3.5: Loans

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto and student loans, and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

Loans

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had $10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests $10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loan Formula

$$A=\frac{d\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}{\left(\frac{r}{n}\right)}$$

- A is the amount in the account at the beginning (the principal, or amount of the loan)
- d is your loan payment (your monthly payment, annual payment, etc)
- r is the annual interest rate in decimal form
- n is the number of compounding periods in one year
- t is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Loans: Many payments

Example 3.5.1

You can afford $200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with $200 per month?

Solution

In this example,

$$\begin{array}{ll}
d=\$200 & \text{the monthly loan payment} \\
r=0.03 & 3\% \text{ annual rate} \\
n=12 & \text{since we're doing monthly payments, we'll compound monthly} \\
t=5 & \text{since we're making monthly payments for 5 years}
\end{array}$$

We're looking for A, the starting amount of the loan.

$$A=\frac{200\left[1-\left(1+\frac{0.03}{12}\right)^{-12\times 5}\right]}{\left(\frac{0.03}{12}\right)}$$

$$A=\frac{200\left[1-(1.0025)^{-60}\right]}{0.0025}$$

$$A=\frac{200(1-0.861)}{0.0025}=\$11,120$$

You can afford a $\$11,120$ loan.

You will pay a total of $12,000 ($200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying $\$12,000-\$11,120=\$880$ interest total.

Similar to the previous section, the amount of payments is the same as the withdrawal formula.

Loan Payment Formula

$$d=\frac{A\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$$

- d is the loan payment
- A is the loan amount
- r is the annual interest rate in decimal form
- n is the number of compounding periods in one year
- t is the length of the loan, in years

Example $3.5.2$

You want to take out a $400,000 mortgage (home loan). The interest rate on the loan is 5%, and the loan is for 30 years. How much will your monthly payments be?

Solution

In this example, we're looking for d.

$$\begin{array}{ll} r=0.05 & 5\% \text{ annual rate} \\ n=12 & \text{since we're doing monthly payments, we'll compound monthly} \\ t=30 & \text{since we're making monthly payments for 30 years} \\ A=\$400,000 & \text{the starting loan amount} \end{array}$$

Using the formula,

$$d=\frac{400,000\left(\frac{0.05}{12}\right)}{\left[1-\left(1+\frac{0.05}{12}\right)^{-12\times 30}\right]}$$

$$d=\frac{400,000(0.0041667)}{\left[1-(1.0041667)^{-360}\right]}$$

$$d=\frac{1666.68}{0.776176}=\$2147.30$$

You will make payments of $2147.30 per month for 30 years.

You're paying a total of $\$773,028$ to the loan company: $\$2147.30$ per month for 360 months. You are paying a total of $\$773,028-\$400,000=\$373,028$ in interest over the life of the loan.

Try It 3.5.1

Janine bought $3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Answer

$$\begin{array}{ll} d= \text{ unknown} & \\ r=0.16 & 16\% \text{ annual rate} \\ n=12 & \text{since we're doing monthly payments, we'll compound monthly} \\ t=2 & 2 \text{ year to repay} \\ A=3,000 & \text{the starting loan amount } \$3{,}000 \text{ loan} \end{array}$$

$$d=\frac{3000\left(\frac{0.16}{12}\right)}{\left[1-\left(1+\frac{0.16}{12}\right)^{-12\times 2}\right]}$$

Solving for d gives $\$146.89$ as monthly payments.

In total, she will pay $\$3,525.36$ to the store, meaning she will pay $\$525.36$ in interest over the two years.

Amortization for a loan

Amortization

Amortization is the process of paying off a loan with equal payments over time. Part of each payment will go toward paying off the principal or loan amount, and part of the payment will go toward interest.

An amortization schedule is a table that lists all payments on a loan, splits them into the portion devoted to interest and the portion that is applied to repay principal, and calculates the outstanding balance on the loan after each payment is made.

Example 3.5.3

An amount of $500 is borrowed for 6 months at a rate of 12%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment contributing toward reducing the debt, and the outstanding balance.

Solution

Use the loan payment formula to verify that the monthly payment is $86.27.

The first month, the outstanding balance is $500, and therefore, the monthly interest on the outstanding balance is the simple interest calculated by $I=Pr$ where P = $500 and the monthly interest rate is $\frac{r}{n}=\frac{0.12}{12}=0.01$:

$$I=Pr= \text{(outstanding balance)(the monthly interest rate) = \$500(0.01) = \$5}$$

This means, the first month, out of the $86.27 payment, $5 goes toward the interest and the remaining $81.27 toward the balance leaving a new balance of $500 - $81.27 = $418.73.

Similarly, the second month, the outstanding balance is $418.73, and the monthly interest on the outstanding balance is ($418.73)(.01) = $4.19. Again, out of the $86.27 payment, $4.19 goes toward the interest and the remaining $82.08 toward the balance leaving a new balance of $418.73 - $82.08 = $336.65. The process continues in the table below.

Payment #	Payment	Interest	Principal Payment	Balance

Payment #	Payment	Interest	Principal Payment	Balance
1	$86.27	$5	$81.27	$418.73
2	$86.27	$4.19	$82.08	$336.65
3	$86.27	$3.37	$82.90	$253.75
4	$86.27	$2.54	$83.73	$170.02
5	$86.27	$1.70	$84.57	$85.45
6	$86.27	$0.85	$85.42	$0.03

Note that the last balance of 3 cents is due to error in rounding off.

The total interest paid on the loan is $17.65. Notice that the interest payments decrease over the course of the loan payments, while the principal payment increases over the course of the loan. You will pay more interest in the beginning of the loan, and pay almost no interest by the time that you're done paying off the loan (if you make payments for the entire life of the loan).

An amortization schedule is usually lengthy and tedious to calculate by hand. For example, an amortization schedule for a 30-year mortgage loan with monthly payments would have $12 \times 30 = 360$ rows of calculations in the amortization schedule table. A car loan with 5 years of monthly payments would have $12 \times 5 = 60$ rows of calculations in the amortization schedule table. However it would be straightforward to use a spreadsheet application on a computer to do these repetitive calculations by inputting and copying formulas for the calculations into the cells.

Remaining Loan Balance

One of the most common loan problems deals with finding the balance owed at a given time during the life of a loan. Suppose a person buys a house and amortizes the loan over 30 years, but decides to sell the house a few years later. At the time of the sale, he is obligated to pay off his lender, therefore, he needs to know the balance he owes. Since most long-term loans are paid off prematurely, we are often confronted with this problem.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments goes towards paying down the loan balance; a portion is going to go towards interest. For example, if your payments were $1,000 a month, after a year you will *not* have paid off $12,000 of the loan balance.

To determine the remaining loan balance, we can think "how much loan will these loan payments be able to pay off in the *remaining* time on the loan?" Use the loan amount formula, but in this case, t does not represent the entire term of the loan. Instead:

- t represents the time that still remains on the loan
- nt represents the total number of future payments

Example 3.5.4

A couple purchases a home with a $\$180,000$mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

Solution

First we will calculate their monthly payments.

We're looking for d, the loan payment amount.

$$\begin{array}{ll} r = 0.04 & 4\% \text{ annual rate} \\ n = 12 & \text{since they're paying monthly} \\ t = 30 & 30 \text{ years} \\ A = \$180,000 & \text{the starting loan amount} \end{array}$$

Using the loan payment formula to calculate d:

$$d = \frac{180,000\left(\frac{0.04}{12}\right)}{\left[1-\left(1+\frac{0.04}{12}\right)^{-12\times 30}\right]}$$

$$d = \frac{180,000(0.00333)}{[1-(1.00333)^{-360}]}$$

$$d = \frac{600}{0.698204}$$

$$= \$859.35$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

$d = \$859.35$	the monthly loan payment we calculated above
$r = 0.04$	4% annual rate
$n = 12$	since they're paying monthly
$t = 25$	since they'll be making monthly payments for 25 more years

$$A = \frac{859.35\left[1-\left(1+\frac{0.04}{12}\right)^{-12\times 25}\right]}{\left(\frac{0.04}{12}\right)}$$

$$= \frac{859.35\left[1-(1.0033333)^{-300}\right]}{0.0033333}$$

$$= \frac{859.35(1-0.3684954)}{0.0033333}$$

$$\approx \$162,807$$

The loan balance after 5 years, with 25 years remaining on the loan, will be $\$162,807$

Over that 5 years, the couple has paid off $\$180,000 - \$162,807 = \$17,193$ of the loan balance. They have paid a total of $\$859.35$ a month for 5 years (60 months), for a total of $\$51,561$, so $\$51,561 - \$17,193 = \$34,368$ of what they have paid so far has been interest.

Try It 3.5.2

If a mortgage at a 6% interest rate has payments of $1000 a month, how much will the loan balance be 10 years from the end of the loan?

Answer

To determine this, we are looking for the amount of a loan that can be paid off by $1000 monthly payments for 10 years. In other words, we're looking for A when

$d = \$1000$	the monthly loan payment
$r = 0.06$	6% annual rate
$n = 12$	since we're doing monthly payments, we'll compound monthly
$t = 10$	since we're making monthly payments for 10 more years

$$
\begin{aligned}
A &= \frac{1000\left[1-\left(1+\frac{0.06}{12}\right)^{-12\times 10}\right]}{\left(\frac{0.06}{12}\right)} \\
&= \frac{1000\left[1-(1.005)^{-120}\right]}{0.005} \\
&= \frac{1000(1-0.5496)}{0.005} \\
&= \$90,073.45
\end{aligned}
$$

The loan balance with 10 years left of the loan will be $\$90,073.45$