3.4: Annuities

> **Learning Objectives**
>
> - Understand the difference between an annuity and a compound interest account
> - Calculate the accumulated amount of a savings or payout annuity
> - Calcluate the deposit or withdrawal amount for a savings or payout annuity

In the last two sections of this chapter, we examined problems where an amount of money was deposited as a lump sum (one-time deposit) in an account and was left there for the entire time period. Now we will do problems where timely payments are made in an account. When a sequence of payments of some fixed amount are made in an account at equal intervals of time, we call that an **annuity**.

Savings Annuity

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

> **Annuity**
>
> A **savings annuity** is a savings account in which equal deposits are made at equal intervals of time, and earns compound interest.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m=\left(1+\frac{r}{n}\right)P_{m-1}$$

For a savings annuity, we simply need to add a deposit, d, to the account with each compounding period:

$$P_m=\left(1+\frac{r}{n}\right)P_{m-1}+d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit \$100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$r=0.06$ (6%)

$n=12$ (12 compounds/deposits per year)

$d=\$100$ (our deposit per month)

Writing out the recursive equation gives

$$P_m=\left(1+\frac{0.06}{12}\right)P_{m-1}+100=(1.005)P_{m-1}+100$$

Assuming we start with an empty account, we can begin using this relationship:

$P_0=0$

$P_1=(1.005)P_0+100=100$

$P_2=(1.005)P_1+100=(1.005)(100)+100=100(1.005)+100$

$P_3=(1.005)P_2+100=(1.005)(100(1.005)+100)+100=100(1.005)^2+100(1.005)+100$

Continuing this pattern, after m deposits, we'd have saved:

$P_m=100(1.005)^{m-1}+100(1.005)^{m-2}+\cdots+100(1.005)+100$

In other words, after m months, the first deposit will have earned compound interest for $m-1$ months. The second deposit will have earned interest for $m-2$ months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005P_m = 1.005\left(100(1.005)^{m-1} + 100(1.005)^{m-2} + \cdots + 100(1.005) + 100\right)$$

Distributing on the right side of the equation gives

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \cdots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$\begin{array}{llll} 1.005P_m & = 100(1.005)^m + & 100(1.005)^{m-1} + \cdots + & 100(1.005) \\ \quad P_m & = & 100(1.005)^{m-1} + \cdots + & 100(1.005) + 100 \end{array}$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005P_m - P_m = 100(1.005)^m - 100$$

Solving for P_m

$$0.005P_m = 100\left((1.005)^m - 1\right)$$

$$P_m = \frac{100\left((1.005)^m - 1\right)}{0.005}$$

Replacing m months with $12t$, where t is measured in years, gives

$$P_t = \frac{100\left((1.005)^{12\mathrm{t}} - 1\right)}{0.005}$$

Recall 0.005 was $\frac{r}{n}$ and 100 was the deposit d. 12 was n, the number of deposits each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula

$$A = \frac{d\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}{\left(\frac{r}{n}\right)}$$

- A is the balance in the account (accumulated amount) after t years
- d is the regular deposit (the amount you deposit each year, each month, etc.)
- r is the annual interest rate in decimal form
- n is the number of compounding periods in one year

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, $n = 12$.

If you make your deposits every year, use yearly compounding, $n = 1$.

If you make your deposits every quarter, use quarterly compounding, $n = 4$.

When do you use this

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Example 3.4.1

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

Solution

In this example,

$$\begin{array}{ll} d=\$100 & \text{the monthly deposit} \\ r=0.06 & 6\% \text{ annual rate} \\ n=12 & \text{since we're doing monthly deposits, we'll compound monthly} \\ t=20 & \text{we want the amount after 20 years} \end{array}$$

Putting this into the equation:

$$A=\frac{100\left[\left(1+\frac{0.06}{12}\right)^{12\times 20}-1\right]}{\left(\frac{0.06}{12}\right)}$$

$$A=\frac{100\left((1.005)^{240}-1\right)}{(0.005)}$$

$$A=\frac{100(3.310-1)}{(0.005)}$$

$$A=\frac{100(2.310)}{(0.005)}=\$46,200$$

The account will grow to $46,200 after 20 years.

Notice that you deposited into the account a total of $24,000 ($100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $\$46,200-\$24,000=\$22,200$

Try It 3.4.1

A more conservative investment account pays 3% interest. If you deposit $5 a day into this account, how much will you have after 10 years? How much is from interest?

Answer

$$\begin{array}{ll} d=\$5 & \text{the daily deposit} \\ r=0.03 & 3\% \text{ annual rate} \\ n=365 & \text{since we're doing daily deposits, we'll compound daily} \\ t=10 & \text{we want the amount after 10 years} \end{array}$$

$$A=\frac{5\left[\left(1+\frac{0.03}{365}\right)^{365\times 10}-1\right]}{\frac{0.03}{365}}=\$21,282.07$$

We would have deposited a total of $\$5\cdot 365\cdot 10=\$18,250$ so $3,032.07 is from interest.

We can solve the annuity formula for d in order to calculate the amount of money that must be deposited regularly in order to reach a savings goal in a specific amount of time.

Deposit Formula

$$d=\frac{A\left(\frac{r}{n}\right)}{\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}$$

- d is the regular deposit (the amount you deposit each year, each month, etc.)
- A is the desired balance in the account (accumulated amount) after t years
- r is the annual interest rate in decimal form
- n is the number of compounding periods in one year

Example 3.4.2

You want to have $200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Solution

In this example, we're looking for d.

$$\begin{array}{ll} r=0.08 & 8\% \text{ annual rate} \\ n=12 & \text{since we're doing monthly deposits, we'll compound monthly} \\ t=30 & 30 \text{ years} \\ A=\$200,000 & \text{The amount we want to have in } 30 \text{ years} \end{array}$$

In this case, we're going to have to set up the equation, and solve for d.

$$d=\frac{200,000\left(\frac{0.08}{12}\right)}{\left[\left(1+\frac{0.08}{12}\right)^{12\times 30}-1\right]}$$

$$d=\frac{200,000(0.00667)}{[(1.00667)^{360}-1]}$$

$$d=\frac{1333.3333}{9.9357}=\$134.20$$

So you would need to deposit $\$134.20$each month to have $\$200,000$in 30 years if your account earns 8% interest.

Try It 3.4.2

A business needs $450,000 in five years. How much should be deposited each quarter in an account that earns 9% compounded quarterly to have this amount in five years?

Answer

$$\begin{array}{ll} r=0.09 & 9\% \text{ annual rate} \\ n=4 & \text{compounded quarterly} \\ t=5 & 5 \text{ years} \\ A=\$450,000 & \text{The amount we want to have in } 5 \text{ years} \end{array}$$

$$d=\frac{450,000\left(\frac{0.09}{4}\right)}{\left[\left(1+\frac{0.09}{4}\right)^{4\times 5}-1\right]}$$

$$d=\frac{450,000(0.0225)}{\left[(1.0225)^{20}-1\right]}$$

$$d=\frac{10,125}{0.5605092}=\$18,063.93$$

The business needs to deposit $18,063.93 at the end of each quarter for 5 years into an sinking fund earning interest of 9% compounded quarterly in order to have $450,000 at the end of 5 years.

This type of investment is called a *sinking fund*, when a business deposits money at regular intervals into an account in order to save for a future purchase of equipment.

Payout Annuities

There are other types of annuities besides savings annuities. In a savings annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

Another common type of annuity is a **payout annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved $500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$A=\frac{d\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}{\left(\frac{r}{n}\right)}$$

- A is the amount in the account at the beginning (starting amount, or principal)
- d is the regular withdrawal (the amount you take out each year, each month, etc.)
- r is the annual interest rate in decimal form
- n is the number of compounding periods in one year
- t is the number of years we plan to take withdrawals

Like with savings annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Example 3.4.3

After retiring, you want to be able to take $1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Solution

In this example,

$d = \$1000$ the monthly withdrawal
$r = 0.06$ 6% annual rate
$n = 12$ since we're doing monthly withdrawals, we'll compound monthly
$t = 20$ since were taking withdrawals for 20 years

We're looking for A; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$A = \frac{1000\left[1-\left(1+\frac{0.06}{12}\right)^{-12\times 20)}\right]}{\left(\frac{0.06}{12}\right)}$$

$$A = \frac{1000\left[1-(1.005)^{-240}\right]}{0.005}$$

$$A = \frac{1000\,[1-0.302]}{0.005} = \$139,600$$

You will need to have $139,600 in your account when you retire.

Notice that you withdrew a total of $240,000 ($1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is $\$240,000 - \$139,600 = \$100,400$ in interest.

Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this [(-)], some with [+/-] . The button is often near the = key or the decimal point.

If your calculator displays operations on it (typically a calculator with multi-line display), to calculate 1.005^{-240} you'd type something like: $1.005[\wedge][(-)]240$

If your calculator only shows one value at a time, then usually you hit the (-) key after a number to negate it, so you'd hit: $1.005[y^x]240[(-)] =$

Give it a try - you should get $1.005^{-240} = 0.302096$

We can solve the payout annuity formula for d, which will tell us how much can be withdrawn at regular intervals over a set number of years if we have a specific amount in the annuity account.

Withdrawal Amount Formula

$$d = \frac{A\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$$

- d is the regular withdrawal (the amount you take out each year, each month, etc.)
- A is the amount in the account at the beginning (starting amount, or principal)
- r is the annual interest rate in decimal form

- n is the number of compounding periods in one year
- t is the number of years we plan to take withdrawals

Example 3.4.4

You know you will have $500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

Solution

In this example, we're looking for d.

$$\begin{array}{ll} r=0.08 & 8\% \text{ annual rate} \\ n=12 & \text{since we're doing monthly withdrawals} \\ t=30 & \text{ since were taking withdrawals for 30 years} \\ A=\$500,000 & \text{we are beginning with } \$500,000 \end{array}$$

Using the formula:

$$d=\frac{500,000\left(\frac{0.08}{12}\right)}{\left[1-\left(1+\frac{0.08}{12}\right)^{-12\times 30}\right]}$$

$$d=\frac{500,000(0.00667)}{\left[1-(1.00667)^{-360}\right]}$$

$$d=\frac{3335}{0.90856}$$

$$d=\frac{500,000}{0.9085566}=\$3670.64$$

You would be able to withdraw $3670.64 each month for 30 years.

Notice that you withdrew a total of $3670.64 x 12 months x 30 years = $1,321, 430.40. This amounts to $821,430.40 in interest over 30 years when the interest rate is 8%.

Try It 3.4.3

A donor gives $100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Answer

$$\begin{array}{ll} d= & \text{unknown} \\ r=0.04 & 4\% \text{ annual rate} \\ n=1 & \text{since we're doing annual scholarships} \\ t=20 & \text{ since were taking withdrawals for 20 years} \\ A=\$100,000 & \text{we are starting with } \$100,000 \end{array}$$

$$d=\frac{100,000\left(\frac{0.04}{1}\right)}{\left[1-\left(1+\frac{0.04}{1}\right)^{-1\times 20}\right]}$$

Solving for d gives $7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, $\$100,000(0.04) = \$4,000$ a year would have been available.