3.5: Poisson Distribution

In this section, we consider our final family of discrete probability distributions. We begin with the definition.

Definition 3.5.1

A random variable X has a ***Poisson distribution***, with parameter $\lambda > 0$, if its probability mass function is given by

$$p(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad \text{for } x = 0, 1, 2, \dots. \tag{3.5.1}$$

We write $X \sim \text{Poisson}(\lambda)$.

The main application of the Poisson distribution is to count the number of times some event occurs over a fixed interval of time or space. More specifically, if the random variable X denotes the number of times the event occurs during an interval of length T, and r denotes the average rate at which the event occurs per unit interval, then X has a Poisson distribution with parameter $\lambda = rT$. Consider the following examples:

- The number of customers arriving at McDonald's between 8 a.m. and 9 a.m.
- The number of calls made to 911 in South Bend on a Saturday.
- The number of accidents at a particular intersection during the month of June.

All of the examples above count the number of times something occurs over an interval of *time*. The next example gives an example where the interval is in *space*.

Example 3.5.1

Suppose typos occur at an average rate of $r = 0.01$ per page in the Friday edition of the *New York Times*, which is 45 pages long. Let X denote the number of typos on the front page. Then X has a Poisson distribution with parameter

$$\lambda = 0.01 \times 1 = 0.01,$$

since we are considering an interval of length one page ($T = 1$). Thus, the probability that there is at least one typo on the front page is given by

$$P(X \geq 1) = P(\{X = 0\}^c) = 1 - P(X = 0) = 1 - \frac{e^{-0.01}(0.01)^0}{0!} = 1 - e^{-0.01} \approx 0.00995.$$

Now, if we let random variable Y denote the number of typos in the entire paper, then Y has a Poisson distribution with parameter

$$\lambda = 0.01 \times 45 = 0.45,$$

since we are considering an interval of $T = 45$ pages. The probability that there are less than three typos in the entire paper is

$$P(Y < 3) = P(Y = 0, 1, \text{ or } 2) = \frac{e^{-0.45}(0.45)^0}{0!} + \frac{e^{-0.45}(0.45)^1}{1!} + \frac{e^{-0.45}(0.45)^2}{2!} \approx 0.98912.$$

The Poisson distribution is similar to all previously considered families of discrete probability distributions in that it counts the number of times something happens. However, the Poisson distribution is different in that there is not an act that is being repeatedly performed. In other words, there are no set trials, but rather a set window of time or space to observe.