5.3: Truth Tables- Conditional, Biconditional

Learning Objectives

- Construct a truth table for a conditional statement
- Construct a truth table for a biconditional statement

Conditional statements are those in which we take an action based on the value of the condition. We are now going to look at a version of a conditional statement, sometimes called an implication, which states that the second part must logically follow from the first.

Conditional Statement

A **conditional statement** is a logical compound statement in which a statement p, called the antecedent, implies a statement q, called the consequent.

A conditional statement is written as $p \rightarrow q$ and is translated as "if p, then q".

Example $5.3.1$

The English statement "If it is raining, then there are clouds in the sky" is a conditional statement. It makes sense because if the antecedent "it is raining" is true, then the consequent "there are clouds in the sky" must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consequent becomes irrelevant.

Example $5.3.2$

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday's game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

There are four possible outcomes:

1. You pay for expedited shipping and receive the jersey by Friday.
2. You pay for expedited shipping and don't receive the jersey by Friday.
3. You don't pay for expedited shipping and receive the jersey by Friday.
4. You don't pay for expedited shipping and don't receive the jersey by Friday.

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don't receive the jersey by Friday. The first outcome is exactly what was promised, so there's no problem with that. The third outcome is not a lie because the website never said what would happen if you didn't pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn't make any promises about when the jersey would arrive if you didn't pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the antecedent is false, we cannot make any judgment about the consequent. The website never said that paying for expedited shipping was the *only* way to receive the jersey by Friday.

Truth Table for the Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Again, if the antecedent p is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true. The conditional statement is only false when the antecedent p is true and the consequent q is false.

Example $5.3.3$

Construct a truth table for the statement $(m \wedge \sim p) \rightarrow r$

Solution

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the antecedent, $m \wedge \sim p$.

m	p	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

m	p	r	$\sim p$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

m	p	r	$\sim p$	$m \wedge \sim p$
T	T	T	F	F
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	F	F
F	T	F	F	F
F	F	T	T	F
F	F	F	T	F

Now we can create a column for the conditional. Because it can be confusing to keep track of all the Ts and Fs, why don't we copy the column for r to the right of the column for $m \wedge \sim p$? This makes it a lot easier to read the conditional from left to right.

m	p	r	$\sim p$	$m\wedge\sim p$	r	$(m\wedge\sim p)\to r$
T	T	T	F	F	T	T
T	T	F	F	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	T	F	F	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	F	T

When m is true, p is false, and r is false- -the fourth row of the table-then the antecedent $m\wedge\sim p$ will be true but the consequent false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by $(m\wedge\sim p)\to r$, consider this: let $m =$ we order meatballs, $p =$ we order pasta, and $r =$ Rob is happy. The statement $(m\wedge\sim p)\to r$ is "if we order meatballs and don't order pasta, then Rob is happy". If m is true (we order meatballs), p is false (we don't order pasta), and r is false (Rob is not happy), then the statement is false, because we satisfied the antecedent but Rob did not satisfy the consequent.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider "If you submit your hours today, then you will be paid next Friday." What the payroll rep really means is "If you submit your hours today, then you will be paid next Friday, and if you don't submit your hours today, then you won't be paid next Friday." The conditional statement if *t*, then *p* also includes the inverse of the statement: if not *t*, then not *p*. A more compact way to express this statement is "You will be paid next Friday *if and only if* you submit your timesheet today." A statement of this form is called a **biconditional**.

Biconditional

A **biconditional statement** is a logical conditional statement in which the antecedent and consequent are interchangeable.

A biconditional is written as $p\leftrightarrow q$ and is translated as " p if and only if q."

Because a biconditional statement $p\leftrightarrow q$ actually means $(p\to q)\wedge(q\to p)$, we may think of it as a compound conditional statement: if p, then q <u>and</u> if q, then p. The double-headed arrow shows that the conditional statement goes from left to right <u>and</u> from right to left. A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

Truth Table for the Biconditional

p	q	$p\leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

A biconditional is true as long as the antecedent and the consequent have the same truth value. If they are either both true or both false, the biconditional is true.

Notice that the fourth row, where both components are false, is true; if you don't submit your timesheet and you don't get paid, the person from payroll told you the truth.

Example 5.3.4

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?

a. It is noon on Thursday and the garbage truck did not come down my street this morning.

b. It is Monday and the garbage truck is coming down my street.
c. It is Wednesday at 11:59PM and the garbage truck did not come down my street today.

Solution

Let p be "it is Thursday morning," and let q be "the garbage truck comes down my street."

a. This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.
b. This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.
c. This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

Try It 5.3.1

Suppose this statement is true: "I wear my running shoes if and only if I am exercising." Determine whether each of the following statements must be true or false.

a. I am exercising and I am not wearing my running shoes.
b. I am wearing my running shoes and I am not exercising.
c. I am not exercising and I am not wearing my running shoes.

Answer

Choices a & b are false; c is true.

Example 5.3.5

Create a truth table for the statement $(A \vee B) \leftrightarrow \sim C$

Solution

Whenever we have three component statements, we start by listing all the possible truth value combinations for A, B, and C.

A	B	C
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

After creating those three columns, we can create a fourth column for the antecedent, $A \vee B$. Now we will temporarily ignore the column for C and focus on A and B, writing the truth values for $A \vee B$.

A	B	C	$A \vee B$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

Next we can create a column for the negation of C. (Ignore the $A \vee B$ column and simply negate the values in the C column.)

A	B	C	$A \vee B$	$\sim C$
T	T	T	T	F
T	T	F	T	T
T	F	T	T	F
T	F	F	T	T
F	T	T	T	F
F	T	F	T	T
F	F	T	F	F
F	F	F	F	T

Finally, we find the truth values of $(A \vee B) \leftrightarrow \sim C$. Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.

A	B	C	$A \vee B$	$\sim C$	$(A \vee B) \leftrightarrow \sim C$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	F

To illustrate this situation, suppose your boss needs you to do either project A or project B (or both, if you have the time). If you do one of the projects, you will not get a crummy review (C is for crummy). So $(A \vee B) \leftrightarrow \sim C$ means "You will not get a crummy review if and only if you do project A or project B." Looking at a few of the rows of the truth table, we can see how this works out. In the first row, $A, B,$ and C are all true: you did both projects and got a crummy review, which is not what your boss told you would happen! That is why the final result of the first row is false. In the fourth row, A is true, B is false, and C is false: you did project A and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row, $A, B,$ and C are all false: you didn't do either project and did not get a crummy review. This is not what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)