4.7: Chi-Squared Distributions

In this section, we introduce the chi-squared distributions, which are very useful in statistics.

Chi-Squared Distributions

Definition 4.7.1

A random variable X has a ***chi-squared distribution*** with k degrees of freedom, where k is a positive integer, write $X \sim \chi^2(k)$, if X has pdf given by

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(k/2) 2^{k/2}} x^{k/2-1} e^{-x/2}, & \text{for } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

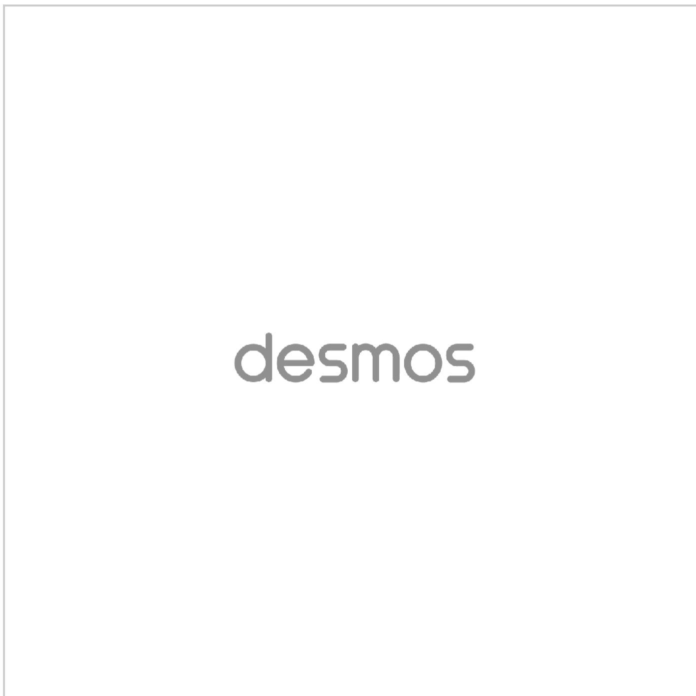

Figure 1: Graph of pdf for $\chi^2(1)$ distribution.

The chi-squared distributions are a special case of the gamma distributions with $\alpha = \frac{k}{2}, \lambda = \frac{1}{2}$, which can be used to establish the following properties of the chi-squared distribution.

Properties of Chi-Squared Distributions

If $X \sim \chi^2(k)$, then X has the following properties.

1. The mgf of X is given by

$$M_X(t) = \frac{1}{(1-2t)^{k/2}}, \quad \text{for } t < \frac{1}{2}$$

2. The mean of X is $\mathrm{E}[X] = k$, i.e., the degrees of freedom.
3. The variance of X is $\mathrm{Var}(X) = 2k$, i.e., twice the degrees of freedom.

Note that there is no closed form equation for the cdf of a chi-squared distribution in general. But most graphing calculators have a built-in function to compute chi-squared probabilities. On the TI-84 or 89, this function is named "χ^2cdf".

The main applications of the chi-squared distributions relate to their importance in the field of statistics, which result from the following relationships between the chi-squared distributions and the normal distributions.

Relationships of Chi-Squared Distributions

1. If Z is a standard normal random variable, i.e., $Z \sim N(0,1)$, then the distribution of Z^2 is chi-squared with $k=1$ degree of freedom.
2. If $X_1, \ldots, X_n$ is a collection of independent, chi-squared random variables each with 1 degree of freedom, i.e., $X_i \sim \chi^2(1)$, for each $i = 1, \ldots, n$, then the sum $X_1 + \cdots + X_n$ is also chi-squared with $k=n$ degrees of freedom.
3. If $X \sim \chi^2(k_1)$ and $Y \sim \chi^2(k_2)$ are independent random variables, then $X + Y \sim \chi^2(k_1 + k_2)$.