4.6: Weibull Distributions

In this section, we introduce the Weibull distributions, which are very useful in the field of actuarial science.

Weibull Distributions

Definition $4.6.1$

A random variable X has a ***Weibull distribution*** with parameters $\alpha, \beta > 0$, write $X \sim \text{Weibull}(\alpha, \beta)$, if X has pdf given by

$$f(x) = \begin{cases} \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha}, & \text{for } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

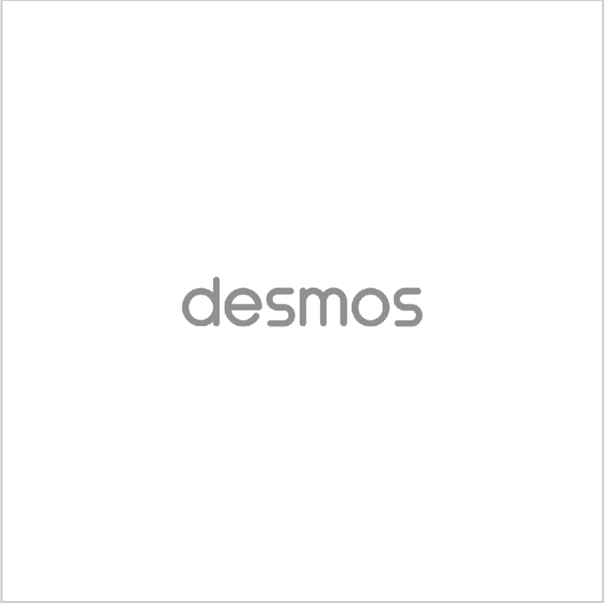

Figure 1: Graph of pdf for Weibull$(\alpha = 2, \beta = 5)$ distribution.

Example $4.6.1$

A typical application of Weibull distributions is to model *lifetimes* that are not “memoryless”. For example, each of the following gives an application of the Weibull distribution.

- modeling the lifetime of a car battery
- modeling the probability that someone survives past the age of 80 years old

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The parameter α is referred to as the ***shape parameter***, and β is the ***scale parameter***. When $\alpha = 1$, the Weibull distribution is an exponential distribution with $\lambda = 1/\beta$, so the exponential distribution is a special case of both the Weibull distributions and the gamma distributions. We can see the similarities between the Weibull and exponential distributions more readily when comparing the cdf's of each. The cdf of the Weibull distribution is given below, with proof, along with other important properties, stated without proof.

Properties of Weibull Distributions

If $X \sim \text{Weibull}(\alpha, \beta)$, then the following hold.

1. The cdf of X is given by

$$F(x)=\begin{cases}0 & \text{for } x<0, \\ 1-e^{-(x/\beta)^\alpha}, & \text{for } x\geq 0.\end{cases}$$

2. For any $0<p<1$, the $(100p)^{\text{th}}$ percentile is $\pi_p=\beta(-\ln(1-p))^{1/\alpha}$.
3. The mean of X is $\mathrm{E}[X]=\beta\Gamma\left(1+\frac{1}{\alpha}\right)$.
4. The variance of X is $\mathrm{Var}(X)=\beta^2\left[\Gamma\left(1+\frac{2}{\alpha}\right)-\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right]$.

Partial Proof

We prove Property #1, but leave #2 as an exercise. Properties #3 and #4 are rather tricky to prove, so we state them without proof.

For the first property, we consider two cases based on the value of x. First, if $x<0$, then the pdf is constant and equal to 0, which gives the following for the cdf:

$$F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{x} 0dt=0$$

Second, if $x\geq 0$, then the pdf is $\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-(x/\beta)^\alpha}$, and the cdf is given by the following integral, which is solved by making the substitution $u=\left(\frac{t}{\beta}\right)^\alpha$:

$$F(x)=\int_{-\infty}^{x} f(t)dt=\int_0^x \frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-(x/\beta)^\alpha}dt=\int_0^{(x/\beta)^\alpha} e^{-u}du=-e^{-u}\Big|_0^{(x/\beta)^\alpha}=-e^{-(x/\beta)^\alpha}-(-e^0)=1-e^{-(x/\beta)^\alpha}.$$