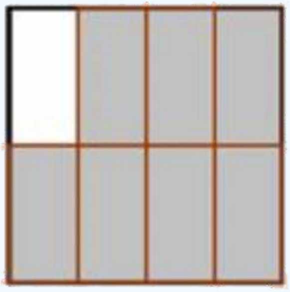
You can use a fraction to represent the quantity in a container. This measuring cup is $\frac{3}{4}$ filled with a liquid. Note that if the cup were $\frac{4}{4}$ full, it would be a whole cup.

You can also use fractions in measuring the length, width, or height of something that is not a full unit. Using a 12-inch ruler, you measure a shell that is 6 inches long. You know that 12 inches equals one foot. So, the length of this shell is $\frac{6}{12}$ of a foot; the 12-

inch ruler is the “whole,” and the length of the shell is the “part.”



**Exercise**



Which fraction represents the portion of the shape that is shaded?

A. $\frac{8}{7}$
B. $\frac{7}{8}$

**Answer**

A. Incorrect. The total number of parts that make up the whole, 8, is the denominator. That is the number *below* the fraction bar. The number of shaded parts, 7, is the numerator. That is the number *above* the fraction bar. The correct answer is $\frac{7}{8}$.

B. Correct. The total number of parts that make up the whole, 8, is the denominator (below the fraction bar). The number of shaded parts, 7, is the numerator (above the fraction bar).

## Summary

Fractions are used to represent parts of a whole. You can use fractions when describing substances, quantities, or diagrams that are not complete. You also use fractions to describe numbers of people or objects that do not make up a complete group. Fractions are written with a numerator and denominator. The numerator (above the fraction bar) tells the number of parts being described, and the denominator (below the fraction bar) tells the number of parts that make up the whole.

---

# 10.2.1.2: Proper and Improper Fractions

**Learning Objectives**

- Learn the rules for rounding.
- Round whole numbers to specific place values, including tens, hundreds, and thousands.

## Introduction

In some situations, you don't need an exact answer. In these cases, **rounding** the number to a specific **place value** is possible. For example, if you travelled 973 miles, you might want to round the distance to 1,000 miles, which is easier to think about. Rounding also comes in handy to see if a calculation is reasonable.

## Rounding Whole Numbers

These are the rules for rounding **whole numbers**:

First, identify the digit with the place value to which you are rounding. You might circle or highlight the digit so you can focus on it better.

Then, determine the possible numbers that you would obtain by rounding. These possible numbers are close to the number that you're rounding to, but have zeros in the digits to the right.

If you are rounding 186 to the nearest ten, then 180 and 190 are the two possible numbers to round to, as 186 is between 180 and 190. But how do you know whether to round to 180 or 190?

> Usually, round a number to the number that is closest to the original number.
>
> When a number is halfway between the two possible numbers, round up to the greater number.

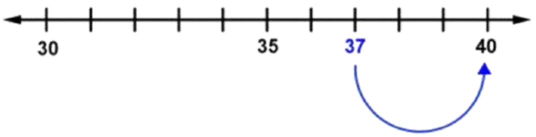
Since 186 is between 180 and 190, and 186 is closer to 190, you round up to 190.

You can use a number line to help you round numbers.

**Example**

**A camera is dropped out of a boat, and sinks to the bottom of a pond that is 37 feet deep. Round 37 to the nearest ten.**

**Solution**

| | |
|---|---|
| 37 | The digit you're rounding to is the tens digit, 3. |
| 30. . .37 . . . 40 | 37 is between 30 and 40. |
|  | 37 is only 3 away from 40, but it's 7 away from 30. So, 37 is closer to 40. |

To the nearest ten, 37 rounds to 40.

**Example**

**Round 33 to the nearest ten.**

**Solution**

33 rounds to 30 because 33 is closer to 30.

To the nearest ten, 33 rounds to 30.

You can determine where to round without using a number line by looking at the digit to the right of the one you're rounding to. If that digit is less than 5, round down. If it's 5 or greater, round up. In the example above, you can see without a number line that 33 is rounded to 30 because the ones digit, 3, is less than 5.

✓ Example

**Round 77 to the nearest ten.**

**Solution**

77 rounds to 80 because the ones digit, 7, is 5 or greater.

77 rounded to the nearest ten is 80.

✓ Example

**There are 576 jellybeans in a jar. Round this number to the nearest ten.**

**Solution**

576 rounds to 580 because the ones digit, 6, is 5 or greater.

576 rounded to the nearest ten is 580.

In the previous examples, you rounded to the tens place. The rounded numbers had a 0 in the ones place. If you round to the nearest hundred, the rounded number will have zeros in the tens and ones places. The rounded number will resemble 100, 500, or 200.

✓ Example

**A runner ran 1,539 meters, but describes the distance he ran with a rounded number. Round 1,539 to the nearest hundred.**

**Solution**

1,539rounds to 1,500 because the next digit is less than 5.

1,539 rounded to the nearest hundred is 1,500.

If you round to the nearest thousand, the rounded number will have zeros in the hundreds, tens, and ones places. The rounded number will resemble 1,000, 2,000, or 14,000.

✓ Example

**A plane's altitude increased by 2,721 feet. Round this number to the nearest thousand.**

**Solution**

2,721 rounds to 3,000 because the next digit, 7, is 5 or greater.

2,721 rounded to the nearest thousand is 3,000.

Now that you know how to round to the nearest ten, hundred, and thousand, try rounding to the nearest ten thousand.

## ✓ Example

**Round 326,749 to the nearest ten thousand.**

### Solution

326,749 rounds to 330,000 because the next digit, 6, is 5 or greater.

326,749 rounded to the nearest ten thousand is 330,000.

## ? Exercise

A record number of 23,386 people voted in a city election. Round this number to the nearest hundred.

A. 23,300
B. 23,400
C. 23,000
D. 23,390

**Answer**

A. Incorrect. The two possible numbers are 23,300 and 23,400, but 23,386 is closer to 23,400. The tens digit, 8, is 5 or greater, so you should round up. The correct answer is 23,400.
B. Correct. The two possible numbers are 23,300 and 23,400, and 23,386 is closer to 23,400. The tens digit, 8, is 5 or greater, so you should round up.
C. Incorrect. This number is rounded to the nearest thousand, not the nearest hundred. The correct answer is 23,400.
D. Incorrect. This number is rounded to the nearest ten, not the nearest hundred. The correct answer is 23,400.

## Summary

In situations when you don't need an exact answer, you can round numbers. When you round numbers, you are always rounding to a particular place value, such as the nearest thousand or the nearest ten. Whether you round up or round down usually depends on which number is closest to your original number. When a number is halfway between the two possible numbers, round up to the larger number.

---

# 10.2.1.3: Factors and Primes

## Learning Objectives

- Recognize (by using the divisibility rule) if a number is divisible by 2, 3, 4, 5, 6, 9, or 10.
- Find the factors of a number.
- Determine whether a number is prime, composite, or neither.
- Find the prime factorization of a number.

## Introduction

**Natural numbers**, also called counting numbers (1, 2, 3, and so on), can be expressed as a product of their **factors**. When working with a fraction, you often need to make the fraction as simple as possible. This means that the **numerator** and the **denominator** have no common factors other than . It will help to find factors, so that later you can simplify and compare fractions.

## Tests of Divisibility

When a natural number is expressed as a product of two other natural numbers, those other numbers are factors of the original number. For example, two factors of 12 are 3 and 4, because $3 \cdot 4=12$.

When one number can be divided by another number with no remainder, we say the first number is **divisible** by the other number. For example, 20 is divisible by $4(20 \div 4=5)$. If a number is divisible by another number, it is also a **multiple** of that number. For example, 20 is divisible by 4, so 20 is a multiple of 4.

**Divisibility tests** are rules that let you quickly tell if one number is divisible by another. There are many divisibility tests. Here are some of the most useful and easy to remember:

- A number is divisible by 2 if the last (ones) digit is divisible by 2. That is, the last digit is 0, 2, 4, 6, or 8. (We then say the number is an **even number**.) For example, in the number 236, the last digit is 6. Since 6 is divisible by $2(6 \div 2=3)$, 236 is divisible by 2.
- A number is divisible by 3 if the sum of all the digits is divisible by 3. For example, the sum of the digits of 411 is 4+1+1=6. Since 6 is divisible by $3(6 \div 3=2)$, 411 is divisible by 3.
- A number is divisible by 5 if the last digit is 0 or 5. For example, 275 and 1,340 are divisible by 5 because the last digits are 5 and 0.
- A number is divisible by 10 if the last digit is 0. For example, 520 is divisible by 10 (last digit is 0).

## Other useful divisibility tests:

1. : A number is divisible by 4 if the last two digits are divisible by 4.
2. : A number is divisible by 6 if it is divisible by both 2 and 3.
3. : A number is divisible by 9 if the sum of its digits is divisible by 9

Here is a summary of the most commonly used divisibility rules.

| A number is divisible by | Example |
|---|---|
| 2 if the last digit is even (0, 2, 4, 6, 8). | 426 yes<br>273 no |
| 3 if the sum of the digits is divisible by 3. | 642 yes<br>(6+4+2=12, 12 is divisible by 3)<br>721 no<br>(7+2+1=10, 10 is not divisible by 3) |
| 4 if the last two digits form a number that is divisible by 4. | 164 yes<br>(64 is divisible by 4)<br>135 no<br>(35 is not divisible by 4) |

| | |
|---|---|
| 5 if the last digit is 0 or 5. | 685 yes<br>432 no |
| 6 if the number is divisible by 2 and 3. | 324 yes<br>(it is even and 3+2+4=9)<br>411 no<br>(although divisible by 3, it is not even) |
| 9 if the sum of the digits is divisible by 9. | 279 yes<br>(2+7+9=18)<br>512 no<br>(5+1+2=8) |
| 10 if the last digit is a 0. | 620 yes.<br>238 no |

If you need to check for divisibility of a number without a rule, divide (either using a calculator or by hand). If the result is a number without any fractional part or remainder, then the number is divisible by the **divisor**. If you forget a rule, you can also use this strategy.

### ? Exercise

Determine whether 522 is divisible by 2, 3, 4, 5, 6, 9, or 10.

A. 2 and 3 only
B. 4 only
C. 2, 3, 6, and 9 only
D. 4, 5, and 10 only

**Answer**

A. Incorrect. Although 522 is divisible by 2 (the last digit is even) and 3 (5+2+2=9, which is a multiple of 3), it is also divisible by 6 and 9. The correct answer is 2, 3, 6, and 9 only.
B. Incorrect. The last two digits (22) are not divisible by 4, so 522 is not divisible by 4. 522 has a last digit divisible by 2, so 522 is divisible by 2. The sum of the digits is divisible by 3 (5+2+2=9) and by 9, so 522 is divisible by 3 and 9. Since 522 is divisible by 2 and 3, it is divisible by 6. Since the last digit is not 0 or 5, 522 is not divisible by 5 or 10. The correct answer is 2, 3, 6, and 9 only.
C. Correct. 522 is divisible by 2 (the last digit is even) and 3 (5+2+2=9, which is a multiple of 3). Since it is divisible by 2 and 3, it is also divisible by 6. Also, the sum of the digits is divisible by 9, so 522 is divisible by 9. Since the last digit is not 0 or 5, 522 is not divisible by 5 or 10. The number formed by the last two digits, 22, is not divisible by 4, so 522 is not divisible by 4.
D. Incorrect. The last two digits are not divisible by 4, so 522 is not divisible by 4. The last digit is not 0 or 5 so 522 is not divisible by 5. The last digit is not 0, so 522 is not divisible by 10. However, 522 is divisible by 2 (the last digit is even) and 3 (5+2+2=9, which is a multiple of 3). Since it is divisible by 2 and 3, it is also divisible by 6. Also, the sum of the digits is divisible by 9, so 522 is divisible by 9. The correct answer is 2, 3, 6, and 9 only.

## Factoring Numbers

To find all the factors of a number, you need to find all numbers that can divide into the original number without a remainder. The divisibility rules from above will be extremely useful!

Suppose you need to find the factors of 30. Since 30 is a number you are familiar with, and small enough, you should know many of the factors without applying any rules. You can start by listing the factors as they come to mind:

$2 \cdot 15$

$3 \cdot 10$

$5 \cdot 6$

Is that it? Not quite. All natural numbers except 1 also have 1 and the number itself as factors:

$1 \cdot 30$

The factors of 30 are 1,2,3,5,6,10,15, and 30.

When you find one factor of a number, you can easily find another factor: it is the quotient using that first factor as the divisor. For example, once you know 2 is a factor of 30, then $30 \div 2$ is another factor. A pair of factors whose product is a given number is a **factor pair** of the original number. So, 2 and 15 are a factor pair for 30.

What do you do if you need to factor a greater number and you can't easily see its factors? That's where the divisibility rules will come in quite handy. Here is a general set of steps that you may follow:

1. Begin with and check the numbers sequentially, using divisibility rules or division.
2. When you find a factor, find the other number in the factor pair.
3. Keep checking sequentially, until you reach the second number in the last factor pair you found, or until the result of dividing gives a number less than the divisor.

Note that you can stop checking when the result of dividing is less than the number you're checking. This means that you have already found all factor pairs, and continuing the process would find pairs that have been previously found.

If a number has exactly two factors, 1 and itself, the number is a **prime number**. A number that has more factors than itself and 1 is called a **composite number**. The number 1 is considered neither prime nor composite, as its only factor is 1. To determine whether a number is prime, composite, or neither, check factors. Here are some examples.

| Number | Composite, Prime, or Neither? | Explanation |
|---|---|---|
| 1 | Neither | 1 does not have two different factors, so it is not prime. |
| 2 | Prime | 2 has only the factors 2 and 1. |
| 3 | Prime | 3 has only the factors 3 and 1. |
| 4 | Composite | 4 has more than two factors: 1, 2, and 4, so it is composite. |
| 5, 7, 11, 13 | Prime | Each number has only two factors: 1 and itself. |
| 6, 8, 9, 10, 50, 63 | Composite | Each number has more than two factors. |

**? Exercise**

Find all the factors of 48.

**Answer**

1, 2, 3, 4, 6, 8, 12, 16, 24, 48

## Prime Factorization

A composite number written as a product of only prime numbers is called the **prime factorization** of the number. One way to find the prime factorization of a number is to begin with the prime numbers 2, 3, 5, 7, 11 and so on, and determine whether the number is divisible by the primes.

For example, if you want to find the prime factorization of 20, start by checking if 20 is divisible by 2. Yes, $2 \cdot 10 = 20$.

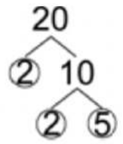
Then factor 10, which is also divisible by $2(2 \cdot 5 = 10)$.

Both of those factors are prime, so you can stop. The prime factorization of 20 is $2 \cdot 2 \cdot 5$, which you can write using **exponential notation** as $2^2 \cdot 5$.

One way to find the prime factorization of a number is to use successive divisions.

| | |
|---|---|
| 10<br>2\longdiv20 | Divide 20 by 2 to get 10. 2 is being used because it is a prime number and a factor of 20. You could also have started with 5. |
| 5<br>2\longdiv10<br>2\longdiv20 | Then divide 10 by 2 to get 5. |
| $2 \cdot 2 \cdot 5$ | Multiplying these divisors forms the prime factorization of 20. |

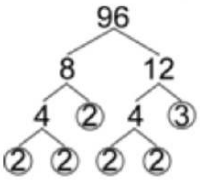
To help you organize the factoring process, you can create a **factor tree**. This is a diagram that shows a factor pair for a composite number. Then, each factor that isn't prime is also shown as a factor pair. You can continue showing factor pairs for composite factors, until you have only prime factors. When a prime number is found as a factor, circle it so you can find it more easily later.

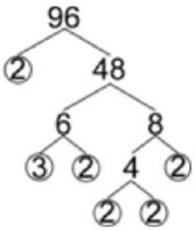


Written using exponential notation, the prime factorization of 20 is again $2^2 \cdot 5$.

Notice that you don't *have* to start checking the number using divisibility of prime numbers. You can factor 20 to $4 \cdot 5$, and then factor 4 to $2 \cdot 2$, giving the same prime factorization: $2 \cdot 2 \cdot 5$.

Now look at a more complicated factorization.





Notice that there are two different trees, but they both produce the same result: five 2s and one 3. Every number will only have one, unique prime factorization. You can use any sets of factor pairs you wish, as long as you keep factoring composite numbers.

When you rewrite the prime factorization of $96(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3)$ in exponential notation, the five 2s can be written as $2^5$. So, $96 = 2^5 \cdot 3$.

### ? Exercise

When finding the prime factorization of 72, Marie began a tree diagram using the two factors 9 and 8. Which of the following statements are true?

1. Marie started the diagram incorrectly and should have started the tree diagram using the factors 2 and 36.
2. Marie's next set of factor pairs could be 3, 3 and 2, 4.
3. Marie's next set of factor pairs could be 3, 3 and 9, 8.
4. Marie didn't have to use a tree diagram.

A. 1 only
B. 2 only
C. 3 and 4 only
D. 2 and 4 only

**Answer**

A. Incorrect. Marie could have started her tree diagram with the factors 2 and 36, but she does not have to start with those factors. Starting with 9 and 8 is fine. The correct answer is D.

B. Incorrect. Yes, Marie’s next set of factor pairs could contain 3, 3, and 2, 4, but statement 4 is also correct. The correct answer is D.

C. Incorrect. Statement 4 is correct: Marie does not have to use a tree diagram, but 3 is not true. When creating a prime factorization, the factors are not combined with the previous composite. The correct answer is D.

D. Correct. Marie’s next set of factor pairs could read 3, 3, and 2, 4, as $3 \cdot 3$ is a factorization of 9 and $2 \cdot 4$ is a factorization of 8. Marie could also find the prime factorization by using successive divisions.

## Summary

Finding the factors of a natural number means that you find all the possible numbers that will divide into the given number without a remainder. There are many rules of divisibility to help you to find factors more quickly. A prime number is a number that has exactly two factors. A composite number is a number that has more than two factors. The prime factorization of a number is the product of the number’s prime factors.

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# 10.2.1.4: Simplifying Fractions

**Learning Objectives**

- Find an equivalent fraction with a given denominator.
- Simplify a fraction to lowest terms.

## Introduction

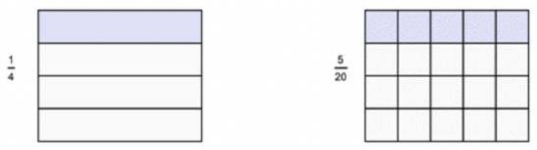
Fractions are used to represent a part of a whole. Fractions that represent the same part of a whole are called **equivalent fractions**. Factoring, multiplication, and division are all helpful tools for working with equivalent fractions.

## Equivalent Fractions

We use equivalent fractions every day. Fifty cents can be 2 quarters, and we have $\frac{2}{4}$ of a dollar, because there are 4 quarters in a dollar. Fifty cents is also 50 pennies out of 100 pennies, or $\frac{50}{100}$ of a dollar. Both of these fractions are the same amount of money, but written with a different numerator and denominator.

Think about a box of crackers that contains 3 packets of crackers. Two of these packets are $\frac{2}{3}$ of the box. Suppose each packet has 30 crackers in it. Two packets are also $60(30 \cdot 2)$ crackers out of $90(30 \cdot 3)$ crackers. This is $\frac{60}{90}$ of the box. The fractions $\frac{2}{3}$ and $\frac{60}{90}$ both represent two packets of crackers, so they are equivalent fractions.

Equivalent fractions represent the same part of a whole, even if the numerator and denominator are different. For example, $\frac{1}{4} = \frac{5}{20}$ . In these diagrams, both fractions represent one of four rows in the rectangle.



Since $\frac{1}{4}$ and $\frac{5}{20}$ are naming the same part of a whole, they are equivalent.

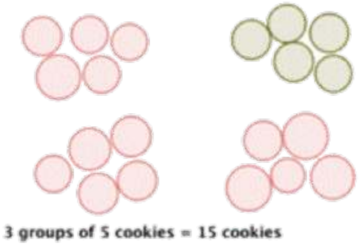
There are many ways to name the same part of a whole using equivalent fractions.

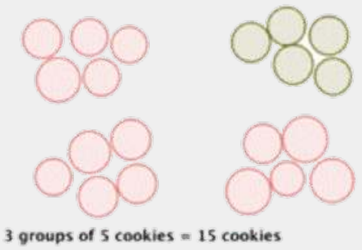
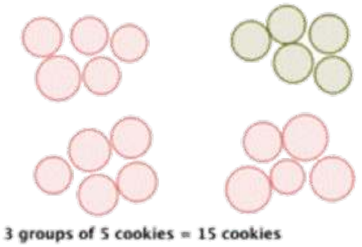
Let’s look at an example where you need to find an equivalent fraction.

**Example**

**John is making cookies for a bake sale. He made 20 large cookies, but he wants to give away only $\frac{3}{4}$ of them for the bake sale. What fraction of the cookies does he give away, using 20 as the denominator?**

### Solution

| | |
|---|---|
|  | Start with 20 cookies. |

| | |
|---|---|
|  | Because the denominator of $\frac{3}{4}$ is 4, make 4 groups of cookies, 5 in each group. |
|  | $\frac{3}{4}=\frac{3\cdot5}{4\cdot5}$ because there are 5 cookies in each group.<br>$\frac{3}{4}=\frac{3\cdot5}{4\cdot5}=\frac{15}{20}$ |

He gives away $\frac{15}{20}$ of the cookies.

When you regroup and reconsider the parts and whole, you are multiplying the numerator and denominator by the same number. In the above example, you multiply 4 by 5 to get the needed denominator of 20, so you also need to multiply the numerator 3 by 5, giving the new numerator of 15.

### Finding Equivalent Fractions

To find equivalent fractions, multiply or divide *both* the numerator and the denominator by the same number.

Examples:

$$\frac{20}{25}=\frac{20\div5}{25\div5}=\frac{4}{5}$$

$$\frac{2}{7}=\frac{2\cdot6}{7\cdot6}=\frac{12}{42}$$

### Exercise

Write an equivalent fraction to $\frac{2}{3}$ that has a denominator of 27.

A. $\frac{26}{27}$
B. $\frac{11}{27}$
C. $\frac{18}{27}$
D. $\frac{12}{18}$

**Answer**

A. Incorrect. You may have added the difference between the two denominators to the numerator (27-3=24, 24+2=26). Instead, you need to use a multiplying factor of $9(27\div3)$. The correct answer is $\frac{18}{27}$.
B. Incorrect. You may have added the multiplying factor of 9 to the numerator. Instead, multiply the numerator by this factor. The correct answer is $\frac{18}{27}$.
C. Correct. The multiplying factor is 9, so the denominator is $3\cdot9=27$ and the numerator is $2\cdot9=18$.
D. Incorrect. Although this is an equivalent fraction to $\frac{2}{3}$, the denominator is not 27. The correct answer is $\frac{18}{27}$.

## Simplifying Fractions

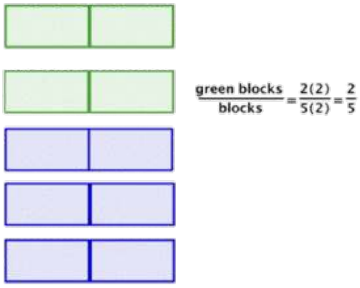
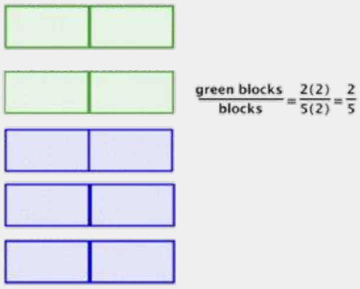
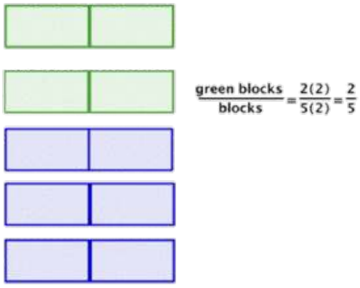
A fraction is in its **simplest form**, or **lowest terms**, when it has the least numerator and the least denominator possible for naming this part of a whole. The numerator and denominator have no common factor other than 1.

Here are 10 blocks, 4 of which are green. So, the fraction that is green is $\frac{4}{10}$. To simplify, you find a common factor and then regroup the blocks by that factor.

✓ Example

**Simplify $\frac{4}{10}$.**

**Solution**

| | |
|---|---|
|  | We start with 4 green blocks out of 10 total blocks. |
|  | Group the blocks in twos, since 2 is a common factor. You have 2 groups of green blocks and a total of 5 groups, each group containing 2 blocks. |
|  | Now, consider the groups as the part and you have 2 green groups out of 5 total groups. |

$\frac{4}{10} = \frac{2}{5}$ The simplified fraction is $\frac{2}{5}$.

Once you have determined a common factor, you can divide the blocks into the groups by dividing both the numerator and denominator to determine the number of groups that you have.

For example, to simplify $\frac{6}{9}$ you find a common factor of 3, which will divide evenly into both 6 and 9. So, you divide 6 and 9 into groups of 3 to determine how many groups of 3 they contain. This gives $\frac{6 \div 3}{9 \div 3} = \frac{2}{3}$, which means 2 out of 3 groups, and $\frac{2}{3}$ is equivalent to $\frac{6}{9}$.

It may be necessary to group more than one time. Each time, determine a common factor for the numerator and denominator using the tests of divisibility, when possible. If both numbers are even numbers, start with 2. For example:

✓ Example

**Simplify $\frac{32}{48}$.**

**Solution**

| | |
|---|---|
| $\frac{32}{48} = \frac{32 \div 2}{48 \div 2} = \frac{16}{24}$ | 32 and 48 have a common factor of 2. Divide each by 2. |

| | |
|---|---|
| $\frac{16}{24}=\frac{16\div 2}{24\div 2}=\frac{8}{12}$ | 16 and 24 have a common factor of 2. Divide each by 2. |
| $\frac{8}{12}=\frac{8\div 4}{12\div 4}=\frac{2}{3}$ | 8 and 12 have a common factor of 4. Divide each by 4. |
| $\frac{32}{48}=\frac{2}{3}$ | $\frac{2}{3}$ is the simplified fraction equivalent to $\frac{32}{48}$. |

In the example above, 16 is a factor of both 32 and 48, so you could have shortened the solution.

$$\frac{32}{48}=\frac{2\cdot 16}{3\cdot 16}=\frac{2}{3}$$

You can also use **prime factorization** to help regroup the numerator and denominator.

### ✓ Example

**Simplify $\frac{54}{72}$.**

**Solution**

| | |
|---|---|
| $\frac{54}{72}=\frac{2\cdot 3\cdot 3\cdot 3}{2\cdot 2\cdot 2\cdot 3\cdot 3}$ | The prime factorization of 54 is $2\cdot 3\cdot 3\cdot 3$.<br>The prime factorization of 72 is $2\cdot 2\cdot 2\cdot 3\cdot 3$. |
| $\frac{3\cdot(2\cdot 3\cdot 3)}{2\cdot 2\cdot(2\cdot 3\cdot 3)}$ | Rewrite, finding common factors. |
| $\frac{3}{2\cdot 2}\cdot 1$ | $\frac{2\cdot 3\cdot 3}{2\cdot 3\cdot 3}=1$ |
| $\frac{3}{4}$ | Multiply: $2\cdot 2$. |
| $\frac{54}{72}=\frac{3}{4}$ | $\frac{3}{4}$ is the simplified fraction equivalent to $\frac{54}{72}$. |

Notice that when you *simplify* a fraction, you *divide* the numerator and denominator by the same number, in the same way you *multiply* by the same number to find an *equivalent* fraction with a greater denominator. In the example above, you could have divided the numerator and denominator by 9, a common factor of 54 and 72.

$$\frac{54\div 9}{72\div 9}=\frac{6}{8}$$

Since the numerator (6) and the denominator (8) still have a common factor, the fraction is not yet in lowest terms. So, again divide by the common factor 2.

$$\frac{6\div 2}{8\div 2}=\frac{3}{4}$$

Repeat this process of dividing by a common factor until the only common factor is 1.

### Simplifying Fractions to Lowest Terms

To simplify a fraction to lowest terms, divide both the numerator and the denominator by their common factors. Repeat as needed until the only common factor is 1.

### ? Exercise

Simplify $\frac{36}{72}$.

A. $\frac{3}{6}$
B. $\frac{9}{18}$
C. $\frac{18}{38}$
D. $\frac{1}{2}$

**Answer**

A. Incorrect. Although $\frac{3}{6}$ is an equivalent fraction to $\frac{36}{72}$, it is not in lowest terms. There is a common factor of 3 in the numerator and denominator. The correct answer is $\frac{1}{2}$.
B. Incorrect. Although $\frac{9}{18}$ is an equivalent fraction to $\frac{36}{72}$, it is not in lowest terms. There are other common factors (9 and 3). The correct answer is $\frac{1}{2}$.
C. Incorrect. You may have divided 72 by 2 and got 38 rather than 36. $\frac{18}{36}$ is an equivalent fraction to $\frac{36}{72}$, but it is not in lowest terms. The correct answer is $\frac{1}{2}$.
D. Correct. $\frac{1}{2}$ is in lowest terms, as 1 is the only common factor of 1 and 2.

## Summary

Equivalent fractions do not always have the same numerator and denominator, but they have the same value. A fraction is in lowest terms when the numerator and denominator of the fraction share no common factors other than . A fraction written in lowest terms is called a simplified fraction.

---

# 10.2.1.5: Comparing Fractions

**Learning Objectives**

- Determine whether two fractions are equivalent.
- Use > or < to compare fractions.

## Introduction

You often need to know when one fraction is greater or less than another fraction. Since a fraction is a part of a whole, to find the greater fraction you need to find the fraction that contains more of the whole. If the two fractions simplify to fractions with a **common denominator**, you can then compare numerators. If the denominators are different, you can find a common denominator first and then compare the numerators.

## Determining Equivalent Fractions

Two fractions are **equivalent fractions** when they represent the same part of a whole. Since equivalent fractions do not always have the same numerator and denominator, one way to determine if two fractions are equivalent is to find a common denominator and rewrite each fraction with that denominator. Once the two fractions have the same denominator, you can check to see if the numerators are equal. If they are equal, then the two fractions are equal as well.

One way to find a common denominator is to check to see if one denominator is a factor of the other denominator. If so, the greater denominator can be used as the common denominator.

**Example**

**Are $\frac{2}{6}$ and $\frac{8}{18}$ equivalent fractions?**

**Solution**

| | |
|---|---|
| Does $\frac{2}{6}=\frac{8}{18}$? | To solve this problem, find a common denominator for the two fractions. This will help you compare the two fractions.<br>Since 6 is a factor of 18, you can write both fractions with 18 as the denominator. |
| $\frac{2 \cdot 3}{6 \cdot 3}=\frac{6}{18}$ | Start with the fraction $\frac{2}{6}$. Multiply the denominator, 6, by 3 to get a new denominator of 18. Since you multiply the denominator by 3, you must also multiply the numerator by 3. |
| $\frac{8}{18}$ | The fraction $\frac{8}{18}$ already has a denominator of 18, so you can leave it as is. |
| $\frac{6}{18}$ does not equal $\frac{8}{18}$ | Compare the fractions. Now that both fractions have the same denominator, 18, you can compare numerators. |

$\frac{2}{6}$ and $\frac{8}{18}$ are not equivalent fractions.

When one denominator is not a factor of the other denominator, you can find a common denominator by multiplying the denominators together.

**Example**

**Determine whether $\frac{3}{6}$ and $\frac{5}{10}$ are equivalent fractions.**

**Solution**

| | |
|---|---|
| $6 \cdot 10=60$ | Use 60 as a common denominator. |
| $\frac{3}{6}=\frac{3 \cdot 10}{6 \cdot 10}=\frac{30}{60}$ | Multiply the numerator and denominator of $\frac{3}{6}$ by 10 to get 60 in the denominator. |

| | |
|---|---|
| $\frac{5}{10} = \frac{5.6}{10 \cdot 6} = \frac{30}{60}$ | Multiply the numerator and denominator of $\frac{5}{10}$ by 6. |
| $\frac{30}{60} = \frac{30}{60}$ | Now that the denominators are the same, compare the numerators. |
| Yes, $\frac{3}{6}$ and $\frac{5}{10}$ are equivalent fractions. | Since 30 is the value of the numerator for both fractions, the two fractions are equal. |

Notice in the above example that you can use 30 as the least common denominator since both 6 and 10 are factors of 30. Any common denominator will work.

In some cases, you can simplify one or both of the fractions, which can result in a common denominator.

### ✓ Example

**Determine whether $\frac{2}{3}$ and $\frac{40}{60}$ are equivalent fractions.**

**Solution**

| | |
|---|---|
| $\frac{40}{60} = \frac{40 \div 10}{60 \div 10} = \frac{4}{6}$ | Simplify $\frac{40}{60}$.<br>Divide the numerator and denominator by the common factor 10. |
| $\frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$ | $\frac{4}{6}$ is still not in lowest terms, so divide the numerator and the denominator again, this time by the common factor 2. |
| $\frac{2}{3} = \frac{2}{3}$ | Compare the fractions. The numerators and denominators are the same. |

Yes, $\frac{2}{3}$ and $\frac{40}{60}$ are equivalent fractions.

### Note

In the example above, you could have used the common factor of 20 to simplify $\frac{40}{60}$ directly to $\frac{2}{3}$.

### Determining Equivalent Fractions

To determine whether or not two fractions are equivalent:

Step 1: Rewrite one or both of the fractions so that they have common denominators.

Step 2: Compare the numerators to see if they have the same value. If so, then the fractions are equivalent.

### ? Exercise

Which of the following fraction pairs are equivalent?

A. $\frac{5}{7}$ and $\frac{7}{5}$
B. $\frac{12}{30}$ and $\frac{6}{10}$
C. $\frac{4}{20}$ and $\frac{1}{5}$
D. $\frac{8}{11}$ and $\frac{8}{22}$

**Answer**

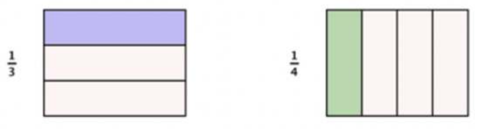
A. Incorrect. Although the same numbers, 5 and 7, are used in each fraction, the numerators and denominators are not equal, so the fractions cannot be equivalent. The correct answer is $\frac{4}{20}$ and $\frac{1}{5}$.
B. Incorrect. 30 is divisible by 10, and 12 is divisible by 6. However, they do not share a common multiple: $6 \cdot 2 = 12$, and $10 \cdot 3 = 30$. This means the fractions are not equivalent. The correct answer is $\frac{4}{20}$ and $\frac{1}{5}$.

C. Correct. Take the fraction $\frac{1}{5}$ and multiply both the numerator and denominator by 4. You are left with the fraction $\frac{4}{20}$. This means that the two fractions are equivalent.

D. Incorrect. The numerators of the two fractions are the same, but the denominators are different. This means the fractions are not equivalent. The correct answer is $\frac{4}{20}$ and $\frac{1}{5}$

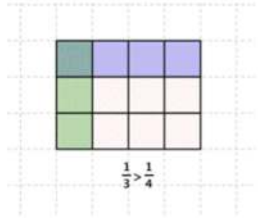
## Comparing Fractions Using < and >

When given two or more fractions, it is often useful to know which fraction is greater than or less than the other. For example, if the discount in one store is $\frac{1}{3}$ off the original price and the discount in another store is $\frac{1}{4}$ off the original price, which store is offering a better deal? To answer this question, and others like it, you can compare fractions.



To determine which fraction is greater, you need to find a common denominator. You can then compare the fractions directly. Since 3 and 4 are both factors of 12, you will divide the whole into 12 parts, create equivalent fractions for $\frac{1}{3}$ and $\frac{1}{4}$, and then compare.

Now you see that $\frac{1}{3}$ contains 4 parts of 12, and $\frac{1}{4}$ contains 3 parts of 12. So, $\frac{1}{3}$ is greater than $\frac{1}{4}$.



As long as the denominators are the same, the fraction with the greater numerator is the greater fraction, as it contains more parts of the whole. The fraction with the lesser numerator is the lesser fraction, as it contains fewer parts of the whole.

Recall that the symbol < means “less than”, and the symbol > means “greater than”. These symbols are inequality symbols. So, the true statement 3<8 is read as "3 is less than 8" and the statement 5>3 is read as "5 is greater than 3". One way to help you remember the distinction between the two symbols is to think that the smaller end of the symbol points to the lesser number.

As with comparing whole numbers, the inequality symbols are used to show when one fraction is “greater than” or “less than” another fraction.

### Comparing Fractions

To compare two fractions:

Step 1: Compare denominators. If they are different, rewrite one or both fractions with a common denominator.

Step 2: Check the numerators. If the denominators are the same, then the fraction with the greater numerator is the greater fraction. The fraction with the lesser numerator is the lesser fraction. And, as noted above, if the numerators are equal, the fractions are equivalent.

### ✓ Example

**Use < or > to compare the two fractions $\frac{4}{5}$ and $\frac{14}{20}$.**

**Solution**

| | |
|---|---|
| Is $\frac{4}{5} > \frac{14}{20}$ or is $\frac{4}{5} < \frac{14}{20}$? | You cannot compare the fractions directly because they have different denominators. You need to find a common denominator for the two fractions. |
| $\frac{4}{5} = \frac{?}{20}$ | Since 5 is a factor of 20, you can use 20 as the common denominator. |
| $\frac{4 \cdot 4}{5 \cdot 4} = \frac{16}{20}$ | Multiply the numerator and denominator by 4 to create an equivalent fraction with a denominator of 20. |
| $\frac{16}{20} > \frac{14}{20}$ | Compare the two fractions. $\frac{16}{20}$ is greater than $\frac{14}{20}$. |
| $\frac{4}{5} > \frac{14}{20}$ | If $\frac{16}{20} > \frac{14}{20}$, then $\frac{4}{5} > \frac{14}{20}$, since $\frac{4}{5} = \frac{16}{20}$. |

### ? Exercise

Which of the following is a true statement?

A. $\frac{5}{6} < \frac{24}{30}$
B. $\frac{75}{100} > \frac{9}{12}$
C. $\frac{4}{16} > \frac{1}{3}$
D. $\frac{3}{8} < \frac{20}{40}$

**Answer**

A. Incorrect. $\frac{5}{6}$ is equivalent to $\frac{25}{30}$, and since 25>24, $\frac{25}{30} > \frac{24}{30}$, which means $\frac{5}{6} > \frac{24}{30}$. The correct answer is $\frac{3}{8} < \frac{20}{40}$.

B. Incorrect. Both fractions simplify to $\frac{3}{4}$, so one isn't greater than or less than the other one. They are equivalent. The correct answer is $\frac{3}{8} < \frac{20}{40}$.

C. Incorrect. Finding a common denominator, you can compare $\frac{12}{48}$ to $\frac{16}{48}$, and see that $\frac{12}{48} < \frac{16}{48}$, which means $\frac{4}{16} < \frac{1}{3}$. The correct answer is $\frac{3}{8} < \frac{20}{40}$.

D. Correct. Simplifying $\frac{20}{40}$, you get the equivalent fraction $\frac{1}{2}$. Since you still don't have a common denominator, write $\frac{1}{2}$ as an equivalent fraction with a denominator of 8: $\frac{1}{2} = \frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$. You find that $\frac{3}{8} < \frac{4}{8}$, so $\frac{3}{8} < \frac{20}{40}$ as well.

## Summary

You can compare two fractions with like denominators by comparing their numerators. The fraction with the greater numerator is the greater fraction, as it contains more parts of the whole. The fraction with the lesser numerator is the lesser fraction as it contains fewer parts of the whole. If two fractions have the same denominator, then equal numerators indicate equivalent fractions.

---

# 10.2.2: Multiplying and Dividing Fractions and Mixed Numbers

# 10.2.2.1: Multiplying Fractions and Mixed Numbers

### Learning Objectives

- Multiply two or more fractions.
- Multiply a fraction by a whole number.
- Multiply two or more mixed numbers.
- Solve application problems that require multiplication of fractions or mixed numbers.

## Introduction

Just as you add, subtract, multiply, and divide when working with whole numbers, you also use these operations when working with fractions. There are many times when it is necessary to multiply fractions and **mixed numbers**. For example, this recipe will make 4 crumb piecrusts:

5 cups graham crackers

8 tablespoons sugar

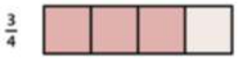
$1\frac{1}{2}$ cups melted butter

$\frac{1}{4}$ teaspoon vanilla

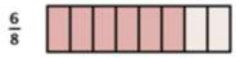
Suppose you only want to make 2 crumb piecrusts. You can multiply all the ingredients by $\frac{1}{2}$, since only half of the number of piecrusts are needed. After learning how to multiply a fraction by another fraction, a whole number or a mixed number, you should be able to calculate the ingredients needed for 2 piecrusts.

## Multiplying Fractions

When you multiply a fraction by a fraction, you are finding a “fraction of a fraction.” Suppose you have $\frac{3}{4}$ of a candy bar and you want to find $\frac{1}{2}$ of the $\frac{3}{4}$:



By dividing each fourth in half, you can divide the candy bar into eighths.



Then, choose half of those to get $\frac{3}{8}$.

In both of the above cases, to find the answer, you can multiply the numerators together and the denominators together.

### Multiplying Two Fractions

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{\text{product of the numerators}}{\text{product of the denominators}}$$

Example:

$$\frac{3}{4} \cdot \frac{1}{2} = \frac{3 \cdot 1}{4 \cdot 2} = \frac{3}{8}$$

### Multiplying More Than Two Fractions

$$\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} = \frac{a \cdot c \cdot e}{b \cdot d \cdot f}$$

Example:

$\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} = \frac{1 \cdot 2 \cdot 3}{3 \cdot 4 \cdot 5} = \frac{6}{60}$

### ✓ Example

| | |
|---|---|
| $\mathbf{\frac{2}{3} \cdot \frac{4}{5}}$ | **Multiply.** |

**Solution**

| | |
|---|---|
| $\frac{2 \cdot 4}{3 \cdot 5}$ | Multiply the numerators and multiply the denominators. |
| $\frac{8}{15}$ | Simplify, if possible. This fraction is already in lowest terms. |

$\frac{8}{15}$

If the resulting **product** needs to be simplified to lowest terms, divide the numerator and denominator by common factors.

### ✓ Example

| | |
|---|---|
| $\mathbf{\frac{2}{3} \cdot \frac{1}{4}}$ | **Multiply. Simplify the answer.** |

**Solution**

| | |
|---|---|
| $\frac{2 \cdot 1}{3 \cdot 4}$ | Multiply the numerators and multiply the denominators. |
| $\frac{2}{12}$ | Simplify, if possible. |
| $\frac{2 \div 2}{12 \div 2}$ | Simplify by dividing the numerator and denominator by the common factor 2. |

$\frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$

You can also simplify the problem before multiplying, by dividing common factors.

### ✓ Example

| | |
|---|---|
| $\mathbf{\frac{2}{3} \cdot \frac{1}{4}}$ | **Multiply. Simplify the answer.** |

**Solution**

| | |
|---|---|
| $\frac{2 \cdot 1}{3 \cdot 4} = \frac{1 \cdot 2}{3 \cdot 4}$ | Reorder the numerators so that you can see a fraction that has a common factor. |
| $\frac{1 \cdot 1}{3 \cdot 2}$ | Simplify.<br>$\frac{2}{4} = \frac{2 \div 2}{4 \div 2} = \frac{1}{2}$ |

$\frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$

You do not have to use the “simplify first” shortcut, but it could make your work easier because it keeps the numbers in the numerator and denominator smaller while you are working with them.

### ? Exercise

$\frac{3}{4} \cdot \frac{1}{3}$ Multiply. Simplify the answer.

A. $\frac{3}{12}$
B. $\frac{4}{7}$
C. $\frac{1}{4}$

D. $\frac{36}{144}$

**Answer**

A. Incorrect. $\frac{3}{12}$ is an equivalent fraction to the correct answer $\frac{1}{4}$, but it is not in lowest terms. You must divide numerator and denominator by the common factor 3. The correct answer is $\frac{1}{4}$.

B. Incorrect. You may have added numerators (3+1) and added denominators (4+3) instead of multiplying. The correct answer is $\frac{1}{4}$.

C. Correct. One way to find this answer is to multiply numerators and denominators, $\frac{3\cdot 1}{4\cdot 3}=\frac{3}{12}$, then simplify: $\frac{3\div 3}{12\div 3}=\frac{1}{4}$.

D. Incorrect. You probably found a common denominator, multiplied correctly, but then forgot to simplify. Finding a common denominator is not necessary and makes the multiplication harder because you are working with greater than necessary numbers. The correct answer is $\frac{1}{4}$.

## Multiplying a Fraction by a Whole Number

When working with both fractions and whole numbers, it is useful to write the whole number as an **improper fraction** (a fraction where the numerator is greater than or equal to the denominator). All whole numbers can be written with a "1" in the denominator. For example: $2=\frac{2}{1}$, $5=\frac{5}{1}$, and $100=\frac{100}{1}$. Remember that the denominator tells how many parts there are in one whole, and the numerator tells how many parts you have.

### Multiplying a Fraction and a Whole Number

$a\cdot\frac{b}{c}=\frac{a}{1}\cdot\frac{b}{c}$

Example:

$4\cdot\frac{2}{3}=\frac{4}{1}\cdot\frac{2}{3}=\frac{8}{3}$

Often when multiplying a whole number and a fraction, the resulting product will be an improper fraction. It is often desirable to write improper fractions as a mixed number for the final answer. You can simplify the fraction before or after rewriting it as a mixed number. See the examples below.

### Example

| | |
|---|---|
| $7\cdot\frac{3}{5}$ | **Multiply. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $\frac{7}{1}\cdot\frac{3}{5}$ | Rewrite 7 as the improper fraction $\frac{7}{1}$. |
| $\frac{7\cdot 3}{1\cdot 5}=\frac{21}{5}$ | Multiply the numerators and multiply the denominators. |
| $4\frac{1}{5}$ | Rewrite as a mixed number. $21\div 5=4$ with a remainder of 1. |

$7\cdot\frac{3}{5}=4\frac{1}{5}$

### Example

| | |
|---|---|
| $4\cdot\frac{3}{4}$ | **Multiply. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $\frac{4}{1}\cdot\frac{3}{4}$ | Rewrite 4 as the improper fraction $\frac{4}{1}$. |
| $\frac{4\cdot 3}{1\cdot 4}$ | Multiply the numerators and multiply the denominators. |
| $\frac{12}{4}=3$ | Simplify. |

$4 \cdot \frac{3}{4} = 3$

**Exercise**

$3 \cdot \frac{5}{6}$ Multiply. Simplify the answer and write it as a mixed number.

A. $1\frac{1}{7}$
B. $2\frac{1}{2}$
C. $\frac{5}{2}$
D. $\frac{8}{6}$

**Answer**

A. Incorrect. You may have added the numerators and added the denominators to get $\frac{8}{7}$, which is the mixed number $1\frac{1}{7}$. Make sure you multiply numerators and multiply denominators. Multiplying the two numbers gives you $\frac{15}{6}$, and since $15 \div 6 = 2\text{R}3$, the mixed number is $2\frac{3}{6}$. The fractional part simplifies to $\frac{1}{2}$. The correct answer is $2\frac{1}{2}$.
B. Correct. Multiplying the two numbers gives $\frac{15}{6}$, and since $15 \div 6 = 2\text{R}3$, the mixed number is $2\frac{3}{6}$. The fractional part simplifies to $\frac{1}{2}$.
C. Incorrect. Multiplying the numerators and multiplying the denominators results in the improper fraction $\frac{5}{2}$, but you need to express this as a mixed number. The correct answer is $2\frac{1}{2}$.
D. Incorrect. You may have added numerators and placed it over the denominator of 6. Make sure you multiply numerators and multiply denominators. Multiplying the two numbers gives $\frac{15}{6}$, and since $15 \div 6 = 2\text{R}3$, the mixed number is $2\frac{3}{6}$. The fractional part simplifies to $\frac{1}{2}$. The correct answer is $2\frac{1}{2}$.

## Multiplying Mixed Numbers

If you want to multiply two mixed numbers, or a fraction and a mixed number, you can again rewrite any mixed number as an improper fraction.

So, to multiply two mixed numbers, rewrite each as an improper fraction and then multiply as usual. Multiply numerators and multiply denominators and simplify. And, as before, when simplifying, if the answer comes out as an improper fraction, then convert the answer to a mixed number.

**Example**

| | |
|---|---|
| $\mathbf{2\frac{1}{5} \cdot 4\frac{1}{2}}$ | **Multiply. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $2\frac{1}{5} = \frac{11}{5}$ | Change 2 $\frac{1}{5}$ to an improper fraction. $5 \cdot 2 + 1 = 11$ , and the denominator is 5. |
| $4\frac{1}{2} = \frac{9}{2}$ | Change $4\frac{1}{2}$ to an improper fraction. $2 \cdot 4 + 1 = 9$ , and the denominator is 2. |
| $\frac{11}{5} \cdot \frac{9}{2}$ | Rewrite the multiplication problem, using the improper fractions. |
| $\frac{11 \cdot 9}{5 \cdot 2} = \frac{99}{10}$ | Multiply numerators and multiply denominators. |
| $\frac{99}{10} = 9\frac{9}{10}$ | Write as a mixed number. $99 \div 10 = 9$ with a remainder of 9. |

$2\frac{1}{5} \cdot 4\frac{1}{2} = 9\frac{9}{10}$

**Example**

| | |
|---|---|
| $\frac{1}{2}\cdot 3\frac{1}{3}$ | **Multiply. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $3\frac{1}{3}=\frac{10}{3}$ | Change $3\frac{1}{3}$ to an improper fraction. $3\cdot 3+1=10$ , and the denominator is 3. |
| $\frac{1}{2}\cdot\frac{10}{3}$ | Rewrite the multiplication problem, using the improper fraction in place of the mixed number. |
| $\frac{1\cdot 10}{2\cdot 3}=\frac{10}{6}$ | Multiply numerators and multiply denominators. |
| $\frac{10}{6}=1\frac{4}{6}$ | Rewrite as a mixed number. $10\div 6=1$ with a remainder of 4. |
| $1\frac{2}{3}$ | Simplify the fractional part to lowest terms by dividing the numerator and denominator by the common factor 2. |

$\frac{1}{2}\cdot 3\frac{1}{3}=1\frac{2}{3}$

As you saw earlier, sometimes it's helpful to look for common factors in the numerator and denominator before you simplify the products.

**Example**

| | |
|---|---|
| $1\frac{3}{5}\cdot 2\frac{1}{4}$ | **Multiply. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $1\frac{3}{5}=\frac{8}{5}$ | Change $1\frac{3}{5}$ to an improper fraction. $5\cdot 1+3=8$ , and the denominator is 5. |
| $2\frac{1}{4}=\frac{9}{4}$ | Change $2\frac{1}{4}$ to an improper fraction. $4\cdot 2+1=9$ , and the denominator is 4. |
| $\frac{8}{5}\cdot\frac{9}{4}$ | Rewrite the multiplication problem using the improper fractions. |
| $\frac{8\cdot 9}{5\cdot 4}=\frac{9\cdot 8}{5\cdot 4}$ | Reorder the numerators so that you can see a fraction that has a common factor. |
| $\frac{9\cdot 8}{5\cdot 4}=\frac{9\cdot 2}{5\cdot 1}$ | Simplify. $\frac{8}{4}=\frac{8\div 4}{4\div 4}=\frac{2}{1}$ |
| $\frac{18}{5}$ | Multiply. |
| $\frac{18}{5}=3\frac{3}{5}$ | Write as a mixed fraction. |

$1\frac{3}{5}\cdot 2\frac{1}{4}=3\frac{3}{5}$

In the last example, the same answer would be found if you multiplied numerators and multiplied denominators without removing the common factor. However, you would get $\frac{72}{20}$ , and then you would need to simplify more to get your final answer.

**? Exercise**

$1\frac{3}{5}\cdot 3\frac{1}{3}$

A. $\frac{80}{15}$<br>
B. $5\frac{5}{15}$<br>
C. $4\frac{14}{15}$

D. $5\frac{1}{3}$

**Answer**

A. Incorrect. You probably wrote both mixed numbers as improper fractions correctly. You probably also correctly multiplied numerators and denominators. However, this improper fraction still needs to be rewritten as a mixed number and simplified. Dividing $80 \div 15 = 5$ with a remainder of 5 or $5\frac{5}{15}$, then simplifying the fractional part, the correct answer is $5\frac{1}{3}$.

B. Incorrect. You probably wrote both mixed numbers as improper fractions correctly. You probably also correctly multiplied numerators and denominators, and wrote the answer as a mixed number. However, the mixed number is not in lowest terms. $\frac{5}{15}$ can be simplified to $\frac{1}{3}$ by dividing numerator and denominator by the common factor 5. The correct answer is $5\frac{1}{3}$.

C. Incorrect. This is the result of adding the two numbers. To multiply, rewrite each mixed number as an improper fraction: $1\frac{3}{5} = \frac{8}{5}$ and $3\frac{1}{3} = \frac{10}{3}$. Next, multiply numerators and multiply denominators: $\frac{8}{5} \cdot \frac{10}{3} = \frac{80}{15}$. Then, write the resulting improper fraction as a mixed number: $\frac{80}{15} = 5\frac{5}{15}$. Finally, simplify the fractional part by dividing both numerator and denominator by the common factor, 5. The correct answer is $5\frac{1}{3}$.

D. Correct. First, rewrite each mixed number as an improper fraction: $1\frac{3}{5} = \frac{8}{5}$ and $3\frac{1}{3} = \frac{10}{3}$. Next, multiply numerators and multiply denominators: $\frac{8}{5} \cdot \frac{10}{3} = \frac{80}{15}$. Then write as a mixed fraction $\frac{80}{15} = 5\frac{5}{15}$. Finally, simplify the fractional part by dividing both numerator and denominator by the common factor 5.

## Solving Problems by Multiplying Fractions and Mixed Numbers

Now that you know how to multiply a fraction by another fraction, by a whole number, or by a mixed number, you can use this knowledge to solve problems that involve multiplication and fractional amounts. For example, you can now calculate the ingredients needed for the 2 crumb piecrusts.

✓ Example

| | |
|---|---|
| **5 cups graham crackers 8 tablespoons sugar $1\frac{1}{2}$ cups melted butter $\frac{1}{4}$ teaspoon vanilla** | **The recipe at left makes 4 piecrusts. Find the ingredients needed to make only 2 piecrusts.** |

**Solution**

| | |
|---|---|
| | Since the recipe is for 4 piecrusts, you can multiply each of the ingredients by $\frac{1}{2}$ to find the measurements for just 2 piecrusts. |
| $5 \cdot \frac{1}{2} = \frac{5}{1} \cdot \frac{1}{2} = \frac{5}{2}$<br>$2\frac{1}{2}$ cups of graham crackers are needed. | 5 cups graham crackers: Since the result is an improper fraction, rewrite $\frac{5}{2}$ as the improper fraction $2\frac{1}{2}$. |
| $8 \cdot \frac{1}{2} = \frac{8}{1} \cdot \frac{1}{2} = \frac{8}{2} = 4$<br>4 tablespoons sugar is needed. | 8 tablespoons sugar: This is another example of a whole number multiplied by a fraction. |
| $\frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}$<br>$\frac{3}{4}$ cup melted butter is needed. | $1\frac{1}{2}$ cups melted butter: You need to multiply a mixed number by a fraction. So, first rewrite $1\frac{1}{2}$ as the improper fraction $\frac{3}{2}$: $2 \cdot 1 + 1$, and the denominator is 2. Then, rewrite the multiplication problem, using the improper fraction in place of the mixed number. Multiply. |
| $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$<br>$\frac{1}{8}$ teaspoon vanilla is needed. | $\frac{1}{4}$ teaspoon vanilla: Here, you multiply a fraction by a fraction. |

The ingredients needed for 2 pie crusts are:

$2\frac{1}{2}$ cups graham crackers

4 tablespoons sugar

$\frac{3}{4}$ cup melted butter

$\frac{1}{8}$ teaspoon vanilla

Often, a problem indicates that multiplication by a fraction is needed by using phrases like “half of,” “a third of,” or " $\frac{3}{4}$ of."

**✓ Example**

**The cost of a vacation is $\$4,500$ and you are required to pay $\frac{1}{5}$ of that amount when you reserve the trip. How much will you have to pay when you reserve the trip?**

**Solution**

| | |
|---|---|
| $4,500 \cdot \frac{1}{5}$ | You need to find $\frac{1}{5}$ of 4,500. “Of” tells you to multiply. |
| $\frac{4,500}{1} \cdot \frac{1}{5}$ | Change 4,500 to an improper fraction by rewriting it with 1 as the denominator. |
| $\frac{4,500}{5}$ | Divide. |
| 900 | Simplify. |

You will need to pay $\$900$ when you reserve the trip.

**✓ Example**

| | |
|---|---|
|  | **The pie chart at left represents the fractional part of daily activities.**<br>**Given a 24-hour day, how many hours are spent sleeping? Attending school? Eating? Use the pie chart to determine your answers.** |

**Solution**

| | |
|---|---|
| $\frac{1}{3} \cdot 24$ = number of hours sleeping | Sleeping is $\frac{1}{3}$ of the pie, so the number of hours spent sleeping is $\frac{1}{3}$ of 24. |
| $\frac{1}{3} \cdot \frac{24}{1} = 8$ | Rewrite 24 as an improper fraction with a denominator of 1. |
| $\frac{24}{3} = 8$<br>8 **hours sleeping** | Multiply numerators and multiply denominators. Simplify $\frac{24}{3}$ to 8. |
| $\frac{1}{6} \cdot 24$ = number of hours spent at school | Attending school is $\frac{1}{6}$ of the pie, so the number of hours spent attending school is $\frac{1}{6}$ of 24. |
| $\frac{1}{6} \cdot \frac{24}{1}$ | Rewrite 24 as an improper fraction with a denominator of 1. |
| $\frac{24}{6} = 4$<br>4 **hours attending school** | Multiply numerators and multiply denominators. Simplify $\frac{24}{6}$ to 4. |
| $\frac{1}{12} \cdot 24$ = number of hours spent eating | Eating is $\frac{1}{12}$ of the pie, so the number of hours spent eating is $\frac{1}{12}$ of 24. |

|  |  |
|---|---|
| $\frac{1}{12}\cdot\frac{24}{1}$ | Rewrite 24 as an improper fraction with a denominator of 1. |
| $\frac{24}{12}=2$<br>2 **hours spent eating** | Multiply numerators and multiply denominators. Simplify $\frac{24}{12}$ to 2. |

Hours spent:

sleeping: 8 hours

attending school: 4 hours

eating: 2 hours

## ❓ Exercise

Neil bought a dozen (12) eggs. He used $\frac{1}{3}$ of the eggs for breakfast. How many eggs are left?

A. 8
B. 4
C. 9
D. 3

**Answer**

A. Correct. $\frac{1}{3}$ of 12 is $4\left(\frac{1}{3}\cdot\frac{12}{1}=\frac{12}{3}=4\right)$, so he used 4 of the eggs. Because $12-4=8$, there are 8 eggs left.
B. Incorrect. $\frac{1}{3}$ of 12 is 4, but that gives how many eggs Neil used, not how many he had left. You need to subtract 4 from 12 to find the number of remaining eggs. The correct answer is 8.
C. Incorrect. You may have incorrectly found $\frac{1}{3}$ of 12 to be $3$. $\frac{1}{3}$ of 12 is 4, and then 12-4 is 8. The correct answer is 8.
D. Incorrect. You need to find $\frac{1}{3}$ of 12, which is 4. Then subtract 4 from 12 to get 8 remaining eggs.

## Summary

You multiply two fractions by multiplying the numerators and multiplying the denominators. Often the resulting product will not be in lowest terms, so you must also simplify. If one or both fractions are whole numbers or mixed numbers, first rewrite each as an improper fraction. Then multiply as usual, and simplify.

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- **2.2.1: Multiplying Fractions and Mixed Numbers** by The NROC Project is licensed CC BY-NC-SA 4.0. Original source: https://content.nroc.org/DevelopmentalMath.HTML5/Common/toc/toc_en.html.

# 10.2.2.2: Dividing Fractions and Mixed Numbers

**Learning Objectives**

- Find the reciprocal of a number.
- Divide two fractions.
- Divide two mixed numbers.
- Divide fractions, mixed numbers, and whole numbers.
- Solve application problems that require division of fractions or mixed numbers.

## Introduction

There are times when you need to use division to solve a problem. For example, if painting one coat of paint on the walls of a room requires 3 quarts of paint and there are 6 quarts of paint, how many coats of paint can you paint on the walls? You divide 6 by 3 for an answer of 2 coats. There will also be times when you need to divide by a fraction. Suppose painting a closet with one coat only required $\frac{1}{2}$ quart of paint. How many coats could be painted with the 6 quarts of paint? To find the answer, you need to divide 6 by the fraction, $\frac{1}{2}$.

## Reciprocals

If the **product** of two numbers is 1, the two numbers are **reciprocals** of each other. Here are some examples:

| **Original number** | **Reciprocal** | **Product** |
|---|---|---|
| $\frac{3}{4}$ | $\frac{4}{3}$ | $\frac{3}{4}\cdot\frac{4}{3}=\frac{3\cdot4}{4\cdot3}=\frac{12}{12}=1$ |
| $\frac{1}{2}$ | $\frac{2}{1}$ | $\frac{1}{2}\cdot\frac{2}{1}=\frac{1\cdot2}{2\cdot1}=\frac{2}{2}=1$ |
| $3=\frac{3}{1}$ | $\frac{1}{3}$ | $\frac{3}{1}\cdot\frac{1}{3}=\frac{3\cdot1}{1\cdot3}=\frac{3}{3}=1$ |
| $2\frac{1}{3}=\frac{7}{3}$ | $\frac{3}{7}$ | $\frac{7}{3}\cdot\frac{3}{7}=\frac{7\cdot3}{3\cdot7}=\frac{21}{21}=1$ |

In each case, the original number, when multiplied by its reciprocal, equals 1.

To create two numbers that multiply together to give an answer of one, the numerator of one is the denominator of the other. You sometimes say one number is the “flip” of the other number: flip $\frac{2}{5}$ to get the reciprocal $\frac{5}{2}$. In order to find the reciprocal of a mixed number, write it first as an **improper fraction** so that it can be “flipped.”

**Example**

Find the reciprocal of $5\frac{1}{4}$.

### Solution

| | |
|---|---|
| $5\frac{1}{4}=\frac{21}{4}$ | Rewrite $5\frac{1}{4}$ as an improper fraction. The numerator is $4\cdot5+1=21$ . |
| $\frac{4}{21}$ | Find the reciprocal by interchanging (“flipping”) the numerator and denominator. |

**Exercise**

What is the reciprocal of $3\frac{2}{5}$?

A. $3\frac{5}{2}$
B. $\frac{17}{5}$
C. $\frac{5}{17}$
D. $\frac{5}{2}$

**Answer**

A. Incorrect. The fractional parts of this answer and the original mixed number are reciprocals, but in order to find the reciprocal of the entire number, you must write the mixed number as an improper fraction before interchanging numerator and denominator. The correct answer is $\frac{5}{17}$.

B. Incorrect. You found the correct improper fraction that represents $3\frac{2}{5}$ but did not find the reciprocal. The reciprocal of $\frac{17}{5}$ is $\frac{5}{17}$.

C. Correct. First, write $3\frac{2}{5}$ as an improper fraction, $\frac{17}{5}$. The reciprocal of $\frac{17}{5}$ is found by interchanging ("flipping") the numerator and denominator.

D. Incorrect. This is the correct reciprocal for the fractional part of the mixed number, but with a mixed number, you first need to write it as an improper fraction. The mixed number $3\frac{2}{5}$ is written as the improper fraction $\frac{17}{5}$. The reciprocal of $\frac{17}{5}$ is found by interchanging ("flipping") the numerator and denominator.

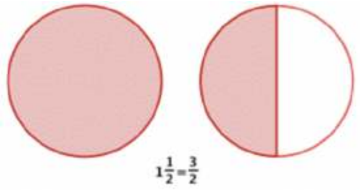
## Dividing a Fraction or a Mixed Number by a Whole Number

When you divide by a whole number, you multiply by the reciprocal of the divisor. In the painting example where you need 3 quarts of paint for a coat and have 6 quarts of paint, you can find the total number of coats that can be painted by dividing 6 by 3, $6 \div 3 = 2$. You can also multiply 6 by the reciprocal of 3, which is $\frac{1}{3}$, so the multiplication problem becomes $\frac{6}{1} \cdot \frac{1}{3} = \frac{6}{3} = 2$.

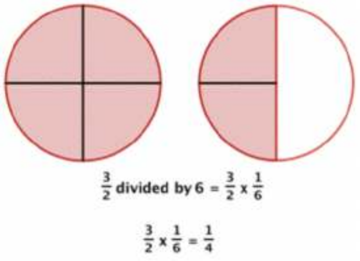
The same idea will work when the divisor is a fraction. If you have $\frac{3}{4}$ of a candy bar and need to divide it among 5 people, each person gets $\frac{1}{5}$ of the available candy: $\frac{1}{5}$ of $\frac{3}{4}$ is $\frac{1}{5} \cdot \frac{3}{4} = \frac{3}{20}$, so each person gets $\frac{3}{20}$ of a whole candy bar.

If you have a recipe that needs to be divided in half, you can divide each ingredient by 2, or you can multiply each ingredient by $\frac{1}{2}$ to find the new amount.

Similarly, with a **mixed number**, you can either divide by the whole number or you can multiply by the reciprocal. Suppose you have $1\frac{1}{2}$ pizzas that you want to divide evenly among 6 people.



Dividing by 6 is the same as multiplying by the reciprocal of, which is $\frac{1}{6}$. Cut the available pizza into six equal-sized pieces.



Each person gets one piece, so each person gets $\frac{1}{4}$ of a pizza.

Dividing a fraction by a whole number is the same as multiplying by the reciprocal, so you can always use multiplication of fractions to solve such division problems.

✓ Example

**Find $2\frac{2}{3} \div 4$. Write your answer as a mixed number with any fraction part in lowest terms.**

**Solution**

| | |
|---|---|
| $2\frac{2}{3}=\frac{8}{3}$ | Rewrite $2\frac{2}{3}$ as an improper fraction. The numerator is $2\cdot 3+2$. The denominator is still 3. |
| $\frac{8}{3}\div 4=\frac{8}{3}\cdot\frac{1}{4}$ | Dividing by 4 or $\frac{4}{1}$ is the same as multiplying by the reciprocal of 4, which is $\frac{1}{4}$. |
| $\frac{8\cdot 1}{3\cdot 4}=\frac{8}{12}$ | Multiply numerators and multiply denominators. |
| $\frac{2}{3}$ | Simplify to lowest terms by dividing numerator and denominator by the common factor 4. |

$2\frac{2}{3}\div 4=\frac{2}{3}$

**? Exercise**

Find $4\frac{3}{5}\div 2$ Simplify the answer and write as a mixed number.

A. $2\frac{3}{10}$
B. $\frac{10}{23}$
C. $\frac{23}{10}$
D. $9\frac{1}{5}$

**Answer**

A. Correct. Write $4\frac{3}{5}$ as the improper fraction $\frac{23}{5}$. Then multiply by $\frac{1}{2}$, the reciprocal of 2. This gives the improper fraction $\frac{23}{10}$, and the mixed number is $23\div 10=2\text{R}3$, which is $2\frac{3}{10}$.

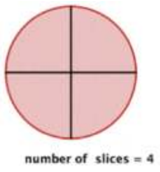
B. Incorrect. After changing the mixed number to an improper fraction, you may have inverted $\frac{23}{5}$ instead of 2. Keep $\frac{23}{5}$, and multiply by the reciprocal of 2, giving you $\frac{23}{5}\cdot\frac{1}{2}=\frac{23}{10}$. Finally, write $\frac{23}{10}$ as a mixed number, $23\div 10=2\text{R}3$, which is $2\frac{3}{10}$.

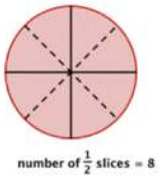
C. Incorrect. This is the correct improper fraction, but you still need to write the final answer as a mixed number, $23\div 10=2\text{R}3$, which is $2\frac{3}{10}$.

D. Incorrect. You may have forgotten to find the reciprocal of 2 before multiplying. Once you have the improper fraction for $4\frac{3}{5}$, which is $\frac{23}{5}$, multiply by the reciprocal of 2, which is $\frac{1}{2}$, giving you $\frac{23}{5}\cdot\frac{1}{2}=\frac{23}{10}$. Finally, write $\frac{23}{10}$ as a mixed number, $23\div 10=2\text{R}3$, which is $2\frac{3}{10}$.

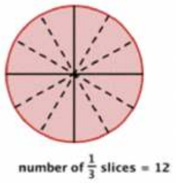
## Dividing by a Fraction

Sometimes you need to solve a problem that requires dividing by a fraction. Suppose you have a pizza that is already cut into 4 slices. How many $\frac{1}{2}$ slices are there?





There are 8 slices. You can see that dividing 4 by $\frac{1}{2}$ gives the same result as multiplying 4 by 2. What would happen if you needed to divide each slice into thirds?



You would have 12 slices, which is the same as multiplying 4 by 3.

**Dividing with Fractions**

Step 1: Find the reciprocal of the number that follows the division symbol.

Step 2: Multiply the first number (the one **before** the division symbol) by the reciprocal of the second number (the one **after** the division symbol).

Examples:

$6 \div \frac{2}{3} = 6 \cdot \frac{3}{2}$ and $\frac{2}{5} \div \frac{1}{3} = \frac{2}{5} \cdot \frac{3}{1}$

An easy way to remember how to divide fractions is the phrase “keep, change, flip”. This means to **KEEP** the first number, **CHANGE** the division sign to multiplication, and then **FLIP** (use the reciprocal) of the second number.

**Example**

| | |
|---|---|
| $\mathbf{\frac{2}{3} \div \frac{1}{6}}$ | **Divide.** |

**Solution**

| | |
|---|---|
| $\frac{2}{3} \cdot \frac{6}{1}$ | Multiply by the reciprocal:<br>Keep $\frac{2}{3}$, change $\div$ to $\cdot$, and flip $\frac{1}{6}$. |
| $\frac{2 \cdot 6}{3 \cdot 1} = \frac{12}{3}$ | Multiply numerators and multiply denominators. |
| $\frac{12}{3} = 4$ | Simplify. |

$\frac{2}{3} \div \frac{1}{6} = 4$

**Example**

| | |
|---|---|
| $\mathbf{\frac{3}{5} \div \frac{2}{3}}$ | **Divide.** |

**Solution**

| | |
|---|---|
| $\frac{3}{5} \cdot \frac{3}{2}$ | Multiply by the reciprocal:<br>Keep $\frac{3}{5}$, change $\div$ to $\cdot$, and flip $\frac{2}{3}$. |
| $\frac{3 \cdot 3}{5 \cdot 2} = \frac{9}{10}$ | Multiply numerators and multiply denominators. |

$\frac{3}{5} \div \frac{2}{3} = \frac{9}{10}$

When solving a division problem by multiplying by the reciprocal, remember to write all whole numbers and mixed numbers as improper fractions. The final answer should be simplified and written as a mixed number.

**Example**

| | |
|---|---|
| $\mathbf{2\frac{1}{4} \div \frac{3}{4}}$ | **Divide.** |

**Solution**

| | |
|---|---|
| $\frac{9}{4} \cdot \frac{4}{3}$ | Write 2 $\frac{1}{4}$ as an improper fraction. |
| $\frac{9}{4} \cdot \frac{4}{3}$ | Multiply by the reciprocal:<br>Keep $\frac{9}{4}$, change $\div$ to $\cdot$, and flip $\frac{3}{4}$. |
| $\frac{9 \cdot 4}{4 \cdot 3} = \frac{36}{12}$ | Multiply numerators and multiply denominators. |

| | |
|---|---|
| $\frac{36}{12}=3$ | Simplify. |

$2\frac{1}{4}\div\frac{3}{4}=3$

**Example**

| | |
|---|---|
| $\mathbf{3\frac{1}{5}\div 2\frac{1}{10}}$ | **Divide. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $\frac{16}{5}\div\frac{21}{10}$ | Write $3\frac{1}{5}$ and $2\frac{1}{10}$ as improper fractions. |
| $\frac{16}{5}\cdot\frac{10}{21}$ | Multiply by the reciprocal of $\frac{21}{10}$. |
| $\frac{16\cdot 10}{5\cdot 21}$ | Multiply numerators, multiply denominators. |
| $\frac{16\cdot 10}{21\cdot 5}$ | Regroup. |
| $\frac{16\cdot 2}{21\cdot 1}$ | Simplify: $\frac{10}{5}=\frac{2}{1}$ |
| $\frac{16\cdot 2}{21\cdot 1}=\frac{32}{21}$ | Multiply. |
| $\frac{32}{21}=1\frac{11}{21}$<br>$3\frac{1}{5}\div 2\frac{1}{10}=1\frac{11}{21}$ | Rewrite as a mixed number. |

$3\frac{1}{5}\div 2\frac{1}{10}=1\frac{11}{21}$

**Exercise**

Find $5\frac{1}{3}\div\frac{2}{3}$. Simplify the answer and write as a mixed number.

A. $4\frac{1}{2}$
B. $3\frac{5}{9}$
C. $\frac{16}{2}$
D. $8$

**Answer**

A. Incorrect. You may have incorrectly written $5\frac{1}{3}$ as the improper fraction $\frac{9}{3}$. To write $5\frac{1}{3}$ as an improper fraction, multiply 5 times 3 and add 1. The denominator is 3. $5\frac{1}{3}=\frac{16}{3}$. Then change division to multiplication and multiply by the reciprocal of $\frac{2}{3}$, which is $\frac{3}{2}$, giving you $\frac{16}{3}\cdot\frac{3}{2}=\frac{3}{3}\cdot\frac{16}{2}=\frac{16}{2}=8$ .
B. Incorrect. You may have forgotten to use the reciprocal of $\frac{2}{3}$. After finding the improper fraction $\frac{16}{3}$ that represents $5\frac{1}{3}$, change division to multiplication and use the reciprocal of $\frac{2}{3}$, giving you $\frac{16}{3}\cdot\frac{3}{2}=\frac{3}{3}\cdot\frac{16}{2}=\frac{16}{2}=8$ .
C. Incorrect. This is the correct improper fraction, but the final answer needs to be a mixed number. Divide 16 by 2, which is 8 and no remainder. There is no fractional part, so the answer is a whole number, 8.
D. Correct. Write 5 $\frac{1}{3}$ as an improper fraction, $\frac{16}{3}$. Then multiply by the reciprocal of $\frac{2}{3}$, which is $\frac{3}{2}$, giving you $\frac{16}{3}\cdot\frac{3}{2}=\frac{3}{3}\cdot\frac{16}{2}=\frac{16}{2}=8$ .

## Dividing Fractions or Mixed Numbers to Solve Problems

Using multiplication by the reciprocal instead of division can be very useful to solve problems that require division and fractions.

## ✓ Example

**A cook has $18\frac{3}{4}$ pounds of ground beef. How many quarter-pound burgers can he make?**

**Solution**

| | |
|---|---|
| $18\frac{3}{4} \div \frac{1}{4}$ | You need to find how many quarter pounds there are $18\frac{3}{4}$, in so use division. |
| $\frac{75}{4} \div \frac{1}{4}$ | Write $18\frac{3}{4}$ as an improper fraction. |
| $\frac{75}{4} \cdot \frac{4}{1}$ | Multiply by the reciprocal. |
| $\frac{75 \cdot 4}{4 \cdot 1}$ | Multiply numerators and multiply denominators. |
| $\frac{4}{4} \cdot \frac{75}{1}$ | Regroup and simplify $\frac{4}{4}$, which is 1. |

75 burgers

## ✓ Example

**A child needs to take $2\frac{1}{2}$ tablespoons of medicine per day in 4 equal doses. How much medicine is in each dose?**

**Solution**

| | |
|---|---|
| $2\frac{1}{2} \div 4$ | You need to make 4 equal doses, so you can use division. |
| $\frac{5}{2} \div 4$ | Write $2\frac{1}{2}$ as an improper fraction. |
| $\frac{5}{2} \cdot \frac{1}{4}$ | Multiply by the reciprocal. |
| $\frac{5 \cdot 1}{2 \cdot 4} = \frac{5}{8}$ | Multiply numerators and multiply denominators. Simplify, if possible. |

$\frac{5}{8}$ tablespoon in each dose.

## ? Exercise

How many $\frac{2}{5}$-cup salt shakers can be filled from cups of salt?

A. $4\frac{4}{5}$
B. $\frac{60}{2}$
C. $30$
D. $\frac{1}{30}$

**Answer**

A. Incorrect. You probably forgot to find the reciprocal of $\frac{2}{5}$. Find $12 \div \frac{2}{5}$. Keep the 12, change division to multiplication, and use the reciprocal ("flip") of $\frac{2}{5}$, giving you: $12 \cdot \frac{5}{2} = \frac{12}{1} \cdot \frac{5}{2} = \frac{60}{2} = 30$ .

B. Incorrect. You still need to find the mixed number that represents $\frac{60}{2}$. In this case, 60 divided by 2 is $30\mathrm{R}0$, so the answer is a whole number, 30.

C. Correct. $12 \div \frac{2}{5}$ will show how many salt shakers can be filled. Write 12 as $\frac{12}{1}$ and multiply by the reciprocal ("flip") of $\frac{2}{5}$, giving you $\frac{12}{1} \cdot \frac{5}{2} = \frac{60}{2} = 30$ .

D. Incorrect. You incorrectly used the expression $\frac{2}{5} \div 12$. This will show how many groups of 12 there are in $\frac{2}{5}$. You need to find how many groups of $\frac{2}{5}$ are in 12, which is $12 \div \frac{2}{5}$. Then write 12 as $\frac{12}{1}$ and multiply by the reciprocal ("flip") of $\frac{2}{5}$, giving you $12 \cdot \frac{5}{2} = \frac{12}{1} \cdot \frac{5}{2} = \frac{60}{2} = 30$ .

## Summary

Division is the same as multiplying by the reciprocal. When working with fractions, this is the easiest way to divide. Whether you divide by a number or multiply by the reciprocal of the number, the result will be the same. You can use these techniques to help you solve problems that involve division, fractions, and/or mixed numbers.

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# 10.2.3: Adding and Subtracting Fractions and Mixed Numbers

# 10.2.3.1: Adding Fractions and Mixed Numbers

### Learning Objectives

- Add fractions with like denominators.
- Find the least common multiple (LCM) of two or more numbers.
- Find the common denominator of fractions with unlike denominators.
- Add fractions with unlike denominators.
- Add mixed numbers with like and unlike denominators.
- Solve application problems that require the addition of fractions or mixed numbers.

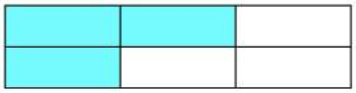
## Introduction

Fractions are used in many areas of everyday life: recipes, woodworking, rainfall, timecards, and measurements, to name just a few. Sometimes you have parts of wholes that you need to combine. Just as you can add whole numbers, you can add fractions and mixed numbers. Consider, for example, how to determine the monthly rainfall if you know the daily rainfall in inches. You have to add fractions. Also, consider several painters who are working to paint a house together with multiple cans of paint. They might add the fractions of what remains in each can to determine if there is enough paint to finish the job or if they need to buy more.

## Adding Fractions with Like Denominators

When the pieces are the same size, they can easily be added. Consider the pictures below showing the fractions $\frac{3}{6}$ and $\frac{2}{6}$.

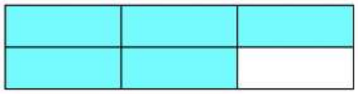
This picture represents $\frac{3}{6}$ shaded because 3 out of 6 blocks are shaded.



This picture represents $\frac{2}{6}$ shaded because 2 out of 6 blocks are shaded.

If you add these shaded blocks together, you are adding $\frac{3}{6}+\frac{2}{6}$ .

You can create a new picture showing 5 shaded blocks in a rectangle containing 6 blocks.



So, $\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$ .

Without drawing rectangles and shading boxes, you can get this answer simply by adding the numerators, 3+2, and keeping the denominator, 6, the same. This procedure works for adding any fractions that have the same denominator, called **like denominators**.

### Example

| | |
|---|---|
| $\frac{3}{5}+\frac{1}{5}$ | **Add.** |

**Solution**

| | |
|---|---|
| $\frac{3+1}{5}$ | Since the denominator of each fraction is 5, these fractions have like denominators. |
| $\frac{4}{5}$ | So, add the numerators and write the sum over the denominator, 5. |

$\frac{3}{5}+\frac{1}{5}=\frac{4}{5}$

### ✓ Example

| | |
|---|---|
| $\mathbf{\frac{3}{8}+\frac{5}{8}}$ | **Add. Simplify the answer.** |

**Solution**

| | |
|---|---|
| $\frac{3}{8}+\frac{5}{8}=\frac{3+5}{8}=\frac{8}{8}$ | The denominators are alike, so add the numerators. |
| $\frac{8}{8}=1$ | Simplify the fraction. |

$\frac{3}{8}+\frac{5}{8}=1$

### ✓ Example

| | |
|---|---|
| $\mathbf{\frac{11}{12}+\frac{5}{12}}$ | **Add. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $\frac{11}{12}+\frac{5}{12}=\frac{11+5}{12}=\frac{16}{12}$ | The denominators are alike, so add the numerators. |
| $\frac{16}{12}=\frac{16\div4}{12\div4}=\frac{4}{3}$ | Simplify the fraction. 16 and 12 have a common factor of 4. |
| $\frac{4}{3}=1\frac{1}{3}$ | Write the improper fraction as a mixed number, by dividing:<br>$4\div3=1$ with a remainder of 1. |

$\frac{11}{12}+\frac{5}{12}=1\frac{1}{3}$

In the previous example, the fraction was simplified and then converted to a mixed number. You could just as easily have first converted the improper fraction to a mixed number and then simplified the fraction in the mixed number. Notice that the same answer is reached with both methods.

$\frac{16}{12}=1\frac{4}{12}$

The fraction $\frac{4}{12}$ can be simplified. $\frac{4}{12}=\frac{4\div4}{12\div4}=\frac{1}{3}$

But, don't forget about the 1 that is part of the mixed number! The final answer is $1\frac{1}{3}$.

### Adding Fractions with Like Denominators

1. Add the numerators (the number in the top of each fraction).
2. Keep the denominator (the bottom number) the same.
3. Simplify to lowest terms.

### ? Exercise

$\frac{7}{10}+\frac{8}{10}$ Add. Simplify the answer and write as a mixed number.

A. $\frac{15}{20}$
B. $\frac{9}{10}$
C. $\frac{3}{2}$
D. $1\frac{1}{2}$

**Answer**

A. Incorrect. When you add fractions with like denominators, only add the numerators, not the denominators. Keep the denominator the same. The correct answer is $1\frac{1}{2}$.

B. Incorrect. To add two fractions with the same denominator, add the two numerators together. Then simplify the fraction. The correct answer is $1\frac{1}{2}$.

C. Incorrect. The answer is to be expressed as a simplified mixed number, not as an improper fraction. The correct answer is $1\frac{1}{2}$.

D. Correct. $\frac{7+8}{10}=\frac{15}{10}=1\frac{5}{10}=1\frac{1}{2}$ .

## Finding Least Common Multiples

Sometimes fractions do not have the same denominator. They have **unlike denominators**. Think about the example of the house painters. If one painter has $\frac{2}{3}$ can of paint and his painting partner has $\frac{1}{2}$ can of paint, how much do they have in total? How can you add these fractions when they do not have like denominators?

The answer is that you can rewrite one or both of the fractions so that they have the same denominator. This is called finding a **common denominator**. While any common denominator will do, it is helpful to find the **least common multiple** of the two numbers in the denominator because this will save having to simplify at the end. The least common multiple is the least number that is a multiple of two or more numbers. Least common multiple is sometimes abbreviated LCM.

There are several ways to find common multiples, some of which you used when comparing fractions. To find the least common multiple (LCM), you can list the multiples of each number and determine which multiples they have in common. The least of these numbers will be the least common multiple. Consider the numbers 4 and 6. Some of their multiples are shown below. You can see that they have several common multiples, and the *least* of these is 12.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 | 44 | 48 | 52 | 56 | 60 | 64 |
| 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 | 66 | 68 | | | | |

### ✓ Example

**Find the least common multiple of 30 and 50.**

**Solution**

| | |
|---|---|
| 30, 60, 90, 120, 150, 180, 210, 240 | List some multiples of 30. |
| 50, 100, 150, 200, 250 | List some multiples of 50. |
| 150 is found on both lists of multiples. | Look for the least number found on both lists. |

The least common multiple of 30 and 50 is 150.

The other method for finding the least common multiple is to use **prime factorization**. This is the method you need for working with rational expressions. The following shows how the factor method works with the numeric example, 4 and 6.

Start by finding the prime factorization of each denominator:

$$4=2\cdot 2$$

$$6=3\cdot 2$$

Identify the greatest number of times any factor appears in either factorization and multiply those factors to get the least common multiple. For 4 and 6, it would be:

$$3\cdot 2\cdot 2=12$$

Notice that 2 is included twice, because it appears twice in the prime factorization of 4. 12 is the least common multiple of 4 and 6.

The next example also shows how to use prime factorization.

✓ Example

**Find the least common multiple of 28 and 40.**

**Solution**

| | |
|---|---|
| $28 = 2 \cdot 2 \cdot 7$ | Write the prime factorization of 28. |
| $40 = 2 \cdot 2 \cdot 2 \cdot 5$ | Write the prime factorization of 40. |
| $2 \cdot 2 \cdot 2 \cdot 5 \cdot 7 = 280$ | Write the factors the greatest number of times they appear in either factorization and multiply. |

The least common multiple of 28 and 40 is 280.

? Exercise

Find the least common multiple of 12 and 80.

A. 240
B. 120
C. 960
D. 480

**Answer**

A. Correct. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 240$
B. Incorrect. 120 is not a multiple of 80. The correct answer is 240.
C. Incorrect. 960 is a common multiple, but it is not the least common multiple. The correct answer is 240.
D. Incorrect. 480 is a common multiple but it is not the least common multiple. The correct answer is 240.

## Finding Least Common Denominators

You can use the least common multiple of two denominators as the **least common denominator** for those fractions. Then you rewrite each fraction using the same denominator.

The example below shows how to use the least common multiple as the least common denominator.

✓ Example

**Rewrite the fractions $\frac{2}{3}$ and $\frac{1}{2}$ as fractions with a least common denominator.**

**Solution**

| | |
|---|---|
| Multiples of 3 include 3, 6, 9, 12<br>Multiples of 2 include 2, 4, 6<br>6 is the least common denominator. | Find the least common multiple of the denominators. This is the least common denominator. |
| $\frac{2}{3} \cdot \frac{2}{2} = \frac{4}{6}$ | Rewrite $\frac{2}{3}$ with a denominator of 6. |
| $\frac{1}{2} \cdot \frac{3}{3} = \frac{3}{6}$ | Rewrite $\frac{1}{2}$ with a denominator of 6. |

The fraction $\frac{2}{3}$ can be rewritten as $\frac{4}{6}$.

The fraction $\frac{1}{2}$ can be rewritten as $\frac{3}{6}$.

**Exercise**

Find the least common denominator. Then express each fraction using the least common denominator: $\frac{3}{4}$ and $\frac{1}{6}$.

A. $\frac{18}{24}, \frac{4}{24}$
B. $\frac{3}{12}, \frac{1}{12}$
C. $24$
D. $\frac{9}{12}, \frac{2}{12}$

**Answer**

A. Incorrect. 24 is a common denominator, but not the *least* common denominator for the fractions. The correct answer is $\frac{9}{12}, \frac{2}{12}$.
B. Incorrect. 12 is the least common denominator, but these fractions are not equivalent to the original fractions. The numerators are incorrect. The correct answer is $\frac{9}{12}, \frac{2}{12}$.
C. Incorrect. This is a multiple of the denominators, but the task was to rewrite the fractions with a common denominator. The correct answer is $\frac{9}{12}, \frac{2}{12}$.
D. Correct. $\frac{3}{4} \cdot \frac{3}{3} = \frac{9}{12}$, $\frac{1}{6} \cdot \frac{2}{2} = \frac{2}{12}$.

## Adding Fractions with Unlike Denominators

To add fractions with unlike denominators, first rewrite them with like denominators. Then, you know what to do! The steps are shown below.

**Adding Fractions with Unlike Denominators**

1. Find a common denominator.
2. Rewrite each fraction using the common denominator.
3. Now that the fractions have a common denominator, you can add the numerators.
4. Simplify to lowest terms, expressing improper fractions as mixed numbers.

You can always find a common denominator by multiplying the two denominators together. See the example below.

**Example**

| | |
|---|---|
| $\frac{2}{3} + \frac{1}{5}$ | **Add. Simplify the answer.** |

**Solution**

| | |
|---|---|
| $3 \cdot 5 = 15$ | Since the denominators are not alike, find a common denominator by multiplying the denominators. |
| $\frac{2}{3} \cdot \frac{5}{5} = \frac{10}{15}$<br>$\frac{1}{5} \cdot \frac{3}{3} = \frac{3}{15}$ | Rewrite each fraction with a denominator of 15. |
| $\frac{10}{15} + \frac{3}{15} = \frac{13}{15}$ | Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified. |

$\frac{2}{3} + \frac{1}{5} = \frac{13}{15}$

You can find a common denominator by finding the common multiples of the denominators. The least common multiple is the easiest to use.

**Example**

| | |
|---|---|
| $\mathbf{\frac{3}{7}+\frac{2}{21}}$ | **Add. Simplify the answer.** |

**Solution**

| | |
|---|---|
| Multiples of 7 include 7, 14, 21<br>Multiples of 21 include **21** | Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 7 and 21. |
| $\frac{3}{7}\cdot\frac{3}{3}=\frac{9}{21}$<br>$\frac{2}{21}$ | Rewrite each fraction with a denominator of 21. |
| $\frac{9}{21}+\frac{2}{21}=\frac{11}{21}$ | Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified. |

$\frac{3}{7}+\frac{2}{21}=\frac{11}{21}$

You can also add more than two fractions as long as you first find a common denominator for all of them. An example of a sum of three fractions is shown below. In this example, you will use the prime factorization method to find the LCM.

**Example**

| | |
|---|---|
| $\frac{3}{4}+\frac{1}{6}+\frac{5}{8}$ | **Add. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $4=2\cdot 2$<br>$6=3\cdot 2$<br>$8=2\cdot 2\cdot 2$<br>LCM: $2\cdot 2\cdot 2\cdot 3=24$ | Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 4, 6, and 8. |
| $\frac{3}{4}\cdot\frac{6}{6}=\frac{18}{24}$<br>$\frac{1}{6}\cdot\frac{4}{4}=\frac{4}{24}$<br>$\frac{5}{8}\cdot\frac{3}{3}=\frac{15}{24}$ | Rewrite each fraction with a denominator of 24. |
| $\frac{18}{24}+\frac{4}{24}+\frac{15}{24}=\frac{37}{24}$ | Add the fractions by adding the numerators and keeping the denominator the same. |
| $\frac{37}{24}=1\frac{13}{24}$ | Write the improper fraction as a mixed number and simplify the fraction. |

$\frac{3}{4}+\frac{1}{6}+\frac{5}{8}=1\frac{13}{24}$

**Exercise**

$\frac{2}{3}+\frac{4}{5}+\frac{1}{12}$ Add. Simplify the answer and write as a mixed number.

A. $1\frac{33}{60}$
B. $1\frac{11}{20}$
C. $\frac{31}{20}$
D. $\frac{7}{20}$

**Answer**

A. Incorrect. The fraction in the mixed number can be simplified. The correct answer is $1\frac{11}{20}$.

B. Correct. $\frac{40}{60}+\frac{48}{60}+\frac{5}{60}=\frac{93}{60}=1\frac{33}{60}=1\frac{11}{20}$ .

C. Incorrect. Express the improper fraction as a mixed number. The correct answer is $1\frac{11}{20}$.

D. Incorrect. Find a common denominator; express each fraction using the common denominator; add the numerators and simplify. The correct answer is $1\frac{11}{20}$.

## Adding Mixed Numbers

Just as you can add whole numbers and proper fractions, you can also add mixed numbers. To add mixed numbers, add the whole numbers together and the fraction parts of the mixed numbers together and then recombine to express the value as a mixed number. The steps for adding two mixed numbers are shown in the examples below.

You can keep the whole numbers and the fractions together using a vertical method for adding mixed numbers as shown below.

### ✓ Example

| | |
|---|---|
| $2\frac{1}{8}+3\frac{3}{8}$ | **Add. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $\begin{array}{r} 2\frac{1}{8} \\ +3\frac{3}{8} \\ \hline \end{array}$ | Arrange the mixed numbers vertically so the whole numbers align and the fractions align. |
| $\begin{array}{r} 2\frac{1}{8} \\ +3\frac{3}{8} \\ \hline 5\frac{4}{8} \end{array}$ | Add whole numbers. Add fractions. |
| $5\frac{4}{8}=5\frac{1}{2}$ | Simplify the fraction. |

$2\frac{1}{8}+3\frac{3}{8}=5\frac{1}{2}$

When adding mixed numbers, you may also need to find a common denominator first. Consider the example below.

### ✓ Example

| | |
|---|---|
| $8\frac{5}{6}+7\frac{4}{9}$ | **Add. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| Multiples of 6 include 6, 12, 18<br>Multiples of 9 include 9, 18 | Find a least common denominator for the fractions. |
| $\frac{5}{6}\cdot\frac{3}{3}=\frac{15}{18}$<br>$\frac{4}{9}\cdot\frac{2}{2}=\frac{8}{18}$ | Express each fraction with a denominator of 18. |
| $\begin{array}{r} 8\frac{15}{18} \\ +7\frac{8}{18} \\ \hline \end{array}$ | Arrange the mixed numbers vertically so the whole numbers align and the fractions align. |
| $\begin{array}{r} 8\frac{15}{18} \\ +7\frac{8}{18} \\ \hline 15\frac{23}{18} \end{array}$ | Add whole numbers. Add fractions. |
| $\frac{23}{18}=1\frac{5}{18}$ | Write the improper fraction as a mixed number. |

| | |
|---|---|
| $15+1+\frac{5}{18}$<br>$16\frac{5}{18}$ | Combine whole numbers and fraction to write a mixed number. |

$8\frac{5}{6}+7\frac{4}{9}=16\frac{5}{18}$

**? Exercise**

$3\frac{3}{5}+1\frac{4}{9}$ Add. Simplify the answer and write as a mixed number.

A. $\frac{2}{45}$
B. $5\frac{2}{45}$
C. $4\frac{1}{2}$
D. $4\frac{47}{45}$

**Answer**

A. Incorrect. The whole numbers must be added also. The correct answer is $5\frac{2}{45}$.
B. Correct. $3+1+\frac{3}{5}+\frac{4}{9}$ , $4+\frac{27}{45}+\frac{20}{45}=4+\frac{47}{45}=4+1\frac{2}{45}=5\frac{2}{45}$ .
C. Incorrect. To add fractions with unlike denominators, you must first find a common denominator. The correct answer is $5\frac{2}{45}$.
D. Incorrect. A simplified mixed number is a whole number and a proper fraction. The correct answer is $5\frac{2}{45}$.

## Adding Fractions to Solve Problems

Knowing how to add fractions is useful in a variety of situations. When reading problems, look for phrases that help you know you want to add the fractions.

**✓ Example**

A stack of pamphlets is placed on top of a book. If the stack of pamphlets is $3\frac{1}{4}$ inches thick and the book is $5\frac{3}{4}$ inches thick, how high is the pile?

**Solution**

| | |
|---|---|
| $3\frac{1}{4}+5\frac{3}{4}$ | Find the total height of the pile by adding the thicknesses of the stack of pamphlets and the book. |
| $3+5+\frac{1}{4}+\frac{3}{4}$ | Group the whole numbers and fractions to make adding easier. |
| $8+\frac{1}{4}+\frac{3}{4}$ | Add whole numbers. |
| $\frac{1}{4}+\frac{3}{4}=\frac{4}{4}=1$ | Add fractions. |
| $8+1=9$ | Combine whole number and fraction. |

The pile is 9 inches high.

**✓ Example**

**A cake recipe requires $2\frac{1}{4}$ cups of milk and $1\frac{1}{2}$ cups of melted butter. If these are the only liquids, how much liquid is in the recipe?**

**Solution**

| | |
|---|---|
| $2\frac{1}{4}+1\frac{1}{2}$ | Find the total amount of liquid by adding the quantities. |
| $2+1+\frac{1}{4}+\frac{1}{2}$ | Group the whole numbers and fractions to make adding easier. |

| | |
|---|---|
| $3+\frac{1}{4}+\frac{1}{2}$ | Add whole numbers. |
| $3+\frac{1}{4}+\frac{2}{4}=3+\frac{3}{4}$ | Add fractions. Recall that $\frac{1}{2}=\frac{2}{4}$. |
| $3\frac{3}{4}$ | Combine whole number and fraction. |

There are $3\frac{3}{4}$ cups of liquid in the recipe.

**Exercise**

What is the total rainfall in a three-day period if it rains $3\frac{1}{4}$ inches the first day, $\frac{3}{8}$ inch the second day, and $2\frac{1}{2}$ inches on the third day?

A. $6$ inches
B. $6\frac{1}{8}$ inches
C. $5\frac{1}{8}$ inches
D. $5\frac{9}{8}$ inches

**Answer**

A. Incorrect. The answer is not a whole number because the fractions have a sum of $\frac{9}{8}$. The correct answer is $6\frac{1}{8}$ inches.
B. Correct. $5\frac{9}{8}=6\frac{1}{8}$ inches.
C. Incorrect. When you add the fractions you get $5+\frac{9}{8}$. The $5+\frac{9}{8}$ still contains an improper fraction. The $\frac{9}{8}$ needs to be changed to $1\frac{1}{8}$ and then recombined with the 5. You need to combine the 1 from the $1\frac{1}{8}$ with the 5 to get $6\frac{1}{8}$, not $5\frac{1}{8}$. The correct answer is $6\frac{1}{8}$.
D. Incorrect. $5\frac{9}{8}$ is not the correct mixed number; the fractional part of the number is still improper. The correct answer is $6\frac{1}{8}$.

## Summary

Adding fractions with like denominators involves adding the numerators and keeping the denominator the same. Always simplify the answer. To add fractions with unlike denominators, first find a common denominator. The least common denominator is easiest to use. The least common multiple can be used as the least common denominator. Adding mixed numbers involves adding the fractional parts, adding the whole numbers, and then recombining them as a mixed number.

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# 10.2.3.2: Subtracting Fractions and Mixed Numbers

**Learning Objectives**

- Subtract fractions with like and unlike denominators.
- Subtract mixed numbers without regrouping.
- Subtract mixed numbers with regrouping.
- Solve application problems that require the subtraction of fractions or mixed numbers.

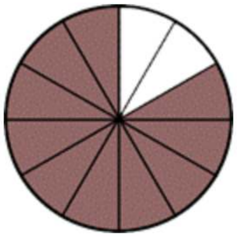
## Introduction

Sometimes subtraction, rather than addition, is required to solve problems that involve fractions. Suppose you are making pancakes and need $4\frac{1}{2}$ cups of flour but you only have $2\frac{3}{4}$ cups. How many additional cups will you have to get to make the pancakes? You can solve this problem by subtracting the mixed numbers.

## Subtracting Fractions

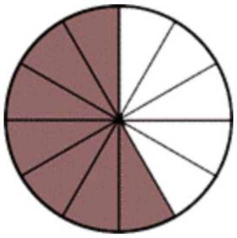
The most simple fraction subtraction problems are those that have two proper fractions with a **common denominator**. That is, each denominator is the same. The process is just as it is for addition of fractions with **like denominators**, except you subtract! You subtract the second numerator from the first and keep the denominator the same.

Imagine that you have a cake with equal-sized pieces. Some of the cake has already been eaten, so you have a fraction of the cake remaining. You could represent the cake pieces with the picture below.



The cake is cut into 12 equal pieces to start. Two are eaten, so the remaining cake can be represented with the fraction $\frac{10}{12}$. If 3 more pieces of cake are eaten, what fraction of the cake is left? You can represent that problem with the expression $\frac{10}{12}-\frac{3}{12}$.

If you subtract 3 pieces, you can see below that $\frac{7}{12}$ of the cake remains.



You can solve this problem without the picture by subtracting the numerators and keeping the denominator the same:

$$\frac{10}{12}-\frac{3}{12}=\frac{7}{12}$$

**Subtracting Fractions with Like Denominators**

If the denominators (bottoms) of the fractions are the same, subtract the numerators (tops) and keep the denominator the same. Remember to simplify the resulting fraction, if possible.

**Example**

| | |
|---|---|
| $\mathbf{\frac{6}{7}-\frac{1}{7}}$ | **Subtract.** |

**Solution**

| | |
|---|---|
| $\frac{6-1}{7}=\frac{5}{7}$ | Both fractions have a denominator of 7, so subtract the numerators and keep the same denominator. |

$\frac{6}{7}-\frac{1}{7}=\frac{5}{7}$

**Example**

| | |
|---|---|
| $\mathbf{\frac{5}{9}-\frac{2}{9}}$ | **Subtract. Simplify the answer.** |

**Solution**

| | |
|---|---|
| $\frac{5}{9}-\frac{2}{9}=\frac{3}{9}$ | The fractions have a **like denominator**, also known as a common denominator, so subtract the numerators. |
| $\frac{3\div 3}{9\div 3}=\frac{1}{3}$ | Simplify the fraction. |

$\frac{5}{9}-\frac{2}{9}=\frac{1}{3}$

If the denominators are not the same (they have **unlike denominators**), you must first rewrite the fractions with a common denominator. The **least common denominator**, which is the least common multiple of the denominators, is the most efficient choice, but any common denominator will do. Be sure to check your answer to be sure that it is in simplest form. You can use prime factorization to find the **least common multiple** (LCM), which will be the least common denominator (LCD). See the example below.

**Example**

| | |
|---|---|
| $\mathbf{\frac{1}{5}-\frac{1}{6}}$ | **Subtract. Simplify the answer.** |

**Solution**

| | |
|---|---|
| $5\cdot 6=30$ | The fractions have unlike denominators, so you need to find a common denominator. Recall that a common denominator can be found by multiplying the two denominators together. |
| $\frac{1}{5}\cdot\frac{6}{6}=\frac{6}{30}$<br>$\frac{1}{6}\cdot\frac{5}{5}=\frac{5}{30}$ | Rewrite each fraction as an equivalent fraction with a denominator of 30. |
| $\frac{6}{30}-\frac{5}{30}=\frac{1}{30}$ | Subtract the numerators. Simplify the answer if needed. |

$\frac{1}{5}-\frac{1}{6}=\frac{1}{30}$

The example below shows using multiples to find the least common multiple, which will be the least common denominator.

**Example**

| | |
|---|---|
| $\mathbf{\frac{5}{6}-\frac{1}{4}}$ | **Subtract. Simplify the answer.** |

**Solution**

| | |
|---|---|
| Multiples of 6 include 6, **12**, 18, 24<br>Multiples of 4 include 4, 8, **12**, 16, 20 | Find the least common multiple of the denominators. |

| | |
|---|---|
| 12 is the least common multiple of 6 and 4. | This is the least common denominator. |
| $\frac{5}{6}\cdot\frac{2}{2}=\frac{10}{12}$<br>$\frac{1}{4}\cdot\frac{3}{3}=\frac{3}{12}$ | Rewrite each fraction with a denominator of 12. |
| $\frac{10}{12}-\frac{3}{12}=\frac{7}{12}$ | Subtract the fractions. Simplify the answer if needed. |

$\frac{5}{6}-\frac{1}{4}=\frac{7}{12}$

### Exercise

$\frac{2}{3}-\frac{1}{6}$ Subtract and simplify the answer.

A. $\frac{1}{3}$
B. $\frac{3}{6}$
C. $\frac{5}{6}$
D. $\frac{1}{2}$

**Answer**

A. Incorrect. Find a least common denominator and subtract; then simplify. The correct answer is $\frac{1}{2}$.
B. Incorrect. Simplify the fraction. The correct answer is $\frac{1}{2}$.
C. Incorrect. Subtract, don't add, the fractions. The correct answer is $\frac{1}{2}$
D. Correct. $\frac{4}{6}-\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$

## Subtracting Mixed Numbers

Subtracting mixed numbers works much the same way as adding mixed numbers. To subtract mixed numbers, subtract the whole number parts of the mixed numbers and then subtract the fraction parts in the mixed numbers. Finally, combine the whole number answer and the fraction answer to express the answer as a mixed number.

### Example

| | |
|---|---|
| $6\frac{4}{5}-3\frac{1}{5}$ | **Subtract. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $6-3=3$<br>$\frac{4}{5}-\frac{1}{5}=\frac{3}{5}$ | Subtract the whole numbers and subtract the fractions. |
| $3\frac{3}{5}$ | Combine the fraction and the whole number. Make sure the fraction in the mixed number is simplified. |

$6\frac{4}{5}-3\frac{1}{5}=3\frac{3}{5}$

Sometimes it might be easier to express the mixed number as an improper fraction first and then solve. Consider the example below.

### Example

| | |
|---|---|
| $8\frac{1}{3}-4\frac{2}{3}$ | **Subtract. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $8\frac{1}{3}=\frac{8\cdot 3+1}{3}=\frac{24+1}{3}=\frac{25}{3}$<br>$4\frac{2}{3}=\frac{4\cdot 3+2}{3}=\frac{12+2}{3}=\frac{14}{3}$ | Write each mixed number as an improper fraction. |
| $\frac{25}{3}-\frac{14}{3}=\frac{11}{3}$ | Since the fractions have a like denominator, subtract the numerators. |
| $\frac{11}{3}=3\frac{2}{3}$ | Write the answer as a mixed number. Divide 11 by 3 to get 3 with a remainder of 2. |

$8\frac{1}{3}-4\frac{2}{3}=3\frac{2}{3}$

Since addition is the inverse operation of subtraction, you can check your answer to a subtraction problem with addition. In the example above, if you add $4\frac{2}{3}$ to your answer of $3\frac{2}{3}$, you should get $8\frac{1}{3}$.

$$\begin{aligned} 4\frac{2}{3}+3\frac{2}{3}\\ 4+3+\frac{2}{3}+\frac{2}{3}\\ 7+\frac{4}{3}\\ 7+1\frac{1}{3}\\ 8\frac{1}{3} \end{aligned}$$

Sometimes you have to find a common denominator in order to solve a mixed number subtraction problem.

✓ Example

| | |
|---|---|
| $\mathbf{7\frac{1}{2}-2\frac{1}{3}}$ | **Subtract.Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $2\cdot 3=6$ | Recall that a common denominator can easily be found by multiplying the denominators together. |
| $\frac{1}{2}\cdot\frac{3}{3}=\frac{3}{6}$<br>$\frac{1}{3}\cdot\frac{2}{2}=\frac{2}{6}$ | Rewrite each fraction using the common denominator 6. |
| $\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$ | Subtract the fractions. |
| $7-2=5$ | Subtract the whole numbers. |
| $5\frac{1}{6}$ | Combine the whole number and the fraction. |

$7\frac{1}{2}-2\frac{1}{3}=5\frac{1}{6}$

? Exercise

$9\frac{4}{5}-4\frac{2}{3}$

Subtract. Simplify the answer and write it as a mixed number.

A. $\frac{2}{15}$
B. $5\frac{2}{15}$
C. $4\frac{7}{15}$
D. 5

**Answer**

A. Incorrect. Subtract the whole numbers, too. The correct answer is $5\frac{2}{15}$.
B. Correct. $9-4=5$ ; $\frac{4}{5}-\frac{2}{3}=\frac{12}{15}-\frac{10}{15}=\frac{2}{15}$ . Combining them gives $5\frac{2}{15}$.

C. Incorrect. Subtract, don’t add, the fractions. The correct answer is $5\frac{2}{15}$.
D. Incorrect. Subtract the fractions as well as the whole numbers. The correct answer is $5\frac{2}{15}$.

## Subtracting Mixed Numbers with Regrouping

Sometimes when subtracting mixed numbers, the fraction part of the second mixed number is larger than the fraction part of the first number. Consider the problem: $7\frac{1}{6}-3\frac{5}{6}$. The standard procedure would be to subtract the fractions, but $\frac{1}{6}-\frac{5}{6}$ would result in a negative number. You don’t want that! You can regroup one of the whole numbers from the first number, writing the first mixed number in a different way:

$$7\frac{1}{6}=7+\frac{1}{6}=6+1+\frac{1}{6}$$
$$6+\frac{6}{6}+\frac{1}{6}=6+\frac{7}{6}=6\frac{7}{6}$$

Now, you can write an equivalent problem to the original:

$$6\frac{7}{6}-3\frac{5}{6}$$

Then, you just subtract like you normally subtract mixed numbers:

$$6-3=3$$
$$\frac{7}{6}-\frac{5}{6}=\frac{2}{6}=\frac{1}{3}$$

So, the answer is $3\frac{1}{3}$.

As with many fraction problems, you may need to find a common denominator. Remember that a key part of adding and subtracting fractions and mixed numbers is making sure to have a common denominator as a first step! In the example below, the original fractions do not have a like denominator. You need to find one before proceeding with the next steps.

✓ Example

| | |
|---|---|
| $\mathbf{7\frac{1}{5}-3\frac{1}{4}}$ | **Subtract. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| Multiples of 5 include 5, 10, 15, **20**, 25<br>Multiples of 4 include 4, 8, 12, 16, **20**, 24 | Find a least common denominator. 20 is the least common multiple, so use it for the least common denominator. |
| $\frac{1}{5}\cdot\frac{4}{4}=\frac{4}{20}$<br>$\frac{1}{4}\cdot\frac{5}{5}=\frac{5}{20}$ | Rewrite each fraction using the common denominator. |
| $7\frac{4}{20}-3\frac{5}{20}$ | Write the expression using the mixed numbers with the like denominator. |
| $7\frac{4}{20}=6+1+\frac{4}{20}$<br>$6+\frac{20}{20}+\frac{4}{20}$<br>$6+\frac{24}{20}$<br>$6\frac{24}{20}$ | Since the second fraction part, $\frac{5}{20}$, is larger than the first fraction part, $\frac{4}{20}$, regroup one of the whole numbers and write it as $\frac{20}{20}$. |
| $7\frac{4}{20}-3\frac{5}{20}$<br>$6\frac{24}{20}-3\frac{5}{20}$ | Rewrite the subtraction expression using the equivalent fractions. |
| $6-3=3$<br>$\frac{24}{20}-\frac{5}{20}=\frac{19}{20}$ | Subtract the whole numbers, subtract the fractions. |
| $3\frac{19}{20}$ | Combine the whole number and the fraction. |

$$7\frac{1}{5}-3\frac{1}{4}=3\frac{19}{20}$$

Sometimes a mixed number is subtracted from a whole number. In this case, you can also rewrite the whole number as a mixed number in order to perform the subtraction. You use an equivalent mixed number that has the same denominator as the fraction in the other mixed number.

✓ Example

| | |
|---|---|
| $\mathbf{8-4\frac{2}{5}}$ | **Subtract. Simplify the answer and write as a mixed number.** |

**Solution**

| | |
|---|---|
| $8=7+1$<br>$7+\frac{5}{5}$<br>$7\frac{5}{5}$ | Regroup one from the whole number and write it as $\frac{5}{5}$. |
| $7\frac{5}{5}-4\frac{2}{5}$ | Rewrite the subtraction expression using the equivalent fractions. |
| $7-4=3$<br>$\frac{5}{5}-\frac{2}{5}=\frac{3}{5}$ | Subtract the whole numbers, subtract the fractions. |
| $3\frac{3}{5}$ | Combine the whole number and the fraction. |

$8-4\frac{2}{5}=3\frac{3}{5}$

Subtracting Mixed Numbers

If the fractional part of the mixed number being subtracted is larger than the fractional part of the mixed number from which it is being subtracted, or if a mixed number is being subtracted from a whole number, follow these steps:

1. Subtract 1 from the whole number part of the mixed number being subtracted.
2. Add that 1 to the fraction part to make an improper fraction. For example: $7\frac{2}{3}=6+\frac{3}{3}+\frac{2}{3}=6\frac{5}{3}$
3. Then, subtract as with any other mixed numbers.

Alternatively, you can change both numbers to improper fractions and then subtract.

? Exercise

$15-13\frac{1}{4}$ Subtract. Simplify the answer and write as a mixed number.

A. $2\frac{1}{4}$
B. $28\frac{1}{4}$
C. $1\frac{3}{4}$
D. $2\frac{3}{4}$

**Answer**

A. Incorrect. This is the answer to $15\frac{1}{4}-13$. The fraction has to be subtracted from the 15. The correct answer is $1\frac{3}{4}$.
B. Incorrect. Subtract, don't add, the quantities. The correct answer is $1\frac{3}{4}$.
C. Correct. $14\frac{4}{4}-13\frac{1}{4}=1\frac{3}{4}$
D. Incorrect. Subtract 1 from the whole number when rewriting it as a mixed number. The correct answer is $1\frac{3}{4}$.

## Subtracting Fractions and Mixed Numbers to Solve Problems

Knowing how to subtract fractions and mixed numbers is useful in a variety of situations. When reading problems, look for key words that indicate that the problem can be solved using subtraction.

### Example

**Sherry loves to quilt, and she frequently buys fabric she likes when she sees it. She had purchased 5 yards of blue print fabric and decided to use $2\frac{3}{8}$ yards of it in a quilt. How much of the blue print fabric will she have left over after making the quilt?**

#### Solution

| | |
|---|---|
| $5-2\frac{3}{8}$ | Write an expression using subtraction to describe the situation. |
| $4\frac{8}{8}-2\frac{3}{8}$ | Rewrite the whole number as a mixed number. |
| $4\frac{8}{8}-2\frac{3}{8}=2\frac{5}{8}$ | Subtract. Check that the mixed number is simplified. |

Sherry has $2\frac{5}{8}$ yards of blue print fabric left over.

### Example

**Pilar and Farouk are training for a marathon. On a recent Sunday, they both completed a run. Farouk ran $12\frac{7}{8}$ miles and Pilar ran $14\frac{3}{4}$ miles. How many more miles did Pilar run than Farouk?**

#### Solution

| | |
|---|---|
| $14\frac{3}{4}-12\frac{7}{8}$ | Write an expression using subtraction to describe the situation. |
| $14\frac{6}{8}-12\frac{7}{8}$ | Rewrite the mixed numbers using the least common denominator. |
| $14\frac{6}{8}$<br>$13+1+\frac{6}{8}$<br>$13+\frac{8}{8}+\frac{6}{8}$<br>$13\frac{14}{8}$ | Since the fraction part of the second mixed number is larger than the fraction part of the first mixed number, regroup one as a fraction and rewrite the first mixed number. |
| $13\frac{14}{8}-12\frac{7}{8}$ | Write the subtraction expression in its new form. |
| $1\frac{7}{8}$ | Subtract. |

Pilar ran $1\frac{7}{8}$ miles more than Farouk.

### Example

**Mike and Jose are painting a room. Jose used $\frac{2}{3}$ of a can of paint and Mike used $\frac{1}{2}$ of a can of paint. How much more paint did Jose use? Write the answer as a fraction of a can.**

#### Solution

| | |
|---|---|
| $\frac{2}{3}-\frac{1}{2}$ | Write an expression using subtraction to describe the situation. |
| $\frac{2\cdot2}{3\cdot2}=\frac{4}{6}$<br>$\frac{1\cdot3}{2\cdot3}=\frac{3}{6}$ | Rewrite the fractions using a common denominator. |
| $\frac{4}{6}-\frac{3}{6}=\frac{1}{6}$ | Subtract. Check that the fraction is simplified. |

Jose used $\frac{1}{6}$ of a can more paint than Mike.

**? Exercise**

Mariah's sunflower plant grew $18\frac{2}{3}$ inches in one week. Her tulip plant grew $3\frac{3}{4}$ inches in one week. How many more inches did the sunflower grow in a week than the tulip?

A. $22\frac{5}{12}$ inches
B. $15\frac{1}{12}$ inches
C. $15$ inches
D. $14\frac{11}{12}$ inches

**Answer**

A. Incorrect. Subtract, don't add, the fractions. The correct answer is $14\frac{11}{12}$ inches.
B. Incorrect. Subtract $\frac{2}{3}-\frac{3}{4}$, not $\frac{3}{4}-\frac{2}{3}$. The correct answer is $14\frac{11}{12}$ inches.
C. Incorrect. Subtract the fractions as well as the whole numbers in the mixed numbers. The correct answer is $14\frac{11}{12}$ inches.
D. Correct. $17\frac{20}{12}-3\frac{9}{12}=14\frac{11}{12}$

## Summary

Subtracting fractions and mixed numbers combines some of the same skills as adding whole numbers and adding fractions and mixed numbers. When subtracting fractions and mixed numbers, first find a common denominator if the denominators are not alike, rewrite each fraction using the common denominator, and then subtract the numerators. When subtracting mixed numbers, if the fraction in the second mixed number is larger than the fraction in the first mixed number, rewrite the first mixed number by regrouping one whole as a fraction. Alternatively, rewrite all fractions as improper fractions and then subtract. This process is also used when subtracting a mixed number from a whole number.

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## SECTION OVERVIEW

# 10.3: Decimals

## 10.3.1: Introduction to Decimals

## 10.3.2: Decimal Operations

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# 10.3.1: Introduction to Decimals

# 10.3.1.1: Decimals and Fractions

**Learning Objectives**

- Read and write numbers in decimal notation.
- Write decimals as fractions.
- Write fractions as decimals.

## Introduction

In addition to fraction notation, decimal notation is another way to write numbers between 0 and 1. Decimals can also be used to write numbers between any two whole numbers. For example, you may have to write a check for $2,003.38. Or, in measuring the length of a room, you may find that the length is between two whole numbers, such as 35.24 feet. In this topic, you will focus on reading and writing decimal numbers, and rewriting them in fraction notation.

To read or write numbers written in decimal notation, you need to know the **place value** of each digit, that is, the value of a digit based on its position within a number. With decimal numbers, the position of a numeral in relation to the decimal point determines its place value. For example, the place value of the 4 in 45.6 is in the *tens* place, while the place value of 6 in 45.6 is in the *tenths* place.

## Decimal Notation

**Decimal numbers** are numbers whose place values are based on 10s. Whole numbers are actually decimal numbers that are greater than or equal to zero. The place-value chart can be extended to include numbers less than one, which are sometimes called **decimal fractions**. A decimal point is used to separate the whole number part of the number and the fraction part of the number.

Let’s say you are measuring the length of a driveway and find that it is 745 feet. You would say this number as seven hundred forty-five. Then, a more accurate measurement shows that it is 745.36 feet. Let’s place this number in a place-value chart.

What you want to examine now are the place values of the decimal part, which are the numbers 3 and 6 in the chart below.

| Decimal Numbers | Hundreds | Tens | Ones | Decimal Point | Tenths | Hundredths |
|---|---|---|---|---|---|---|
| 745.36 | 7 | 4 | 5 | . | **3** | **6** |

Notice how the place-value names start from the decimal point. To the left of the decimal point are the ones, tens, and hundreds places, where you put digits that represent whole numbers that are greater than or equal to zero. To the right of the decimal point are the tenths and hundredths, where you put digits that represent numbers that are fractional parts of one, numbers that are more than zero and less than one.

Again, the place value of a number depends on how far away it is from the decimal point. This is evident in the chart below, where each number has the digit "4" occupying a different place value.

| Decimal Numbers | Thousands | Hundreds | Tens | Ones | Decimal Point | Tenths | Hundredths | Thousandths |
|---|---|---|---|---|---|---|---|---|
| 0.004 | | | | 0 | . | 0 | 0 | 4 |
| 0.04 | | | | 0 | . | 0 | 4 | |
| 0.4 | | | | 0 | . | 4 | | |
| 4 | | | | 4 | . | | | |
| 40 | | | 4 | 0 | . | | | |
| 400 | | 4 | 0 | 0 | . | | | |
| 4000 | 4 | 0 | 0 | 0 | . | | | |

Imagine that as a large balloon deflates, the volume of air inside it goes from 1,000 liters, to 100 liters, to 10 liters, to 1 liter. Notice that you're dividing a place value by ten as you go to the right. You divide 100 by 10 to get to the tens place. This is because there are 10 tens in 100. Then, you divide 10 by 10 to get to the ones place, because there are 10 ones in 10.

Now, suppose the balloon continues to lose volume, going from 1 liter, to 0.1 liters, to 0.01 liters, and then to 0.001 liters. Notice that you continue to divide by 10 when moving to decimals. You divide 1 by $10\left(\frac{1}{10}\right)$ to get to the tenths place, which is basically breaking one into 10 pieces. And to get to the hundreds place, you break the tenth into ten more pieces, which results in the fraction $\frac{1}{100}$. The relationship between decimal places and fractions is captured in the table below.

| Word Form | Decimal Notation | Fraction Notation |
|---|---|---|
| one thousand | 1,000 | $\frac{1,000}{1}$ |
| one hundred | 100 | $\frac{100}{1}$ |
| ten | 10 | $\frac{10}{1}$ |
| one | 1 | $\frac{1}{1}$ |
| one tenth | 0.1 | $\frac{1}{10}$ |
| one hundredth | 0.01 | $\frac{1}{100}$ |
| one thousandth | 0.001 | $\frac{1}{1,000}$ |

Consider a number with more digits. Suppose a fisherman has a net full of fish that weighs 1,357.924 kilograms. To write this number, you need to use the thousands place, which is made up of 10 hundreds. You also use the thousandths place, which is $\frac{1}{10}$ of a hundredth. In other words, there are ten thousandths in one hundredth.

| Decimal Numbers | no *th* side | | | | | *th* side | | |
|---|---|---|---|---|---|---|---|---|
| | Thousands | Hundreds | Tens | Ones | Decimal point | Tenths | Hundredths | Thousandths |
| 1,357.924 | 1 | 3 | 5 | 7 | . | **9** | **2** | **4** |

As you can see, moving from the decimal point to the left is ones, tens, hundreds, thousands, etc. This is the "no *th* side," which are the numbers greater than or equal to one. Moving from the decimal point to the right is tenths, hundredths, thousandths. This is the "*th* side," which are the numbers less than 1.

1,357.924

no *th* side *th* side

> The pattern going to the right or the left from the decimal point is the same, but there are two big differences:
>
> 1. The place values to the right of the decimal point all end in "th".
> 2. There is no such thing as "one-ths." From your work with fractions, you know that 5 and $\frac{5}{1}$ are the same.

**✓ Example**

**What is the place value of 8 in 4,279.386?**

### Solution

Write the number in a place-value chart. Read the value of the 8 from the chart.

| Decimal Numbers | no *th* side | | | | | *th* side | | |
|---|---|---|---|---|---|---|---|---|
| | Thousands | Hundreds | Tens | Ones | Decimal point | Tenths | Hundredths | Thousandths |

|  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|
| 4,279.386 | 4 | 2 | 7 | 9 | . | 3 | **8** | 6 |

In the number 4,279.386, the 8 is in the hundredths place.

**Exercise**

What is the place value of the 7 in 324.2671?

A. thousands
B. thousandths
C. hundreds
D. hundredths

**Answer**

A. Incorrect. The digit 7 is to the right of the decimal point, which means that it is less than one and on the *th* side. The correct answer is thousandths.
B. Correct. The digit 7 is three decimal places to the right of the decimal point, which means that it is in the thousandths place.
C. Incorrect. The digit 7 is three decimal places to the right of the decimal point, which means that it is in the thousandths place.
D. Incorrect. The digit 7 is three decimal places to the right of the decimal point, which means that it is in the thousandths place.

## Reading Decimals

The easiest way to read a decimal number is to read the decimal fraction part as a fraction. (Don't simplify the fraction though.) Suppose you have 0.4 grams of yogurt in a cup. You would say, "4 tenths of a gram of yogurt," as the 4 is in the tenths place.

Note that the denominator of the fraction written in fraction form is always a power of ten, and the number of zeros in the denominator is the same as the number of decimal places to the right of the decimal point. See the examples in the table below for further guidance.

| Decimal Notation | Fraction Notation | Word Form |
|---|---|---|
| 0.5 | $\frac{5}{10}$ | five tenths |
| 0.34 | $\frac{34}{100}$ | thirty-four hundredths |
| 0.896 | $\frac{896}{1{,}000}$ | eight hundred ninety-six thousandths |

Notice that 0.5 has *one* decimal place. Its equivalent fraction, $\frac{5}{10}$, has a denominator of 10—which is 1 followed by *one* zero. In general, when you are converting decimals to fractions, the denominator is always 1, followed by the number of zeros that correspond to the number of decimal places in the original number.

Another way to determine which number to place in the denominator is to use the place value of the last digit without the "ths" part. For example, if the number is 1.458, the 8 is in the thousandths place. Take away the "ths" and you have a thousand, so the number is written as $1\frac{458}{1000}$.

**Example**

**Write 0.68 in word form.**

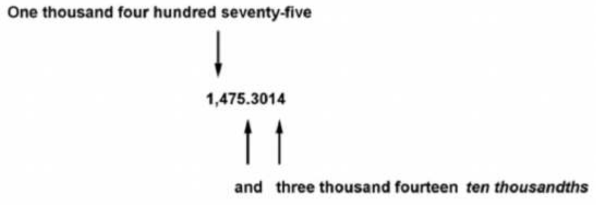
**Solution**

|  |  |
|---|---|
| $0.68 = \frac{68}{100}$ =sixty-eight hundredths | Note that the number is read as a fraction.<br>Also note that the denominator has 2 zeros, the same as the number of decimal places in the original number. |

The number 0.68 in word form is sixty-eight hundredths.

Recall that a **mixed number** is a combination of a whole number and a fraction. In the case of a decimal, a mixed number is also a combination of a whole number and a fraction, where the fraction is written as a decimal fraction.

To read mixed numbers, say the whole number part, the word “and” (representing the decimal point), and the number to the right of the decimal point, followed by the name and the place value of the last digit. You can see this demonstrated in the diagram below, in which the last digit is in the ten thousandths place.



Another way to think about this is with money. Suppose you pay $ 15,264.25 for a car. You would read this as *fifteen thousand, two hundred sixty-four dollars and twenty-five cents*. In this case, the “cents” means “hundredths of a dollar,” so this is the same as saying *fifteen thousand, two hundred sixty-four and twenty-five hundredths*. A few more examples are shown in the table below.

| Decimal Notation | Fraction Notation | Word Form |
|---|---|---|
| 9.4 | $9\frac{4}{10}$ | Nine and four tenths |
| 87.49 | $87\frac{49}{100}$ | Eighty-seven and forty-nine hundredths |
| 594.236 | $594\frac{236}{1000}$ | Five hundred ninety-four and two hundred thirty-six thousandths |

## ✓ Example

**Write 4.379 in word form.**

### Solution

| | |
|---|---|
| $4.379 = 4\frac{379}{1,000}$ = four and three hundred seventy-nine thousandths | The decimal fraction is read as a fraction.<br>Note that the denominator has 3 zeros, the same as the number of decimal places in the original number. |

The number 4.379 in word form is four and three hundred seventy-nine thousandths.

## ? Exercise

Write 2.364 in word form.

A. two and three hundred sixty-four hundredths
B. two and three hundred sixty-four thousandths
C. two thousand three hundred sixty-four
D. three hundred sixty-four tenths and two

**Answer**

A. two and three hundred sixty-four hundredths

Incorrect. You indicated the wrong decimal place in your answer. The correct answer is two and three hundred sixty-four thousandths.

B. Correct. 2.364 is the same as $2\frac{364}{1,000}$, so in addition to the whole number 2, you have three hundred sixty-four thousandths.

C. Incorrect. You ignored the decimal point. The correct answer is a decimal; in this case, two and three hundred sixty-four thousandths.
D. Incorrect. You indicated the wrong decimal place in your answer, and the whole number part should be mentioned before the decimal part. The correct answer is two and three hundred sixty-four thousandths.

## Writing Decimals as Simplified Fractions

As you have seen above, every decimal can be written as a fraction. To convert a decimal to a fraction, place the number after the decimal point in the numerator of the fraction and place the number 10,100, or 1,000, or another power of 10 in the denominator. For example, 0.5 would be written as $\frac{5}{10}$. You'll notice that this fraction can be further simplified, as $\frac{5}{10}$ reduces to $\frac{1}{2}$, which is the final answer.

Let's get more familiar with this relationship between decimal places and zeros in the denominator by looking at several examples. Notice that in each example, the number of decimal places is different.

✓ Example

**Write 0.6 as a simplified fraction.**

**Solution**

| | |
|---|---|
| $0.6=\frac{6}{10}$ | The last decimal place is tenths, so use 10 for your denominator. The number of zeros in the denominator is always the same as the number of decimal places in the original decimal. |
| $\frac{6\div 2}{10\div 2}=\frac{3}{5}$ | Simplify the fraction. |

$0.6=\frac{3}{5}$

Let's look at an example in which a number with two decimal places is written as a fraction.

✓ Example

**Write 0.64 as a simplified fraction.**

**Solution**

| | |
|---|---|
| $0.64=\frac{64}{100}$ | The last decimal place is hundredths, so use 100 for your denominator. The number of zeros in the denominator is always the same as the number of decimal places in the original decimal. |
| $\frac{64\div 4}{100\div 4}=\frac{16}{25}$ | Simplify the fraction. |

$0.64=\frac{16}{25}$

Now, examine how this is done in the example below using a decimal with digits in three decimal places.

✓ Example

**Write 0.645 as a simplified fraction.**

**Solution**

| | |
|---|---|
| $0.645=\frac{645}{1{,}000}$ | Note that there are 3 zeros in the denominator, which is the same as the number of decimal places in the original decimal. |
| $\frac{645\div 5}{1{,}000\div 5}=\frac{129}{200}$ | Simplify the fraction. |

$0.645 = \frac{129}{200}$

You can write a fraction as a decimal even when there are zeros to the right of the decimal point. Here is an example in which the only digit greater than zero is in the thousandths place.

✓ Example

**Write 0.007 as a simplified fraction.**

**Solution**

| | |
|---|---|
| $0.007 = \frac{7}{1{,}000}$ | Note that 7 is in the thousandths place, so you write 1,000 in the denominator. The number of zeros in the denominator is always the same as the number of decimal places in the original decimal.<br>The fraction cannot be simplified further. |

$0.007 = \frac{7}{1{,}000}$

When writing decimals greater than 1, you only need to change the decimal part to a fraction and keep the whole number part. For example, 6.35 can be written as $6\frac{35}{100}$.

✓ Example

**Write 8.65 as a simplified mixed fraction.**

**Solution**

| | |
|---|---|
| $8.65 = 8\frac{65}{100} = 8\frac{13}{20}$ | Rewrite 0.65 as $\frac{65}{100}$.<br>Note that the number of zeros in the denominator is two, which is the same as the number of decimal places in the original decimal.<br>Then simplify $\frac{65}{100}$ by dividing numerator and denominator by 5. |

$8.65 = 8\frac{13}{20}$

? Exercise

Write 0.25 as a fraction.

A. $\frac{2}{5}$
B. $\frac{1}{4}$
C. $\frac{4}{1}$
D. $\frac{5}{2}$

**Answer**

A. Incorrect. You may have put the digit from the tenths place in the numerator, and the digit from the hundredths place in the denominator. The correct answer is $\frac{1}{4}$.

B. Correct. The number 0.25 can be written as $\frac{25}{100}$, which reduces to $\frac{1}{4}$.

C. Incorrect. You probably confused the numerator and the denominator. The correct answer is $\frac{1}{4}$.

D. Incorrect. You may have put the digit from the tenths place in the denominator, and the digit from the hundredths place in the numerator. The correct answer is $\frac{1}{4}$.

## Writing Fractions as Decimals

Just as you can write a decimal as a fraction, every fraction can be written as a decimal. To write a fraction as a decimal, divide the numerator (top) of the fraction by the denominator (bottom) of the fraction. Use long division, if necessary, and note where to place the decimal point in your answer. For example, to write $\frac{3}{5}$ as a decimal, divide 3 by 5, which will result in 0.6.

**✓ Example**

**Write $\frac{1}{2}$ as a decimal.**

**Solution**

| | |
|---|---|
| 0.5<br>2\longdiv1.0<br>$-1.0$<br>0 | Using long division, you can see that dividing 1 by 2 results in 0.5. |

$\frac{1}{2}=0.5$

Note that you could also have thought about the problem like this: $\frac{1}{2}=\frac{?}{10}$, and then solved for?. One way to think about this problem is that 10 is five times greater than 2, so will have to be five times greater than 1. What number is five times greater than 1? Five is, so the solution is $\frac{1}{2}=\frac{5}{10}$.

Now look at a more complex example, where the final digit of the answer is in the thousandths place.

**✓ Example**

**Write $\frac{3}{8}$ as a decimal.**

**Solution**

| | |
|---|---|
| 0.375<br>8\longdiv3.000<br>$-24$<br>60<br>$-56$<br>40<br>$-40$<br>0 | Using long division, you can see that dividing 3 by 8 results in 0.375. |

$\frac{3}{8}=0.375$

Converting from fractions to decimals sometimes results in answers with decimal numbers that begin to repeat. For example, $\frac{2}{3}$ converts to 0.666, a repeating decimal, in which the 6 repeats infinitely. You would write this as $0.\bar{6}$, with a bar over the first decimal digit to indicate that the 6 repeats. Look at this example of a problem in which two consecutive digits in the answer repeat.

**✓ Example**

**Convert $\frac{4}{11}$ to a decimal.**

**Solution**

| | |
|---|---|
| $0.3636$<br>$11$\longdiv$4.0000$<br>$-33$<br>$70$<br>$-66$<br>$40$<br>$-33$<br>$70$<br>$-66$<br>$4$ | Using long division, you can see that dividing 4 by 11 results in 0.36 repeating. As a result, this is written with a line over it as $0.\overline{36}$. |

$\frac{4}{11} = 0.\overline{36}$

With numbers greater than 1, keep the whole number part of the mixed number as the whole number in the decimal. Then use long division to convert the fraction part to a decimal. For example, $2\frac{3}{20}$ can be written as 2.15.

### ✓ Example

**Convert $2\frac{1}{4}$ to a decimal.**

**Solution**

| | |
|---|---|
| $0.25$<br>$4$\longdiv$1.00$<br>$-8$<br>$20$<br>$-20$<br>$0$ | Knowing that the whole number 2 will remain the same during the conversion, focus only on the decimal part. Using long division, you can see that dividing 1 by 4 results in 0.25. |
| $2 + 0.25 = 2.25$ | Now bring back the whole number 2, and the resulting fraction is 2.25. |

$2\frac{1}{4} = 2.25$

### Tips on Converting Fractions to Decimals

To write a fraction as a decimal, divide the numerator (top) of the fraction by the denominator (bottom) of the fraction.

In the case of repeating decimals, write the repeating digit or digits with a line over it. For example, 0.333 repeating would be written as $0.\overline{3}$.

## Summary

Decimal notation is another way to write numbers that are less than 1 or that combine whole numbers with decimal fractions, sometimes called mixed numbers. When you write numbers in decimal notation, you can use an extended place-value chart that includes positions for numbers less than one. You can write numbers written in fraction notation (fractions) in decimal notation (decimals), and you can write decimals as fractions. You can always convert between fractional notation and decimal notation.

---

# 10.3.1.2: Ordering and Rounding Decimals

**Learning Objectives**

- Use a number line to assist with comparing decimals.
- Compare decimals, beginning with their digits from left to right.
- Use < or > to compare decimals.
- Round a given decimal to a specified place.

## Introduction

**Decimal numbers** are a combination of whole numbers and numbers between whole numbers. It is sometimes important to be able to compare decimals to know which is greater. For example, if someone ran the 100-meter dash in 10.57 seconds, and someone else ran the same race in 10.67 seconds, you can compare the decimals to determine which time is faster. Knowing how to compare decimals requires an understanding of decimal place value, and is similar to comparing whole numbers.

When working with decimals, there are times when a precise number isn't needed. When that's true, rounding decimal numbers is helpful. For example, if the pump at the gas station shows that you filled a friend's car with 16.478 gallons of gasoline, you may want to round the number and just tell her that you filled it with 16.5 gallons.

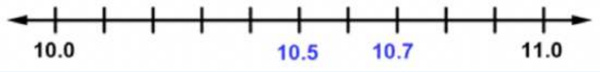
## Comparing Decimals

You can use a number line to compare decimals. The number that is further to the right is greater. Examine this method in the example below.

**Example**

**Use < or > for [] to write a true sentence: 10.5 [] 10.7.**

**Solution**



Here, the digits in the *tenths* places differ. The numbers are both plotted on a number line ranging from 10.0 to 11.0. Because 10.5 is to the left of 10.7 on the number line, 10.5 is less than (<) 10.7.

10.5 < 10.7

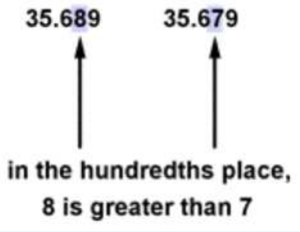
Another approach to comparing decimals is to compare the digits in each number, beginning with the greatest place value, which is on the left. When one digit in a decimal number is greater than the corresponding digit in the other number, then that decimal number is greater.

For example, first compare the tenths digits. If they are equal, move to the hundredths place. If these digits are not equal, the decimal with the greater digit is the greater decimal number. Observe how this is done in the examples below.

**Example**

**Use < or > for [] to write a true sentence: 35.689 [] 35.679.**

**Solution**



Here, the numbers in the tens, ones, and tenths places, 35.6, are the same. However, the digits in the hundredths places differ. Because 8 is greater than 7, 35.689 is greater than 35.679.

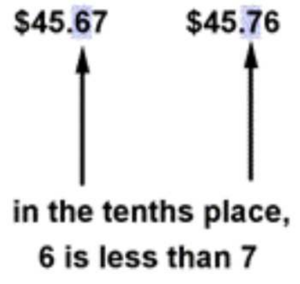
35.689 > 35.679

If more than two digits in the two numbers differ, focus on the digit in the greatest place value. Look at this example in which two sums of money are compared.

✓ Example

**Use < or > for [] to write a true sentence: $ 45.67 [] $ 45.76.**

**Solution**



Here, the place value that determines which amount is greater is not the hundredths place, but the tenths place. Because 6 tenths is less than 7 tenths, $45.67 is less than $ 45.76.

$ 45.67 < $ 45.76

If one number has more decimal places than another, you may use 0s as placeholders in the number with fewer decimal places to help you compare. For example, if you are comparing 4.75 and 4.7, you may find it helpful to write 4.7 as 4.70 so that each number has three digits. Note that adding this extra 0 does not change the value of the decimal; you are adding 0 hundredths to the number. You can add placeholder 0s as long as you remember to add the zeros at the end of the number, to the right of the decimal point.

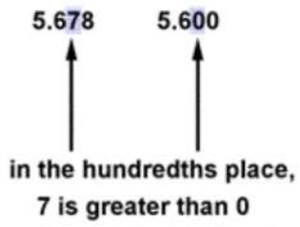
This is demonstrated in the example below.

✓ Example

**Use < or > for [] to write a true sentence: 5.678 [] 5.6.**

**Solution**

Here, you can insert two zeros in the hundredths and thousandths places of 5.6 so that you have the same number of digits for each. Then, you can compare digits. Look for the greatest place value that has different values in each number. Here, it is the hundredths place.



Because 7 is greater than 0, 5.678 is greater than 5.600.

5.678 > 5.6

**Strategies for Comparing Decimals**

- Use a number line to assist with comparing decimals, as you did with whole numbers.
- Compare decimals beginning with their digits from left to right. When two digits are not equal, the one with the greater digit is the larger number.

**Exercise**

Use < or > for [] to write a true sentence: 45.675 [] 45.645.

A. 45.675 < 45.645
B. 45.675 > 45.645

**Answer**

A. Incorrect. Comparing the digits in the hundredths places shows that 7 is greater than 4. As a result, 45.675 > 45.645.
B. Correct. 45.675 > 45.645, because the 7 in 45.675 is greater than the corresponding 4 in 45.645.

## Rounding Decimals

Rounding with decimals is like rounding with whole numbers. As with whole numbers, you round a number to a given place value. Everything to the right of the given place value becomes a zero, and the digit in the given place value either stays the same or increases by one.

With decimals, you can "drop off" the zeros at the end of a number without changing its value. For example, 0.20=0.2, as $\frac{20}{100}$ simplifies to $\frac{2}{10}$. Of course, you cannot drop zeros before the decimal point: $200 \neq 20$.

The zeros that occur at the end of a decimal number are called **trailing zeros**.

Dropping Zeros with Whole Numbers and Decimals

| | |
|---|---|
| Dropping zeros at the end of a whole number changes the value of a number. | $7200 \neq 72$<br>$200,000 \neq 2$ |
| Dropping zeros at the end of a decimal does not change the number's value. | $36.00 = 36$<br>$1.00000 = 1$ |

One way to think of it is to consider the number "thirty-six dollars." This amount can be written equally well one of two ways:

$\$36 = \$36.00$

Any zero at the very end of a decimal number can be dropped:

18.25000=18.2500=18.250=18.25

**Example**

**A sprinter ran a race in 7.354 seconds. What was the sprinter's time, rounded to the nearest tenth of a second?**

**Solution**

| | |
|---|---|
| 7.354 | Look at the first digit to the right of the tenths digit. |
| $7.354 \rightarrow 7.400$ | Since 5=5, round the 3 up to 4. |
| 7.400 | Change all digits to the right of the given place value into zeros. This is an intermediate step that you don't actually write down. |
| 7.4 | Since 0.400=0.4, the zeros are not needed and should be dropped. |

7.354 rounded to the nearest tenth is 7.4

In the example above, the digit next to the selected place value is 5, so you round up. Let’s look at a case in which the digit next to the selected place value is less than 5.

✓ Example

**Round 7.354 to the nearest hundredth.**

**Solution**

| | |
|---|---|
| $7.354$ | Look at the first digit to the right of the hundredths digit. |
| $7.354 \rightarrow 7.350$ | Since 4<5, leave 5 as is. |
| $7.35$ | The zeros to the right of the given place value are not needed and should be dropped. |

7.354 rounded to the nearest hundredth is 7.35.

Sometimes you’re asked to round a decimal number to a place value that is in the whole number part. Remember that you may not drop zeros to the left of the decimal point.

✓ Example

**Round 1,294.6374 to the nearest hundred.**

**Solution**

| | |
|---|---|
| $1,294.6374$ | Look at the first digit to the right of the hundreds place. |
| $1,294.6374 \rightarrow 1,300.000$ | 9 is greater than 5, so round the 2 up to 3. |
| $1,300$ | Zeros to the left of the decimal must be included.<br>Zeros to the right of the decimal can be dropped and should be dropped. |

1,294.6374 rounded to the nearest hundred is 1,300.

Tip on Rounding Decimals

When you round, everything to the right of the given place value becomes a zero, and the digit in the given place value either stays the same or rounds up one. Trailing zeros after the decimal point should be dropped.

? Exercise

Round 10.473 to the nearest tenth.

A. 10.47
B. 10.4
C. 10.473
D. 10.5

**Answer**

A. Incorrect. You rounded to the nearest hundredth, when you should have rounded to the nearest tenth. The correct answer is 10.5.
B. Incorrect. You rounded down when you should have rounded up. The correct answer is 10.5.
C. Incorrect. You did not round this number at all. If you had rounded to the nearest tenth, you would have found the correct answer, 10.5.
D. Correct. You rounded to the nearest tenth, and were right to round up, because the 7 in 10.473 is greater than 5.

## Summary

Ordering, comparing, and rounding decimals are important skills. You can figure out the relative sizes of two decimal numbers by using number lines and by comparing the digits in the same place value of the two numbers. To round a decimal to a specific place value, change all digits to the right of the given place value to zero, and then round the digit in the given place value either up or down. Trailing zeros after the decimal point should be dropped.

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# 10.3.2: Decimal Operations

# 10.3.2.1: Adding and Subtracting Decimals

**Learning Objectives**

- Add two or more decimals.
- Subtract two or more decimals, with and without regrouping.
- Solve application problems that require decimal addition or subtraction.

## Introduction

Since dollars and cents are typically written as decimals, you often need to work with decimals. Knowing how to add and subtract **decimal numbers** is essential when you deposit money to (and withdraw money from) your bank account; perform an incorrect calculation, and you may be costing yourself some cash!

When adding or subtracting decimals, it is essential that you pay attention to the **place value** of the digits in the numbers you are adding or subtracting. This will be the key idea in the discussion that follows. Let's begin with an everyday example that illustrates this idea before moving into more general techniques.

## Adding Decimals

Suppose Celia needs $0.80 to ride the bus from home to her office. She reaches into her purse and pulls out the following coins: 3 quarters, 1 dime and 2 pennies. Does she have enough money to ride the bus?

Take a moment to think about this problem. Does she have enough money? Some people may solve it like this: "I know each quarter is 25¢, so three quarters is 75¢. Adding a dime brings me to 85¢, and then another two pennies is 87¢. So, Celia does have enough money to ride the bus."

This problem provides a good starting point for our conversation because you can use your knowledge about pocket change to understand the basics about how to add decimals. The coins you use every day can all be represented as whole cent values, as shown above. But they can also be represented as decimal numbers, too, because quarters, dimes, nickels, and pennies are each worth less than one whole dollar.

| **Currency** | **Value (cents)** | **Value (dollars)** |
|---|---|---|
| Dollar bill | 100¢ | $1 |
| Quarter | 25¢ | $0.25 |
| Dime | 10¢ | $0.10 |
| Nickel | 5¢ | $0.05 |
| Penny | 1¢ | $0.01 |

Celia has 87¢. You can also write this amount in terms of the number of dollars she has: $0.87. The table below shows a step-by-step approach to adding the coins in terms of cents and also as dollars. As you review the table, pay attention to the place values.

| **Coin Combination** | **Value (cents)** | **Value (dollars)** |
|---|---|---|
| Quarter | 25¢ | $0.25 |
| Quarter | 25¢ | $0.25 |
| Quarter | 25¢ | $0.25 |
| Dime | 10¢ | $0.10 |
| Penny | 1¢ | $0.01 |
| +Penny | 1¢ | $0.01 |
| Eighty-seven cents | 87¢ | $0.87 |

When you add whole numbers, as shown in the **Value (cents)** column above, you line up the numbers so that the digits in the ones place-value column are aligned.

In order to keep the numbers in the proper place-value column when adding decimals, align the decimal points. This will keep the numbers aligned; ones to ones, tenths to tenths, hundredths to hundredths, and so on. Look at the column titled **Value (dollars)**. You will see that place value is maintained, and that the decimal points align from top to bottom.

### Adding Decimals

To add decimals:

- Align the decimal points, which will allow all the digits to be aligned according to their place values.
- Add just as you would add whole numbers, beginning on the right and progressing to the left.
- Write the decimal point in the sum, aligned with the decimal points in the numbers being added.

### Example

**Add. 0.23+4.5+20.32**

**Solution**

| | |
|---|---|
| $\begin{array}{r} 0.23 \\ 4.5 \\ +20.32 \\ \hline \end{array}$ | Write the numbers so that the decimal points are aligned. |
| $\begin{array}{r} 0.23 \\ 4.50 \\ +20.32 \\ \hline \end{array}$ | Optional: Write an extra 0 at the end of 4.5 to keep the numbers in the correct position. (Adding this zero does not change the value of the decimal or the sum of the three numbers.) |
| $\begin{array}{r} 0.23 \\ 4.50 \\ +20.32 \\ \hline 25.05 \end{array}$ | Add. Begin at the right and move left.<br>Align the decimal point in the sum with the decimal points in the numbers being added. |

0.23+4.5+20.32=25.05

### Example

**Add. 4.041+8+510.042**

**Solution**

| | |
|---|---|
| $\begin{array}{r} 4.041 \\ 8. \\ +510.042 \\ \hline \end{array}$ | Write the numbers so that the decimal points are aligned. |
| $\begin{array}{r} 4.041 \\ 8.000 \\ +510.042 \\ \hline \end{array}$ | As presented in the problem, the number 8 does not have a decimal point. You can rewrite this number as 8.0, 8.00, or 8.000 without changing the value of the number. Using 8.000 will allow you to align it with the other two numbers. |
| $\begin{array}{r} 4.041 \\ 8.000 \\ +510.042 \\ \hline 522.083 \end{array}$ | Add. Begin at the right and move left.<br>Align the decimal point in the sum with the decimal points in the numbers being added. |

4.041+8+510.042=522.083

**Exercise**

Add: 0.08+0.156

A. 0.956
B. 0.236
C. 0.164
D. 0.1568

**Answer**

A. Incorrect. Pay attention to the locations of the decimal points. An answer of 0.956 would have been correct for the problem 0.8+0.156. The correct answer is 0.236.
B. Correct. Line up the decimal points and then add. The correct answer is 0.236.
C. Incorrect. Pay attention to the locations of the decimal points. An answer of 0.164 would have been correct for the problem 0.008+0.156. You can only add zeros at the end of the number. The correct answer is 0.236.
D. Incorrect. Pay attention to the locations of the decimal points. An answer of 0.1568 would have been correct for the problem 0.0008+0.156. You can only add zeros at the end of the number. The correct answer is 0.236.

## Subtracting Decimals

Subtracting decimals uses the same setup as adding decimals: line up the decimal points, and then subtract.

In cases where you are subtracting two decimals that extend to different place values, it often makes sense to add extra zeros to make the two numbers line up—this makes the subtraction a bit easier to follow.

**Subtracting Decimals**

To subtract decimals:

- Align the decimal points, which will allow all of the digits to be aligned according to their place values.
- Subtract just as you would subtract whole numbers, beginning on the right and progressing to the left.
- Align the decimal point in the difference directly below the decimal points in the numbers that were subtracted.

**Example**

**Subtract. 39.672-5.431**

### Solution

| | |
|---|---|
| $\begin{array}{r} 39.672 \\ -\quad 5.431 \\ \hline \end{array}$ | Write the numbers so that the decimal points are aligned. |
| $\begin{array}{r} 39.672 \\ -\quad 5.431 \\ \hline 34.241 \end{array}$ | Subtract. Begin at the right and move left.<br>Align the decimal point in the difference with the decimal points in the numbers being subtracted. |

39.672-5.431=34.241

**Example**

**Subtract. 0.9-0.027**

### Solution

| | |
|---|---|
| $\begin{array}{r} 0.9 \\ -0.027 \\ \hline \end{array}$ | Write the numbers so that the decimal points are aligned. |

| | |
|---|---|
| $\begin{array}{r} 0.900 \\ -0.027 \\ \hline \end{array}$ | Optional: Write two extra 0s after 9. This will help you line up the numbers and perform the subtraction. |
| $\begin{array}{r} 0.900 \\ -0.027 \\ \hline 0.873 \end{array}$ | Subtract. Regroup as needed. |

0.9-0.027=0.873

### Exercise

Subtract. 43.21-8.1

A. 35.11
B. 42.40
C. 37.79
D. 35.2

**Answer**

A. Correct. Line up the two numbers so that the decimal points are aligned, and then subtract. The difference is 35.11.
B. Incorrect. Pay attention to the location of the decimal point. An answer of 42.40 would have been correct for the problem 43.21-0.81. The correct answer is 35.11.
C. Incorrect. To subtract, you need to align the decimal points first. The correct answer is 35.11.
D. Incorrect. Pay attention to the location of the decimal point. An answer of 35.2 would have been correct for the problem 43.21-8.01. The correct answer is 35.11.

## Solving Problems

In adding and subtracting decimals, you may have noticed that as long as you line up the decimal points in the numbers you are adding or subtracting, you can operate upon them as you would whole numbers.

Determining whether you need to add or subtract in a given situation is also straightforward. If two quantities are being combined, then add them. If one is being withdrawn from the other, then subtract them.

### Example

**Javier has a balance of $1,800.50 in his personal checking account. He pays two bills out of this account: a $50.23 electric bill, and a $70.80 cell phone bill.**

**How much money is left in Javier’s checking account after he pays these bills?**

#### Solution

| | |
|---|---|
| $\begin{array}{r} 1800.50 \\ -\quad 50.23 \\ \hline \end{array}$ | Since Javier is paying out money, you will subtract, starting with the electric bill. |
| $\begin{array}{r} 1800.50 \\ -\quad 50.23 \\ \hline 1750.27 \end{array}$ | Align the decimals and subtract, regrouping as needed. |
| $\begin{array}{r} 1750.27 \\ -\quad 70.80 \\ \hline \end{array}$ | Javier has $1,750.27 remaining after he pays his electric bill. Next, subtract his cell phone bill, $70.80, from this new amount. |
| $\begin{array}{r} 1750.27 \\ -\quad 70.80 \\ \hline 1679.47 \end{array}$ | Align the decimals and subtract, regrouping as needed. |

Javier has $1,679.47 left in his checking account after paying his bills.

**? Exercise**

Helene ran the 100-meter dash twice on Saturday. The difference between her two times was 0.3 seconds. Which pair of numbers below could have been her individual race times?

A. 14.22 and 14.25 seconds
B. 14.22 and 17.22 seconds
C. 14.22 and 14.58 seconds
D. 14.22 and 13.92 seconds

**Answer**

A. Incorrect. The difference between these times is 0.03 seconds, not 0.3 seconds. The correct answer is 14.22 and 13.92 seconds.
B. Incorrect. The difference between these times is 3 seconds, not 0.3 seconds. The correct answer is 14.22 and 13.92 seconds.
C. Incorrect. The difference between these times is 0.36 seconds, not 0.3 seconds. The correct answer is 14.22 and 13.92 seconds.
D. Correct. 14.22-13.92=0.3; the difference between Helene’s two race times is 0.3 seconds.

## Summary

When adding or subtracting decimals, you must always align the decimal points, which will allow the place-value positions to fall in place. Then add or subtract as you do with whole numbers, regrouping as necessary. You can use these operations to solve real-world problems involving decimals, especially those with money.

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# 10.3.2.2: Multiplying and Dividing Decimals

**Learning Objectives**

- Multiply two or more decimals.
- Multiply a decimal by a power of 10.
- Divide by a decimal.
- Divide a decimal by a power of 10.
- Solve application problems that require decimal multiplication or division.

## Introduction

As with whole numbers, sometimes you run into situations where you need to multiply or divide decimals. And just as there is a correct way to multiply and divide whole numbers, so, too, there is a correct way to multiply and divide decimals.

Imagine that a couple eats dinner at a Japanese steakhouse. The bill for the meal is $58.32 —which includes a tax of $4.64. To calculate the tip, they can double the tax. So if they know how to multiply $4.64 by 2, the couple can figure out how much they should leave for the tip.

Here's another problem. Andy just sold his van that averaged 20 miles per gallon of gasoline. He bought a new pickup truck and took it on a trip of 614.25 miles. He used 31.5 gallons of gas to make it that far. Did Andy get better gas mileage with the new truck?

Both of these problems can be solved by multiplying or dividing decimals. Here's how to do it.

## Multiplying Decimals

Multiplying decimals is the same as multiplying whole numbers except for the placement of the decimal point in the answer. When you multiply decimals, the decimal point is placed in the **product** so that the number of decimal places in the product is the sum of the decimal places in the **factors**.

Let's compare two multiplication problems that look similar: $214 \cdot 36$ and $21.4 \cdot 3.6$.

$$\begin{array}{rr} & 214 \\ \times & 36 \\ \hline & 1284 \\ & 6420 \\ \hline & \mathbf{7,704} \end{array} \qquad \begin{array}{rr} & 21.4 \\ \times & 3.6 \\ \hline & 1284 \\ & 6420 \\ \hline & \mathbf{77.04} \end{array}$$

Notice how the digits in the two solutions are exactly the same: 7, 7, 0, and 4. The multiplication does not change at all. The difference lies in the placement of the decimal point in the final answers: $214 \cdot 36 = 7,704$, and $21.4 \cdot 3.6 = 77.04$.

To find out where to put the decimal point in a decimal multiplication problem, count the total number of decimal places in each of the factors.

21.4 the first factor has one decimal place

3.6 the second factor has one decimal place

77.04 the product will have 1+1=2 decimal places

Note that the decimal points do not have to be aligned as for addition and subtraction.

**Example**

$3.04 \cdot 6.1 = ?$

**Solution**

| | |
|---|---|
| $\begin{array}{r} 3.04 \\ \times\ \ 6.1 \\ \hline 304 \\ 18240 \\ \hline 18544 \end{array}$ | Set up the problem.<br>Multiply 3.04 by 6.1.<br>Add 304 and 18240. |
| | Count the total number of decimal places in the factors and insert the decimal point in the product. |
| $\begin{array}{r} 3.04 \\ \times\ \ 6.1 \\ \hline 304 \\ 18240 \\ \hline 18.544 \end{array}$ | ← 2 decimal places in 3.04.<br>← 1 decimal place in 6.1.<br>← 3 total decimal places. |

$3.04 \cdot 6.1 = 18.544$

Sometimes you may need to insert zeros in front of the product so that you have the right number of decimal places. See the final answer in the example below:

**Example**

0.037 \cdot 0.08=?

**Solution**

| | |
|---|---|
| $\begin{array}{r} 0.037 \\ \times\ \ 0.08 \\ \hline 296 \end{array}$ | Set up the problem.<br>Multiply 0.037 by 0.08.<br>Count the total number of decimal places in the factors and insert the decimal point in the product. |
| $\begin{array}{r} 0.037 \\ \times\ \ 0.08 \\ \hline 0.00296 \end{array}$ | 3 decimal places in 0.037.<br>2 decimal places in 0.08.<br>5 decimal places in the product. |
| $0.037 \cdot 0.08 = 0.00296$ | Note that you needed to add zeros before 296 to get the 5 decimal places. |

If one or more zeros occur on the right in the product, they are not dropped until after the decimal point is inserted.

**Example**

$2.04 \cdot 1.95 = ?$

**Solution**

| | |
|---|---|
| $\begin{array}{r} 2.04 \\ \times\ \ 1.95 \\ \hline 1020 \\ 18360 \\ 20400 \\ \hline 39780 \end{array}$ | Set up the problem.<br>Multiply 2.04 by 1.95.<br>Add<br>1020,18360, and 20400. |

| | |
|---|---|
| $\begin{array}{r} 2.04 \\ \times\quad 1.95 \\ \hline 1020 \\ 18360 \\ 20400 \\ \hline 3.9780 \end{array}$ | ← 2 decimal places in 2.04<br>← 2 decimal places 1.95<br>← 4 decimal places. |
| $2.04 \cdot 1.95 = 3.978$ | Answer can omit the final trailing 0. |

### Multiplying Decimals

To multiply decimals:

- Set up and multiply the numbers as you do with whole numbers.
- Count the total number of decimal places in both of the factors.
- Place the decimal point in the product so that the number of decimal places in the product is the sum of the decimal places in the factors.
- Keep all zeros in the product when you place the decimal point. You can drop the zeros on the right once the decimal point has been placed in the product. If the number of decimal places is greater than the number of digits in the product, you can insert zeros in front of the product.

### Exercise

Multiply. $51.2 \cdot 3.08$

A. 15769.6
B. 1576.96
C. 157.696
D. 15.7696

**Answer**

A. Incorrect. Pay attention to the placement of the decimal point. The correct answer is 157.696.
B. Incorrect. Pay attention to the placement of the decimal point. The correct answer is 157.696.
C. Correct. To find the product, multiply $512 \cdot 308 = 157696$ Count the total number of decimal places in the factors, 3, and then place a decimal point in the product so that the product has three decimal places as well. The answer is 157.696.
D. Incorrect. Pay attention to the placement of the decimal point. The correct answer is 157.696.

## Multiplying by Tens

Take a moment to multiply 4.469 by 10. Now do $4.469 \cdot 100$. Finally, do $4.469 \cdot 1,000$. Notice any patterns in your products?

$$\begin{array}{r} 4.469 \\ \times\quad 10 \\ \hline 44.690 \end{array} \qquad \begin{array}{r} 4.469 \\ \times\quad 100 \\ \hline 446.900 \end{array} \qquad \begin{array}{r} 4.469 \\ \times\quad 1000 \\ \hline 4469.000 \end{array}$$

Notice that the products keep getting greater by one place value as the multiplier (10, 100, and 1,000) increases. In fact, the decimal point moves to the right by the same number of zeros in the power of ten multiplier.

$$4.469 \cdot 10 = 44.69 \qquad 4.469 \cdot 100 = 446.9 \qquad 4.469 \cdot 1,000 = 4469$$

You can use this observation to help you quickly multiply any decimal by a power of ten (10, 100, 1,000, etc).

**✓ Example**

$0.03 \cdot 100 = ?$

**Solution**

| | |
|---|---|
| $0.03 \cdot 100 = ?$ | 100 has two zeros. |
| $0.03 \cdot 100 = 3$ | Move the decimal point two places to the right to find the product. |

$0.03 \cdot 100 = 3$

**Multiplying a Decimal by a Power of Ten**

To multiply a decimal number by a power of ten (such as 10, 100, 1,000, etc.), count the number of zeros in the power of ten. Then move the decimal point that number of places to the right.

For example, $0.054 \cdot 100 = 5.4$. The multiplier 100 has two zeros, so you move the decimal point in 0.054 two places to the right—for a product of 5.4.

## Dividing Decimals

To divide decimals, you will once again apply the methods you use for dividing whole numbers. Look at the two problems below. How are the methods similar?

$$
\begin{array}{rr}
867 & 8.67 \\
3\backslash\text{longdiv}2601 & 3\backslash\text{longdiv}26.01 \\
-24 & -24 \\
\hline
20 & 2\ 0 \\
-18 & -1\ 8 \\
\hline
21 & 21 \\
-21 & -21 \\
\hline
0 & 0
\end{array}
$$

Notice that the division occurs in the same way—the only difference is the placement of the decimal point in the **quotient**.

**✓ Example**

$18.32 \div 8 = ?$

**Solution**

| | |
|---|---|
| $8\backslash\text{longdiv}18.32$ | Set up the problem. |
| $2.29$<br>$8\backslash\text{longdiv}18.32$<br>$-16$<br>$23$<br>$-16$<br>$72$<br>$-72$<br>$0$ | Divide. |
| \(\ \begin{array}{r}<br>2.29\ \\<br>8\longdiv{18.32}\\<br>\end{array}\) | Place the decimal point in the quotient. It should be placed directly above the decimal point in the dividend. |

$18.32 \div 8 = 2.29$

But what about a case where you are dividing by a decimal, as in the problem below?

$0.3\longdiv 260.1$

In cases like this, you can use powers of 10 to help create an easier problem to solve. In this case, you can multiply the **divisor**, 0.3, by 10 to move the decimal point 1 place to the right. If you multiply the divisor by 10, then you also have to multiply the **dividend** by 10 to keep the quotient the same. The new problem, with its solution, is shown below.

✓ Example

$260.1 \div 0.3 = ?$

**Solution**

| | |
|---|---|
| $0.3\longdiv 260.1$ | Set up the problem. |
| $3.\longdiv 2601.$ | Multiply divisor and dividend by 10 to create a whole number divisor. |
| $867$<br>$3\longdiv 2601$<br>$-24$<br>$20$<br>$-18$<br>$21$<br>$-21$<br>$0$ | Divide. |

$260.1 \div 0.3 = 867$

Often, the dividend will still be a decimal after multiplying by a power of 10. In this case, the placement of the decimal point must align with the decimal point in the dividend.

✓ Example

$15.275 \div 3.25 = ?$

**Solution**

| | |
|---|---|
| $3.25\longdiv 15.275$ | Set up the problem. |
| $325.\longdiv 1527.5$ | Multiply divisor and dividend by 100 to create a whole number divisor. |
| $4.7$<br>$325.\longdiv 1527.5$<br>$-1300$<br>$2275$<br>$-2275$<br>$0$ | Divide. 325 goes into 1527 four times, so the number 4 is placed above the digit 7.<br>The decimal point in the quotient is placed directly above the decimal point in the dividend. |

$15.275 \div 3.25 = 4.7$

### Dividing Decimals

**Dividing Decimals by Whole Numbers**

Divide as you would with whole numbers. Then place the decimal point in the quotient directly above the decimal point in the dividend.

**Dividing by Decimals**

To divide by a decimal, multiply the divisor by a power of ten to make the divisor a whole number. Then multiply the dividend by the same power of ten. You can think of this as moving the decimal point in the dividend the same number of places to the right as you move the decimal point in the divisor.

Then place the decimal point in the quotient directly over the decimal point in the dividend. Finally, divide as you would with whole numbers.

### Exercise

Divide: $25.095 \div 0.5$.

A. 5,019
B. 501.9
C. 50.19
D. 0.5019

**Answer**

A. Incorrect. Multiply both the divisor and the dividend by 10 (this will change 0.5 into a whole number), and then divide. Then place the decimal point in the quotient directly over the decimal point in the dividend. The correct answer is 50.19.
B. Incorrect. Multiply both the divisor and the dividend by 10 (this will change 0.5 into a whole number), and then divide. Then place the decimal point in the quotient directly over the decimal point in the dividend. The correct answer is 50.19.
C. Correct. This problem can be set up as $250.95 \div 5$; the quotient is 50.19.
D. Incorrect. Remember that when you divide, you do not count the total number of decimal places in the divisor and dividend. You change the divisor to a whole number, then move the decimal point in the dividend the same number of places and divide. Finally, place the decimal point in the quotient directly over the decimal point in the dividend. The correct answer is 50.19.

## Dividing by Tens

Recall that when you multiply a decimal by a power of ten (10, 100, 1,000, etc.), the placement of the decimal point in the product will move to the right according to the number of zeros in the power of ten. For instance, $4.12 \cdot 10 = 41.2$.

Multiplication and division are inverse operations, so you can expect that if you divide a decimal by a power of ten, the decimal point in the quotient will also correspond to the number of zeros in the power of ten. The difference is that the decimal point moves to the right when you multiply; it moves to the left when you divide.

$$\begin{array}{r} .4469 \\ 10\backslash\text{longdiv}4.4690 \\ -4\ 0 \\ \hline 46 \\ -40 \\ \hline 69 \\ -60 \\ \hline 90 \\ -90 \\ \hline 0 \end{array} \quad \begin{array}{r} .04469 \\ 100\backslash\text{longdiv}4.46900 \\ -4\ 00 \\ \hline 469 \\ -400 \\ \hline 690 \\ -600 \\ \hline 900 \\ -900 \\ \hline 0 \end{array} \quad \begin{array}{r} .004469 \\ 1000\backslash\text{longdiv}4.469000 \\ -4\ 000 \\ \hline 4690 \\ -4000 \\ \hline 6900 \\ -6000 \\ \hline 9000 \\ -9000 \\ \hline 0 \end{array}$$

In the examples above, notice that each quotient still contains the digits 4469—but as another 0 is added to the end of each power of ten in the divisor, the decimal point moves an additional place to the left in the quotient.

**Dividing by Powers of Ten**

To divide a decimal by a power of ten (10, 100, 1,000, ect.), count the number of zeros in the divisor. Then move the decimal point in the dividend that number of decimal places to the left; this will be your quotient.

**Example**

$31.05 \div 10 = ?$

**Solution**

| | |
|---|---|
| $31.05 \div 10 = ?$ | 10 has one zero. |
| $31.05 \div 10 = 3.105$ | Move the decimal point one place to the left in the dividend; this is the quotient. |

$31.05 \div 10 = 3.105$

**Exercise**

Divide. $0.045 \div 100$

A. 0.00045
B. 0.045
C. 4.5
D. 4,500

**Answer**

A. Correct. There are two zeros in the divisor, 100, so to find the quotient, take the dividend, 0.045, and move the decimal point two places to the left. The quotient is 0.00045.
B. Incorrect. 0.045 is the dividend in the problem; it cannot be the quotient unless the divisor is 1. The correct answer is 0.00045.
C. Incorrect. 4.5 would be the correct answer if you *multiplied* 0.045 by 100, not divided it by 100. The correct answer is 0.00045.
D. Incorrect. 4,500 would be the correct answer if you *multiplied* 0.045 by 100,000, not divided it by 100. The correct answer is 0.00045.

## Solving Problems by Multiplying or Dividing Decimals

Now let's return to the two problems from the beginning of this section. You know how to multiply and divide with decimals now. Let's put that knowledge to the test.

**Example**

**A couple eats dinner at a Japanese steakhouse. The bill for the meal totals $58.32—which includes a tax of $4.64. To calculate the tip, they can double the tax. How much tip should the couple leave?**

**Solution**

| | |
|---|---|
| $\begin{array}{r} 4.64 \\ \times \quad 2 \\ \hline \end{array}$ | Set up a multiplication problem. |

| | |
|---|---|
| 4.64<br>× 2<br>928 | Multiply 4.64 by 2. |
| 4.64<br>× 2<br>9.28 | Count the number of decimal places in the two factors, and place the decimal point accordingly. |

The couple should leave a tip of $9.28.

### ✓ Example

**Andy just sold his van that averaged 20 miles per gallon of gasoline. He bought a new pickup truck and took it on a trip of 614.25 miles. He used 31.5 gallons of gas for the trip. Did Andy get better gas mileage with the new truck?**

#### Solution

| | |
|---|---|
| 31.5\longdiv614.25 | Set up a division problem. |
| 315.\longdiv6142.5 | Make the divisor a whole number by multiplying by 10; do the same to the dividend. |
| 19.5<br>315.\longdiv6142.5<br>−315<br>2992<br>−2835<br>1575<br>−1575<br>0 | Divide. Insert a decimal point in the quotient so that it is directly above the decimal point in the dividend. |

Andy gets 19.5 miles per gallon now. He used to get 20 miles per gallon. He does not get better gas mileage with the new truck.

## Summary

Learning to multiply and divide with decimals is an important skill. In both cases, you work with the decimals as you have worked with whole numbers, but you have to figure out where the decimal point goes. When multiplying decimals, the number of decimal places in the product is the sum of the decimal places in the factors. When dividing by decimals, move the decimal point in the dividend the same number of places to the right as you move the decimal point in the divisor. Then place the decimal point in the quotient above the decimal point in the dividend.

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# 10.3.2.3: Estimation with Decimals

**Learning Objectives**

- Estimate the answer to a decimal problem.

## Introduction

Being able to **estimate** your answer is a very useful skill. Not only will it help you decide if your answer is reasonable when doing homework problems or answering test questions, it can prove to be very helpful in everyday life. When shopping, you can estimate how much money you have spent, the tip for a restaurant bill, or the price of an item on sale. By **rounding** and then doing a quick calculation, you will at least know if you are close to the exact answer.

## Estimating with Decimals

Consider this problem. Stewart wanted to buy a DVD home theater system that cost $345.23. He also wanted a universal remote priced at $32.90. He used a calculator to add the costs and the sum that he got was $674.23. He was surprised!

Whether Stewart can afford to spend $674.23 or not is not really the problem here. Rather, the problem is whether the system and remote would total $674.23. Round both item costs to the tens place: $345.23 is about $350, and $32.90 is about $30. Is $350+$30 close to $ 674.23? Of course not!

Estimating an answer is a good skill to have. Even when using a calculator, you can get an incorrect answer by accidentally pressing a wrong button. When decimals are involved, it's very easy to put the decimal point in the wrong place, and then your numbers can be drastically wrong.

**Example**

**Hakim wrote checks for $64.20, $47.89, and $95.80. Estimate the total of all three checks.**

**Solution**

| | |
|---|---|
| | To estimate the total, first round each of the check values. You want to round to the nearest $10 in this example. |
| $64.20 \rightarrow 60$<br>$47.89 \rightarrow 50$<br>$95.80 \rightarrow 100$ | Since 4<5, round 64.20 to 60.<br>Since 7>5, round 47.89 up to 50. Since 5=5, round 95.80 up to 100. |
| $60+50+100=210$ | Add the estimates to find the estimated total. |

The total estimate for the three checks is $210.

You can use estimation to see if you have enough money for a purchase. In this case, it is best to round all the numbers up to make sure that you have enough money.

**Example**

**Sherry has $50 and wants to buy CDs that cost $11.50 each. About how many CDs can she buy?**

**Solution**

| | |
|---|---|
| $11.50 \rightarrow 12.00$ | Round $11.50 to the nearest whole number.<br>Since you want to make sure that Sherry has enough money, round up to 12.00 or 12. |
| $50 \div 12 = 4\text{R}2$ | Divide 50 by 12. The amount of the remainder is not important. |

Sherry can buy about 4 CDs with $50.

You will generally estimate when you compute the amount of tip to leave when you eat at a restaurant. Recall that an easy way to compute the tip is to double the tax. You can probably do this in your head, if you estimate this product by rounding to the nearest \$1. You can round up if the service is good or round down if not.

### Example

**After a delicious meal at a restaurant, the bill for two is \$45.36, which includes tax of \$3.74. The service was very good. How much tip would you leave if you follow the rule to double the amount of tax?**

#### Solution

| | |
|---|---|
| $3.74 \rightarrow 4.00$ | Round up to \$4.00, or \$4, as the service was good. |
| $4 \cdot 2 = 8$ | Multiply the rounded number by 2 to double this number. |

The tip for good service would be \$8.

When using rounding in addition and subtraction problems, you usually round all numbers to the same place value—this makes adding or subtracting a bit easier. However, when you use rounding to help you multiply or divide numbers, it's usually better to round the numbers so they each have only one or two digits that are not 0. This is shown in the example below.

### Example

**Jin is building a model ship based on a real one. Each length of the model is 0.017 times the actual length of the ship. The real ship is 132 feet long. Estimate the length of the model, then use a calculator to find the actual length of the model.**

#### Solution

| | |
|---|---|
| $0.017 \rightarrow 0.02$ | The scale of the model is 0.017. It wouldn't make sense to round this value to a whole number or even tenths, but you can round it to hundredths.<br>Since 7>5, round up to 0.020 or just 0.02. |
| $132 \rightarrow 130$ | You will round 132 to the tens to make it easier to work with.<br>Since 2<5, round to 130. |
| $130 \cdot 0.02$<br>$130 \cdot 2 = 260$<br>$130 \cdot 0.02 = 2.60$ | Multiply $130 \cdot 2 = 260$.<br>Then place the decimal point to get 2.60.<br>Use a calculator to find the exact length. The estimate is fairly accurate! |

The model length is about 2.6ft, or exactly 2.244ft.

### ? Exercise

Evelyn is purchasing 287 ceramic tiles for her new kitchen. Each one costs \$0.21. Which of the following is the most accurate estimate for the cost of purchasing the tiles for her kitchen?

A. \$90
B. \$60
C. \$50
D. \$40

**Answer**

A. Incorrect. You may have rounded 287 to 300 and then 0.21 to 0.30, to arrive at $300 \cdot 0.30 = 90$, or \$90. This is too high; 0.21 rounds down to 0.20, not up to 0.30. The correct answer is \$60.
B. Correct. 287 rounds to 300, and 0.21 rounds to 0.20, to arrive at $300 \cdot 0.20 = 60$ or \$60.

C. Incorrect. You may have rounded 287 to 250 and then 0.21 to 0.20, to arrive at $250 \cdot 0.20 = 50$, or \$50. This is too low; to the tens place, 287 rounds up to 290, not down to 250. Then $290 \cdot 0.20 = \$58$; the closest answer to this is the \$60. The correct answer is \$60.

D. Incorrect. You may have rounded 287 to 200 and then 0.21 to 0.20, to arrive at $200 \cdot 0.20 = 40$, or \$40. This is too low; to the hundreds place, 287 rounds up to 300, not down to 200. The correct answer is \$60.

## Summary

Estimation is useful when you don’t need an exact answer. It also lets you check to be sure an exact answer is close to being correct. Estimating with decimals works just the same as estimating with whole numbers. When rounding the values to be added, subtracted, multiplied, or divided, it helps to round to numbers that are easy to work with.

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# SECTION OVERVIEW

## 10.4: Ratios, Rates, and Proportions

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# 10.4.1: Ratio and Rates

# 10.4.1.1: Simplifying Ratios and Rates

**Learning Objectives**

- Write ratios and rates as fractions in simplest form.
- Find unit rates.
- Find unit prices.

## Introduction

**Ratios** are used to compare amounts or quantities or describe a relationship between two amounts or quantities. For example, a ratio might be used to describe the cost of a month's rent as compared to the income earned in one month. You may also use a ratio to compare the number of elephants to the total number of animals in a zoo, or the amount of calories per serving in two different brands of ice cream.

**Rates** are a special type of ratio used to describe a relationship between different units of measure, such as speed, wages, or prices. A car can be described as traveling 60 miles per hour; a landscaper might earn $35 per lawn mowed; gas may be sold at $3 per gallon.

## Ratios

Ratios compare quantities using division. This means that you can set up a ratio between two quantities as a division expression between those same two quantities.

Here is an example. If you have a platter containing 10 sugar cookies and 20 chocolate chip cookies, you can compare the cookies using a ratio.

The ratio of sugar cookies to chocolate chip cookies is:

$$\frac{\text{sugar cookies}}{\text{chocolate chip cookies}}=\frac{10}{20}$$

The ratio of chocolate chip cookies to sugar cookies is:

$$\frac{\text{chocolate chip cookies}}{\text{sugar cookies}}=\frac{20}{10}$$

You can write the ratio using words, a fraction, and also using a colon as shown below.

$$\begin{gathered}\text{ratio of sugar cookies to}\\ \text{chocolate chip cookies}\\ 10 \text{ to } 20\\ \frac{10}{20}\\ 10:20\end{gathered}$$

Some people think about this ratio as: "For every 10 sugar cookies I have, I have 20 chocolate chip cookies."

You can also simplify the ratio just as you simplify a fraction.

$$\frac{10}{20}=\frac{10\div 10}{20\div 10}=\frac{1}{2}$$

So we can also say that:

$$\begin{gathered}\text{ratio of sugar cookies to}\\ \text{chocolate chip cookies}\\ 1 \text{ to } 2\\ \frac{1}{2}\\ 1:2\end{gathered}$$

### How to Write a Ratio

A ratio can be written in three different ways:

- with the word "to": 3 to 4
- as a fraction: $\frac{3}{4}$
- with a colon: 3:4

A ratio is simplified if it is equivalent to a fraction that has been simplified.

Below are two more examples that illustrate how to compare quantities using a ratio, and how to express the ratio in simplified form.

### ✓ Example

**A basketball player takes 50 jump shots during a practice. She makes 28 of them. What is the ratio of shots made to shots taken? Simplify the ratio.**

**Solution**

| | |
|---|---|
| $\frac{\text{shots made}}{\text{shots taken}}$ | Identify the relationship. |
| $\frac{28}{50}$ | Express the two quantities in fraction form. |
| $\frac{28 \div 2}{50 \div 2} = \frac{14}{25}$ | Simplify the fraction to express the ratio in simplest form. |
| $14 : 25$<br>14 to 25 | Consider the two other ways to write a ratio. You'll want to express your answer in a particular format if required. |

The ratio of shots made to shots taken is $\frac{14}{25}$, 14:15, or 14 to 15.

Often, one quantity in the ratio is greater than the second quantity. You do not have to write the ratio so that the lesser quantity comes first; the important thing is to keep the relationship consistent.

### ✓ Example

**Paul is comparing the amount of calories in a large order of French fries from his two favorite fast food restaurants. Fast Foodz advertises that an order of fries has 450 calories, and Beef Stop states that its fries have 300 calories. Write a ratio that represents the amount of calories in the Fast Foodz fries compared to the calories in Beef Stop fries.**

**Solution**

| | |
|---|---|
| $\frac{\text{calories in Fast Foodz fries}}{\text{calories in Beef Stop fries}}$ | Identify the relationship. |
| $\frac{450}{300}$ | Write a ratio comparing calories. |
| $\frac{450 \div 150}{300 \div 150} = \frac{3}{2}$ | Simplify the ratio.<br>450 and 300 have a common factor of 150. |

The ratio of calories in Fast Foodz fries to Beef Stop fries is \frac{3}{2}, 3:2, or 3 to 2.

Ratios can compare a *part to a part* or a *part to a whole*. Consider the example below that describes guests at a party.

### ✓ Example

**Luisa invites a group of friends to a party. Including Luisa, there are a total of 22 people, 10 of whom are women.**

**Which is greater: the ratio of women to men at the party, or the ratio of women to the total number of people present?**

**Solution**

| | |
|---|---|
| $\frac{\text{number of women}}{\text{number of men}}$ | Identify the first relationship. |
| $\frac{10}{12}$ | Write a ratio comparing women to men. Since there are 22 people and 10 are women, 12 must be men. |
| $\frac{10 \div 2}{12 \div 2} = \frac{5}{6}$ | Simplify the ratio. 10 and 12 have a common factor of 2; the ratio of women to men at the party is $\frac{5}{6}$. |
| $\frac{\text{number of women}}{\text{number of people}}$ | Identify the next relationship. |
| $\frac{10}{22}$ | Write a ratio comparing the number of women to the total number of people at the party. |
| $\frac{10 \div 2}{22 \div 2} = \frac{5}{11}$ | Simplify the ratio. 10 and 22 have a common factor of 2. |
| $\frac{5 \cdot 11}{6 \cdot 11} = \frac{55}{66}$<br>$\frac{5 \cdot 6}{11 \cdot 6} = \frac{30}{66}$ | Rewrite each fraction with a common denominator, 66. |
| $\frac{5}{6} > \frac{5}{11}$ | Since $\frac{55}{66} > \frac{30}{66}$. |

The ratio of women to men at the party, $\frac{5}{6}$, is greater than the ratio of women to the total number of people, $\frac{5}{11}$.

### ? Exercise

A poll at Forrester University found that 4,000 out of 6,000 students are unmarried. Find the ratio of unmarried to married students. Express as a simplified ratio.

A. 3 to 2
B. 1 to 3
C. 2 to 1
D. 2 to 3

**Answer**

A. Incorrect. The ratio 3 to 2 compares the total number of students to the number of unmarried students. The correct answer is 2 to 1.
B. Incorrect. The ratio 1 to 3 compares the number of married students to the total number of students. The correct answer is 2 to 1.
C. Correct. If 4,000 students out of 6,000 are unmarried, then 2,000 must be married. The ratio of unmarried to married students can be represented as 4,000 to 2,000, or simply 2 to 1.
D. Incorrect. The ratio 2 to 3 compares the number of unmarried students to the total number of students. The correct answer is 2 to 1.

## Rates

A rate is a ratio that compares two different quantities that have different units of measure. A rate is a comparison that provides information such as dollars per hour, feet per second, miles per hour, and dollars per quart, for example. The word “per” usually indicates you are dealing with a rate. Rates can be written using words, using a colon, or as a fraction. It is important that you know which quantities are being compared.

For example, an employer wants to rent 6 buses to transport a group of 300 people on a company outing. The rate to describe the relationship can be written using words, using a colon, or as a fraction; and you must include the units.

six buses per 300 people

6 buses : 300 people

$$\frac{6 \text{ buses}}{300 \text{ people}}$$

As with ratios, this rate can be expressed in simplest form by simplifying the fraction.

$$\frac{6 \text{ buses} \div 6}{300 \text{ people} \div 6} = \frac{1 \text{ bus}}{50 \text{ people}}$$

This fraction means that the rate of buses to people is 6 to 300 or, simplified, 1 bus for every 50 people.

## ✓ Example

**Write the rate as a simplified fraction: 8 phone lines for 36 employees.**

### Solution

| | |
|---|---|
| $\frac{8 \text{ phone lines}}{36 \text{ employees}}$ | Write as a fraction. |
| $\frac{8 \text{ phone lines} \div 4}{36 \text{ employees} \div 4} = \frac{2 \text{ phone lines}}{9 \text{ employees}}$ | Simplify the fraction by using the common factor of 4. |

The rate of phone lines for employees can be expressed as $\frac{2 \text{ phone lines}}{9 \text{ employees}}$.

## ✓ Example

**Write the rate as a simplified fraction: 6 flight attendants for 200 passengers.**

### Solution

| | |
|---|---|
| $\frac{6 \text{ flight attendants}}{200 \text{ passengers}}$ | Write as a fraction. |
| $\frac{6 \text{ flight attendants} \div 2}{200 \text{ passengers} \div 2} = \frac{3 \text{ flight attendants}}{100 \text{ passengers}}$ | Simplify the fraction by using the common factor of 2. |

The rate of flight attendants to passengers is $\frac{3 \text{ flight attendants}}{100 \text{ passengers}}$.

## ? Exercise

Anyla rides her bike 18 blocks in 20 minutes. Express her rate as a simplified fraction.

A. 18 : 20

B. $\frac{9 \quad \text{blocks}}{10 \quad \text{minutes}}$

C. $\frac{9 \text{ minutes}}{10 \text{ blocks}}$

D. $\frac{18 \text{ blocks}}{20 \text{ minutes}}$

**Answer**

A. 18 : 20

Incorrect. Anyla's trip compares quantities with different units, so it can be described as a rate. Since rates compare two quantities measured in different units of measurement, they must include their units. The correct answer is $\frac{9 \text{ blocks}}{10 \text{ minutes}}$.

B. $\frac{9 \quad \text{blocks}}{10 \quad \text{minutes}}$

Correct. Anyla's trip compares quantities with different units (blocks and minutes), so it is a rate and can be written $\frac{18 \quad \text{blocks}}{20 \quad \text{minutes}}$. This fraction can be simplified by dividing both the numerator and the denominator by 2.

C. $\frac{9 \text{ minutes}}{10 \text{ blocks}}$

Incorrect. 18 blocks in 20 minutes is not equivalent to 10 blocks in 9 minutes. Check the units again in your answer. The correct answer is $\frac{9\text{ blocks}}{10\text{ minutes}}$.

D. $\frac{18\text{ blocks}}{20\text{ minutes}}$

Incorrect. Anyla’s trip compares quantities with different units, so it can be described as a rate. This is a correct representation and includes the units, but the fraction can be simplified. The correct answer is $\frac{9\text{ blocks}}{10\text{ minutes}}$.

## Finding Unit Rates

A **unit rate** compares a quantity to one unit of measure. You often see the speed at which an object is traveling in terms of its unit rate.

For example, if you wanted to describe the speed of a boy riding his bike—and you had the measurement of the distance he traveled in miles in 2 hours—you would most likely express the speed by describing the distance traveled in *one* hour. This is a unit rate; it gives the distance traveled per one hour. The denominator of a unit rate will always be one.

Consider the example of a car that travels 300 miles in 5 hours. To find the unit rate, you find the number of miles traveled in ***one*** hour.

$$\frac{300\text{ miles}\div 5}{5\text{ hours}\div 5}=\frac{60\text{ miles}}{1\text{ hour}}$$

A common way to write this unit rate is 60 miles per hour.

✓ Example

**A crowded subway train has 375 passengers distributed evenly among 5 cars. What is the unit rate of passengers per subway car?**

### Solution

| | |
|---|---|
| $\frac{\text{passengers}}{\text{subway cars}}$ | Identify the relationship. |
| $\frac{375\text{ passengers}}{5\text{ subway cars}}$ | Write the rate as a fraction. |
| $\frac{375\text{ passengers }\div 5}{5\text{ subway cars }\div 5}=\frac{75\text{ passengers}}{1\text{ subway car}}$ | Express the fraction with 1 in the denominator to find the number of passengers in one subway car. |

The unit rate of the subway car is 75 riders per subway car.

## Finding Unit Prices

A **unit price** is a unit rate that expresses the price of something. The unit price always describes the price of *one* unit, so that you can easily compare prices.

You may have noticed that grocery shelves are marked with the unit price (as well as the total price) of each product. This unit price makes it easy for shoppers to compare the prices of competing brands and different package sizes.

Consider the two containers of blueberries shown below. It might be difficult to decide which is the better buy just by looking at the prices; the container on the left is cheaper, but you also get fewer blueberries. A better indicator of value is the price *per single ounce* of blueberries for each container.

Look at the unit prices—the container on the right is actually a better deal, since the price per ounce is *lower* than the unit price of the container on the left. You pay more money for the larger container of blueberries, but you also get more blueberries than you would with the smaller container. Put simply, the container on the right is a better value than the container on the left.

So, how do you find the unit price?

Imagine a shopper wanted to use unit prices to compare a 3-pack of tissue for $4.98 to a single box of tissue priced at $1.60. Which is the better deal?

Find the unit price of the 3-pack: $\frac{\$4.98}{3\text{ boxes}}$

Since the price given is for 3 boxes, divide both the numerator and the denominator by 3 to get the price of 1 box, the unit price. The unit price is $1.66 per box.

The unit price of the 3-pack is $1.66 per box; compare this to the price of a single box at $1.60. Surprisingly, the 3-pack has a higher unit price! Purchasing the single box is the better value.

Like rates, unit prices are often described with the word “per.” Sometimes, a slanted line / is used to mean “per.” The price of the tissue might be written $1.60/box, which is read "$1.60 per box."

### ✓ Example

3 **pounds of sirloin tips cost $21. What is the unit price per pound?**

**Solution**

| | |
|---|---|
| $\frac{\$21.00}{3\text{ pounds}}$ | Write a rate to represent the cost per number of pounds. |
| $\frac{\$21.00 \div 3}{3\text{ pounds} \div 3} = \frac{\$7.00}{1\text{ pound}}$ | Express the fraction with 1 in the denominator by dividing both the numerator and the denominator by 3. |

The unit price of the sirloin tips is $7.00/pound.

The following example shows how to use unit price to compare two products and determine which has the lower price.

### ✓ Example

**Sami is trying to decide between two brands of crackers. Which brand has the lower unit price?**

**Brand A: $1.12 for 8 ounces**

**Brand B: $1.56 for 12 ounces**

**Solution**

| | |
|---|---|
| Brand A | |
| $\frac{\$1.12}{8\text{ ounces}}$ | Write a rate to represent the cost per ounce for Brand A. |

| | |
|---|---|
| $\frac{\$1.12 \div 8}{8 \text{ ounces} \div 8} = \frac{\$0.14}{1 \text{ ounce}}$ | Find the unit price of Brand A by dividing both the numerator and the denominator by 8. |
| Brand B | |
| $\frac{\$1.56}{12 \text{ ounces}}$ | Write a rate to represent the cost per ounce for Brand B. |
| $\frac{\$1.56 \div 12}{12 \text{ ounces } \div 12} = \frac{\$0.13}{1 \text{ ounce}}$ | Find the unit price of Brand B by dividing both the numerator and the denominator by 12. |
| $\frac{\$0.14}{1 \text{ ounce}} > \frac{\$0.13}{1 \text{ ounce}}$ | Compare unit prices. |

The unit price of Brand A crackers is 14 cents/ounce and the unit price of Brand B is 13 cents/ounce. Brand B has a lower unit price and represents the better value.

### ? Exercise

A shopper is comparing two packages of rice at the grocery store. A 10-pound package costs $9.89 and a 2-pound package costs $1.90. Which package has the lower unit price to the nearest cent? What is its unit price?

A. The 2-pound bag has a lower unit price of $.95/pound.
B. The 10-pound bag has a lower unit price of $0.99/pound.
C. The 10-pound bag has a lower unit price of $.95/pound.
D. The 2-pound bag has a lower price of $1.89/2pound.

**Answer**

A. Correct. The unit price per pound for the 2-pound bag is $\$1.90 \div 2 = \$0.95$. The unit price per pound for the 10-pound bag is $\$9.89 \div 10 = \$0.989$, which rounds to $0.99.
B. Incorrect. $\$9.89 \div 10 = \$0.989$, which rounds to $0.99. $\$1.90 \div 2 = \$0.95$. The 2-pound bag has a lower unit price. The correct answer is A.
C. Incorrect. $\$9.89 \div 10 = \$0.989$, which rounds to $0.99. The correct answer is A.
D. Incorrect. A unit price is the price for one unit; in this case, you need to find the cost of one pound, not two pounds. The correct answer is A.

## Summary

Ratios and rates are used to compare quantities and express relationships between quantities measured in the same units of measure and in different units of measure. They both can be written as a fraction, using a colon, or using the words “to” or “per”. Since rates compare two quantities measured in different units of measurement, such as dollars per hour or sick days per year, they must include their units. A unit rate or unit price is a rate that describes the rate or price for one unit of measure.

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# 10.4.2: Proportions

# 10.4.2.1: Understanding Proportions

**Learning Objectives**

- Determine whether a proportion is true or false.
- Find an unknown in a proportion.
- Solve application problems using proportions.

## Introduction

A true **proportion** is an equation that states that two **ratios** are equal. If you know one ratio in a proportion, you can use that information to find values in the other equivalent ratio. Using proportions can help you solve problems such as increasing a recipe to feed a larger crowd of people, creating a design with certain consistent features, or enlarging or reducing an image to scale.

For example, imagine you want to enlarge a 5-inch by 8-inch photograph to fit a wood frame that you purchased. If you want the shorter edge of the enlarged photo to measure 10 inches, how long does the photo have to be for the image to scale correctly? You can set up a proportion to determine the length of the enlarged photo.

## Determining Whether a Proportion is True or False

A proportion is usually written as two equivalent fractions. For example:

$$\frac{12\text{ inches}}{1\text{ foot}}=\frac{36\text{ inches}}{3\text{ feet}}$$

Notice that the equation has a ratio on each side of the equal sign. Each ratio compares the same units, inches and feet, and the ratios are equivalent because the units are consistent, and $\frac{12}{1}$ is equivalent to $\frac{36}{3}$.

Proportions might also compare two ratios with the same units. For example, Juanita has two different-sized containers of lemonade mix. She wants to compare them. She could set up a proportion to compare the number of ounces in each container to the number of servings of lemonade that can be made from each container.

$$\frac{40\text{ ounces}}{84\text{ ounces}}=\frac{10\text{ servings}}{21\text{ servings}}$$

Since the units for each ratio are the same, you can express the proportion without the units:

$$\frac{40}{84}=\frac{10}{21}$$

When using this type of proportion, it is important that the numerators represent the same situation. In the example above they represent 40 ounces for 10 servings. The denominators should also represent the same situation, which is ounces for servings.

Juanita could also have set up the proportion to compare the ratios of the container sizes to the number of servings of each container.

$$\frac{40\text{ ounces}}{10\text{ servings}}=\frac{84\text{ ounces}}{21\text{ servings}}$$

Sometimes you will need to figure out whether two ratios are, in fact, a true or false proportion. Below is an example that shows the steps of determining whether a proportion is true or false.

**Example**

**Is the proportion true or false?**

$$\frac{\mathbf{100}\text{ miles}}{\mathbf{4}\text{ gallons}}=\frac{\mathbf{50}\text{ miles}}{\mathbf{2}\text{ gallons}}$$

### Solution

| | |
|---|---|
| miles (check) | The units are consistent across the numerators. |
| gallons (check) | The units are consistent across the denominators.<br>Write each ratio in simplest form. |

| | |
|---|---|
| $\frac{100 \div 4}{4 \div 4} = \frac{25}{1}$<br>$\frac{50 \div 2}{2 \div 2} = \frac{25}{1}$<br>$\frac{25}{1} = \frac{25}{1}$ | Since the simplified fractions are equivalent, the proportion is true. |

The proportion is true.

### Identifying True Proportions

To determine if a proportion compares equal ratios or not, you can follow these steps.

1. Check to make sure that the units in the individual ratios are consistent either vertically or horizontally. For example, $\frac{\text{miles}}{\text{hour}} = \frac{\text{miles}}{\text{hour}}$ or $\frac{\text{miles}}{\text{miles}} = \frac{\text{hour}}{\text{hour}}$ are valid setups for a proportion.
2. Express each ratio as a simplified fraction.
3. If the simplified fractions are the same, the proportion is *true*; if the fractions are different, the proportion is *false*.

Sometimes you need to create a proportion before determining whether it is true or not. An example is shown below.

### Example

**One office has 3 printers for 18 computers. Another office has 20 printers for 105 computers. Is the ratio of printers to computers the same in these two offices?**

#### Solution

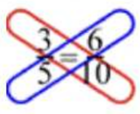
| | |
|---|---|
| $\frac{\text{printers}}{\text{computers}} = \frac{\text{printers}}{\text{computers}}$ | Identify the relationship. |
| $\frac{3\,\text{printers}}{18\,\text{computers}} = \frac{20\,\text{printers}}{105\,\text{computers}}$ | Write ratios that describe each situation, and set them equal to each other. |
| Printers (check) | Check that the units in the numerators match. |
| Computers (check) | Check that the units in the denominators match.<br>Simplify each fraction and determine if they are equivalent. |
| $\frac{3 \div 3}{18 \div 3} = \frac{1}{6}$<br>$\frac{20 \div 5}{105 \div 5} = \frac{4}{21}$<br>$\frac{1}{6} \neq \frac{4}{21}$ | Since the simplified fractions are not equal (designated by the $\neq$ sign), the proportion is not true. |

The ratio of printers to computers is **not** the same in these two offices.

There is another way to determine whether a proportion is true or false. This method is called "finding the cross product" or "cross multiplying."

To cross multiply, you multiply the numerator of the first ratio in the proportion by the denominator of the other ratio. Then multiply the denominator of the first ratio by the numerator of the second ratio in the proportion. If these products are equal, the proportion is true; if these products are not equal, the proportion is not true.

This strategy for determining whether a proportion is true is called cross-multiplying because the pattern of the multiplication looks like an "x" or a criss-cross. Below is an example of finding a cross product, or cross multiplying.



In this example, you multiply $3 \cdot 10 = 30$, and then multiply $5 \cdot 6 = 30$. Both products are equal, so the proportion is true.

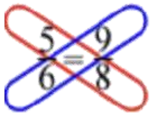
Below is another example of determining if a proportion is true or false by using cross products.

**✓ Example**

**Is the proportion true or false?**

$\frac{5}{6} = \frac{9}{8}$

**Solution**

| | |
|---|---|
|  | Identify the cross product relationship. |
| $5 \cdot 8 = 40$<br>$6 \cdot 9 = 54$ | Use cross products to determine if the proportion is true or false. |
| $40 \neq 54$ | Since the products are not equal, the proportion is false. |

The proportion is false.

**? Exercise**

Is the proportion $\frac{3}{5} = \frac{24}{40}$ true or false?

A. True
B. False

**Answer**

A. Correct. Using cross products, you find that $3 \cdot 40 = 120$ and $5 \cdot 24 = 120$, so the cross products are equal and the proportion is true.
B. Incorrect. The cross products are equal, so the proportion is true. The correct answer is true.

## Finding an Unknown Quantity in a Proportion

If you know that the relationship between quantities is proportional, you can use proportions to find missing quantities. Below is an example.

**✓ Example**

**Solve for the unknown quantity, $n$.**

$\frac{n}{4} = \frac{25}{20}$

**Solution**

| | |
|---|---|
| $20 \cdot n = 4 \cdot 25$ | Cross multiply. |
| $20n = 100$ | You are looking for a number that when you multiply it by 20 you get 100. |
| 5<br>20\longdiv100 | You can find this value by dividing 100 by 20. |

$n = 5$

Now back to the original example. Imagine you want to enlarge a 5-inch by 8-inch photograph to make the length 10 inches and keep the proportion of the width to length the same. You can set up a proportion to determine the width of the enlarged photo.

**Example**

**Find the length of a photograph whose width is 10 inches and whose proportions are the same as a 5-inch by 8-inch photograph.**

**Solution**

| | |
|---|---|
| $\frac{\text{width}}{\text{length}}$ | Determine the relationship. |
| Original photo:<br>$\frac{5 \text{ inches wide}}{8 \text{ inches long}}$<br>Enlarged photo:<br>$\frac{10 \text{ inches wide}}{n \text{ inches long}}$ | Write a ratio that compares the length to the width of each photograph.<br>Use a letter to represent the quantity that is not known (the width of the enlarged photo). |
| $\frac{5}{8}=\frac{10}{n}$ | Write a proportion that states that the two ratios are equal. |
| $5 \cdot n=8 \cdot 10$ | Cross multiply. |
| $5n=80$ | You are looking for a number that, when it is multiplied by 5, will give you 80. |
| $\frac{5n}{5}=\frac{80}{5}$<br>$n=\frac{80}{5}$ | Divide both sides by 5 to isolate the variable. |
| $n=16$ | 16<br>5\longdiv80 |

The length of the enlarged photograph is 16 inches.

## Solving Application Problems Using Proportions

Setting up and solving a proportion is a helpful strategy for solving a variety of proportional reasoning problems. In these problems, it is always important to determine what the unknown value is, and then identify a proportional relationship that you can use to solve for the unknown value. Below are some examples.

**Example**

**Among a species of tropical birds, 30 out of every 50 birds are female. If a certain bird sanctuary has a population of 1,150 of these birds, how many of them would you expect to be female?**

**Solution**

| | |
|---|---|
| Let $x$=the number of female birds in the sanctuary. | Determine the unknown item: the number of female birds in the sanctuary. Assign a letter to this unknown quantity. |
| $\frac{30 \text{ female birds}}{50 \text{ birds}}=\frac{x \text{ female birds in sanctuary}}{1{,}150 \text{ birds in sanctuary}}$ | Set up a proportion, setting the ratios equal. |
| $\frac{30 \div 10}{50 \div 10}=\frac{3}{5}$<br>$\frac{3}{5}=\frac{x}{1150}$ | Simplify the ratio on the left to make the upcoming cross multiplication easier. |

| | |
|---|---|
| $3 \cdot 1150 = 5 \cdot x$<br>$3,450 = 5x$ | Cross multiply. |
| 690<br>5\longdiv3, 450 | What number, when multiplied by 5, gives a product of 3,450? You can find this value by dividing 3,450 by 5. |
| $x = 690$ birds | |

You would expect 690 birds in the sanctuary to be female.

### ✓ Example

**It takes Sandra 1 hour to word process 4 pages. At this rate, how long will she take to complete 27 pages?**

### Solution

| | |
|---|---|
| $\frac{4 \text{ pages}}{1 \text{ hour}} = \frac{27 \text{ pages}}{x \text{ hours}}$ | Set up a proportion comparing the pages she types and the time it takes to type them. |
| $4 \cdot x = 1 \cdot 27$ | Cross multiply. |
| $4x = 27$ | You are looking for a number that, when it is multiplied by 4, will give you 27. |
| 6.75<br>4\longdiv27.00 | You can find this value by dividing 27 by 4. |
| $x = 6.75$ hours | |

It will take Sandra 6.75 hours to complete 27 pages.

### ? Exercise

A map uses a scale where 2 inches represents 5 miles. If the distance between two cities is shown on a map as 20 inches, how many miles apart are the two cities?

A. 50 inches
B. 50 miles
C. 8 miles
D. 100 miles

**Answer**

A. Incorrect. The distance between the cities is measured in inches on the map, but in miles in actuality. The correct answer is 50 miles.
B. Correct. Setting up the proportion $\frac{2 \text{ inches}}{5 \text{ miles}} = \frac{20 \text{ inches}}{x}$, you find that $x = 50$ miles.
C. Incorrect. When you solve a proportion, you cross multiply: $\frac{2 \text{ inches}}{5 \text{ miles}} = \frac{20 \text{ inches}}{x}$, $2x = 5 \cdot 20$. The correct answer is 50 miles.
D. Incorrect. When you solve a proportion, you cross multiply: $\frac{2 \text{ inches}}{5 \text{ miles}} = \frac{20 \text{ inches}}{x}$, $2x = 5 \cdot 20$. The correct answer is 50 miles.

## Summary

A proportion is an equation comparing two ratios. If the ratios are equivalent, the proportion is true. If not, the proportion is false. Finding a cross product is another method for determining whether a proportion is true or false. Cross multiplying is also helpful for finding an unknown quantity in a proportional relationship. Setting up and solving proportions is a skill that is useful for solving a variety of problems.

# 10.5: Percent

We have to work with money every day. Calculating your monthly expenses, splitting the tab on a restaurant bill, or figuring out how much the tip should be, requires only arithmetic. But when we start saving for the future, planning for retirement, or need a loan, then we need more mathematics.

When one hears the word “percent,” other words come immediately to mind, words such as “century,” “cents,” or “centimeters.” A century equals 100 years. There are one hundred cents in a dollar and there are 100 centimeters in a meter. Thus, it should come as no surprise that percent means “parts per hundred.”

**The Meaning of Percent**

In the square shown in Figure $10.5.1$, a large square has been partitioned into ten rows of ten little squares in each row. We have shaded 20 of 100 possible little squares, or 20% of the total number of little squares.

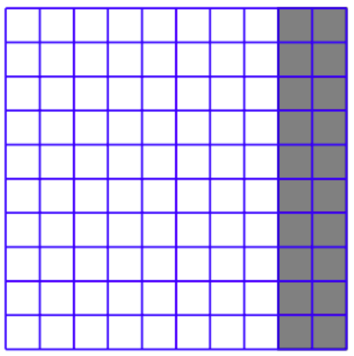


Figure $10.5.1$: Shading 20 of 100 little squares, or 20% of the total number of little squares.

Notice in the figure that 80 out of a possible 100 squares are left unshaded. Thus, 80% of the little squares are unshaded. If instead we shaded 35 out of the 100 squares, then 35% of the little squares would be shaded. If we shaded all of the little squares, then 100% of the little squares would be shaded (100 out of 100).

So, when you hear the word “percent,” think “parts per hundred.”

## Changing a Percent to a Fraction

Based on the discussion above, it is fairly straightforward to change a percent to a fraction.

**Percent to Fraction**

To change a percent to a fraction, drop the percent sign and put the number over 100.

**Example 1**

Change 24% to a fraction.

**Solution**

Drop the percent symbol and put 24 over 100.

$$\begin{aligned} 24\% &= \frac{24}{100} && \text{Percent: Parts per hundred.} \\ &= \frac{6}{25} && \text{Reduce.} \end{aligned}$$

Hence, 24% = 6/25.

**Exercise**

Change 36% to a fraction reduced to lowest terms.

**Answer**

9/25

**Example 2**

Change $14\frac{2}{7}\%$ to a fraction.

**Solution**

Drop the percent symbol and put $14\frac{2}{7}$ over 100.

$$\begin{aligned} 14\frac{2}{7}\% &= \frac{14\frac{2}{7}}{100} && \text{Percent: Parts per hundred.} \\ &= \frac{\frac{100}{7}}{100} && \text{Mixed to improper fraction.} \\ &= \frac{100}{7} \cdot \frac{1}{100} && \text{Invert and multiply.} \\ &= \frac{\cancel{100}}{7} \cdot \frac{1}{\cancel{100}} && \text{Cancel.} \\ &= \frac{1}{7} \end{aligned}$$

Hence, $14\frac{2}{7}\% = 1/7$.

**Exercise**

Change $11\frac{1}{9}\%$ to a fraction reduced to lowest terms.

**Answer**

1/9

**Example 3**

Change 28.4% to a fraction.

**Solution**

Drop the percent symbol and put 28.4 over 100.

$$\begin{aligned} 28.4\% &= \frac{28.4}{100} && \text{Percent: Parts per hundred.} \\ &= \frac{28.4 \cdot 10}{100 \cdot 10} && \text{Multiply numerator and denominator by 10.} \\ &= \frac{284}{1000} && \text{Multiplying by 10 moves decimal point one place right.} \\ &= \frac{71 \cdot 4}{250 \cdot 4} && \text{Factor.} \\ &= \frac{71}{250} && \text{Cancel common factor.} \end{aligned}$$

**Exercise**

Change 87.5% to a fraction reduced to lowest terms.

**Answer**

7/8

## Changing a Percent to a Decimal

To change a percent to a decimal, we need only remember that percent means “parts per hundred.”

**Example 4**

Change 23.25% to a decimal.

**Solution**

Drop the percent symbol and put 23.25 over 100.

$$\begin{aligned} 23.25\% &= \frac{23.25}{100} && \text{Percent: Parts per hundred.} \\ &= 0.2325 && \text{Dividing by 100 moves decimal point 2 places left.} \end{aligned}$$

Therefore, 23.25% = 0.2325.

**Exercise**

Change 2.4% to a decimal.

**Answer**

0.024

This last example motivates the following simple rule.

**Percent to a Decimal**

To change a percent to a decimal, drop the percent symbol and move the decimal point two places to the left.

**Example 5**

Change $5\frac{1}{2}\%$ to a decimal.

**Solution**

Note that 1/2=0.5, then move the decimal 2 places to the left.

$$\begin{aligned} 5\frac{1}{2}\% &= 5.5\% && 1/2 = 0.5. \\ &= 0.055 && \begin{array}{l}\text{Drop \% symbol.} \\ \text{Move decimal point 2 places left.}\end{array} \\ &= 0.055 \end{aligned}$$

Thus, $5\frac{1}{2}\% = 0.055$.

**Exercise 10.5.1**

Change $6\frac{3}{4}\%$ to a decimal.

**Answer**

0.0675

## Changing a Decimal to a Percent

Changing a decimal to a percent is the exact opposite of changing a percent to a decimal. In the latter case, we drop the percent symbol and move the decimal point 2 places to the left. The following rule does just the opposite.

**Decimal to a Percent**

To change a decimal to a percent, move the decimal point two places to the right and add a percent symbol.

**Example 6**

Change 0.0725 to a percent.

**Solution**

Move the decimal point two places to the right and add a percent symbol.

$$\begin{aligned} 0.0725 &= 007.25\% \\ &= 7.25\% \end{aligned}$$

**Exercise**

Change to 0.0375 to a percent.

**Answer**

3.75%

**Example 7**

Change 1.025 to a percent.

**Solution**

Move the decimal point two places to the right and add a percent symbol.

$$\begin{aligned} 1.025 &= 102.5\% \\ &= 102.5\% \end{aligned}$$

**Exercise**

Change 0.525 to a percent.

**Answer**

52.5%

## Changing a Fraction to a Percent

One way to proceed is to first change the fraction to a decimal, then change the resulting decimal to a percent.

### Fractions to Percents: Technique #1

To change a fraction to a percent, follow these steps:

1. Divide numerator by the denominator to change the fraction to a decimal.
2. Move the decimal point in the result two places to the right and append a percent symbol.

### Example 8

Use Technique #1 to change 5/8 to a percent.

**Solution**

Change 5/8 to a decimal, then change the decimal to a percent.

To change 5/8 to a decimal, divide 5 by 8. Since the denominator is a product of twos, the decimal should terminate.

$$\begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ \underline{4\,8}\phantom{00} \\ 20\phantom{0} \\ \underline{16}\phantom{0} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

To change 0.625 to a percent, move the decimal point 2 places to the right and append a percent symbol.

0.625 = 0 62.5% = 62.5%

### Exercise

Change 5/16 to a percent.

**Answer**

31.35%

A second technique is to create an equivalent fraction with a denominator of 100.

### Fractions to Percents: Technique #2

To change a fraction to a percent, create an equivalent fraction with a denominator of 100.

### Example 9

Use Technique #2 to change 5/8 to a percent.

**Solution**

Create an equivalent fraction for 5/8 with a denominator of 100.

$$\frac{5}{8} = \frac{x}{100}$$

Solve this proportion for *x*.

$$\begin{aligned} 8x &= 500 && \text{Cross multiply.} \\ \frac{8x}{8} &= \frac{500}{8} && \text{Divide both sides by } 8. \\ x &= \frac{125}{2} && \text{Reduce: Divide numerator and denominator by } 4. \\ x &= 62.5 && \text{Divide.} \end{aligned}$$

Thus,

$$\frac{5}{8} = \frac{62.5}{100} = 62.5\%.$$

**Alternate Ending**

We could also change 125/2 to a mixed fraction; i.e., 125/2 = 62 1 2 . Then,

$$\frac{5}{8} = \frac{62\frac{1}{2}}{100} = 62\frac{1}{2}\%.$$

Same answer.

**Exercise**

Change 4/9 to a percent.

**Answer**

$44\frac{4}{9}\%$

Sometimes we will be content with an approximation.

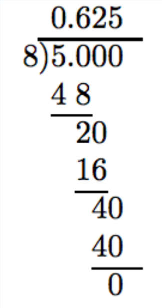
**Example 10**

Change 4/13 to a percent. Round your answer to the nearest tenth of a percent.

**Solution**

We will use Technique #1.

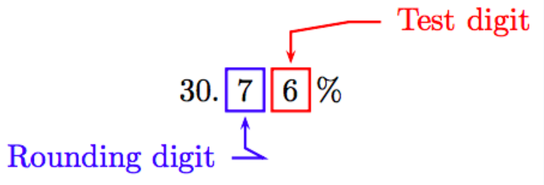
To change 4/13 to a decimal, divide 4 by 13. Since the denominator has factors other than 2’s and 5’s, the decimal will repeat. However, we intend to round to the nearest tenth of a percent, so we will carry the division to four decimal places only. (*Four places are necessary because we will be moving the decimal point two places to the right.*)



To change the decimal to a percent, move the decimal point two places to the right.

0.3076 ≈ 0 30.76% ≈ 30.76%

To round to the nearest tenth of a percent, identify the rounding and test digits.



Because the test digit is greater than or equal to 5, add 1 to the rounding digit and truncate. Thus,

$$0.03076 \approx 30.8\%.$$

**? Exercise**

Change 4/17 to a percent. Round your answer to the nearest tenth of a percent.

**Answer**

23.5%

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## SECTION OVERVIEW

# 10.6: Problem Solving

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# 10.6.1: Introduction

In previous math courses, you’ve no doubt run into the infamous “word problems.” Unfortunately, these problems rarely resemble the type of problems we actually encounter in everyday life. In math books, you usually are told exactly which formula or procedure to use, and are given exactly the information you need to answer the question. In real life, problem solving requires identifying an appropriate formula or procedure, and determining what information you will need (and won’t need) to answer the question.

In this chapter, we will review several basic but powerful algebraic ideas: percents, rates, and proportions. We will then focus on the problem solving process, and explore how to use these ideas to solve problems where we don’t have perfect information.

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# 10.6.2: Percents

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties[1]. Who is correct? How can we make sense of these numbers?

**Percent** literally means "per 100," or "parts per hundred." When we write 40%, this is equivalent to the fraction $\frac{40}{100}$ or the decimal 0.40. Notice that 80 out of 200 and 10 out of 25 are also 40%, since $\frac{80}{200}=\frac{10}{25}=\frac{40}{100}$ .

**✓ Example 1**

243 people out of 400 state that they like dogs. What percent is this?

**Solution**

$\frac{243}{400}=0.6075=\frac{60.75}{100}$. This is 60.75%.

Notice that the percent can be found from the equivalent decimal by moving the decimal point two places to the right.

**✓ Example 2**

Write each as a percent: a) $\frac{1}{4}$ b) 0.02 c) 2.35

**Solution**

a) $\frac{1}{4}=0.25=25\%$ b) $0.02=2\%$ c) $2.35=235\%$

**Percents**

If we have a *part* that is some *percent* of a *whole*, then

$\text{percent} = \frac{\text{part}}{\text{whole}}$ , or equivalently, $\text{part} = \text{percent}\cdot\text{whole}$

To do the calculations, we write the percent as a decimal.

**✓ Example 3**

The sales tax in a town is 9.4%. How much tax will you pay on a $140 purchase?

**Solution**

Here, $140 is the whole, and we want to find 9.4% *of* $140. We start by writing the percent as a decimal by moving the decimal point two places to the left (which is equivalent to dividing by 100). We can then compute:

$\text{tax}=0.094(140)=\$13.16$in tax.

**✓ Example 4**

In the news, you hear "tuition is expected to increase by 7% next year." If tuition this year was $1200 per quarter, what will it be next year?

**Solution**

The tuition next year will be the current tuition plus an additional 7%, so it will be 107% of this year's tuition:

$\$1200(1.07)=\$1284$

Alternatively, we could have first calculated 7% of $1200: $\$1200(0.07)=\$84$

Notice this is *not* the expected tuition for next year (we could only wish). Instead, this is the expected *increase*, so to calculate the expected tuition, we'll need to add this change to the previous year's tuition:

$\$1200+\$84=\$1284$

## ? Try it Now 1

A TV originally priced at $799 is on sale for 30% off. There is then a 9.2% sales tax. Find the price after including the discount and sales tax.

**Answer**

The sale price is $\$799(0.70) = \$559.30$After tax, the price is $\$559.30(1.092) = \$610.76$

## ✓ Example 5

The value of a car dropped from $7400 to $6800 over the last year. What percent decrease is this?

### Solution

To compute the percent change, we first need to find the dollar value change: $\$6800 - \$7400 = -\$600$ Often we will take the absolute value of this amount, which is called the **absolute change**: $|-600| = 600$.

Since we are computing the decrease relative to the starting value, we compute this percent out of $\$7400$:

$\frac{600}{7400} = 0.081 = 8.1\%$ decrease. This is called a **relative change**.

## Absolute and Relative Change

Given two quantities,

Absolute change = $|\text{ending quantity } - \text{ starting quantity}|$

Relative change: $\frac{\text{absolute change}}{\text{starting quantity}}$

Absolute change has the same units as the original quantity.

Relative change gives a percent change.

The starting quantity is called the **base** of the percent change.

The base of a percent is very important. For example, while Nixon was president, it was argued that marijuana was a “gateway” drug, claiming that 80% of marijuana smokers went on to use harder drugs like cocaine. The problem is, this isn’t true. The true claim is that 80% of harder drug users first smoked marijuana. The difference is one of base: 80% of marijuana smokers using hard drugs, vs. 80% of hard drug users having smoked marijuana. These numbers are not equivalent. As it turns out, only one in 2,400 marijuana users actually go on to use harder drugs[2].

## ✓ Example 6

There are about 75 QFC supermarkets in the U.S. Albertsons has about 215 stores. Compare the size of the two companies.

### Solution

When we make comparisons, we must ask first whether an absolute or relative comparison. The absolute difference is $215 - 75 = 140$. From this, we could say “Albertsons has 140 more stores than QFC.” However, if you wrote this in an article or paper, that number does not mean much. The relative difference may be more meaningful. There are two different relative changes we could calculate, depending on which store we use as the base:

Using QFC as the base, $\frac{140}{75} = 1.867$.

This tells us Albertsons is 186.7% larger than QFC.

Using Albertsons as the base, $\frac{140}{215} = 0.651$.

This tells us QFC is 65.1% smaller than Albertsons.

Notice both of these are showing percent *differences*. We could also calculate the size of Albertsons relative to QFC: , which tells us Albertsons is 2.867 times the size of QFC. Likewise, we could calculate the size of QFC relative to Albertsons: , which tells us that QFC is 34.9% of the size of Albertsons.

## ✓ Example 7

Suppose a stock drops in value by 60% one week, then increases in value the next week by 75%. Is the value higher or lower than where it started?

### Solution

To answer this question, suppose the value started at $100. After one week, the value dropped by 60%:

$$\$100-\$100(0.60)=\$100-\$60=\$40.$$

In the next week, notice that base of the percent has changed to the new value, $40. Computing the 75% increase:

$$\$40+\$40(0.75)=\$40+\$30=\$70.$$

In the end, the stock is still $30 lower, or $\frac{\$30}{\$100}=30\%$ lower, valued than it started.

## ? Try it Now 2

The U.S. federal debt at the end of 2001 was $5.77 trillion, and grew to $6.20 trillion by the end of 2002. At the end of 2005 it was $7.91 trillion, and grew to $8.45 trillion by the end of 2006[3]. Calculate the absolute and relative increase for 2001-2002 and 2005-2006. Which year saw a larger increase in federal debt?

**Answer**

2001-2002: Absolute change: $0.43 trillion. Relative change: 7.45%

2005-2006: Absolute change: $0.54 trillion. Relative change: 6.83%

2005-2006 saw a larger absolute increase, but a smaller relative increase.

## ✓ Example 8

A Seattle Times article on high school graduation rates reported “The number of schools graduating 60 percent or fewer students in four years – sometimes referred to as “dropout factories” – decreased by 17 during that time period. The number of kids attending schools with such low graduation rates was cut in half.”

a) Is the “decrease by 17” number a useful comparison?

b) Considering the last sentence, can we conclude that the number of “dropout factories” was originally 34?

### Solution

a) This number is hard to evaluate, since we have no basis for judging whether this is a larger or small change. If the number of “dropout factories” dropped from 20 to 3, that’d be a very significant change, but if the number dropped from 217 to 200, that’d be less of an improvement.

b) The last sentence provides relative change which helps put the first sentence in perspective. We can estimate that the number of “dropout factories” was probably previously around 34. However, it’s possible that students simply moved schools rather than the school improving, so that estimate might not be fully accurate.

## ✓ Example 9

In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties. Who is correct?

### Solution

Without more information, it is hard for us to judge who is correct, but we can easily conclude that these two percents are talking about different things, so one does not necessarily contradict the other. Edward's claim was a percent with coalition forces as the base of the percent, while Cheney's claim was a percent with both coalition and Iraqi security forces as the base of the percent. It turns out both statistics are in fact fairly accurate.

## ? Try it Now 3

In the 2012 presidential elections, one candidate argued that "the president's plan will cut $716 billion from Medicare, leading to fewer services for seniors," while the other candidate rebuts that "our plan does not cut current spending and actually expands benefits for seniors, while implementing cost saving measures." Are these claims in conflict, in agreement, or not comparable because they're talking about different things?

**Answer**

> Without more information, it is hard to judge these arguments. This is compounded by the complexity of Medicare. As it turns out, the $716 billion is not a cut in current spending, but a cut in future increases in spending, largely reducing future growth in health care payments. In this case, at least the numerical claims in both statements could be considered at least partially true. Here is one source of more information if you're interested: http://factcheck.org/2012/08/a-campaign-full-of-mediscare/

We'll wrap up our review of percents with a couple cautions. First, when talking about a change of quantities that are already measured in percents, we have to be careful in how we describe the change.

## ✓ Example 10

A politician's support increases from 40% of voters to 50% of voters. Describe the change.

### Solution

We could describe this using an absolute change: $|50\% - 40\%| = 10\%$. Notice that since the original quantities were percents, this change also has the units of percent. In this case, it is best to describe this as an increase of 10 **percentage points**.

In contrast, we could compute the percent change: $\frac{10\%}{40\%} = 0.25 = 25\%$ increase. This is the relative change, and we'd say the politician's support has increased by 25%.

Lastly, a caution against averaging percents.

## ✓ Example 11

A basketball player scores on 40% of 2-point field goal attempts, and on 30% of 3-point of field goal attempts. Find the player's overall field goal percentage.

### Solution

It is very tempting to average these values, and claim the overall average is 35%, but this is likely not correct, since most players make many more 2-point attempts than 3-point attempts. We don't actually have enough information to answer the question. Suppose the player attempted 200 2-point field goals and 100 3-point field goals. Then they made 200(0.40) = 80 2-point shots and $100(0.30) = 30$ 3-point shots. Overall, they made 110 shots out of 300, for a $\frac{110}{300} = 0.367 = 36.7\%$ overall field goal percentage.

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[1] www.factcheck.org/cheney_edwards_mangle_facts.html

[2] http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics

[3] www.whitehouse.gov/sites/defa...s/hist07z1.xls

## 10.6.3: Proportions and Rates

If you wanted to power the city of Seattle using wind power, how many windmills would you need to install? Questions like these can be answered using rates and proportions.

### Rates

A rate is the ratio (fraction) of two quantities.

A **unit rate** is a rate with a denominator of one.

### Example 12

Your car can drive 300 miles on a tank of 15 gallons. Express this as a rate.

**Solution**

Expressed as a rate, $\frac{300 \text{ miles}}{15 \text{ gallons}}$. We can divide to find a unit rate: $\frac{20 \text{ miles}}{1\text{gallon}}$, which we could also write as $20 \frac{\text{miles}}{\text{gallon}}$, or just 20 miles per gallon.

### Proportion Equation

A proportion equation is an equation showing the equivalence of two rates or ratios.

### Example 13

Solve the proportion $\frac{5}{3} = \frac{x}{6}$ for the unknown value $x$.

**Solution**

This proportion is asking us to find a fraction with denominator 6 that is equivalent to the fraction $\frac{5}{3}$. We can solve this by multiplying both sides of the equation by 6, giving $x = \frac{5}{3} \cdot 6 = 10$ .

### Example 14

A map scale indicates that $\frac{1}{2}$ inch on the map corresponds with 3 real miles. How many miles apart are two cities that are $2\frac{1}{4}$ inches apart on the map?

**Solution**

We can set up a proportion by setting equal two $\frac{\text{map inches}}{\text{real miles}}$ rates, and introducing a variable, $x$, to represent the unknown quantity – the mile distance between the cities.

$$\begin{array}{ll}
\frac{\frac{1}{2}\text{ map inch}}{3\text{ miles}} = \frac{2\frac{1}{4}\text{ map inches}}{x\text{ miles}} & \text{Multiply both sides by } x \text{ and rewriting the mixed number} \\
\frac{\frac{1}{2}}{3} \cdot x = \frac{9}{4} & \text{Multiply both sides by 3} \\
\frac{1}{2}x = \frac{27}{4} & \text{Multiply both sides by 2 (or divide by } \frac{1}{2}) \\
x = \frac{27}{2} = 13\frac{1}{2} \text{ miles} &
\end{array}$$

Many proportion problems can also be solved using **dimensional analysis**, the process of multiplying a quantity by rates to change the units.

### Example 15

Your car can drive 300 miles on a tank of 15 gallons. How far can it drive on 40 gallons?

**Solution**

We could certainly answer this question using a proportion: $\$\frac{300 \text{ miles}}{15 \text{ gallons}} = \frac{x \text{ miles}}{40 \text{ gallons}}\$$.

However, we earlier found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit “cancels” and we’re left with a number of miles:

$$40\text{gallons} \cdot \frac{20 \text{ miles}}{\text{gallon}} = \frac{40 \text{ gallons}}{1} \cdot \frac{20 \text{ miles}}{\text{gallon}} = 800 \text{ miles}$$

Notice if instead we were asked “how many gallons are needed to drive 50 miles?” we could answer this question by inverting the 20 mile per gallon rate so that the *miles* unit cancels and we’re left with gallons:

$$50\text{miles} \cdot \frac{\text{lgallon}}{20 \text{ miles}} = \frac{50 \text{ miles}}{1} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ gallons}}{20} = 2.5 \text{ gallons}$$

Dimensional analysis can also be used to do unit conversions. Here are some unit conversions for reference.

### Unit Conversions

**Length**

$$\begin{array}{ll}
1 \text{ foot (ft)} = 12 \text{ inches (in)} & 1 \text{ yard (yd) } = 3 \text{ feet (ft)} \\
1 \text{ mile } = 5,280 \text{ feet} & \\
1000 \text{ millimeters } mm = 1 \text{ meter (m)} & 100 \text{ centimeters (cm)} = 1 \text{ meter} \\
1000 \text{ meters (m)} = 1 \text{ kilometer (km)} & 2.54 \text{ centimeters (cm) } = 1 \text{ inch}
\end{array}$$

**Weight and Mass**

$$\begin{array}{ll}
1 \text{ pound (lb) } = 16 \text{ ounces (oz)} & 1 \text{ ton } = 2000 \text{ pounds} \\
1000 \text{ milligrams (mg) } = 1 \text{ gram (g)} & 1000 \text{ grams } = 1 \text{ kilogram (kg)} \\
1 \text{ kilogram } = 2.2 \text{ pounds (on earth)} &
\end{array}$$

**Capacity**

$$\begin{array}{ll}
1 \text{ cup } = 8 \text{ fluid ounces (fl oz)}^* & 1 \text{ pint } = 2 \text{ cups} \\
1 \text{ quart } = 2 \text{ pints } = 4 \text{ cups} & 1 \text{ gallon } = 4 \text{ quarts } = 16 \text{ cups} \\
1000 \text{ milliliters (ml) } = 1 \text{ liter (L)} &
\end{array}$$

*Fluid ounces are a capacity measurement for liquids. 1 fluid ounce ≈ 1 ounce (weight) for water only.

### Example 16

A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

**Solution**

To answer this question, we need to convert 20 seconds into feet. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don’t, we will need to do additional unit conversions. We will need to know that 5280 ft = 1 mile. We might start by converting the 20 seconds into hours:

$$\begin{array}{ll}
20 \text{ seconds } \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{180} \text{ hour} & \text{Now we can multiply by the 15 miles/hr} \\
\frac{1}{180} \text{ hour } \cdot \frac{15 \text{ miles}}{\text{lhour}} = \frac{1}{12} \text{ mile} & \text{Now we can convert to feet} \\
\frac{1}{12} \text{ mile } \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet} &
\end{array}$$

We could have also done this entire calculation in one long set of products:

$$20 \text{ seconds } \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{15 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

### Try it Now 4

A 1000 foot spool of bare 12-gauge copper wire weighs 19.8 pounds. How much will 18 inches of the wire weigh, in ounces?

**Answer**

$$18 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{19.8 \text{ pounds}}{1000 \text{ feet}} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} \approx 0.475 \text{ ounces}$$

Notice that with the miles per gallon example, if we double the miles driven, we double the gas used. Likewise, with the map distance example, if the map distance doubles, the real-life distance doubles. This is a key feature of proportional relationships, and one we must confirm before assuming two things are related proportionally.

### ✓ Example 17

Suppose you're tiling the floor of a 10 ft by 10 ft room, and find that 100 tiles will be needed. How many tiles will be needed to tile the floor of a 20 ft by 20 ft room?

**Solution**

In this case, while the width the room has doubled, the area has quadrupled. Since the number of tiles needed corresponds with the area of the floor, not the width, 400 tiles will be needed. We could find this using a proportion based on the areas of the rooms:

$$\frac{100 \text{ tiles}}{100 \text{ft}^2} = \frac{n \text{ tiles}}{400 \text{ft}^2}$$

Other quantities just don't scale proportionally at all.

### ✓ Example 18

Suppose a small company spends \$1000 on an advertising campaign, and gains 100 new customers from it. How many new customers should they expect if they spend \$10,000?

**Solution**

While it is tempting to say that they will gain 1000 new customers, it is likely that additional advertising will be less effective than the initial advertising. For example, if the company is a hot tub store, there are likely only a fixed number of people interested in buying a hot tub, so there might not even be 1000 people in the town who would be potential customers.

Sometimes when working with rates, proportions, and percents, the process can be made more challenging by the magnitude of the numbers involved. Sometimes, large numbers are just difficult to comprehend.

### ✓ Example 19

Compare the 2010 U.S. military budget of \$683.7 billion to other quantities.

Here we have a very large number, about \$683,700,000,000 written out. Of course, imagining a billion dollars is very difficult, so it can help to compare it to other quantities.

If that amount of money was used to pay the salaries of the 1.4 million Walmart employees in the U.S., each would earn over \$488,000.

There are about 300 million people in the U.S. The military budget is about \$2,200 per person.

If you were to put \$683.7 billion in \$100 bills, and count out 1 per second, it would take 216 years to finish counting it.

### ✓ Example 20

Compare the electricity consumption per capita in China to the rate in Japan.

To address this question, we will first need data. From the CIA[1] website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank[2], we can find the population of China is 1,344,130,000, or 1.344 billion, and the population of Japan is 127,817,277, or 127.8 million.

**Solution**

Computing the consumption per capita for each country:

China: $\frac{4{,}693{,}000{,}000{,}000\text{KWH}}{1{,}344{,}130{,}000\text{ people}} \approx 3491.5$ KWH per person

Japan: $\frac{859{,}700{,}000{,}000\text{KWH}}{127{,}817{,}277\text{ people}} \approx 6726$ KWH per person

While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

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[1] www.cia.gov/library/publicat.../2042rank.html

[2] http://data.worldbank.org/indicator/SP.POP.TOT

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# 10.6.4: Geometry

Geometric shapes, as well as area and volumes, can often be important in problem solving.

### ✓ Example 21

You are curious how tall a tree is, but don’t have any way to climb it. Describe a method for determining the height.

There are several approaches we could take. We’ll use one based on triangles, which requires that it’s a sunny day. Suppose the tree is casting a shadow, say 15 ft long. I can then have a friend help me measure my own shadow. Suppose I am 6 ft tall, and cast a 1.5 ft shadow. Since the triangle formed by the tree and its shadow has the same angles as the triangle formed by me and my shadow, these triangles are called **similar triangles** and their sides will scale proportionally. In other words, the ratio of height to width will be the same in both triangles. Using this, we can find the height of the tree, which we’ll denote by $h$:

**Solution**

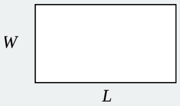
$$\frac{6 \text{ft tall}}{1.5 \text{ft shadow}} = \frac{h \text{ft tall}}{15 \text{ft shadow}}$$

Multiplying both sides by 15, we get $h = 60$. The tree is about 60 ft tall.

It may be helpful to recall some formulas for areas and volumes of a few basic shapes.

### Areas

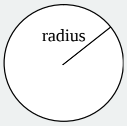
**Rectangle**



Area: $L \cdot W$

Perimeter: $2L + 2W$

**Circle**, radius $r$

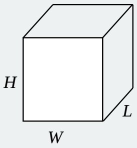


Area: $\pi r^2$

Circumference = $2\pi r$

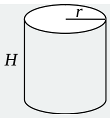
### Volumes

**Rectangular Box**



Volume: $L \cdot W \cdot H$

**Cylinder**



Volume: $\pi r^2 H$

## Example 22

If a 12 inch diameter pizza requires 10 ounces of dough, how much dough is needed for a 16 inch pizza?

**Solution**

To answer this question, we need to consider how the weight of the dough will scale. The weight will be based on the volume of the dough. However, since both pizzas will be about the same thickness, the weight will scale with the area of the top of the pizza. We can find the area of each pizza using the formula for area of a circle, $A = \pi r^2$:

A 12” pizza has radius 6 inches, so the area will be $\pi 6^2$ = about 113 square inches.

A 16” pizza has radius 8 inches, so the area will be $\pi 8^2$ = about 201 square inches.

Notice that if both pizzas were 1 inch thick, the volumes would be 113 in$^3$ and 201 in$^3$ respectively, which are at the same ratio as the areas. As mentioned earlier, since the thickness is the same for both pizzas, we can safely ignore it.

We can now set up a proportion to find the weight of the dough for a 16” pizza:

$\frac{10 \text{ ounces}}{113 \text{in}^2} = \frac{x \text{ ounces}}{201 \text{in}^2}$ Multiply both sides by 201

$x = 201 \cdot \frac{10}{113} =$ about 17.8 ounces of dough for a 16” pizza.

It is interesting to note that while the diameter is $\frac{16}{12} = 1.33$ times larger, the dough required, which scales with area, is $1.33^2 = 1.78$ times larger.

## Example 23

A company makes regular and jumbo marshmallows. The regular marshmallow has 25 calories. How many calories will the jumbo marshmallow have?

We would expect the calories to scale with volume. Since the marshmallows have cylindrical shapes, we can use that formula to find the volume. From the grid in the image, we can estimate the radius and height of each marshmallow.

Photo courtesy Christopher Danielson

**Solution**

The regular marshmallow appears to have a diameter of about 3.5 units, giving a radius of 1.75 units, and a height of about 3.5 units. The volume is about $\pi(1.75)^2(3.5) = 33.7 \text{ units}^3$.

The jumbo marshmallow appears to have a diameter of about 5.5 units, giving a radius of 2.75 units, and a height of about 5 units. The volume is about $\pi(2.75)^2(5) = 118.8 \text{ units}^3$.

We could now set up a proportion, or use rates. The regular marshmallow has 25 calories for 33.7 cubic units of volume. The jumbo marshmallow will have:

$118.8 \text{ units }^3 \cdot \frac{25 \text{ calories}}{33.7 \text{ units }^3} = 88.1 \text{ calories}$

It is interesting to note that while the diameter and height are about 1.5 times larger for the jumbo marshmallow, the volume and calories are about $1.5^3 = 3.375$ times larger.

**Try it Now 5**

A website says that you’ll need 48 fifty-pound bags of sand to fill a sandbox that measure 8ft by 8ft by 1ft. How many bags would you need for a sandbox 6ft by 4ft by 1ft?

**Answer**

The original sandbox has volume $64\mathrm{ft}^3$. The smaller sandbox has volume $24\mathrm{ft}^3$.

$\frac{48\text{bags}}{64\mathrm{ft}^3} = \frac{x\text{bags}}{24\mathrm{ft}^3}$ results in $x = 18$ bags.

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# 10.6.5: Problem Solving and Estimating

Finally, we will bring together the mathematical tools we've reviewed, and use them to approach more complex problems. In many problems, it is tempting to take the given information, plug it into whatever formulas you have handy, and hope that the result is what you were supposed to find. Chances are, this approach has served you well in other math classes.

This approach does not work well with real life problems. Instead, problem solving is best approached by first starting at the end: identifying exactly what you are looking for. From there, you then work backwards, asking "what information and procedures will I need to find this?" Very few interesting questions can be answered in one mathematical step; often times you will need to chain together a solution pathway, a series of steps that will allow you to answer the question.

### Problem Solving Process

1. Identify the question you're trying to answer.
2. Work backwards, identifying the information you will need and the relationships you will use to answer that question.
3. Continue working backwards, creating a solution pathway.
4. If you are missing necessary information, look it up or estimate it. If you have unnecessary information, ignore it.
5. Solve the problem, following your solution pathway.

In most problems we work, we will be approximating a solution, because we will not have perfect information. We will begin with a few examples where we will be able to approximate the solution using basic knowledge from our lives.

### Example 24

How many times does your heart beat in a year?

**Solution**

This question is asking for the rate of heart beats per year. Since a year is a long time to measure heart beats for, if we knew the rate of heart beats per minute, we could scale that quantity up to a year. So the information we need to answer this question is heart beats per minute. This is something you can easily measure by counting your pulse while watching a clock for a minute.

Suppose you count 80 beats in a minute. To convert this beats per year:

$$\frac{80 \text{ beats}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{365 \text{ days}}{1 \text{ year}} = 42,048,000 \text{ beats per year}$$

### Example 25

How thick is a single sheet of paper? How much does it weigh?

**Solution**

While you might have a sheet of paper handy, trying to measure it would be tricky. Instead we might imagine a stack of paper, and then scale the thickness and weight to a single sheet. If you've ever bought paper for a printer or copier, you probably bought a ream, which contains 500 sheets. We could estimate that a ream of paper is about 2 inches thick and weighs about 5 pounds. Scaling these down,

$$\frac{2 \text{ inches}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.004 \text{ inches per sheet}$$

$$\frac{5 \text{ pounds}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.01 \text{ pounds per sheet, or 0.16 ounces per sheet.}$$

### Example 26

A recipe for zucchini muffins states that it yields 12 muffins, with 250 calories per muffin. You instead decide to make mini-muffins, and the recipe yields 20 muffins. If you eat 4, how many calories will you consume?

**Solution**

There are several possible solution pathways to answer this question. We will explore one.

To answer the question of how many calories 4 mini-muffins will contain, we would want to know the number of calories in each mini-muffin. To find the calories in each mini-muffin, we could first find the total calories for the entire recipe, then divide it by the number of mini-muffins produced. To find the total calories for the recipe, we could multiply the calories per standard muffin by the number per muffin. Notice that this produces a multi-step solution pathway. It is often easier to solve a problem in small steps, rather than trying to find a way to jump directly from the given information to the solution.

We can now execute our plan:

$12\text{muffins\$} \cdot \frac{250 \text{ calories}}{\text{muffin}} = 3000\$$ calories for the whole recipe

$\frac{3000 \text{ calories}}{20 \text{ mini} - \text{muffins}}$ gives 150 calories per mini-muffin

$4 \text{ mini muffins} \cdot \frac{150 \text{ calories}}{\text{mini - muffin}}$ totals 600 calories consumed.

### Example 27

You need to replace the boards on your deck. About how much will the materials cost?

#### Solution

There are two approaches we could take to this problem: 1) estimate the number of boards we will need and find the cost per board, or 2) estimate the area of the deck and find the approximate cost per square foot for deck boards. We will take the latter approach.

For this solution pathway, we will be able to answer the question if we know the cost per square foot for decking boards and the square footage of the deck. To find the cost per square foot for decking boards, we could compute the area of a single board, and divide it into the cost for that board. We can compute the square footage of the deck using geometric formulas. So first we need information: the dimensions of the deck, and the cost and dimensions of a single deck board.

Suppose that measuring the deck, it is rectangular, measuring 16 ft by 24 ft, for a total area of $384 \text{ft}^2$.

From a visit to the local home store, you find that an 8 foot by 4 inch cedar deck board costs about $7.50. The area of this board, doing the necessary conversion from inches to feet, is:

$8 \text{ feet } \cdot 4 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = 2.667 \text{ft}^2$ . The cost per square foot is then

$\frac{\$7.50}{2.667 \text{ft}^2} = \$2.8125 \text{ per ft}^2$.

This will allow us to estimate the material cost for the whole $384 \text{ft}^2$ deck

$\$384 \text{ft}^2 \cdot \frac{\$2.8125}{\text{ft}^2} = \$1080$ total cost.

Of course, this cost estimate assumes that there is no waste, which is rarely the case. It is common to add at least 10% to the cost estimate to account for waste.

### Example 28

Is it worth buying a Hyundai Sonata hybrid instead the regular Hyundai Sonata?

#### Solution

To make this decision, we must first decide what our basis for comparison will be. For the purposes of this example, we'll focus on fuel and purchase costs, but environmental impacts and maintenance costs are other factors a buyer might consider.

It might be interesting to compare the cost of gas to run both cars for a year. To determine this, we will need to know the miles per gallon both cars get, as well as the number of miles we expect to drive in a year. From that information, we can find the number of gallons required from a year. Using the price of gas per gallon, we can find the running cost.

From Hyundai's website, the 2013 Sonata will get 24 miles per gallon (mpg) in the city, and 35 mpg on the highway. The hybrid will get 35 mpg in the city, and 40 mpg on the highway.

An average driver drives about 12,000 miles a year. Suppose that you expect to drive about 75% of that in the city, so 9,000 city miles a year, and 3,000 highway miles a year.

We can then find the number of gallons each car would require for the year.

Sonata:

$$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{24 \text{ city miles}} + 3000 \text{ hightway miles.} \frac{1 \text{ gallon}}{35 \text{ highway miles}} = 460.7 \text{ gallons}$$

Hybrid:

$$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{35 \text{ city miles}} + 3000 \text{ hightway miles.} \frac{1 \text{ gallon}}{40 \text{ highway miles}} = 332.1 \text{ gallons}$$

If gas in your area averages about $3.50 per gallon, we can use that to find the running cost:

Sonata: $460.7 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1612.45$

Hybrid: $332.1 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1162.35$

The hybrid will save $450.10 a year. The gas costs for the hybrid are about $\frac{\$450.10}{\$1612.45} = 0.279 = 27.9\%$ lower than the costs for the standard Sonata.

While both the absolute and relative comparisons are useful here, they still make it hard to answer the original question, since “is it worth it” implies there is some tradeoff for the gas savings. Indeed, the hybrid Sonata costs about $25,850, compared to the base model for the regular Sonata, at $20,895.

To better answer the “is it worth it” question, we might explore how long it will take the gas savings to make up for the additional initial cost. The hybrid costs $4965 more. With gas savings of $451.10 a year, it will take about 11 years for the gas savings to make up for the higher initial costs.

We can conclude that if you expect to own the car 11 years, the hybrid is indeed worth it. If you plan to own the car for less than 11 years, it may still be worth it, since the resale value of the hybrid may be higher, or for other non-monetary reasons. This is a case where math can help guide your decision, but it can’t make it for you.

### Try it Now 6

If traveling from Seattle, WA to Spokane WA for a three-day conference, does it make more sense to drive or fly?

**Answer**

There is not enough information provided to answer the question, so we will have to make some assumptions, and look up some values.

Assumptions:

a) We own a car. Suppose it gets 24 miles to the gallon. We will only consider gas cost.

b) We will not need to rent a car in Spokane, but will need to get a taxi from the airport to the conference hotel downtown and back.

c) We can get someone to drop us off at the airport, so we don’t need to consider airport parking.

d) We will not consider whether we will lose money by having to take time off work to drive.

Values looked up (your values may be different)

a) Flight cost: $\$184$

b) Taxi cost: $\$25$ each way (estimate, according to hotel website)

c) Driving distance: $280$ miles each way

d) Gas cost: $\$3.79$ a gallon

Cost for flying: $\$184 \text{ flight cost } + \$50 \text{ in taxi fares } = \$234.$

Cost for driving: $560$ miles round trip will require 23.3 gallons of gas, costing $\$88.31$

Based on these assumptions, driving is cheaper. However, our assumption that we only include gas cost may not be a good one. Tax law allows you deduct $\$0.55$ (in 2012) for each mile driven, a value that accounts for gas as well as a portion of

> the car cost, insurance, maintenance, etc. Based on this number, the cost of driving would be $\$319$.

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## 10.6.6: Exercises

1. Out of 230 racers who started the marathon, 212 completed the race, 14 gave up, and 4 were disqualified. What percentage did not complete the marathon?

2. Patrick left an $8 tip on a $50 restaurant bill. What percent tip is that?

3. Ireland has a 23% VAT (value-added tax, similar to a sales tax). How much will the VAT be on a purchase of a €250 item?

4. Employees in 2012 paid 4.2% of their gross wages towards social security (FICA tax), while employers paid another 6.2%. How much will someone earning $45,000 a year pay towards social security out of their gross wages?

5. A project on Kickstarter.com was aiming to raise $15,000 for a precision coffee press. They ended up with 714 supporters, raising 557% of their goal. How much did they raise?

6. Another project on Kickstarter for an iPad stylus raised 1,253% of their goal, raising a total of $313,490 from 7,511 supporters. What was their original goal?

7. The population of a town increased from 3,250 in 2008 to 4,300 in 2010. Find the absolute and relative (percent) increase.

8. The number of CDs sold in 2010 was 114 million, down from 147 million the previous year[1]. Find the absolute and relative (percent) decrease.

9. A company wants to decrease their energy use by 15%.

   a. If their electric bill is currently $2,200 a month, what will their bill be if they're successful?

   b. If their next bill is $1,700 a month, were they successful? Why or why not?

10. A store is hoping an advertising campaign will increase their number of customers by 30%. They currently have about 80 customers a day.

   a. How many customers will they have if their campaign is successful?

   b. If they increase to 120 customers a day, were they successful? Why or why not?

11. An article reports "attendance dropped 6% this year, to 300." What was the attendance before the drop?

12. An article reports "sales have grown by 30% this year, to $200 million." What were sales before the growth?

13. The Walden University had 47,456 students in 2010, while Kaplan University had 77,966 students. Complete the following statements:

   a. Kaplan's enrollment was ___% larger than Walden's.

   b. Walden's enrollment was ___% smaller than Kaplan's.

   c. Walden's enrollment was ___% of Kaplan's.

14. In the 2012 Olympics, Usain Bolt ran the 100m dash in 9.63 seconds. Jim Hines won the 1968 Olympic gold with a time of 9.95 seconds.

   a. Bolt's time was ___% faster than Hines'.

   b. Hine' time was ___% slower than Bolt's.

   c. Hine' time was ___% of Bolt's.

15. A store has clearance items that have been marked down by 60%. They are having a sale, advertising an additional 30% off clearance items. What percent of the original price do you end up paying?

16. Which is better: having a stock that goes up 30% on Monday than drops 30% on Tuesday, or a stock that drops 30% on Monday and goes up 30% on Tuesday? In each case, what is the net percent gain or loss?

17. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?

   a. "16.3% of Americans are without health insurance"[2]

   b. "only 55.9% of adults receive employer provided health insurance"[3]

18. Are these two claims equivalent, in conflict, or not comparable because they’re talking about different things?

a. “We mark up the wholesale price by 33% to come up with the retail price”

b. “The store has a 25% profit margin”

19. Are these two claims equivalent, in conflict, or not comparable because they’re talking about different things?

a. “Every year since 1950, the number of American children gunned down has doubled.”

b. “The number of child gunshot deaths has doubled from 1950 to 1994.”

20. Are these two claims equivalent, in conflict, or not comparable because they’re talking about different things?[4]

a. “75 percent of the federal health care law’s taxes would be paid by those earning less than $120,000 a year”

b. “76 percent of those who would pay the penalty [health care law’s taxes] for not having insurance in 2016 would earn under $120,000”

21. Are these two claims equivalent, in conflict, or not comparable because they’re talking about different things?

a. “The school levy is only a 0.1% increase of the property tax rate.”

b. “This new levy is a 12% tax hike, raising our total rate to $9.33 per $1000 of value.”

22. Are the values compared in this statement comparable or not comparable? “Guns have murdered more Americans here at home in recent years than have died on the battlefields of Iraq and Afghanistan. In support of the two wars, more than 6,500 American soldiers have lost their lives. During the same period, however, guns have been used to murder about 100,000 people on American soil”[5]

23. A high school currently has a 30% dropout rate. They’ve been tasked to decrease that rate by 20%. Find the equivalent percentage point drop.

24. A politician’s support grew from 42% by 3 percentage points to 45%. What percent (relative) change is this?

25. Marcy has a 70% average in her class going into the final exam. She says "I need to get a 100% on this final so I can raise my score to 85%." Is she correct?

26. Suppose you have one quart of water/juice mix that is 50% juice, and you add 2 quarts of juice. What percent juice is the final mix?

27. Find a unit rate: You bought 10 pounds of potatoes for $4.

28. Find a unit rate: Joel ran 1500 meters in 4 minutes, 45 seconds.

29. Solve: $\frac{2}{5} = \frac{6}{x}$.

30. Solve: $\frac{n}{5} = \frac{16}{20}$.

31. A crepe recipe calls for 2 eggs, 1 cup of flour, and 1 cup of milk. How much flour would you need if you use 5 eggs?

32. An 8ft length of 4 inch wide crown molding costs $14. How much will it cost to buy 40ft of crown molding?

33. Four 3-megawatt wind turbines can supply enough electricity to power 3000 homes. How many turbines would be required to power 55,000 homes?

34. A highway had a landslide, where 3,000 cubic yards of material fell on the road, requiring 200 dump truck loads to clear. On another highway, a slide left 40,000 cubic yards on the road. How many dump truck loads would be needed to clear this slide?

35. Convert 8 feet to inches.

36. Convert 6 kilograms to grams.

37. A wire costs $2 per meter. How much will 3 kilometers of wire cost?

38. Sugar contains 15 calories per teaspoon. How many calories are in 1 cup of sugar?

39. A car is driving at 100 kilometers per hour. How far does it travel in 2 seconds?

40. A chain weighs 10 pounds per foot. How many ounces will 4 inches weigh?

41. The table below gives data on three movies. Gross earnings is the amount of money the movie brings in. Compare the net earnings (money made after expenses) for the three movies.[6]

Movie Earnings

| Movie | Release Date | Budget | Gross Earnings |
|---|---|---|---|
| Saw | 10/29/2004 | \$1,200,000 | \$103,096,345 |
| Titanic | 10/29/2004 | \$200,000,000 | \$1,842,879,955 |
| Jurassic Park | 6/11/1993 | \$63,000,000 | \$923,863,984 |

42. For the movies in the previous problem, which provided the best return on investment?

43. The population of the U.S. is about 309,975,000, covering a land area of 3,717,000 square miles. The population of India is about 1,184,639,000, covering a land area of 1,269,000 square miles. Compare the population densities of the two countries.

44. The GDP (Gross Domestic Product) of China was \$5,739 billion in 2010, and the GDP of Sweden was \$435 billion. The population of China is about 1,347 million, while the population of Sweden is about 9.5 million. Compare the GDP per capita of the two countries.

45. In June 2012, Twitter was reporting 400 million tweets per day. Each tweet can consist of up to 140 characters (letter, numbers, etc.). Create a comparison to help understand the amount of tweets in a year by imagining each character was a drop of water and comparing to filling something up.

46. The photo sharing site Flickr had 2.7 billion photos in June 2012. Create a comparison to understand this number by assuming each picture is about 2 megabytes in size, and comparing to the data stored on other media like DVDs, iPods, or flash drives.

47. Your chocolate milk mix says to use 4 scoops of mix for 2 cups of milk. After pouring in the milk, you start adding the mix, but get distracted and accidentally put in 5 scoops of mix. How can you adjust the mix if:

a. There is still room in the cup?

b. The cup is already full?

48. A recipe for sabayon calls for 2 egg yolks, 3 tablespoons of sugar, and $\frac{1}{4}$ cup of white wine. After cracking the eggs, you start measuring the sugar, but accidentally put in 4 tablespoons of sugar. How can you compensate?

49. The Deepwater Horizon oil spill resulted in 4.9 million barrels of oil spilling into the Gulf of Mexico. Each barrel of oil can be processed into about 19 gallons of gasoline. How many cars could this have fueled for a year? Assume an average car gets 20 miles to the gallon, and drives about 12,000 miles in a year.

50. The store is selling lemons at 2 for \$1. Each yields about 2 tablespoons of juice. How much will it cost to buy enough lemons to make a 9-inch lemon pie requiring $\frac{1}{2}$ cup of lemon juice?

51. A piece of paper can be made into a cylinder in two ways: by joining the short sides together, or by joining the long sides together[7]. Which cylinder would hold more? How much more?

52. Which of these glasses contains more liquid? How much more?

In the next 4 questions, estimate the values by making reasonable approximations for unknown values, or by doing some research to find reasonable values.

53. Estimate how many gallons of water you drink in a year.

54. Estimate how many times you blink in a day.

55. How much does the water in a 6-person hot tub weigh?

56. How many gallons of paint would be needed to paint a two-story house 40 ft long and 30 ft wide?

57. During the landing of the Mars Science Laboratory *Curiosity*, it was reported that the signal from the rover would take 14 minutes to reach earth. Radio signals travel at the speed of light, about 186,000 miles per second. How far was Mars from Earth when *Curiosity* landed?

58. It is estimated that a driver takes, on average, 1.5 seconds from seeing an obstacle to reacting by applying the brake or swerving. How far will a car traveling at 60 miles per hour travel (in feet) before the driver reacts to an obstacle?

59. The flash of lightning travels at the speed of light, which is about 186,000 miles per second. The sound of lightning (thunder) travels at the speed of sound, which is about 750 miles per hour.

a. If you see a flash of lightning, then hear the thunder 4 seconds later, how far away is the lightning?

b. Now let's generalize that result. Suppose it takes $n$ seconds to hear the thunder after a flash of lightning. How far away is the lightning, in terms of $n$?

60. Sound travels about 750 miles per hour. If you stand in a parking lot near a building and sound a horn, you will hear an echo.

a. Suppose it takes about $\frac{1}{2}$ a second to hear the echo. How far away is the building[8]?

b. Now let's generalize that result. Suppose it takes $n$ seconds to hear the echo. How far away is the building, in terms of $n$?

61. It takes an air pump 5 minutes to fill a twin sized air mattress (39 by 8.75 by 75 inches). How long will it take to fill a queen sized mattress (60 by 8.75 by 80 inches)?

62. It takes your garden hose 20 seconds to fill your 2-gallon watering can. How long will it take to fill

a. An inflatable pool measuring 3 feet wide, 8 feet long, and 1 foot deep.[9]

b. A circular inflatable pool 13 feet in diameter and 3 feet deep.[10]

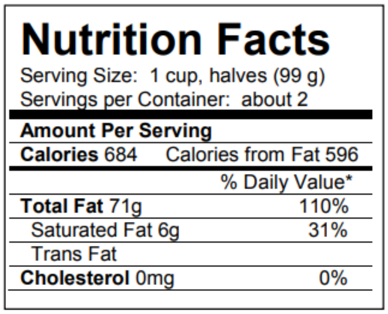
63. You want to put a 2" thick layer of topsoil for a new 20'x30' garden. The dirt store sells by the cubic yards. How many cubic yards will you need to order?

64. A box of Jell-O costs $0.50, and makes 2 cups. How much would it cost to fill a swimming pool 4 feet deep, 8 feet wide, and 12 feet long with Jell-O? (1 cubic foot is about 7.5 gallons)

65. You read online that a 15 ft by 20 ft brick patio would cost about $2,275 to have professionally installed. Estimate the cost of having a 18 by 22 ft brick patio installed.

66. I was at the store, and saw two sizes of avocados being sold. The regular size sold for $0.88 each, while the jumbo ones sold for $1.68 each. Which is the better deal?

67. The grocery store has bulk pecans on sale, which is great since you're planning on making 10 pecan pies for a wedding. Your recipe calls for 1¾ cups pecans per pie. However, in the bulk section there's only a scale available, not a measuring cup. You run over to the baking aisle and find a bag of pecans, and look at the nutrition label to gather some info. How many pounds of pecans should you buy?

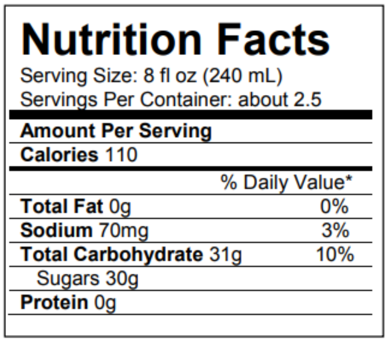


**Nutrition Facts**

Serving Size: 1 cup, halves (99 g)
Servings per Container: about 2

| | |
|---|---|
| **Amount Per Serving** | |
| **Calories** 684 | Calories from Fat 596 |
| | % Daily Value* |
| **Total Fat** 71g | 110% |
| Saturated Fat 6g | 31% |
| Trans Fat | |
| **Cholesterol** 0mg | 0% |

68. Soda is often sold in 20 ounce bottles. The nutrition label for one of these bottles is shown to the right. A packet of sugar (the kind they have at restaurants for your coffee or tea) typically contain 4 grams of sugar in the U.S. Drinking a 20 oz soda is equivalent to eating how many packets of sugar?[11]



**Nutrition Facts**

Serving Size: 8 fl oz (240 mL)
Servings Per Container: about 2.5

| | |
|---|---|
| **Amount Per Serving** | |
| **Calories** 110 | |
| | % Daily Value* |
| **Total Fat** 0g | 0% |
| **Sodium** 70mg | 3% |
| **Total Carbohydrate** 31g | 10% |
| Sugars 30g | |
| **Protein** 0g | |

For the next set of questions, *first* identify the information you need to answer the question, and *then* turn to the end of the section to find that information. The details may be imprecise; answer the question the best you can with the provided information. Be sure to justify your decision.

69. You're planning on making 6 meatloafs for a party. You go to the store to buy breadcrumbs, and see they are sold by the canister. How many canisters do you need to buy?

70. Your friend wants to cover their car in bottle caps, like in this picture.[12] How many bottle caps are you going to need?

71. You need to buy some chicken for dinner tonight. You found an ad showing that the store across town has it on sale for $2.99 a pound, which is cheaper than your usual neighborhood store, which sells it for $3.79 a pound. Is it worth the extra drive?

72. I have an old gas furnace, and am considering replacing it with a new, high efficiency model. Is upgrading worth it?

73. Janine is considering buying a water filter and a reusable water bottle rather than buying bottled water. Will doing so save her money?

74. Marcus is considering going car-free to save money and be more environmentally friendly. Is this financially a good decision?

For the next set of problems, research or make educated estimates for any unknown quantities needed to answer the question.

75. You want to travel from Tacoma, WA to Chico, CA for a wedding. Compare the costs and time involved with driving, flying, and taking a train. Assume that if you fly or take the train you'll need to rent a car while you're there. Which option is best?

76. You want to paint the walls of a 6ft by 9ft storage room that has one door and one window. You want to put on two coats of paint. How many gallons and/or quarts of paint should you buy to paint the room as cheaply as possible?

77. A restaurant in New York tiled their floor with pennies[13]. Just for the materials, is this more expensive than using a more traditional material like ceramic tiles? If each penny has to be laid by hand, estimate how long it would take to lay the pennies for a 12ft by 10ft room. Considering material and labor costs, are pennies a cost-effective replacement for ceramic tiles?

78. You are considering taking up part of your back yard and turning it into a vegetable garden, to grow broccoli, tomatoes, and zucchini. Will doing so save you money, or cost you more than buying vegetables from the store?

79. Barry is trying to decide whether to keep his 1993 Honda Civic with 140,000 miles, or trade it in for a used 2008 Honda Civic. Consider gas, maintenance, and insurance costs in helping him make a decision.

80. Some people claim it costs more to eat vegetarian, while some claim it costs less. Examine your own grocery habits, and compare your current costs to the costs of switching your diet (from omnivore to vegetarian or vice versa as appropriate). Which

diet is more cost effective based on your eating habits?

**Info for the breadcrumbs question**

How much breadcrumbs does the recipe call for?

It calls for 1½ cups of breadcrumbs.

How many meatloafs does the recipe make?

It makes 1 meatloaf.

How many servings does that recipe make?

It says it serves 8.

How big is the canister?

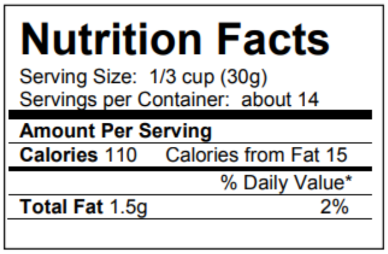
It is cylindrical, 3.5 inches across and 7 inches tall.

What is the net weight of the contents of 1 canister?

15 ounces.

How much does a cup of breadcrumbs weigh?

I’m not sure, but maybe something from the nutritional label will help.



**Nutrition Facts**

Serving Size: 1/3 cup (30g)
Servings per Container: about 14

**Amount Per Serving**

**Calories** 110 Calories from Fat 15

% Daily Value*

**Total Fat** 1.5g 2%

How much does a canister cost?

$2.39

**Info for bottle cap car**

What kind of car is that?

A 1993 Honda Accord.

How big is that car / what are the dimensions? Here is some details from MSN autos:

Weight: 2800lb Length: 185.2 in Width: 67.1 in Height: 55.2 in

How much of the car was covered with caps?

Everything but the windows and the underside.

How big is a bottle cap?

Caps are 1 inch in diameter.

**Info for chicken problem**

How much chicken will you be buying?

Four pounds

How far are the two stores?

My neighborhood store is 2.2 miles away, and takes about 7 minutes. The store across town is 8.9 miles away, and takes about 25 minutes.

What kind of mileage does your car get?

It averages about 24 miles per gallon in the city.

How many gallons does your car hold?

About 14 gallons

How much is gas?

About $3.69/gallon right now.

**Info for furnace problem**

How efficient is the current furnace?

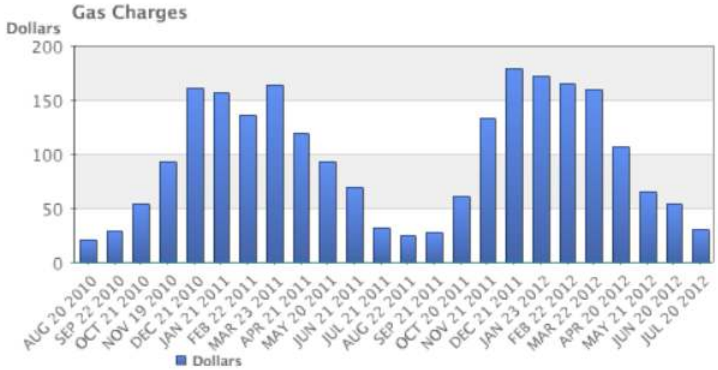
It is a 60% efficient furnace.

How efficient is the new furnace?

It is 94% efficient.

What is your gas bill?

Here is the history for 2 years:

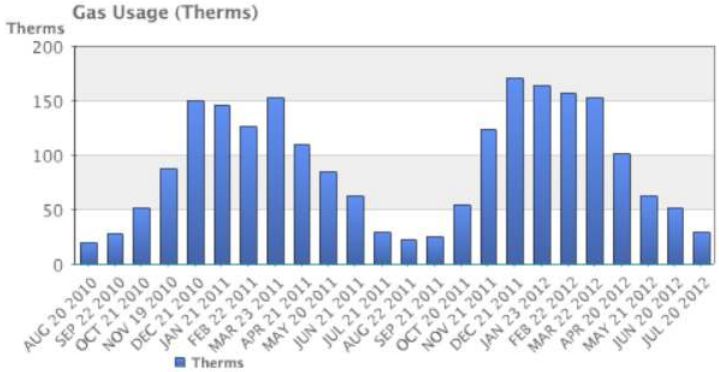


How much do you pay for gas?

There is $10.34 base charge, plus $0.39097 per Therm for a delivery charge, and $0.65195 per Therm for cost of gas.

How much gas do you use?

Here is the history for 2 years:



How much does the new furnace cost?

It will cost $7,450.

How long do you plan to live in the house?

Probably at least 15 years.

**Info for water filter problem**

How much water does Janine drink in a day?

She normally drinks 3 bottles a day, each 16.9 ounces.

How much does a bottle of water cost?

She buys 24-packs of 16.9 ounce bottles for $3.99.

How much does a reusable water bottle cost?

About $10.

How long does a reusable water bottle last?

Basically forever (or until you lose it).

How much does a water filter cost? How much water will they filter?

- A faucet-mounted filter costs about $28. Refill filters cost about $33 for a 3-pack. The box says each filter will filter up to 100 gallons (378 liters)
- A water filter pitcher costs about $22. Refill filters cost about $20 for a 4-pack. The box says each filter lasts for 40 gallons or 2 months
- An under-sink filter costs $130. Refill filters cost about $60 each. The filter lasts for 500 gallons.

**Info for car-free problem**

Where does Marcus currently drive? He:

- Drives to work 5 days a week, located 4 miles from his house.
- Drives to the store twice a week, located 7 miles from his house.
- Drives to other locations on average 5 days a week, with locations ranging from 1 mile to 20 miles.
- Drives to his parent's house 80 miles away once a month.

How will he get to these locations without a car?

- For work, he can walk when it's sunny and he gets up early enough. Otherwise he can take a bus, which takes about 20 minutes
- For the store, he can take a bus, which takes about 35 minutes.
- Some of the other locations he can bus to. Sometimes he'll be able to get a friend to pick him up. A few locations he is able to walk to. A couple locations are hard to get to by bus, but there is a ZipCar (short term car rental) location within a few blocks.
- He'll need to get a ZipCar to visit his parents.

How much does gas cost?

> About $3.69/gallon.

How much does he pay for insurance and maintenance?

- He pays $95/month for insurance.
- He pays $30 every 3 months for an oil change, and has averaged about $300/year for other maintenance costs.

How much is he paying for the car?

- He's paying $220/month on his car loan right now, and has 3 years left on the loan.
- If he sold the car, he'd be able to make enough to pay off the loan.
- If he keeps the car, he's planning on trading the car in for a newer model in a couple years.

What mileage does his car get?

> About 26 miles per gallon on average.

How much does a bus ride cost?

> $2.50 per trip, or $90 for an unlimited monthly pass.

How much does a ZipCar rental cost?

- The "occasional driving plan": $25 application fee and $60 annual fee, with no monthly commitment. Monday-Thursday the cost is $8/hour, or $72 per day. Friday-Sunday the cost is $8/hour or $78/day. Gas, insurance, and 180 miles are included in the cost. Additional miles are $0.45/mile.
- The "extra value plan": Same as above, but with a $50 monthly commitment, getting you a 10% discount on the usage costs.

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[1] http://www.cnn.com/2010/SHOWBIZ/Musi...les/index.html

[2] http://www.cnn.com/2012/06/27/politi...are/index.html

[3] http://www.politico.com/news/stories/0712/78134.html

[4] http://factcheck.org/2012/07/twistin...th-care-taxes/

[5] www.northjersey.com/news/opin...tml?c=y&page=2

[6] http://www.the-numbers.com/movies/records/budgets.php

[7] vimeo.com/42501010

[8] vimeo.com/40377128

[9] http://www.youtube.com/watch?v=DIkwefReHZc

[10] http://www.youtube.com/watch?v=p9SABH7Yg9M

[11] http://www.youtube.com/watch?v=62JMfv0tf3Q

[12] Photo credit: http://www.flickr.com/photos/swayze/, CC-BY

[13] http://www.notcot.com/archives/2009/...-of-pennie.php

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# 10.7: Leases

Average vehicle costs in North America have risen from $10,668 in 1982 to $25,683 in 2009,1 a whopping 141% increase over 27 years. With vehicle purchase prices running well ahead of inflation, leasing now represents an attractive option for many Canadians with many added conveniences. On the upside, lease payments are substantially lower than purchase payments. As a bonus you are able to return your vehicle after a certain number of years and upgrade to the most recent models available. And while you are in possession of the vehicle, it usually remains under the manufacturer warranty, which minimizes your maintenance and repair expenses. On the downside, you never actually own the vehicle and always make car payments, while various research has also calculated that leasing is actually much more expensive than vehicle ownership in the long run.

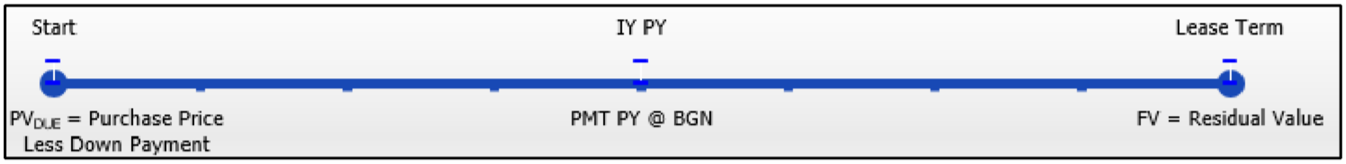
Businesses take advantage of leasing not only for vehicles, but for real estate and office space along with production equipment and office equipment. Businesses constantly need to change to avoid obsolescence, particularly in their fleet of computers. As a business grows, it must expand or even change its office space. Leasing allows organizations to do all of this. Leasing also offers significant tax benefits to most companies as well.

This section explains the basic principles of a lease and then illustrates leasing calculations such as purchase prices, residual values, lease payments, and leasing interest rates.

## How Leases Work

A **lease** is a contract by which the owner of an asset gives another party an exclusive right to possess and use the asset under specified conditions for a specific period of time in return for agreed-upon payments. In other words, you are able to borrow something for a period of time and make payments on it. At the end of the lease, the asset is usually returned to the owner, or it can be purchased outright for a depreciated price. The owner of the asset is known as the **lessor**, while the borrowing party is known as the **lessee**.

A typical leasing timeline is illustrated below. Important lease components include the present value, annuity timing, term, interest rate, and future value.



1. Present Value of the Lease. The purchase price of an asset is known by many names, such as its selling price, list price, or even the manufacturer's suggested retail price (MSRP). A down payment, which is a portion of the purchase price required up front, may or may not be required depending on the asset being leased. If it is required, it is deducted from the purchase price to determine the amount of money being financed.
2. Annuity Timing. All leases take the form of annuities due since you must first pay for the item before you are allowed to borrow it. Car dealerships will not let you drive that new car home until you demonstrate the financial capability to pay for it!
3. Lease Term. A lease term is no different from an annuity term. It is the specified period of time for which the lessor leases the asset.
4. Lease Interest Rate. The nominal interest rate and compounding frequency are determined by the owner of the asset. Most vehicle leases default to rates that are compounded monthly; however, advertisements typically present effective rates.
5. Future Value of the Lease. The future value is more commonly known as the residual value, which is the projected value of an asset at the end of its lease term. A residual value does not necessarily occur in every lease calculation, as some assets, like computer equipment, may only have an insubstantial value upon lease expiration. Vehicle leases always have a residual value. Thus, the lessee has a choice to give the car dealership either the keys to the car or a cheque in the amount of the residual value, thereby purchasing the car.

In accounting, under certain circumstances, when an officer of the company signs a lease for the acquisition of a capital good (i.e., a lease of an asset, not a rental of property like a building), this creates a debt. Most leases are closed and cannot be canceled without severe financial penalties; therefore, a liability is recorded on the balance sheet.

The amount recorded is the present value of the remaining lease payments, and it is calculated using a discount rate equivalent to the interest rate the business would have had to pay if the business had purchased the asset instead. The residual value, if any, is included in the amount of this debt, which is known as a **capitalized lease liability**.2

For example, assume a company signed a two-year lease with quarterly payments of $1,000 and no residual value. Alternatively, it could have purchased the asset and financed it through a bank loan for 8% quarterly. Therefore, applying Formula 11.5, the present value of the eight $1,000 payments discounted at 8% quarterly is $7,471.99. This is the amount that appears on the balance sheet.

### The Formula

A lease is an annuity due involving some special characteristics. To solve a leasing situation you can use any of the Chapter 11 annuity due formulas.

### How It Works

The typical unknown variable in a leasing situation is the present value, future value, lease payment amount, or lease interest rate. To solve a lease for those variables you follow the exact same steps as found in Section 11.2, Section 11.3, Section 11.4, and Section 11.6.

### Things To Watch Out For

Here are some of the most common mistakes made when working with leases:

1. **Cash Flow Violation**. This mistake occurs when the cash flow sign convention on the calculator or in Excel is violated. From the perspective of the lessee, the present value of the lease is a positive number since the lessee is receiving the asset. Payments on the lease are therefore negative. The future value or residual value of the lease (if other than zero) is also a negative number, since it represents the amount still owing on the asset if the lessee wishes to purchase it.
2. **Confusing the Present Value and the Purchase Price**. When calculating the present value of the lease, be careful to answer the question that is asked. If you need to know how much money was financed for the lease, this is your present value, whereas if you need to know the purchase price of the asset, you must add any down payment to the present value to arrive at the right answer.

### ? Exercise 10.7.1: Give It Some Thought

Assume two cars are financed at the same amount at the same interest rate for the same period of time, but one is leased and the other is purchased. Are the lease payments larger than, smaller than, or equal to the purchase payments?

**Answer**

Smaller. The lease has a residual value at the end, meaning the lease payments do not have to pay as much principal as would be the case under the purchase payments.

### ✓ Example 10.7.1: What Will the Lease Payments Be?

Mazda is advertising a new Mazda3 Sport with an MSRP inclusive of all taxes, deliveries, and fees for $39,091.41. It offers a 48-month lease with a lease rate of 3.9% effectively. The residual value of the vehicle is $13,354.00. Determine the monthly lease payments if a $2,500 down payment is made.

**Solution**

**Step 1**:

This is a leasing question, so payments are made at the beginning of each period. Monthly payments with annual compounding create a general annuity due. Calculate the annuity payment amount ($PMT$).

**What You Already Know**

**Step 1 (continued)**:

The timeline for the vehicle lease appears below.

**Step 2**:

$PV_{DUE}$ =\$39,091.41−\$2,500.00 = \$36,591.41, $IY$ = 3.9%, $CY$ = 1, $PY$ = 12, Years = 4, $FV$ = \$13,354.00

**How You Will Get There**

**Step 3**:

Apply Formula 9.1.

**Step 4**:

The future value needs to be moved to the start, the same date as $PV_{DUE}$. Apply Formulas 9.2 and 9.3 (rearranged for $PV$). The calculated $PV$ is money the lease is not required to pay off, so it is subtracted from $PV_{DUE}$ to arrive at the amount required by the annuity.

**Step 5**:

Apply Formulas 11.1 and 11.5 (rearranged for $PMT$) using the calculated step 4 $PV_{DUE}$ amount.

**Perform**

**Step 3**:

$i = 3.9\%/1 = 3.9\%$

**Step 4**:

$N = 1 \times 4 = 4$ compounds, $\$13,354 = PV(1+0.039)^4$

new $PV_{DUE} = \$36,591.41 - \$11,459.06497 = \$25,132.34503$

**Step 5**:

$N = 12 \times 4 = 48$ payments

$$\$25,132.34503 = PMT\left[\frac{1-\left[\frac{1}{(1+0.039)^{\frac{1}{12}}}\right]^{48}}{(1+0.039)^{\frac{1}{12}}-1}\right]\times(1+0.039)^{\frac{1}{12}}$$

$$\$25,132.34503 = PMT\left[\frac{0.141900}{0.003193}\right]\times 1.003193$$

$$PMT = \frac{\$25,132.34503}{44.578554} = \$563.78$$

**Calculator Instructions**

| Mode | N | I/Y | PV | PMT | FV | P/Y | C/Y |
|---|---|---|---|---|---|---|---|
| BGN | 48 | 3.9 | 36591.41 | Answer:<br>-563.776579 | -13354 | 12 | 1 |

With \$2,500 down on this vehicle, a four-year lease has beginning-of-month payments of \$563.78.

### ✓ Example 10.7.2: Recording the Lease Liability

Bradford Enterprises just updated its computer mainframe systems and leased its new equipment under a two-year contract requiring fixed quarterly payments of $17,300. Alternatively, it could have borrowed the money through a financial institution at 6.5% compounded semi-annually. What liability amount (the capitalized lease liability) should be recorded on the company's balance sheet

a. on the day the contract was signed?
b. after five payments have been made?

#### Solution

**Step 1**:

This is a leasing question, so payments are made at the beginning of the period. Quarterly payments with semi-annual compounding create a general annuity due. Calculate the present value ($PV_{DUE}$) at two time points— today and after five payments.

**What You Already Know**

**Step 1 (continued)**:

The timeline for the computer equipment lease appears below.

**Step 2**:

$FV$ = $0, $IY$ = 6.5%, $CY$ =2, $PMT$ = $17,300, $PY$ = 4, Years = 2

**How You Will Get There**

**Step 3**:

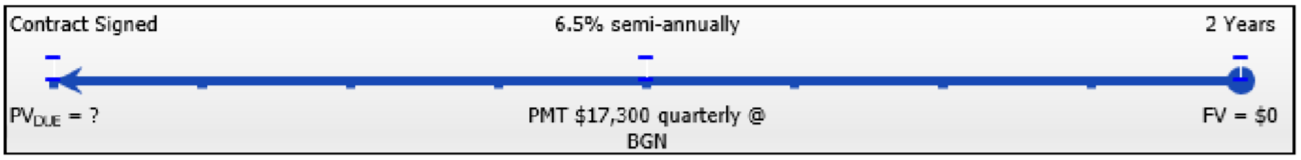
Apply Formula 9.1.

**Step 4**:

Since $FV$ = $0, skip this step.

**Step 5**:

Apply Formulas 11.1 and 11.5 twice (for both today and after five payments).



**Perform**

**Step 3**:

$i = 6.5\%/2 = 3.25\%$

**Step 5**:

a. For today, $N = 4 \times 2 = 8$ payments

$$PV_{DUE} = \$17,300 \left[ \frac{1 - \left[ \frac{1}{(1+0.0325)^{\frac{2}{4}}} \right]^8}{(1+0.0325)^{\frac{2}{4}} - 1} \right] \times (1+0.0325)^{\frac{2}{4}} = \$130,954.37$$

b. After five payments, $N = 4 \times 3/4 = 3$ payments left

$$PV_{DUE} = \$17,300\left[\frac{1-\left[\frac{1}{(1+0.0325)^{\frac{2}{4}}}\right]^{3}}{(1+0.0325)^{\frac{2}{4}}-1}\right] \times (1+0.0325)^{\frac{2}{4}} = \$51,080.99$$

**Calculator Instructions**

| Calculation | Mode | N | I/Y | PV | PMT | FV | P/Y | C/Y |
|---|---|---|---|---|---|---|---|---|
| Today | BGN | 8 | 6.5 | Answer: 130,954.37 | -17300 | 0 | 4 | 2 |
| After 5 payments | √ | 3 | √ | Answer: 51,080.99 | √ | √ | √ | √ |

On the day that the computer equipment is leased, the accountants will record a capital lease liability of $130,954.37. This amount will reduce with each payment, such that after five payments the liability would be $51,080.99.

### ✓ Example 10.7.3: What Is It Worth at the End?

A Nissan 370Z Roadster A7 advertised in the local paper offers monthly payments of $728.70 for 48 months at 3.83% compounded monthly. The lease requires a $5,000 down payment, and the vehicle has an MSRP (including all fees and taxes) of $53,614.83. What residual value is used in the calculation?

**Solution**

**Step 1**:

This is a leasing question, so payments are made at the beginning of each period. Monthly payments with monthly compounding create a simple annuity due. Calculate its residual value, which is the same thing as the future value ($FV_{DUE}$).

**What You Already Know**

**Step 1 (continued)**:

The timeline for the vehicle lease appears below.

**Step 2**:

$PV$ = $53,614.83 − $5,000 = $48,614.83, $IY$ = 3.83%, $CY$ =12, $PMT$ = $728.70, $PY$ = 12, Years = 4

**How You Will Get There**

**Step 3**:

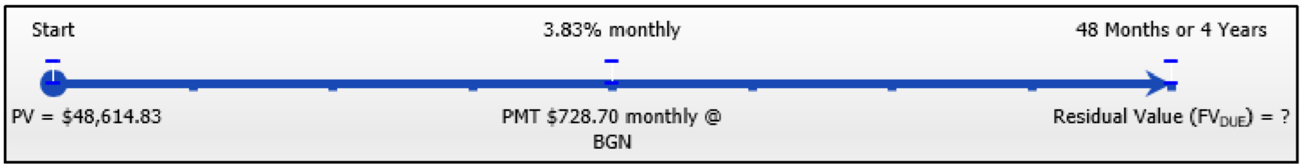
Apply Formula 9.1.

**Step 4**:

For the $PV$, apply Formulas 9.2 and 9.3.

**Step 5**:

Apply Formulas 11.1 and 11.2. The residual value is the difference between the answers to step 4 (what the lessee owes) and step 5 (what the lessee paid).



**Perform**

**Step 3**:

$i = 3.83\%/12 = 0.3191\bar{6}\%$

**Step 4**:

$N = 12 \times 4 = 48$ compounds; $FV = \$48,614.83(1+0.003191)^{48} = \$56,649.58522$

**Step 5**:

$N = 12 \times 4 = 48$ payments

$$FV_{DUE} = \$728.70 \left[ \frac{\left[(1+0.003191)^{\frac{12}{12}}\right]^{48} - 1}{(1+0.003191)^{\frac{12}{12}} - 1} \right] \times (1+0.003191)^{\frac{12}{12}}$$

$$= \$37,854.63271$$

$$\text{Residual Value } = \$56,649.58522 - \$37,854.63721 = \$18,794.95$$

**Calculator Instructions**

| Mode | N | I/Y | PV | PMT | FV | P/Y | C/Y |
|---|---|---|---|---|---|---|---|
| BGN | 48 | 3.83 | 48614.83 | -728.7 | Answer: -18,794.95252 | 12 | 12 |

In determining the lease payments, Nissan expects its Roadster to have a residual value of $18,794.95 after the four year lease term expires. The owner will either have to pay this amount or give the keys back.

### ✓ Example 10.7.4: What Rate Are They Using?

An advertisement during The Bachelorette tells you that you can lease a Dodge Challenger SE for $161.00 biweekly for five years. The MSRP of the vehicle is $26,480, and a $1,500 down payment is required. If the residual value of the vehicle is $8,146.16, what annually compounded interest rate does the calculation use?

#### Solution

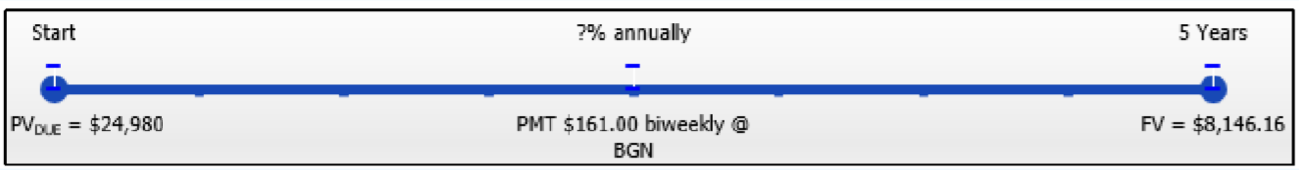
**Step 1**:

This is a leasing question, so payments are made at the beginning of each period. Monthly payments with monthly compounding create a simple annuity due. Calculate the annually compounded interest rate ($IY$) with $CY$ = 1.

**What You Already Know**

**Step 1 (continued)**:

The timeline for the vehicle lease appears below.



**Step 2**:

$PV_{DUE}$ = $26,480 − $1,500 = $24,980, $CY$ = 1, $PMT$ = $161, $PY$ = 26, Years = 5, $FV$ = $8,146.16

**How You Will Get There**

**Step 3**:

Apply Formulas 11.1 and 11.5. Enter this information into the calculator to solve for $IY$ (which automatically applies Formula 9.1 to convert the $i$ to an $IY$).

**Perform**

**Step 3**:

$N = 26 \times 5 = 130$ payments

$$\$24,980 = \$161\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{1}{26}}}\right]^{130}}{(1+i)^{\frac{1}{26}}-1}\right] \times (1+i)^{\frac{1}{26}}$$

See calculator instructions for the solution. $IY$ = 4.99%

**Calculator Instructions**

| Mode | N | I/Y | PV | PMT | FV | P/Y | C/Y |
|---|---|---|---|---|---|---|---|
| BGN | 130 | Answer: 4.989999 | 24980 | -161 | -8146.16 | 26 | 1 |

To arrive at payments of $161 biweekly on a Dodge Challenger with $1,500 down and a residual value of $8,146.16, Dodge is using a 4.99% compounded annually interest rate.

## References

1. Statistics Canada, New Motor Vehicle Sales, June 2011 (Catalogue no. 63-007-X, p. 15), publications.gc.ca/collection...llection_2011/statcan/63-007-X/63-007-x2011006-eng.pdf (accessed August 18, 2013).
2. For illustrative purposes this textbook makes some simplifying assumptions:
    i. Accounting rules require the asset to be recorded at the present value of the minimum lease obligations or its fair market value, whichever is less. The examples in this book assume that the present value is the lesser number.
    ii. Accounting rules require at least one of four specific conditions to be met for a lease to qualify as a capital lease. The examples assume that this criterion is met.
    iii. Issues involving capital leases such as depreciation or executory costs are ignored.

## Contributors and Attributions

- Template:ContribBusiMathOlivier

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# 10.8: Application - How To Purchase A Vehicle

What is the cheapest way to acquire a vehicle? With 20 million cars on the road for 12 million Canadian households, that is an average of 1.7 cars per household in Canada. In fact, on average Canadians purchase about 10 to 12 cars in their lifetime. With average prices in the $25,000 to $30,000 range per vehicle, and factoring in rising inflation over your lifetime, you will spend approximately $550,000 on acquiring vehicles! Yet despite this huge amount of money being spent, shockingly few consumers take the time to analyze their financing options critically. Would you spend $550,000 without doing some research? I thought not.

This section guides you through the practical task of finding the cheapest of the alternatives for obtaining a vehicle. It lays out the four primary means of vehicle ownership and develops a procedure through which you can make a smart purchase decision.

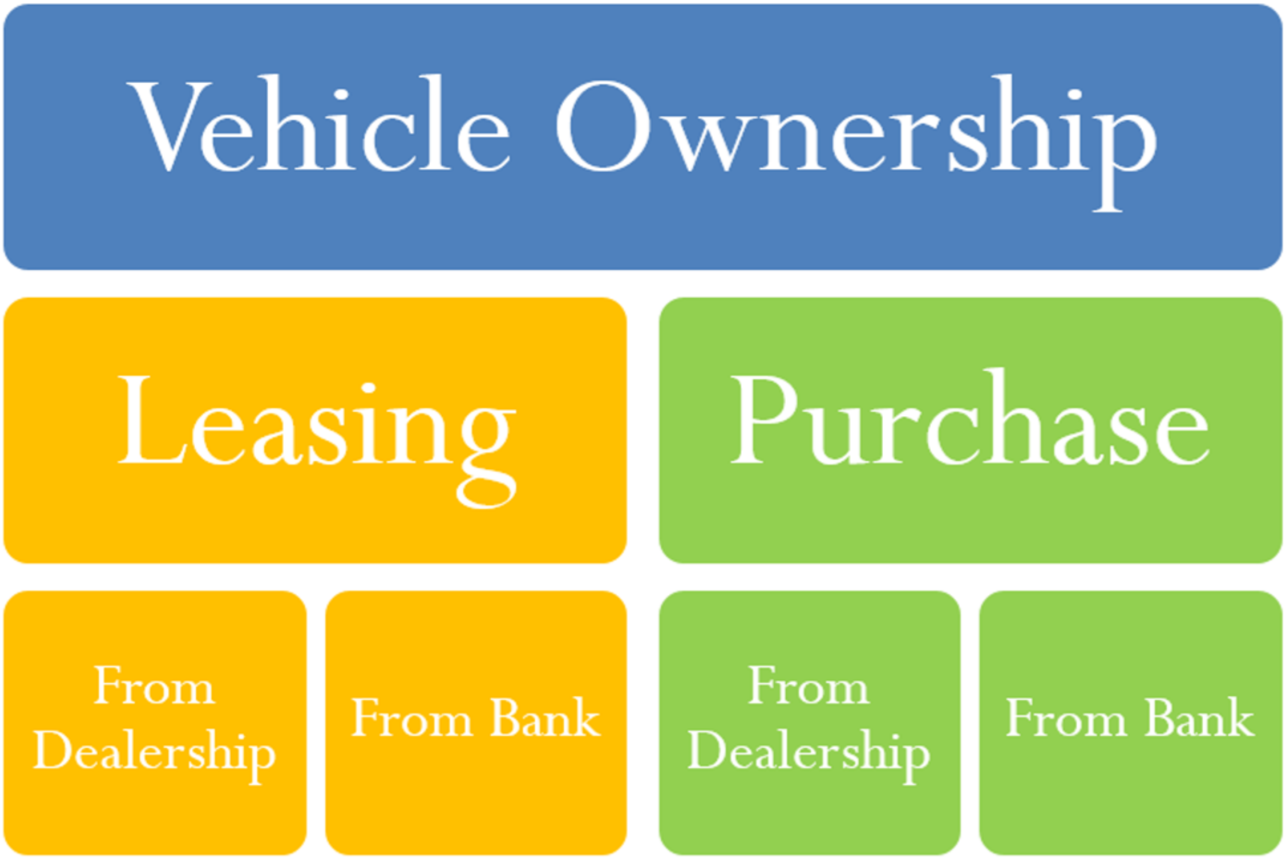
## What Are the Choices?

A few matters need to be cleared up from the outset. This section sticks to examining vehicle leases aimed at eventual ownership. Studies of vehicle leases have revealed that leasing a vehicle without intent to own is much more costly than ownership, even after factoring in maintenance and repairs. Give it some thought: Leasing a vehicle with no intent ever to own it means that you will be making monthly car payments for the rest of your driving life. That is called the "forever-in-debt" club. That does not sound like much fun, does it?

Smart vehicle ownership means exploring the various financial paths to vehicle ownership and ultimately choosing the lowest cost option that results both in 100% ownership of the vehicle for as long as you decide to retain the vehicle and in eventual termination of your car payments. That definitely sounds much more financially appealing.

One of the ways to purchase the vehicle is to pay cash for it. While this is technically possible, very few Canadians have $30,000 lying around that they can just write a cheque against. Therefore, this option is not pursued as a normal alternative. If you cannot pay cash, then you must finance the purchase by borrowing the money from an organization.

The figure below illustrates the four paths available for a typical vehicle acquisition.



The two basic options are to lease the vehicle or purchase it. You can borrow the money from either the dealership or a financial institution of your choice (from here on just called a bank for simplicity), such as RBC or a credit union. Note that if you lease the vehicle, you intend to pay the residual value upon the termination of the lease term.

Through the Internet and publications such as newspapers and flyers, you can usually obtain a lot of information about each of these four options without having to visit the dealership or a financial institution. Sometimes you may have to look in the fine print. The goal is to explore your monthly payment under equal conditions for each of the four options and then select the lowest payment.

### The Formula

Borrowing funds for a car requires either a loan or a lease. Both options represent annuities and have previously been discussed in this textbook. Therefore, to figure out your lowest cost alternative to vehicle ownership you do not need any new formulas, just a new process.

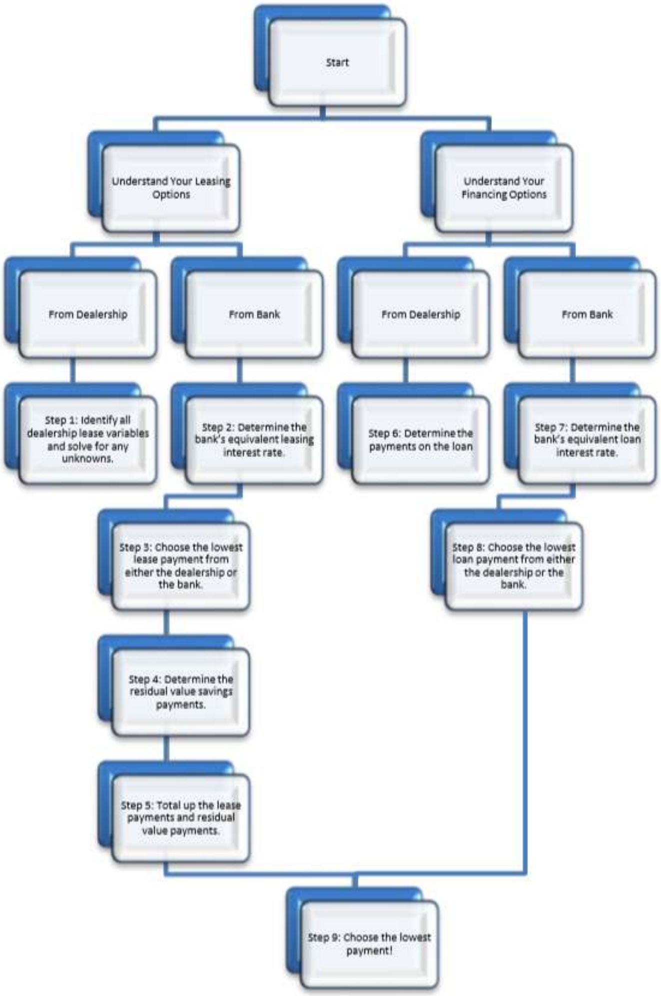
### How It Works

The secret to choosing well is that most vehicles actually have two prices at the dealership. One is the financing price and the other is a cash price, which results from a cash rebate that lowers the price. To be eligible to receive this cash rebate, the dealership must receive payment in full for the vehicle on the date of purchase. If you elect to have a bank be your source of funds, then the bank will provide a cheque to the dealership for the vehicle on your behalf, and therefore you will have paid cash! Thus, you need to borrow less money from the bank to acquire the vehicle since the cash price is lower. If you elect for the dealership to be your source of funds, then the dealership is not paid in full right away, and you are not eligible for the cash rebate. Thus, you pay the financing price, which is higher.

Of course, the dealerships want you to borrow from them since you pay the higher price and they receive an additional revenue stream from the interest they charge you. Dealerships put very tempting offers in front of consumers. Because most consumers do not understand the time value of money, they fail to compare these offers with other alternatives.

Do you pay more for the car but at a lower interest rate from the dealership, or do you pay less for the car but at a higher interest rate from the bank? Only a time value of money calculation can determine that. The consumer's best decision is based on which alternative has the lowest payment under equal conditions.

The steps for figuring out the lowest cost automotive option are illustrated in the figure below.



To understand the process illustrated, it is much easier to obtain information on vehicles from dealership advertisements or websites. Most commonly, the price of the vehicle is the same regardless of which dealership you might visit. Therefore, acquiring the vehicle from a dealership commonly represents a singular path with known information. On the other hand, approximately 75 banks and 485 credit unions are available to consumers in Canada. This represents up to 560 different choices, from which it would be extremely time consuming to gather all required information. The process establishes a threshold against which the banks are compared. You need contact only those financial institutions that pass the initial screening.

**Step 1**: Understand your leasing terms from the dealership. You must identify your leasing variables, including the lease interest rate, lease term, leasing payment structure, any down payment required, the purchase price, and the residual value. If any of these variables is unknown, identify its value by applying the appropriate annuity due formula.

**Step 2**: Determine the equivalent leasing interest rate from your bank. Banks advertise interest rates, not payments. Thus, you calculate the interest rate a bank must charge to be equivalent to the dealership using the identical leasing structure using the same lease term, lease payments, compounding, and residual value. The present value used in this calculation is the dealer's purchase price from step 1 less the cash rebate. Solve for the bank's equivalent nominal interest rate ($IY$).

**Step 3**: Make your leasing decision. After conducting some research, if you find a bank with a lower interest rate than the calculated $IY$ from step 2, then choose the bank lease (you should contact the bank to confirm the rate). Recalculate the leasing payments at the bank's interest rate. If you cannot find a lower interest rate from a bank, then select the dealership lease payments from step 1.

**Step 4**: A lease requires you to save up for the residual value. In choosing a leasing option, you also must save enough funds to pay off the residual value at the lease end. Set up an annuity due savings account using the same term and payment frequency. Use an appropriate interest rate from your savings institution. Calculate the annuity payment required to meet your residual value savings goal.

**Step 5**: Total the lease payment. Your lease payment from step 3 plus the residual value annuity payment from step 4 is the total out-of-pocket commitment required per payment interval to own the vehicle through leasing.

That completes the analysis of the leasing alternative. The remainder of the steps now involve examining your financing options.

**Step 6**: Figure out the loan payment from the dealership. Using the same term and payment frequency as the lease, calculate the payments to purchase the vehicle. Remember that a vehicle loan is an ordinary annuity. Use the financing price of the vehicle and the financing interest rate (not the lease rate) offered by the dealership.

**Step 7**: Figure out the equivalent loan interest rate from your bank. Using the same process as in step 2, aim to determine the equivalent interest rate from a bank loan that matches the payments with the dealership. With an identical structure and using the cash price, calculate the bank's equivalent nominal interest rate ($IY$).

**Step 8**: Make the loan decision. If a bank offers an interest rate lower than the calculated $IY$ from step 7, then select the bank loan (and you should probably contact the bank to confirm the rate). Recalculate the loan payments at the bank's interest rate. If you are unable to find a bank with a lower interest rate, then select the dealership loan payments from step 6.

**Step 9**: The final decision. In step 5, you calculated the total lease payment. From step 8, you have the lowest loan payment. The obvious choice is the lowest payment.

### Important Notes

1. At any given point in time, numerous vehicle rebates that adjust the purchase price of the vehicle are available (subject to various conditions). If such rebates apply to your purchase situation, modify the vehicle's purchase price accordingly in your calculations.
2. While leases are generally non-negotiable, the purchase price of a vehicle may be negotiable depending on the dealership and automotive manufacturer's policies. If haggling is allowed, estimate the amount by which you could reduce the purchase price and use the reduced amount for the calculations in steps 6 and 7.
3. If you have a trade-in for your vehicle acquisition, the value of the trade-in is treated in the same manner as a down payment toward the vehicle. It reduces the purchase price, so you modify your calculations accordingly.
4. The fundamental concept of time value of money says that to decide between the different leasing and financing timelines, all money must be moved to the same focal date. Note that in the process explained above you are making the final decision by comparing the payment amounts on the annuity due against the payment amounts on the ordinary annuity. A clear discrepancy in the timing of the payments is evident. In technically correct financial theory, to obey the fundamental concept of time value of money these payment amounts should then be present valued to the start of the transaction (the focal date) and the lower present value should be chosen. However, the reality is that most consumers base their purchase decision on the payment amount and what they can afford rather than on the total amount. Therefore, it is more logical to base the car acquisition decision strictly on the physical out-of-pocket annuity payment amounts.

### ✓ Example 10.8.1: How Should You Acquire This Vehicle?

Haber Suzuki advertises a Suzuki SX4 Hatchback for sale. The following information is known:

- Leasing information: Seven-year term, 0.9% compounded annually lease rate, \$90.75 biweekly payments, \$8,250 residual value, no down payment required.
- Purchase information: Cash rebate of \$3,500 available, purchase financing at 2.99% compounded annually.
- Bank options: The lowest vehicle lease rate from a bank is 4.4% compounded annually, while the lowest loan rate is 8.25% compounded annually.
- Savings accounts: The best rate available is 2% compounded annually. What is the lowest cost method to purchase this vehicle?

### Solution

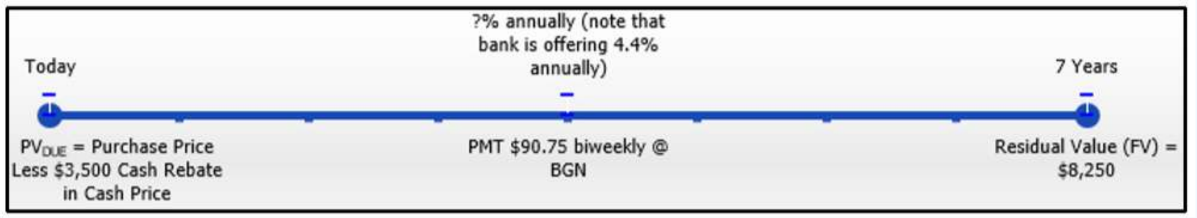
Examine the four paths to vehicle ownership and choose the path with the lowest biweekly payment.

**What You Already Know**

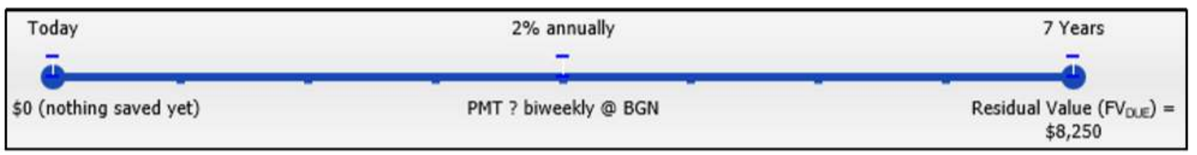
There are five timelines, one for each of the options (two leases, one savings account, two purchases):

Dealership Lease: $IY$ = 0.9%, $CY$ = 1, $PMT$ = \$90.75, $PY$ = 26, $FV$ = \$8,250, Years = 7

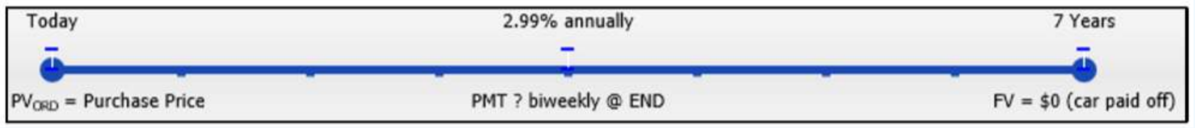
Bank Lease: $PV_{DUE}$ = Purchase Price − \$3,500, $CY$ = 1, $PMT$ = \$90.75, $PY$ = 26, $FV$ = \$8,250, Years = 7



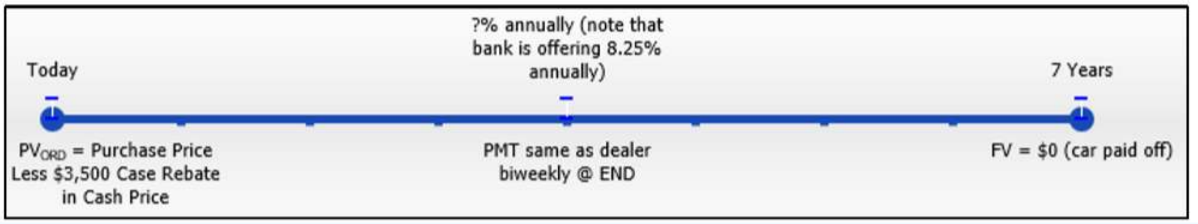
Residual Savings Plan: $PV$ = \$0, $IY$ = 2%, $CY$ = 1, $PY$ = 26, $FV$ = \$8,250, Years = 7



Dealership Loan: $PV_{ORD}$ = Purchase Price, $IY$ = 2.99%, $CY$ = 1, $PY$ = 26, $FV$ = \$0, Years = 7



Bank Loan: $PV_{ORD}$ = Purchase Price − \$3,500, $CY$ = 1, $PMT$ = Dealer loan payment, $PY$ = 26, $FV$ = \$0, Years =



**How You Will Get There**

**Step 1**:

In the dealership lease, the purchase price of the vehicle for the lease is missing. Find the present value of the residual value by applying Formulas 9.1, 9.2, and 9.3 (rearranged for $PV$). Then apply Formulas 11.1 and 11.5 to find the present value of the lease payments. Add the $PV$ and $PV_{DUE}$ to arrive at the purchase price.

**Step 2**:

In the bank lease, calculate the equivalent bank leasing interest rate. Adjust the purchase price to reflect the cash price and use your calculator to solve Formula 11.5 for $IY$.

**Step 3**:

If the bank rate is higher, choose the payment from step 1. If the bank rate is lower, then recalculate the lease payment using Formulas 9.1, 9.2, and 9.3 (rearranged for PV) to adjust the $PV_{DUE}$ and apply Formula 11.5 (rearranged for $PMT$).

**Step 4**:

In the residual savings plan, save to pay off the residual value using the savings account interest rate. Apply Formulas 9.1 and 11.3 (rearranged for $PMT$).

**Step 5**:

Add the payments from steps 3 and 4. This is the total lease payment.

**Step 6**:

In the dealership loan, calculate the loan payments from the dealership by applying Formulas 9.1 and 11.4 (rearranged for $PMT$).

**Step 7**:

In the bank loan, calculate the equivalent bank loan interest rate. Adjust the purchase price and use your calculator to solve Formula 11.4 for $IY$.

**Step 8**:

If the bank rate is higher, choose the payment from step 6. If the bank rate is lower, recalculate the loan payment, first using Formulas 9.1 and 9.3 (rearranged for $PV$) to adjust the $PV_{ORD}$ and then using Formula 11.4 (rearranged for $PMT$).

**Step 9**:

Choose the lowest payment from step 5 or step 8.

**Perform**

**Step 1**:

Residual Value:

$i = 0.9\%/1 = 0.9\%; N = 1 \times 7 = 7 \text{ compounds }\;;$

$\$8,250 = PV(1+0.009)^{7};$

$$PV = \$8,250 \div (1+0.009)^{7} = \$7,748.466949$$

Lease Payments:

$N = 26 \times 7 = 182 \text{ payments}$

$$PV_{DUE} = \$90.75 \left[ \frac{1 - \left[ \frac{1}{(1+0.009)^{\frac{1}{26}}} \right]^{182}}{(1+0.009)^{\frac{1}{26}} - 1} \right] \times (1+0.009)^{\frac{1}{26}} = \$16,011.97657$$

$$\text{Total } PV = \$7,748.466949 + \$16,011.97657 = \$23,760.44 = \text{ purchase price of the vehicle}$$

**Step 2**:

$\text{Cash price of vehicle } = \$23,760.44 - \$3,500.00 = \$20,260.44$

Solve Formula 11.5 (adjusting for residual value):

$$\$20,260.44 - \frac{\$8,250}{\left((1+i)^{\frac{1}{26}}\right)^{182}} = \$90.75\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{1}{26}}}\right]^{182}}{(1+i)^{\frac{1}{26}}-1}\right] \times (1+i)^{\frac{1}{26}}$$

From calculator, $IY$ = 4.5331% compounded annually.

**Step 3**:

The bank lease rate of 4.4% is lower than the equivalent lease rate of 4.5331%. Choose the lease from the bank and recalculate the payment.

$i = 4.4\%/1 = 4.4\%; N = 1 \times 7 = 7 \text{ compounds}$

$$\begin{aligned}\$8,250 &= PV(1+0.044)^7 \\ PV &= \$8,250 \div (1+0.044)^7 \\ &= \$6,103.099605\end{aligned}$$

$$\text{new } PV_{DUE} = \$20,260.44 - \$6,103.099605 = \$14,157.34039$$

$N = 182$ payments (same as step 1)

$$\$14,157.34039 = PMT\left[\frac{1-\left[\frac{1}{(1+0.044)^{\frac{1}{26}}}\right]^{182}}{(1+0.044)^{\frac{1}{26}}-1}\right] \times (1+0.044)^{\frac{1}{26}}$$

$$\$14,157.34039 = PMT\left[\frac{0.260230}{0.001657}\right] \times 1.001657$$

$$PMT = \frac{\$14,157.34039}{157.261348} = \$90.02$$

**Step 4**:

$i = 2\%/1 = 2\%; N = 182 \text{ payments } \; \text{(same as step 1)}$

$$\$8,250 = PMT\left[\frac{\left[(1+0.02)^{\frac{1}{26}}\right]^{182}-1}{(1+0.02)^{\frac{1}{26}}-1}\right] \times (1+0.02)^{\frac{1}{26}}$$

$$\$8,250 = PMT\left[\frac{0.148685}{0.000761}\right] \times 1.000761$$

$$PMT = \frac{\$8,250}{195.292254} = \$42.24$$

**Step 5**:

The total lease payment is $90.02 + $42.24 = $132.26 biweekly.

**Step 6**:

$\text{Purchase price } = \$23,760.44 \text{ (from step 1) }; i = 2.99\%/1 = 2.99\%; N = 182 \text{ payments } \text{ (same as in step 1)}$

$$\$23,760.44 = PMT\left[\frac{1-\left[\frac{1}{(1+0.0299)^{\frac{1}{26}}}\right]^{182}}{(1+0.0299)^{\frac{1}{26}}-1}\right]$$

$$\$23,760.44 = PMT\left[\frac{0.186355}{0.001133}\right]$$

**Step 7**:

Cash price of vehicle =$20,260.44 (from step 2).

$$\$20,260.44 = \$144.56\left[\frac{1-\left[\frac{1}{(1+i)^{\frac{1}{26}}}\right]^{182}}{(1+i)^{\frac{1}{26}}-1}\right]$$

From calculator, $IY$ = 8.0841% compounded annually.

**Step 8**:

Since the bank loan rate of 8.25% is higher than the equivalent loan rate of 8.0841%, remain with the dealership loan and make payments of $144.56.

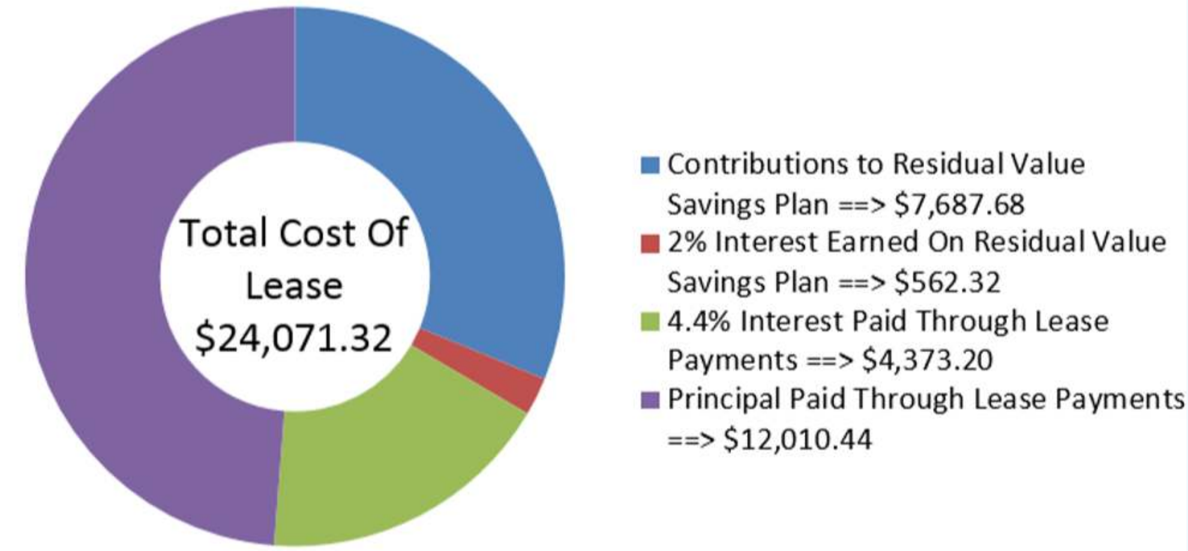
**Step 9**:

The lowest payment is from step 5, which is $132.26.

**Calculator Instructions**

| Step | Mode | N | I/Y | PV | PMT | FV | P/Y | C/Y |
|---|---|---|---|---|---|---|---|---|
| 1 | BGN | 182 | 0.9 | Answer: 23,760.44352 | -90.75 | -8250 | 26 | 1 |
| 2 | √ | √ | Answer: 4.533113 | 20260.44 | √ | √ | √ | √ |
| 3 | √ | √ | 4.4 | √ | Answer: -90.024284 | √ | √ | √ |
| 4 | √ | √ | 2 | 0 | Answer: -42.244378 | 8250 | √ | √ |
| 6 | END | √ | 2.99 | 23760.44 | Answer: -144.558127 | 0 | √ | √ |
| 7 | √ | √ | Answer: 8.084126 | 20260.44 | -144.56 | √ | √ | √ |

The best option to acquire this vehicle is to lease it from the bank. This requires biweekly lease payments of $90.02 and residual savings payments of $42.24 for a total of $132.26 per period for seven years. At the end of the lease, the accumulated savings is used to pay the $8,250 residual value.



## Contributors and Attributions

- Template:ContribBusiMathOlivier

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# 10.9: Application - Planning Your RRSP

How and when should you start planning your RRSP? When you are in your late teens or early twenties, you are probably not thinking much about your retirement income. You are at a stage in life when, besides paying for education or paying off student loans, you move out on your own and need to acquire many possessions ranging from appliances to furniture to transportation. Your income is probably at its lowest point in your career. You find yourself on a limited budget with high demands.

Ironically, though, this is the best time to start your RRSP. Throughout the last few chapters you have already witnessed the incredible power of compound interest. Through discussion, examples, and even some of the exercises, it should be clear that the earlier you invest your money, the less you have to pay out of your pocket to reach your retirement income goals.

But why should you go to the trouble and expense of contributing to an RRSP? For those counting on the government to provide retirement income, it was pointed out that the typical Canadian in retirement earns $529.09 per month from the Canada Pension Plan (CPP) and $514.74 from Old Age Security (OAS). Both of these amounts are pre-tax, which means your net income from these sources will be even less. This is not much to live off of.

Are companies not supposed to provide pension plans for their employees? While some companies do offer pension plans, these public and private-sector companies are few and far between. The Pension Investment Association of Canada (PIAC) manages approximately 130 pension funds representing approximately 80% of Canada's pension industry by asset size. In a 2007 study conducted by the PIAC, it found that approximately 20.5% of Canadian membership companies are either closing or considering closing their defined benefit pension funds. In Ontario, the percentage is much higher at 36.2%. These closures are in the private sector. Additionally approximately 20% of Canadian organizations are reducing future benefits.1

The bottom line is that you should not count on somebody else to create your retirement income for you. Also, do not bank on being lucky enough to work for a company that has a good pension plan. You need to take care of yourself.

This section introduces a simplified model of RRSP planning that reasonably approximates the financial commitment that will provide a satisfactory retirement income. Ultimately, you should always consult a financial adviser as you plan your RRSP.

## Income Planning

Income planning requires you to project future earnings, determine income needs in retirement, and then develop a savings plan toward that goal. Look at these details:

1. Projected Retirement Income. The first step is to determine your annual retirement income at the age of retirement. The general consensus is that a retiree needs approximately 70% of pre-retirement gross income to live comfortably and maintain the same lifestyle. It is difficult to know what that amount will be 30 or 40 years from now. However, today's typical retirement income is known. The average Canadian needs about $40,000 of gross income. You can adjust this number higher if necessary. You can then project today's amount forward to the age of retirement using an estimated rate of inflation based on historical data.
2. Principal Required at Age of Retirement. The second step requires you to determine the principal needed to fund the annual income requirement at the age of retirement. You must estimate the number of years for which withdrawals are to be made from the RRSP balance. In essence, how long are you going to live? While no one knows the answer, the average Canadian male lives to age 78 while the average Canadian female lives to age 82. These are good starting points. It would not make sense to lower these numbers since you do not want to be caught short, but you could raise these numbers if necessary. A look at your parents, grandparents, and other ancestors combined with your own lifestyle choices might indicate the typical life span you may expect. While you are not guaranteed to live that long, at least you can base a reasonable financial decision upon it. You must also factor in inflation during the retirement years. Each year, your income must rise to keep up with the cost of living. Finally, you should use a

conservative, low-risk interest rate in these calculations because you certainly cannot afford to lose your savings to another "Black Monday" (which refers to October 19, 1987, when stock markets around the world crashed).

3. Annuity Savings Payments. The final step is to determine the payments necessary to reach the principal required for retirement. You should consider that your income generally rises over your lifetime, which means as you get older you could afford to contribute a larger amount to your RRSP. Thus, contributions start small when you are young and increase as you grow older. The interest rate you use must reflect the market and your risk tolerance. Since 1980, the Toronto Stock Exchange (TSX) has averaged approximately 6% annual growth, which is a good starting point for determining what interest rate to use in your calculations.

### The Formula

RRSP planning does not involve any new formulas. Instead, you must combine previously studied single payment concepts from Chapter 9 and annuity concepts from both Chapter 11 and Chapter 12.

### How It Works

Your end goal is to calculate the regular contributions to your RRSP necessary to achieve a balance in your RRSP from which you can regularly withdraw in retirement to form your income. The calculations presented ignore other sources of retirement income such as the CPP or OAS, as if you were solely responsible for your own financial well-being. You can then treat any other income as bonus income.

Follow the steps:

**Step 1**: Calculate the annual retirement income you will need. Choose a value of annual income in today's dollars along with an annual rate of inflation to use. Then, by applying Formula 9.1 (Periodic Interest Rate), Formula 9.2 (Number of Compound Periods for Single Payments), and Formula 9.3 (Compound Interest for Single Payments) you can move that income to your required age of retirement.

**Step 2**: Calculate the present value of the retirement income. Most people receive their retirement income monthly, so you divide the result from step 1 by 12. Retirement income usually starts one month after retirement, thus forming an ordinary annuity. Using a reasonable annual rate of inflation, you also divide the inflation rate by 12 to approximate the growth rate to be used in calculating a constant growth annuity required during retirement. Estimate the number of years the retirement fund must sustain and select a low-risk conservative interest rate. Then apply Formula 9.1 (Periodic Interest Rate), Formula 11.1 (Number of Annuity Payments), and Formula 12.3 (Present Value of a Constant Growth Ordinary Annuity) to arrive at the principal required at the age of retirement.

**Step 3**: Calculate the annuity payment required to achieve your goal. If a single payment is already invested today, deduct its future value at the age of retirement (using Formula 9.1, Formula 9.2, and Formula 9.3) from the amount of money determined in step two. The principal at the age of retirement (from step 2) now becomes the future value for the ordinary constant growth annuity. Use an interest rate that reflects market rates and your risk level, along with an appropriate growth rate you select for the annuity payments you will make. This growth rate may or may not match the growth rate determined in step 2. Thus, arriving at the annuity payment amount requires Formula 9.1 (Periodic Interest Rate), Formula 11.1 (Number of Annuity Payments), and Formula 12.1 (Future Value of a Constant Growth Ordinary Annuity) rearranged for PMT.

### Important Notes

This procedure approximates the required regular RRSP contribution. However, recognize that the procedure is somewhat simplified and certainly does not factor in all variables that may apply in an individual situation. It is always best to consult a certified financial planner to ensure that your RRSP plan works under current market conditions and under any applicable restrictions.

Also, the steps listed above also assume an ordinary annuity structure. In the event of an annuity due, substitute the appropriate annuity due formula as required.

### Paths To Success

One of the hardest tasks in planning an RRSP is to guess interest rates and rates of inflation. If you are unsure of what numbers to use, some safe values are as follows:

1. A rate of inflation or growth rate of 3% compounded annually, equaling 0.25% per month
2. During retirement, an interest rate of 4% compounded annually
3. While you contribute toward your RRSP, an interest rate of 6% compounded annually, based on the approximate historical average over the past 31 years (1980–2011) for the TSX.

### ? Exercise 10.9.1: Give It Some Thought

Assume that two people—Person A and Person B—are saving for the same retirement amount under equal conditions except as noted. In each of the following cases, determine whose RRSP contributions are higher:

a. Person A starts contributing at age 18 while Person B starts contributing at age 25.
b. Person A earns 7% compounded annually while Person B earns 6% compounded annually.
c. Person A contributes monthly while Person B contributes semi-monthly.
d. Person A contributes through an ordinary annuity while Person B uses an annuity due.
e. Person A earns 4.5% interest during retirement while Person B earns 4% interest during retirement.
f. Person A has quarterly compounded earnings while Person B has monthly compounded earnings.

**Answer**

a. Person B, because he has a shorter period to accumulate the same amount as Person A.
b. Person B, because a lower interest rate requires more principal, hence larger contributions.
c. Person A, because the principal does not increase as often as Person B’s does.
d. Person A, because there is one fewer compounds than for Person B.
e. Person B earns less interest, so a larger principal is required at retirement than Person A requires.
f. Person A, because the interest is not converted to principal as often as it is for Person B.

### ✓ Example 10.9.1: An 18-Year Old with No Savings Planning for Retirement

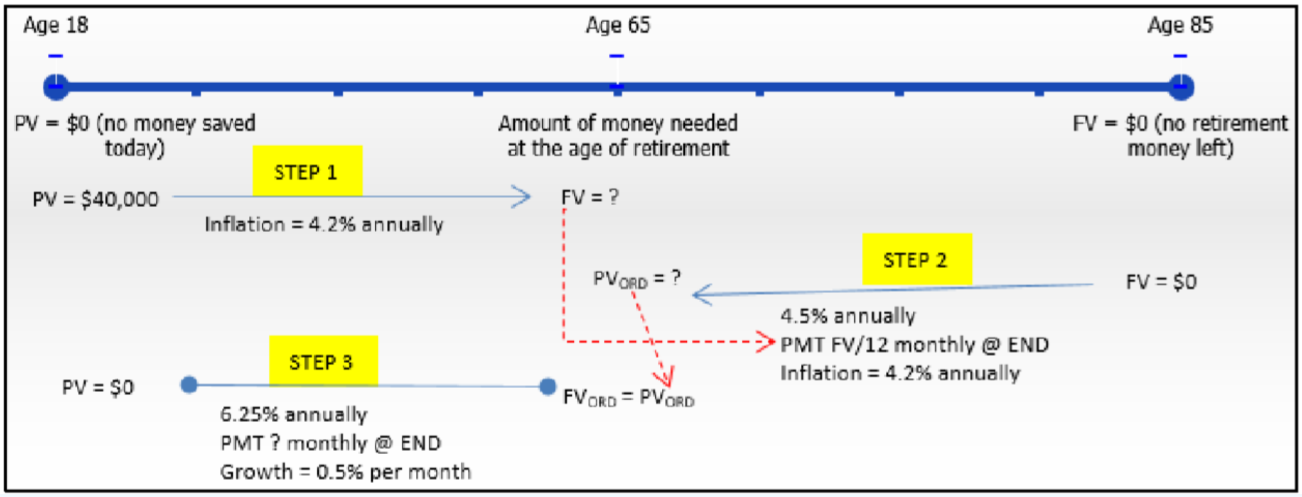
Jesse just turned 18 and plans on retiring when he turns 65. In today's funds, he wants to earn $40,000 annually in retirement. Based on the past 50 years, he estimates inflation at 4.2% compounded annually. He would like to receive 20 years of monthly payments from his retirement money, which is forecasted to earn 4.5% compounded annually. He believes his RRSP can earn 6.25% compounded annually during his contributions, he has no money saved to date, and he would like to increase each payment by 0.5%. One month from now, what is the amount of his first monthly RRSP contribution?

#### Solution

Jesse wants to make regularly increasing contributions to his RRSP at the end of the monthly payment interval with annual compounding. Therefore, this is a constant growth general ordinary annuity. Calculate his first payment ($PMT$).

**What You Already Know**

Jesse's RRSP plan appears in the timeline.



**Step 1**:

$PV$ = $40,000, $IY$ = 4.2%, $CY$ = 1, Years = 47

**Step 2**:

$FV$ = $0, $IY$ = 4.5%, $CY$ = 1, $PY$ = 12, Years = 20, $\Delta\%$ = 4.2%/12 = 0.35% per payment

**Step 3**:

$PV$ = $0, $FV_{ORD}$ = Step 2 answer, $IY$ = 6.25%, $CY$ = 1, $PY$ = 12, Years = 47, $\Delta\%$ = 0.5%

**How You Will Get There**

**Step 1**:

Move Jesse's desired annual income from today to the age of retirement by applying Formula 9.1, Formula 9.2, and Formula 9.3.

**Step 2**:

Calculate the amount of money needed at the age of retirement by applying Formula 9.1, Formula 11.1, and Formula 12.3.

**Step 3**:

Calculate Jesse's regular monthly payment by applying Formula 9.1, Formula 11.1, and Formula 12.1.

**Perform**

**Step 1**:

$$i = 4.2\%/1 = 4.2\%; N = 1 \times 47 = 47 \text{ compounds}; FV = \$40,000(1+0.042)^{47} = \$276,594.02$$

**Step 2**:

$$i = 4.5\%/1 = 4.5\%; N = 12 \times 20 = 240 \text{ payments}$$

Note that the monthly payment is $276,594.02 ÷ 12 = $23,049.50

$$PV_{ORD} = \frac{\$23,049.50}{1+0.0035}\left[\frac{1-\left[\frac{1+0.0035}{(1+0.045)^{\frac{1}{12}}}\right]^{240}}{\frac{(1+0.045)^{\frac{1}{12}}}{1+0.0035}-1}\right] = \$5,398,479.88$$

**Step 3**:

$$i = 6.25\%/1 = 6.25\%; N = 12 \times 47 = 564 \text{ payments}$$

$$\$5,398,479.88 = PMT(1+0.005)^{564-1}\left[\frac{\left[\frac{(1+0.0625)^{\frac{1}{12}}}{1+0.005}\right]^{564}-1}{\frac{(1+0.0625)^{\frac{1}{12}}}{1+0.005}-1}\right]$$

$$\begin{aligned}\$5,398,479.88 &= PMT(16.576497)[574.367284]\\ \$5,398,479.88 &= PMT(9,520.997774)\\ \$567.01 &= PMT\end{aligned}$$

**Calculator Instructions**

| Step | Mode | N | I/Y | PV | PMT | FV | P/Y | C/Y |
|---|---|---|---|---|---|---|---|---|
| 1 | END | 47 | 4.2 | -40000 | 0 | Answer: 276,594.0176 | 1 | 1 |

On the basis that the rate of inflation is relatively accurate, Jesse's first RRSP contribution one month from now is \$567.01. Each subsequent payment increases by 0.5% resulting in a balance of \$5,398,479.88 at retirement, from which he can make his first withdrawal of \$23,049.50 increasing at 0.35% per month for 20 years.

## ✓ Example 10.9.2: A 30-Year Old with Savings Planning for Retirement

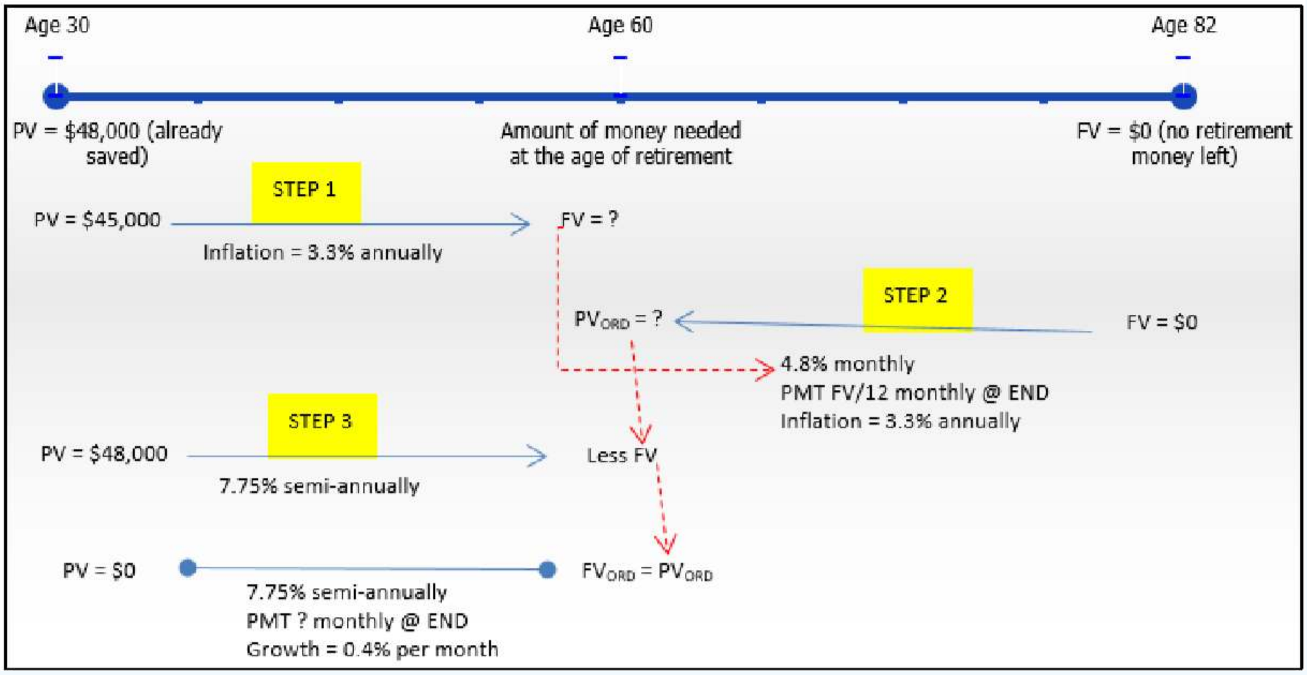
Marilyn just turned 30 and plans on retiring when she turns 60. In today's funds, she wants to earn \$45,000 in retirement. Based on the past 30 years, she estimates inflation at 3.3% compounded annually. She would like to receive 22 years of month-end payments from her retirement fund, which is forecasted to earn 4.8% interest compounded monthly. Based on the results to date, she believes her RRSP can earn 7.75% compounded semiannually during her contributions. She has already saved \$48,000, and she will increase each contribution by 0.4%. One month from now, what is the amount of her first monthly RRSP contribution?

### Solution

She will make regularly increasing contributions to her RRSP at the end of the monthly payment interval with semi-annual compounding. Therefore, this is a constant growth general ordinary annuity. Calculate her first payment ($PMT$).

**What You Already Know**

Marilyn's RRSP plan appears in the timeline.



**Step 1**:

$PV$ = $45,000, $IY$ = 3.3%, $CY$ = 1, Years = 30

**Step 2**:

$FV$ = $0, $IY$ = 4.8%, $CY$ = 12, $PY$ = 12, Years = 22, $\Delta\%$ = 3.3%/12 = 0.275% per payment

**Step 3**:

$PV$ = $48,000, $FV_{ORD}$ = Step 2 answer, $IY$ = 7.75%, $CY$ = 2, $PY$ = 12, Years = 30, $\Delta\%$ = 0.4%

**How You Will Get There**

**Step 1**:

Move her desired annual income from today to the age of retirement by applying Formula 9.1, Formula 9.2, and Formula 9.3.

**Step 2**:

Calculate the amount of money needed at the age of retirement by applying Formula 9.1, Formula 11.1, and Formula 12.3.

**Step 3**:

Deduct her current savings from her required future contributions by calculating the future value of her present savings and deducting it from the step 2 answer by applying Formula 9.1, Formula 9.2, and Formula 9.3. Calculate her regular monthly payment by applying Formula 9.1, Formula 11.1, and Formula 12.1.

**Perform**

**Step 1**:

$$i = 3.3\%/1 = 3.3\%; N = 1 \times 30 = 30 \text{ compounds}; FV = \$45,000(1 + 0.033)^{30} = \$119,185.15$$

**Step 2**:

$$i = 4.8\%/12 = 0.4\%; N = 12 \times 22 = 264 \text{ payments}$$

Note that the monthly payment is $119,185.15/12 = $9,932.10

$$PV_{ORD}=\frac{\$9,932.10}{1+0.00275}\left[\frac{1-\left[\frac{1+0.00275}{(1+0.004)^{\frac{12}{12}}}\right]^{264}}{\frac{(1+0.004)^{\frac{12}{12}}}{1+0.00275}-1}\right]=\$2,226,998.08$$

**Step 3**:

Current Savings:

$$i=7.75\%/2=3.875\%; N=2\times30=60\text{ compounds }; FV=\$48,000(1+0.03875)^{60}=\$469,789.65$$

$$\text{new }FV_{ORD}=\$2,226,998.08-\$469,789.65=\$1,757,208.43$$

RRSP Payments:

$$i=7.75\%/2=3.875\%; N=12\times30=360\text{ payments}$$

$$\$1,757,208.43=PMT(1+0.004)^{360-1}\left[\frac{\left[\frac{(1+0.03875)^{\frac{2}{12}}}{1+0.004}\right]^{360}-1}{\frac{(1+0.03875)^{\frac{2}{12}}}{1+0.004}-1}\right]$$

$$\begin{aligned}\$1,757,208.43&=PMT(4.191822)[564.766990]\\ \$1,757,208.43&=PMT(2,367.403054)\\ \$742.25&=PMT\end{aligned}$$

**Calculator Instructions**

| Step | Mode | N | I/Y | PV | PMT | FV | P/Y | C/Y |
|---|---|---|---|---|---|---|---|---|
| 1 | END | 30 | 3.3 | -45000 | 0 | Answer: 119,185.1533 | 1 | 1 |
| 3(Savings) | √ | 60 | 7.75 | -48000 | √ | Answer: 469,789.6504 | 2 | 2 |

On the basis that the rate of inflation is relatively accurate, Marilyn's first RRSP contribution one month from now is $742.25. Each subsequent payment increases by 0.4% resulting in a balance of $2,226,998.08 (including her current savings) at retirement, from which she makes her first withdrawal of $9,932.10 increasing at 0.275% per month for 22 years.

## References

1. Pension Investment Association of Canada, Pension Plan Funding Challenges: 2007 PIAC Survey, December 2007, www.piacweb.org/files/Pension%20Plan%20Funding%20Challenges%202007%20PIAC%20Survey%20Dec%2007.pdf (accessed September 26, 2010).

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# 10.10: Counting Rules (Webb)

There are times when the sample space is very large and is not feasible to write out. In that case, it helps to have mathematical tools for counting the size of the sample space. These tools are known as counting techniques or counting rules.

**Fundamental Counting Rule:** If task 1 can be done m1 ways, task 2 can be done $m_2$ ways, and so forth to task n being done mn ways. Then the number of ways to do task 1, 2,…, n together would be to multiply the number of ways for each task $m_1 \cdot m_2 \cdots m_n$.

## ✓ Example 4-25

A menu offers a choice of 3 salads, 8 main dishes, and 5 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

### Solution

There are three tasks, picking a salad, a main dish, and a dessert. The salad task can be done 3 ways, the main dish task can be done 8 ways, and the dessert task can be done 5 ways. The ways to pick a salad, main dish, and dessert are: $\frac{3}{\text{salad}} \cdot \frac{2}{\text{main}} \cdot \frac{1}{\text{dessert}}$ = 120 different meals.

## ✓ Example 4-26

How many 4-digit debit card personal identification numbers (PIN) can be made?

### Solution

Four tasks must be done in this example. The tasks are to pick the first number, then the second number, then the third number, and then the fourth number. The first task can be done 10 ways since there are digits 1 through 9 or a zero. We can use the same numbers over again (repeats are allowed) to find that the second task can also be done 10 ways. The same with the third and fourth tasks, which also have 10 ways.

There are $\frac{10}{\text{first number}} \cdot \frac{10}{\text{second number}} \cdot \frac{10}{\text{third number}} \cdot \frac{10}{\text{fourth number}}$ = 10,000 possible PINs.

## ✓ Example 4-27

How many ways can the three letters a, b, and c be arranged with no letters repeating?

### Solution

Three tasks must be done in this case. The tasks are to pick the first letter, then the second letter, and then the third letter. The first task can be done 3 ways since there are 3 letters. The second task can be done 2 ways, since the first task took one of the letters (repeats are not allowed). The third task can be done 1 way, since the first and second task took two of the letters.

There are $\frac{3}{1^{\text{st}} \text{ letter}} \cdot \frac{2}{2^{\text{nd}} \text{ letter}} \cdot \frac{1}{3^{\text{rd}} \text{ letter}}$

You can also look at this example in a tree diagram, see Figure 4-17. There are 6 different arrangements of the letters. The solution was found by multiplying 3·2·1 = 6.

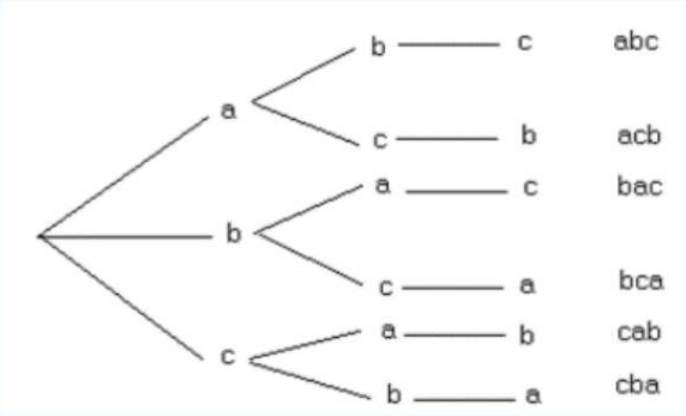

Figure 4-17

If we have 10 different letters for, say, a password, the tree diagram would be very time-consuming to make because of the length of options and tasks, so we have some shortcut formulas that help count these arrangements.

Many counting problems involve multiplying a list of decreasing numbers, which is called a **factorial**. The factorial is represented mathematically by the starting number followed by an exclamation point, in this case 3! = 3·2·1 = 6. There is a special symbol for this and a special button on your calculator or computer.

**Factorial Rule:** The number of different ways to arrange n objects is n! = n·(n – 1)·(n – 2) ···3·2·1, where repetitions are not allowed.

Zero factorial is defined to be 0! = 1, and 1 factorial is defined to be 1! = 1.

**TI-84:** On the home screen, enter the number of which you would like to find the factorial. Press [MATH]. Use cursor keys to move to the PRB menu. Press 4 (4:!) Press [ENTER] to calculate.

**TI-89:** On the home screen, enter the number of which you would like to find the factorial. Press [2$^{nd}$] [Math] > 7:Probability > 1:!. Press [ENTER] to calculate.

**Excel:** In an empty cell type in =FACT(n) where n is the number so 4! would be =FACT(4).

✓ Example 4-28

How many ways can you arrange five people standing in line?

### Solution

No repeats are allowed since you cannot reuse a person twice. Order is important since the first person is first in line and will be selected first. This meets the requirements for the factorial rule,

5! = 5·4·3·2·1 = 120 ways.

Sometimes we do not want to select the entire group but only select *r* objects from n total objects. The number of ways to do this depends on if the order you choose the *r* objects matters or if it does not matter. As an example, if you are trying to call a person on the phone, you have to have the digits of their number in the correct order. In this case, the order of the numbers matters. If you were picking random numbers for the lottery, it does not matter which number you pick first since they always arrange the numbers from the smallest to largest once the numbers are drawn. As long as you have the same numbers that the lottery officials pick, you win. In this case, the order does not matter.

A **permutation** is an arrangement of items with a specific order. You use permutations to count items when the order matters.

**Permutation Rule:** The number of different ways of picking r objects from n total objects when repeats are not allowed and order matters $_{\mathrm{n}}\mathrm{P}_{\mathrm{r}} = \frac{n!}{(n-r)!}$ .

When the order does not matter, you use combinations. A **combination** is an arrangement of items when order is not important. When you do a counting problem, the first thing you should ask yourself is “are repeats allowed,” then ask yourself “does order matter?”

> **Combination Rule:** The number of ways to select r objects from n total objects when repeats are not allowed and order does not matter ${}_{\mathrm{n}}\mathrm{C}_{\mathrm{r}} = \frac{n!}{(r!(n-r)!)}$.

**TI-84:** Enter the number “trials” (n) on the home screen. Press [MATH]. Use cursor keys to move to the PRB menu. Press 2 for permutation (2: ${}_{\mathrm{n}}\mathrm{P}_{\mathrm{r}}$), 3 for combination (3: ${}_{\mathrm{n}}\mathrm{C}_{\mathrm{r}}$). Enter the number of “successes” (r). Press [ENTER] to calculate.

**TI-89:** Press [2nd] Math > 7:Probability > Press 2 for permutation (2: ${}_{\mathrm{n}}\mathrm{P}_{\mathrm{r}}$), 3 for combination (3: ${}_{\mathrm{n}}\mathrm{C}_{\mathrm{r}}$). Enter the sample size on the home screen, then a comma, then enter the number of “successes,” then end the parenthesis. Press [ENTER] to calculate.

**Excel:** In a blank cell type in the formula =COMBIN(n, r) or =PERMUT(n, r) where n is the total number of objects and r is the smaller number of objects that you are selecting out of n. For example =COMBIN(8, 3).

The following flow chart in Figure 4-18 may help with deciding which counting rule to use.

Start on the left; ask yourself if the same item can be repeated. For instance, a person on a committee cannot be counted as two distinct people; however, a number on a car license plate may be used twice. If repeats are not allowed, then ask, does the order in which the item is chosen matter? If it does not then we use the combinations, if it does then ask are you ordering the entire group, use factorial, or just some of the group, use permutation.

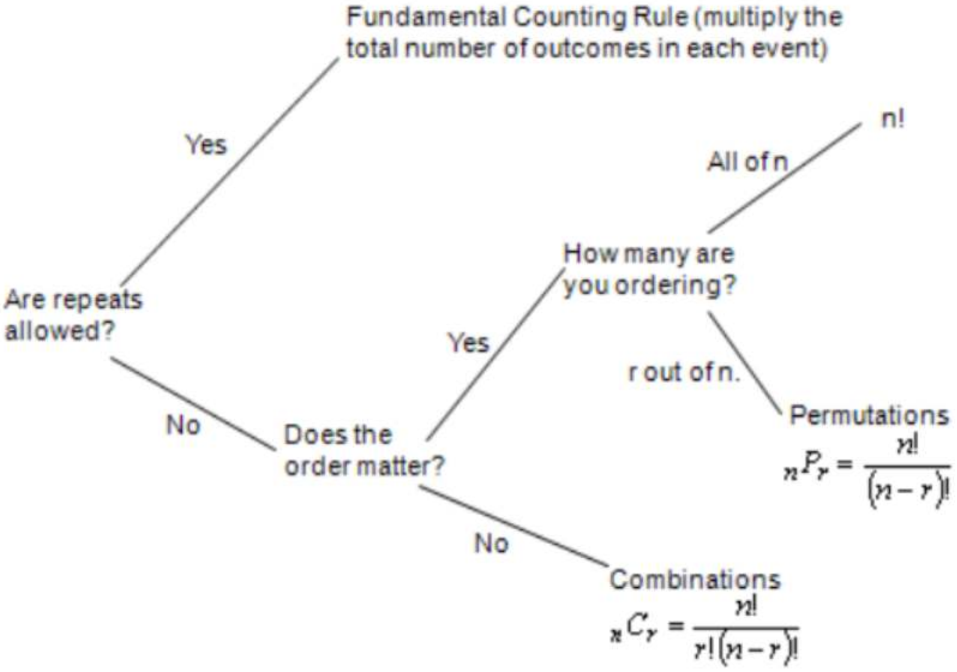

Figure 4-18

**Example 4-30**

Critical Miss, PSU's Tabletop Gaming Club, has 15 members this term. How many ways can a slate of 3 officers consisting of a president, vice-president, and treasurer be chosen?

**Solution**

In this case, repeats are not allowed since we don’t want the same member to hold more than one position. The order matters, since if you pick person 1 for president, person 2 for vice-president, and person 3 for treasurer, you would have different members in those positions than if you picked person 2 for president, person 1 for vice-president, and person 3 for treasurer. This is a permutation problem with n = 15 and r = 3.

$_{15}\mathrm{P}_3 = \frac{15!}{(15-3)!} = \frac{15!}{12!} = 2730$
There are 2,730 ways to elect these three positions.

In general, if you were selecting items that involve rank, a position title, 1st, 2nd, or 3rd place or prize, etc. then the order in which the items are arranged is important and you would use permutation.

### ✓ Example 4-31

Critical Miss, PSU's Tabletop Gaming Club, has 15 members this term. They need to select 3 members to have keys to the game office. How many ways can the 3 members be chosen?

**Solution**
In this case, repeats are not allowed, because we don't want one person to have more than one key. The order in which the keys are handed out does not matter. This is a combination problem with n = 15 and r = 3.

$_{15}\mathrm{C}_3 = \frac{15!}{(3!(15-3)!)} = \frac{15!}{(3! \cdot 12!)} = 455 \quad \text{There are 455 ways to hand out the three keys.}$

We can use these counting rules in finding probabilities. For instance, the probability of winning the lottery can be found using these counting rules.

### ✓ Example 4-32

What is the probability of winning the jackpot in the Pick-4 Lottery? To play Pick-4, you choose 4 numbers from 0 to 9. This will give you a number between 0000 and 9999. You win the jackpot if you match your 4 numbers in the exact order they are drawn.

https://www.oregonlottery.org/jackpot/pick-4/

**Solution**
There is only one winning number, so the numerator for the probability will just be 1. The denominator will be all the ways to select the 4 numbers. Repeat numbers are allowed, for example you can have 4242 with repeating 4s and 2s. The order in which the balls are selected does matter by the rules of the game. Use the fundamental counting rule combined with the fundamental counting rule and we would get 10·10·10·10 = 10,000.

Thus, the probability of winning the jackpot would be $\frac{1}{10000} = 0.0001$

### ✓ Example 4-33

What is the probability of getting a full house if 5 cards are randomly dealt from a standard deck of cards?

**Solution**
A full house is a combined three of a kind and pair, for example, QQQ22. There are $_{13}\mathrm{C}_1$ ways to choose a card between Ace, 2, 3, … , King. Once a number is chosen, there are 4 cards with that rank and there are $_4\mathrm{C}_3$ ways to choose a three of kind from that rank. Once we use up one of the ranks, such as the three queens, there are $_{12}\mathrm{C}_1$ ways to choose the rank for the pair. Once the pair is chosen there are $_4\mathrm{C}_2$ ways to choose a pair from that rank. All together there are $_{52}\mathrm{C}_5$ ways to randomly deal out 5 cards. The probability of getting a full house with then be

$\frac{_{13}C_1 \cdot {_4C_3} \cdot {_{12}C_1} \cdot {_4C_2}}{_{52}C_5} = \frac{3744}{2598960} = 0.00144$

### ✓ Example 4-34

What is the probability of winning the Powerball jackpot? As of 2021, the Powerball lottery consists of drawing five white balls in any order numbered 1 through 69, and one red Powerball numbered 1 through 26.

https://www.oregonlottery.org/jackpot/powerball/

**Solution**

There is only one winning number, so the numerator for the probability will just be 1. The denominator will be all the ways to select the 5 white balls and 1 red ball. The order in which the balls are selected does not matter and repeat numbers are not allowed. Using the combination rule combined with the fundamental counting rule we would get $_{69}C_5 \cdot {}_{26}C_1$.

Thus, the probability of winning the jackpot would be $\frac{1}{_{69}C_5 \cdot {}_{26}C_1} = 0.000000003707$.

**Example 4-35**

An iPhone has a 6 digit numerical password to unlock the phone. What is the probability of guessing the password on the first try?

**Solution**

There is only one correct password, so the numerator for the probability will just be 1. The denominator will be all the ways to select the 6 numbers. Repeat numbers are allowed. The order in which the numbers are entered into the phone matters. Using the fundamental counting rule, we would get $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^6 = 1{,}000{,}000$ possible passwords.

Thus, the probability of guessing the password on the first try would be $\frac{1}{1000000} = 1\mathrm{E} - 6 = 0.000001$

> *“The chances of this happening are more or less one to infinity against. Little is known of how this came about because none of the geophysicists, probability statisticians, meteoranalysts or bizzarrologists who are so keen to research it can afford to stay there.”*
>
> *(Adams, 2002)*

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# Glossary

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