6.3: Independent Events and Conditional Probabilities

Learning Objectives

- Understand the difference between independent and dependent events
- Use the Multiplication Rule to find the probability of independent and dependent events
- Find conditional probabilities

Independent events

Sometimes we need to calculate probabilities for compound events that are connected by the word "and." Tossing a coin multiple times or rolling dice are independent events. Each time you toss a fair coin the probability of getting heads is $\frac{1}{2}$. It does not matter what happened the last time you tossed the coin. It's similar for dice. If you rolled double sixes last time that does not change the probability that you will roll double sixes this time. Drawing two cards without replacement is not an independent event. When you draw the first card and set it aside, the probability for the second card is now out of 51 cards not 52 cards.

Independent Events

Events A and B are **independent events** if the occurrence of A has no effect on the probability of the occurrence of B. In other words, the probability of event B occurring is the same whether or not event A occurs.

If 2 events are not independent, then they are **dependent** events.

Example 6.3.1

Are these events independent?

a. A fair coin is tossed two times. The two events are (1) first toss is heads and (2) second toss is heads.
b. The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).
c. You draw a card from a deck, then draw a second card without replacing the first.

Solution

a. The probability that heads comes up on the second toss is $\frac{1}{2}$ regardless of whether or not heads came up on the first toss, so these events are independent.
b. These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.
c. The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

Multiplication Rule for Independent Events

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring.

Example 6.3.2

Suppose we flipped a coin and rolled a die. What is the probability of getting heads on the coin and a 6 on the die?

Solution

Let A = getting heads and B = rolling 6. We know that $P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{6}$. A and B are independent since the probability of rolling a 6 does not depend on the outcome of the coin toss. Using the multiplication rule, we get:

$$P(A \text{ and } B)=P(A)\cdot P(B)=\frac{1}{2}\cdot\frac{1}{6}=\frac{1}{12}.$$

To confirm this answer, we could list all possible outcomes: $S=\{H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6\}$ Notice there are $2\cdot 6=12$ total outcomes. Out of these, only 1 is the desired outcome $\{H6\}$, so the probability is $\frac{1}{12}$.

Example 6.3.3

A bag contains 5 red and 4 white marbles. A marble is drawn from the bag, its color recorded and the marble is returned to the bag. A second marble is then drawn. What is the probability that the first marble is red and the second marble is white?

Solution

Since the first marble is put back in the bag before the second marble is drawn these are independent events.

$$\begin{aligned} P(\text{1st red and 2nd white}) &= P(\text{1st red})\cdot P(\text{2nd white}) \\ &= \frac{5}{9}\cdot\frac{4}{9}=\frac{20}{81} \end{aligned}$$

The probability that the first marble is red and the second marble is white is $\frac{20}{81}$.

Try It 6.3.1

A card is drawn from a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is drawn and noted. What is the probability that both cards are Aces?

Answer

Since the second draw is made after replacing the first card, these events are independent. The probability of an Ace on each draw is $P(\text{Ace})=\frac{4}{52}=\frac{1}{13}$, so the probability of an Ace on both draws is $P(\text{Ace and Ace})=\frac{1}{13}\cdot\frac{1}{13}=\frac{1}{169}$.

Conditional Probability

What do you think the probability is that a man is over six feet tall? If you knew that both his parents were tall would you change your estimate of the probability? A conditional probability is a probability that is based on some prior knowledge.

Conditional Probability

A **conditional probability** is the probability that event B will occur if event A has already occurred. This is denoted by $P(B|A)$, which is read "the probability of B given A."

Example 6.3.4

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a. has a speeding ticket *given* they have a red car
b. has a red car *given* they have a speeding ticket

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Solution

a. Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so P(ticket | red car) = $\frac{15}{150}=\frac{1}{10}=0.1$

b. Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so P(red car | ticket) = $\frac{15}{60}=\frac{1}{4}=0.25$.

Notice from the last example that $P(B|A)$ is **not** equal to $P(A|B)$.

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

Multiplication Rule for Dependent Events

For events dependent events A and B, $P(A \text{ and } B)=P(A)\cdot P(B|A)$.

Example 6.3.5

What is the probability that two cards drawn at random from a deck of playing cards will both be Aces?

Solution

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52}\cdot\frac{4}{52}=\frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an Ace, then the probability that the second card is also an Ace would be lower because there would only be 3 Aces left in the deck out of 51 total cards left. This means that the conditional probability of drawing an Ace after one Ace has already been drawn is $P(\text{Ace on the second draw} \mid \text{Ace on the first draw})=\frac{3}{51}=\frac{1}{17}$.

Thus, the probability of both cards being aces is $P(\text{Ace and Ace})=\frac{4}{52}\cdot\frac{3}{51}=\frac{12}{2652}=\frac{1}{221}$.

Try It 6.3.2

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

Answer

$$P(\text{both white})=\frac{6}{10}\cdot\frac{5}{9}=\frac{30}{90}=\frac{1}{3}$$

Conditional Probability Rule

For events A and B, $P(B|A)=\frac{P(A \text{ and } B)}{P(A)}$

This formula basically reduces the sample space to A instead of all events in S.

Example 6.3.6

One card is drawn from a deck of cards. What is the probability that it is a face card, if we know the card is black?

Solution

We need to find the conditional probability of drawing a face card, given it is black, $P(\text{face card} \mid \text{black})$. The formula is:

$$P(\text{face card} \mid \text{black}) = \frac{P(\text{face card and black})}{P(\text{black})}$$

There are 6 cards that are black face cards (the Jack, Queen and King of spades and clubs). So the probability of face card and black is $P(\text{face card and black}) = \frac{6}{52}$. There are 26 black cards, so the probability of drawing a black card is $P(\text{black}) = \frac{26}{52}$. Plugging these probabilities in the formula we get:

$$P(\text{face card} \mid \text{black}) = \frac{\frac{6}{52}}{\frac{26}{52}} = \frac{6}{\cancel{52}} \cdot \frac{\cancel{52}}{26} = \frac{6}{26} = \frac{3}{13}$$

This answer should make sense: 3 out of every 13 black cards are face cards.

Try It 6.3.3

One card is drawn from a deck of cards. What is the probability that it is a black card, if we know the card is a face card?

Answer

We need to find the conditional probability of drawing a black card, given it is a face card, $P(\text{black} \mid \text{face})$. The formula is:

$$P(\text{black} \mid \text{face}) = \frac{P(\text{black and face})}{P(\text{face})}$$

There are 6 cards that are black face cards (the Jack, Queen and King of spades and clubs). So the probability of face card and black is $P(\text{black and face}) = \frac{6}{52}$. There are 12 face cards, so the probability of drawing a face card is $P(\text{face}) = \frac{12}{52}$. Plugging these probabilities in the formula we get:

$$P(\text{black} \mid \text{face}) = \frac{\frac{6}{52}}{\frac{12}{52}} = \frac{6}{\cancel{52}} \cdot \frac{\cancel{52}}{12} = \frac{6}{12} = \frac{1}{2}$$

This answer should make sense: half of the face cards are black.

Example 6.3.7

Two hundred fifty people who recently purchased a car were questioned and the results are summarized in the following table.

Satisfaction of Car Buyers

	Satisfied	Not Satisfied	Total
New Car	92	28	120
Used Car	83	47	130

	Satisfied	Not Satisfied	Total
Total	175	75	250

What is the probability that a person is satisfied if it is known that the person bought a used car?

Solution

This is a conditional probability because we already know that the person bought a used car.

$$P(\text{satisfied} \mid \text{used car}) = \frac{P(\text{satisfied and used})}{P(\text{used})}$$

$$= \frac{\frac{83}{250}}{\frac{130}{250}} = \frac{83}{\cancel{250}} \cdot \frac{\cancel{250}}{130} = \frac{83}{130} \approx 0.638$$

The probability that a person is satisfied if it is known that the person bought a used car is approximately 0.638 or 63.8%

Note: it is faster to do a table problem like this using the method from Example $6.3.4$. There are 83 people who bought a used car and are satisfied out of the 130 people who bought a used car.

Try It 6.3.4

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

a. $P(\text{not pregnant} \mid \text{positive test result})$
b. $P(\text{positive test result} \mid \text{not pregnant})$

	Positive test	Negative test	Total
Pregnant	70	4	74
Not Pregnant	5	14	19
Total	75	18	93

Answer

a. Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant.

   $P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} \approx 0.067.$

b. Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test.

   $P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263.$

   This result is what is usually called a false positive: a positive result when the woman is not actually pregnant.

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