10.1.5.2: Order of Operations

Learning Objectives

- Use the order of operations to simplify expressions, including those with parentheses.
- Use the order of operations to simplify expressions containing exponents and square roots.

Introduction

People need a common set of rules for performing computation. Many years ago, mathematicians developed a standard **order of operations** that tells you which calculations to make first in an expression with more than one **operation**. Without a standard procedure for making calculations, two people could get two different answers to the same problem. For example, $3+5\cdot 2$ has only one correct answer. Is it 13 or 16?

The Order of Addition, Subtraction, Multiplication, and Division Operations

First, consider expressions that include one or more of the arithmetic operations: addition, subtraction, multiplication, and division. The order of operations requires that all multiplication and division be performed first, going from left to right in the **expression**. The order in which you compute multiplication and division is determined by which one comes first, reading from left to right.

After multiplication and division has been completed, add or subtract in order from left to right. The order of addition and subtraction is also determined by which one comes first when reading from left to right.

Below are three examples showing the proper order of operations for expressions with addition, subtraction, multiplication, and/or division.

Example

Simplify $3+5\cdot 2$.

Solution

$3+5\cdot 2$	Order of operations tells you to perform multiplication before addition.
3+10	Then add.

$3+5\cdot 2=13$

Example

Simplify $20-16\div 4$.

Solution

$20-16\div 4$	Order of operations tells you to perform division before subtraction.
20-4 16	Then subtract.

$20-16\div 4=16$

Example

Simplify $60-30\div 3\cdot 5+7$.

Solution

$60-30\div3\cdot5+7$	Order of operations tells you to perform multiplication and division first, working from left to right, before doing addition and subtraction.
$60-10\cdot5+7$ $60-50+7$	Continue to perform multiplication and division from left to right.
10+7 17	Next, add and subtract from left to right. (Note that addition is not necessarily performed before subtraction.)

$60-30\div3\cdot5+7=17$

Grouping Symbols and the Order of Operations

Grouping symbols such as parentheses ( ), brackets [ ], braces { }, and fraction bars can be used to further control the order of the four basic arithmetic operations. The rules of the order of operations require computation within grouping symbols to be completed first, even if you are adding or subtracting within the grouping symbols and you have multiplication outside the grouping symbols. After computing within the grouping symbols, divide or multiply from left to right and then subtract or add from left to right.

Example

Simplify $900\div(6+3\cdot8)-10$.

Solution

$900\div((6+3\cdot8))-10$	Order of operations tells you to perform what is inside the parentheses first.
$900\div(6+(3\cdot8))-10$ $900\div(6+24)-10$	Simplify the expression in the parentheses. Multiply first.
$900\div30-10$	Then add 6+24.
$900\div30-10$ $30-10$ 20	Now perform division; then subtract.

$900\div(6+3\cdot8)-10=20$

When there are grouping symbols within grouping symbols, compute from the inside to the outside. That is, begin simplifying the innermost grouping symbols first. Two examples are shown.

Example

Simplify $4-3[20-3\cdot4-(2+4)]\div2$.

Solution

$4-3\div2$	There are brackets and parentheses in this problem. Compute inside the innermost grouping symbols first.
$4-3[20-3\cdot4-((2+4)]]\div2$ $4-3\div2$	Simplify within parentheses.
$4-3\div2$ $4-3\div2$ $4-3\div2$	Then, simplify within the brackets by multiplying and then subtracting from left to right.
$4-3(2)\div2$	

$4-3((2)\div 2$ $4-6\div 2$ $4-3$	Multiply and divide from left to right.
$4-3$ 1	Subtract.

$4-3[20-3\cdot 4-(2+4)]\div 2=1$

Remember that parentheses can also be used to show multiplication. In the example that follows, the parentheses are not a grouping symbol; they are a multiplication symbol. In this case, since the problem only has multiplication and division, we compute from left to right. Be careful to determine what parentheses mean in any given problem. Are they a grouping symbol or a multiplication sign?

Example

Simplify $6\div(3)(2)$.

Solution

$6\div 3\cdot 2$	This expression has multiplication and division only. The multiplication operation can be shown with a dot.
$6\div 3\cdot 2$ $2\cdot 2$ 4	Since this expression has only division and multiplication, compute from left to right.

$6\div(3)(2)=4$

Consider what happens if braces are added to the problem above: $6\div\{(3)(2)\}$. The parentheses still mean multiplication; the additional braces are a grouping symbol. According to the order of operations, compute what is inside the braces first. This problem is now evaluated as $6\div 6=1$. Notice that the braces caused the answer to change from 1 to 4.

Exercise

Simplify $40-(4+6)\div 2+3$.

A. 18
B. 38
C. 24
D. 32

Answer

A. Incorrect. Compute the addition in parentheses first. $40-10\div 2+3$. Then, perform division. $40-5+3$. Finally, add and subtract from left to right. The correct answer is 38.
B. Correct. Compute the addition in parentheses first. $40-10\div 2+3$. Then, perform division. $40-5+3$. Finally, add and subtract from left to right.
C. Incorrect. Compute the addition in parentheses first. $40-10\div 2+3$. Then, perform division. $40-5+3$. Finally, add and subtract from left to right. The correct answer is 38.
D. Incorrect. Compute the addition in parentheses first. $40-10\div 2+3$. Then, perform division. $40-5+3$. Finally, with only subtraction and addition left, add and subtract from left to right. The correct answer is 38.

The Order of Operations

1. Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), braces { }, brackets [ ], and fraction bars.
2. Multiply and Divide, from left to right.
3. Add and Subtract, from left to right.

Performing the Order of Operations with Exponents and Square Roots

So far, our rules allow us to simplify expressions that have multiplication, division, addition, subtraction or grouping symbols in them. What happens if a problem has **exponents** or **square roots** in it? We need to expand our order of operation rules to include exponents and square roots.

If the expression has exponents or square roots, they are to be performed after parentheses and other grouping symbols have been simplified and *before* any multiplication, division, subtraction and addition that are outside the parentheses or other grouping symbols.

Note that you compute from more complex operations to more basic operations. Addition and subtraction are the most basic of the operations. You probably learned these first. Multiplication and division, often thought of as repeated addition and subtraction, are more complex and come before addition and subtraction in the order of operations. Exponents and square roots are repeated multiplication and division, and because they're even more complex, they are performed before multiplication and division. Some examples that show the order of operations involving exponents and square roots are shown below.

Example

Simplify $14 + 28 \div 2^2$.

Solution

$14 + 28 \div 2^2$	This problem has addition, division, and exponents in it. Use the order of operations.
$14 + 28 \div 4$	Simplify 2^2
$14 + 7$	Perform division before addition.
21	Add.

$14 + 28 \div 2^2 = 21$

Example

Simplify $3^2 \cdot 2^3$.

Solution

$3^2 \cdot 2^3$	This problem has exponents and multiplication in it.
$9 \cdot 8$	Simplify 3^2 and 2^3.
72	Perform multiplication.

$3^2 \cdot 2^3 = 72$

Example

Simplify $(3 + 4)^2 + (8)(4)$.

Solution

$(3+4)^2+(8)(4)$	This problem has parentheses, exponents, and multiplication in it. The first set of parentheses is a grouping symbol. The second set indicates multiplication. Grouping symbols are handled first.
$7^2+(8)(4)$ $49+(8)(4)$	Add the numbers inside the parentheses that are serving as grouping symbols. Simplify 7^2.
$49+32$	Perform multiplication.
81	Add.

$(3+4)^2+(8)(4)=81$

? Exercise

Simplify $77-(1+4-2)^2$.

A. 68
B. 28
C. 71
D. 156

Answer

A. Correct. $77-(1+4-2)^2=77-(3)^2=77-9=68$
B. Incorrect. Simplify the expression in parentheses first. $77-(1+4-2)^2=77-(3)^2=77-9=68$
C. Incorrect. The exponent of 2 tells you to multiply the number by itself, not by 2; $77-(3)^2=77-9$, not $77-6$. The correct answer is 68.
D. Incorrect. Parentheses are a grouping symbol, and numbers inside them should be computed first. The exponent of 2 tells you to multiply the number by itself, not by $2.77-(1+4-2)^2=77-(3)^2=77-9=68$. The correct answer is 68.

The Order of Operations

1. Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), braces { }, brackets [ ], and fraction bars.
2. Evaluate exponents and roots of numbers, such as square roots.
3. Multiply and Divide, from left to right.
4. Add and Subtract, from left to right.

Some people use a saying to help them remember the order of operations. This saying is called PEMDAS or “Please Excuse My Dear Aunt Sally.” The first letter of each word begins with the same letter of an arithmetic operation.

The P in Please stands for Parentheses (and other grouping symbols).

The E in Excuse stands for Exponents.

The M and D in My Dear stand for Multiplication and Division (from left to right).

The A and S in Aunt Sally stand for Addition and Subtraction (from left to right).

Note: Even though multiplication comes before division in the saying, division could be performed first. Which is performed first, between multiplication and division, is determined by which comes first when reading from left to right. The same is true of addition and subtraction. Don’t let the saying confuse you about this!

Summary

The order of operations gives us a consistent sequence to use in computation. Without the order of operations, you could come up with different answers to the same computation problem. (Some of the early calculators, and some inexpensive ones, do NOT use the order of operations. In order to use these calculators, the user has to input the numbers in the correct order.)
