6.6: Odds and Expected Value

Learning Objectives

- Find the odds for and against an event
- Find the probability of an event given the odds of the event
- Find the expected value of an event

Odds

Probabilities are always numbers between 0 and 1. Many people are not comfortable working with such small values. Another way of describing the likelihood of an event happening is to use the ratio of how often it happens to how often it does not happen. The ratio is called the odds of the event happening. There are two types of odds, odds for and odds against. Casinos, race tracks and other types of gambling usually state the odds against an event happening.

Odds

If the probability of an event A is $P(A)$, then the **odds for event A**, $O(A)$, are given by:

$$O(A)=\frac{P(A)}{P(A')} \text{ or } O(A)=\frac{\text{number of ways for } A \text{ to occur}}{\text{number of ways for } A \text{ to not occur}}$$

Also, the **odds against event A**, $O(A')$, are given by:

$$O(A')=\frac{P(A')}{P(A)} \text{ or } O(A')=\frac{\text{number of ways for } A \text{ to not occur}}{\text{number of ways for } A \text{ to occur}}$$

Example 6.6.1

A single card is drawn from a well shuffled deck of 52 cards. Find the odds that the card is a red 8.

There are two red 8s in the deck.

$P(\text{red } 8)=\frac{2}{52}=\frac{1}{26}$.

$P(\text{not a red } 8)=\frac{50}{52}=\frac{25}{26}$.

$$O(\text{red } 8)=\frac{P(\text{red } 8)}{P(\text{not a red } 8)}=\frac{\frac{1}{26}}{\frac{25}{26}}=\frac{1}{\cancel{26}}\cdot\frac{\cancel{26}}{25}=\frac{1}{25}$$

The odds of drawing a red 8 are 1 to 25. This can also be written as 1:25.

Note: Do not write odds as a decimal or a percent.

Example 6.6.2

Many roulette wheels have slots numbered 0, 00, and 1 through 36. The slots numbered 0 and 00 are green. Half of the remaining slots are red and the other half are black. The game is played by spinning the wheel one direction and rolling a marble around the outer edge the other direction. Players bet on which slot the marble will fall into. What are the odds the marble will land in a red slot?

Solution

There are 38 slots in all. The slots 2, 4, 6, …, 36 are red so there are 18 red slots. The other 20 slots are not red.

$P(\text{red})=\frac{18}{38}=\frac{9}{19}$.

$$P(\text{not red}) = 1 - \frac{9}{19} = \frac{19}{19} - \frac{9}{19} = \frac{10}{19} \quad .$$

$$O(\text{red}) = \frac{P(\text{red})}{P(\text{not red})} = \frac{\frac{9}{19}}{\frac{10}{19}} = \frac{9}{\cancel{19}} \cdot \frac{\cancel{19}}{10} = \frac{9}{10}$$

The odds of the marble landing in a red slot are 9 to 10. This can also be written as 9:10.

Try It 6.6.1

Two fair dice are tossed and the sum is recorded. Find the odds against rolling a sum of 9.

Answer

The event A, roll a sum of 9 is: A = {(3, 6), (4, 5), (5, 4), (6, 3)}

There are 36 ways to roll 2 dice and 4 ways to roll a sum of 9. That means there are 32 ways to roll a sum that is not 9.

$$P(A) = \frac{4}{36} = \frac{1}{9} \quad .$$

$$P(A') = \frac{32}{36} = \frac{8}{9} \quad .$$

$$O(\text{against sum is } 9) = \frac{P(\text{sum is not } 9)}{P(\text{sum is } 9)} = \frac{P(A')}{P(A)} = \frac{\frac{8}{9}}{\frac{1}{9}} = \frac{8}{\cancel{9}} \cdot \frac{\cancel{9}}{1} = \frac{8}{1}$$

The odds against rolling a sum of 9 are 8 to 1 or 8:1.

We can also find the probability of an event happening based on the odds for the event. Saying that the odds of an event are 3 to 5 means that the event happens 3 times for every 5 times it does not happen. If we add up the possibilities of both we get a sum of 8. So the event happens about 3 out of every 8 times. We would say the probability is $\frac{3}{8}$.

Odds to Probability

If the odds favoring event A are a to b, then:

$$P(A) = \frac{a}{a+b} \text{ and } P(A') = \frac{b}{a+b}.$$

Example 6.6.3

A local little league baseball team is going to a tournament. The odds of the team winning the tournament are 3 to 7. Find the probability of the team winning the tournament.

Solution

$$P(\text{winning}) = \frac{3}{3+7} = \frac{3}{10} = 0.3$$

Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 6.6.4

In the casino game roulette [1], a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Solution

Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won" $\frac{-\$2}{\$38} \approx -\$0.053$. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

Outcome	Probability of outcome
$\$35$	$\frac{1}{38}$
$-\$1$	$\frac{37}{38}$

Notice that if we multiply each outcome by its corresponding probability we get $\$35 \cdot \frac{1}{38} = 0.9211$ and $-\$1 \cdot \frac{37}{38} = -0.9737$, and if we add these numbers we get $0.9211 + (-0.9737) \approx -0.053$, which is the expected value we computed above.

Expected Value

Expected value is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products. We will use E as the notation for the expected value.

Try It 6.6.2

You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

Answer

Outcome	Probability of outcome
$\$3995$	$\frac{1}{2000}$
$-\$5$	$\frac{1999}{2000}$

$$E = (\$3995) \cdot \frac{1}{2000} + (-\$5) \cdot \frac{1999}{2000} \approx -\$3.00$$

In the long run, you should expect to lose \$3.00 on average each time you buy a \$5 raffle ticket.

Example 6.6.5

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. If they match 5 numbers, then win $1,000. It costs $1 to buy a ticket. Find the expected value.

Solution

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$$P(\text{match all 6 numbers}) = \frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12,271,512} \approx 0.0000000815$$

$$P(\text{match 5 numbers}) = \frac{\left({}_6C_5\right)\left({}_{42}C_1\right)}{{}_{48}C_6} = \frac{252}{12,271,512} \approx 0.0000205$$

Our probabilities and outcome values are:

Outcome	Probability of outcome
$\$999,999$	$\frac{1}{12,271,512}$
$\$999$	$\frac{252}{12,271,512}$
$-\$1$	$1 - \frac{253}{12,271,512} = \frac{12,271,259}{12,271,512}$

The expected value, then is:

$$E = (\$999,999) \cdot \frac{1}{12,271,512} + (\$999) \cdot \frac{252}{12,271,512} + (-\$1) \cdot \frac{12,271,259}{12,271,512} \approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose $1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

Try It 6.6.3

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him $1. If any 2 dice match values, you get $2. What is the expected value of this game? Would you play?

Answer

Suppose you roll the first die. The probability the second will be different is $\frac{5}{6}$. The probability that the third roll is different than the previous 2 is $\frac{4}{6}$, so the probability that the 3 dice are different is $\frac{5}{6} \cdot \frac{4}{6} = \frac{20}{36}$. The probability that 2 dice will match is the complement, $1 - \frac{20}{36} = \frac{16}{36}$.

Our probabilities and outcome values are:

Outcome	Probability of outcome
$\$2$	$\frac{16}{36}$
$-\$1$	$\frac{20}{36}$

The expected value is: $E=(\$2)\cdot\frac{16}{36}+(-\$1)\cdot\frac{20}{36}=\frac{12}{36}\approx\$0.33$. Yes, it is in your advantage to play. On average, you'd win \$0.33 per play.

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example 6.6.6

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year [2]. An insurance company charges \$275 for a life-insurance policy that pays a \$100,000 death benefit. What is the expected value for the person buying the insurance?

Solution

The probabilities and outcomes are

Outcome	Probability of outcome
$\$100,000-\$275=\$99,725$	0.00242
$-\$275$	$1-0.00242=0.99758$

The expected value is $E=(\$99,725)(0.00242)+(-\$275)(0.99758)=-\$33$

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

[1] Photo CC-BY-SA http://www.flickr.com/photos/stoneflower/

[2] According to the estimator at www.numericalexample.com/inde...=article&id=91