1.1: Sample Spaces and Events

Introduction

We begin with a definition.

Definition 1.1.1

Probability theory provides a mathematical model for chance (or random) phenomena.

While this is not a very informative definition, it does indicate the overall goal of this course, which is to develop a *formal, mathematical structure* for the fairly intuitive concept of probability. While most everyone is familiar with the notion of "chance" -- we informally talk about the chance of it raining tomorrow, or the chance of getting what you want for your birthday -- when it comes to *quantifying* the chance of something happening, we need to develop a mathematical model to make things precise and calculable.

Sample Spaces and Events

Before we can formally define what the mathematical model is that we will use to make probability precise, we first establish the *structure* on which the model operates: *sample spaces* and *events*.

Definition 1.1.2

The ***sample space*** for a probability experiment (i.e., an experiment with random outcomes) is the set of all possible outcomes.

- The sample space is denoted S.
- An ***outcome*** is an *element* of S, generally denoted $s \in S$.

Example 1.1.1

Suppose we toss a coin twice and record the sequence of heads (h) and tails (t). A possible outcome of this experiment is then given by

$$s = ht$$

and the sample space is

$$S = \{hh, ht, th, tt\}. \tag{1.1.1}$$

Example 1.1.2

Suppose we record the time (t), in minutes, that a car spends waiting for a green light at a particular intersection. A possible outcome of this experiment is then given by

$$t = 1.5,$$

indicating that a particular car waited one and a half minutes for the light to turn green. The sample space consists of all non-negative numbers, since a measurement of time cannot be negative and, in theory, there is no limit on how a long a car could wait for a green light. We can then write the sample space as follows:

$$S = \{t \in \mathbb{R} \mid t \geq 0\} = [0, \infty). \tag{1.1.2}$$

Definition 1.1.3

An ***event*** is a particular subset of the sample space.

Example 1.1.3

Continuing in the context of Example 1.1.1, define A to be the event that at least one heads is recorded. We can write event A as the following subset of the sample space:

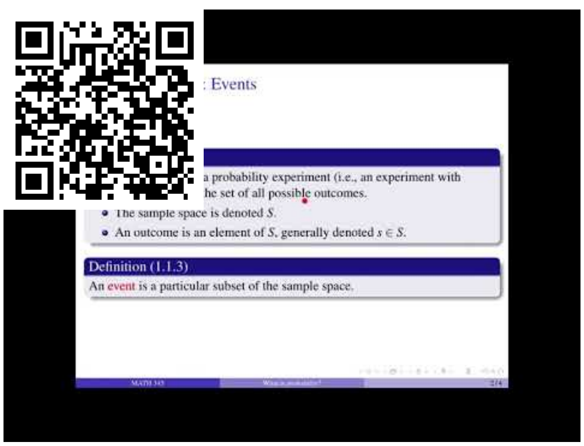
$$A = \{hh, ht, th\}.$$

Note that A is a subset of S given in Equation 1.1.1.

Example 1.1.4

Continuing in the context of Example 1.1.2, define B to be the event that a car waits at most 2 minutes for the light to turn green. We can write the event B as the following interval, i.e., a subset of the sample space S given in Equation 1.1.2:

$$B = [0, 2] = \{t \in \mathbb{R} \mid 0 \leq t \leq 2\}.$$



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Set Theory: A Brief Review

As we see from the above definitions of sample spaces and events, *sets* play the primary role in the structure of probability experiments. So, in this section, we review some of the basic definitions and notation from *set theory*. We do this in the context of sample spaces, outcomes, and events.

Definition 1.1.4

1. The ***union*** of two events A and B, denoted $A \cup B$, is the set of all outcomes in A or B (*or both*).
2. The ***intersection*** of two events A and B, denoted $A \cap B$, is the set of all outcomes in both A and B.

3. The ***complement*** of an event A, denoted A^c, is the set of all outcomes in the sample space that are not in A. This may also be written as follows:

$$A^c = \{s \in S \mid s \notin A\}.$$

4. The ***empty set***, denoted $\varnothing$, is the set containing no outcomes.
5. Two events A and B are ***disjoint*** (or ***mutually exclusive***) if their intersection is the empty set, i.e., $A \cap B = \varnothing$.

Example 1.1.5

Continuing in the context of both Examples 1.1.1 & 1.1.3, define B to be the event that exactly one heads is recorded:

$$B = \{ht, th\}.$$

Now we can apply the set operations just defined to the events A and B:

$$A \cup B = \{hh, ht, th\} = A$$

$$A \cap B = \{ht, th\} = B$$

$$A^c = \{tt\}$$

$$B^c = \{hh, tt\}$$

Note the relationship between events A and B: every outcome in B is an outcome in A. In this case, we say that B is a ***subset*** of A, and write

$$B \subseteq A.$$

Note also that events A and B are not disjoint, since their intersection is not the empty set. However, if we let C be the event that no heads are recorded, then

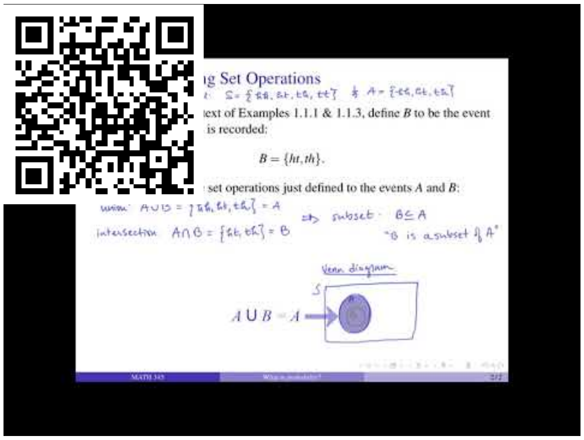
$$C = \{tt\},$$

and

$$A \cap C = \varnothing$$

$$B \cap C = \varnothing.$$

Thus, events A and C are disjoint, and events B and C are disjoint.



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