6.2: Probability Rules with "Not," "Or" and "And"

Learning Objectives

- Find the probability of the complement of an event
- Determine if 2 events are mutually exclusive
- Find the probability of the intersection of events
- Use the Addition Rule to find the probability of "or" events

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a 6-sided die and first compute the probability of rolling a 6: the answer is $P(6)=\frac{1}{6}$. Now consider the probability that we do *not* roll a 6: there are 5 outcomes that are not a 6, so the answer is $P(\text{not a } 6)=\frac{5}{6}$. Notice that

$$P(6)+P(\text{not a } 6)=\frac{1}{6}+\frac{5}{6}=\frac{6}{6}=1$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event A that corresponds to m of these outcomes. Then the remaining $n-m$ outcomes correspond to A not happening, thus

$$P(\text{not } A)=\frac{n-m}{n}=\frac{n}{n}-\frac{m}{n}=1-\frac{m}{n}=1-P(A)$$

Complement of an Event

The **complement** of event A is the event "A does not happen." It is the set of outcomes in the sample space S that are not in event A.

- The notation A' is used for the complement of event A.
- The probability of the complement of event A is $P(A')=1-P(A)$.
- Notice also that $P(A)=1-P(A')$ since event A and its complement make up the entire sample space, $P(A)+P(A')=1$.

Note that the complement of an event in probability is essentially the same as the complement of a set, since an event is a subset of the sample space.

Example 6.2.1

If you pick a random card from a deck of playing cards, what is the probability it is not a heart?

Solution

There are 13 hearts in the deck, so

$$P(\text{heart})=\frac{13}{52}=\frac{1}{4}.$$

The probability of *not* drawing a heart is the complement:

$$P(\text{not heart})=1-P(\text{heart})=1-\frac{1}{4}=\frac{3}{4}.$$

Try It 6.2.1

A 6-sided die is rolled. Find the probability that the number rolled is not a multiple of 3.

Answer

Let A = rolling a multiple of 3 = $\{3,6\}$. The probability of A is $P(A)=\frac{2}{6}=\frac{1}{3}$. The probability of not rolling a multiple of 3 is $P(A')=1-P(A)=1-\frac{1}{3}=\frac{2}{3}$.

Many probabilities in real life involve more than one event. If we draw a single card from a deck we might want to know the probability that it is either red or a jack. If we look at a group of students, we might want to know the probability that a single student has brown hair and blue eyes. When we combine two events we make a single event called a **compound event**. To create a compound event, we can use the word "and" or the word "or" to combine events. It is very important in probability to pay attention to the words "and" and "or" if they appear in a problem. The word "and" restricts the field of possible outcomes to only those outcomes that simultaneously describe all events. The word "or" broadens the field of possible outcomes to those that describe one or more events.

Example 6.2.2

Suppose a teacher wants to know the probability that a student in her class of 30 students is taking either Art or English. She asks the class to raise their hands if they are taking Art and counts 13 hands. Then she asks the class to raise their hands if they are taking English and counts 21 hands. The teacher then calculates

$$P(\text{Art or English})=\frac{13+21}{30}=\frac{33}{30}$$

The teacher knows that this is wrong because probabilities must be between 0 and 1, inclusive. After thinking about it she remembers that nine students are taking both Art and English. These students raised their hands each time she counted, so the teacher counted them twice. When we calculate probabilities, we have to be careful to count each outcome only once.

Mutually Exclusive Events

An experiment consists of drawing one card from a well shuffled deck of 52 cards. Consider the events E = the card is red, F = the card is a 5, and G = the card is a spade. It is possible for a card to be both red and a 5 at the same time but it is not possible for a card to be both red and a spade at the same time. It would be easy to accidentally count a red 5 twice by mistake. It is not possible to count a red spade twice.

If two events do have events in common or they can happen at the same time, the overlap is called the **intersection** of the events. The intersection of events is denoted as A "and" B, and is the same as the intersection of two sets A and B.

Mutually Exclusive

Two events are **mutually exclusive** if they have no outcomes in common. They cannot occur at the same time.

The probability of 2 mutually exclusive events A and B is $P(A \text{ and } B)=0$.

Example 6.2.3

Two fair dice are tossed and different events are recorded. Let the events A, B and C be as follows:

- $A=\text{the sum is }5=\{(1,4),(2,3),(3,2),(4,1)\}$
- $B=\text{both numbers are even}=\{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$
- $C=\text{both numbers are less than }5=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),$ $(3,4),(4,1),(4,2),(4,3),(4,4)\}$

a. Are events A and B mutually exclusive?
b. Are events A and C mutually exclusive?
c. Are events B and C mutually exclusive?

Solution

a. Yes. A and B are mutually exclusive because they have no outcomes in common. It is not possible to add 2 even numbers to get a sum of 5.

b. No. A and C are not mutually exclusive because they have some outcomes in common. The pairs (1, 4), (2, 3), (3, 2) and (4, 1) all have sums of 5 and both numbers are less than 5.
c. No. B and C are not mutually exclusive because they have some outcomes in common. The pairs (2, 2), (2, 4), (4, 2) and (4, 4) all have 2 even numbers that are less than 5.

Addition Rule for "Or" Probabilities

The addition rule for probabilities is used when the events are connected by the word "or". Remember our teacher in Example $6.2.2$? She wanted to know the probability that her students were taking either Art or English. Her problem was that she counted some students twice. She needed to add the number of students taking Art to the number of students taking English and then subtract the number of students she counted twice. After dividing the result by the total number of students she will find the desired probability. The calculation is as follows:

$$\begin{aligned} P(\text{Art or English}) &= \frac{\#\text{ taking Art} + \#\text{ taking English - }\#\text{ taking both}}{\text{total number of students}} \\ &= \frac{13+21-9}{30} \\ &= \frac{25}{30} \approx 0.833 \end{aligned}$$

The probability that a student is taking Art or English is 0.833 or 83.3%.

When we calculate the probability for compound events connected by the word "or" we need to be careful not to count the same thing twice. If we want the probability of drawing a red card or a 5 we cannot count the red 5s twice. If we want the probability a person is blonde-haired or blue-eyed we cannot count the blue-eyed blondes twice. The addition rule for probabilities adds the number of blonde-haired people to the number of blue-eyed people then subtracts the number of people we counted twice.

Addition Rule for "Or" Probabilities

If A and B are any events, then the probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

If A and B are mutually exclusive events then $P(A \text{ and } B) = 0$, so then

$$P(A \text{ or } B) = P(A) + P(B).$$

Example $6.2.4$

A single card is drawn from a well shuffled deck of 52 cards. Find the probability that the card is a club or a face card.

Solution

There are 13 cards that are clubs, 12 face cards (J, Q, K in each suit) and 3 face cards that are clubs.

$$\begin{aligned} P(\text{club or face card}) &= P(\text{club}) + P(\text{face card}) - P(\text{club and face card}) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} = \frac{11}{26} \approx 0.423 \end{aligned}$$

The probability that the card is a club or a face card is approximately 0.423 or 42.3%.

A simple way to check this answer is to take the 52 card deck and count the number of physical cards that are either clubs or face cards. If you were to set aside all of the clubs and face cards in the deck, you would end up with the following:

{2 Clubs, 3 Clubs, 4 Clubs, 5 Clubs, 6 Clubs, 7 Clubs, 8 Clubs, 9 Clubs, 10 Clubs, J Clubs, Q Clubs, K Clubs, A Clubs, J Hearts, Q Hearts, K Hearts, J Spades, Q Spades, K Spades, J Diamonds, Q Diamonds, K Diamonds}

That is 22 cards out of the 52 card deck, which gives us a probably of:

$$\frac{22}{52}=\frac{11}{26}\approx 0.423$$

This confirms our earlier answer using the formal Addition Rule.

Try It 6.2.2

An experiment consists of tossing a coin then rolling a die. Find the probability that the coin lands on heads or the number is 5 is rolled.

Answer

Let H represent heads and T represent tails. The sample space for this experiment is $S=\{H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6\}$

There are 6 ways the coin can land on heads, $\{H1,H2,H3,H4,H5,H6\}$There are 2 ways the die can land on 5, $\{H5,T5\}$. There is 1 way for the coin to land on heads and the die to land on 5, $\{H5\}$.

$$\begin{aligned} P(\text{heads or } 5) &= P(\text{heads}) + P(5) - P(\text{both heads and } 5) \\ &= \frac{6}{12}+\frac{2}{12}-\frac{1}{12} \\ &= \frac{7}{12} = \approx 0.583 \end{aligned}$$

The probability that the coin lands on heads or the number is 5 is approximately 0.583 or 58.3%.

Example 6.2.5

250 people who recently purchased a car were questioned and the results are summarized in the following table.

Satisfaction of Car Buyers

	Satisfied	Not Satisfied	Total
New Car	92	28	120
Used Car	83	47	130
Total	175	75	250

Find the probability that a person bought a new car or was not satisfied.

Solution

$$\begin{aligned} P(\text{new car or not satisfied}) &= P(\text{new car}) + P(\text{not satisfied}) - P(\text{new car and not satisfied}) \\ &= \frac{120}{250}+\frac{75}{250}-\frac{28}{250}=\frac{167}{250}\approx 0.668 \end{aligned}$$

The probability that a person bought a new car or was not satisfied is approximately 0.668 or 66.8%.

Example 6.2.6

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

Solution

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{ King or Queen}) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen})=0$. Using the addition rule, we could have said:

$$P(\text{ King or Queen})=P(\text{ King })+P(\text{ Queen})-P(\text{ King and Queen})=\frac{4}{52}+\frac{4}{52}-0=\frac{8}{52}=\frac{2}{13}$$

In the last example, the events were **mutually exclusive**, so $P(A \text{ or } B)=P(A)+P(B)$.

Try It 6.2.3

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Answer

- Half the cards are red, so $P(\text{red})=\frac{26}{52}$
- There are 4 kings, so $P(\text{King})=\frac{4}{52}$
- There are 2 red kings, so $P(\text{red and King})=\frac{2}{52}$

We can then calculate

$$P(\text{red or King})=P(\text{red})+P(\text{King})-P(\text{red and King})=\frac{26}{52}+\frac{4}{52}-\frac{2}{52}=\frac{28}{52}=\frac{7}{13}$$

Example 6.2.7

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a. Has a red car *and* got a speeding ticket
b. Has a red car *or* got a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Solution

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $P(\text{red car and ticket})=\frac{15}{665}\approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665}\approx 0.2932$.

We also could have found this probability by:

$$\begin{aligned}\mathrm{P}(\text{had a red car})+\mathrm{P}(\text{got a speeding ticket})-\mathrm{P}(\text{had a red car and got a speeding ticket}) &=\frac{150}{665}+\frac{60}{665}-\frac{15}{665}\\ &=\frac{195}{665}\end{aligned}$$