10.6.3: Proportions and Rates

If you wanted to power the city of Seattle using wind power, how many windmills would you need to install? Questions like these can be answered using rates and proportions.

Rates

A rate is the ratio (fraction) of two quantities.

A **unit rate** is a rate with a denominator of one.

Example 12

Your car can drive 300 miles on a tank of 15 gallons. Express this as a rate.

Solution

Expressed as a rate, $\frac{300 \text{ miles}}{15 \text{ gallons}}$. We can divide to find a unit rate: $\frac{20 \text{ miles}}{1 \text{gallon}}$, which we could also write as $20 \frac{\text{miles}}{\text{gallon}}$, or just 20 miles per gallon.

Proportion Equation

A proportion equation is an equation showing the equivalence of two rates or ratios.

Example 13

Solve the proportion $\frac{5}{3} = \frac{x}{6}$ for the unknown value x.

Solution

This proportion is asking us to find a fraction with denominator 6 that is equivalent to the fraction $\frac{5}{3}$. We can solve this by multiplying both sides of the equation by 6, giving $x = \frac{5}{3} \cdot 6 = 10$.

Example 14

A map scale indicates that $\frac{1}{2}$ inch on the map corresponds with 3 real miles. How many miles apart are two cities that are $2\frac{1}{4}$ inches apart on the map?

Solution

We can set up a proportion by setting equal two $\frac{\text{map inches}}{\text{real miles}}$ rates, and introducing a variable, x, to represent the unknown quantity – the mile distance between the cities.

$$\frac{\frac{1}{2} \text{ map inch}}{3 \text{ miles}} = \frac{2\frac{1}{4} \text{ map inches}}{x \text{ miles}} \quad \text{Multiply both sides by } x \text{ and rewriting the mixed number}$$

$$\frac{\frac{1}{2}}{3} \cdot x = \frac{9}{4} \quad \text{Multiply both sides by 3}$$

$$\frac{1}{2}x = \frac{27}{4} \quad \text{Multiply both sides by 2 (or divide by } \tfrac{1}{2}\text{)}$$

$$x = \frac{27}{2} = 13\frac{1}{2} \text{ miles}$$

Many proportion problems can also be solved using **dimensional analysis**, the process of multiplying a quantity by rates to change the units.

Example 15

Your car can drive 300 miles on a tank of 15 gallons. How far can it drive on 40 gallons?

Solution

We could certainly answer this question using a proportion: $\frac{300 \text{ miles}}{15 \text{ gallons}} = \frac{x \text{ miles}}{40 \text{ gallons}}$.

However, we earlier found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit "cancels" and we're left with a number of miles:

$$40\text{gallons} \cdot \frac{20 \text{ miles}}{\text{gallon}} = \frac{40 \text{ gallons}}{1} \cdot \frac{20 \text{ miles}}{\text{gallon}} = 800 \text{ miles}$$

Notice if instead we were asked "how many gallons are needed to drive 50 miles?" we could answer this question by inverting the 20 mile per gallon rate so that the *miles* unit cancels and we're left with gallons:

$$50\text{miles} \cdot \frac{\text{lgallon}}{20 \text{ miles}} = \frac{50 \text{ miles}}{1} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ gallons}}{20} = 2.5 \text{ gallons}$$

Dimensional analysis can also be used to do unit conversions. Here are some unit conversions for reference.

Unit Conversions

Length

$$\begin{array}{ll} 1 \text{ foot (ft)} = 12 \text{ inches (in)} & 1 \text{ yard (yd)} = 3 \text{ feet (ft)} \\ 1 \text{ mile } = 5,280 \text{ feet} & \\ 1000 \text{ millimeters } mm = 1 \text{ meter (m)} & 100 \text{ centimeters (cm)} = 1 \text{ meter} \\ 1000 \text{ meters (m)} = 1 \text{ kilometer (km)} & 2.54 \text{ centimeters (cm) } = 1 \text{ inch} \end{array}$$

Weight and Mass

$$\begin{array}{ll} 1 \text{ pound (lb) } = 16 \text{ ounces (oz)} & 1 \text{ ton } = 2000 \text{ pounds} \\ 1000 \text{ milligrams (mg) } = 1 \text{ gram (g)} & 1000 \text{ grams } = 1 \text{ kilogram (kg)} \\ 1 \text{ kilogram } = 2.2 \text{ pounds (on earth)} & \end{array}$$

Capacity

$$\begin{array}{ll} 1 \text{ cup } = 8 \text{ fluid ounces (fl oz)}^* & 1 \text{ pint } = 2 \text{ cups} \\ 1 \text{ quart } = 2 \text{ pints } = 4 \text{ cups} & 1 \text{ gallon } = 4 \text{ quarts } = 16 \text{ cups} \\ 1000 \text{ milliliters (ml) } = 1 \text{ liter (L)} & \end{array}$$

*Fluid ounces are a capacity measurement for liquids. 1 fluid ounce ≈ 1 ounce (weight) for water only.

Example 16

A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

Solution

To answer this question, we need to convert 20 seconds into feet. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don't, we will need to do additional unit conversions. We will need to know that 5280 ft = 1 mile. We might start by converting the 20 seconds into hours:

$$\begin{array}{ll} 20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{180} \text{ hour} & \text{Now we can multiply by the 15 miles/hr} \\ \frac{1}{180} \text{ hour} \cdot \frac{15 \text{ miles}}{\text{lhour}} = \frac{1}{12} \text{ mile} & \text{Now we can convert to feet} \\ \frac{1}{12} \text{ mile} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet} & \end{array}$$

We could have also done this entire calculation in one long set of products:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{15 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

Try it Now 4

A 1000 foot spool of bare 12-gauge copper wire weighs 19.8 pounds. How much will 18 inches of the wire weigh, in ounces?

Answer

$$18 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{19.8 \text{ pounds}}{1000 \text{ feet}} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} \approx 0.475 \text{ ounces}$$

Notice that with the miles per gallon example, if we double the miles driven, we double the gas used. Likewise, with the map distance example, if the map distance doubles, the real-life distance doubles. This is a key feature of proportional relationships, and one we must confirm before assuming two things are related proportionally.

Example 17

Suppose you're tiling the floor of a 10 ft by 10 ft room, and find that 100 tiles will be needed. How many tiles will be needed to tile the floor of a 20 ft by 20 ft room?

Solution

In this case, while the width the room has doubled, the area has quadrupled. Since the number of tiles needed corresponds with the area of the floor, not the width, 400 tiles will be needed. We could find this using a proportion based on the areas of the rooms:

$$\frac{100 \text{ tiles}}{100 \text{ft}^2} = \frac{n \text{ tiles}}{400 \text{ft}^2}$$

Other quantities just don't scale proportionally at all.

Example 18

Suppose a small company spends $1000 on an advertising campaign, and gains 100 new customers from it. How many new customers should they expect if they spend $10,000?

Solution

While it is tempting to say that they will gain 1000 new customers, it is likely that additional advertising will be less effective than the initial advertising. For example, if the company is a hot tub store, there are likely only a fixed number of people interested in buying a hot tub, so there might not even be 1000 people in the town who would be potential customers.

Sometimes when working with rates, proportions, and percents, the process can be made more challenging by the magnitude of the numbers involved. Sometimes, large numbers are just difficult to comprehend.

Example 19

Compare the 2010 U.S. military budget of $683.7 billion to other quantities.

Here we have a very large number, about $683,700,000,000 written out. Of course, imagining a billion dollars is very difficult, so it can help to compare it to other quantities.

If that amount of money was used to pay the salaries of the 1.4 million Walmart employees in the U.S., each would earn over $488,000.

There are about 300 million people in the U.S. The military budget is about $2,200 per person.

If you were to put $683.7 billion in $100 bills, and count out 1 per second, it would take 216 years to finish counting it.

Example 20

Compare the electricity consumption per capita in China to the rate in Japan.

To address this question, we will first need data. From the CIA[1] website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank[2], we can find the population of China is 1,344,130,000, or 1.344 billion, and the population of Japan is 127,817,277, or 127.8 million.

Solution

Computing the consumption per capita for each country:

China: $\frac{4{,}693{,}000{,}000{,}000 \mathrm{KWH}}{1{,}344{,}130{,}000 \text{ people}} \approx 3491.5$ KWH per person

Japan: $\frac{859{,}700{,}000{,}000 \mathrm{KWH}}{127{,}817{,}277 \text{ people}} \approx 6726$ KWH per person

While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

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[1] www.cia.gov/library/publicat.../2042rank.html

[2] http://data.worldbank.org/indicator/SP.POP.TOT
