Sampling Distribution of the Sample Proportion, p-hat

CO-6: Apply basic concepts of probability, random variation, and commonly used statistical probability distributions.

Behavior of Sample Proportions

Learning Objectives

LO 6.21: Apply the sampling distribution of the sample proportion (when appropriate). In particular, be able to identify unusual samples from a given population.

EXAMPLE 6: Behavior of Sample Proportions

Approximately 60% of all part-time college students in the United States are female. (In other words, the population proportion of females among part-time college students is p = 0.6.) What would you expect to see in terms of the behavior of a sample proportion of females (p-hat) if random samples of size 100 were taken from the population of all part-time college students?

As we saw before, due to sampling variability, sample proportion in random samples of size 100 will take numerical values which vary according to the laws of chance: in other words, sample proportion is a **random variable**. To summarize the behavior of any random variable, we focus on three features of its distribution: the center, the spread, and the shape.

Based only on our intuition, we would expect the following:

Center: Some sample proportions will be on the low side — say, 0.55 or 0.58 — while others will be on the high side — say, 0.61 or 0.66. It is reasonable to expect all the sample proportions in repeated random samples to average out to the underlying population proportion, 0.6. In other words, the mean of the distribution of p-hat should be p.

Spread: For samples of 100, we would expect sample proportions of females not to stray too far from the population proportion 0.6. Sample proportions lower than 0.5 or higher than 0.7 would be rather surprising. On the other hand, if we were only taking samples of size 10, we would not be at all surprised by a sample proportion of females even as low as 4/10 = 0.4, or as high as 8/10 = 0.8. Thus, sample size plays a role in the spread of the distribution of sample proportion: there should be less spread for larger samples, more spread for smaller samples.

Shape: Sample proportions closest to 0.6 would be most common, and sample proportions far from 0.6 in either direction would be progressively less likely. In other words, the shape of the distribution of sample proportion should bulge in the middle and taper at the ends: it should be somewhat **normal.**

Comment:

- The **distribution** of the values of the sample proportions (p-hat) in repeated **samples** (of the same size) is called the **sampling distribution of p-hat**.

The purpose of the next video and activity is to check whether our intuition about the center, spread and shape of the sampling distribution of p-hat was correct via simulations.

Video

Video: Simulation #1 (p-hat) (4:13)

Did I Get This?: Simulation #1 (p-hat)

At this point, we have a good sense of what happens as we take random samples from a population. Our simulation suggests that our initial intuition about the shape and center of the sampling distribution is correct. If the population has a proportion of p, then random samples of the same size drawn from the population will have sample proportions close to p. More specifically, the distribution of sample proportions will have a mean of p.

We also observed that for this situation, the sample proportions are approximately normal. We will see later that this is not always the case. But if sample proportions are normally distributed, then the distribution is centered at p.

Now we want to use simulation to help us think more about the variability we expect to see in the sample proportions. Our intuition tells us that larger samples will better approximate the population, so we might expect less variability in large samples.

In the next walk-through we will use simulations to investigate this idea. After that walk-through, we will tie these ideas to more formal theory.

Video

Video: Simulation #2 (p-hat) (4:55)

Did I Get This?: Simulation #2 (p-hat)

The simulations reinforced what makes sense to our intuition. Larger random samples will better approximate the population proportion. When the sample size is large, sample proportions will be closer to p. In other words, the sampling distribution for large samples has less variability. Advanced probability theory confirms our observations and gives a more precise way to describe the standard deviation of the sample proportions. This is described next.

The Sampling Distribution of the Sample Proportion

If repeated random samples of a given size n are taken from a population of values for a categorical variable, where the proportion in the category of interest is p, then the mean of all sample proportions (p-hat) is the population proportion (p).

As for the spread of all sample proportions, theory dictates the behavior much more precisely than saying that there is less spread for larger samples. In fact, the standard deviation of all sample proportions is directly related to the sample size, n as indicated below.

$$\text{The standard deviation of all sample proportions } (\hat{p}) \text{ is exactly } \sqrt{\frac{p(1-p)}{n}}$$

Since the sample size n appears in the denominator of the square root, the standard deviation does decrease as sample size increases. Finally, the shape of the distribution of p-hat will be approximately normal as long as the sample size n is large enough. The convention is to require both np and n(1 – p) to be at least 10.

We can summarize all of the above by the following:

$$\hat{p} \text{ is normally distributed with a mean of } \mu_{\hat{p}} = p$$

$$\text{and a standard deviation } \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

$$\text{as long as } np \geq 10 \text{ and } n(1-p) \geq 10$$

Let’s apply this result to our example and see how it compares with our simulation.

In our example, n = 25 (sample size) and p = 0.6. Note that np = 15 ≥ 10 and n(1 – p) = 10 ≥ 10. Therefore we can conclude that p-hat is approximately a normal distribution with mean p = 0.6 and standard deviation

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.6(1-0.6)}{25}} = 0.097$$

(which is very close to what we saw in our simulation).

Comment:

- These results are similar to those for binomial random variables (X) discussed previously. Be careful not to confuse the results for the mean and standard deviation of X with those of p-hat.

Learn by Doing: Sampling Distribution of p-hat

Did I Get This?: Sampling Distribution of p-hat

If a sampling distribution is normally shaped, then we can apply the Standard Deviation Rule and use z-scores to determine probabilities. Let's look at some examples.

EXAMPLE 7: Using the Sample Distribution of p-hat

A random sample of 100 students is taken from the population of all part-time students in the United States, for which the overall proportion of females is 0.6.

(a) There is a 95% chance that the sample proportion (p-hat) falls between what two values?

First note that the distribution of p-hat has mean p = 0.6, standard deviation

$$\sigma_{\hat{p}}=\sqrt{\frac{p(1-p)}{n}}=\sqrt{\frac{0.6(1-0.6)}{100}}=0.05$$

and a shape that is close to normal, since np = 100(0.6) = 60 and n(1 – p) = 100(0.4) = 40 are both greater than 10. The Standard Deviation Rule applies: the probability is approximately 0.95 that p-hat falls within 2 standard deviations of the mean, that is, between 0.6 – 2(0.05) and 0.6 + 2(0.05). There is roughly a 95% chance that p-hat falls in the interval (0.5, 0.7) for samples of this size.

(b) What is the probability that sample proportion p-hat is less than or equal to 0.56?

To find

$$P(\hat{p} \leq 0.56)$$

we standardize 0.56 into a z-score by subtracting the mean and dividing the result by the standard deviation. Then we can find the probability using the standard normal calculator or table.

$$P(\hat{p} \leq 0.56)=P\left(Z \leq \frac{0.56-0.6}{0.05}\right)=P(Z \leq -0.80)=0.2119$$

To see the impact of the sample size on these probability calculations, consider the following variation of our example.

EXAMPLE 8: Using the Sample Distribution of p-hat

A random sample of **2500** students is taken from the population of all part-time students in the United States, for which the overall proportion of females is 0.6.

(a) There is a 95% chance that the sample proportion (p-hat) falls between what two values?

First note that the distribution of p-hat has mean p = 0.6, standard deviation

$$\sigma_{\hat{p}}=\sqrt{\frac{p(1-p)}{n}}=\sqrt{\frac{0.6(1-0.6)}{2500}}=0.01$$

and a shape that is close to normal, since np = 2500(0.6) = 1500 and n(1 – p) = 2500(0.4) = 1000 are both greater than 10. The Standard Deviation Rule applies: the probability is approximately 0.95 that p-hat falls within 2 standard deviations of the mean, that is, between 0.6 – 2(0.01) and 0.6 + 2(0.01). There is roughly a 95% chance that p-hat falls in the interval (0.58, 0.62) for samples of this size.

(b) What is the probability that sample proportion p-hat is less than or equal to 0.56?

To find

$$P(\hat{p} \leq 0.56)$$

we standardize 0.56 to into a z-score by subtracting the mean and dividing the result by the standard deviation. Then we can find the probability using the standard normal calculator or table.

$$P(\hat{p} \leq 0.56)=P\left(Z \leq \frac{0.56-0.6}{0.01}\right)=P(Z \leq -4) \approx 0$$

Comment:

- As long as the sample is truly random, the distribution of p-hat is centered at p, no matter what size sample has been taken. Larger samples have less spread. Specifically, when we multiplied the sample size by 25, increasing it from 100 to 2,500, the standard deviation was reduced to 1/5 of the original standard deviation. Sample proportion strays less from population proportion 0.6 when the sample is larger: it tends to fall anywhere between 0.5 and 0.7 for samples of size 100, whereas it tends to fall between 0.58 and 0.62 for samples of size 2,500. It is not so improbable to take a value as low as 0.56 for samples of 100 (probability is more than 20%) but it is almost impossible to take a value as low as 0.56 for samples of 2,500 (probability is virtually zero).

Applet: Sampling Distribution for a Sample Proportion