3.3: Compound Interest

Compound Interest

Learning Objectives

- Calculate compound interest, accumulated amount and present value on an account
- Find the effective rate of an account
- Use the Rule of 70s to estimate doubling time

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit $1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3\%}{12} = 0.25\%$ per month.

In the first month,

$P = \$1000$

$r = 0.25\% = 0.0025$

$I = \$1000(0.0025) = \$2.50$

$A = \$1000 + \$2.50 = \$1002.50$

In the first month, we will earn $2.50 in interest, raising our account balance to $1002.50.

In the second month,

$P = \$1002.50$

$I = \$1002.50(0.0025) = \$2.51$ (rounded)

$A = \$1002.50 + \$2.51 = \$1005.01$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original $1000 we deposited, but we also earned interest on the $2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:

Month	Starting balance	Interest earned	Ending Balance
1	$1000.00	$2.50	$1002.50
2	$1002.50	$2.51	$1005.01
3	$1005.01	$2.51	$1007.52
4	$1007.52	$2.52	$1010.04
5	$1010.04	$2.53	$1012.57
6	$1012.57	$2.53	$1015.10
7	$1015.10	$2.54	$1017.64
8	$1017.64	$2.54	$1020.18
9	$1020.18	$2.55	$1022.73
10	$1022.73	$2.56	$1025.29
11	$1025.29	$2.56	$1027.85
12	$1027.85	$2.57	$1030.42

To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

$P_0 = \$1000$

$P_m = (1 + 0.0025)P_{m-1}$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$P_0 = \$1000$

$P_1 = 1.0025P_0 = 1.0025(1000)$

$P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = 1.0025^2(1000)$

$P_3 = 1.0025P_2 = 1.0025\left(1.0025^2(1000)\right) = 1.0025^3(1000)$

$P_4 = 1.0025P_3 = 1.0025\left(1.0025^3(1000)\right) = 1.0025^4(1000)$

Observing a pattern, we could conclude

$P_m = (1.0025)^m(\$1000)$

Notice that the \$1000 in the equation was P_0, the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

$P_m = P_0\left(1 + \frac{r}{n}\right)^m$

In this formula:

m is the number of compounding periods (months in our example)

r is the annual interest rate

n is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If t is the number of years, then $m = nt$. Making this change gives us the standard formula for compound interest.

Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

- A is the balance in the account (accumulated amount) after t years (also called the future value)
- P is the starting balance of the account (also called the initial deposit, principal or present value)
- r is the annual interest rate in decimal form
- n is the number of compounding periods in one year

If the compounding is done annually (once a year), $n = 1$.

If the compounding is done quarterly, $n = 4$.

If the compounding is done monthly, $n = 12$.

If the compounding is done daily, $n = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

✓ Example 3.3.1

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit $3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

Solution

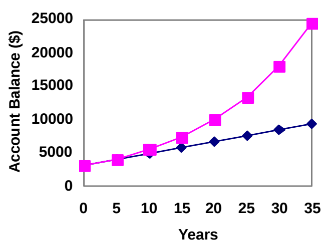
In this example,

$P = \$3000$ the initial deposit
$r = 0.06$ 6% annual rate
$n = 12$ 12 months in 1 year
$t = 20$ since we're looking for how much we'll have after 20 years

So $A = 3000\left(1+\frac{0.06}{12}\right)^{12\times 20} = \$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest.

Years	Simple Interest ($15 per month)	6% compounded monthly = 0.5% each month
5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13,394.91
30	$8400	$18,067.73
35	$9300	$24,370.65



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Evaluating exponents on the calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5\cdot 5\cdot 5 = 125$. But when we need to calculate something like 1.005^{240}, it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

$[\wedge]$, $[y^x]$, or $[x^y]$

To evaluate 1.005^{240} we'd type 1.005 $[\wedge]$ 240, or 1.005 $[y^x]$ 240. Try it out - you should get something around 3.3102044758.

✓ Example 3.3.2

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Solution

We're looking for P, the present value.

$r = 0.04$	4%
$n = 4$	4 quarters in 1 year
$t = 18$	Since we know the balance in 18 years
$A = \$40,000$	The accumulated amount we will have in 18 years

In this case, we're going to have to set up the equation, and solve for P.

$$40,000 = P\left(1 + \frac{0.04}{4}\right)^{4\times18}$$

$$40,000 = P(2.0471)$$

$$P = \frac{40,000}{2.0471} = \$19,539.84$$

So you would need to deposit \$19,539.84 now to have \$40,000 in 18 years.

Note: this value is also known as the *present value* of the investment--the amount that must be invested now to have a specific amount later with interest.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better.

Example 3.3.3

To see why not over-rounding is so important, suppose you were investing \$1000 at 5% interest compounded monthly for 30 years.

Solution

$P = \$1000$	the initial deposit
$r = 0.05$	5%
$n = 12$	12 months in 1 year
$t = 30$	since we're looking for the amount after 30 years

If we first compute $\frac{r}{n}$, we find $\frac{0.05}{12} = 0.00416666666667$

Here is the effect of rounding this to different values:

$\frac{r}{n}$ rounded to:	Gives A to be:	Error
0.004	\$4208.59	\$259.15
0.0042	\$4521.45	\$53.71
0.00417	\$4473.09	\$5.35
0.004167	\$4468.28	\$0.54
0.0041667	\$4467.80	\$0.06
no rounding	\$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough - \$5 off of \$4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$A = 1000\left(1+\frac{0.05}{12}\right)^{12\times 30}$

We can quickly calculate $12\times 30 = 360$, giving $A = 1000\left(1+\frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.

Type this	Calculator shows
$0.05[\div]12[=]$	0.00416666666667
$[+]1[=]$	1.00416666666667
$[y^x]360[=]$	4.46774431400613
$[\times]1000[=]$	4467.74431400613

Using your calculator continued

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 $[\times]$ (1 $[+]$ 0.05 $[\div]$ 12) $[y^x]$ 360 $[=]$

Effective Interest Rate

Banks are required to state their interest rate in terms of an **"effective yield"** " or **"effective interest rate"**, for comparison purposes. The effective rate is also called the Annual Percentage Yield (APY) or Annual Percentage Rate (APR).

Effective rate

The **effective rate** is the annual simple interest rate that would be equivalent to the annual compound interest accrued for the stated rate and number of compounding periods.

Formula:

$$r_{EFF} = \left(1+\frac{r}{n}\right)^n - 1$$

- r_{EFF} is the effective rate, in decimal form
- r is the stated annual interest rate, in decimal form
- n is the number of compounding periods in one year

Be sure to change the answer back to a percent from decimal form.

The formula is derived from the compound interest formula by depositing P = $1 in an account and calculating how much interest it will accrue in a year (t = 1). The original $1 deposit must be subtracted out to find the total interest. To examine several investments to see which has the best rate, we find and compare the effective rate for each investment.

Example 3.3.4

If Bank A pays 7.2% interest compounded monthly, what is the effective interest rate?
If Bank B pays 7.25% interest compounded semiannually, what is the effective interest rate?
Which bank pays more interest?

Solution

Bank A: Using the formula with n = 12, we will get:

$$\mathbf{r}_{\text{EFF}} = \left(1 + \frac{0.072}{12}\right)^{12} - 1 = 1.0744 - 1 = 0.0744$$

We earned interest of $.0744 on an investment of $1. The effective interest rate is 7.44%, often referred to as the APY or APR.

Bank B: The effective rate is calculated with n = 2:

$$\mathbf{r}_{\text{EFF}} = \left(1 + \frac{0.072}{2}\right)^{2} - 1 = .0738$$

The effective interest rate is 7.38%.

Bank A pays slightly higher interest, with an effective rate of 7.44%, compared to Bank B with effective rate 7.38%.

Rule of 70

The rule of 70 is a useful tool for estimating the time needed for an investment to double in value. It is an approximation and is not exact, but is a quick-and-easy way to estimate how long it will take for an investment to double.

Rule of 70

Rule of 70: The approximate number of years required to double an investment with compound interest

Formula:

$$t \approx \frac{70}{r}$$

- t is the approximate number of years to double the investment
- r is the annual interest rate as a percent (not in decimal form)

Note this formula does not take into account the number of compounding periods per year. This is because it is based on the idea of interest that is continuously compounded, a concept that makes calculations easier. In the real world, however, there is no bank or financial institution that offers interest that is continuously compounded.

Here are some approximate doubling times for various interest rates using the Rule of 70.

Approximate Doubling Time in Years By Interest Rate

Annual interest rate	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
Number of years to double money	70	35	23.3	17.5	14	11.7	10	8.75	7.8	7

The rule of 70 can be useful to help quickly estimate many “doubling time” problems mentally, which can be useful in compound interest applications as well as other applications involving exponential growth.

Example $3.3.5$

As of 2015, the world population’s annual growth rate was approximately 1.14%. Based on that rate, find the approximate doubling time.

Solution

According to the rule of 70,

$$t \approx \frac{70}{1.14} \approx 61.4 \text{ years}$$

If the world population were to continue to grow at the annual growth rate of 1.14% , it would take approximately 61.4 years for the population to double.