FCC DS 21 FINITE MATHEMATICS - SPRING 2023

JoEllen Green
Fresno City College

LibreTexts™

New FCC DS 21 Finite Mathematics - Spring 2023

The LibreTexts libraries are Powered by NICE CXOne and are supported by the Department of Education Open Textbook Pilot Project, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

This text was compiled on 03/18/2025

TABLE OF CONTENTS

Licensing

1: Linear Equations

- 1.1: Graphing a Linear Equation
  - 1.1.1: Graphing a Linear Equation (Exercises)
- 1.2: Slope of a Line
  - 1.2.1: Slope of a Line (Exercises)
- 1.3: Determining the Equation of a Line
  - 1.3.1: Determining the Equation of a Line (Exercises)
- 1.4: Applications
  - 1.4.1: Applications (Exercises)
- 1.5: More Applications
  - 1.5.1: More Applications (Exercises)
- 1.6: Chapter Review

2: Inequalities

- 2.1: Properties of Inequalities
- 2.2: Solving and Graphing Inequalities, and Writing Answers in Interval Notation
- 2.3: Set Builder Notation
- 2.4: Rational Inequalities

3: Functions

- 3.1: Functions and Function Notation
- 3.2: Domain and Range
- 3.3: Piecewise-Defined Functions
- 3.4: Absolute Value
- 3.5: Absolute Value Equations
- 3.6: Break-Even Analysis (Sink or Swim)
- 3.7: Applications
  - 3.7.1: Applications (Exercises)
- 3.8: More Applications
  - 3.8.1: More Applications (Exercises)
- 3.9: Quadratic Functions
  - 3.9E: Quadratic Functions (Exercises)
- 3.10: Power Functions and Polynomial Functions
- 3.11: Graphs of Polynomial Functions
- 3.12: Graphing Rational Functions

4: Exponential and Logarithmic Functions

- 4.1: Prelude to Exponential and Logarithmic Functions including Exponential properties
- 4.2: Definition of Exponential Function, Exponential Growth and Decay Models
  - 4.2.1: Exponential Growth and Decay Models (Exercises)
- 4.3: Graphs and Properties of Exponential Growth and Decay Functions/Use TI-84 calculator

- 4.3.1: Graphs and Properties of Exponential Growth and Decay Functions (Exercises)
- 4.4: Logarithms and Logarithmic Functions
  - 4.4.1: Logarithms and Logarithmic Functions (Exercises)
- 4.5: Graphs and Properties of Logarithmic Functions
  - 4.5.1: Graphs and Properties of Logarithmic Functions (Exercises)
- 4.6: Exponential and Logarithmic Equations
- 4.7: Chapter Review
- 4.E: Exponential and Logarithmic Functions (Exercises)

5: Introduction to Calculus

- 5.1: Prelude to Calculus
- 5.2: Finding Limits - Numerical and Graphical Approaches
- 5.3: Finding Limits - Properties of Limits
- 5.4: Continuity
- 5.5: Average rate of Change and Derivatives
- 5.E: Introduction to Calculus (Exercises)
  - 1.1: 5.6 Derivatives of Polynomials
- 5.R: Introduction to Calculus (Review)

6: Mathematics of Finance

- 6.1: Simple and Compound Interest
  - 6.1.1: Simple and Compound Interest (Exercises)
- 6.2: Annuities and Sinking Funds
  - 6.2.1: Annuities and Sinking Funds (Exercises)
- 6.3: Present Value of an Annuity and Installment Payment
  - 6.3.1: Present Value of an Annuity and Installment Payment (Exercises)
- 6.4: Miscellaneous Application Problems
  - 6.4.1: Miscellaneous Application Problems (Exercises)
- 6.5: Classification of Finance Problems
  - 6.5.1: Classification of Finance Problems (Exercises)
- 6.6: Chapter Review

7: Matrices

- 7.1: Introduction to Matrices
  - 7.1.1: Introduction to Matrices (Exercises)
- 7.2: Systems of Linear Equations and the Gauss-Jordan Method
  - 7.2.1: Systems of Linear Equations and the Gauss-Jordan Method (Exercises)
- 7.3: Systems of Linear Equations – Special Cases
  - 7.3.1: Systems of Linear Equations – Special Cases (Exercises)
- 7.4: Inverse Matrices
  - 7.4.1: Inverse Matrices (Exercises)
- 7.5: Application of Matrices in Cryptography
  - 7.5.1: Application of Matrices in Cryptography (Exercises)
- 7.6: Applications – Leontief Models
  - 7.6.1: Applications – Leontief Models (Exercises)

- 7.7: Chapter Review

8: Linear Programming - A Geometric Approach

- 8.1: Graph Linear Inequalities in Two Variables
  - 8.1E: Exercises
- 8.2: Maximization Applications
  - 8.2.1: Maximization Applications (Exercises)
- 8.3: Minimization Applications
  - 8.3.1: Minimization Applications (Exercises)
- 8.4: Chapter Review

9: Linear Programming - The Simplex Method

- 9.1: Introduction to Linear Programming Applications in Business, Finance, Medicine, and Social Science
- 9.2: Maximization By The Simplex Method
  - 9.2.1: Maximization By The Simplex Method (Exercises)
- 9.3: Minimization By The Simplex Method
  - 9.3.1: Minimization By The Simplex Method (Exercises)
- 9.4: Chapter Review

10: Sets and Counting

- 10.1: Sets and Venn Diagrams
  - 10.1.1: Sets and Venn Diagrams (Exercises)
- 10.2: Tree Diagrams and the Multiplication Axiom
  - 10.2.1: Tree Diagrams and the Multiplication Axiom (Exercises)
- 10.3: Permutations
  - 10.3.1: Permutations (Exercises)
- 10.4: Circular Permutations and Permutations with Similar Elements
  - 10.4.1: Circular Permutations and Permutations with Similar Elements (Exercises)
- 10.5: Combinations
  - 10.5.1: Combinations (Exercises)
- 10.6: Combinations- Involving Several Sets
  - 10.6.1: Combinations- Involving Several Sets (Exercises)
- 10.7: Binomial Theorem
  - 10.7.1: Binomial Theorem (Exercises)
- 10.8: Chapter Review

11: Probability

- 11.1: Sample Spaces and Probability
  - 11.1.1: Sample Spaces and Probability (Exercises)
- 11.2: Mutually Exclusive Events and the Addition Rule
  - 11.2.1: Mutually Exclusive Events and the Addition Rule (Exercises)
- 11.3: Probability Using Tree Diagrams and Combinations
  - 11.3.1: Probability Using Tree Diagrams and Combinations (Exercises)
- 11.4: Conditional Probability

- 11.4.1: Conditional Probability (Exercises)
- 11.5: Independent Events
  - 11.5.1: Independent Events (Exercises)
- 11.6: Chapter Review

12: More Probability

- 12.1: Binomial Probability
  - 12.1.1: Binomial Probability (Exercises)
- 12.2: Bayes' Formula
  - 12.2.1: Bayes' Formula (Exercises)
- 12.3: Expected Value
  - 12.3.1: Expected Value (Exercises)
- 12.4: Probability Using Tree Diagrams
  - 12.4.1: Probability Using Tree Diagrams (Exercises)
- 12.5: Chapter Review

Index

Glossary

Detailed Licensing

Licensing

A detailed breakdown of this resource's licensing can be found in ***Back Matter/Detailed Licensing***.

CHAPTER OVERVIEW

1: Linear Equations

Learning Objectives

In this chapter, you will learn to:

1. Graph a linear equation.
2. Find the slope of a line.
3. Determine an equation of a line.
4. Solve linear systems.
5. Do application problems using linear equations.

1.1: Graphing a Linear Equation
1.1.1: Graphing a Linear Equation (Exercises)
1.2: Slope of a Line
1.2.1: Slope of a Line (Exercises)
1.3: Determining the Equation of a Line
1.3.1: Determining the Equation of a Line (Exercises)
1.4: Applications
1.4.1: Applications (Exercises)
1.5: More Applications
1.5.1: More Applications (Exercises)
1.6: Chapter Review

1.1: Graphing a Linear Equation

Learning Objectives

In this section, you will learn to:

1. Graph a line when you know its equation
2. Graph a line when you are given its equation in parametric form
3. Graph and find equations of vertical and horizontal lines

Graphing a Line from it Equation

Equations whose graphs are straight lines are called **linear equations.** The following are some examples of linear equations:

$$2x-3y=6, \quad 3x=4y-7, \quad y=2x-5, \quad 2y=3, \quad \text{and} \quad x-2=0$$

A line is completely determined by two points. Therefore, to graph a linear equation we need to find the coordinates of two points. This can be accomplished by choosing an arbitrary value for x or y and then solving for the other variable.

Example 1.1.1

Graph the line: $y=3x+2$

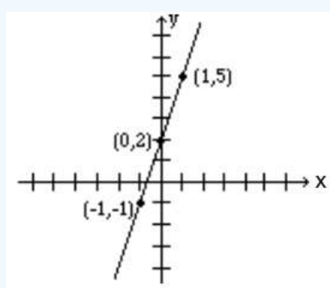
Solution

We need to find the coordinates of at least two points. We arbitrarily choose x = - 1, x = 0, and x = 1.

- If x = -1, then y = 3(-1) + 2 or -1. Therefore, (-1, -1) is a point on this line.
- If x = 0, then y = 3(0) + 2 or y = 2. Hence the point (0, 2).
- If x = 1, then y = 5, and we get the point (1, 5).

Below, the results are summarized, and the line is graphed.

x	-1	0	1
y	-1	2	5



Example 1.1.2

Graph the line: $2x+y=4$

Solution

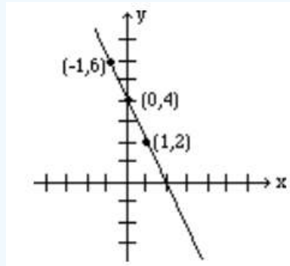
Again, we need to find coordinates of at least two points.

We arbitrarily choose x = -1, x = 0, and y = 2.

- If x = -1, then 2(-1) + y = 4 which results in y = 6. Therefore, (-1, 6) is a point on this line.
- If x = 0, then 2(0) + y = 4, which results in y = 4. Hence the point (0, 4).
- If y = 2, then 2x + 2 = 4, which yields x = 1, and gives the point (1, 2).

The table below shows the points, and the line is graphed.

x	-1	0	1
y	6	4	2



Intercepts

The points at which a line crosses the coordinate axes are called the **intercepts**.
When graphing a line by plotting two points, using the intercepts is often preferred because they are easy to find.

- To find the value of the x-intercept, we let y = 0
- To find the value of the y-intercept, we let x = 0.

Example 1.1.3

Find the intercepts of the line: $2x - 3y = 6$, and graph.

Solution

To find the x-intercept, let y = 0 in the equation, and solve for x.

$$\begin{aligned} 2x - 3(0) &= 6 \\ 2x - 0 &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

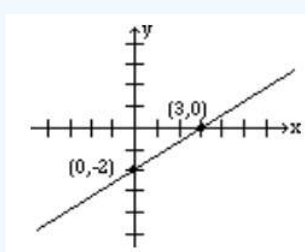
Therefore, the x-intercept is the point (3,0).

To find the y-intercept, let x = 0 in the equation, and solve for y.

$$\begin{aligned} 2(0) - 3y &= 6 \\ 0 - 3y &= 6 \\ -3y &= 6 \\ y &= -2 \end{aligned}$$

Therefore, the y-intercept is the point (0, -2).

To graph the line, plot the points for the x-intercept (3,0) and the y-intercept (0, -2), and use them to draw the line.



Graphing a Line from its Equation in Parametric Form

In higher math, equations of lines are sometimes written in parametric form. For example, $x = 3 + 2t$, $y = 1 + t$. The letter t is called the **parameter**, or the dummy variable.

Parametric lines can be graphed by finding values for x and y by substituting numerical values for t. Plot the points using their (x,y) coordinates and use the points to draw the line.

Example 1.1.4

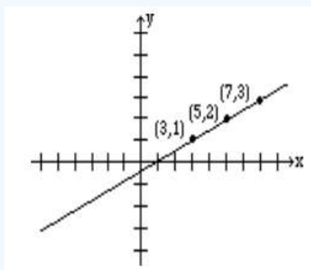
Graph the line given by the parametric equations: $x = 3 + 2t$, $y = 1 + t$

Solution

Let t = 0, 1 and 2; for each value of t find the corresponding values for x and y.

The results are given in the table below.

t	x	y	Point on Line
0	3	1	(3, 1)
1	5	2	(5, 2)
2	7	3	(7, 3)



Horizontal and Vertical Lines

When an equation of a line has only one variable, the resulting graph is a horizontal or a vertical line.

- The graph of the line $x = a$, where a is a constant, is a vertical line that passes through the point $(a, 0)$. Every point on this line has the x-coordinate equal to a, regardless of the y-coordinate.
- The graph of the line $y = b$, where b is a constant, is a horizontal line that passes through the point $(0, b)$. Every point on this line has the y-coordinate equal to b, regardless of the x-coordinate.

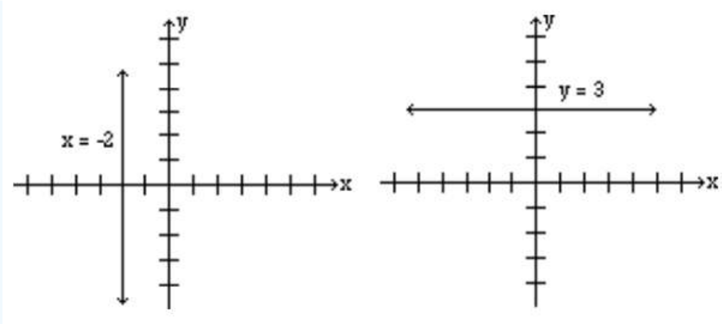
Example 1.1.5

Graph the lines: x = -2 , and y = 3.

Solution

The graph of the line x = -2 is a vertical line that has the x-coordinate -2 no matter what the y-coordinate is. The graph is a vertical line passing through point (-2, 0).

The graph of the line y = 3, is a horizontal line that has the y-coordinate 3 regardless of what the x-coordinate is. Therefore, the graph is a horizontal line that passes through point (0, 3).



Note: Most students feel that the coordinates of points must always be integers. This is not true, and in real life situations, not always possible. Do not be intimidated if your points include numbers that are fractions or decimals.

1.1.1: Graphing a Linear Equation (Exercises)

SECTION 1.1 PROBLEM SET: GRAPHING A LINEAR EQUATION

Work the following problems.

1) Is the point (2, 3) on the line 5x - 2y = 4?	2) Is the point (1, - 2) on the line 6x - y = 4?
3) For the line 3x - y = 12, complete the following ordered pairs. (2,) (, 6) (0,) (, 0)	4) For the line 4x + 3y = 24, complete the following ordered pairs. (3,) (, 4) (0,) (, 0)
5) Graph y = 2x + 3	6) Graph y = - 3x + 5
7) Graph y = 4x - 3	8) Graph x - 2y = 8
9) Graph 2x + y = 4	10) Graph 2x - 3y = 6

SECTION 1.1 PROBLEM SET: GRAPHING A LINEAR EQUATION

11) Graph 2x + 4 = 0	12) Graph 2y - 6 = 0
13) Graph the following three equations on the same set of coordinate axes. $y=x+1 \quad y=2x+1 \quad y=-x+1$	14) Graph the following three equations on the same set of coordinate axes. $y=2x+1 \quad y=2x \quad y=2x-1$
15) Graph the line using the parametric equations x = 1 + 2t, y = 3 + t	16) Graph the line using the parametric equations x = 2 - 3t, y = 1 + 2t

1.2: Slope of a Line

Learning Objectives

In this section, you will learn to:

1. Find the slope of a line.
2. Graph the line if a point and the slope are given.

In the last section, we learned to graph a line by choosing two points on the line. A graph of a line can also be determined if one point and the "steepness" of the line is known. The number that refers to the steepness or inclination of a line is called the **slope** of the line.

From previous math courses, many of you remember slope as the "rise over run," or "the vertical change over the horizontal change" and have often seen it expressed as:

$$\frac{\text{rise}}{\text{run}}, \frac{\text{vertical change}}{\text{horizontal change}}, \frac{\Delta y}{\Delta x} \text{ etc.}$$

We give a precise definition.

Definition: Slope

If (x_1, y_1) and (x_2, y_2) are two different points on a line, the **slope** of the line is

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.2.1}$$

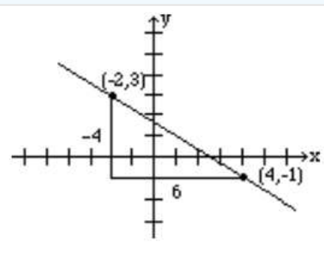
Example 1.2.1

Find the slope of the line passing through points (−2, 3) and (4, −1), and graph the line.

Solution

Let (x_1, y_1) = (−2, 3) and (x_2, y_2) = (4, −1), then the slope (via Equation 1.2.1) is

$$m = \frac{-1-3}{4-(-2)} = \frac{-4}{6} = -\frac{2}{3}$$



To give the reader a better understanding, both the vertical change, -4, and the horizontal change, 6, are shown in the above figure.

When two points are given, it does not matter which point is denoted as (x_1, y_1) and which (x_2, y_2). The value for the slope will be the same.

In Example 1.2.1, if we instead choose (x_1, y_1) = (4, −1) and (x_2, y_2) = (−2, 3), then we will get the same value for the slope as we obtained earlier.

The steps involved are as follows.

$$m = \frac{3-(-1)}{-2-4} = \frac{4}{-6} = -\frac{2}{3}$$

The student should further observe that

- if a line rises when going from left to right, then it has a positive slope. In this situation, as the value of x increases, the value of y also increases
- if a line falls going from left to right, it has a negative slope; as the value of x increases, the value of y decreases.

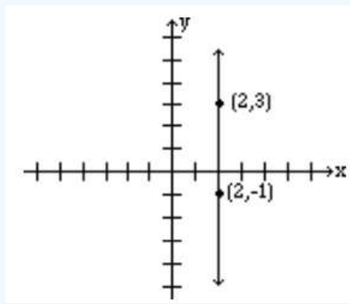
Example 1.2.2

Find the slope of the line that passes through the points (2, 3) and (2, −1), and graph.

Solution

Let (x_1, y_1) = (2, 3) and (x_2, y_2) = (2, −1) then the slope is

$$m = \frac{-1-3}{2-2} = \frac{4}{0} = \text{undefined.}$$



Note: The slope of a vertical line is undefined.

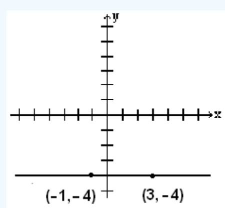
Example 1.2.3

Find the slope of the line that passes through the points (−1, −4) and (3, −4)

Solution

Let (x_1, y_1) = (−1, −4) and (x_2, y_2) = (3, −4), then the slope is

$$m = \frac{-4-(-4)}{3-(-1)} = \frac{0}{4} = 0$$



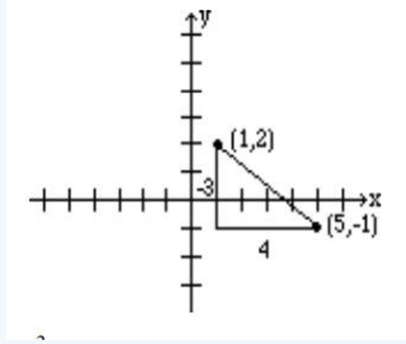
Note: The slope of a horizontal line is 0

Example 1.2.4

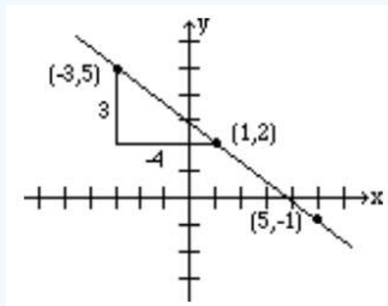
Graph the line that passes through the point (1, 2) and has slope $-\frac{3}{4}$.

Solution

Slope equals $\frac{\text{rise}}{\text{run}}$. The fact that the slope is $\frac{-3}{4}$, means that for every rise of -3 units (fall of 3 units) there is a run of 4. So if from the given point (1, 2) we go down 3 units and go right 4 units, we reach the point (5, -1). The graph is obtained by connecting these two points.



Alternatively, since $\frac{3}{-4}$ represents the same number, the line can be drawn by starting at the point (1,2) and choosing a rise of 3 units followed by a run of -4 units. So from the point (1, 2), we go up 3 units, and to the left 4, thus reaching the point (-3, 5) which is also on the same line. See figure below.



✔ Example 1.2.5

Find the slope of the line $2x + 3y = 6$.

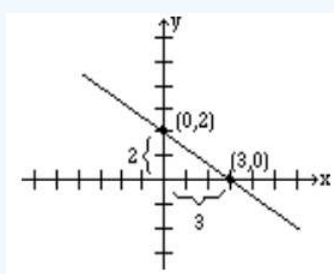
Solution

In order to find the slope of this line, we will choose any two points on this line.

Again, the selection of x and y intercepts seems to be a good choice. The x-intercept is (3, 0), and the y-intercept is (0, 2). Therefore, the slope is

$$m = \frac{2-0}{0-3} = -\frac{2}{3}.$$

The graph below shows the line and the x-intercepts and y-intercepts:



✔ Example 1.2.6

Find the slope of the line $y = 3x + 2$.

Solution

We again find two points on the line, e.g., (0, 2) and (1, 5). Therefore, the slope is

$$m = \frac{5-2}{1-0} = \frac{3}{1} = 3.$$

Look at the slopes and the y-intercepts of the following lines.

The line	slope	y-intercept
$y = 3x + 2$	3	2
$y = -2x + 5$	-2	5
$y = \frac{3}{2}x - 4$	$\frac{3}{2}$	-4

It is no coincidence that when an equation of the line is solved for y, the coefficient of the x term represents the slope, and the constant term represents the y-intercept.

In other words, for the line $y = mx + b$, m is the slope, and b is the y-intercept.

Example 1.2.7

Determine the slope and y-intercept of the line $2x + 3y = 6$.

Solution

We solve for y:

$$\begin{aligned} 2x + 3y &= 6 \\ 3y &= -2x + 6 \\ y &= (-2/3)x + 2 \end{aligned}$$

The slope = the coefficient of the x term = − 2/3.

The y-intercept = the constant term = 2.

1.2.1: Slope of a Line (Exercises)

SECTION 1.2 PROBLEM SET: SLOPE OF A LINE

Find the slope of the line passing through the following pair of points.

1) (2, 3) and (5, 9)	2) (4, 1) and (2, 5)
3) (- 1, 1) and (1, 3)	4) (4, 3) and (- 1, 3)
5) (6, - 5) and (4, - 1)	6) (5, 3) and (- 1, - 4)
7) (3, 4) and (3, 7)	8) (- 2, 4) and (- 3, - 2)
9) (- 3, - 5) and (- 1, - 7)	10) (0, 4) and (3, 0)

SECTION 1.2 PROBLEM SET: SLOPE OF A LINE

Determine the slope of the line from the given equation of the line.

11) y = - 2x + 1	12) y = 3x - 2
13) 2x - y = 6	14) x + 3y = 6
15) 3x - 4y = 12	16) What is the slope of the x-axis? What is the slope of the y-axis?

Graph the line that passes through the given point and has the given slope.

17) (1, 2) and m = - 3/4	18) (2, - 1) and m = 2/3
19) (0, 2) and m = - 2	20) (2, 3) and m = 0

1.3: Determining the Equation of a Line

Learning Objectives

In this section, you will learn to:

1. Find an equation of a line if a point and the slope are given.
2. Find an equation of a line if two points are given.

So far, we were given an equation of a line and were asked to give information about it. For example, we were asked to find points on the line, find its slope and even find intercepts. Now we are going to reverse the process. That is, we will be given either two points, or a point and the slope of a line, and we will be asked to find its equation.

An equation of a line can be written in three forms, the **slope-intercept form**, the **point-slope form,** or the **standard form**. We will discuss each of them in this section.

A line is completely determined by two points, or by a point and slope. The information we are given about a particular line will influence which form of the equation is most convenient to use. Once we know any form of the equation of a line, it is easy to re-express the equation in the other forms if needed.

THE SLOPE-INTERCEPT FORM OF A LINE: $y = mx + b$

In the last section we learned that the equation of a line whose slope = m and
y-intercept = b is

$$\mathbf{y = mx + b}.$$

This is called the **slope-intercept form** of the line and is the most commonly used form.

Example 1.3.1

Find an equation of a line whose slope is 5, and y-intercept is 3.

Solution

Since the slope is $m = 5$, and the y- intercept is $b = 3$, the equation is $y = 5x + 3$.

Example 1.3.2

Find the equation of the line that passes through the point $(2, 7)$ and has slope 3.

Solution

Since $m = 3$, the partial equation is $y = 3x + b$.

Now b can be determined by substituting the point $(2, 7)$ in the equation $y = 3x + b$.

$$\begin{aligned} 7 &= 3(2) + b \\ b &= 1 \end{aligned}$$

Therefore, the equation is $y = 3x + 1$.

Example 1.3.3

Find an equation of the line that passes through the points (-1, 2), and (1, 8).

Solution

$m = \frac{8-2}{1-(-1)} = \frac{6}{2} = 3$. So the partial equation is $y = 3x + b$.

We can use either of the two points (-1, 2) or (1, 8), to find b. Substituting (-1, 2) gives

$$2 = 3(-1) + b$$
$$5 = b$$

So the equation is $y = 3x + 5$.

Example 1.3.4

Find an equation of the line that has x-intercept 3, and y-intercept 4.

Solution

x-intercept = 3, and y-intercept = 4 correspond to the points (3, 0), and (0, 4), respectively.

$$m = \frac{4-0}{0-3} = -\frac{4}{3}$$

We are told the y-intercept is 4; thus b = 4

Therefore, the equation is $y = -\frac{4}{3}x + 4$.

THE POINT-SLOPE FORM OF A LINE: $y - y_1 = m(x - x_1)$

The **point-slope** form is useful when we know two points on the line and want to find the equation of the line.

Let L be a line with slope m, and known to contain a specific point (x_1, y_1). If (x, y) is any other point on the line L, then the definition of a slope leads us to the **point-slope form** or point-slope formula.

The slope is $\frac{y-y_1}{x-x_1} = m$

Multiplying both sides by $(x - x_1)$ gives the point-slope form:

$$\mathbf{y - y_1 = m(x - x_1)}$$

Example 1.3.5

Find the point-slope form of the equation of a line that has slope 1.5 and passes through the point (12,4).

Solution

Substituting the point $(x_1, y_1) = (12, 4)$ and $m = 1.5$ in the point-slope formula, we get

$$\mathbf{y - y_1 = m(x - x_1)}$$

$$y - 4 = 1.5(x - 12)$$

The student may be tempted to simplify this into the slope intercept form $y = mx + b$. But since the problem specifically requests **point-slope** form we will not simplify it.

THE STANDARD FORM OF A LINE: $Ax + By = C$

Another useful form of the equation of a line is the standard form.

If we know the equation of a line in point-slope form, $y - y_1 = m(x - x_1)$, or if we know the equation of the line in slope-intercept form $y = mx + b$, we can simplify the formula to have all terms for the x and y variables on one side of the equation, and the constant on the other side of the equation.

The result is referred to as the **standard form** of the line:

$$\mathbf{Ax + By = C}.$$

Example 1.3.6

Using the point-slope formula, find the standard form of an equation of the line that passes through the point (2, 3) and has slope $-3/5$.

Solution: Substituting the point (2, 3) and $m=-3/5$ in the point-slope formula, we get

$$y-3=-3/5(x-2)$$

Multiplying both sides by 5 gives us

$$\begin{aligned}&5(y-3)=-3(x-2)\\&5y-15=-3x+6\\&3x+5y=21 \quad \text{Standard Form}\end{aligned}$$

Example 1.3.7

Find the standard form of the line that passes through the points (1, -2), and (4, 0).

Solution

First we find the slope: $m=\frac{0-(-2)}{4-1}=\frac{2}{3}$

Then, the point-slope form is: $y-(-2)=\frac{2}{3}(x-1)$

Multiplying both sides by 3 gives us

$$\begin{aligned}&3(y+2)=2(x-1)\\&3y+6=2x-2\\&-2x+3y=-8\\&2x-3y=8 \quad \text{Standard Form}\end{aligned}$$

We should always be able to convert from one form of an equation to another. For example, if we are given a line in the slope-intercept form, we should be able to express it in the standard form, and vice versa.

Example 1.3.8

Write the equation $y=-\frac{2}{3}x+3$ in the standard form.

Solution

Multiplying both sides of the equation by 3, we get

$$\begin{aligned}&3y=-2x+9\\&2x+3y=9 \quad \text{Standard Form}\end{aligned}$$

Example 1.3.9

Write the equation $3x-4y=10$ in the slope-intercept form.

Solution

Solving for y, we get

$$\begin{aligned}&-4y=-3x+10\\&y=\frac{3}{4}x-\frac{5}{2} \quad \text{Standard Form}\end{aligned}$$

Finally, we learn a very quick and easy way to write an equation of a line in the standard form. But first we must learn to find the slope of a line in the standard form by inspection.

By solving for y, it can easily be shown that the slope of the line $Ax+By=C$ is $-A/B$.
The reader should verify this.

Example 1.3.10

Find the slope of the following lines, by inspection.

a. $3x - 5y = 10$
b. $2x + 7y = 20$
c. $4x - 3y = 8$

Solution

a. $A = 3$, $B = -5$, therefore, $m = -\frac{3}{-5} = \frac{3}{5}$
b. $A = 2$, $B = 7$, therefore, $m = -\frac{2}{7}$
c. $m = -\frac{4}{-3} = \frac{4}{3}$

Now that we know how to find the slope of a line in the standard form by inspection, our job in finding the equation of a line is going to be easy.

Example 1.3.11

Find an equation of the line that passes through (2, 3) and has slope - 4/5.

Solution

Since the slope of the line is - 4/5, we know that the left side of the equation is $4x + 5y$, and the partial equation is going to be

$$4x + 5y = c$$

Of course, c can easily be found by substituting for x and y.

$$\begin{aligned}4(2) + 5(3) &= c\\ 23 &= c\end{aligned}$$

The desired equation is

$$4x + 5y = 23.$$

If you use this method often enough, you can do these problems very quickly.

We summarize the forms for equations of a line below:

Slope Intercept form: $\mathbf{y = mx + b}$,
where m = slope, b = y-intercept

Point Slope form: $\mathbf{y - y_1 = m(x - x_1)}$,
where m = slope, (x_1, y_1) is a point on the line

Standard form: $\mathbf{Ax + By = C}$

Horizontal Line: $\mathbf{y = b}$
where b = y-intercept

Vertical Line: $\mathbf{x = a}$
where a = x-intercept

1.3.1: Determining the Equation of a Line (Exercises)

SECTION 1.3 PROBLEM SET: DETERMINING THE EQUATION OF A LINE

Write an equation of the line satisfying the following conditions.
Write the equation in the form y = mx + b.

1) It passes through the point (3, 10) and has slope = 2.	2) It passes through point (4,5) and has m = 0.
3) It passes through (3, 5) and (2, - 1).	4) It has slope 3, and its y-intercept equals 2.
5) It passes through (5, - 2) and m = 2/5.	6) It passes through (- 5, - 3) and (10, 0).
7) It passes through (4, - 4) and (5, 3).	8) It passes through (7, - 2) ; its y-intercept is 5.
9) It passes through (2, - 5) and its x-intercept is 4.	10) Its a horizontal line through the point (2, - 1).

SECTION 1.3 PROBLEM SET: DETERMINING THE EQUATION OF A LINE

Write an equation of the line satisfying the following conditions.
Write the equation in the form y = mx + b.

11) It passes through (5, - 4) and (1, - 4).	12) It is a vertical line through the point (3, - 2).
13) It passes through (3, - 4) and (3, 4).	14) It has x-intercept = 3 and y-intercept = 4.

Write an equation of the line satisfying the following conditions.
Write the equation in the form Ax + By = C.

15) It passes through (3, - 1) and m = 2.	16) It passes through (- 2, 1) and m = - 3/2.
17) It passes through (- 4, - 2) and m = 3/4.	18) Its x-intercept equals 3, and m = - 5/3.

SECTION 1.3 PROBLEM SET: DETERMINING THE EQUATION OF A LINE

Write an equation of the line satisfying the following conditions.
Write the equation in the form Ax + By = C.

19) It passes through (2, - 3) and (5, 1).	20) It passes through (1, - 3) and (- 5, 5).

Write an equation of the line satisfying the following conditions.
Write the equation in point slope form $y-y_1 = m(x-x_1)$

21) It passes through (2, - 3) and (5, 1).	22) It passes through (1, - 3) and (- 5, 2).
23) It passes through (6, -2) and (0, 2).	24) It passes through (8, 2) and (-7, -4).
25) It passes through (-12, 7) and has slope = -1/3.	26) It passes through (8, - 7) and has slope 3/4.

1.4: Applications

Learning Objectives

In this section, you will learn to use linear functions to model real-world applications

Now that we have learned to determine equations of lines, we get to apply these ideas in a variety of real-life situations.

Read the problem carefully. Highlight important information. Keep track of which values correspond to the independent variable (x) and which correspond to the dependent variable (y).

Example 1.4.1

A taxi service charges $0.50 per mile plus a $5 flat fee. What will be the cost of traveling 20 miles? What will be cost of traveling x miles?

Solution

x = distance traveled, in miles and y = cost in dollars

The cost of traveling 20 miles is

$$y=(0.50)(20)+5=10+5=15$$

The cost of traveling x miles is

$$y=(0.50)(x)+5=0.50x+5$$

In this problem, $0.50 per mile is referred to as the **variable cost**, and the flat charge $5 as the **fixed cost**. Now if we look at our cost equation $y=.50x+5$, we can see that the variable cost corresponds to the slope and the fixed cost to the y-intercept.

Example 1.4.2

The variable cost to manufacture a product is $10 per item and the fixed cost $2500. If x represents the number of items manufactured and y represents the total cost, write the cost function.

Solution

- The variable cost of $10 per item tells us that $m=10$.
- The fixed cost represents the y-intercept. So $b=2500$.

Therefore, the cost equation is $y=10x+2500$.

Example 1.4.3

It costs $750 to manufacture 25 items, and $1000 to manufacture 50 items. Assuming a linear relationship holds, find the cost equation, and use this function to predict the cost of 100 items.

Solution

We let x = the number of items manufactured, and let y = the cost.

Solving this problem is equivalent to finding an equation of a line that passes through the points (25, 750) and (50, 1000).

$$m=\frac{1000-750}{50-25}=10$$

Therefore, the partial equation is $y=10x+b$

By substituting one of the points in the equation, we get $b=500$

Therefore, the cost equation is $y=10x+500$

To find the cost of 100 items, substitute $x=100$ in the equation $y=10x+500$

So the cost is

$$y = 10(100) + 500 = 1500$$

It costs $1500 to manufacture 100 items.

Example 1.4.4

The freezing temperature of water in Celsius is 0 degrees and in Fahrenheit 32 degrees. And the boiling temperatures of water in Celsius, and Fahrenheit are 100 degrees, and 212 degrees, respectively. Write a conversion equation from Celsius to Fahrenheit and use this equation to convert 30 degrees Celsius into Fahrenheit.

Solution

Let us look at what is given.

Celsius	Fahrenheit
0	32
100	212

Again, solving this problem is equivalent to finding an equation of a line that passes through the points (0, 32) and (100, 212).

Since we are finding a linear relationship, we are looking for an equation $y = mx + b$, or in this case $F = mC + b$, where x or C represent the temperature in Celsius, and y or F the temperature in Fahrenheit.

$$\text{slope m } = \frac{312 - 32}{100 - 0} = \frac{9}{5}$$

The equation is $F = \frac{9}{5}C + b$

Substituting the point (0, 32), we get

$$F = \frac{9}{5}C + 32.$$

To convert 30 degrees Celsius into Fahrenheit, substitute $C = 30$ in the equation

$$\begin{aligned} \mathrm{F} &= \frac{9}{5}\mathrm{C} + 32 \\ \mathrm{F} &= \frac{9}{5}(30) + 32 = 86 \end{aligned}$$

Example 1.4.5

The population of Canada in the year 1980 was 24.5 million, and in the year 2010 it was 34 million. The population of Canada over that time period can be approximately modelled by a linear function. Let x represent time as the number of years after 1980 and let y represent the size of the population.

a. Write the linear function that gives a relationship between the time and the population.
b. Assuming the population continues to grow linearly in the future, use this equation to predict the population of Canada in the year 2025.

Solution

The problem can be made easier by using 1980 as the base year, that is, we choose the year 1980 as the year zero. This will mean that the year 2010 will correspond to year 30. Now we look at the information we have:

Year	Population
0 (1980)	24.5 million

30 (2010)	34 million

a. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 24.5) and (30, 34). We use these two points to find the slope:

$$m=\frac{34-24.5}{30-0}=\frac{9.5}{30}=0.32$$

The y-intercept occurs when $x=0$, so $b=24.5$

$$y=0.32x+24.5$$

b. Now to predict the population in the year 2025, we let $x=2025-1980=45$

$$\begin{array}{l}y=0.32x+24.5\\ y=0.32(45)+24.5=38.9\end{array}$$

In the year 2025, we predict that the population of Canada will be 38.9 million people.

Note that we assumed the population trend will continue to be linear. Therefore if population trends change and this assumption does not continue to be true in the future, this prediction may not be accurate.

1.4.1: Applications (Exercises)

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

1) The variable cost to manufacture a product is \$25 per item, and the fixed costs are \$1200. If x is the number of items manufactured and y is the cost, write the cost function.	2) It costs \$90 to rent a car driven 100 miles and \$140 for one driven 200 miles. If x is the number of miles driven and y the total cost of the rental, write the cost function.
3) The variable cost to manufacture an item is \$20 per item, and it costs a total of \$750 to produce 20 items. If x represents the number of items manufactured and y is the cost, write the cost function.	4) To manufacture 30 items, it costs \$2700, and to manufacture 50 items, it costs \$3200. If x represents the number of items manufactured and y the cost, write the cost function.
5) To manufacture 100 items, it costs \$32,000, and to manufacture 200 items, it costs \$40,000. If x is the number of items manufactured and y is the cost, write the cost function.	6) It costs \$1900 to manufacture 60 items, and the fixed costs are \$700. If x represents the number of items manufactured and y the cost, write the cost function.

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

7) A person who weighs 150 pounds has 60 pounds of muscles; a person that weighs 180 pounds has 72 pounds of muscles. If x represents body weight and y is muscle weight, write an equation describing their relationship. Use this relationship to determine the muscle weight of a person that weighs 170 pounds.	8) A spring on a door stretches 6 inches if a force of 30 pounds is applied. It stretches 10 inches if a 50 pound force is applied. If x represents the number of inches stretched, and y is the force, write a linear equation describing the relationship. Use it to determine the amount of force required to stretch the spring 12 inches.
9). A male college student who is 64 inches tall weighs 110 pounds. Another student who is 74 inches tall weighs 180 pounds. Assuming the relationship between male students' heights (x), and weights (y) is linear, write a function to express weights in terms of heights, and use this function to predict the weight of a student who is 68 inches tall.	10) EZ Clean company has determined that if it spends \$30,000 on advertising, it can hope to sell 12,000 of its Minivacs a year, but if it spends \$50,000, it can sell 16,000. Write an equation that gives a relationship between the number of dollars spent on advertising (x) and the number of minivacs sold(y).
11) The freezing temperatures for water for Celsius and Fahrenheit scales are 0ºC and 32ºF. The boiling temperatures for water are 100 ºC and 212 ºF. Let C denote the temperature in Celsius and F in Fahrenheit. Write the conversion function from Celsius to Fahrenheit. Use the function to convert 25 ºC into ºF.	12) By reversing the coordinates in the previous problem, find a conversion function that converts Fahrenheit into Celsius, and use this conversion function to convert 72 ºF into an equivalent Celsius measure.

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

13) California’s population was 29.8 million in the year 1990, and 37.3 million in 2010. Assume that the population trend was and continues to be linear, write the population function. Use this function to predict the population in 2025. *Hint: Use 1990 as the base year (year 0); then 2010 and 2025 are years 20, and 35, respectively.)*	14) Use the population function for California in the previous problem to find the year in which the population will be 40 million people.

15) A college's enrollment was 13,200 students in the year 2000, and 15,000 students in 2015. Enrollment has followed a linear pattern. Write the function that models enrollment as a function of time. Use the function to find the college's enrollment in the year 2010. *Hint: Use year 2000 as the base year.*	16) If the college's enrollment continues to follow this pattern, in what year will the college have 16,000 students enrolled.
17) The cost of electricity in residential homes is a linear function of the amount of energy used. In Grove City, a home using 250 kilowatt hours (kwh) of electricity per month pays $55. A home using 600 kwh per month pays $118. Write the cost of electricity as a function of the amount used. Use the function to find the cost for a home using 400 kwh of electricity per month.	18) Find the level of electricity use that would correspond to a monthly cost of $100.

SECTION 1.4 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

19) At ABC Co., sales revenue is $170,000 when it spends $5000 on advertising. Sales revenue is $254,000 when $12,000 is spent on advertising. a) Find a linear function for y = amount of sales revenue as a function of x = amount spent on advertising. b) Find revenue if $10,000 is spent on advertising. c) Find the amount that should be spent on advertising to achieve $200,000 in revenue.	20) For problem 19, explain the following: a. Explain what the slope of the line tells us about the effect on sales revenue of money spent on advertising. Be specific, explaining both the number and the sign of the slope in the context of this problem. b. Explain what the y intercept of the line tells us about the sales revenue in the context of this problem
21) Mugs Café sells 1000 cups of coffee per week if it does not advertise. For every $50 spent in advertising per week, it sells an additional 150 cups of coffee. a) Find a linear function that gives y = number of cups of coffee sold per week x = amount spent on advertising per week. b) How many cups of coffee does Mugs Café expect to sell if $100 per week is spent on advertising?	22) Party Sweets makes baked goods that can be ordered for special occasions. The price is $24 to order one dozen (12 cupcakes) and $9 for each additional 6 cupcakes. a. Find a linear function that gives the total price of a cupcake order as a function of the number of cupcakes ordered b. Find the price for an order of 5 dozen (60) cupcakes

1.5: More Applications

Learning Objectives

In this section, you will learn to:

1. Solve a linear system in two variables.
2. Find the equilibrium point when a demand and a supply equation are given.
3. Find the break-even point when the revenue and the cost functions are given.

Finding the Point of Intersection of Two Lines

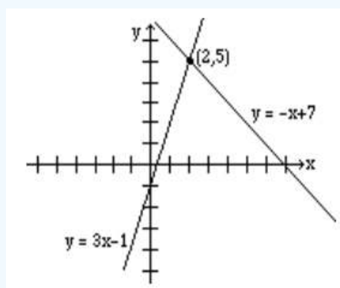
In this section, we will do application problems that involve the intersection of lines. Therefore, before we proceed any further, we will first learn how to find the intersection of two lines.

Example 1.5.1

Find the intersection of the line $y = 3x - 1$ and the line $y = -x + 7$.

Solution

We graph both lines on the same axes, as shown below, and read the solution (2, 5).



Finding an intersection of two lines graphically is not always easy or practical; therefore, we will now learn to solve these problems algebraically.

At the point where two lines intersect, the x and y values for both lines are the same. So in order to find the intersection, we either let the x-values or the y-values equal.

If we were to solve the above example algebraically, it will be easier to let the y-values equal. Since $y = 3x - 1$ for the first line, and $y = -x + 7$ for the second line, by letting the y-values equal, we get

$$\begin{aligned} 3x - 1 &= -x + 7 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

By substituting $x = 2$ in any of the two equations, we obtain $y = 5$.

Hence, the solution (2, 5).

A common algebraic method used to solve systems of equations is called the **elimination method**. The object is to eliminate one of the two variables by adding the left and right sides of the equations together. Once one variable is eliminated, we have an equation with only one variable for can be solved. Finally, by substituting the value of the variable that has been found in one of the original equations, we get the value of the other variable.

Example 1.5.2

Find the intersection of the lines $2x+y=7$ and $3x-y=3$ by the elimination method.

Solution

We add the left and right sides of the two equations.

$$\begin{array}{r} 2x+y=7 \\ 3x-y=3 \\ \hline 5x=10 \\ x=2 \end{array}$$

Now we substitute $x=2$ in any of the two equations and solve for y.

$$\begin{aligned} 2(2)+y &= 7 \\ y &= 3 \end{aligned}$$

Therefore, the solution is (2, 3).

Example 1.5.3

Solve the system of equations $x+2y=3$ and $2x+3y=4$ by the elimination method.

Solution

If we add the two equations, none of the variables are eliminated. But the variable x can be eliminated by multiplying the first equation by -2, and leaving the second equation unchanged.

$$\begin{array}{r} -2x-4y=-6 \\ 2x+3y=4 \\ \hline -y=-2 \\ y=2 \end{array}$$

Substituting $y=2$ in $x+2y=3$, we get

$$\begin{aligned} x+2(2)=3 \\ x=-1 \end{aligned} \tag{1.5.1}$$

Therefore, the solution is (-1, 2).

Example 1.5.4

Solve the system of equations $3x-4y=5$ and $4x-5y=6$.

Solution

This time, we multiply the first equation by - 4 and the second by 3 before adding. (The choice of numbers is not unique.)

$$\begin{array}{r} -12x+16y=-20 \\ 12x-15y=18 \\ \hline y=-2 \end{array}$$

By substituting y = - 2 in any one of the equations, we get x = -1.
Hence the solution is (-1, -2).

SUPPLY, DEMAND AND THE EQUILIBRIUM MARKET PRICE

In a free market economy the supply curve for a commodity is the number of items of a product that can be made available at different prices, and the demand curve is the number of items the consumer will buy at different prices.

As the price of a product increases, its demand decreases and supply increases. On the other hand, as the price decreases the demand increases and supply decreases. The **equilibrium price** is reached when the demand equals the supply.

Example 1.5.5

The supply curve for a product is $y = 3.5x - 14$ and the demand curve for the same product is $y = -2.5x + 34$, where x is the price and y the number of items produced. Find the following.

a. How many items will be supplied at a price of $10?
b. How many items will be demanded at a price of $10?
c. Determine the equilibrium price.
d. How many items will be produced at the equilibrium price?

Solution

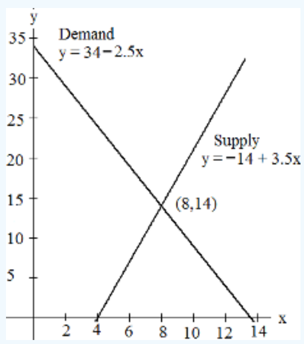
a) We substitute $x = 10$ in the supply equation, $y = 3.5x - 14$; the answer is $y = 3.5(10) - 14 = 21$.

b) We substitute $x = 10$ in the demand equation, $y = -2.5x + 34$; the answer is $y = -2.5(10) + 34 = 9$.

c) By letting the supply equal the demand, we get

$$\begin{aligned} 3.5x - 14 &= -25x + 34 \\ 6x &= 48 \\ x &= \$8 \end{aligned}$$

d) We substitute $x = 8$ in either the supply or the demand equation; we get $y = 14$.



The graph shows the intersection of the supply and the demand functions and their point of intersection, (8,14).

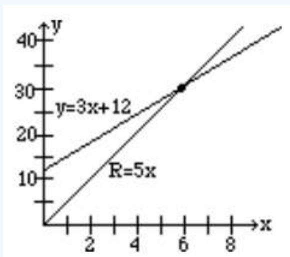
Interpretation: At equilibrium, the price is $8 per item, and 14 items are produced by suppliers and purchased by consumers.

The Break-Even Point

In a business, the profit is generated by selling products.

- If a company sells x number of items at a price P, then the **revenue R** is the price multiplied by number of items sold: $R = P \cdot x$.
- The **production costs C** are the sum of the variable costs and the fixed costs, and are often written as C = mx + b, where x is the number of items manufactured.
-
  - The slope m is the called marginal cost and represents the cost to produce one additional item or unit.
  - The variable cost, mx, depends on how much is being produced
  - The fixed cost b is constant; it does not change no matter how much is produced.
- **Profit** is equal to Revenue minus Cost: Profit = R − C

A company makes a profit if the revenue is greater than the cost. There is a loss if the cost is greater than the revenue. The point on the graph where the revenue equals the cost is called the **break-even point**. At the break-even point, profit is 0.

> **Example 1.5.6**
>
> If the revenue function of a product is $R=5x$ and the cost function is $y=3x+12$, find the following.
>
> a. If 4 items are produced, what will the revenue be?
> b. What is the cost of producing 4 items?
> c. How many items should be produced to break even?
> d. What will be the revenue and the cost at the break-even point?
>
> **Solution**
>
> a) We substitute $x=4$ in the revenue equation $R=5x$, and the answer is $R=20$.
>
> b) We substitute $x=4$ in the cost equation $C=3x+12$, and the answer is $C=24$.
>
> c) By letting the revenue equal the cost, we get
>
> $$\begin{aligned} 5x &= 3x+12 \\ x &= 6 \end{aligned}$$
>
> d) Substitute $x=6$ in either the revenue or the cost equation: we get $R=C=30$.
>
> The graph below shows the intersection of the revenue and cost functions and their point of intersection, (6, 30).
>
> 
>

1.5.1: More Applications (Exercises)

SECTION 1.5 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

1) Solve for x and y. $y = 3x + 4$ $y = 5x - 2$	2) Solve for x and y. $2x - 3y = 4$ $3x - 4y = 5$
3) The supply and demand curves for a product are: Supply $y = 2000x - 6500$ Demand $y = -1000x + 28000$, where x is price and y is the number of items. At what price will supply equal demand and how many items will be produced at that price?	4) The supply and demand curves for a product are Supply $y = 300x - 18000$ and Demand $y = -100x + 14000$, where x is price and y is the number of items. At what price will supply equal demand, and how many items will be produced at that price?
5) A car rental company offers two plans for one way rentals. Plan I charges \$36 per day and 17 cents per mile. Plan II charges \$24 per day and 25 cents per mile. a. If you were to drive 300 miles in a day, which plan is better? b. For what mileage are both rates equal?	

SECTION 1.5 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

6) A demand curve for a product is the number of items the consumer will buy at different prices. At a price of \$2 a store can sell 2400 of a particular type of toy truck. At a price of \$8 the store can sell 600 such trucks. If x represents the price of trucks and y the number of items sold, write an equation for the demand curve.	7) A supply curve for a product is the number of items that can be made available at different prices. A manufacturer of toy trucks can supply 2000 trucks if they are sold for \$8 each; it can supply only 400 trucks if they are sold for \$4 each. If x is the price and y the number of items, write an equation for the supply curve.
8) The equilibrium price is the price where the supply equals the demand. From the demand and supply curves obtained in the previous two problems, find the equilibrium price, and determine the number of items that can be sold at that price.	9) A break-even point is the intersection of the cost function and the revenue function, that is, where total cost equals revenue, and profit is zero. Mrs. Jones Cookies Store's cost and revenue, in dollars, for x number of cookies is given by $C = .05x + 3000$ and $R = .80x$. Find the number of cookies that must be sold to break even.

SECTION 1.5 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

10) A company's revenue and cost in dollars are given by $R = 225x$ and $C = 75x + 6000$, where x is the number of items. Find the number of items that must be produced to break-even.	11) A firm producing socks has a fixed cost of \$20,000 and variable cost of \$2 per pair of socks. Let x = the number of pairs of socks. Find the break-even point if the socks sell for \$4.50 per pair.

12) Whackemhard Sports is planning to introduce a new line of tennis rackets. The fixed costs for the new line are $25,000 and the variable cost of producing each racket is $60. x is the number of rackets; y is in dollars. If the racket sells for $80, how many rackets must be sold in order to break even?	13) It costs $1,200 to produce 50 pounds of a chemical and it costs $2,200 to produce 150 pounds. The chemical sells for $15 per pound x is the amount of chemical; y is in dollars. a. Find the cost function. b. What is the fixed cost? c. How many pounds must be sold to break even? d. Find the cost and revenue at the break-even point.

1.6: Chapter Review

SECTION 1.6 PROBLEM SET: CHAPTER REVIEW

1) Find an equation of the x-axis.

2) Find the slope of the line whose equation is $2x + 3y = 6$.

3) Find the slope of the line whose equation is $y = -3x + 5$.

4) Find both the x and y intercepts of the line $3x - 2y = 12$.

5) Find an equation of the line whose slope is 3 and y-intercept 5.

6) Find an equation of the line whose x-intercept is 2 and y-intercept 3.

7) Find an equation of the line that has slope 3 and passes through the point (2, 15).

8) Find an equation of the line that has slope -3/2 and passes through the point (4, 3).

9) Find an equation of the line that passes through the points (0, 32) and (100, 212).

10) Find an equation of the line that passes through the point (2, 5) and is parallel to the line $y = 3x + 4$.

11) Find the point of intersection of the lines $2x - 3y = 9$ and $3x + 4y = 5$.

12) Is the point (3, - 2) on the line $5x - 2y = 11$?

13) Find two points on the line given by the parametric equations, $x = 2 + 3t$, $y = 1 - 2t$.

14) Find two points on the line $2x - 6 = 0$.

15) Graph the line $2x - 3y + 6 = 0$.

16) Graph the line $y = -2x + 3$.

17) A female college student who is 60 inches tall weighs 100 pounds. Another female student who is 66 inches tall weighs 124 pounds. Assume the relationship between the female students' weights and heights is linear. Find an equation for weight as a function of height. Use this relationship to predict the weight of a female student who is 70 inches tall.

18) In deep-sea diving, the pressure exerted by water plays a great role in designing underwater equipment. If at a depth of 10 feet there is a pressure of 21 lb/in^2, and at a depth of 50 ft there is a pressure of 75 lb/in^2, write a linear equation giving a relationship between depth and pressure. Use this relationship to predict pressure at a depth of 100 ft.

19) The variable cost to manufacture an item is \$30 per item; the fixed costs are \$2750. Find the cost function.

20) The variable cost to manufacture an item is \$10 per item, and it costs \$2,500 to produce 100 items. Write the cost function, and use this function to estimate the cost of manufacturing 300 items.

21) It costs \$2,700 to manufacture 100 items of a product, and \$4,200 to manufacture 200 items.
x= the number of items; y= cost. Find the cost function; use it to predict the cost to produce 1000 items.

22) In 1990, the average house in Emerald City cost \$280,000 and in 2007 the same house cost \$365,000. Assuming a linear relationship, write an equation that will give the price of the house in any year, and use this equation to predict the price of a similar house in the year 2020.

23) The population of Mexico in 1995 was 95.4 million and in 2010 it was 117.9 million. Assuming a linear relationship, write an equation that will give the population of Mexico in any year, and use this equation to predict the population of Mexico in the year 2025.

SECTION 1.6 PROBLEM SET: CHAPTER REVIEW Word Problems

24) At Nuts for Soup Lunch Bar, they sell 150 bowls of soup if the high temperature for the day is 40 ºF. For every 5 ºF increase in high temperature for the day, they sell 10 fewer bowls of soup.

a. Assuming a linear relationship, write an equation that will give y = the number of bowls of soup sold as a function of x = the daily high temperature.
b. How many bowls of soup are sold when the temperature is 75 ºF?

c. What is the temperature when 100 bowls of soup are sold?

25) Two hundred items are demanded at a price of \$5, and 300 items are demanded at a price of \$3. If x represents the price, and y the number of items, write the demand function.

26) A supply curve for a product is the number of items of the product that can be made available at different prices. A doll manufacturer can supply 2000 dolls if the dolls are sold for \$30 each, but he can supply only 400 dolls if the dolls are sold for \$10 each. If x represents the price of dolls and y the number of items, write an equation for the supply curve.

27) Suppose you are trying to decide on a price for your latest creation - a coffee mug that never tips. Through a survey, you have determined that at a price of \$2, you can sell 2100 mugs, but at a price of \$12 you can only sell 100 mugs. Furthermore, your supplier can supply you 3100 mugs if you charge your customers \$12, but only 100 mugs if you charge \$2. What price should you charge so that the supply equals demand, and at that price how many coffee mugs will you be able to sell?

28) A car rental company offers two plans. Plan I charges \$16 a day and 25 cents a mile, while Plan II charges \$45 a day but no charge for miles. If you were to drive 200 miles in a day, which plan is better? For what mileage are both rates the same?

29) The supply curve for a product is $y = 250x - 1000$. The demand curve for the same product is
$y = -350x + 8,000$, where x is the price and y the number of items produced. Find the following.

a. At a price of \$10, how many items will be in demand?
b. At what price will 4,000 items be supplied?
c. What is the equilibrium price for this product?
d. How many items will be manufactured at the equilibrium price?

30) The supply curve for a product is $y = 625x - 600$ and the demand curve for the same product is
$y = -125x + 8,400$, where x is the price and y the number of items produced.
Find the equilibrium price and determine the number of items that will be produced at that price.

31) Both Jenny and Masur work in the sales department for Sports Supply. Jenny gets paid \$120 per day plus 4% commission on the sales. Masur gets paid \$132 per day plus 8% commission on the sales in excess of \$1,000. For what sales amount would they both earn the same daily amounts?

32) A company's revenue and cost in dollars are given by $R = 25x$ and $C = 10x + 9,000$, where x represents the number of items. Find the number of items that must be produced to break-even.

33) A firm producing a certain type of CFL lightbulb has fixed costs of \$6,800, and a variable cost of \$2.30 per bulb. The bulbs sell for \$4 each. How many bulbs must be produced to break-even?

34) A company producing tire pressure gauges has fixed costs of \$7,500, and variable cost of \$1.50 cents per item. If the gauges sell for \$4.50, how many must be produced to break-even?

35) A company is introducing a new cordless travel shaver before the Christmas holidays. It hopes to sell 15,000 of these shavers in December. The variable cost is \$11 per item and the fixed costs \$100,000. If the shavers sell for \$19 each, how many must be produced and sold to break-even?

CHAPTER OVERVIEW

2: Inequalities

2.1: Properties of Inequalities

2.2: Solving and Graphing Inequalities, and Writing Answers in Interval Notation

2.3: Set Builder Notation

2.4: Rational Inequalities

2.1: Properties of Inequalities

Here are some important properties of inequalities:

Properties of Inequalities

If a, b, and c are real numbers, then:

Transitive Property if $a < b$ and $b < c$ then $a < c$

Addition Property if $a < b$ then $a + c < b + c$

Subtraction Property if $a < b$ then $a - c < b - c$

Multiplication Property (Multiplying by a positive number) if $a < b$ and $c > 0$ then $ac < bc$

Multiplication Property (Multiplying by a negative number) if $a < b$ and $c < 0$ then $ac > bc$

Division Property (Dividing by a positive number) if $a < b$ and $c > 0$ then $\frac{a}{c} < \frac{b}{c}$

Division Property (Dividing by a negative number) if $a < b$ and $c < 0$ then $\frac{a}{c} > \frac{b}{c}$

Example 2.1.1

Transitive Property

If 3 < 7 and 7 < 14 then...

Solution

3 < 14

Example 2.1.2

Addition Property

If 3 < 7, then add 4 to both sides.

Solution

3 + 4 < 7 + 4

7 < 11

Example 2.1.3

Subtraction Property

If 3 < 7, subtract 6 on both sides

Solution

3 < 7

3 - 6 < 7 - 6

-3 < 1

Example 2.1.4

Multiplication Property (Multiplying by a positive number)

If 3 < 7, multiply both sides by 5.

Solution

3 < 7

3 * 5 < 5 * 7

15 < 35

✓ Example 2.1.5

Multiplication Property (Multiplying by a negative number)

If 3 < 7, multiply both sides by -4.

Solution

3 < 7

3 * -4 ? -4 * 7

-12 ? -28

-12 > -28 **The direction of the inequality is changed.**

✓ Example 2.1.6

Division Property (Dividing by a positive number)

If 6 < 8, divide both sides by 2.

Solution

6/2 < 8/2

3 < 4

✓ Example 2.1.7

Division Property (Dividing by a negative number)

If 9 < 15, divide both sides by -3.

Solution

9 < 15

9/-3 ? 15/-3

-3 ? -5

-3 > -5 **The direction of the inequality is changed.**

2.2: Solving and Graphing Inequalities, and Writing Answers in Interval Notation

2.2.1: Interval Notation

Definition: How to write inequalities in Interval Notation

- **Symbol to not include a value are (or).**
- **Symbol to include a value are [or].**
- **Use the Symbol (or) when using infinity (-∞, ∞).**

---

1. **x < -4** The region is from negative infinity (-∞) up to and not including -4. Therefore, the **Interval Notation is (-∞, -4)**.
2. **x ≤ -4** The region is from negative infinity (-∞) up to and including -4. There for , the **Interval Notation is (-∞, -4].**
3. **-3 < x < 2** The region is from -3 up to and not including 2. Therefore, the **Interval Notation is (-3, 2).**
4. **-3 ≤ x ≤ 2** The region is including -3 and up to and including 2. Therefore, the **Interval Notation is [-3, 2].**
5. **x > 5** The region is greater than and not including 5. Therefore, the **Interval Notation is (5, ∞).**
6. **x ≥ 5** The region is greater than and including 5. Therefore, the **Interval Notation is [5, ∞).**
7. **-3 < x ≤ 10** The region is greater than and not including 3 but less than or equal to 10. Therefore, the **Interval Notation is (-3, 10].**
8. **4 ≤ x < 7** The region is greater than and including 4 and less than not including 7. Therefore, the **Interval Notation is [4, 7).**

2.2.2: To solve and graph inequalities:

1. Solve the inequality using the Properties of Inequalities from the previous section.
2. Graph the solution set on a number line.
3. Write the solution set in interval notation.

Example 2.2.1

Solve the inequality, graph the solution set on a number line and show the solution set in interval notation:

a. $-1 \leq 2x-5<7$
b. $x^2+7x+10<0$
c. $-6<x-2<4$

Solution

a.
$$\begin{array}{ll}
-1 \leq 2x-5<7 & \text{Example problem} \\
-1+5 \leq 2x-5+5<7+5 & \text{The goal is to isolate the variable } x\text{, so start by adding 5 to all three regions in the inequality.} \\
4 \leq 2x<12 & \text{Simplify.} \\
\frac{4}{2} \leq 2x<\frac{4}{2} & \text{Divide all by 2 to isolate the variable } x. \\
2 \leq x<6 & \text{Final answer written in inequality/solution set form.} \\
[2,6) & \text{Final answer written in interval notation (see section on Interval Notation for more details)}
\end{array}$$

b.
$$\begin{array}{ll}
x^2+7x+10<0 & \text{Example problem} \\
(x+5)(x+2)<0 & \text{Factor the polynomial.} \\
(x+5)(x+2)<0 & \text{Set each factor equal to 0 and solve for x. x = -5 and x = -2.}
\end{array}$$

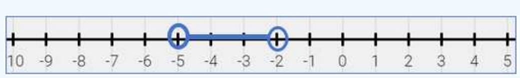
Since the inequality is a strict inequality($<$ or $>$), -5 and -2 is not included in the solution set.

Pick a point less than -5, x = -6. Evaluate the expression (-6+5)(-6+2) = (-1)(-4) = 4. It is positive, x < -5 is not in the solution set.

Pick a point between -5 and x = -2. Evaluate the expression (-4 + 5)(-4+2) = (1)(-2) = -2. It is negative, -5 < x < -2 is in the solution set.

Finally, pick a point greater than -2, x = 0. Evaluate the expression (0 + 5)(0 + 2) = 5(2) = 10. It is positive, x > -2 is not the solution set.

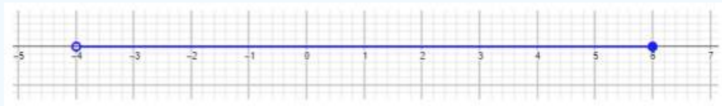
Interval notation: (-5, -2)



-5 < x < -2

c.

$-6 < x - 2 \leq 4$	Example problem
$-6 + 2 < x - 2 + 2 \leq 4 + 2$	The goal is to isolate the variable x, so start by adding 2 to all three regions in the inequality.
$-4 < x \leq 6$	Final answer written in inequality/solution set form.
$(-4, 6]$	Final answer written in interval notation (see section on Interval Notation for more details).



Exercise Problems 2.2.1

Solve the inequalities, graph the solution sets on a number line and show the solution sets in interval notation:

1. $0 \leq x + 1 \leq 4$
2. $0 < 2(x - 1) \leq 4$
3. $6 < 2(x - 1) < 12$
4. $x^2 - 6x - 16 < 0$
5. $2x^2 - x - 15 > 0$

Detail answers below

Exercise Answers 2.2.1

1. $0 \leq x + 1 \leq 4$

Answer

$0 \leq x + 1 \leq 4$

subtract 1 from all three sides

$0 - 1 \leq x + 1 - 1 \leq 4 - 1$

$-1 \leq x + 0 \leq 3$

$-1 \leq x \leq 3$

[-1, 3] Interval Notation

Exercise Answers 2.2.2

2. $0 < 2(x - 1) \leq 4$

Answer

Multiply the 2 through the parenthesis in the middle section.

$0 < 2x - 2 \leq 4$

Add 2 to all three sides

$0 + 2 < 2x - 2 + 2 \leq 4 + 2$

$2 < 2x + 0 \leq 6$

$2 < 2x \leq 6$

Divide all three sides by 2

$2/2 < 2x/2 \leq 6/2$

$1 < x \leq 3$

(1, 3] Interval Notation

Exercise 2.2.3

3. $6 < 2(x - 1) < 12$

Answer

Add texts here. Do not delete this text first.

Multiply the 2 through the parenthesis in the middle section.

$6 < 2x - 2 < 12$

Add 2 to all three sides

$6+2<2x-2+2<12+2$

$8<2x+0<14$

$8<2x<14$

Divide all sides by 2

$8/2<2x/2<14/2$

$4<x<7$

(4, 7) Interval Notation

? Exercise 2.2.4

4. $x^2-6x-16<0$

Answer

Factor $x^2-6x-16$ to (x - 8)(x + 2).

Set x - 8 = 0 and solve for x

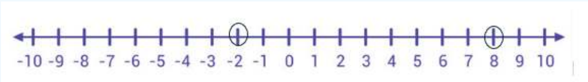
x - 8 = 0   Add 8 to both sides

x - 8 + 8 = 0 + 8
x + 0 = 8
x = 8

Set x + 2 = 0 and solve for x

x + 2 = 0   Subtract 2 from both sides
x + 2 - 2 = 0 - 2
x + 0 = -2
x = -2

Create a number line with the two points selected. Since it is a strict inequality, <, the points will not be included.



Select a point to the left of x = -2, -3. Substitute it in the factored form of the quadratic equation.

(-3 - 8)(-3 +2) = (-11)(-1) = 11 > 0. For values to the left of x = -2 the function is positive.

Select a point between x = -2 and x = 8, 0. Substitute it in the factored form of the quadratic equation.

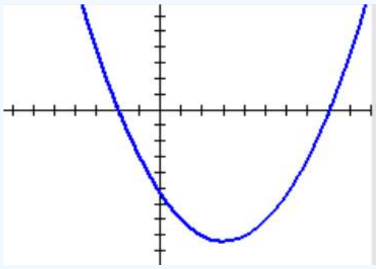
(0 -8)(0 + 2) = (-8)(2) = -16 < 0.   For values between -2 and 8, the function is negative.

Select a point to the right of x = 8, 9. Substitute the value in the factored form of the quadratic equation.

(9 - 8)(9 + 2) = (1)(11) = 11 > 0. For values to the right of x = 8 the function is positive.

The solution is in the interval -2 < x < 8, or (-2, 8) Interval notation.

2.2.3: Graph of quadratic equation



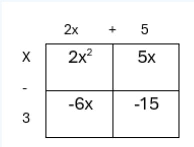
Note the equation is negative between -2 and 8. Therefore, **the solution interval is -2 < x < 8 or (-2, 8) in Interval notation.**

Exercise 2.2.5

5. $2x^2 - x - 15 > 0$

Answer

Using the box method we can factor the $2x^2 - x - 15 > 0$.



$2x^2 - x - 15$ = (2x + 5)(x - 3)

Set each factor equal to 0 and solve for x

2x + 5 = 0 subtract 5 from both sides

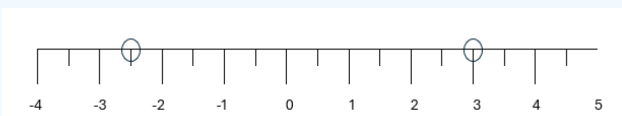
2x + 5 - 5 = 0 - 5
2x + 0 = -5
2x = -5 divide both sides by 2

2x/2 = -5/2

x = -5/2 or -2.5

x - 3 = 0 add 3 to both sides
x - 3 + 3 = 0 + 3
x + 0 = 3
x = 3

Divide the number line into three regions



Select a number to the left of -2.5, -3. Substitute the value in the factored form of the quadratic equation.

(2(-3) + 5)(-3 - 3) = (-6 + 5)(-3 - 3) = (-1)(-6) = 6 > 0. The region to the left of -2.5 is positive.

Select a number between -2.5 and 3, 0. Substitute the value in the factored form of the quadratic equation.

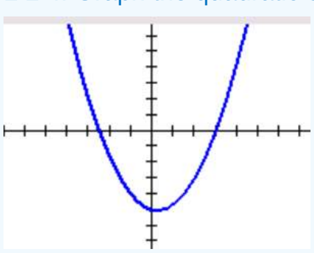
(2(0) + 5)(0 - 3) = (0 + 5)(-3) = (5)(-3) = -15 < 0. The region between -2.5 and 3 is negative.

Select a number to the right of 3, 4. Substitute the value in the factored form of the quadratic equation.

(2(4) + 5)(5 - 3) = (8 + 5)(2) = (13)(2) = 26 > 0. The region to the right of 3 is positive.

The solution is $(-\infty, -2.5) \cup (3, \infty)$.

2.2.4: Graph the quadratic equation



You can see the graph is positive to the left of -2.5 and to the right of 3. The solution in interval notation is $(-\infty, -2.5) \cup (3, \infty)$.

2.3: Set Builder Notation

Definition: Set Builder Notation

There is another way we can write interval notation. It is called set-builder notation. Set builder notation is the descriptive definition of the set.

- x < 4 Interval Notation: (-∞, 4) Set Builder Notation: {x| x is a real number less than 4}
- 3 < x ≤ 8 Interval Notation: (3, 8] Set Builder Notation: {x| x is a real number greater than 3 and less than or equal to 8}
- x ≥ 100 Interval Notation: [100, ∞) Set Builder Notation: {x | x is a real number greater than or equal to 100}

Exercise 2.3.1

Write the following in Set Builder Notation.

1. x > 6 Interval Notation: (6, ∞)

Answer

Set Builder Notation: {x| x is a real number greater than 6}

Exercise 2.3.2

Write the following in Set Builder Notation.

1. -4 ≤ x < 12 Interval Notation: [-4, 12)

Answer

Set Builder Notation: {x|x is a real number greater than or equal to -4 and less than 12}

Exercise 2.3.3

Write the following in Set Builder Notation.

1. x ≤ 30 Interval Notation: (-∞, 30]

Answer

Add texts here. Do not delete this text first.

Set Builder Notation: {x|x is a real number less than or equal to 30}

2.4: Rational Inequalities

Solving rational inequalities involves finding the zeroes of the numerator and denominator, then using these values to investigate solution set regions on the number line.

Example 2.4.1

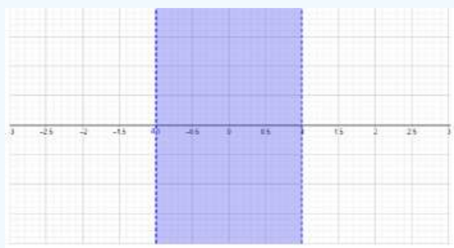
Solve the inequalities and write the solution sets in interval notation:

a. $\frac{x-1}{x+1} \geq 0$
b. $\frac{2x-3}{x+1} \leq 0$
c. $\frac{x+2}{x-2} \geq 0$

Solution

a.

$\frac{x-1}{x+1} \geq 0$	Example problem
$\frac{x-1}{x+1} \geq 0$	The quotient must be greater than or equal to 0.
$x-1=0,\ x=1$	Find the zeroes of the numerator
$x+1=0, x=\ -1$	Find the zeroes of the denominator



The zeroes divide the number line into 3 regions, $x<-1$, $-1<x<1$, $x>1$

For $x<-1$, choose $x=-2$. $\frac{-2-1}{-2+1}=\frac{-3}{-1}=3\geq 0$
Replacing -2 for x results in the answer 3, which is greater than or equal to 0. This region $x<-1$ is included in the solution set.

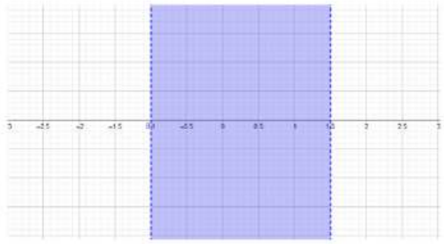
For $-1<x<1$, choose $x=0$. $\frac{0-1}{0+1}=\frac{-1}{1}=-1<0$
Replacing 0 for x results in the answer -1, which is less than 0, not fulfilling the given inequality in the problem. This region $-1<x<1$ is excluded from the solution set.

For $x>1$, choose $x=2$. $\frac{2-1}{2+1}=\frac{1}{3}\geq 0$
Replacing 2 for x results in the answer $\frac{1}{3}$, which is greater than or equal to 0. This region $x>1$ is included in the solution set.

$(-\infty,-1)\cup(1,\infty)$
Final answer written in interval notation (see section on Interval Notation for more details).

b.

$\frac{2x-3}{x+1} \leq 0$	Example problem
$\frac{2x-3}{x+1} \leq 0$	The quotient must be less than or equal to 0.
$2x-3=0,\ x=1.5$	Find the zeroes of the numerator
$x+1=0,\ x=-1$	Find the zeroes of the denominator



The zeroes divide the number line into 3 regions, $x<-1$, $-1<x<1.5$, $x>1.5$

For $-1<x$, choose $x=-2$. $\dfrac{2(-2)-3}{-2+1}=\dfrac{-7}{-1}=7\geq 0$

Replacing -2 for x results in the answer 7, which is greater than or equal to 0. This region $-1<x$ is excluded in the solution set.

For $-1<x<1.5$, choose $x=0$. $\dfrac{2(0)-3}{0+1}=\dfrac{-3}{1}=-3\leq 0$

Replacing 0 for x results in the answer -3, which is less than or equal to 0. This region $-1<x<1.5$ is included in the solution set.

For $x>1.5$, choose $x=2$. $\dfrac{2(2)-3}{2+1}=\dfrac{1}{3}\geq 0$

Replacing 2 for x results in the answer $\dfrac{1}{3}$, which is greater than or equal to 0. This region $x>1.5$ is excluded in the solution set.

(-1, 1.5)

Final answer written in interval notation (see section on Interval Notation for more details).

c.

$\dfrac{x+2}{x-2}\geq 0$	Example problem
$\dfrac{x+2}{x-2}\geq 0$	The quotient must be greater than or equal to 0.
$x+2=0,\quad x=-2$	Find the zeroes of the numerator
$x-2=0,\quad x=2$	Find the zeroes of the denominator

The zeroes divide the number line into 3 regions, $x<-2$, $-2<x<2$, $x>2$

For $x<-2,$ choose $x=-3. \dfrac{-3+2}{-3-2}=\dfrac{-1}{-5}=\dfrac{1}{5}\geq 0$

Replacing -3 for x results in the answer $\dfrac{1}{5}$, which is greater than or equal to 0. This region $x<-2$ is included in the solution set.

For $-2<x<2,$ choose $x=0.$ $\dfrac{0+2}{0-2}=\dfrac{2}{-2}=-1<0$

Replacing 0 for x results in the answer -1, which is less than 0, not fulfilling the given inequality in the problem. This region $-2<x<2$ is not included in the solution set.

For $x>2,$ choose $x=3.$ $\dfrac{3+2}{3-2}=\dfrac{5}{1}=5\geq 0$

Replacing 3 for x results in the answer 5, which is greater than or equal to 0. This region $x>2$ is included in the solution set.

$(-\infty,-2)\cup(2,\infty)$

Final answer written in interval notation (see section on Interval Notation for more details).

Exercise 2.4.1

1. $\dfrac{x+3}{x-2}\geq 0$
2. $\dfrac{x-2}{x-1}\leq 0$
3. $\dfrac{8}{x+2}\leq\dfrac{1}{x+2}$
4. $\dfrac{2x-3}{x+1}\geq 0$

Exercise Solutions 2.4.1

$\dfrac{x+3}{x-2}\geq 0$

Solution

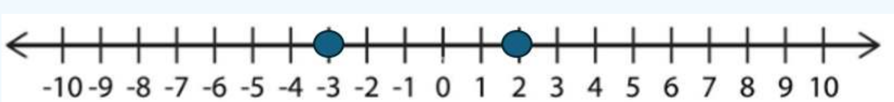
Set x + 3 = 0 and solve for x.

x + 3 = 0
 -3   -3
x = -3

Set x - 2 = 0 and solve for x.
x - 2= 0
 2   2
X + 0 = 2
x = 2

Because we have the line under the inequality, ≥, we will include -3 and 2 in the solution.

The number line is divided into three regions. Left of -3, between -3 and 2, to the right of 2. Pick a point in each region.



x = -4

$\dfrac{-4+3}{-4-2}=\dfrac{-1}{-6}=\dfrac{1}{6}\geq 0.$ The area to the left of -3 is a solution. The interval notation is $(-\infty,-3]$

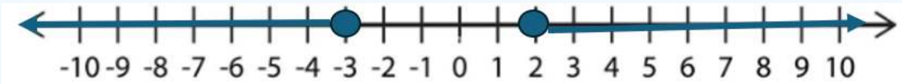
x = 0

$\dfrac{0+3}{0-2}=\dfrac{3}{-2}<0.$ The area between -3 and 2 is not a solution.

x = 3

$\frac{3+3}{3-2}$ = $\frac{6}{1}$ = 6 ≥ 0. The area to the right of 2 is a solution. The interval notation is $[2, \infty)$

The solution is $(-\infty, -3] \cup [2, \infty)$ in Interval Notation.



? Exercise 2.4.1

$\frac{8}{x+2} \leq \frac{1}{x+2}$

Answer

Add texts here. Do not delete this text first.

Subtract $\frac{1}{x-2}$ from both sides

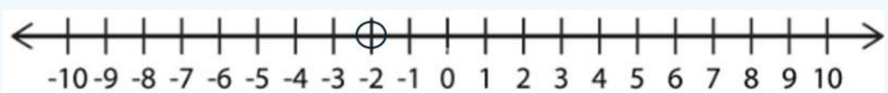
$\frac{8}{x+2}$ - $\frac{1}{x+2}$ ≤ 0

$\frac{7}{x+2}$ ≤ 0

There is only a variable in the denominator so we set x + 2 = 0 and solve for x.

x + 2 = 0
  -2   -2
x + 0 = -2
x = -2

Since x = -2 results in 0 in the denominator x = -2 is not a part of the solution. -2 breaks the number line into two parts: to the left of -2 and to the right of -2.



Pick a number both regions.

x = -3 and substitute it in the rational expression.

$\frac{8}{-3+2}$ = $\frac{8}{-1}$ = -8

$\frac{1}{-3+2}$ = $\frac{1}{-1}$ = -1

-8 ≤ -1 The region to the left of -2 is part of the solution. Interval Notation $(-\infty, -2)$.

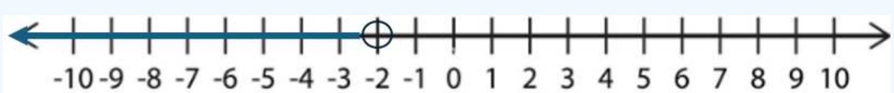
x = 0

$\frac{8}{0+2}$ = $\frac{8}{2}$ = 4

$\frac{1}{0+2}$ = $\frac{1}{2}$

4 ≥ $\frac{1}{2}$ Therefore, this region is not a part of the solution.

The solution is $(-\infty, -2)$.



CHAPTER OVERVIEW

3: Functions

3.1: Functions and Function Notation
3.2: Domain and Range
3.3: Piecewise-Defined Functions
3.4: Absolute Value
3.5: Absolute Value Equations
3.6: Break-Even Analysis (Sink or Swim)
3.7: Applications
3.7.1: Applications (Exercises)
3.8: More Applications
3.8.1: More Applications (Exercises)
3.9: Quadratic Functions
3.9E: Quadratic Functions (Exercises)
3.10: Power Functions and Polynomial Functions
3.11: Graphs of Polynomial Functions
3.12: Graphing Rational Functions

3.1: Functions and Function Notation

Learning Objectives

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\} \tag{1.1.1}$$

The domain is $\{1,2,3,4,5\}$. The range is $\{2,4,6,8,10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no x-values are repeated. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\} \tag{1.1.2}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$. For example, the term "odd" corresponds to three values from the range, $\{1,3,5\}$, and the term "even" corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure $3.1.1$ compares relations that are functions and not functions.

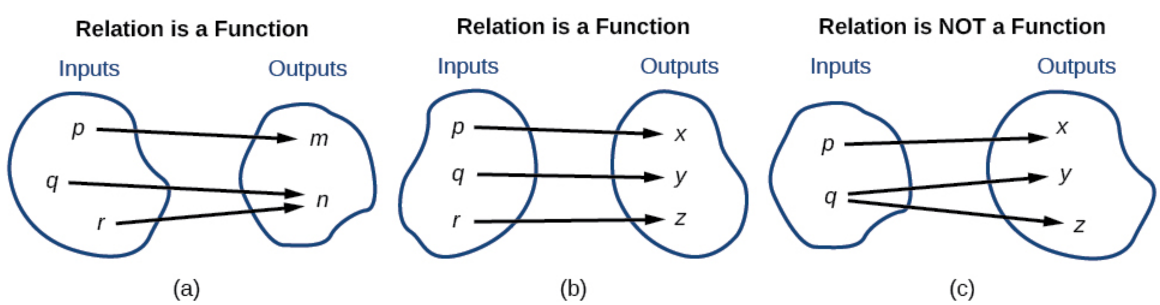


Figure $3.1.1$: (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.

How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example 3.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown inFigure 3.1.2 consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

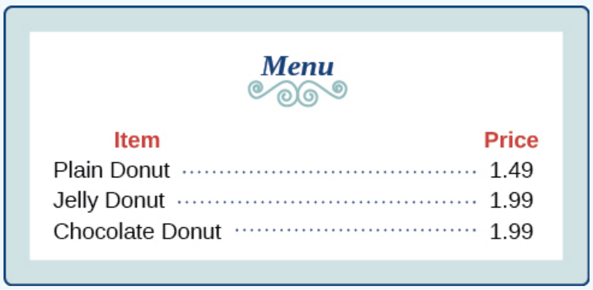


Figure 3.1.2 A menu of donut prices from a coffee shop where a plain donut is \$1.49 and a jelly donut and chocolate donut are \$1.99.

Solution

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure 3.1.3.

Figure 3.1.3 A menu of donut prices from a coffee shop where a plain donut is \$1.49 and a jelly donut and chocolate donut are \$1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure 3.1.4.

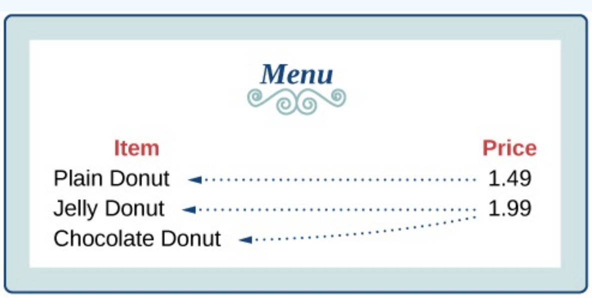


Figure 3.1.4 Association of the prices to the donuts.

Therefore, the item is a not a function of price.

Example 3.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table $3.1.1$ shows a possible rule for assigning grade points.

Table $3.1.1$: Class grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Exercise 3.1.2

Table $3.1.2$ lists the five greatest baseball players of all time in order of rank.

Table $3.1.2$: Five greatest baseball players.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Answer a

Yes

Answer b

yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent “height is a function of age,” we start by identifying the descriptive variables h for height and a for age. The letters f, g,and h are often used to represent functions just as we use x, y,and z to represent numbers and A, B, and C to represent sets.

$$\begin{array}{ll} h \text{ is } f \text{ of } a & \text{We name the function } f\text{; height is a function of age.} \\ h=f(a) & \text{We use parentheses to indicate the function input.} \\ f(a) & \text{We name the function } f\text{ ; the expression is read as “ } f \text{ of } a\text{.”} \end{array} \tag{3.1.1}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means “first add a and b, and the result is the input for the function f.” The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y=f(x)$ defines a function named f. This is read as “y is a function of x.” The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Example 3.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days}=f(\text{month})$ or $d=f(m)$. The name of the month is the input to a “rule” that associates a specific number (the output) with each input.

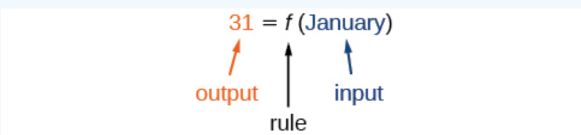


Figure $3.1.5$: The function $31=f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March})=31$, because March has 31 days. The notation $d=f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 3.1.3B: Interpreting Function Notation

A function $N=f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005)=300$ represent?

Solution

When we read $f(2005)=300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N=f(y)$. The statement $f(2005)=300$ tells us that in the year 2005 there were 300 police officers in the town.

Exercise 3.1.3

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Answer

$w = f(d)$

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x), P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table $3.1.3$ lists the input number of each month ($\text{January} = 1, \text{February} = 2$, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Table $3.1.3$: Months and number of days per month.

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table $3.1.4$ defines a function $Q = g(n)$ Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

Table $3.1.4$: Function $Q = g(n)$

n	1	2	3	4	5
Q	8	6	7	6	8

Table $3.1.5$ displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table $3.1.5$: Age of children and their corresponding heights.

Age in years, a (input)	5	5	6	7	8	9	10

Height in inches, h (output)	40	42	44	47	50	52	54

How To: Given a table of input and output values, determine whether the table represents a function

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example $3.1.5$: Identifying Tables that Represent Functions

Which table, Table $3.1.6$, Table $3.1.7$, or Table $3.1.8$, represents a function (if any)?

Table 3.1.6

Input	Output
2	1
5	3
8	6

Table 3.1.7

Input	Output
-3	5
0	1
4	5

Table 3.1.8

Input	Output
1	0
5	2
5	4

Solution

Table $3.1.6$ and Table $3.1.7$ define functions. In both, each input value corresponds to exactly one output value. Table $3.1.8$ does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table $3.1.6$ can be represented by writing

$$f(2)=1,\ f(5)=3, \text{ and } f(8)=6$$

Similarly, the statements

$$g(-3)=5,\ g(0)=1, \text{ and } g(4)=5$$

represent the function in Table $3.1.7$.

Table $3.1.8$ cannot be expressed in a similar way because it does not represent a function.

Exercise $3.1.5$

Does Table $3.1.9$ represent a function?

Table 3.1.9

Input	Output
1	10
2	100
3	1000

Answer

yes

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x)=5-3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example $3.1.6A$: Evaluating Functions at Specific Values

1. Evaluate $f(x)=x^2+3x-4$ at

a. 2
b. a
c. $a+h$
d. Evaluate $\frac{f(a+h)-f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2+3(2)-4 \\ &= 4+6-4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a)=a^2+3a-4$$

c. With an input value of $a+h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2+3(a+h)-4 \\ &= a^2+2ah+h^2+3a+3h-4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2+2ah+h^2+3a+3h-4$$

and we know that

$$f(a) = a^2+3a-4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h)-f(a)}{h} &= \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} \\ &= \frac{(2ah+h^2+3h)}{h} \\ &= \frac{h(2a+h+3)}{h} && \text{Factor out h.} \\ &= 2a+h+3 && \text{Simplify.} \end{aligned}$$

Example 3.1.6B: Evaluating Functions

Given the function $h(p) = p^2+2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2+2p \\ h(4) &= (4)^2+2(4) \\ &= 16+8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Exercise 3.1.6

Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

Answer

$g(5) = 1$

Example 3.1.7: Solving Functions

Given the function $h(p) = p^2+2p$, solve for $h(p) = 3$.

Solution

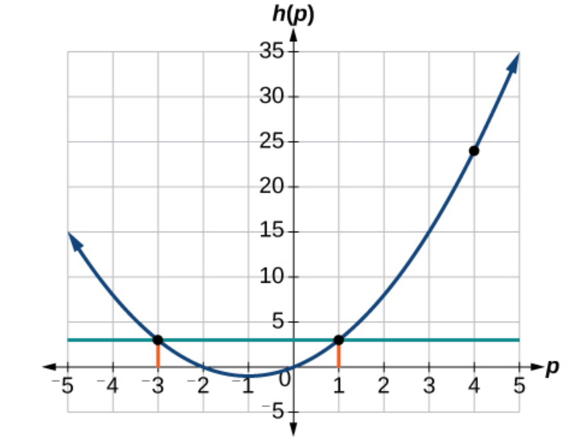
$$\begin{aligned} h(p) &= 3 \\ p^2+2p &= 3 && \text{Substitute the original function} \\ p^2+2p-3 &= 0 && \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 && \text{Factor.} \end{aligned}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3) = 0, \ p = -3$$

$$(p-1)=0,\ p=1$$

This gives us two solutions. The output $h(p)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing as in Figure $3.1.6$. The graph verifies that $h(1)=h(-3)=3$ and $h(4)=24$.



p	−3	−2	0	1	4
$h(p)$	3	0	0	3	24

Figure $3.1.6$: Graph of $h(p)=p^2+2p$

Exercise $3.1.7$

Given the function $g(m)=\sqrt{m-4}$, solve $g(m)=2$.

Answer

$m=8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in **equation** form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

How to: Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example $3.1.8A$: Finding an Equation of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p=[\text{expression involving } n]$.

$$\begin{aligned} 2n+6p &= 12 \\ 6p &= 12-2n && \text{Subtract 2n from both sides.} \\ p &= \frac{12-2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6}-\frac{2n}{6} \\ p &= 2-\frac{1}{3}n \end{aligned}$$

Therefore, p as a function of n is written as

$$p=f(n)=2-\frac{1}{3}n$$

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example 3.1.8B: Expressing the Equation of a Circle as a Function

Does the equation $x^2+y^2=1$ represent a function with x as input and y as output? If so, express the relationship as a function $y=f(x)$.

Solution

First we subtract x^2 from both sides.

$$y^2=1-x^2$$

We now try to solve for y in this equation.

$$y=\pm\sqrt{1-x^2}$$

$$\text{so, } y=\sqrt{1-x^2} \text{ and } y=-\sqrt{1-x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y=f(x)$.

Exercise 3.1.8

If $x-8y^3=0$, express y as a function of x.

Answer

$$y=f(x)=\frac{\sqrt[3]{x}}{2}$$

Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x=y+2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table (Table $3.1.10$).

Table 3.1.10

Pet Memory	span in hours
Puppy	0.008
Adult Dog	0.083
Cat	3
Goldfish	2160
Beta Fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example $3.1.9$: Evaluating and Solving a Tabular Function

Using Table $3.1.11$,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

Table 3.1.11

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Solution

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. Table $3.1.12$ shows two solutions: 2 and 4.

Table 3.1.12

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Exercise 3.1.1

Using Table $3.1.12$, evaluate $g(1)$.

Answer

$g(1) = 8$

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 3.1.10: Reading Function Values from a Graph

Given the graph in Figure $3.1.7$,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

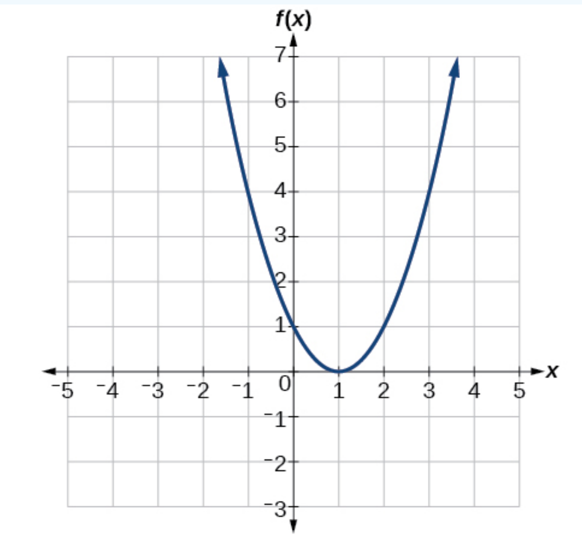
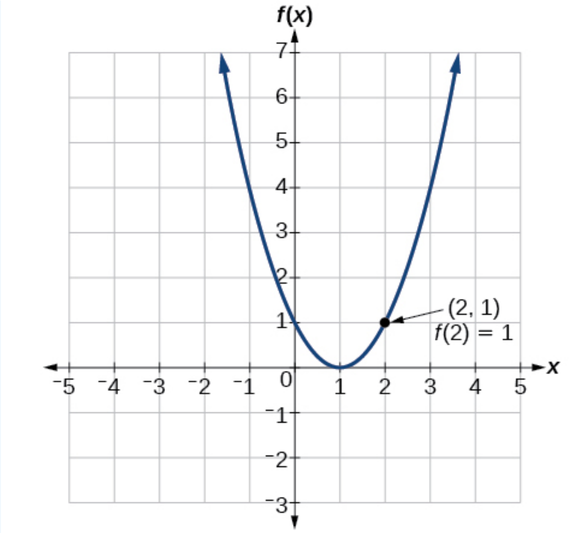


Figure $3.1.7$: Graph of a positive parabola centered at $(1, 0)$.

Solution

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure $3.1.8$.



$3.1.8$ Graph of a positive parabola centered at $(1, 0)$ with the labeled point $(2, 1)$ where $f(2) = 1$.

To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: −1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is −1 or 3, the output is 4. See Figure $3.1.9$.

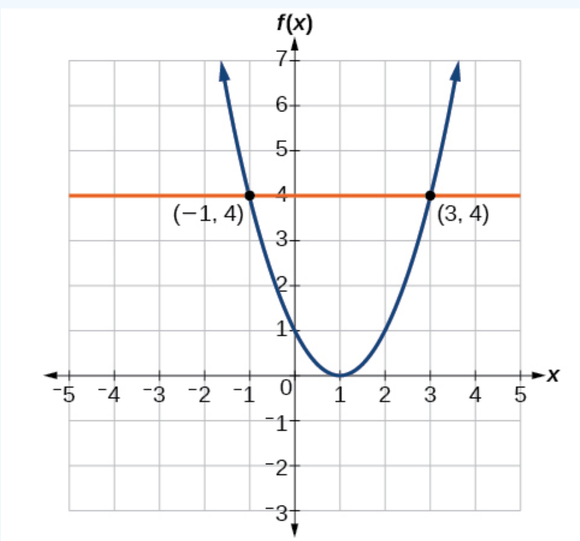


Figure $3.1.9$ Graph of an upward-facing parabola with a vertex at $(0, 1)$ and labeled points at $(-1, 4)$ and $(3, 4)$. A line at $y = 4$ intersects the parabola at the labeled points.

Exercise 3.1.10

Given the graph in Figure $3.1.7$, solve $f(x) = 1$.

Answer

$x = 0$ or $x = 2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the Figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table $3.1.13$.

Table $3.1.13$: Letter grades and decimal equivalents.

Letter Grade	Grade Point Average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figures $3.1.1a$ and $3.1.1b$. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

One-to-One Functions

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Example 3.1.11: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Exercise $3.1.11A$

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Answer

a. yes, because each bank account has a single balance at any given time;

b. no, because several bank account numbers may have the same balance;

c. no, because the same output may correspond to more than one input.

Exercise $3.1.11B$

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Answer

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades,

so there cannot be only one percent number that corresponds to each letter grade.

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure $3.1.10$ tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

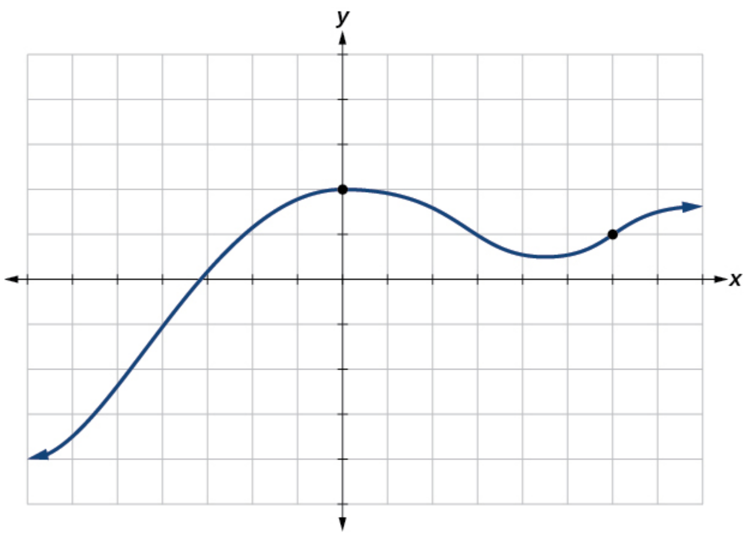


Figure $3.1.10$: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure $3.1.11$.

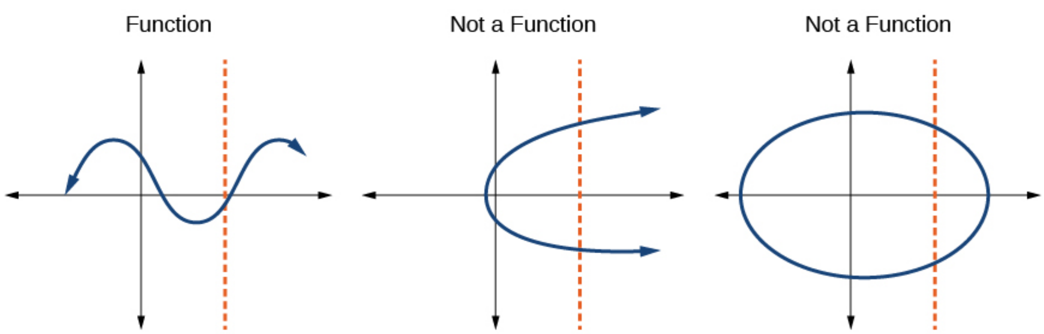


Figure $3.1.11$: Three graphs visually showing what is and is not a function.

Howto: Given a graph, use the vertical line test to determine if the graph represents a function

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example 3.1.12: Applying the Vertical Line Test

Which of the graphs inFigure $3.1.12$represent(s) a function $y = f(x)$?

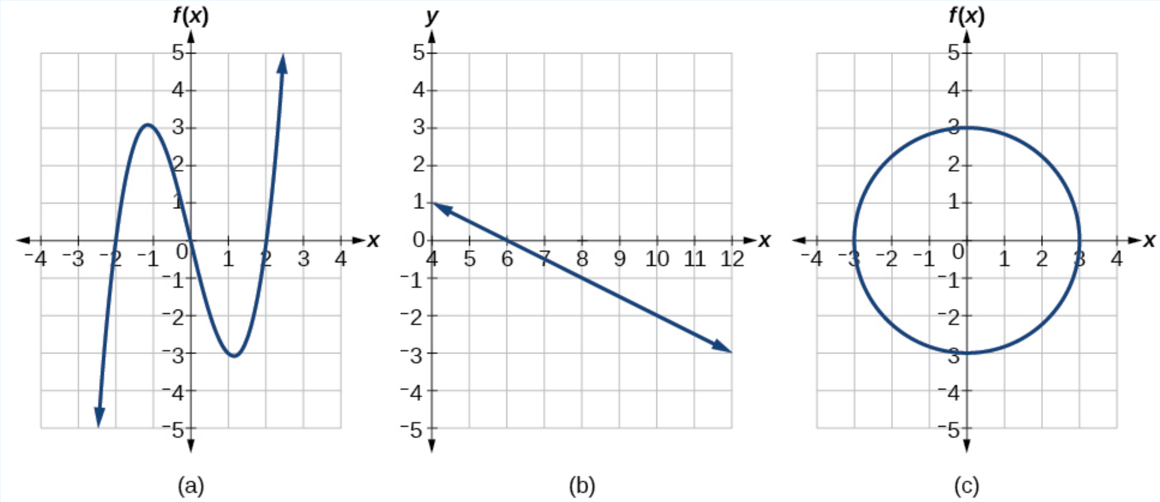


Figure $3.1.12$ Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure $3.1.12$. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure $3.1.13$.

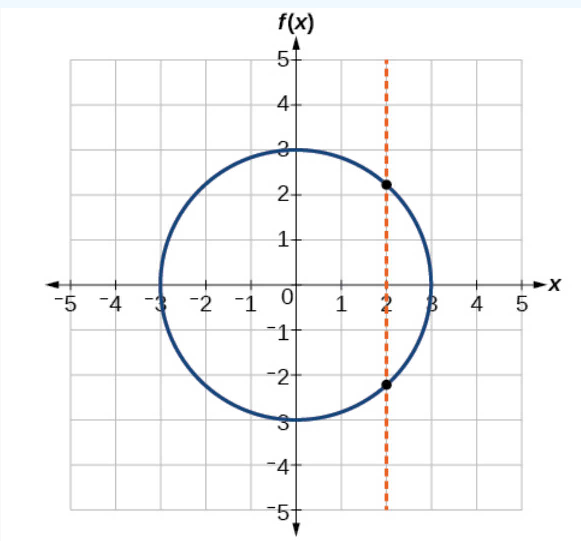


Figure $3.1.13$ Graph of a circle.

Exercise 3.1.12

Does the graph in Figure $3.1.14$represent a function?

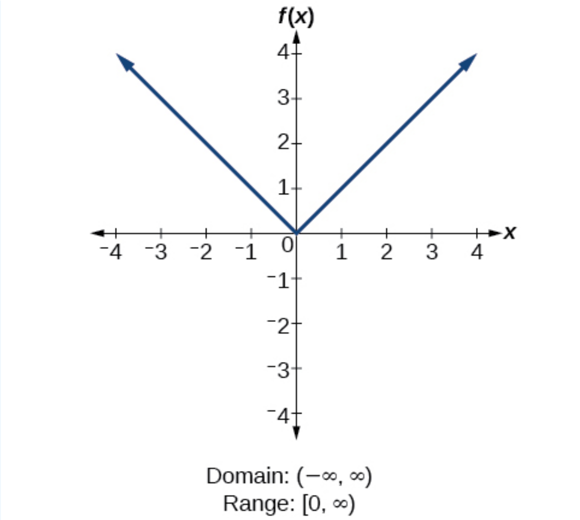


Figure $3.1.14$ Graph of absolute value function.

Answer

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Howto: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

Example $3.1.13$: Applying the Horizontal Line Test

Consider the functions shown in Figure $3.1.12a$ and Figure $3.1.12b$ Are either of the functions one-to-one?

Solution

The function in Figure $3.1.12a$ is not one-to-one. The horizontal line shown in Figure $3.1.15$ intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

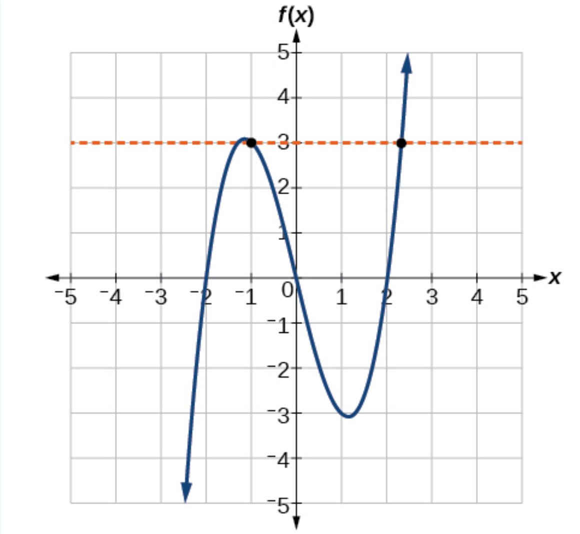


Figure $3.1.15$: Graph of a polynomial with a horizontal line crossing through 2 points

The function in Figure $3.1.12b$ is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Exercise $3.1.13$

Is the graph shown in Figure $3.1.13$ one-to-one?

Answer

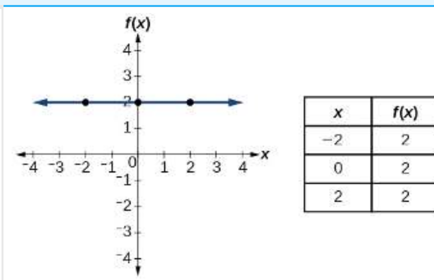
No, because it does not pass the horizontal line test.

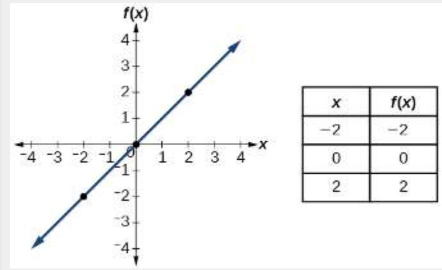
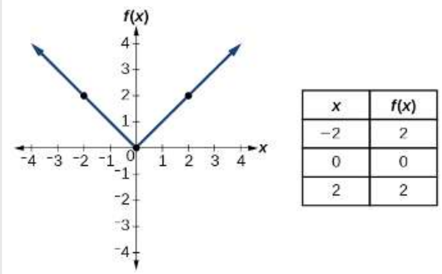
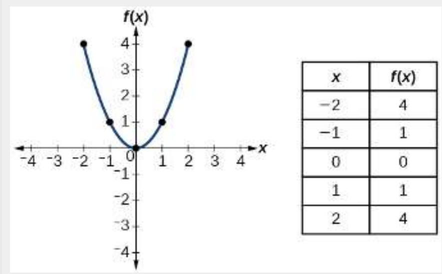
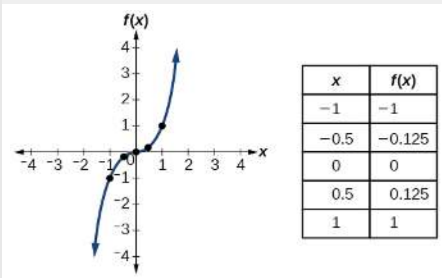
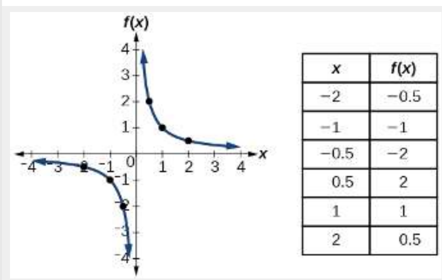
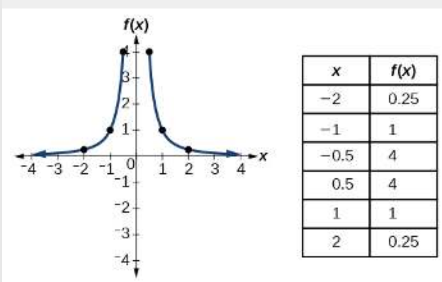
Identifying Basic Toolkit Functions

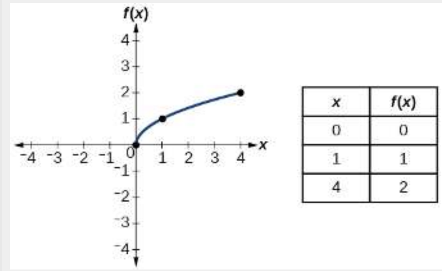
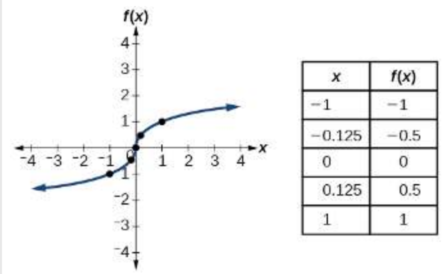
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our “toolkit functions,” which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table $3.1.14$.

Table 3.1.14: Toolkit Functions

Name	Function	Graph
Constant	$f(x) = c$ where c is a constant	

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Identity</td><td>$f(x)=x$</td><td>
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
<tr><td>Absolute Value</td><td>$f(x)=|x|$</td><td>
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
<tr><td>Quadratic</td><td>$f(x)=x^2$</td><td>
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>4</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>4</td></tr>
</table></td></tr>
<tr><td>Cubic</td><td>$f(x)=x^3$</td><td>
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−0.125</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.5</td><td>0.125</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
<tr><td>reciprocal</td><td>$f(x)=\frac{1}{x}$</td><td>
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−0.5</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−2</td></tr>
<tr><td>0.5</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.5</td></tr>
</table></td></tr>
<tr><td>Reciprocal squared</td><td>$f(x)=\frac{1}{x^2}$</td><td>
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>0.25</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>−0.5</td><td>4</td></tr>
<tr><td>0.5</td><td>4</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.25</td></tr>
</table></td></tr>
</table>

Name	Function	Graph
Square root	$f(x)=\sqrt{x}$	
Cube root	$f(x)=\sqrt[3]{x}$	

Key Equations

- Constant function $f(x)=c$, where c is a constant
- Identity function $f(x)=x$
- Absolute value function $f(x)=\|x\|$
- Quadratic function $f(x)=x^2$
- Cubic function $f(x)=x^3$
- Reciprocal function $f(x)=\dfrac{1}{x}$
- Reciprocal squared function $f(x)=\frac{1}{x^2}$
- Square root function $f(x)=\sqrt{x}$
- Cube root function $f(x)=3\sqrt{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form $y=f(x)$.
- In tabular form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- The graph of a one-to-one function passes the horizontal line test.

Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.
2 www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

Glossary

dependent variable
an output variable

domain
the set of all possible input values for a relation

function
a relation in which each input value yields a unique output value

horizontal line test
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
an input variable

input
each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
a function for which each value of the output is associated with a unique input value

output
each object or value in the range that is produced when an input value is entered into a function

range
the set of output values that result from the input values in a relation

relation
a set of ordered pairs

vertical line test
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

---

3.2: Domain and Range

Learning Objectives

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—I am Legend, Hannibal, The Ring, The Grudge, and The Conjuring. Figure $3.2.1$ shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

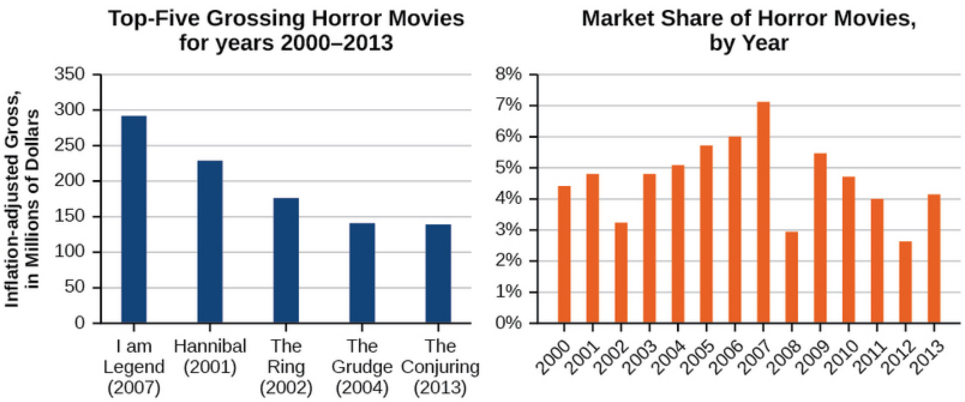


Figure $3.2.1$: Graph of the Top-Five Grossing Horror Movies for years 2000-2003, and a Graph of the Market Share of Horror Movies by Year. Based on data compiled by www.the-numbers.com.

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (Figure $3.2.2$).

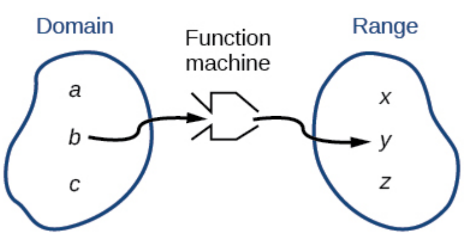


Figure $3.2.2$ Diagram of how a function relates two relations

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has \$100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See Figure $3.2.3$ for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure $3.2.3$

Summary of interval notation.

Example $3.2.1$: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2,10),(3,10),(4,20),(5,30),(6,40)\}$.

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2,3,4,5,6\}$$

Exercse 3.2.1

Find the domain of the function:

$$\{(-5,4),(0,0),(5,-4),(10,-8),(15,-12)\}$$

Answer

$\{-5,0,5,10,15\}$

How To: Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example 3.2.2: Finding the Domain of a Function

Find the domain of the function $f(x)=x^2-1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty,\infty)$.

Exercse 3.2.2

Find the domain of the function:

$$f(x)=5-x+x^3$$

Answer

$(-\infty,\infty)$

Howto: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example 3.2.3: Finding the Domain of a Function Involving a Denominator

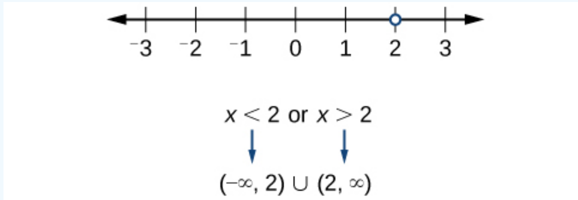
Find the domain of the function $f(x)=\dfrac{x+1}{2-x}$.

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2-x&=0 \\ -x&=-2 \\ x&=2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty, 2) \cup (2, \infty)$.



In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Exercse 3.2.3

Find the domain of the function:

$$f(x) = \frac{1+4x}{2x-1}$$

Answer

$$(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 3.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x) = \sqrt{7-x}.$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7-x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

Exercse 3.2.4

Find the domain of the function

$$f(x) = \sqrt{5+2x}.$$

Answer

$$[-2.5, \infty)$$

> **Q&A: Can there be functions in which the domain and range do not intersect at all?**
>
> Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure $3.2.4$ compares inequality notation, set-builder notation, and interval notation.

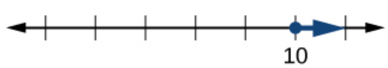
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure $3.2.4$

Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$ It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3\} = (-\infty, -3] \cup [3, \infty) \tag{3.2.1}$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x| \text{ statement about x}\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

How To: Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol ∪ to combine all intervals into one set.

Example 3.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure $3.2.5$ using inequality notation, set-builder notation, and interval notation.

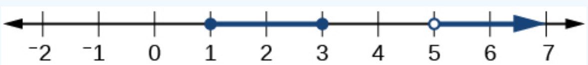


Figure $3.2.5$ Line graph of $1 \leq x \leq 3$ or $x > 5$.

Solution

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x|1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Exercse 3.2.5

Given Figure $3.2.6$, specify the graphed set in

a. words
b. set-builder notation
c. interval notation

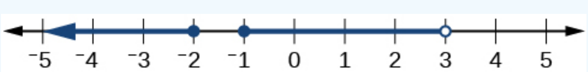


Figure $3.2.6$ Line graph of $-2 \leq x$, $-1 \leq x < 3$ .leq}x\), $-1 \leq x < 3$.

Answer a

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

Answer b

$\{x|x \leq -2 or -1 \leq x < 3\}$

Answer c

$(-\infty, -2] \cup [-1, 3)$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure $3.2.7$.

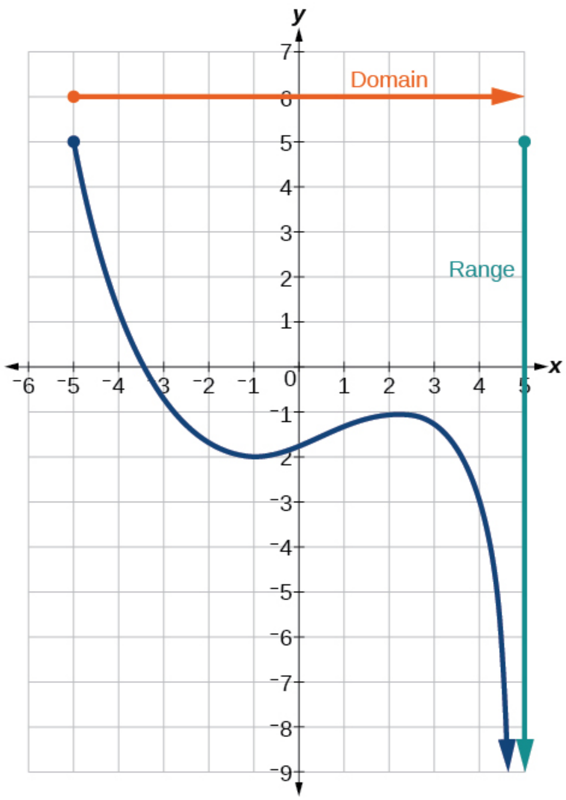


Figure $3.2.7$: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from −5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example $3.2.6A$: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

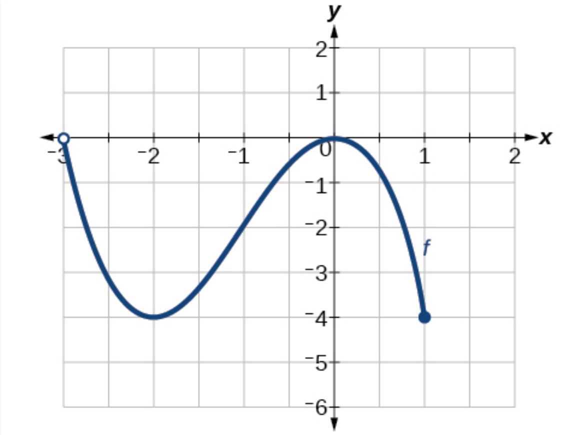


Figure $3.2.8$ Graph of a function from (-3, 1].

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3,1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4,0)$. See Figure $3.2.9$.

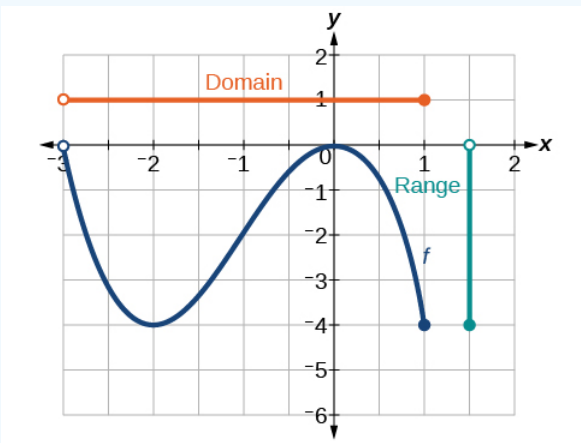


Figure $3.2.9$ Graph of the previous function shows the domain and range

Example $3.2.6B$: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure $3.2.10$.

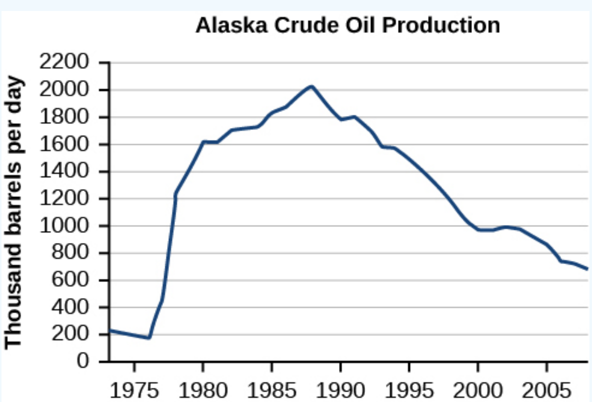


Figure $3.2.10$ Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

Solution

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is $[1973, 2008]$ and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Exercse $3.2.6$

Given Figure $3.2.11$, identify the domain and range using interval notation.

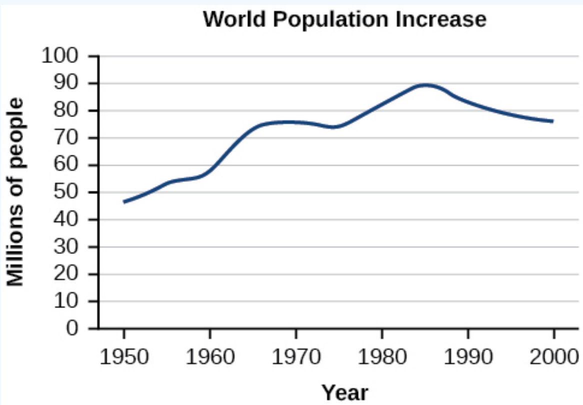


Figure $3.2.11$: Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.

Answer

domain =$[1950, 2002]$

range = $[47,000,000, 89,000,000]$

Can a function's domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.

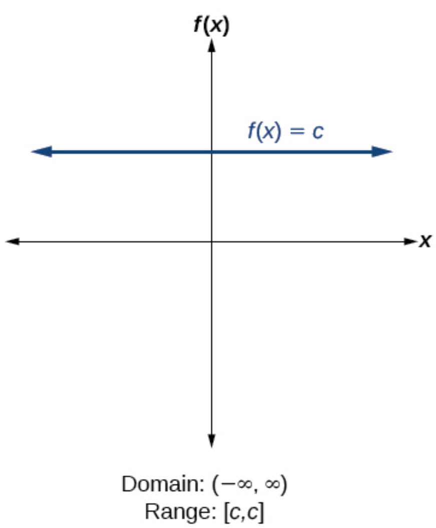


Figure $3.2.12$ Constant function $f(x) = c$.

For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.

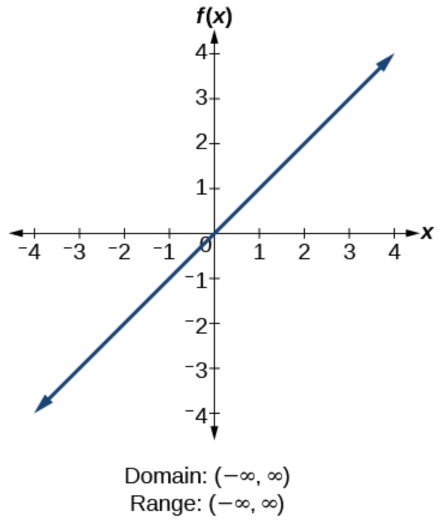


Figure* $3.2.13$*: *Identity function f(x)=x.*

For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.

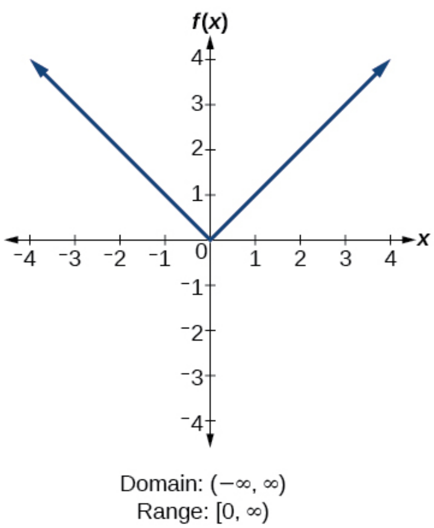


Figure $3.2.14$ Absolute function $f(x) = |x|$.

For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

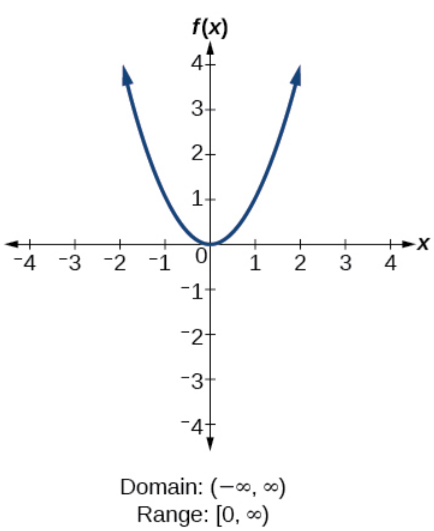


Figure $3.2.15$ Quadratic function $f(x) = x^2$.

For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.

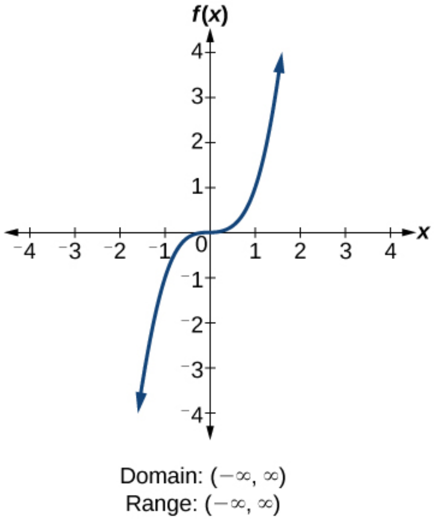


Figure $3.2.16$ Cubic function $f(x) - x^3$.

For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.

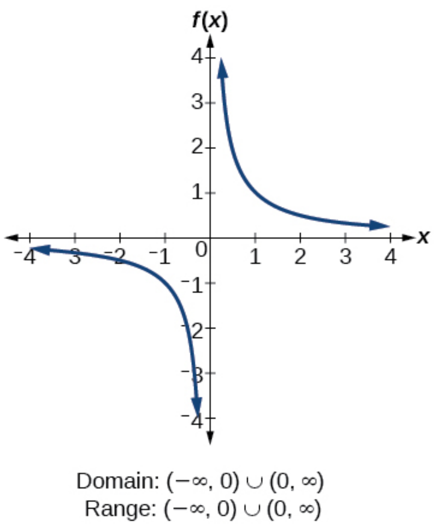


Figure $3.2.17$ Reciprocal function $f(x) = \frac{1}{x}$.

For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x | x \neq 0\}$,the set of all real numbers that are not zero.

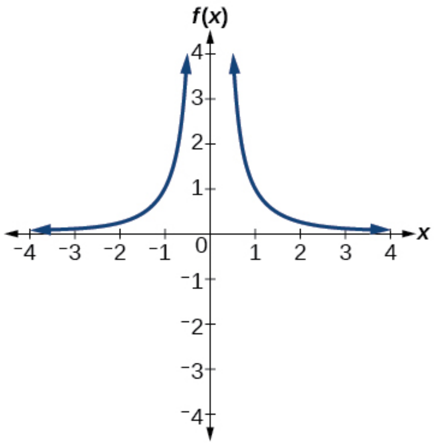


Figure 3.2.18 Reciprocal squared function $f(x) = \frac{1}{x^2}$

For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.

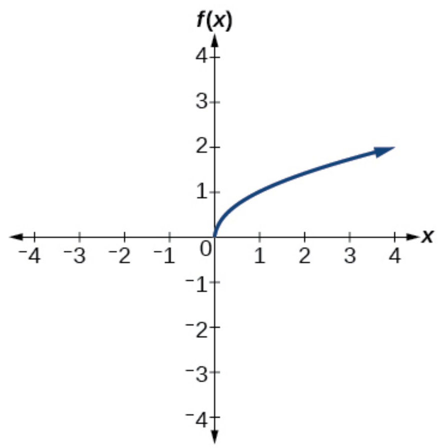


***Figure 3.2.19**: Square root function $f(x) = \sqrt{(x)}$.*

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.

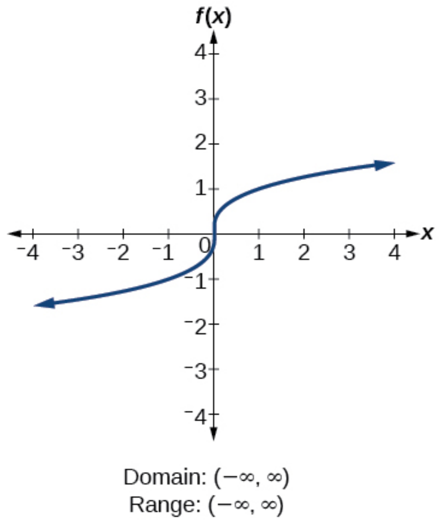


Figure 3.2.20 Cube root function $f(x)=\sqrt[3]{x}$.

For the **cube root function** $f(x)=\sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

How To: Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x)=2x^3-x$.

Solution

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty,\infty)$ and the range is also $(-\infty,\infty)$.

Example $3.2.7B$: Finding the Domain and Range

Find the domain and range of $f(x)=\frac{2}{x+1}$.

Solution

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty,-1)\cup(-1,\infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty,0)\cup(0,\infty)$.

Example $3.2.7C$: Finding the Domain and Range

Find the domain and range of $f(x)=2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$x+4\geq 0$ when $x\geq -4$

The domain of $f(x)$ is $[-4,\infty)$.

We then find the range. We know that $f(-4)=0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0,\infty)$.

Analysis

Figure $3.2.19$ represents the function f.

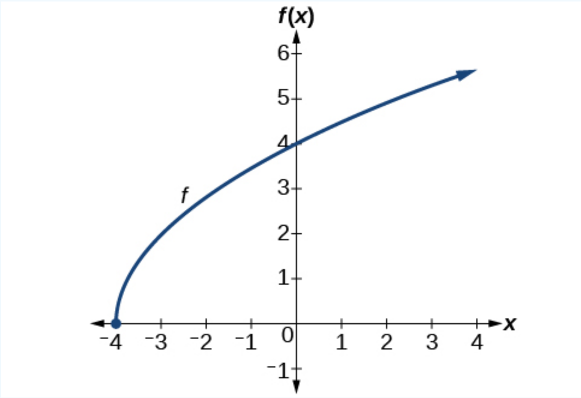


Figure $3.2.19$: Graph of a square root function at $(-4, 0)$.

Exercise $3.2.7$

Find the domain and range of

$f(x) = \sqrt{-2-x}$.

Answer

domain: $(-\infty, -2]$

range: $[0, \infty)$

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, **absolute value** can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

Piecewise Function

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1} & \text{if x is in domain 1} \\ \text{formula 2} & \text{if x is in domain 2} \\ \text{formula 3} & \text{if x is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example $3.2.8A$: Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a **function** relating the number of people, n, to the cost, C.

Solution

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Analysis

The function is represented in Figure $3.2.20$. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

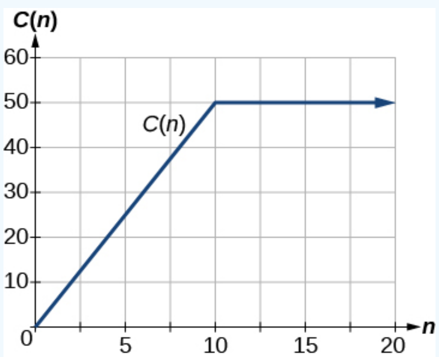


Figure $3.2.20$

Example $3.2.8B$: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g - 2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Soltuion

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, C(4), we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4 - 2) = \$45$$

Analysis

The function is represented in Figure $3.2.21$. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

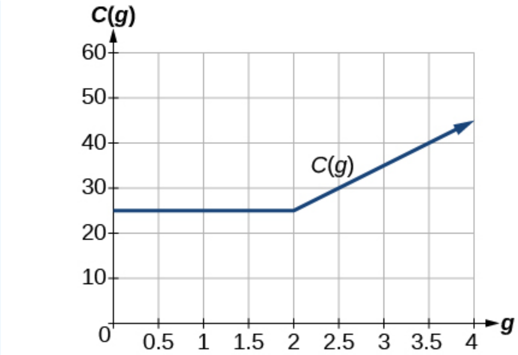


Figure $3.2.21$

Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example $3.2.8C$: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x)=\begin{cases} x^2 & \text{if } x\leq 1 \\ 3 & \text{if } 1<x\leq 2 \\ x & \text{if } x>2 \end{cases}$$

Solution

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure $3.2.20$ shows the three components of the piecewise function graphed on separate coordinate systems.

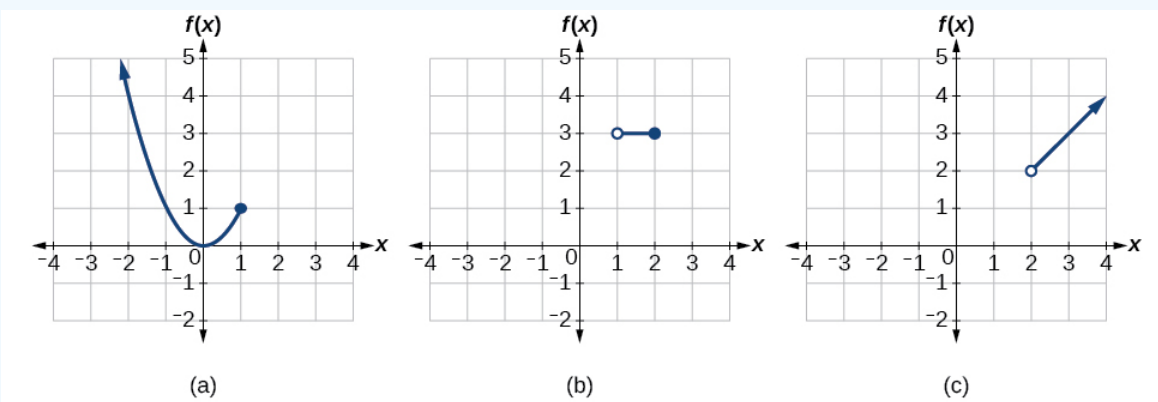


***Figure $3.2.20$:** Graph of each part of the piece-wise function f(x)*

(a)$f(x)=x^2$ if $x\leq 1$; (b) $f(x)=3$ if $1<x\leq 2$; (c) $f(x)=x$ if $x>2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure $3.2.21$.

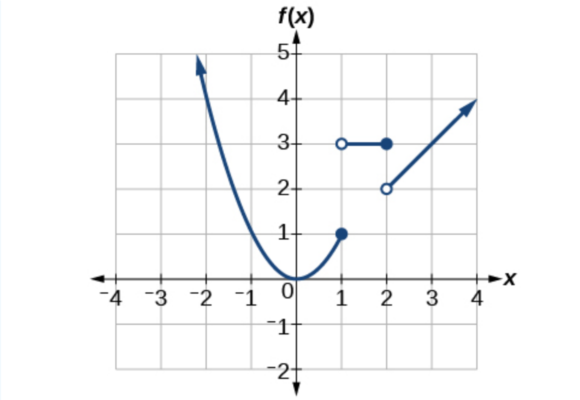


Figure $3.2.21$: Graph of the entire function.

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

Exercise $3.2.8$

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Answer

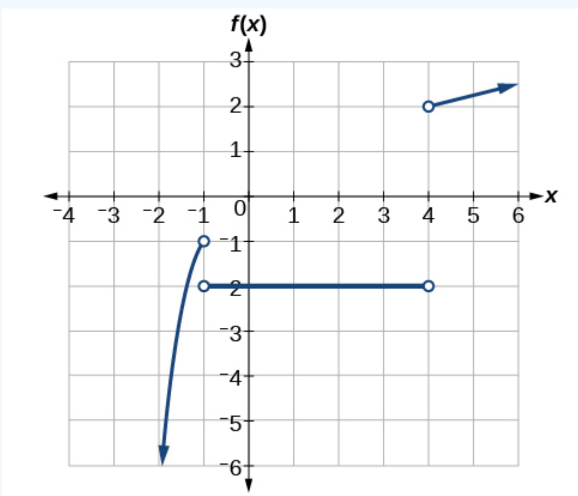


Figure $3.2.22$

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.

- An understanding of toolkit functions can be used to find the domain and range of related functions.
- A piecewise function is described by more than one formula.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain.

Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014
2 www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

Glossary

interval notation

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

piecewise function

a function in which more than one formula is used to define the output

set-builder notation

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

3.3: Piecewise-Defined Functions

In preparation for the definition of the absolute value function, it is extremely important to have a good grasp of the concept of a piecewise-defined function. However, before we jump into the fray, let's take a look at a special type of function called a constant function.

One way of understanding a constant function is to have a look at its graph.

Example 3.3.1

Sketch the graph of the constant function f(x) = 3.

Solution

Because the notation f(x) = 3 is equivalent to the notation y = 3, we can sketch a graph of f by drawing the graph of the horizontal line having equation y = 3, as shown in Figure $3.3.1$.

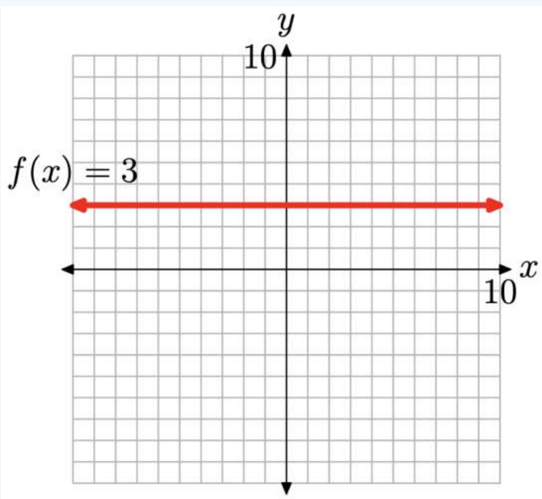


Figure $3.3.1$ The graph of a constant function is a horizontal line.

When you look at the graph in Figure 1, note that every point on the horizontal line having equation f(x) = 3 has a y-value equal to 3. We say that the y-values on this horizontal line are constant, for the simple reason that they are constantly equal to 3.

The function form works in precisely the same manner. Look again at the notation

$$f(x) = 3$$

Note that no matter what number you substitute for x in the left-hand side of f(x) = 3, the right-hand side is constantly equal to 3. Thus,

$$f(-5) = 3, \quad f(-1/2) = 3, \quad f(\sqrt{2}) = 3, \quad \text{or} \quad f(\pi) = 3$$

The above discussion leads to the following definition.

Definition

The function defined by f(x) = c, where c is a constant (fixed real number), is called a constant function.

Two comments are in order:

1. f(x) = c for all real numbers x.
2. The graph of f(x) = c is a horizontal line. It consists of all the points (x, y) having y-value equal to c.

Piecewise Constant Functions

Piecewise functions are a favorite of engineers. Let's look at an example.

Example 3.3.2

Suppose that a battery provides no voltage to a circuit when a switch is open. Then, starting at time $t = 0$, the switch is closed and the battery provides a constant 5 volts from that time forward. Create a piecewise function modeling the problem constraints and sketch its graph.

Solution

This is a fairly simple exercise, but we will have to introduce some new notation. First of all, if the time t is less than zero ($t < 0$), then the voltage is 0 volts. If the time t is greater than or equal to zero ($t \geq 0$), then the voltage is a constant 5 volts. Here is the notation we will use to summarize this description of the voltage.

$$V(t) = \begin{cases} 0, & \text{if } t < 0 \\ 5, & \text{if } t \geq 0 \end{cases}$$

Some comments are in order:

- The voltage difference provide by the battery in the circuit is a function of time. Thus, V (t) represents the voltage at time t.
- The notation used in (4) is universally adopted by mathematicians in situations where the function changes definition depending on the value of the independent variable. This definition of the function V is called a "piecewise definition." Because each of the pieces in this definition is constant, the function V is called a piecewise constant function.
- This particular function has two pieces. The function is the constant function $V(t) = 0$, when $t < 0$, but a different constant function, $V(t) = 5$, when $t \geq 0$.

If $t < 0, V(t) = 0.$ For example, for $t = -1, t = -10,$ and $t = -100$

$$V(-1) = 0, \quad V(-10) = 0, \quad \text{and} \quad V(-100) = 0$$

On the other hand, if $t \geq 0$, then $V(t) = 5.$ For example, for $t = 0, t = 10,$ and $t = 100$

$$V(0) = 5, \quad V(10) = 5, \quad \text{and} \quad V(100) = 5$$

Before we present the graph of the piecewise constant function V , let's pause for a moment to make sure we understand some standard geometrical terms.

Geometrical Terms

- A line stretches indefinitely in two directions, as shown in Figure $3.3.2$(a).
- If a line has a fixed endpoint and stretches indefinitely in only one direction, as shown in Figure $3.3.2$(b), then it is called a ray.
- If a portion of the line is fixed at each end, as shown in Figure $3.3.2$(c), then it is called a line segment.

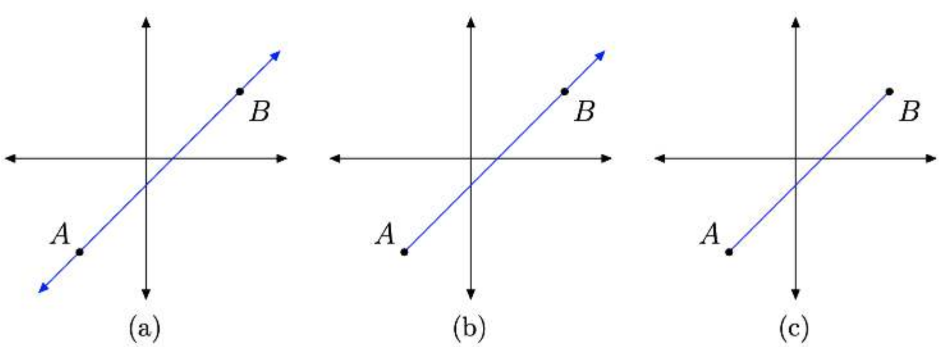


Figure $3.3.2$ Lines, rays, and segments.

With these terms in hand, let's turn our attention to the graph of the voltage defined by equation (4). When $t < 0$, then $V(t) = 0$. Normally, the graph of $V(t) = 0$ would be a horizontal line where each point on the line has V -value equal to zero. However, $V(t) = 0$ only if $t < 0$, so the graph is the horizontal ray that starts at the origin, then moves indefinitely to the left, as shown in Figure 3.3.3. That is, the horizontal line $V = 0$ has been restricted to the domain $\{t : t < 0\}$ and exists only to the left of the origin.

Similarly, when $t \geq 0$, then $V(t) = 5$ is the horizontal ray shown in Figure 3.3.3. Each point on the ray has a V -value equal to 5.

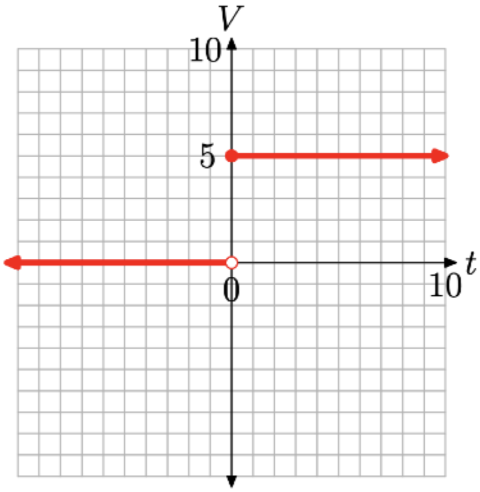


Figure 3.3.3 The voltage as a function of time t.

Two comments are in order:

- Because $V(t) = 0$ only when t < 0, the point (0, 0) is unfilled (it is an open circle). The open circle at (0, 0) is a mathematician's way of saying that this particular point is not plotted or shaded.
- Because $V(t) = 5$ when $t \geq 0$, the point (0, 5) is filled (it is a filled circle). The filled circle at (0, 5) is a mathematician's way of saying that this particular point is plotted or shaded.

Let's look at another example.

Example 3.3.3

Consider the piecewise-defined function

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } 0 \leq x < 2 \\ 2, & \text{if } x \geq 2 \end{cases}$$

Evaluate f(x) at x = −1, 0, 1, 2, and 3. Sketch the graph of the piecewise function f.

Solution

Because each piece of the function in (6) is constant, evaluation of the function is pretty easy. You just have to select the correct piece.

• Note that x = −1 is less than 0, so we use the first piece and write f(−1) = 0.

• Note that x = 0 satisfies $0 \leq x < 2$, so we use the second piece and write f(0) = 1.

• Note that x = 1 satisfies $0 \leq x < 2$, so we use the second piece and write f(1) = 1.

• Note that x = 2 satisfies $x \geq 2$, so we use the third piece and write f(2) = 2.

• Finally, note that x = 3 satisfies $x \geq 2$, so we use the third piece and write f(3) = 2. The graph is just as simple to sketch.

• Because f(x) = 0 for x < 0, the graph of this piece is a horizontal ray with endpoint at x = 0. Each point on this ray will have a y-value equal to zero and the ray will lie entirely to the left of x = 0, as shown in Figure 3.3.4.

• Because f(x) = 1 for $0 \leq x < 2$, the graph of this piece is a horizontal segment with one endpoint at x = 0 and the other at x = 2. Each point on this segment will have a y-value equal to 1, as shown in Figure 3.3.4.

• Because f(x) = 2 for $x \geq 2$, the graph of this piece is a horizontal ray with endpoint at x = 2. Each point on this ray has a y-value equal to 2 and the ray lies entirely to the right of x = 2, as shown in Figure $3.3.4$.

Several remarks are in order:

• The function is zero to the left of the origin (for x < 0), but not at the origin. This is indicated by an empty circle at the origin, an indication that we are not plotting that particular point.

• For $0 \leq x < 2$, the function equals 1. That is, the function is constantly equal to 1 for all values of x between 0 and 2, including zero but not including 2. This is why you see a filled circle at (0, 1) and an empty circle at (2, 1).

• Finally, for $x \geq 2$, the function equals 2. That is, the function is constantly equal to 2 whenever x is greater than or equal to 2. That is why you see a filled circle at (2, 2).

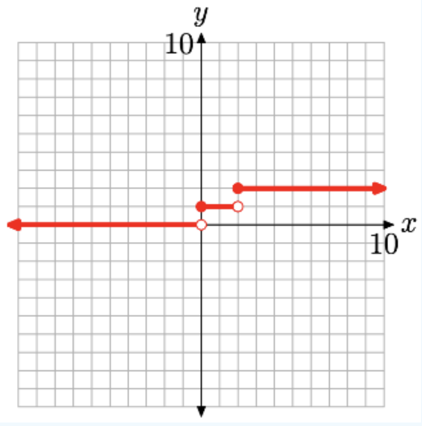


Figure $3.3.4$ Sketching the graph of the piecewise function (6).

Piecewise-Defined Functions

Now, let's look at a more generic situation involving piecewise-defined functions—one where the pieces are not necessarily constant. The best way to learn is by doing, so let's start with an example.

Example $3.3.4$

Consider the piecewise-defined function

$$f(x)=\begin{cases}-x+2, & \text{if } x<2\\ x-2, & \text{if } x\geq 2\end{cases}$$

Evaluate f(x) for x = 0, 1, 2, 3 and 4, then sketch the graph of the piecewise-defined function.

Solution

The function changes definition at x = 2. If x < 2, then f(x) = −x + 2. Because both 0 and 1 are strictly less than 2, we evaluate the function with this first piece of the definition.

$$\begin{array}{lll} f(x)=-x+2 & \text{and} & f(x)=-x+2\\ f(0)=-0+2 & & f(1)=-1+2\\ f(0)=2 & & f(1)=1\end{array}$$

On the other hand, if $x \geq 2$, then $f(x)=x-2$. Because 2, 3, and 4 are all greater than or equal to 2, we evaluate the function with this second piece of the definition.

$$\begin{array}{lllll} f(x)=x-2 & \text{and} & f(x)=x-2 & \text{and} & f(x)=x-2\\ f(2)=2-2 & \text{and} & f(3)=3-2 & \text{and} & f(4)=4-2\\ f(2)=0 & \text{and} & f(3)=1 & \text{and} & f(4)=2\end{array}$$

One possible approach to the graph of f is to place the points we've already calculated, plus a couple extra, in a table (see Figure $3.3.5$(a)), plot them (see Figure $3.3.5$(b)), then intuit the shape of the graph from the evidence provided by the plotted points. This is done in Figure $3.3.5$(c).

x	$f(x)$
-1	3
0	2
1	1
2	0
3	1
4	2
5	3

(a)

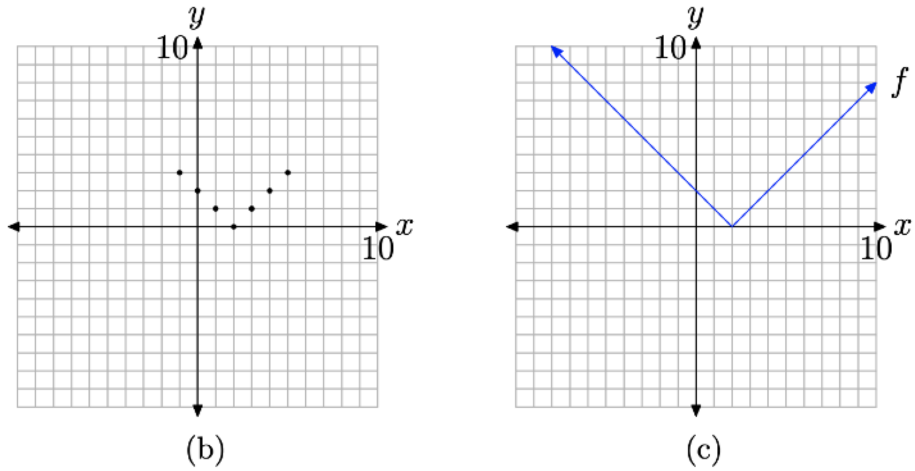


Figure 3.3.5 Plotting the graph of the piecewise function defined in (8).

However pragmatic, this point-plotting approach is a bit tedious; but, more importantly, it does not provide the background necessary for the discussion of the absolute value function in the next section. We need to stretch our understanding to a higher level. Fortunately, all the groundwork is in place. We need only apply what we already know about the equations of lines to fit this piecewise situation.

Alternative approach. Let's use our knowledge of the equation of a line (i.e. y = mx + b) to help sketch the graph of the piecewise function defined in (8).

Let's sketch the first piece of the function f defined in (8). We have f(x) = −x+ 2, provided x < 2. Normally, this would be a line (with slope −1 and intercept 2), but we are to sketch only a part of that line, the part where x < 2 (x is to the left of 2). Thus, this piece of the graph will be a ray, starting at the point where x = 2, then moving indefinitely to the left.

The easiest way to sketch a ray is to first calculate and plot its fixed endpoint (in this case at x = 2), then plot a second point on the ray having x-value less than 2, then use a ruler to draw the ray.

With this thought in mind, to find the coordinates of the endpoint of the ray, substitute x = 2 in f(x) = −x + 2 to get f(2) = 0. Now, technically, we're not supposed to use this piece of the function unless x is strictly less than 2, but we could use it with x = 1.9, or x = 1.99, or x = 1.999, etc. So let's go ahead and use x = 2 in this piece of the function, but indicate that we're not actually supposed to use this point by drawing an "empty circle" at (2, 0), as shown in Figure 3.3.6(a).

To complete the plot of the ray, we need a second point that lies to the left of its endpoint at (2, 0). Note that x = 0 is to the left of x = 2. Evaluate f(x) = −x + 2 at x = 0 to obtain f(0) = −0 + 2 = 2. This gives us the second point (0, 2), which we plot as shown in Figure 3.3.6(a). Finally, draw the ray with endpoint at (2, 0) and second point at (0, 2), as shown in Figure 3.3.6(a).

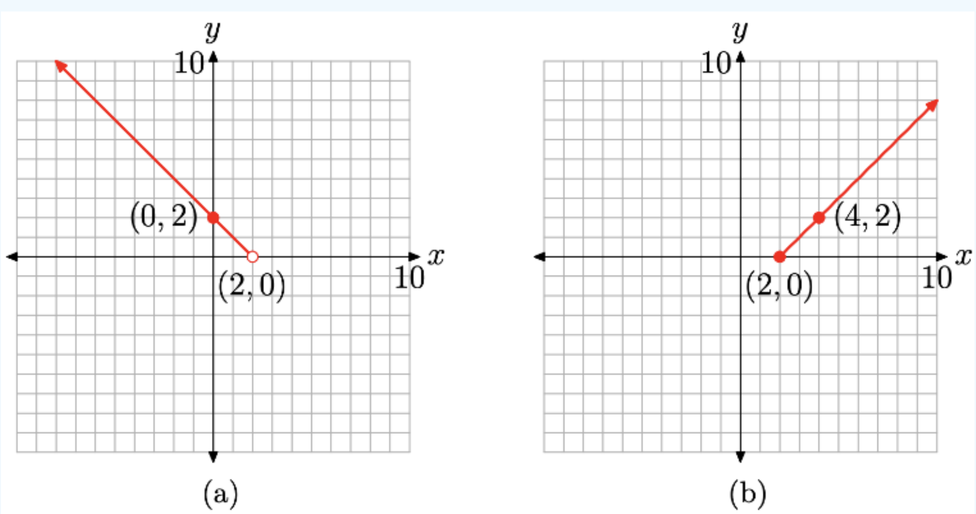


Figure 3.3.6 Sketch each piece separately

We now repeat this process for the second piece of the function defined in (8). The equation of the second piece is f(x) = x − 2, provided $x \geq 2$. Normally, f(x) = x − 2 would be a line (with slope 1 and intercept −2), but we're only supposed to sketch that part of the line that lies to the right of or at x = 2. Thus, the graph of this second piece is a ray, starting at the point with x = 2 and continuing to the right. If we evaluate f(x) = x − 2 at x = 2, then f(2) = 2 − 2 = 0. Thus, the fixed endpoint of the ray is at the point (2, 0). Since we're actually supposed to use this piece with x = 2, we indicate this fact with a filled circle at (2, 0), as shown in Figure $3.3.6$(b).

We need a second point to the right of this fixed endpoint, so we evaluate f(x) = x−2 at x = 4 to get f(4) = 4 − 2 = 2. Thus, a second point on the ray is the point (4, 2). Finally, we simply draw the ray, starting at the endpoint (2, 0) and passing through the second point at (4, 2), as shown in Figure $3.3.6$(b).

To complete the graph of the piecewise function f defined in equation (8), simply combine the two pieces in Figure $3.3.6$(a) and Figure $3.3.6$(b) to get the finished graph in Figure $3.3.7$. Note that the graph in Figure $3.3.7$ is identical to the earlier result in Figure $3.3.5$(c).

Let's try this alternative procedure in another example.

Example $3.3.5$

A source provides voltage to a circuit according to the piecewise definition

$$V(t) = \begin{cases} 0, & \text{if } t < 0 \\ t, & \text{if } t \geq 0 \end{cases}$$

Sketch the graph of the voltage V versus time t.

Solution

For all time t that is less than zero, the voltage V is zero. The graph of V (t) = 0 is a constant function, so its graph is normally a horizontal line. However, we must restrict

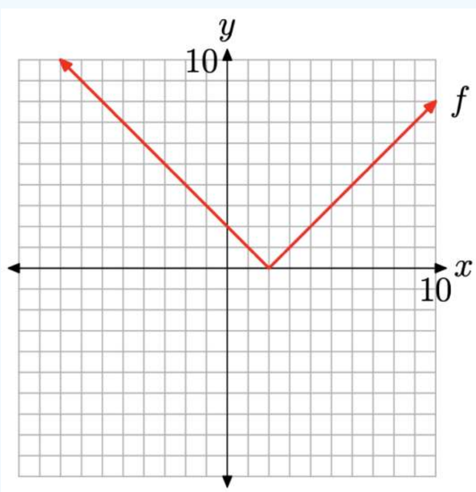
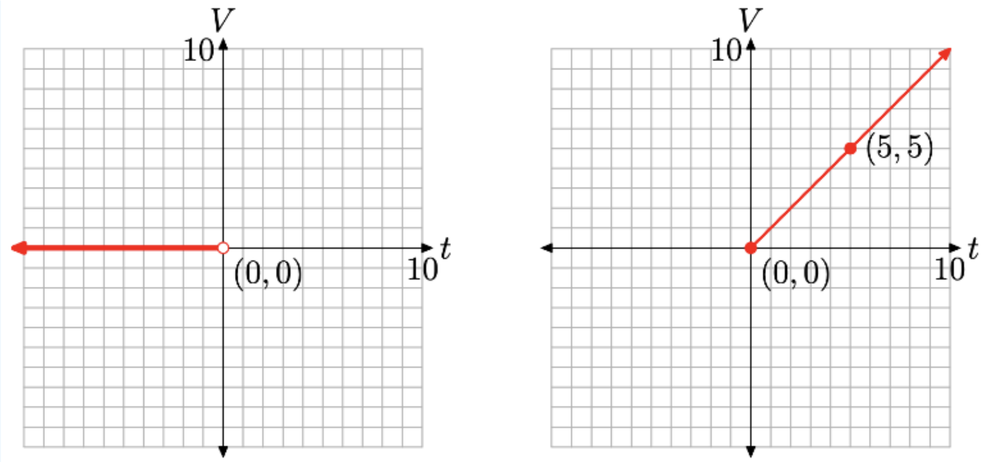


Figure $3.3.7$ Combining both pieces.

the graph to the domain $(-\infty, 0)$, so this piece of equation (10) will be a horizontal ray, starting at the origin and moving indefinitely to the left, as shown in Figure $3.3.8$(a).

On the other hand, V (t) = t for all values of t that are greater than or equal to zero. Normally, this would be a line with slope 1 and intercept zero. However, we must restrict the domain to $[0, \infty)$, so this piece of equation (10) will be a ray, starting at the origin and moving indefinitely to the right.

- The endpoint of this ray starts at t = 0. Because V (t) = t, V (0) = 0. Hence, the endpoint of this ray is at the point (0, 0).
- Choose any value of t that is greater than zero. We'll choose t = 5. Because V (t) = t, V (5) = 5. This gives us a second point on the ray at (5, 5), as shown in Figure $3.3.8$(b).



(a) $V(t) = 0$ for $t < 0$. (b) $V(t) = t$ for $t \geq 0$. Figure $3.3.8$

Finally, to provide a complete graph of the voltage function defined by equation (10), we combine the graphs of each piece of the definition shown in Figures $3.3.8$(a) and (b).

The result is shown in Figure $3.3.9$. Engineers refer to this type of input function as a "ramp function."

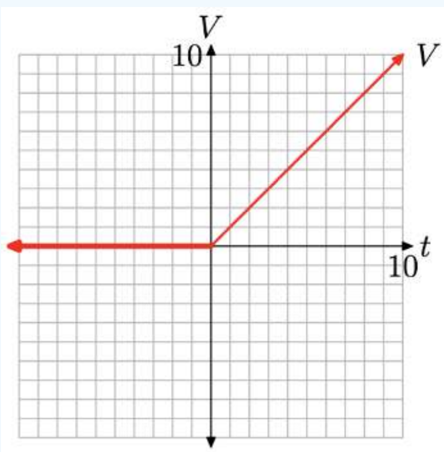


Figure $3.3.9$ The graph of the ramp function defined by equation (10).

Let's look at a very practical application of piecewise functions.

✓ Example $3.3.6$

The federal income tax rates for a single filer in the year 2005 are given in Table $3.3.1$.

Table $3.3.1$. 2005 Federal Income Tax rates for single filer.

Income	Tax Rate
Up to \$7,150	10%
\$7,151-\$29,050	15%
\$29,051-\$70,350	25%
\$70,351-\$146,750	28%
\$146,751-\$319,100	33%
\$319,101 or more	35%

Create a piecewise definition that provides the tax rate as a function of personal income.

Solution

In reporting taxable income, amounts are rounded to the nearest dollar on the federal income tax form. Technically, the domain is discrete. You can report a taxable income of $35,000 or $35,001, but numbers between these two incomes are not used on the federal income tax form. However, we will think of the income as a continuum, allowing the income to be any real number greater than or equal to zero. If we did not do this, then our graph would be a series of dots–one for each dollar amount. We would have to plot lots of dots!

We will let R represent the tax rate and I represent the income. The goal is to define R as a function of I.

- If income I is any amount greater than or equal to zero, and less than or equal to $7,150, the tax rate R is 10% (i.e., R = 0.10). Thus, if $\$0 \leq I \leq \$7,150$, R(I) = 0.10.
- If income I is any amount that is strictly greater than $7,150 but less than or equal to $29,050, then the tax rate R is 15% (i.e., R = 0.15). Thus, if $7, 150 < I ≤ $29, 050, then R(I) = 0.15.

Continuing in this manner, we can construct a piecewise definition of rate R as a function of taxable income I.

$$R(I)=\begin{cases}0.10, & \text{if } \$0 \leq I \leq \$7,150 \\ 0.15, & \text{if } \$7,150 < I \leq \$29,050 \\ 0.25, & \text{if } \$29,050 < I \leq \$70,350 \\ 0.28, & \text{if } \$70,350 < I \leq \$146,750 \\ 0.33, & \text{if } \$146,750 < I \leq \$319,100 \\ 0.35, & \text{if } I > \$319,100\end{cases}$$

Let's turn our attention to the graph of this piecewise-defined function. All of the pieces are constant functions, so each piece will be a horizontal line at a level indicating the tax rate. However, each of the first five pieces of the function defined in equation (12) are segments, because the rate is defined on an interval with a starting and ending income. The sixth and last piece is a ray, as it has a starting endpoint, but the rate remains constant for all incomes above $319,100. We use this knowledge to construct the graph shown in Figure $3.3.10$.

The first rate is 10% and this is assigned to taxable income starting at $0 and ending at $7,150, inclusive. Thus, note the first horizontal line segment in Figure $3.3.10$ that runs from $0 to $7,150 at a height of R = 0.10. Note that each of the endpoints are filled circles.

The second rate is 15% and this is assigned to taxable incomes greater than $7,150, but less than or equal to $29,050. The second horizontal line segment in Figure 10 runs from $7,150 to $29,050 at a height of R = 0.15. Note that the endpoint at the left end of this horizontal segment is an open circle while the endpoint on the right end is a filled circle because the taxable incomes range on $7, 150 < I ≤ $29, 050. Thus, we exclude the left endpoint and include the right endpoint.

The remaining segments are drawn in a similar manner.

The last piece assigns a rate of R = 0.35 to all taxable incomes strictly above $319,100. Hence, the last piece is a horizontal ray, starting at ($319 100, 0.35) and extending indefinitely to the right. Note that the left endpoint of this ray is an open circle because the rate R = 0.35 applies to taxable incomes I > $319, 100.

Let's talk a moment about the domain and range of the function R defined by equation (12). The graph of R is depicted in Figure $3.3.10$. If we project all points on the graph onto the horizontal axis, the entire axis will "lie in shadow." Thus, at first

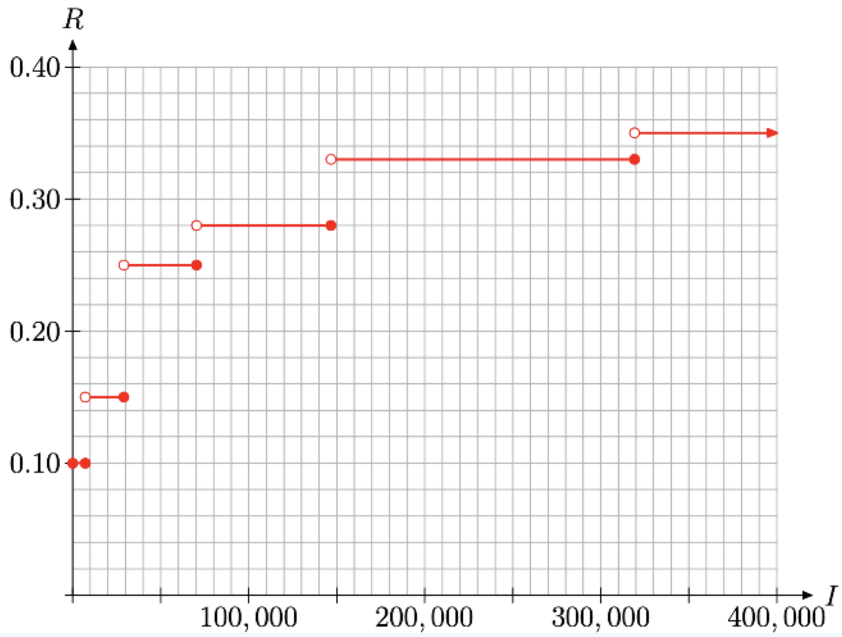


Figure 3.3.10 The graph of the tax rate R versus taxable income I.

glance, one would state that the domain of R is the set of all real numbers that are greater than or equal to zero.

However, remember that we chose to model a discrete situation with a continuum. Taxable income is always rounded to the nearest dollar on federal income tax forms. Therefore, the domain is actually all whole numbers greater than or equal to zero. In symbols,

$$\text{Domain } = \{I \in \mathbb{W} : I \geq 0\}$$

To find the range of R, we would project all points on the graph of R in Figure 3.3.10 onto the vertical axis. The result would be that six points would be shaded on the vertical axis, one each at 0.10, 0.15, 0.25, 0.28, 0.33, and 0.35. Thus, the range is a finite discrete set, so it's best described by simply listing its members.

$$\text{Range } = \{0.10, 0.15, 0.25, 0.28, 0.33, 0.35\}$$

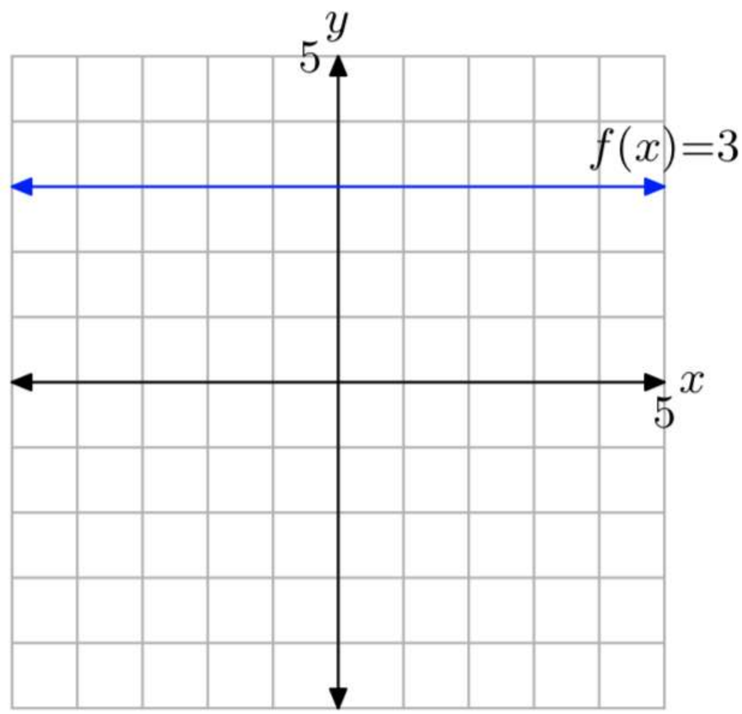
Exercise

Exercise 3.3.1

Given the function defined by the rule f(x) = 3, evaluate f(−3), f(0) and f(4), then sketch the graph of f.

Answer

f(−3) = 3, f(0) = 3, and f(4) = 3.



? Exercise 3.3.2

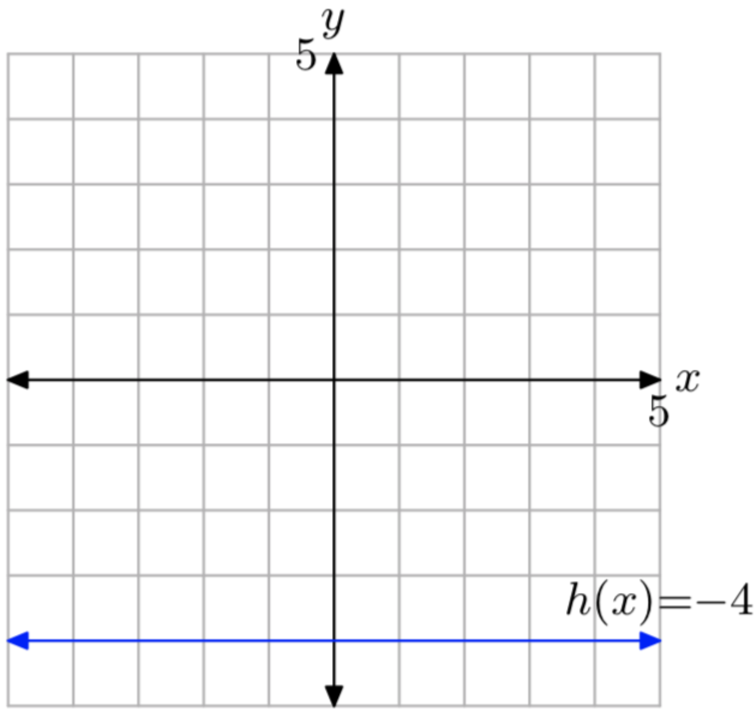
Given the function defined by the rule g(x) = 2, evaluate g(−2), g(0) and g(4), then draw the draw the graph of g.

? Exercise 3.3.3

Given the function defined by the rule h(x) = −4, evaluate h(−2), h(a), and h(2x+3), then draw the graph of h.

Answer

h(−2) = −4, h(a) = −4, and h(2x+3) = −4.



Exercise 3.3.4

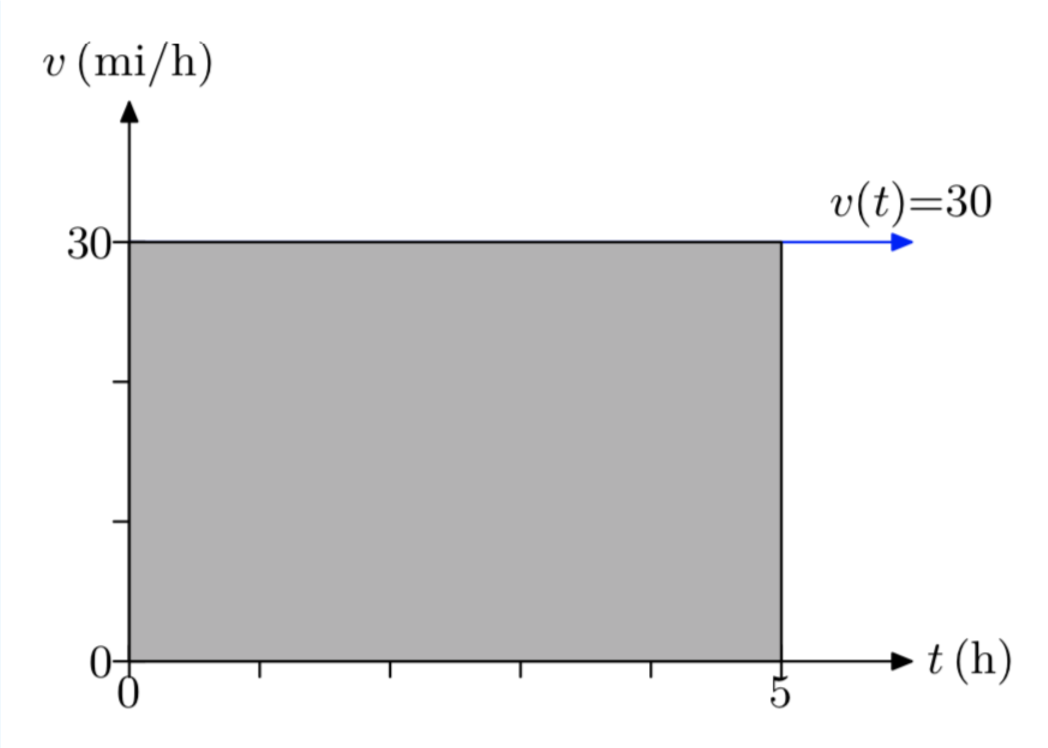
Given the function defined by the rule f(x) = −2, evaluate f(0), f(b), and f(5−4x), then draw the graph of f.

Exercise 3.3.5

The speed of an automobile traveling on the highway is a function of time and is described by the constant function v(t) = 30, where t is measured in hours and v is measured in miles per hour. Draw the graph of v versus t. Be sure to label each axis with the appropriate units. Shade the area under the graph of v over the time interval [0,5] hours. What is the area under the graph of v over this time interval and what does it represent?

Answer

The area under the curve is 150 miles. This is the distance traveled by the car.



? Exercise 3.3.6

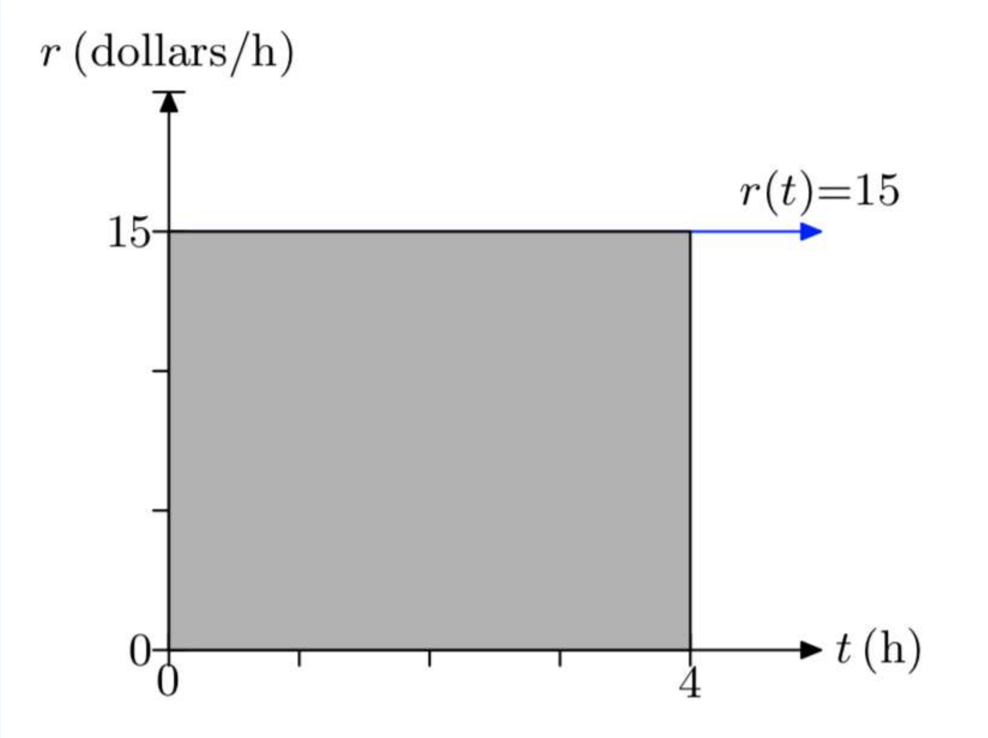
The speed of a skateboarder as she travels down a slope is a function of time and is described by the constant function v(t) = 8, where t is measured in seconds and v is measured in feet per second. Draw the graph of v versus t. Be sure to label each axis with the appropriate units. Shade the area under the graph of v over the time interval [0,60] seconds. What is the area under the graph of v over this time interval and what does it represent?

? Exercise 3.3.7

An unlicensed plumber charges 15 dollars for each hour of labor. Let's define this rate as a function of time by r(t) = 15, where t is measured in hours and r is measured in dollars per hour. Draw the graph of r versus t. Be sure to label each axis with appropriate units. Shade the area under the graph of r over the time interval [0,4] hours. What is area under the graph of r over this time interval and what does it represent?

Answer

The area under the curve is 150 miles. This is the distance traveled by the car.



? Exercise 3.3.8

A carpenter charges a fixed rate for each hour of labor. Let's describe this rate as a function of time by r(t) = 25, where t is measured in hours and r is measured in dollars per hour. Draw the graph of r versus t. Be sure to label each axis with appropriate units. Shade the area under the graph of r over the time interval [0, 5] hours. What is the area under the graph of r over this time interval and what does it represent?

? Exercise 3.3.9

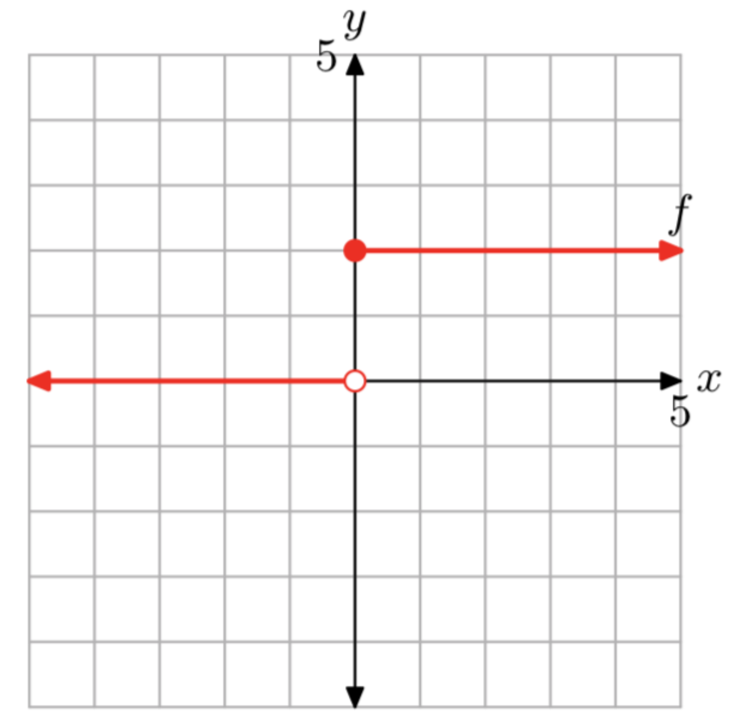
Given the function defined by the rule

$$f(x)=\begin{cases}0, & \text{if } x<0\\ 2, & \text{if } x\geq 0\end{cases}$$

evaluate f(−2), f(0), and f(3), then draw the graph of f on a sheet of graph paper. State the domain and range of f.

Answer

f(−2) = 0, f(0) = 2, and f(3) = 2.



The domain of f is the set of all real numbers. The range of f is {0, 2}.

Exercise 3.3.10

Given the function defined by the rule

$$f(x)=\begin{cases}2, & \text{if } x<0\\ 0, & \text{if } x\geq 0\end{cases}$$

evaluate f(−2), f(0), and f(3), then draw the graph of f on a sheet of graph paper. State the domain and range of f.

Exercise 3.3.11

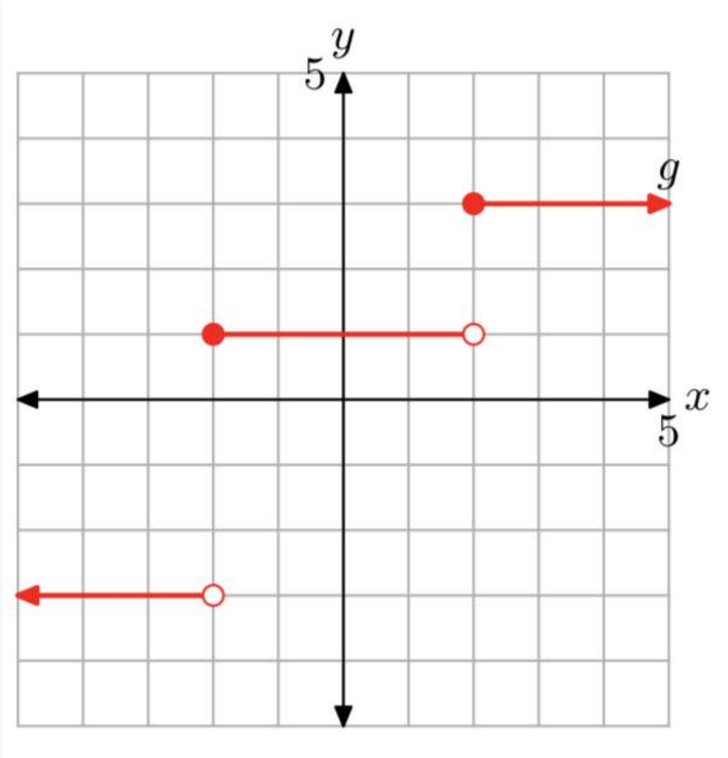
Given the function defined by the rule

$$f(x)=\begin{cases}-3, & \text{if } x<0\\ 1, & \text{if } -2\leq x<2\\ 3, & \text{if } x\geq 2\end{cases}$$

evaluate g(−3), g(−2), and g(5), then draw the graph of g on a sheet of graph paper. State the domain and range of g.

Answer

g(−3) = −3, g(−2) = 1, and g(5) = 3



The domain of g is all real numbers. The range of g is {−3, 1, 3}.

? Exercise 3.3.12

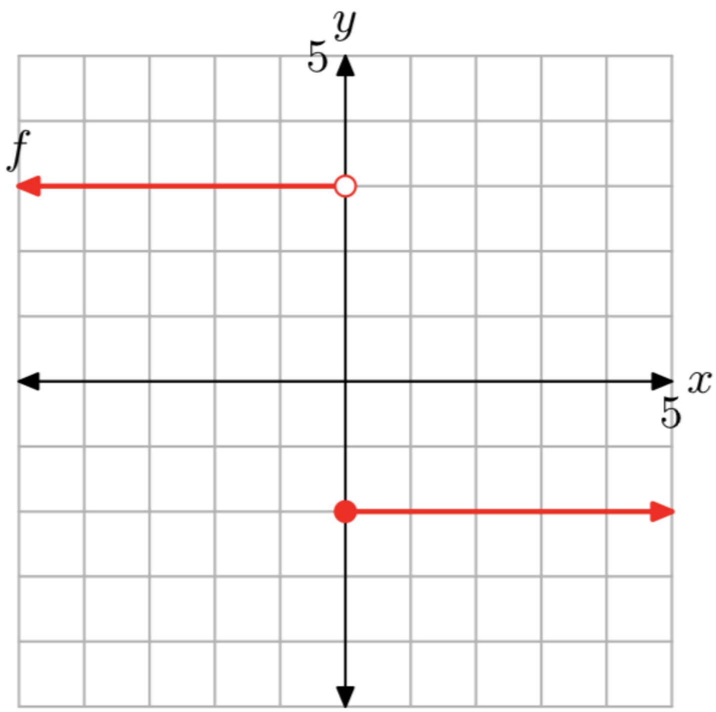
Given the function defined by the rule

$$f(x)=\begin{cases}4, & \text{if } x \leq -1\\ 2, & \text{if } -1<x \leq 2\\ -3, & \text{if } x>2\end{cases}$$

evaluate g(−1), g(2), and g(3), then draw the graph of g on a sheet of graph paper. State the domain and range of g.

In **Exercises 13-16**, determine a piecewise definition of the function described by the graphs, then state the domain and range of the function.

? Exercise 3.3.13

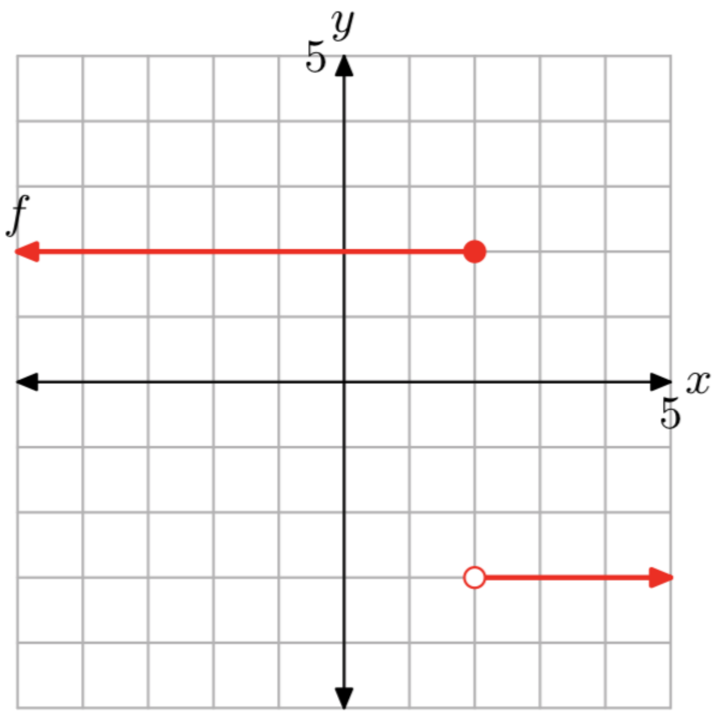


Answer

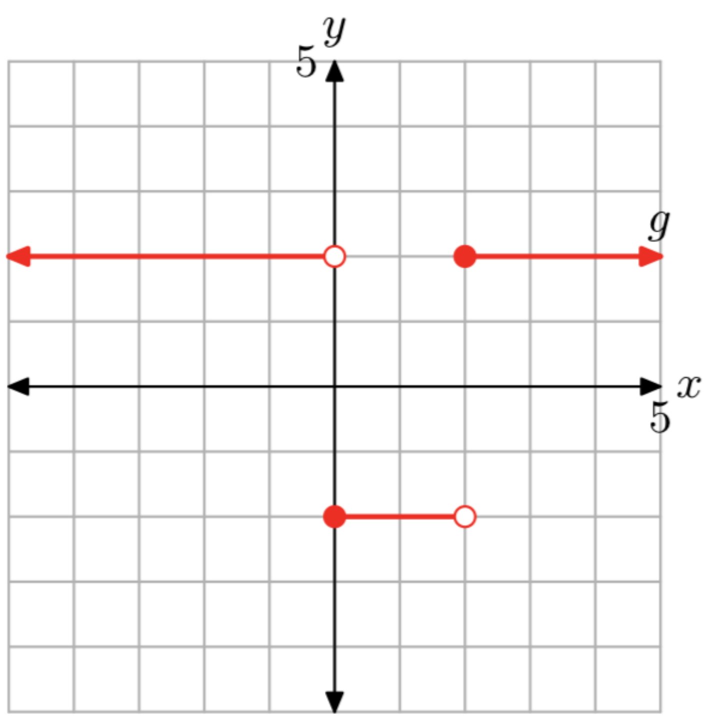
$$f(x)=\begin{cases}3, & \text{if } x<0\\ -2, & \text{if } x\geq 0\end{cases}$$

Domain of f is the set of all real numbers. The range of f is {−2, 3}.

Exercise 3.3.14



? Exercise 3.3.15

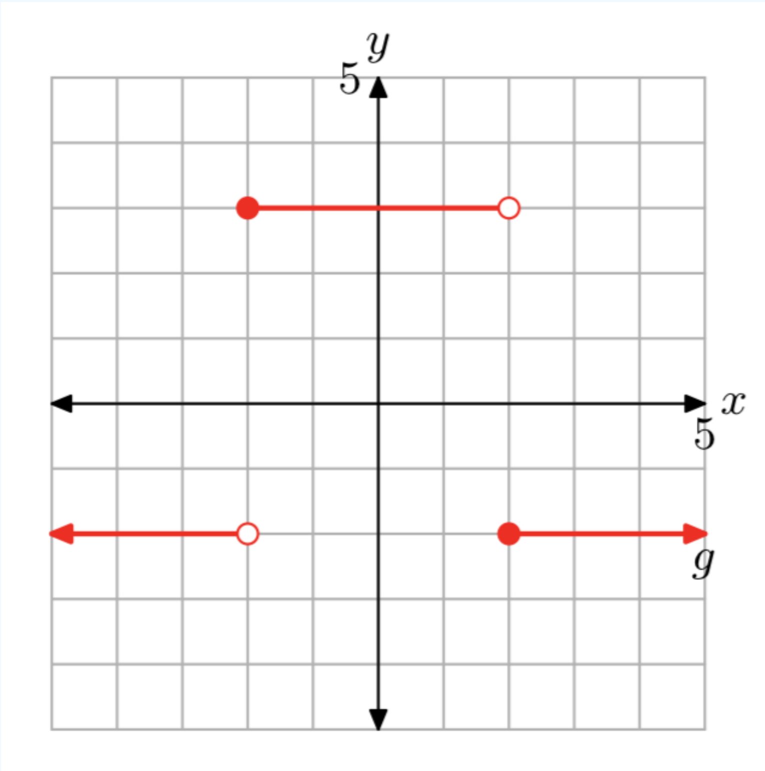


Answer

$$g(x)=\begin{cases}2, & \text{if } x<0\\ -2, & \text{if } 0\leq x<2\\ 2, & \text{if } x>2\end{cases}$$

The domain of f is the set of all real numbers. The range of f is {−2, 2}.

Exercise 3.3.16



Exercise 3.3.17

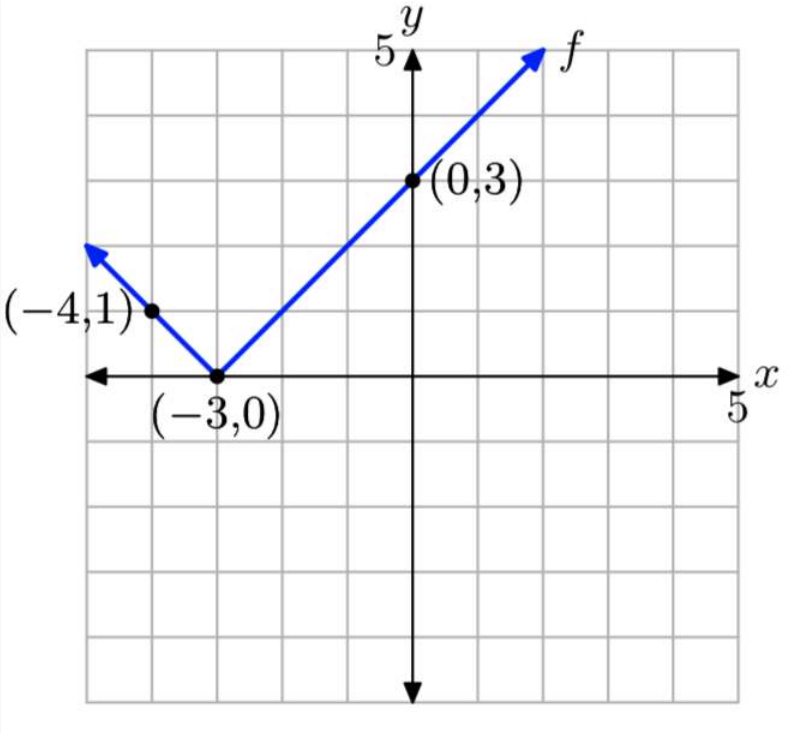
Given the piecewise function

$$f(x)=\begin{cases}-x-3, & \text{if } x<-3\\ x+3, & \text{if } x\geq -3\end{cases}$$

evaluate f(−4) and f(0), then draw the graph of f on a sheet of graph paper. State the domain and range of the function.

Answer

f(−4) = 1 and f(0) = 3.



The domain of f is the set of all real numbers. The range off is$\{$y: $y \geq 0\}$.

Exercise 3.3.18

Given the piecewise function

$$f(x) = \begin{cases} -x+1, & \text{if } x < 1 \\ x-1, & \text{if } x \geq 1 \end{cases}$$

evaluate f(−2) and f(3), then draw the graph of f on a sheet of graph paper. State the domain and range of the function.

EXERICSE 3.3.19

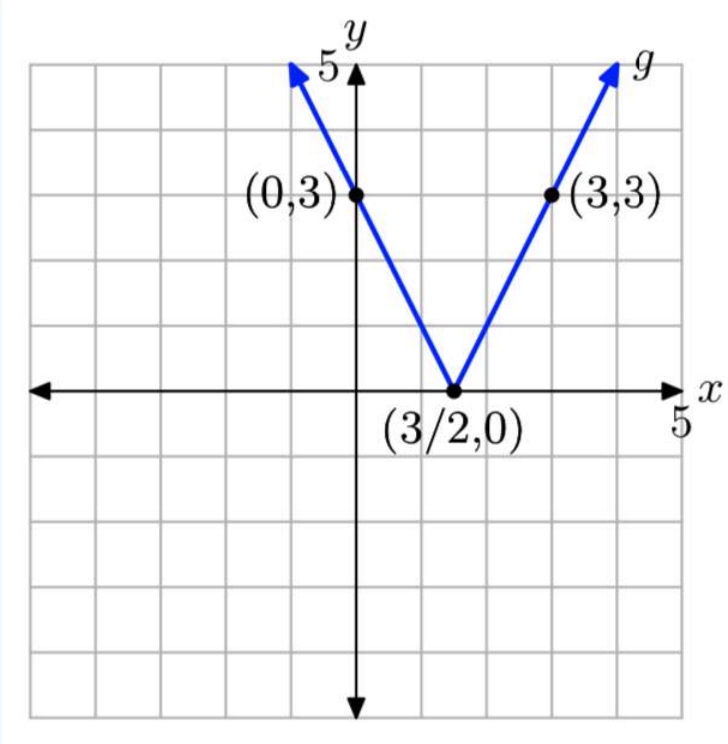
Given the piecewise function

$$g(x) = \begin{cases} -2x+3, & \text{if } x < \frac{3}{2} \\ 2x-3, & \text{if } x \geq \frac{3}{2} \end{cases}$$

evaluate g(0) and g(3), then draw the graph of f on a sheet of graph paper. State the domain and range of the function.

Answer

g(−2) = 7 and g(2) = 1.



The domain of g is the set of all real numbers. The range off is{y: $y \geq 0$}.

Exercise 3.3.20

Given the piecewise function

$$g(x) = \begin{cases} -3x-4, & \text{if } x < -\frac{4}{3} \\ 3x+4, & \text{if } x \geq -\frac{4}{3} \end{cases}$$

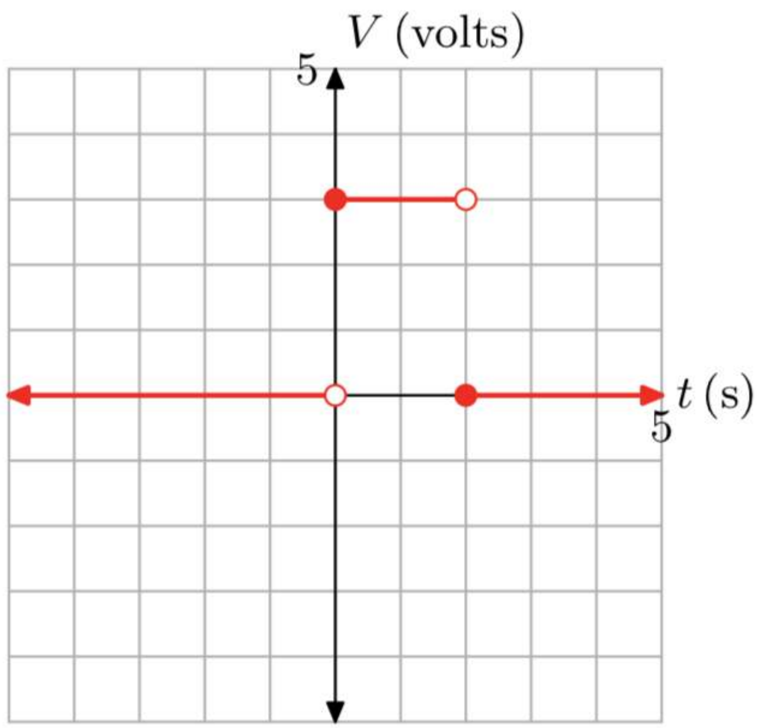
evaluate g(−2) and g(3), then draw the graph of f on a sheet of graph paper. State the domain and range of the function.

Exercise 3.3.21

A battery supplies voltage to an electric circuit in the following manner. Before time t = 0 seconds, a switch is open, so the voltage supplied by the battery is zero volts. At time t = 0 seconds, the switch is closed and the battery begins to supply a constant 3 volts to the circuit. At time t = 2 seconds, the switch is opened again, and the voltage supplied by the battery drops immediately to zero volts. Sketch a graph of the voltage vversus time t, label each axis with the appropriate units, then provide a piecewise definition of the voltage v supplied by the battery as a function of time t.

Answer

The graph follows.



$$g(x)=\begin{cases}0, & \text{if } x<0\\ 3, & \text{if } 0\le x<2\\ 0, & \text{if } x\ge 2\end{cases}$$

? Exercise 3.3.22

Prior to time t = 0 minutes, a drum is empty. At time t = 0 minutes a hose is turned on and the water level in the drum begins to rise at a constant rate of 2 inches every minute. Let h represent water level (in inches) at time t (in minutes). Sketch the graph of h versust, label the axes with appropriate units, then provide a piecewise definition of has a function of t.

3.4: Absolute Value

Now that we have the fundamentals of piecewise-defined functions in place, we are ready to introduce the absolute value function. First, let's state a trivial reminder of what it means to take the absolute value of a real number.

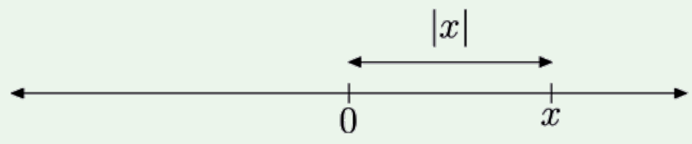
In a sense, the absolute value of a number is a measure of its magnitude, sans (without) its sign. Thus,

$$|7|=7 \qquad \text{and} \qquad |-7|=7$$

Here is the formal definition of the absolute value of a real number.

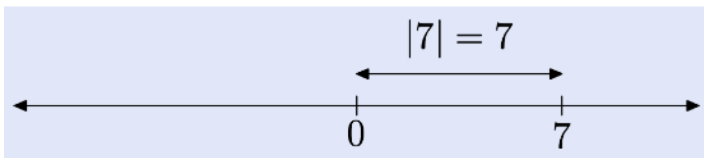
Definition: Absolute Value

To find the absolute value of any real number, first locate the number on the real line.

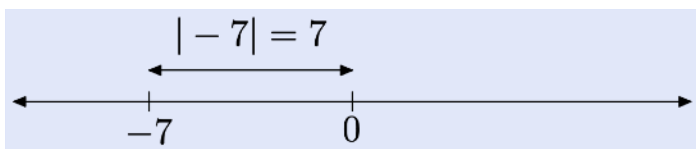


The absolute value of the number is defined as its distance from the origin.

For example, to find the absolute value of 7, locate 7 on the real line and then find its distance from the origin.



To find the absolute value of −7, locate −7 on the real line and then find its distance from the origin.



Some like to say that taking the absolute value "produces a number that is always positive." However, this ignores an important exception, that is,

$$|0|=0$$

Thus, the correct statement is "the absolute value of any real number is either positive or it is zero," i.e., the absolute value of a real number is "not negative."2 Instead of using the phrase "not negative," mathematicians prefer the word "nonnegative." When we take the absolute value of a number, the result is always nonnegative; that is, the result is either positive or zero. In symbols,

$$|x| \geq 0 \text{ for all real numbers } x$$

This makes perfect sense in light of Definition 2. Distance is always nonnegative.

However, the discussion above is not of sufficient depth to handle more sophisticated problems involving absolute value.

A Piecewise Definition of Absolute Value

Because absolute value is intimately connected with distance, mathematicians and scientists find it an invaluable tool for measurement and error analysis. However, we will need a formulaic definition of the absolute value if we want to use this tool in a meaningful way. We need to develop a piecewise definition of the absolute value function, one that will define the absolute value for any arbitrary real number x.

We begin with a few observations. Remember, the absolute value of a number is always nonnegative (positive or zero).

1. If a number is negative, negating that number will make it positive. $|-5| = -(-5) = 5$, and similarly, $|-12| = -(-12) = 12$. Thus, if $x < 0$ (if x is negative), then $|x| = -x$.
2. If $x = 0$, then $|x| = 0$.
3. If a number is positive, taking the absolute value of that number will not change a thing.

$$|5| = 5, \text{ and similarly, } |12| = 12$$

Thus, if $x > 0$ (if x is positive), then $|x| = x$.

We can summarize these three cases with a piecewise definition.

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ x, & \text{if } x > 0 \end{cases}$$

It is the first line in our piecewise definition (4) that usually leaves students scratching their heads. They might say "I thought absolute value makes a number positive (or zero), yet you have $|x| = -x$; that is, you have the absolute value of x equal to a negative x." Try as they might, this seems contradictory. Does it seem so to you?

However, there is no contradiction. If $x < 0$, that is, if x is a negative number, then $-x$ is a positive number, and our intuitive notion of absolute value is not dissimilar to that of our piecewise definition (4). For example, if $x = -8$, then $-x = 8$, and even though we say "negative x," in this case $-x$ is a positive number.

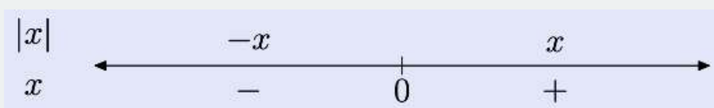
If this still has you running confused, consider the simple fact that x and $-x$ must have "opposite signs." If one is positive, the other is negative, and vice versa. Consequently,

- if x is positive, then $-x$ is negative, but
- if x is negative, then $-x$ is positive.

Let's summarize what we've learned thus far.

Summarizing the Definition on a Number Line

We like to use a number line to help summarize the definition of the absolute value of x.



Some remarks are in order for this summary on the number line.

- We first draw the real line then mark the "critical value" for the expression inside the absolute value bars on the number line. The number zero is a critical value for the expression x, because x changes sign as you move from one side of zero to the other.
- To the left of zero, x is a negative number. We indicate this with the minus sign below the number line. To the right of zero, x is a positive number, indicated with a plus sign below the number line.
- Above the number line, we simplify the expression $|x|$. To the left of zero, x is a negative number (look below the line), so $|x| = -x$. Note how the result $-x$ is placed above the line to the left of zero. Similarly, to the right of zero, x is a positive number (look below the line), so $|x| = x$. Note how the result x is placed above the line to the right of zero.

In the piecewise definition of $|x|$ in (4), note that we have three distinct pieces, one for each case discussed above. However, because $|0| = 0$, we can include this case with the piece $|x| = x$, if we adjust the condition to include zero.

Definition

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Note that this piecewise definition agrees with our discussion to date.

1. In the first line of equation (6), if x is a negative number (i.e., if $x<0$), then the absolute value must change x to a positive number by negating. That is, |x| = −x.
2. In the second line of equation (6), if x is positive or zero (i.e., if $x \geq 0$), then there's nothing to do except remove the absolute value bars. That is, |x| = x.

Because |0| = −0, we could just as well include the case for zero on the left, defining the absolute value with

$$|x| = \begin{cases} -x, & \text{if } x \leq 0 \\ x, & \text{if } x > 0 \end{cases}$$

However, in this text we will always include the critical value on the right, as shown in Definition 5.

Constructing Piecewise Definitions

Let's see if we can determine piecewise definitions for other expressions involving absolute value.

Example 3.4.1

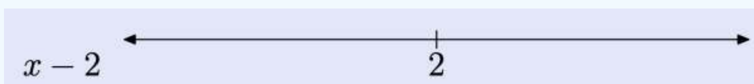
Determine a piecewise definition for |x − 2|.

Solution

First, set the expression inside the absolute value bars equal to zero and solve for x.

$$\begin{aligned} x-2 &= 0 \\ x &= 2 \end{aligned}$$

Note that x − 2 = 0 at x = 2. This is the "critical value" for this expression. Draw a real line and mark this critical value of x on the line. Place the expression x − 2 below the line at its left end.



Next, determine the sign of x − 2 for values of x on each side of 2. This is easily done by "testing" a point on each side of 2 in the expression x − 2.

- Take x = 1, which lies to the left of the critical value 2 on our number line. Substitute this value of x in the expression x − 2, obtaining

$$x-2=1-2=-1$$

which is negative. Indeed, regardless of which x-value you pick to the left of 2, when inserted into the expression x − 2, you will get a negative result (you should check this for other values of x to the left of 2). We indicate that the expression x − 2 is negative for values of x to the left of 2 by placing a minus (−) sign below the number line to the left of 2.

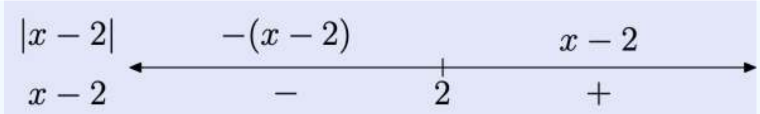
- Next, pick x = 3, which lies to the right of the critical value 2 on the number line. Substitute this value of x into the expression x − 2, obtaining

$$x-2=3-2=1$$

which is positive. Indeed, regardless of which x-value you pick to the right of 2, when inserted into the expression x − 2, you will get a positive result (you should check this for other values of x to the right of 2). We indicate that the expression x − 2 is positive for values of x to the right of 2 by placing a plus (+) sign below the number line to the right of 2 (see the number line above).

The next step is to remove the absolute value bars from the expression |x−2|, depending on the sign of x − 2.

- To the left of 2, the expression x − 2 is negative (note the minus sign (−) below the number line), so |x − 2| = −(x − 2). That is, we have to negate x − 2 to make it positive. This is indicated by placing −(x − 2) above the line to the left of 2.



- To the right of 2, the expression x − 2 is positive (note the plus sign (+) below the line), so |x − 2| = x − 2. That is, we simply remove the absolute value bars because the quantity inside is already positive. This is indicated by placing x − 2 above the line to the right of 2 (see the number line above).

We can use this last number line summary to construct a piecewise definition of the expression |x − 2|.

$$|x-2| = \begin{cases} -(x-2), & \text{if } x < 2, \\ x-2, & \text{if } x \geq 2 \end{cases} = \begin{cases} -x+2, & \text{if } x < 2 \\ x-2, & \text{if } x \geq 2 \end{cases}$$

Our number line and piecewise definition agree: |x − 2| = −(x − 2) to the left of 2 and |x − 2| = x − 2 to the right of 2. Further, note how we've included the critical value of 2 "on the right" in our piecewise definition.

Let's summarize the method we followed to construct the piecewise function above.

Constructing a Piecewise Definition for Absolute Value

When presented with the absolute value of an algebraic expression, perform the following steps to remove the absolute value bars and construct an equivalent piecewise definition.

1. Take the expression that is inside the absolute value bars, and set that expression equal to zero. Then solve for x. This value of x is called a "critical value." (Note: The expression inside the absolute value bars could have more than one critical value. We will not encounter such problems in this text.)
2. Place your critical value on a number line.
3. Place the expression inside the absolute value bars below the number line at the left end.
4. Test the sign of the expression inside the absolute value bars by inserting a value of x from each side of the critical value and marking the result with a plus (+) or minus (−) sign below the number line.
5. Place the original expression, the one including the absolute value bars, above the number line at the left end.
6. Use the sign of the expression inside the absolute value bars (indicated by the plus and minus signs below the number line) to remove the absolute value bars, placing the results above the number line on each side of the critical value.
7. Construct a piecewise definition that mimics the results on the number line.

Let's apply this technique to another example.

Example 3.4.2

Determine a piecewise definition for |3 − 2x|.

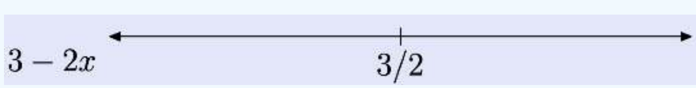
Solution

Step 1: First set the expression inside the absolute value bars equal to zero and solve for x.

$$\begin{aligned} 3-2x &= 0 \\ x &= 3/2 \end{aligned}$$

Note that 3 − 2x = 0 at x = 3/2. This is the "critical value" for this expression.

Steps 2 and 3: Draw a number line and mark this critical value on the line. The next step requires that we place the expression inside the absolute value bars, namely 3 − 2x, underneath the line at its left end.



Step 4: Next, determine the sign of 3 − 2x for values of x on each side of 3/2. This is easily done by "testing" a point on each side of 3/2 in the expression 3 − 2x.

- Take x = 1, which lies to the left of 3/2. Substitute this value of x into the expression 3 − 2x, obtaining

$$3-2x=3-2(1)=1$$

which is positive. Indicate this result by placing a plus sign (+) below the number line to the left of 3/2.

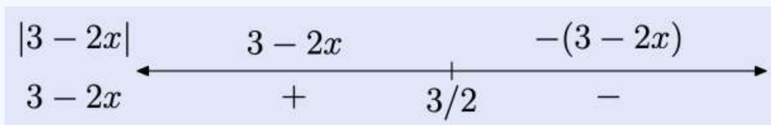
- Next, pick x = 2, which lies to the right of 3/2. Substitute this value of x into the expression 3 − 2x, obtaining

$$3-2x=3-2(2)=-1$$

which is negative. Indicate this result by placing a negative sign (−) below the line to the right of 3/2 (see the number line above).

Steps 5 and 6: Place the original expression, namely |3 − 2x|, above the number line at the left end. The next step is to remove the absolute value bars from the expression |3 − 2x|.

- To the left of 3/2, the expression 3 − 2x is positive (note the plus sign (+) below the number line), so |3−2x| = 3−2x. Indicate this result by placing the expression 3 − 2x above the number line to the left of 3/2.



- To the right of 3/2, the expression 3−2x is negative (note the minus sign (−) below the numberline), so |3−2x| = −(3−2x). That is, we have to negate 3−2x to make it positive. This is indicated by placing the expression −(3 − 2x) above the line to the right of 3/2 (see the number line above).

Step 7: We can use this last number line summary to write a piecewise definition for the expression |3 − 2x|.

$$|3-2x|=\begin{cases}3-2x, & \text{if } x<3/2. \\ -(3-2x), & \text{if } x\geq 3/2\end{cases}=\begin{cases}3-2x, & \text{if } x<3/2 \\ -3+2x, & \text{if } x\geq 3/2\end{cases}$$

Again, note how we've included the critical value of 3/2 "on the right."

Drawing the Graph of an Absolute Value Function

Now that we know how to construct a piecewise definition for an expression containing absolute value bars, we can use what we learned in the previous section to draw the graph.

✓ Example 3.4.3

Sketch the graph of the function f(x) = |3 − 2x|.

Solution

In Example $3.4.2$, we constructed the following piecewise definition.

$$f(x)=|3-2x|=\begin{cases}3-2x, & \text{if } x<3/2 \\ -3+2x, & \text{if } x\geq 3/2\end{cases}$$

We now sketch each piece of this function.

- If x < 3/2, then f(x) = 3 − 2x (see equation (10)). This is a ray, starting at x = 3/2 and extending to the left. At x = 3/2,

$$f(3/2)=3-2(3/2)=3-3=0$$

Thus, the endpoint of the ray is located at (3/2, 0).

Next, pick a value of x that lies to the left of 3/2. At x = 0,

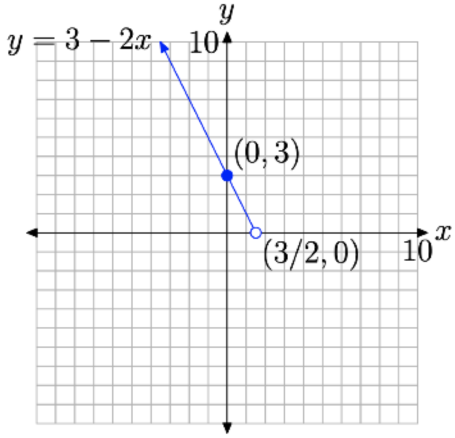
$$f(0)=3-2(0)=3-0=3$$

Thus, a second point on the ray is (0, 3).

A table containing the two evaluated points and a sketch of the accompanying ray are shown in Figure 3.4.1. Because f(x) = 3 − 2x only if x is strictly less than 3/2, the point at (3/2, 0) is unfilled.

x	$f(x) = 3 - 2x$	$(x, f(x))$
3/2	0	(3/2, 0)
0	3	(0, 3)

(a)



(b)

Figure 3.4.1 f(x) = 3 − 2x when x < 3/2.

- If x ≥ 3/2, then f(x) = −3 + 2x (see equation (10)). This is a ray, starting at x = 3/2 and extending to the right. At x = 3/2,

$$f(3/2) = -3 + 2(3/2) = -3 + 3 = 0$$

Thus, the endpoint of the ray is located at (3/2, 0).

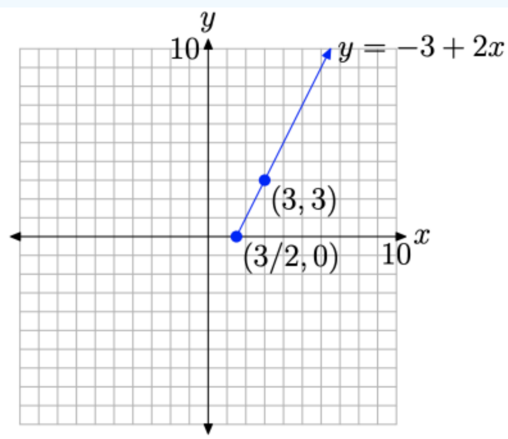
Next, pick a value of x that lies to the right of 3/2. At x = 3,

$$f(3) = -3 + 2(3) = -3 + 6 = 3$$

Thus, a second point on the ray is (3, 3). A table containing the two evaluated points and a sketch of the accompanying ray are shown in Figure 3.4.2. Because f(x) = −3 + 2x for all values of x that are greater than or equal to 3/2, the point at (3/2, 0) is filled in this plot.

x	$f(x) = -3 + 2x$	$(x, f(x))$
3/2	0	(3/2, 0)
3	3	(−3, 3)

(a)



(b)

Figure 3.4.2 f(x) = −3 + 2x when $x \geq 3/2$.

- To sketch the graph of f(x) = |3 − 2x|, we need only combine the two pieces from Figures 3.4.1 and 3.4.2. The result is shown in Figure 3.4.3.

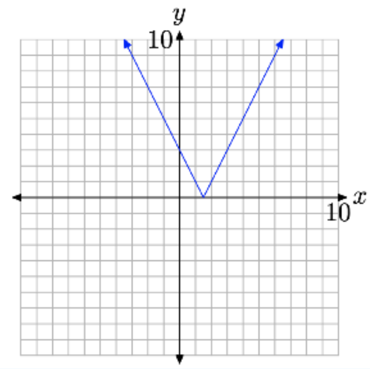


Figure 3.4.3 The graph of f(x) = |3 − 2x|.

Note the “V-shape” of the graph. We will refer to the point at the tip of the “V” as the vertex of the absolute value function.

In Figure 3.4.3, the equation of the left-hand branch of the “V” is y = 3 − 2x. An alternate approach to drawing this branch is to note that its graph is contained in the graph of the full line y = 3 − 2x, which has slope −2 and y-intercept at (0, 3). Thus, one could draw the full line using the slope and y-intercept, then erase that part of the line that lies to the right of x = 3/2. A similar strategy would work for the right-hand branch of y = |3 − 2x|.

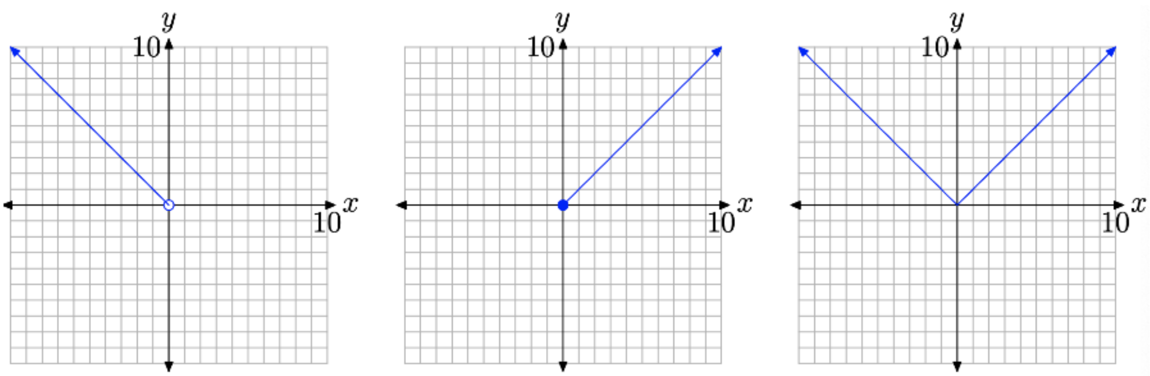
Using Transformations

Consider again the basic definition of the absolute value of x.

$$f(x) = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Some basic observations are:

- If x < 0, then f(x) = −x. This ray starts at the origin and extends to the left with slope −1. Its graph is pictured in Figure 3.4.4(a).
- If $x \geq 0$, then f(x) = x. This ray starts at the origin and extends to the right with slope 1. Its graph is pictured in Figure 3.4.4(b).
- We combine the graphs in Figures 3.4.4(a) and 3.4.4(b) to produce the graph of f(x) = |x| in Figure 3.4.4(c).



(a) $f(x) = -x$, if $x < 0$. (b) $f(x) = x$, if $x \geq 0$. (c) $f(x) = |x|$.

Figure 3.4.4

Combine left and right branches to produce the basic graph of f(x) = |x|.

You should commit the graph of f(x) = |x| to memory. Things to note:

- The graph of f(x) = |x| is “V-shaped.”
- The vertex of the graph is at the point (0, 0).
- The left-hand branch has equation y = −x and slope −1.
- The right-hand branch has equation y = x and slope 1.
- Each branch of the graph of f(x) = |x| forms a perfect 45∘ angle with the x-axis.

Now that we know how to draw the graph of f(x) = |x|, we can use the transformations we learned in Chapter 2 (sections 5 and 6) to sketch a number of simple graphs involving absolute value.

✓ Example 3.4.4

Sketch the graph of f(x) = |x − 3|.

Solution

First, sketch the graph of y = f(x) = |x|, as shown in Figure 3.4.5(a). Note that if f(x) = |x|, then

$$y = f(x-3) = |x-3|$$

To sketch the graph of y = f(x − 3) = |x − 3|, shift the graph of y = f(x) = |x| three units to the right, producing the result shown in Figure 3.4.5(b).

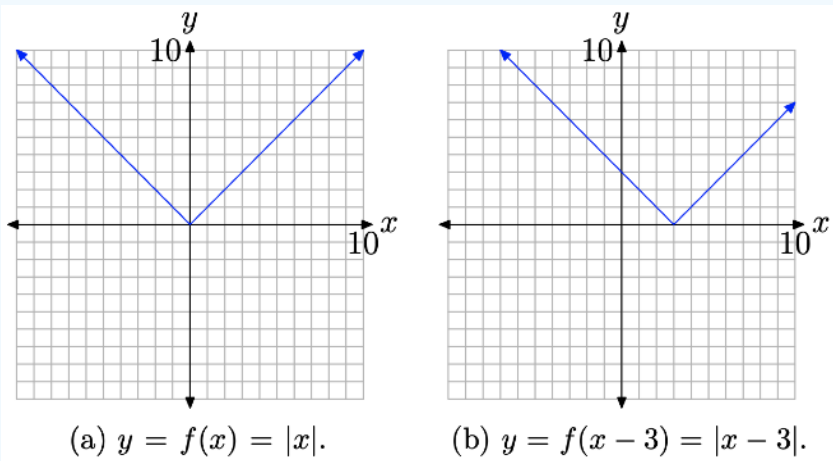


Figure 3.4.5 To draw the graph of y = |x − 3|, shift the graph of y = |x| three units to the right.

We can check this result using the graphing calculator. Load the function f(x) = |x − 3| into Y1 in the Y= menu on your graphing calculator as shown in Figure 3.4.6(a). Push the MATH button, right-arrow to the NUM menu, then select 1:abs((see Figure 3.4.6(b)) to enter the absolute value in Y1. Push the ZOOM button, then select 6:ZStandard to produce the image shown in Figure 3.4.6(c).

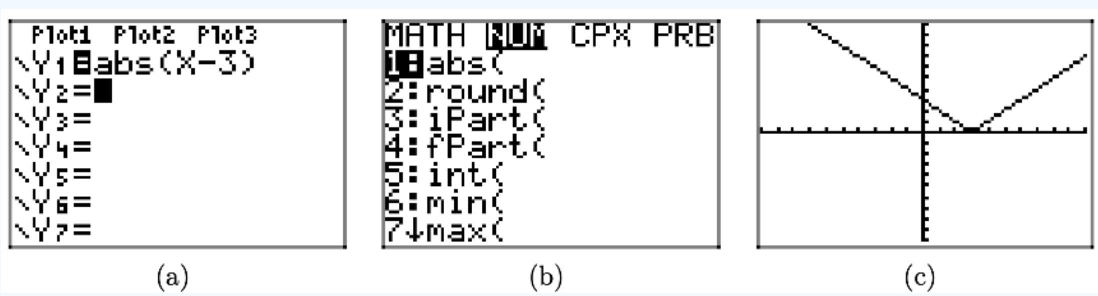


Figure 3.4.6 Using the graphing calculator to draw the graph of f(x) = |x − 3|.

Let's look at another simple example.

✓ Example 3.4.5

Sketch the graph of f(x) = |x| − 4.

Solution

First, sketch the graph of y = f(x) = |x|, as shown in Figure 3.4.7(a). Note that if f(x) = |x|, then

$$y = f(x) - 4 = |x| - 4$$

To sketch the graph of y = f(x) − 4 = |x| − 4, shift the graph of y = f(x) = |x| downward 4 units, producing the result shown in Figure 3.4.5(b).

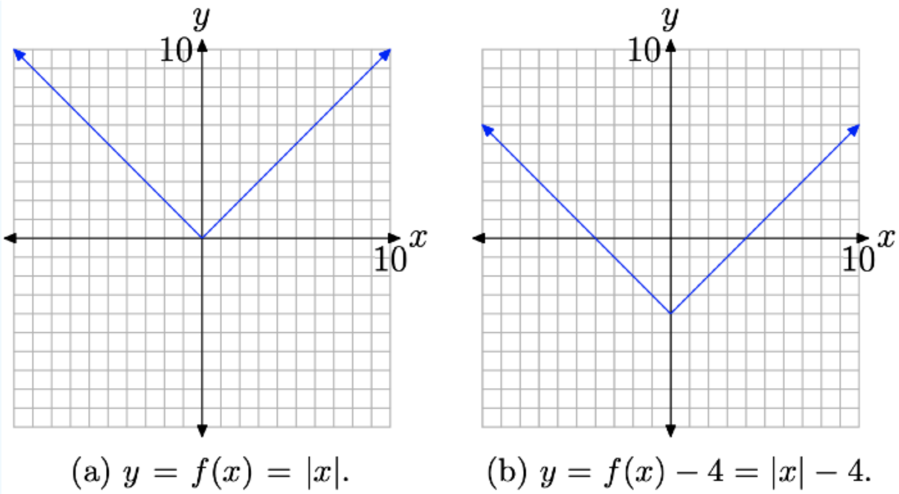


Figure $3.4.7$ To draw the graph of y = |x| − 4, shift the graph of y = |x| downward 4 units.

Let's look at one final example.

Example $3.4.6$

Sketch the graph of f(x) = −|x| + 5. State the domain and range of this function.

Solution

- First, sketch the graph of y = f(x) = |x|, as shown in Figure $3.4.8$(a).
- Next, sketch the graph of y = −f(x) = −|x|, which is a reflection of the graph of y = f(x) = |x| across the x-axis and is pictured in Figure $3.4.8$(b).
- Finally, we will want to sketch the graph of y = −f(x) + 5 = −|x| + 5. To do this, we shift the graph of y = −f(x) = −|x| in Figure $3.4.8$(b) upward 5 units to produce the result in Figure $3.4.8$(c).

To find the domain of f(x) = −|x| + 5, project all points on the graph onto the x-axis, as shown in Figure $3.4.9$(a). Thus, the domain of f is $(-\infty, \infty)$. To find the range, project all points on the graph onto the y-axis, as shown in Figure $3.4.9$(b). Thus, the range is $(-\infty, 5]$.

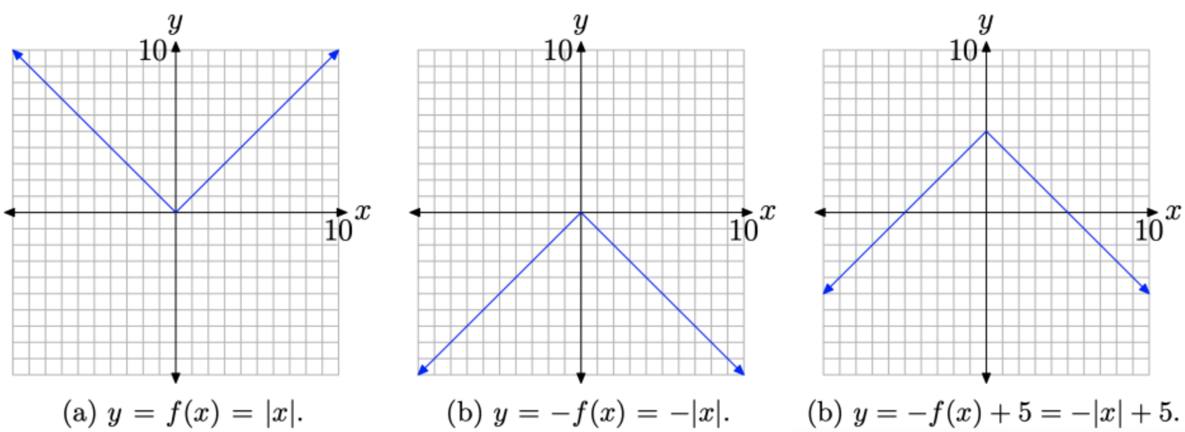
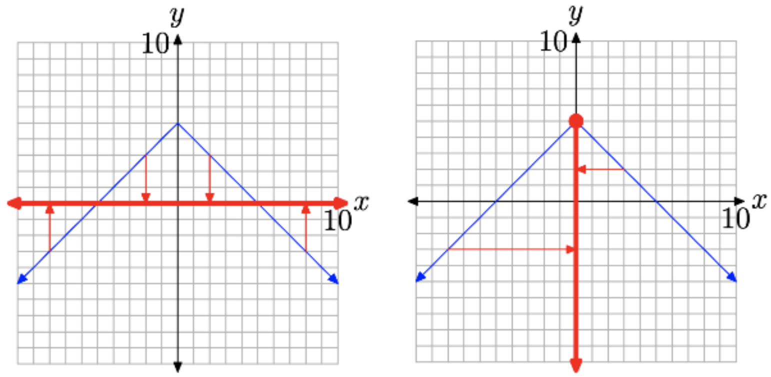


Figure $3.4.8$
To draw the graph of y = −|x| + 5, first reflect the graph of y = |x| across the x-axis to produce the graph of y = −|x|, then shift this



(a) Domain $= (-\infty, \infty)$. (b) Range $= (-\infty, 5]$.

result up 5 units to produce the graph of y = −|x| + 5.

Figure $3.4.9$ Projecting onto the axes to find the domain and range

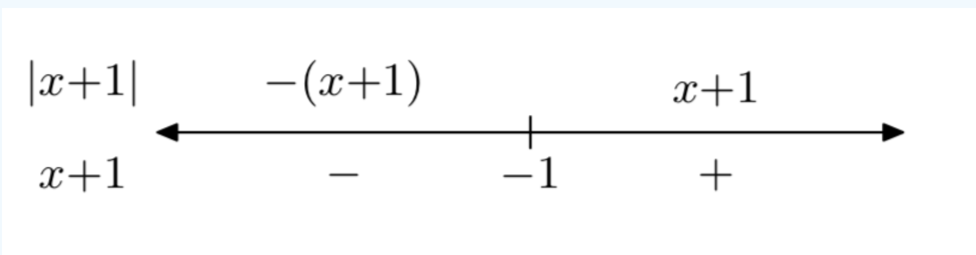
Exercise

For each of the functions in **Exercises 1-8**, as in Examples 7 and 8 in the narrative, mark the “critical value” on a number line, then mark the sign of the expression inside of the absolute value bars below the number line. Above the number line, remove the absolute value bars according to the sign of the expression you marked below the number line. Once your number line summary is finished, create a piecewise definition for the given absolute value function.

Exercise 3.4.1

f(x) = |x+1|

Answer



$$f(x)=\begin{cases}-x-1, & \text{if } x<-1\\ x+1, & \text{if } x\geq -1\end{cases}$$

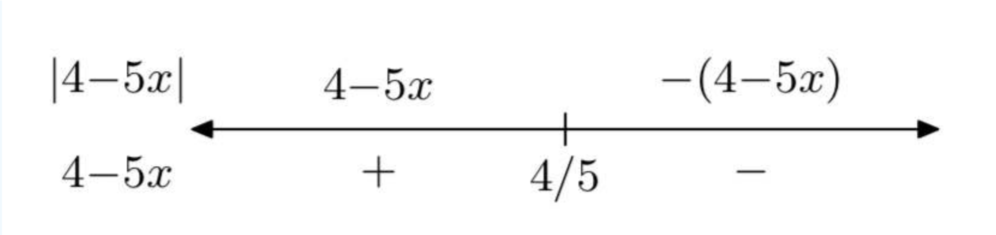
Exercise 3.4.2

f(x) = |x−4|

Exercise 3.4.3

g(x) = |4−5x|

Answer



$$g(x)=\begin{cases}4-5x, & \text{if } x<\frac{4}{5}\\ -4+5x, & \text{if } x\geq\frac{4}{5}\end{cases}$$

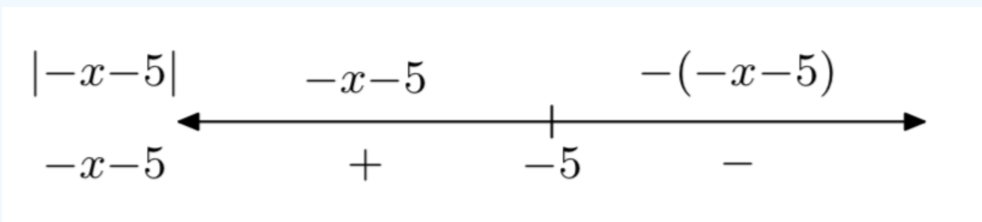
? Exercise 3.4.4

g(x) = |3−2x|

? Exercise 3.4.5

h(x) = |−x−5|

Answer



$$h(x)=\begin{cases}-x-5, & \text{if } x<-5\\ x+5, & \text{if } x\geq-5\end{cases}$$

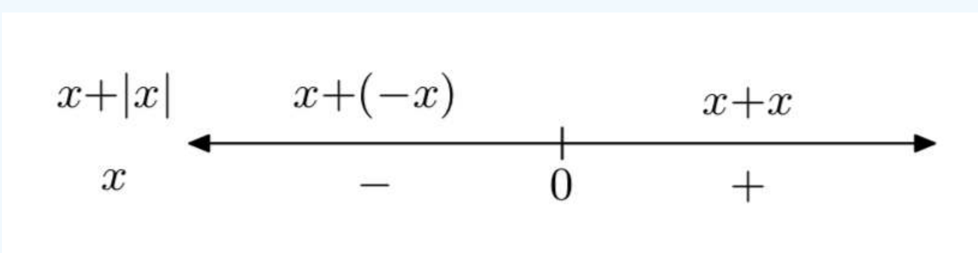
? Exercise 3.4.6

h(x) = |−x−3|

? Exercise 3.4.7

f(x) = x+|x|

Answer



$$f(x)=\begin{cases}0, & \text{if } x<0\\ 2x, & \text{if } x\geq 0\end{cases}$$

Exercise 3.4.8

$f(x)=\frac{|x|}{x}$

For each of the functions in **Exercises 9-16**, perform each of the following tasks.

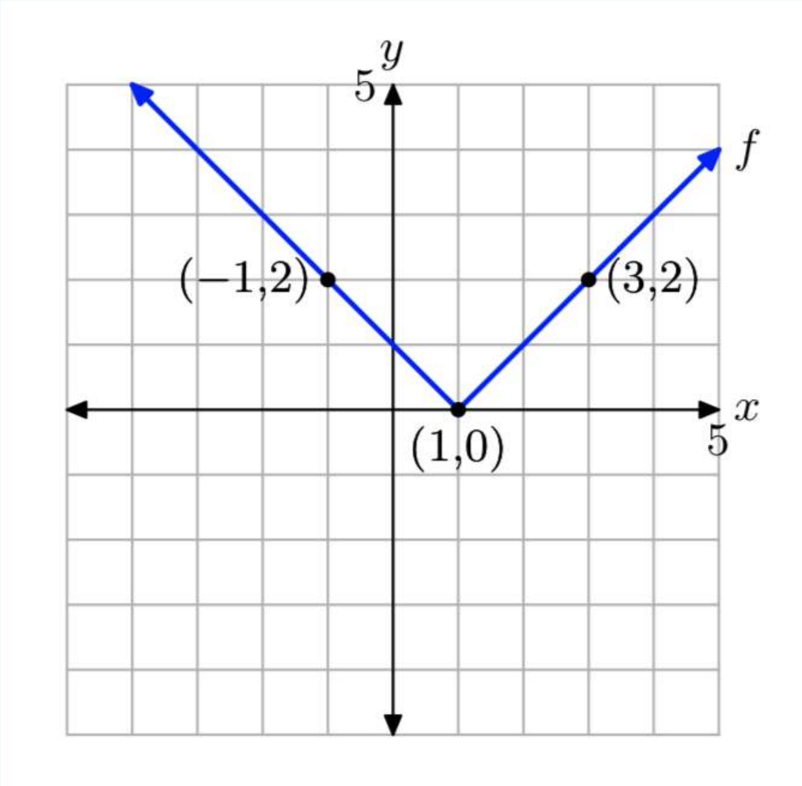
1. Create a piecewise definition for the given function, using the technique in **Exercises 1-8** and Examples 7 and 8 in the narrative.
2. Following the lead in Example 9 in the narrative, use your piecewise definition to sketch the graph of the given function on a sheet of graph paper. Please place each exercise on its own coordinate system.

Exercise 3.4.9

f(x) = |x−1|

Answer

$$f(x)=\begin{cases}-x+1, & \text{if } x<1\\ x-1, & \text{if } x\geq 1\end{cases}$$



Exercise 3.4.10

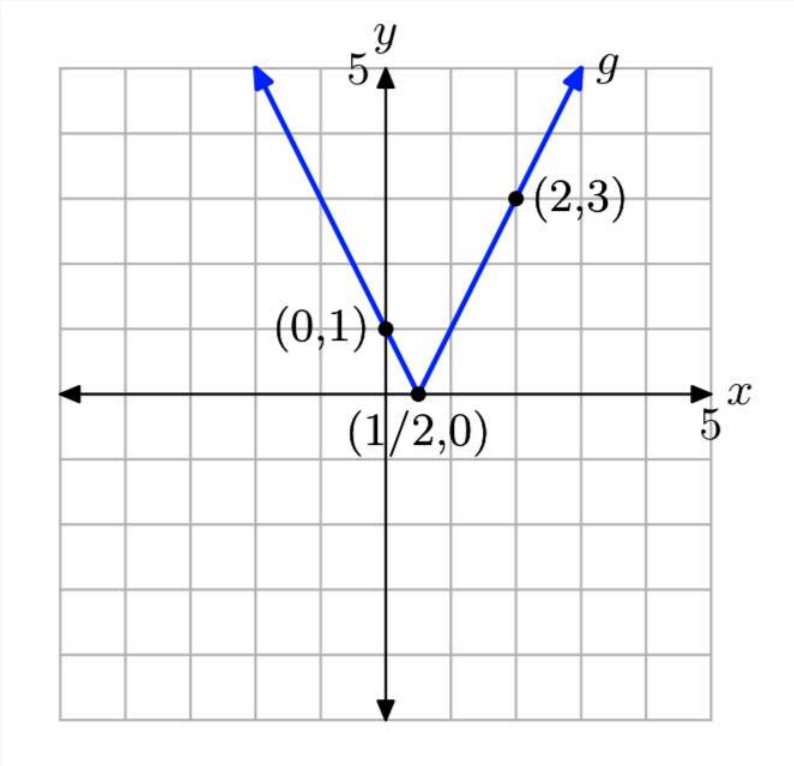
f(x) = |x+2|

Exercise 3.4.11

g(x) = |2x−1|

Answer

$$g(x)=\begin{cases}-2x+1, & \text{if } x<\frac{1}{2}\\ 2x-1, & \text{if } x\geq\frac{1}{2}\end{cases}$$



Exercise 3.4.12

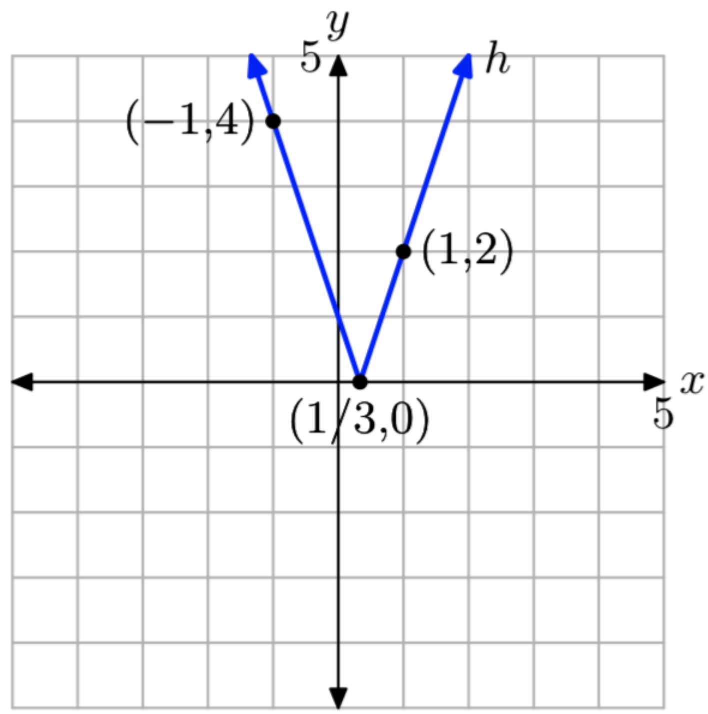
g(x) = |5−2x|

Exercise 3.4.13

h(x) = |1−3x|

Answer

$$h(x)=\begin{cases}1-3x, & \text{if } x<\frac{1}{3}\\ -1+3x, & \text{if } x\geq\frac{1}{3}\end{cases}$$



? Exercise 3.4.14

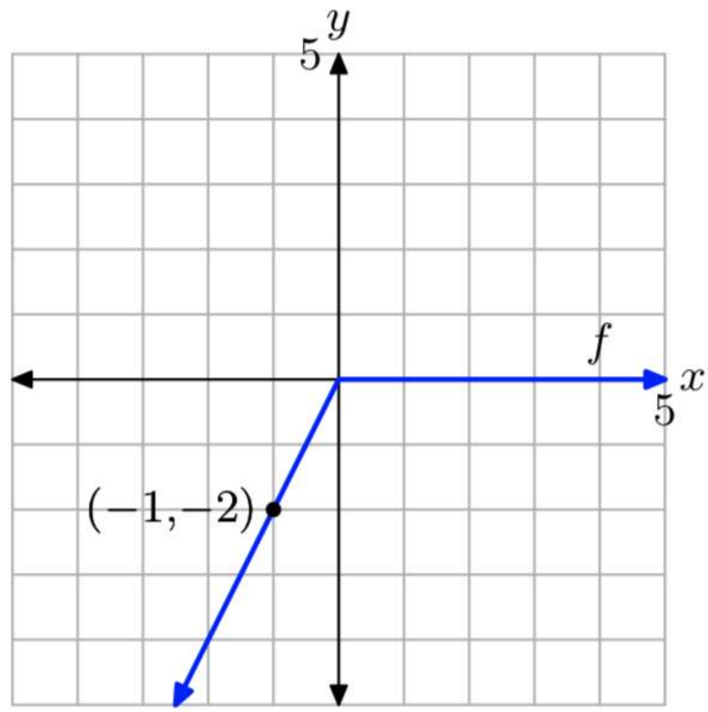
h(x) = |2x+1|

? Exercise 3.4.15

f(x) = x−|x|

Answer

$$f(x)=\begin{cases}2x, & \text{if } x<0\\ 0, & \text{if } x\geq 0\end{cases}$$



? Exercise 3.4.16

$f(x) = x+|x-1|$

? Exercise 3.4.17

Use a graphing calculator to draw the graphs of $y = |x|$, $y = 2|x|$, $y = 3|x|$, and $y = 4|x|$ on the same viewing window. In your own words, explain what you learned in this exercise.

Answer

Multiplying by a factor of $a > 1$, as in $y = a|x|$, stretches the graph of $y = |x|$ vertically by a factor of a. The higher the value of a, the more it stretches vertically.

? Exercise 3.4.18

Use a graphing calculator to draw the graphs of $y = |x|$, $y = (1/2)|x|$, $y = (1/3)|x|$, and $y = (1/4)|x|$ on the same viewing window. In your own words, explain what you learned in this exercise.

? Exercise 3.4.19

Use a graphing calculator to draw the graphs of $y = |x|$, $y = |x-2|$, $y =|x-4|$, and $y = |x-6|$ on the same viewing window. In your own words, explain what you learned in this exercise.

Answer

Subtracting a positive value of a, as in $y = |x-a|$, shifts the graph a units to the right.

? Exercise 3.4.20

Use a graphing calculator to draw the graphs of y = |x|, y = |x+2|, y = |x+4|, and y = |x+6| on the same view- ing window. In your own words, explain what you learned in this exercise.

In **Exercises 21-36**, perform each of the following tasks. Feel free to check your work with your graphing calculator, but you should be able to do all of the work by hand.

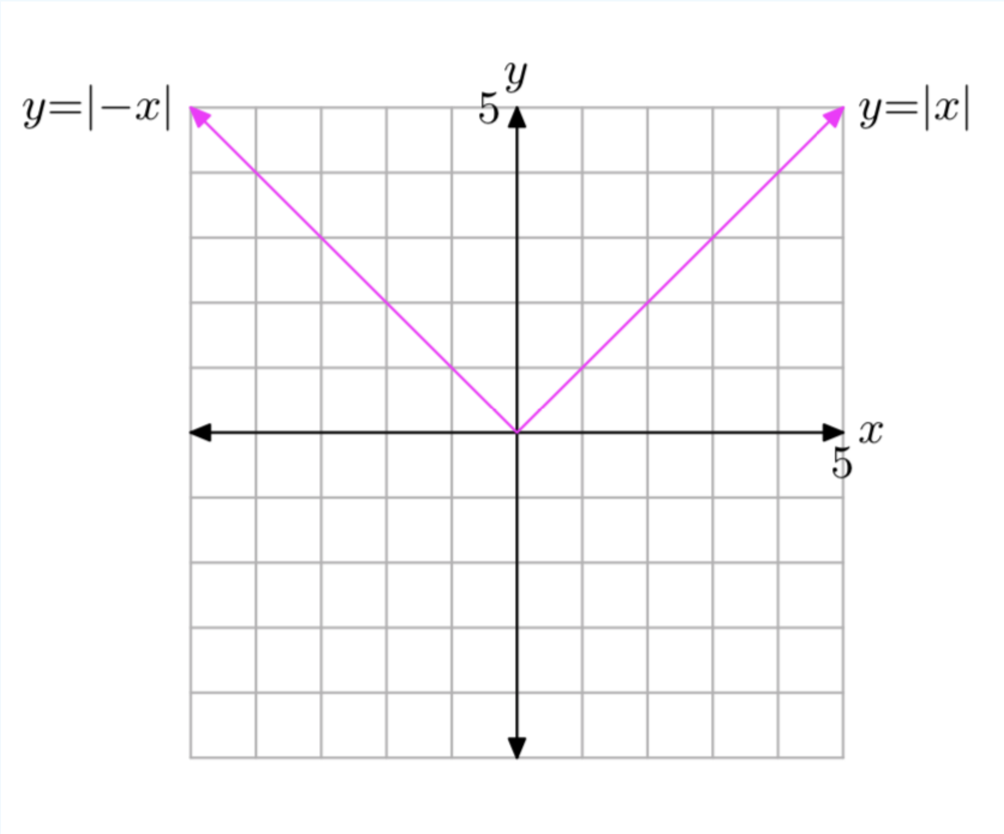
1. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Create an accurate plot of the function y = |x| on your coordinate system and label this graph with its equation.
2. Use the technique of Examples 12,13, and 14 in the narrative to help select the appropriate geometric transformations to transform the equation y = |x| into the form of the function given in the exercise. On the same coordinate system, use a different colored pencil or pen to draw the graph of the function resulting from your applied transformation. Label the resulting graph with its equation.
3. Use interval notation to describe the domain and range of the given function.

? Exercise 3.4.21

f(x) = |−x|

Answer

The graphs of y = |x| and y = |−x| coincide. The domain is $(-\infty, \infty)$ and the range is $[0, \infty)$.



? Exercise 3.4.22

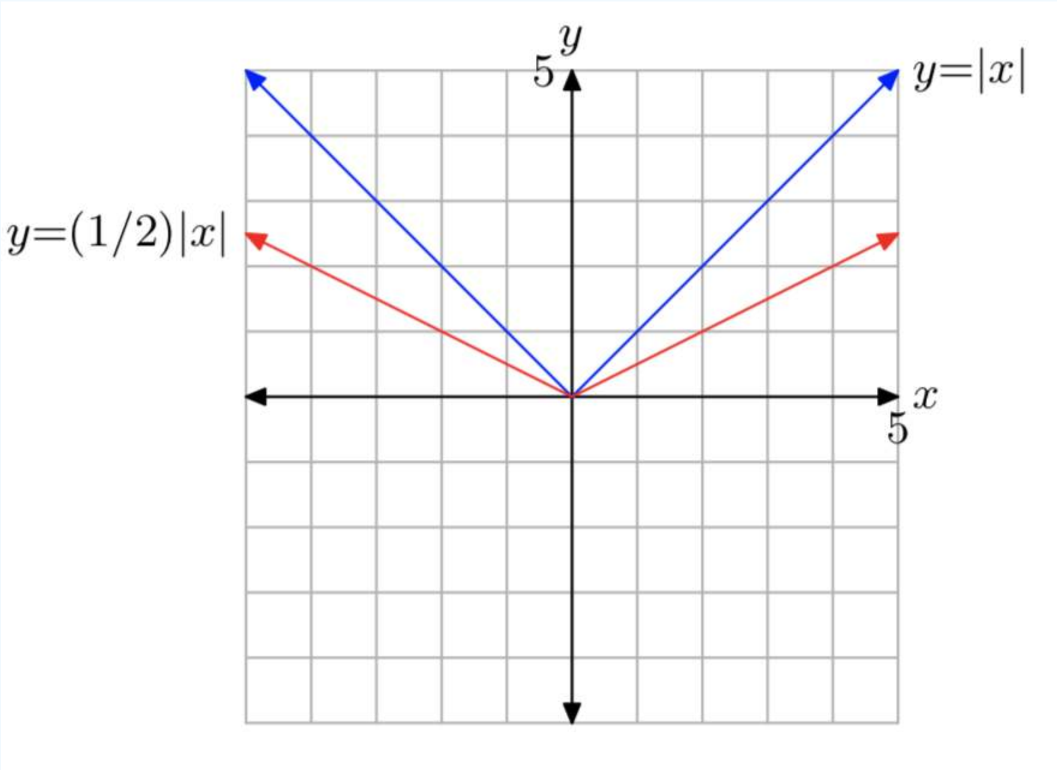
f(x) = −|x|

? Exercise 3.4.23

$f(x) = \frac{1}{2}|x|$

Answer

The domain is $(-\infty, \infty)$ and the range is $[0, \infty)$.



? Exercise 3.4.24

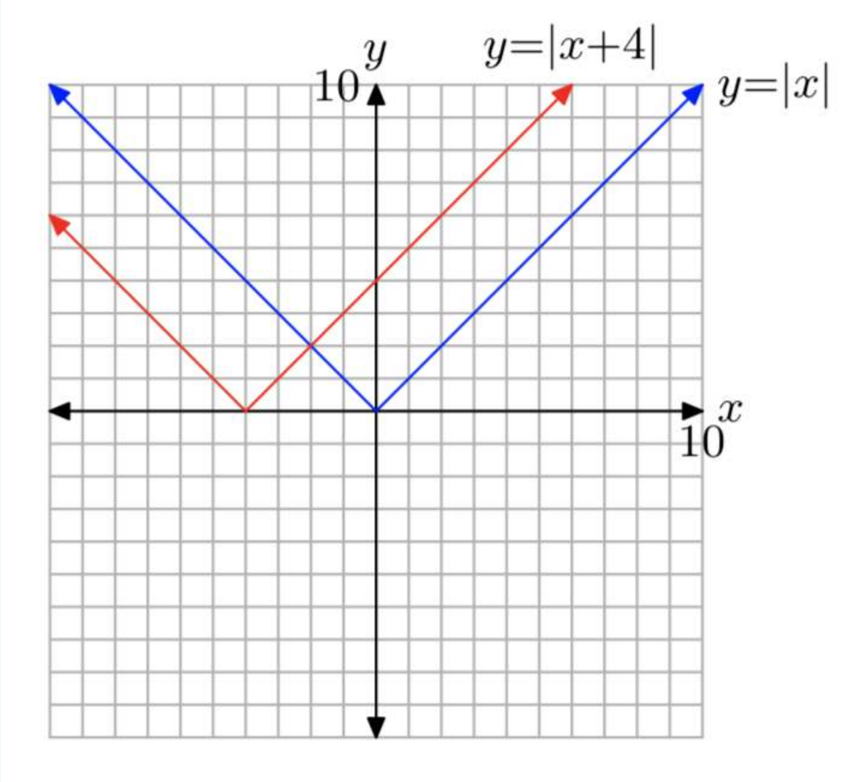
f(x) = −2|x|

? Exercise 3.4.25

f(x) = |x+4|

Answer

The domain is $(-\infty, \infty)$ and the range is $[0, \infty)$.



Exercise 3.4.26

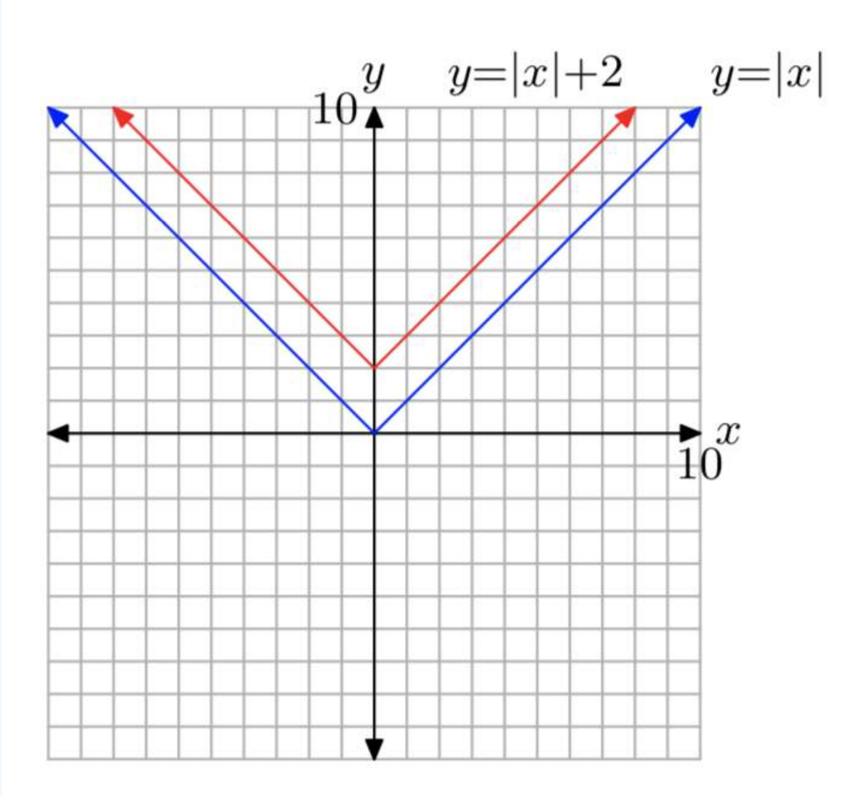
f(x) = |x−2|

Exercise 3.4.27

f(x) = |x|+2

Answer

The domain is $(-\infty, \infty)$ and the range is $[2, \infty)$.



Exercise 3.4.28

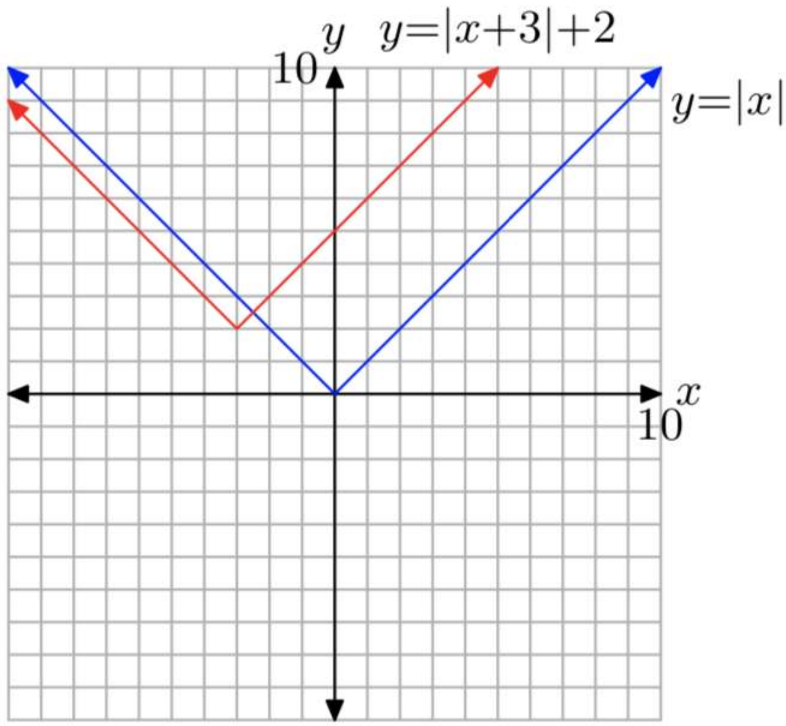
f(x) = |x|−3

Exercise 3.4.29

f(x) = |x+3|+2

Answer

The domain is $(-\infty, \infty)$ and the range is $[2, \infty)$.



? Exercise 3.4.30

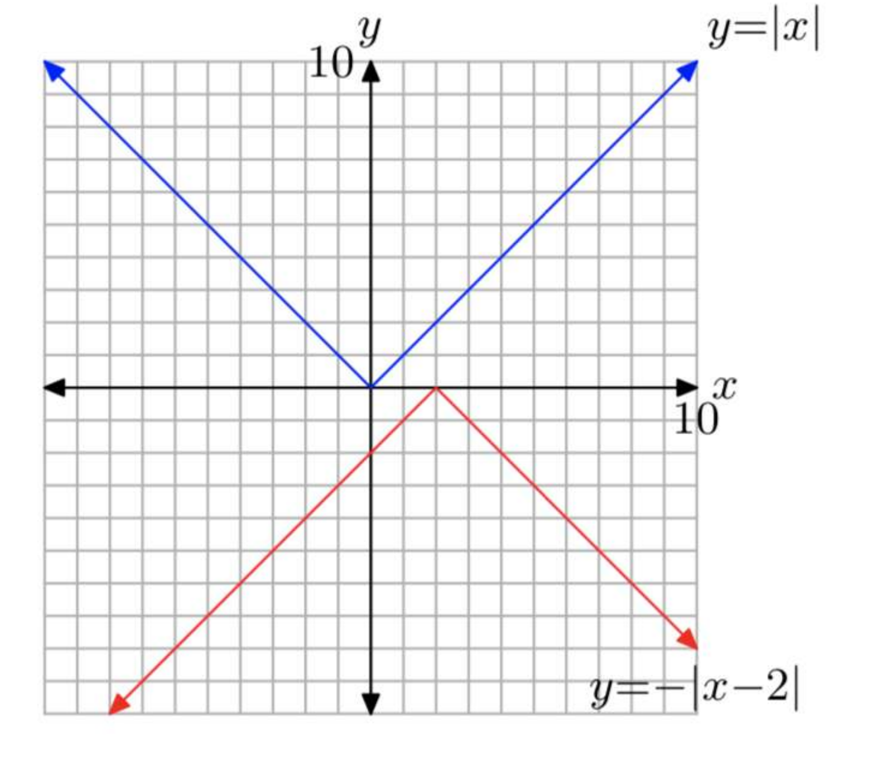
f(x) = |x−3|−4

? Exercise 3.4.31

f(x) = −|x−2|

Answer

The domain is $(-\infty, \infty)$ and the range is $(-\infty, 0]$.



? Exercise 3.4.32

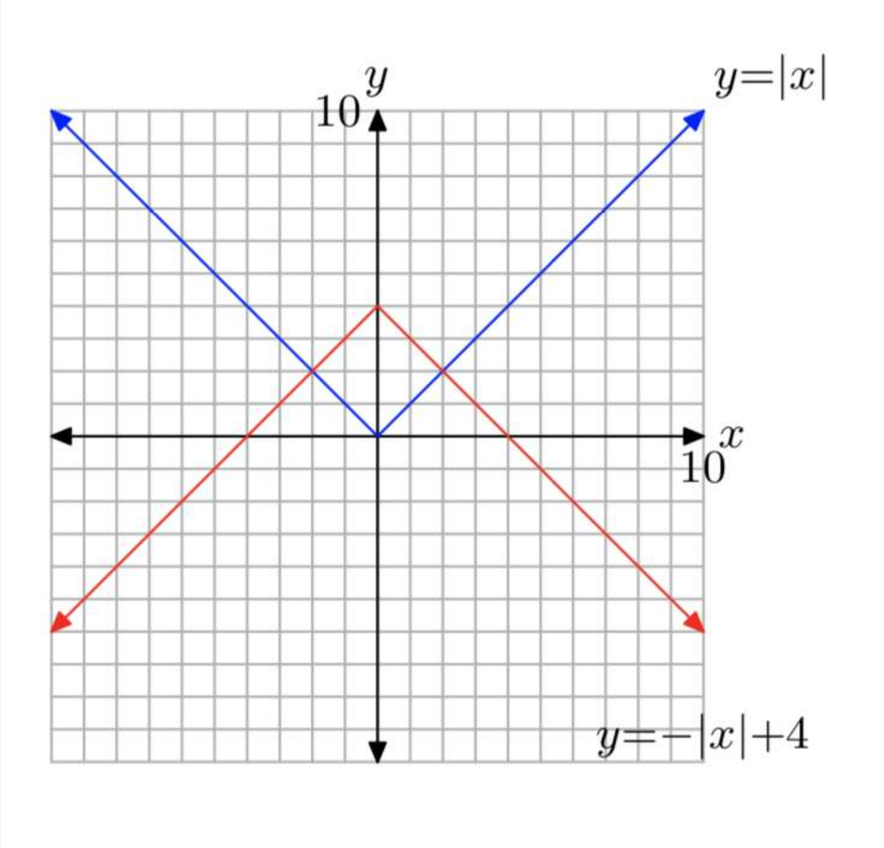
f(x) = −|x|−2

? Exercise 3.4.33

f(x) = −|x|+4

Answer

The domain is $(-\infty, \infty)$ and the range is $(-\infty, 4]$.



? Exercise 3.4.34

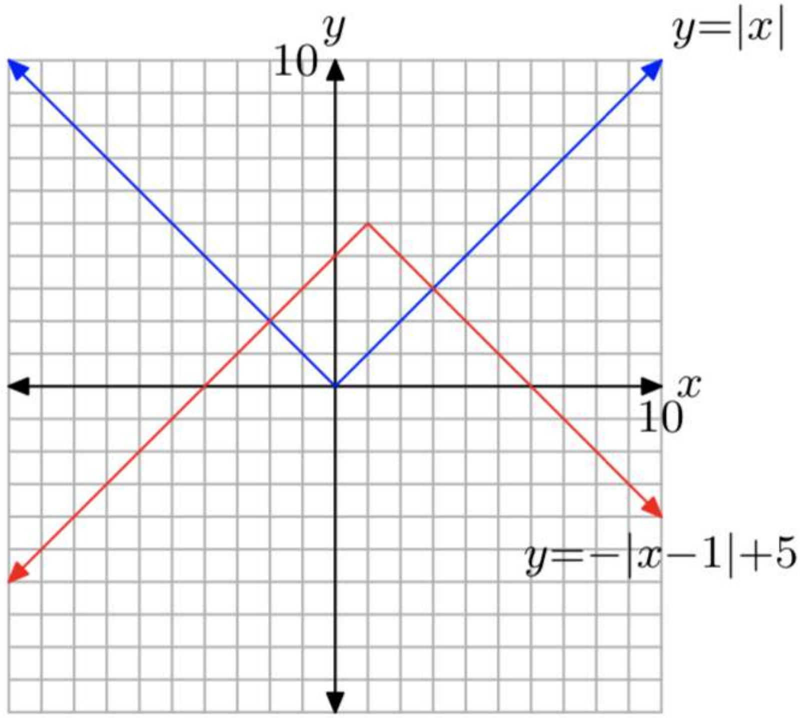
f(x) = −|x+4|

? Exercise 3.4.35

f(x) = −|x−1|+5

Answer

The domain is $(-\infty, \infty)$ and the range is $(-\infty, 5]$.



Exercise 3.4.36

f(x) = −|x+5|+2

3.5: Absolute Value Equations

In the previous section, we defined

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

and we saw that the graph of the absolute value function defined by f(x) = |x| has the "V-shape" shown in Figure $3.5.1$.

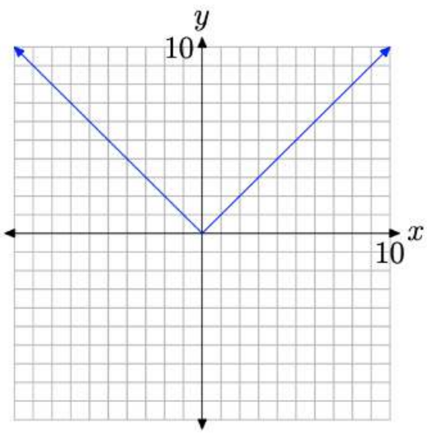


Figure $3.5.1$ The graph of the absolute value function f(x) = |x|.

It is important to note that the equation of the left-hand branch of the "V" is y = −x. Typical points on this branch are (−1, 1), (−2, 2), (−3, 3), etc. It is equally important to note that the right-hand branch of the "V" has equation y = x. Typical points on this branch are (1, 1), (2, 2), (3, 3), etc.

Solving |x| = a

We will now discuss the solutions of the equation

$$|x| = a$$

There are three distinct cases to discuss, each of which depends upon the value and sign of the number a.

- Case I: a < 0

  If a < 0, then the graph of y = a is a horizontal line that lies strictly below the x-axis, as shown in Figure $3.5.2$(a). In this case, the equation |x| = a has no solutions because the graphs of y = a and y = |x| do not intersect.

- Case II: a = 0

  If a = 0, then the graph of y = 0 is a horizontal line that coincides with the x-axis, as shown in Figure $3.5.2$(b). In this case, the equation |x| = 0 has the single solution x = 0, because the horizontal line y = 0 intersects the graph of y = |x| at exactly one point, at x = 0.

- Case III: a > 0

  If a > 0, then the graph of y = a is a horizontal line that lies strictly above the x-axis, as shown in Figure $3.5.2$(c). In this case, the equation |x| = a has two solutions, because the graphs of y = a and y = |x| have two points of intersection.

  Recall that the left-hand branch of y = |x| has equation y = −x, and points on this branch have the form (−1, 1), (−2, 2), etc. Because the point where the graph of y = a intersects the left-hand branch of y = |x| has y-coordinate y = a, the x-coordinate of this point of intersection is x = −a. This is one solution of |x| = a.

  Recall that the right-hand branch of y = |x| has equation y = x, and points on this branch have the form (1, 1), (2, 2), etc. Because the point where the graph of y = a intersects the right-hand branch of y = |x| has y-coordinate y = a, the x-coordinate of this point of intersection is x = a. This is the second solution of |x| = a.

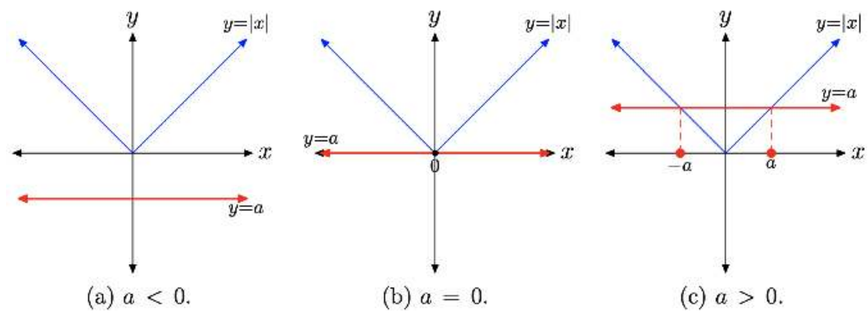


Figure 3.5.2 The solution of |x| = a has three cases.

This discussion leads to the following key result.

Property 2

The solution of |x| = a depends upon the value and sign of a.

- Case I: a < 0

  The equation |x| = a has no solutions.

- Case II: a = 0

  The equation |x| = 0 has one solution, x = 0.

- Case III: a > 0

  The equation |x| = a has two solutions, x = −a or x = a.

Let's look at some examples.

Example 3.5.1

Solve |x| = −3 for x.

Solution

The graph of the left-hand side of |x| = −3 is the "V" of Figure 3.5.2(a). The graph of the right-hand side of |x| = −3 is a horizontal line three units below the x-axis. This has the form of the sketch in Figure 3.5.2(a). The graphs do not intersect. Therefore, the equation |x| = −3 has no solutions.

An alternate approach is to consider the fact that the absolute value of x can never equal −3. The absolute value of a number is always nonnegative (either zero or positive). Hence, the equation |x| = −3 has no solutions.

Example 3.5.2

Solve |x| = 0 for x

Solution

This is the case shown in Figure 3.5.2(b). The graph of the left-hand side of |x| = 0 intersects the graph of the right-hand side of |x| = 0 at x = 0. Thus, the only solution of |x| = 0 is x = 0.

Thinking about this algebraically instead of graphically, we know that 0 = 0, but there is no other number with an absolute value of zero. So, intuitively, the only solution of |x| = 0 is x = 0.

Example 3.5.3

Solve |x| = 4 for x.

Solution

The graph of the left-hand side of |x| = 4 is the “V” of Figure $3.5.2$(c). The graph of the right-hand side is a horizontal line 4 units above the x-axis. This has the form of the sketch in Figure $3.5.2$(c). The graphs intersect at (−4, 4) and (4, 4). Therefore, the solutions of |x| = 4 are x = −4 or x = 4.

Alternatively, | − 4| = 4 and |4| = 4, but no other real numbers have absolute value equal to 4. Hence, the only solutions of |x| = 4 are x = −4 or x = 4.

Example $3.5.4$

Solve the equation |3 − 2x| = −8 for x.

Solution

If the equation were |x| = −8, we would not hesitate. The equation |x| = −8 has no solutions. However, the reasoning applied to the simple case |x| = −8 works equally well with the equation |3 − 2x| = −8. The left-hand side of this equation must be nonnegative, so its graph must lie above or on the x-axis. The right-hand side of |3−2x| = −8 is a horizontal line 8 units below the x-axis. The graphs cannot intersect, so there is no solution.

We can verify this argument with the graphing calculator. Load the left and righthand sides of |3 − 2x| = −8 into Y1 and Y2, respectively, as shown in Figure $3.5.3$(a). Push the MATH button on your calculator, then right-arrow to the NUM menu, as shown in Figure $3.5.3$(b). Use 1:abs(to enter the absolute value shown in Y1 in Figure $3.5.3$(a). From the ZOOM menu, select 6:ZStandard to produce the image shown in Figure $3.5.3$(c).

Note, that as predicted above, the graph of y = |3 − 2x| lies on or above the xaxis and the graph of y = −8 lies strictly below the x-axis. Hence, the graphs cannot intersect and the equation |3 − 2x| = −8 has no solutions.

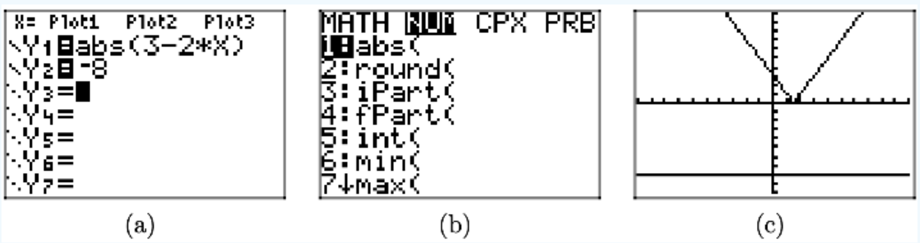


Figure $3.5.3$ Using the graphing calculator to examine the solution of |3 − 2x| = −8.

Alternatively, we can provide a completely intuitive solution of |3 − 2x| = −8 by arguing that the left-hand side of this equation is nonnegative, but the right-hand side is negative. This is an impossible situation. Hence, the equation has no solutions.

Example $3.5.5$

Solve the equation |3 − 2x| = 0 for x.

Solution

We have argued that the only solution of |x| = 0 is x = 0. Similar reasoning points out that |3 − 2x| = 0 only when 3 − 2x = 0. We solve this equation independently.

$$\begin{aligned} 3-2x &= 0 \\ -2x &= -3 \\ x &= \frac{3}{2} \end{aligned}$$

Thus, the only solution of |3 − 2x| = 0 is x = 3/2.

It is worth pointing out that the “tip” or “vertex” of the “V” in Figure $3.5.3$(c) is located at x = 3/2. This is the only location where the graphs of y = |3 − 2x| and y = 0 intersect.

Example 3.5.6

Solve the equation |3 − 2x| = 6 for x.

Solution

In this example, the graph of y = 6 is a horizontal line that lies 6 units above the x-axis, and the graph of y = |3 − 2x| intersects the graph of y = 6 in two locations. You can use the intersect utility to find the points of intersection of the graphs, as we have in Figure $3.5.4$(b) and (c).

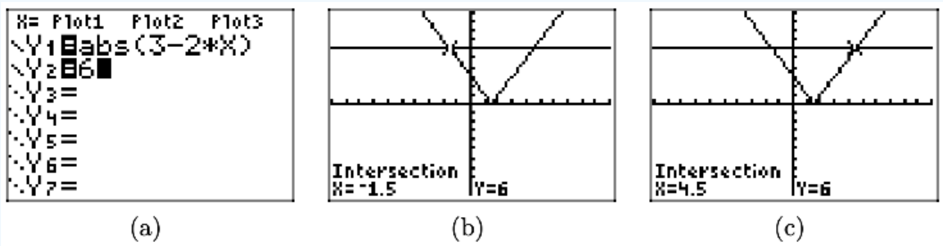


Figure $3.5.4$ Using the graphing calculator to find two solutions of |3 − 2x| = 6.

We need a way of summarizing this graphing calculator approach on our homework paper. First, draw a reasonable facsimile of your calculator's viewing window on your homework paper. Use a ruler to draw all lines. Complete the following checklist.

- Label each axis, in this case with x and y.
- Scale each axis. To do this, press the WINDOW button on your calculator, then report the values of xmin, xmax, ymin, and ymax on the appropriate axis.
- Label each graph with its equation.
- Drop dashed vertical lines from the points of intersection to the x-axis. Shade and label these solutions of the equation on the x-axis.

Following the guidelines in the above checklist, we obtain the image in Figure $3.5.5$.

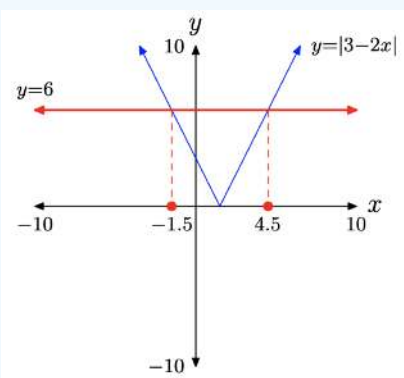


Figure $3.5.5$ Reporting a graphical solution of |3 − 2x| = 6.

Algebraic Approach. One can also use an algebraic technique to find the two solutions of |3 − 2x| = 6. Much as |x| = 6 has solutions x = −6 or x = 6, the equation

$$|3-2x|=6$$

is possible only if the expression inside the absolute values is either equal to −6 or 6. Therefore, write

$$3-2x=-6 \qquad \text{or} \qquad 3-2x=6$$

and solve these equations independently

$$\begin{aligned} 3-2x &= -6 \quad \text{or} \quad & 3-2x &= 6 \\ -2x &= -9 & -2x &= 3 \\ x &= \frac{9}{2} & x &= -\frac{3}{2} \end{aligned}$$

Because −3/2 = −1.5 and 9/2 = 4.5, these exact solutions agree exactly with the graphical solutions in Figure $3.5.4$(b) and (c).

Let’s summarize the technique involved in solving this important case.

Note

Solving |expression| = a, when a > 0. To solve the equation

$$|\text{ expression }|=a, \quad \text{ when } a>0$$

set

$$\text{expression }=-a \qquad \text{or} \qquad \text{expression }=a$$

then solve each of these equations independently.

For example:

• To solve |2x + 7| = 5, set

$$2x+7=-5 \qquad or \qquad 2x+7=5$$

, then solve each of these equations independently.

• To solve |3 − 5x| = 9, set

$$3-5x=-9 \qquad or \qquad 3-5x=9$$

, then solve each of these equations independently.

• Note that this technique should not be applied to the equation |2x + 11| = −10, because the right-hand side of the equation is not a positive number. Indeed, in this case, no values of x will make the left-hand side of this equation equal to −10, so the equation has no solutions.

Sometimes we have to do a little algebra before removing the absolute value bars.

Example 3.5.7

Solve the equation

$$|x+2|+3=8$$

for x.

Solution

First, subtract 3 from both sides of the equation.

$$\begin{aligned} |x+2|+3 &=8 \\ |x+2|+3-3 &=8-3 \end{aligned}$$

This simplifies to

$$|x+2|=5$$

Now, either

$$x+2=-5 \qquad \text{or} \qquad x+2=5$$

each of which can be solved separately.

$$\begin{array}{rclcrcl} x+2 &=& -5 & \quad\text{or}\quad & x+2 &=& 5 \\ x+2-2 &=& -5-2 & & x+2-2 &=& 5-2 \\ x &=& -7 & & x &=& 3 \end{array}$$

Example 3.5.8

Solve the equation

$$3|x-5|=6$$

for x.

Solution

First, divide both sides of the equation by 3

$$\begin{aligned}3|x-5| &= 6\\ \frac{3|x-5|}{3} &= \frac{6}{3}\end{aligned}$$

This simplifies to

$$|x-5|=2$$

Now, either

$$x-5=-2 \qquad \text{or} \qquad x-5=2$$

each of which can be solved separately.

$$\begin{array}{rcl}x-5 &=& -2\\ x-5+5 &=& -2+5\\ x &=& 3\end{array} \qquad \text{or} \qquad \begin{array}{rcl}x-5 &=& 2\\ x-5+5 &=& 2+5\\ x &=& 7\end{array}$$

Properties of Absolute Value

An example will motivate the need for some discussion of the properties of absolute value.

Example 3.5.9

Solve the equation

$$\left|\frac{x}{2}-\frac{1}{3}\right|=\frac{1}{4}$$

for x.

Solution

It is tempting to multiply both sides of this equation by a common denominator as follows.

$$\begin{aligned}\left|\frac{x}{2}-\frac{1}{3}\right| &= \frac{1}{4}\\ 12\left|\frac{x}{2}-\frac{1}{3}\right| &= 12\left(\frac{1}{4}\right)\end{aligned}$$

If it is permissible to move the 12 inside the absolute values, then we could proceed as follows.

$$\begin{aligned}\left|12\left(\frac{x}{2}-\frac{1}{3}\right)\right| &= 3\\ |6x-4| &= 3\end{aligned}$$

Assuming for the moment that this last move is allowable, either

$$6x-4=-3 \qquad \text{or} \qquad 6x-4=3$$

Each of these can be solved separately, first by adding 4 to both sides of the equations, then dividing by 6.

$$\begin{aligned} 6x-4 &= -3 \quad \text{or} \quad & 6x-4 &= 3 \\ 6x &= 1 & 6x &= 7 \\ x &= 1/6 & x &= 7/6 \end{aligned}$$

As we’ve used a somewhat questionable move in obtaining these solutions, it would be wise to check our results. First, substitute x = 1/6 into the original equation.

$$\begin{aligned} \left|\frac{x}{2}-\frac{1}{3}\right| &= \frac{1}{4} \\ \left|\frac{1/6}{2}-\frac{1}{3}\right| &= \frac{1}{4} \\ \left|\frac{1}{12}-\frac{1}{3}\right| &= \frac{1}{4} \end{aligned}$$

Write equivalent fractions with a common denominator and subtract.

$$\begin{aligned} \left|\frac{1}{12}-\frac{4}{12}\right| &= \frac{1}{4} \\ \left|-\frac{3}{12}\right| &= \frac{1}{4} \\ \left|-\frac{1}{4}\right| &= \frac{1}{4} \end{aligned}$$

Clearly, x = 1/6 checks. We’ll leave the check of the second solution to our readers.

Well, we’ve checked our solutions and they are correct, so it must be the case that

$$12\left|\frac{x}{2}-\frac{1}{3}\right| = \left|12\left(\frac{x}{2}-\frac{1}{3}\right)\right|$$

But why? After all, absolute value bars, though they do act as grouping symbols, have a bit more restrictive meaning than ordinary grouping symbols such as parentheses, brackets, and braces.

We state the first property of absolute values.

Property

If a and b are any real numbers, then

$$|ab| = |a||b|$$

We can demonstrate the validity of this property by simply checking cases.

- If a and b are both positive real numbers, then so is ab and $|a||b| = ab$. On the other hand, $|a||b| = ab$. Thus, $|ab| = |a||b|$.
- If a and b are both negative real numbers, then ab is positive and $|ab| = ab$. On the other hand, $|a||b| = (-a)(-b) = ab$. Thus, $|ab| = |a||b|$.

We will leave the proof of the remaining two cases as exercises. We can use $|a||b| = |ab|$ to demonstrate that

$$12\left|\frac{x}{2}-\frac{1}{3}\right| = |12|\left|\frac{x}{2}-\frac{1}{3}\right| = \left|12\left(\frac{x}{2}-\frac{1}{3}\right)\right|$$

This validates the method of attack we used to solve equation (12) in Example $3.5.9$.

Warning 14

On the other hand, it is not permissible to multiply by a negative number and simply slide the negative number inside the absolute value bars. For example,

$$-2|x-3| = |-2(x-3)|$$

is clearly an error (well, it does work for x = 3). For any x except 3, the lefthand side of this result is a negative number, but the right-hand side is a positive number. They are clearly not equal.

In similar fashion, one can demonstrate a second useful property involving absolute value.

Definition

If a and b are any real numbers, then

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

provided, of course, that $b \neq 0$.

Again, this can be proved by checking four cases. For example, if a is a positive real number and b is a negative real number, then a/b is negative and $|a/b| = -a/b$. On the other hand, $|a|/|b| = a/(-b) = -a/b$.

We leave the proof of the remaining three cases as exercises.

This property is useful in certain situations. For example, should you desire to divide $|2x-4|$ by 2, you would proceed as follows.

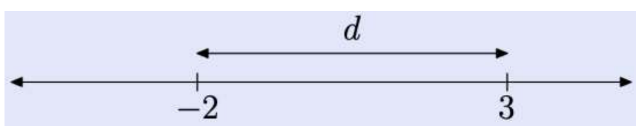
$$\frac{|2x-4|}{2} = \frac{|2x-4|}{|2|} = \left|\frac{2x-4}{2}\right| = |x-2|$$

This technique is useful in several situations. For example, should you want to solve the equation $|2x-4| = 6$, you could divide both sides by 2 and apply the quotient property of absolute values.

Distance Revisited

Recall that for any real number x, the absolute value of x is defined as the distance between the real number x and the origin on the real line. In this section, we will push this distance concept a bit further.

Suppose that you have two real numbers on the real line. For example, in the figure that follows, we've located 3 and −2 on the real line.



You can determine the distance between the two points by subtracting the number on the left from the number on the right. That is, the distance between the two points is d = 3 − (−2) = 5 units. If you subtract in the other direction, you get the negative of the distance, as in −2 − 3 = −5 units. Of course, distance is a nonnegative quantity, so this negative result cannot represent the distance between the two points. Consequently, to find the distance between two points on the real line, you must always subtract the number on the left from the number on the right.

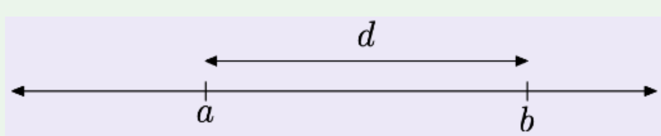
However, if you take the absolute value of the difference, you'll get the correct result regardless of the direction of subtraction.

$$d = |3-(-2)| = |5| = 5 \quad \text{and} \quad d = |-2-3| = |-5| = 5$$

This discussion leads to the following key idea.

Property 16.

Suppose that a and b are two numbers on the real line



You can determine the distance d between a and b on the real line by taking the absolute value of their difference. That is,

$$d = |a - b|$$

Of course, you could subtract in the other direction, obtaining $d = |b - a|$. This is also correct.

Now that this geometry of distance has been introduced, it is useful to pronounce the symbolism |a−b| as "the distance between a and b" instead of saying "the absolute value of a minus b."

Example 3.5.10

Solve the equation

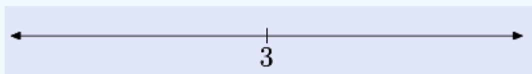
$$|x - 3| = 8$$

for x.

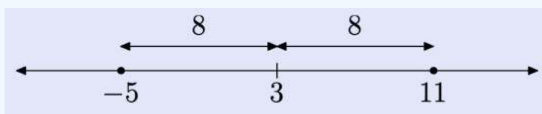
Solution

Here's the ideal situation to apply our new concept of distance. Instead of saying "the absolute value of x minus 3 is 8," we pronounce the equation $|x - 3| = 8$ as "the distance between x and 3 is 8."

Draw a number line and locate the number 3 on the line.



Recall that the "distance between x and 3 is 8." Having said this, mark two points on the real line that are 8 units away from 3.



Thus, the solutions of |x − 3| = 8 are x = −5 or x = 11

Example 3.5.11

Solve the equation

$$|x + 5| = 2$$

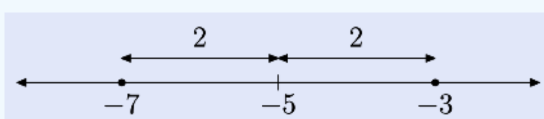
for x.

Solution

Rewrite the equation as a difference.

$$|x - (-5)| = 2$$

This is pronounced "the distance between x and −5 is 2." Locate two points on the number line that are 2 units away from −5.



Hence, the solutions of $|x + 5| = 2$ are x = −7 or x = −3.

Exercise

For each of the equations in **Exercises 1-4**, perform each of the following tasks.

1. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
2. Sketch the graph of each side of the equation without the aid of a calculator. Label each graph with its equation.

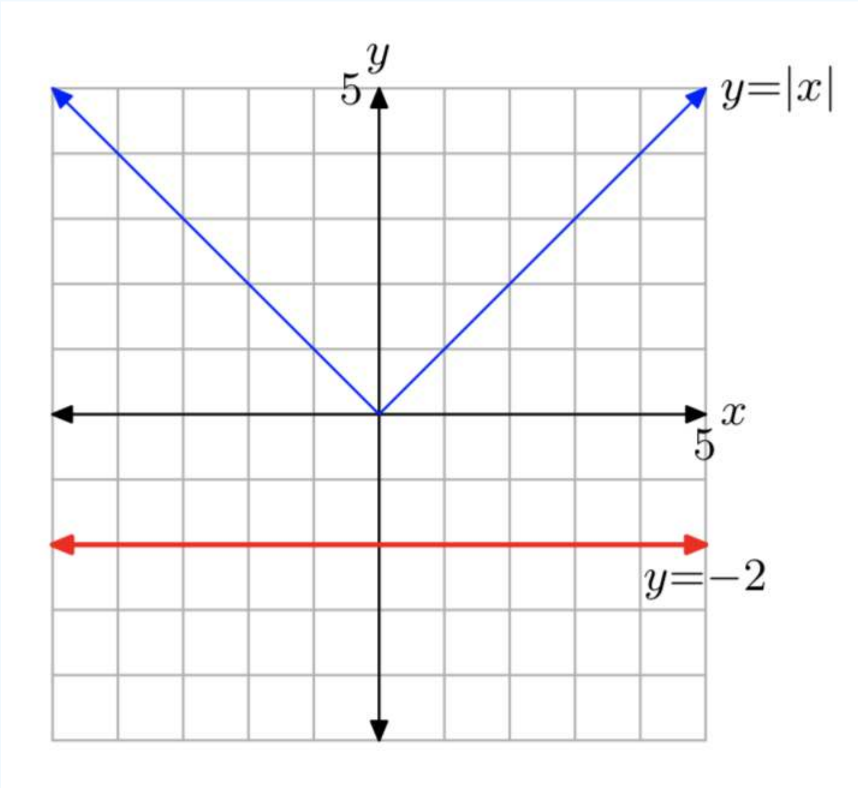
3. Shade the solution of the equation on the x-axis (if any) as shown in Figure 5 (read "Expectations") in the narrative. That is, drop dashed lines from the points of intersection to the axis, then shade and label the solution set on the x-axis.

? Exercise 3.5.1

$|x| = -2$

Answer

No solutions.



? Exercise 3.5.2

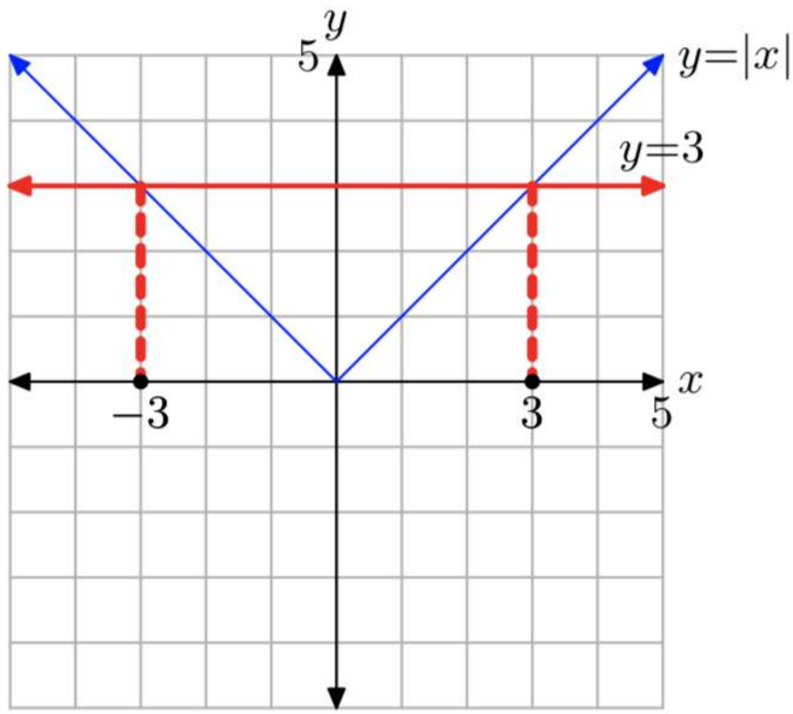
$|x| = 0$

? Exercise 3.5.3

$|x| = 3$

Answer

Solution: $x = -3$ or $x = 3$.



? Exercise 3.5.4

|x| = 2

For each of the equations in **Exercises 5-8**, perform each of the following tasks.

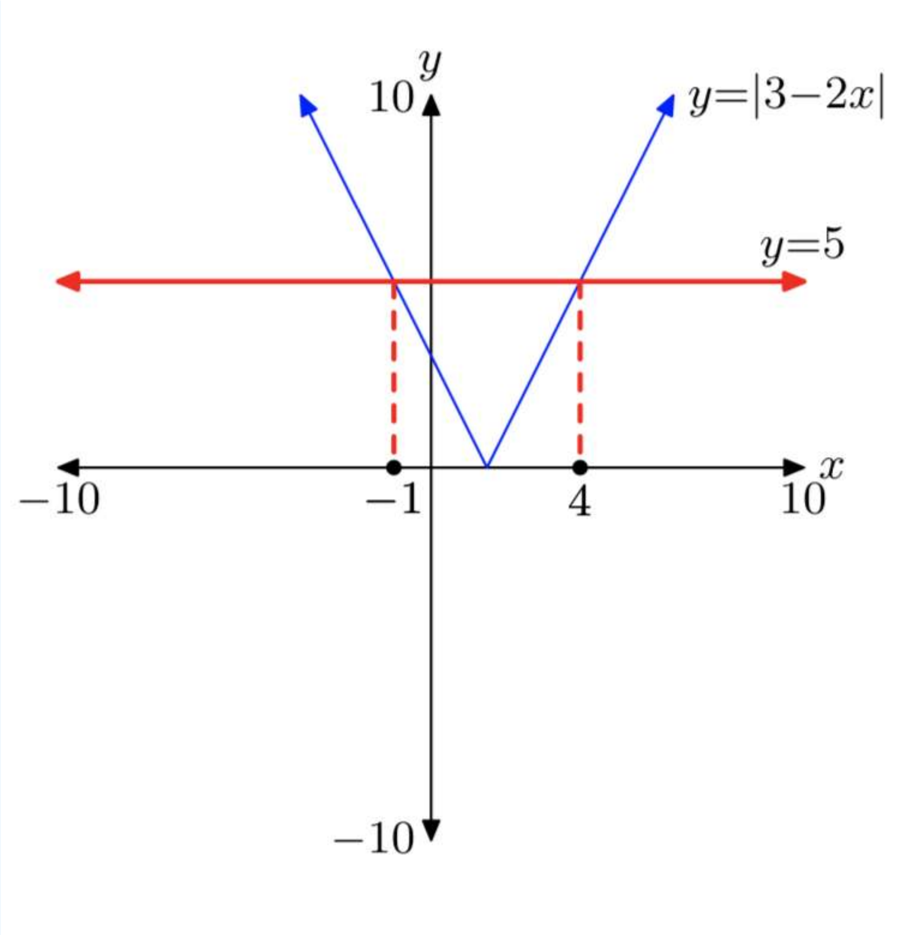
1. Load each side of the equation into the Y= menu of your calculator. Ad- just the viewing window so that all points of intersection of the two graphs are visible in the viewing window.
2. Copy the image in your viewing screen onto your homework paper. Label each axis and scale each axis with xmin, xmax, ymin, and ymax. Label each graph with its equation.
3. Use the intersect utility in the CALC menu to determine the points of intersection. Shade and label each solution as shown in Figure 5 (read "Expectations") in the narrative. That is, drop dashed lines from the points of intersection to the axis, then shade and label the solution set on the x-axis.

? Exercise 3.5.5

|3−2x| = 5

Answer

Solutions: x = −1 or x = 4.



? Exercise 3.5.6

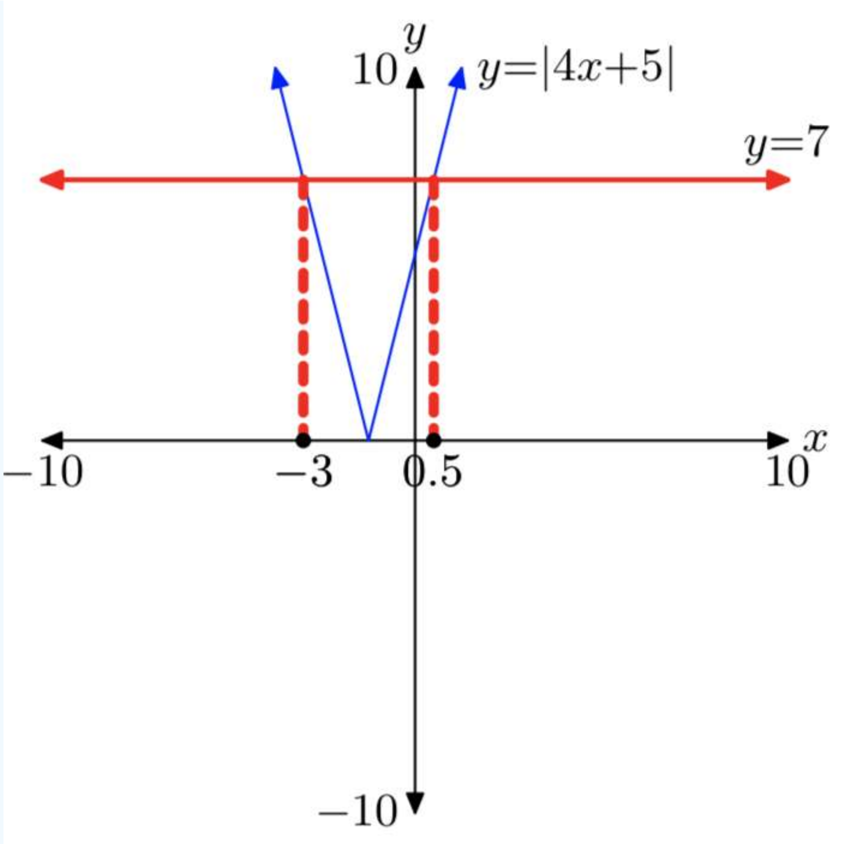
|2x+7| = 4

? Exercise 3.5.7

|4x+5| = 7

Answer

Solutions: x = −3 or x = 0.5.



? Exercise 3.5.8

$|5x-7| = 8$

For each of the equations in **Exercises 9-14**, provide a purely algebraic solution without the use of a calculator. Arrange your work as shown in Examples 6, 7, and 8 in the narrative, but do not use a calculator.

? Exercise 3.5.9

$|4x+3| = 0$

Answer

$x = -\frac{3}{4}$

? Exercise 3.5.10

$|3x-11| = -5$

? Exercise 3.5.11

$|2x+7| = 14$

Answer

$x = -\frac{21}{2}$ or $x = \frac{7}{2}$

Exercise 3.5.12

$|7-4x| = 8$

Exercise 3.5.13

$|3-2x| = -1$

Answer

No solutions.

Exercise 3.5.14

$|4x+9| = 0$

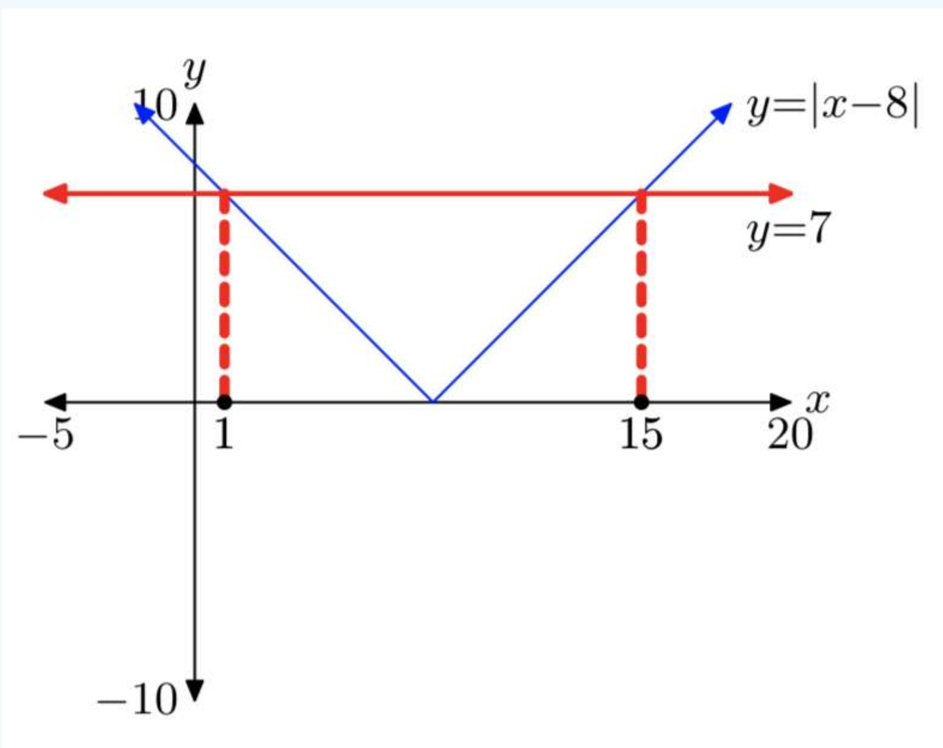
For each of the equations in **Exercises 15-20**, perform each of the following tasks.

1. Arrange each of the following parts on your homework paper in the same location. Do not do place the algebraic work on one page and the graphical work on another.
2. Follow each of the directions given for **Exercises 5-8** to find and record a solution with your graphing calculator.
3. Provide a purely algebraic solution, showing all the steps of your work. Do these solutions compare favorably with those found using your graphing calculator in part (ii)? If not, look for a mistake in your work.

Exercise 3.5.15

$|x-8| = 7$

Answer



$x = 1$ or $x = 15$

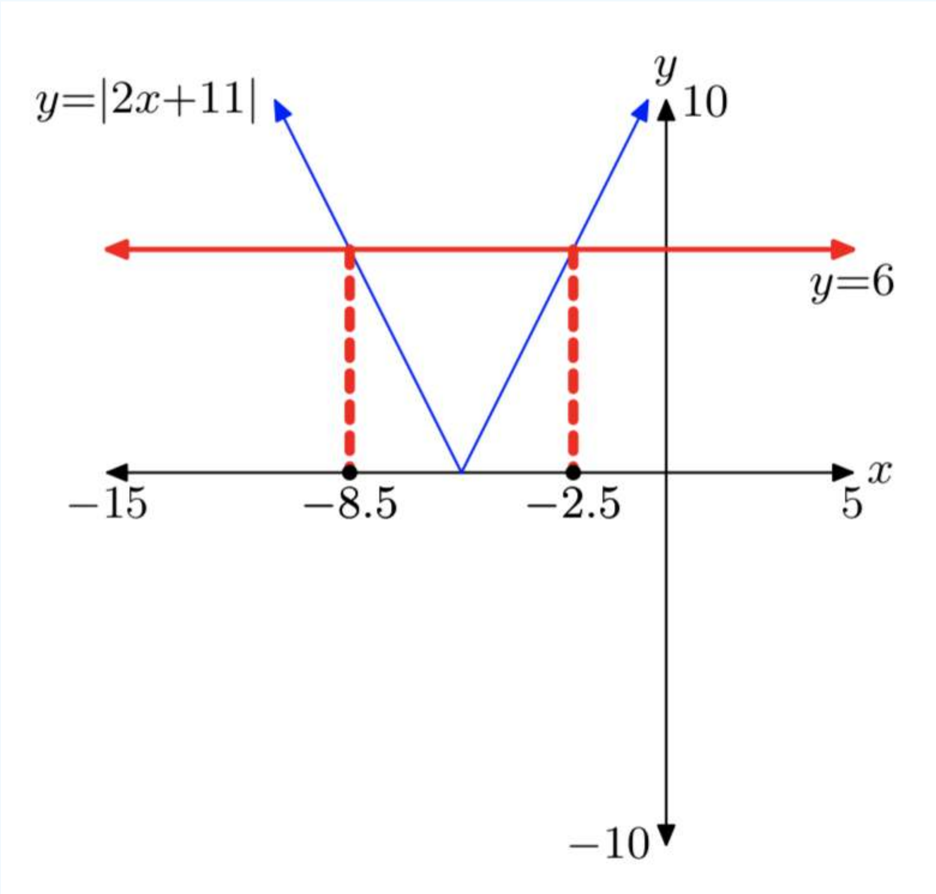
Exercise 3.5.16

$|2x-15| = 5$

Exercise 3.5.17

$|2x+11| = 6$

Answer



$x = -8.5$ or $x = -2.5$

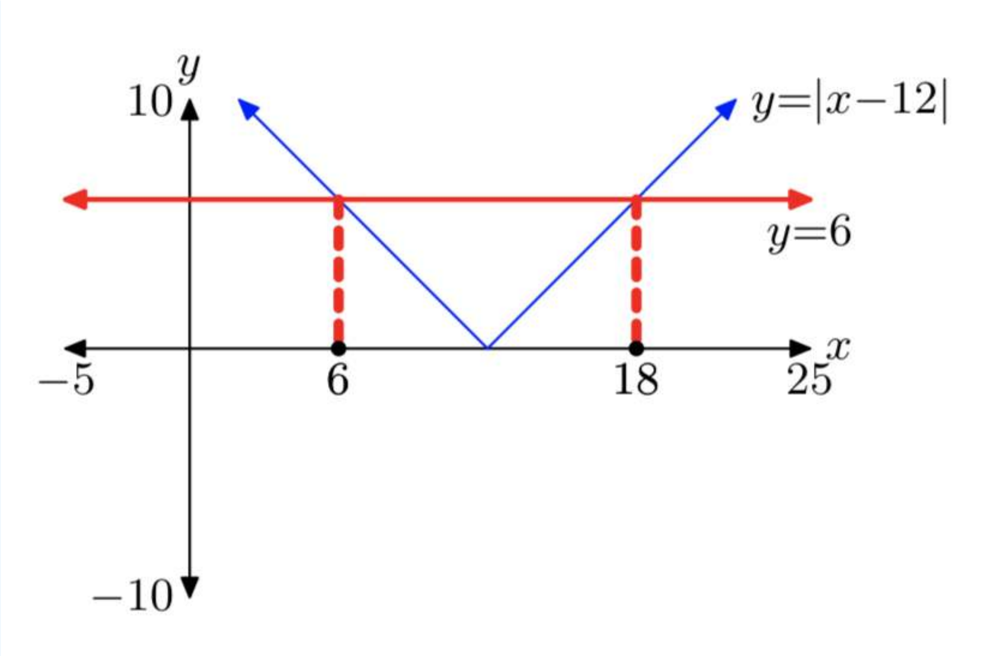
Exercise 3.5.18

$|5x-21| = 7$

Exercise 3.5.19

$|x-12| = 6$

Answer



x = 6 or x = 18

? Exercise 3.5.20

|x+11| = 5

Use a strictly algebraic technique to solve each of the equations in **Exercises 21-28**. Do not use a calculator.

? Exercise 3.5.21

|x+2|−3 = 4

Answer

x = −9 or x = 5

? Exercise 3.5.22

3|x+5| = 6

? Exercise 3.5.23

−2|3−2x| = −6

Answer

x = 0 or x = 3

? Exercise 3.5.24

|4−x|+5 = 12

Exercise 3.5.25

$3|x+2|-5 = |x+2|+7$

Answer

$x = -8$ or $x = 4$

Exercise 3.5.26

$4-3|4-x| = 2|4-x|-1$

Exercise 3.5.27

$\left|\frac{x}{3}-\frac{1}{4}\right|=\frac{1}{12}$

Answer

$x=\frac{1}{2}$ or $x = 1$

Exercise 3.5.28

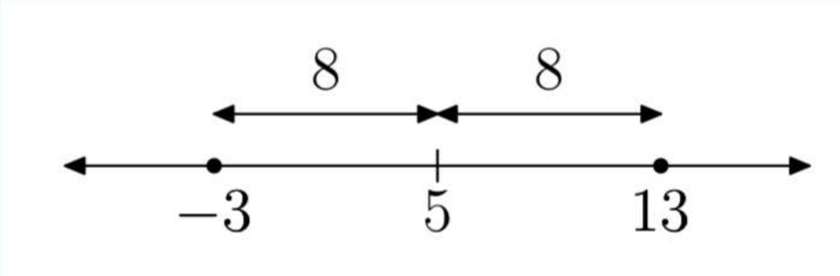
$\left|\frac{x}{4}-\frac{1}{2}\right|=\frac{2}{3}$

Use the technique of distance on the number line demonstrated in Examples 16 and 17 to solve each of the equations in **Exercises 29-32**. Provide number line sketches on your homework paper as shown in Examples 16 and 17 in the narrative.

Exercise 3.5.29

$|x-5| = 8$

Answer



$x = -3$ or $x = 13$

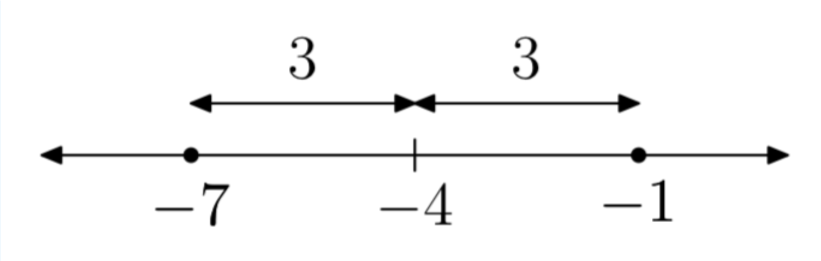
Exercise 3.5.30

$|x-2| = 4$

Exercise 3.5.31

$|x+4| = 3$

Answer



x = −7 or x = −1

Exercise 3.5.32

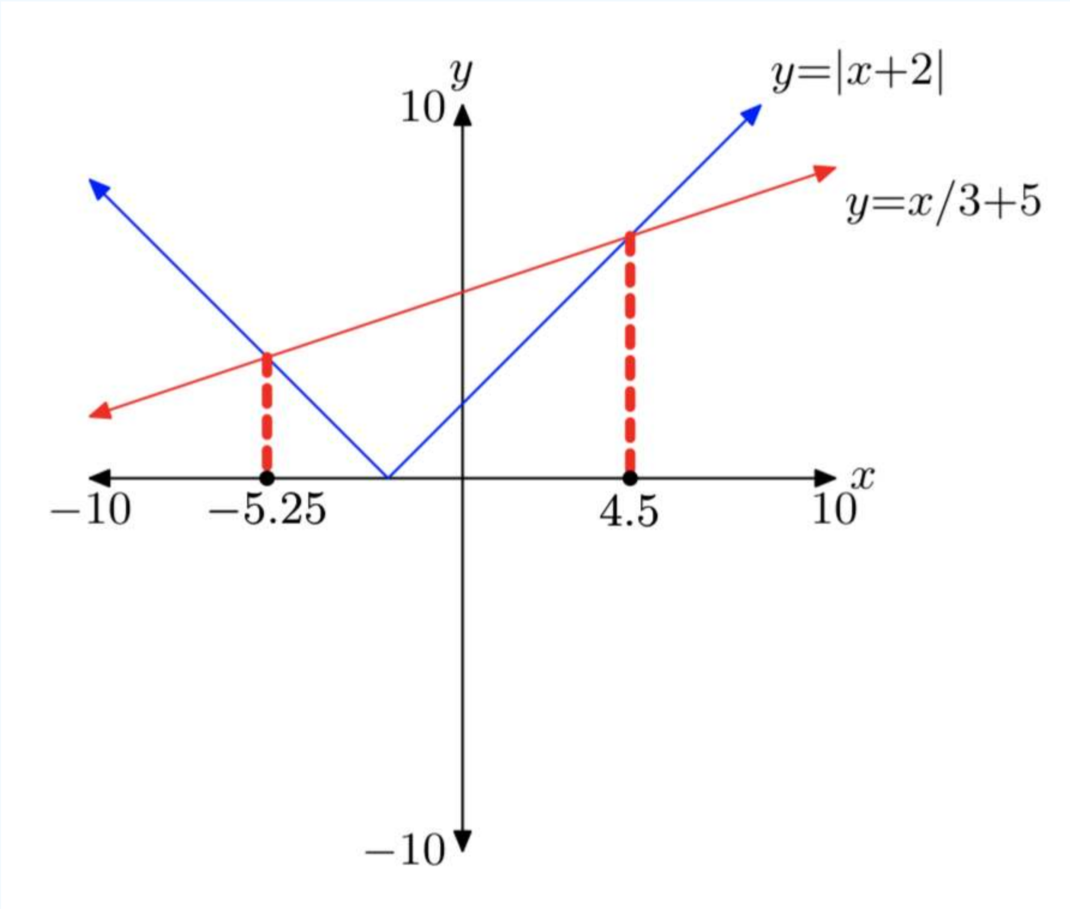
|x+2| = 11

Use the instructions provided in **Exercises 5-8** to solve the equations in **Exercises 33-34**.

Exercise 3.5.33

$|x+2| = \frac{1}{3}x+5$

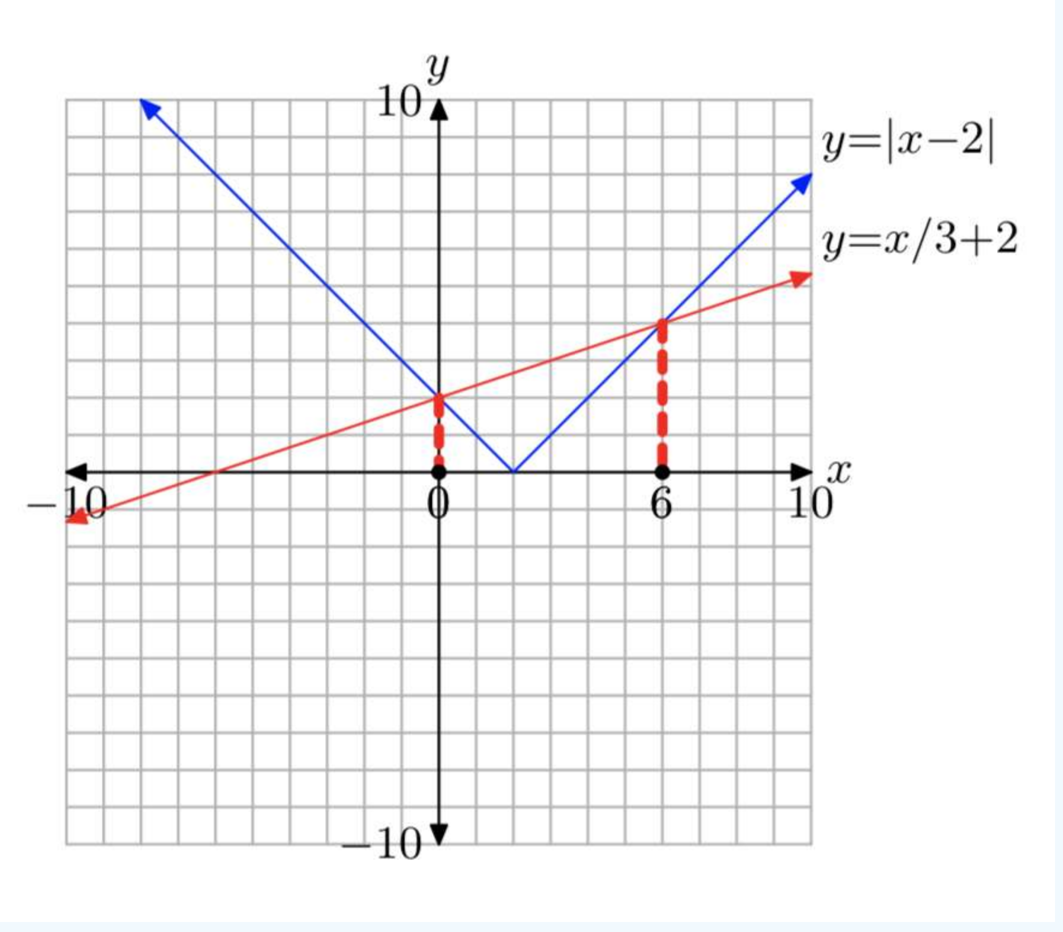
Answer



> **? Exercise 3.5.34**
>
> $|x-3|=5-\frac{1}{2}x$

In **Exercises 35-36**, perform each of the following tasks.

1. Set up a coordinate system on graph paper. Label and scale each axis.
2. Without the use of a calculator, sketch the graphs of the left- and right-hand sides of the given equation. Label each graph with its equation.
3. Drop dashed vertical lines from each point of intersection to the x-axis. Shade and label each solution on the x-axis (you will have to approximate).

> **? Exercise 3.5.35**
>
> $|x-2|=\frac{1}{3}x+2$
>
> **Answer**
>
> 
>

> **? Exercise 3.5.36**
>
> $|x+4|=\frac{1}{3}x+4$

Exercise 3.5.37

Given that a < 0 and b > 0, prove that |ab| = |a||b|.

Answer

If a is a negative real number and b is a positive real number, then ab is negative, so |ab| = −ab. On the other hand, a negative also means that |a| = −a, and b positive means that |b| = b, so that |a||b| = −a(b) = −ab. Comparing these results, we see that |ab| and |a||b| equal the same thing, and so they must be equal to one another.

Exercise 3.5.38

Given that a>0 and b<0, prove that |ab| = |a||b|.

Exercise 3.5.39

In the narrative, we proved that if a > 0 and b < 0, then $|\frac{a}{b}| = \frac{|a|}{|b|}$. Prove the remaining three cases.

Answer

CaseI. (a, b > 0) If a and b are both positive real numbers, then $\frac{a}{b}$ is positive and so $|\frac{a}{b}| = \frac{a}{b}$. On the other hand, a positive also means that |a| = a, and b positive means that |b| = b, so that $\frac{|a|}{|b|} = \frac{a}{b}$. Comparing these two results, we see that $|\frac{a}{b}|$ and $\frac{|a|}{|b|}$ equal the same thing, and so they must be equal to one another.

Case II. (a, b < 0) If a and b are both negative real numbers, then $\frac{a}{b}$ is positive and so $|\frac{a}{b}| = \frac{a}{b}$. On the other hand, a negative also means that |a| = −a, and b negative means that |b| = −b, so that $\frac{|a|}{|b|} = \frac{-a}{(-b)} = \frac{a}{b}$. Comparing these two results, we see that $|\frac{a}{b}|$ and $\frac{|a|}{|b|}$ equal the same thing, and so they must be equal to one another.

Case III. (a < 0, b > 0) If a is a negative real number and b is a positive real number, then $\frac{a}{b}$ is negative and so $|\frac{a}{b}| = -(\frac{a}{b})$. On the other hand, a negative also means that |a| = −a, and b positive means that |b| = b, so that $\frac{|a|}{|b|} = -\frac{a}{b} = -(\frac{a}{b})$. Comparing these two results, we see that $|\frac{a}{b}|$ and $\frac{|a|}{|b|}$ equal the same thing, and so they must be equal to one another.

3.6: Break-Even Analysis (Sink or Swim)

Should you start up the Internet business described in the last section? Right now, all you have are some projected costs and a forecasted level of sales. You imagine you are going to sell 400 units. Is that possible? Is it reasonable to forecast this many sales?

Now you may say to yourself, “400 units a month . . . that's about 13 per day. What's the big deal?” But let’s gather some more information. What if you looked up your industry in Statistics Canada data and learned that the product in question sells just 1,000 units per month in total? Statistics Canada also indicates that there are eight existing companies selling these products. How does that volume of 400 units per month sound now? Unless you are revolutionizing your industry, it is unlikely you will receive a 40% market share in your first month of operations. With so few unit sales in the industry and too many competitors, you might be lucky to sell 100 units. If this is the case, are you still profitable?

Simply looking at the fixed costs, variable costs, potential revenues, contribution margins, and typical net income is not enough. Ultimately, all costs in a business need to be recovered through sales. Do you know how many units have to be sold to pay your bills? The answer to this question helps assess the feasibility of your business idea.

What Is Break-Even Analysis?

If you are starting your own business and head to the bank to initiate a start-up loan, one of the first questions the banker will ask you is your break-even point. You calculate this number through break-even analysis, which is the analysis of the relationship between costs, revenues, and net income with the sole purpose of determining the point at which total revenue equals total cost. This break-even point is the level of output (in units or dollars) at which all costs are paid but no profits are earned, resulting in a net income equal to zero. To determine the break-even point, you can calculate a break-even analysis in two different ways, involving either the number of units sold or the total revenue in dollars. Each of these two methods is discussed in this section.

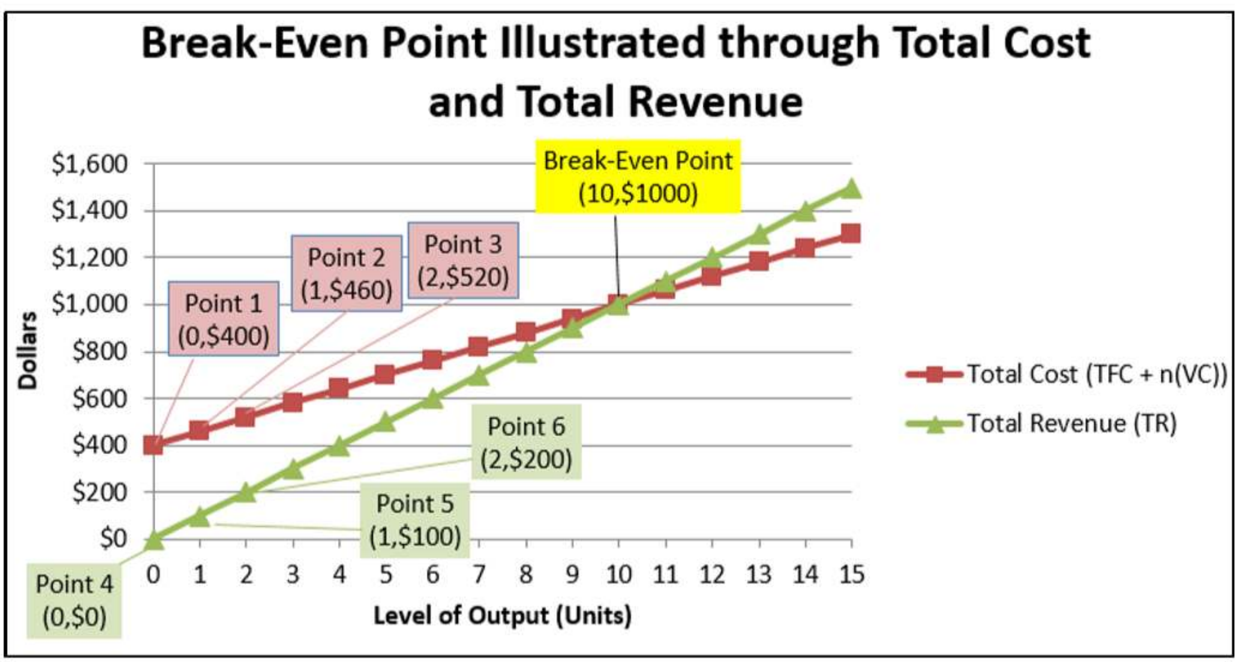
Method 1: Break-Even Analysis in Units

In this method, your goal is to determine the level of output that produces a net income equal to zero. This method requires unit information, including the unit selling price and unit variable cost.

It is helpful to see the relationship of total cost and total revenue on a graph. Assume that a company has the following information:

$$TFC = \$400 \quad S = \$100 \quad VC = \$60$$

The graph shows dollar information on the y-axis and the level of output on the x-axis.



Here is how you construct such a graph:

1. Plot the total costs:
   a. At zero output you incur the total fixed costs of $400. Denote this as Point 1 (0, $400).
   b. As you add one level of output, the total cost rises in the amount of the unit variable cost. Therefore, total cost is $TFC + n(VC) = \$400 + 1(\$60) = \$460$. Denote this as Point 2 (1, $460).
   c. As you add another level of output (2 units total), the total cost rises once again in the amount of the unit variable cost, producing $\$400 + 2(\$60) = \$520$. Denote this as Point 3 (2, $520).
   d. Repeat this process for each subsequent level of output and plot it onto the figure. The red line plots these total costs at all levels of output.
2. Plot the total revenue:
   a. At zero output, there is no revenue. Denote this as Point 4 (0, $0).
   b. As you add one level of output, total revenue rises by the selling price of the product. Therefore, total revenue is $n(S) = 1(\$100) = \$100$. Denote this as Point 5 (1, $100).
   c. As you add another level of output (2 units total), the total revenue rises once again in the amount of the selling price, producing $2(\$100) = \$200$. Denote this as Point 6 (2, $200).
   d. Repeat this process for each subsequent level of output and plot it onto the figure. The green line plots the total revenue at all levels of output.

The purpose of break-even analysis is to determine the point at which total cost equals total revenue. The graph illustrates that the break-even point occurs at an output of 10 units. At this point, the total cost is $\$400 + 10(\$60) = \$1,000$, and the total revenue is $10(\$100) = \$1,000$. Therefore, the net income is $\$1,000 - \$1,000 = \$0$; no money is lost or gained at this point.

The Formula

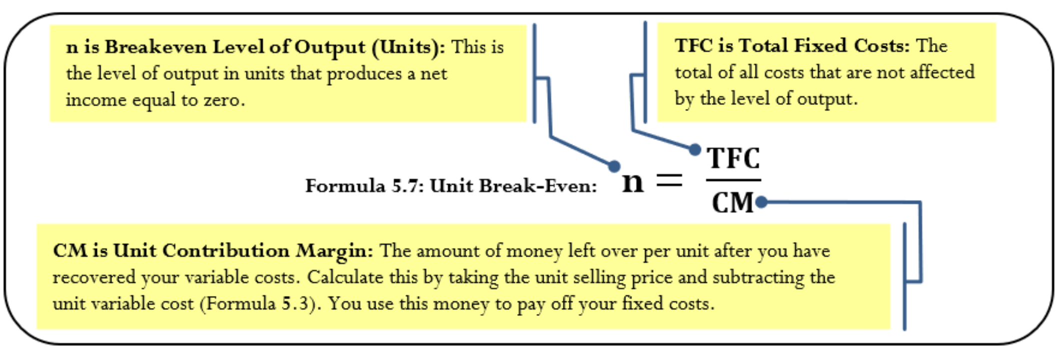
Recall that Formula 5.2 states that the net income equals total revenue minus total costs. In break-even analysis, net income is set to zero, resulting in

$$0 = n(S) - (TFC + n(VC))$$

Rearranging and solving this formula for n gives the following:

$$\begin{aligned} 0 &= n(S) - TFC - n(VC) \\ TFC &= n(S) - n(VC) \\ TFC &= n(S - VC) \\ \frac{TFC}{(S - VC)} &= n \end{aligned}$$

Formula 5.3 states that $CM = S - VC$; therefore, the denominator becomes just CM. The calculation of the break-even point using this method is thus summarized in Formula 5.7.



Formula 5.7

How It Works

Follow these steps to calculate the break-even point in units:

Step 1: Calculate or identify the total fixed costs (TFC).

Step 2: Calculate the unit contribution margin (CM) by applying any needed techniques or formulas.

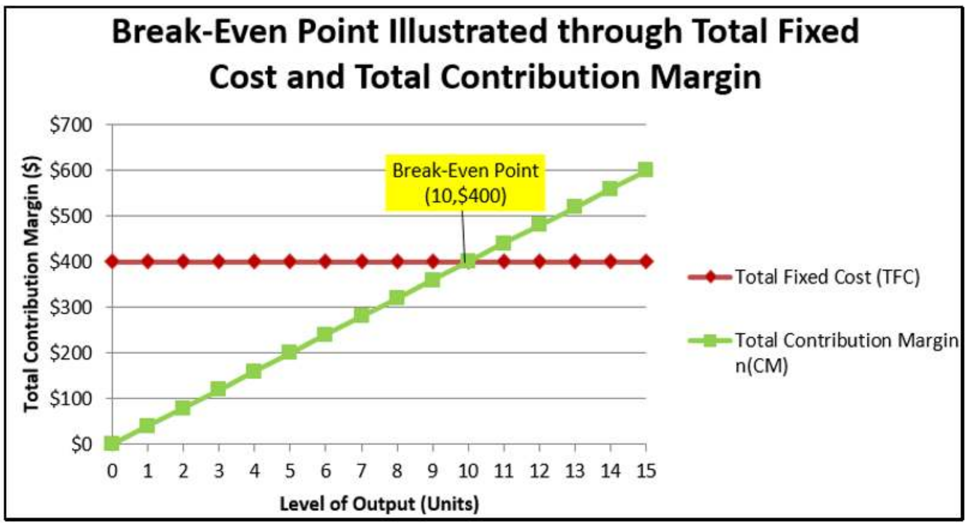
Step 3: Apply Formula 5.7.

Continuing with the example that created the graph on the previous page:

Step 1: Total fixed costs are known, $TFC = \$400$.

Step 2: The unit contribution margin is $\$100 - \$60 = \$40$. For each unit sold this is the amount left over that can be applied against total fixed costs.

Step 3: Applying Formula 5.7 results in $n = \$400 \div \$40 = 10$ units.



Important Notes

When you calculate the break-even units, the formula may produce a number with decimals. For example, a break-even point might be 324.39 units. How should you handle the decimal? A partial unit cannot be sold, so the rule is always to round the level of output up to the next integer, regardless of the decimal. Why? The main point of a break-even analysis is to show the point at which you have recovered all of your costs. If you round the level of output down, you are 0.39 units short of recovering all of your costs. In the long-run, you always operate at a loss, which ultimately puts you out of business. If you round the level of output up to 325, all costs are covered and a tiny dollar amount, as close to zero as possible, is left over as profit. At least at this level of output you can stay in business.

✔ Example 3.6.1: The Break-Even Units for Your Planned Internet Business

Recall from Section 5.1 the Internet business explored throughout Examples 5.1.1 to 5.1.4. Now let’s determine the break-even point in units. As previously calculated, the total fixed costs are $638.03 and the unit contribution margin is $3.57.

Solution

Calculate the break-even point in units sold, or n at break-even.

What You Already Know

Step 1:

The total fixed costs are known: $TFC = \$638.03$

Step 2:

The contribution rate is known: $CM = \$3.57$.

How You Will Get There

Step 3:

Apply Formula 5.7.

Perform

Step 3:

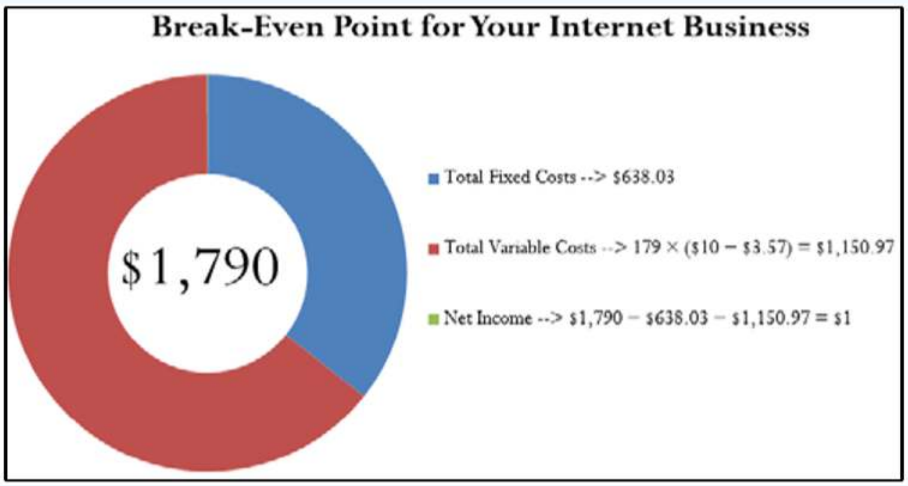
$$n=\frac{\$638.03}{\$3.57}=178.719888$$

Round this up to 179.

Calculator Instructions

FC	VC	P	PFT	Q
638.03	6.43	10	0	Answer: 178.719888

In order for your Internet business to break even, you must sell 179 units. At a price of \$10 per unit, that requires a total revenue of \$1,790. At this level of output your business realizes a net income of \$1 because of the rounding.



Method 2: Break-Even Analysis in Dollars

The income statement of a company does not display unit information. All information is aggregate, including total revenue, total fixed costs, and total variable costs. Typically, no information is listed about unit selling price, unit variable costs, or the level of output. Without this unit information, it is impossible to apply Formula 5.7.

The second method for calculating the break-even point relies strictly on aggregate information. As a result, you cannot calculate the break-even point in units. Instead, you calculate the break-even point in terms of aggregate dollars expressed as total revenue.

The Formula

To derive the break-even point in dollars, once again start with Formula 5.2, where total revenue at break-even less total fixed costs and total variable costs must equal a net income of zero:

$$NI=TR-(TFC+TVC)$$

$$0=TR-(TFC+TVC)$$

Rearranging this formula for total revenue gives:

$$0=TR-TFC-TVC$$

$$TR=TFC+TVC$$

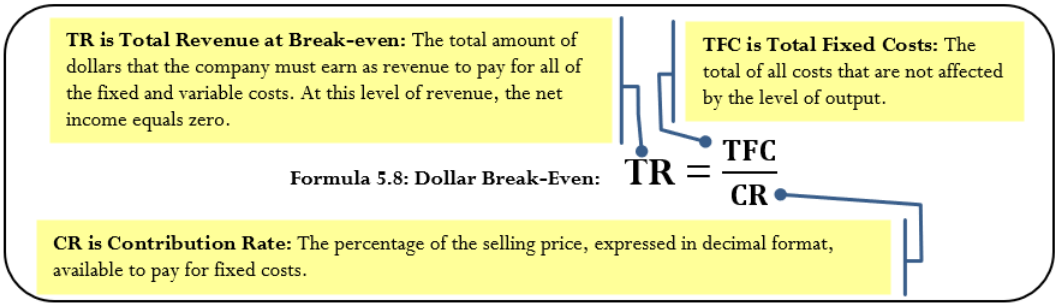
Thus, at the break-even point the total revenue must equal the total cost. Substituting this value into the numerator of Formula 5.6 gives you:

$$CR = (TR - TVC)/TR \times 100$$

$$CR = ((TFC + TVC) - TVC)/TR \times 100$$

$$CR = TFC/TR \times 100$$

A final rearrangement results in Formula 5.8, which expresses the break-even point in terms of total revenue dollars.



Formula 5.8

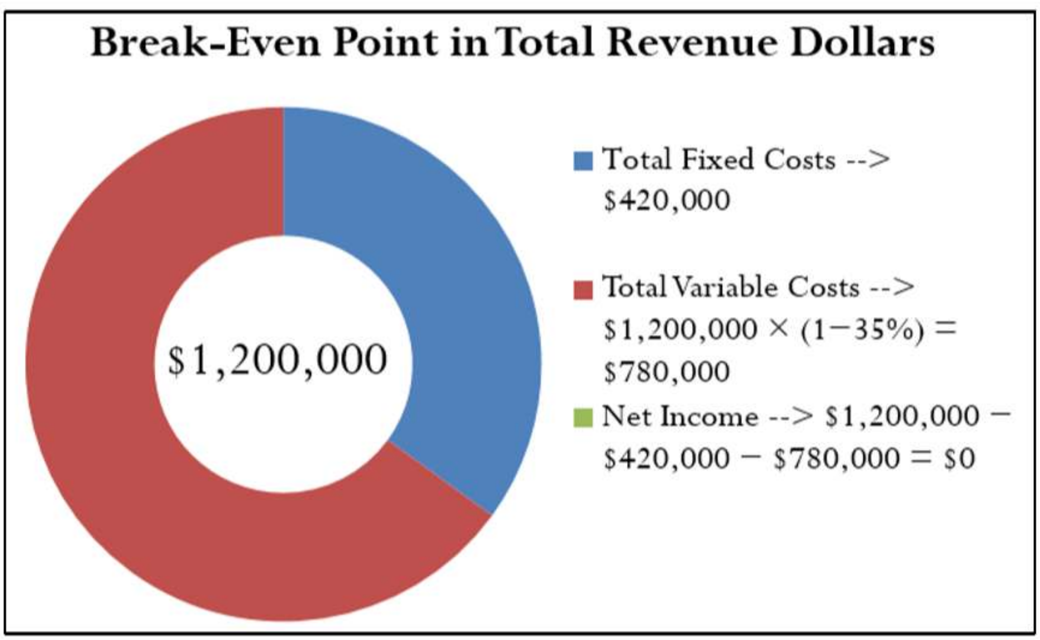
How It Works

Follow these steps to calculate the break-even point in total revenue dollars:

Step 1: Calculate or identify the total fixed costs (TFC).

Step 2: Calculate the contribution rate (CR), by applying any needed techniques or formulas. If not provided, typically the CR is calculated using Formula 5.6, which requires aggregate information only.

Step 3: Apply Formula 5.8 to calculate the break-even point in dollars.



Assume that you are looking at starting your own business. The fixed costs are generally easier to calculate than the variable costs. After running through the numbers, you determine that your total fixed costs are \$420,000, or $TFC = \$420,000$. You are not sure of your variable costs but need to gauge your break-even point. Many of your competitors are publicly traded companies, so you go online and pull up their annual financial reports. After analyzing their financial statements, you see that your competitors have a contribution rate of 35%, or $CR = 0.35$, on average. What is your estimate of your break-even point in dollars?

Step 1: Total fixed costs are $TFC = \$420,000$.

Step 2: The estimated contribution rate is $CR = 0.35$.

Step 3: Applying Formula 5.8 results in

$TR = \$420,000 \div 0.35 = \$1,200,000$. If you average a similar contribution rate, you require total revenue of $1,200,000 to cover all costs, which is your break-even point in dollars.

Important Notes

You need to be very careful with the interpretation and application of a break-even number. In particular, the break-even must have a point of comparison, and it does not provide information about the viability of the business.

Break-Even Points Need to Be Compared. The break-even number by itself, whether in units or dollars, is meaningless. You need to compare it against some other quantity (or quantities) to determine the feasibility of the number you have produced. The other number needs to be some baseline that allows you to grasp the scope of what you are planning. This baseline could include but is not limited to the following:

- Industry sales (in units or dollars)
- Number of competitors fighting for market share in your industry
- Production capacity of your business

For example, in your Internet business the break-even point is 179 units per month. Is that good? In the section opener, you explored a possibility where your industry had total monthly sales of 1,000 units and you faced eight competitors. A basic analysis shows that if you enter the industry and if everyone split the market evenly, you would have sales of 1,000 divided by nine companies, equal to 111 units each. To just pay your bills, you would have to sell almost 61% higher than the even split and achieve a 17.9% market share. This doesn't seem very likely, as these other companies are already established and probably have satisfied customers of their own that would not switch to your business.

Break-Even Points Are Not Green Lights. A break-even point alone cannot tell you to do something, but it can tell you not to do something. In other words, break-even points can put up red lights, but at no point does it give you the green light. In the above scenario, your break-even of 179 units put up a whole lot of red lights since it does not seem feasible to obtain. However, what if your industry sold 10,000 units instead of 1,000 units? Your break-even would now be a 1.79% market share (179 units out of 10,000 units), which certainly seems realistic and attainable. This does not mean “Go for it,” however. It just means that from a strictly financial point of view breaking even seems possible. Whether you can actually sell that many units depends on a whole range of factors beyond just a break-even number. For instance, if your Google ad is written poorly you might not be able to generate that many sales. The break-even analysis cannot factor in this non-quantitative variable, and for that reason it cannot offer a “go ahead.”

? Exercise 3.6.1: Give It Some Thought

What would happen to the break-even point in each of the following situations? Would it increase, decrease, or remain the same?

a. The unit contribution margin increases.
b. The total fixed costs increase.
c. The contribution rate decreases.

Answer

a. Decrease. In Formula 5.7, the denominator is larger, producing a lower break-even.
b. Increase. In both Formula 5.7 and Formula 5.8, the numerator is larger, producing a higher break-even.
c. Increase. In Formula 5.8, the denominator is smaller, producing a higher break-even.

✓ Example 3.6.2: Determining the Break-Even Dollars

In the annual report to shareholders, Borland Manufacturing reported total gross sales of $7,200,000, total variable costs of $4,320,000, and total fixed costs of $2,500,000. Determine Borland's break-even point in dollars.

Solution

Calculate the dollar break-even point, which is the total revenue (TR) at break-even.

What You Already Know

Step 1:

The total fixed costs are known: $TFC = \$2,500,000$

Other known information includes the following:

$TR = \$7,200,000, TVC = \$4,320,000$

How You Will Get There

Step 2:

Calculate the contribution rate by applying Formula 5.6.

Step 3:

Apply Formula 5.8.

Perform

Step 2:

$$CR = \frac{\$7,200,000 - \$4,320,000}{\$7,200,000} \times 100 = \frac{\$2,880,000}{\$7,200,000} \times 100 = 40\%$$

Step 3:

$$TR = \frac{\$2,500,000}{40\%} = \frac{52,500,000}{0.4} = \$6,250,000$$

Borland Manufacturing achieves its break-even point at $6,250,000 in total revenue. At this point, total fixed costs are $2,500,000 and total variable costs are $3,750,000, producing a net income of zero.

Contributors and Attributions

- Template:ContribBusiMathOlivier

3.7: Applications

Learning Objectives

In this section, you will learn to use linear functions to model real-world applications

Now that we have learned to determine equations of lines, we get to apply these ideas in a variety of real-life situations.

Read the problem carefully. Highlight important information. Keep track of which values correspond to the independent variable (x) and which correspond to the dependent variable (y).

Example 3.7.1

A taxi service charges $0.50 per mile plus a $5 flat fee. What will be the cost of traveling 20 miles? What will be cost of traveling x miles?

Solution

x = distance traveled, in miles and y = cost in dollars

The cost of traveling 20 miles is

$$y=(0.50)(20)+5=10+5=15$$

The cost of traveling x miles is

$$y=(0.50)(x)+5=0.50x+5$$

In this problem, $0.50 per mile is referred to as the **variable cost**, and the flat charge $5 as the **fixed cost**. Now if we look at our cost equation $y=.50x+5$, we can see that the variable cost corresponds to the slope and the fixed cost to the y-intercept.

Example 3.7.2

The variable cost to manufacture a product is $10 per item and the fixed cost $2500. If x represents the number of items manufactured and y represents the total cost, write the cost function.

Solution

- The variable cost of $10 per item tells us that $m=10$.
- The fixed cost represents the y-intercept. So $b=2500$.

Therefore, the cost equation is $y=10x+2500$.

Example 3.7.3

It costs $750 to manufacture 25 items, and $1000 to manufacture 50 items. Assuming a linear relationship holds, find the cost equation, and use this function to predict the cost of 100 items.

Solution

We let x = the number of items manufactured, and let y = the cost.

Solving this problem is equivalent to finding an equation of a line that passes through the points (25, 750) and (50, 1000).

$$m=\frac{1000-750}{50-25}=10$$

Therefore, the partial equation is $y=10x+b$

By substituting one of the points in the equation, we get $b=500$

Therefore, the cost equation is $y=10x+500$

To find the cost of 100 items, substitute $x=100$ in the equation $y=10x+500$

So the cost is

$$y = 10(100) + 500 = 1500$$

It costs \$1500 to manufacture 100 items.

Example 3.7.4

The freezing temperature of water in Celsius is 0 degrees and in Fahrenheit 32 degrees. And the boiling temperatures of water in Celsius, and Fahrenheit are 100 degrees, and 212 degrees, respectively. Write a conversion equation from Celsius to Fahrenheit and use this equation to convert 30 degrees Celsius into Fahrenheit.

Solution

Let us look at what is given.

Celsius	Fahrenheit
0	32
100	212

Again, solving this problem is equivalent to finding an equation of a line that passes through the points (0, 32) and (100, 212).

Since we are finding a linear relationship, we are looking for an equation $y = mx + b$, or in this case $F = mC + b$, where x or C represent the temperature in Celsius, and y or F the temperature in Fahrenheit.

$$\text{slope m } = \frac{312 - 32}{100 - 0} = \frac{9}{5}$$

The equation is $F = \frac{9}{5}C + b$

Substituting the point (0, 32), we get

$$F = \frac{9}{5}C + 32.$$

To convert 30 degrees Celsius into Fahrenheit, substitute $C = 30$ in the equation

$$\mathrm{F} = \frac{9}{5}\mathrm{C} + 32$$
$$\mathrm{F} = \frac{9}{5}(30) + 32 = 86$$

Example 3.7.5

The population of Canada in the year 1980 was 24.5 million, and in the year 2010 it was 34 million. The population of Canada over that time period can be approximately modelled by a linear function. Let x represent time as the number of years after 1980 and let y represent the size of the population.

a. Write the linear function that gives a relationship between the time and the population.
b. Assuming the population continues to grow linearly in the future, use this equation to predict the population of Canada in the year 2025.

Solution

The problem can be made easier by using 1980 as the base year, that is, we choose the year 1980 as the year zero. This will mean that the year 2010 will correspond to year 30. Now we look at the information we have:

Year	Population
0 (1980)	24.5 million

30 (2010)	34 million

a. Solving this problem is equivalent to finding an equation of a line that passes through the points (0, 24.5) and (30, 34). We use these two points to find the slope:

$$m=\frac{34-24.5}{30-0}=\frac{9.5}{30}=0.32$$

The y-intercept occurs when $x=0$, so $b=24.5$

$$y=0.32x+24.5$$

b. Now to predict the population in the year 2025, we let $x=2025-1980=45$

$$\begin{aligned}
&y=0.32x+24.5\\
&y=0.32(45)+24.5=38.9
\end{aligned}$$

In the year 2025, we predict that the population of Canada will be 38.9 million people.

Note that we assumed the population trend will continue to be linear. Therefore if population trends change and this assumption does not continue to be true in the future, this prediction may not be accurate.

3.7.1: Applications (Exercises)

SECTION 3.7 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

1) The variable cost to manufacture a product is \$25 per item, and the fixed costs are \$1200. If x is the number of items manufactured and y is the cost, write the cost function.	2) It costs \$90 to rent a car driven 100 miles and \$140 for one driven 200 miles. If x is the number of miles driven and y the total cost of the rental, write the cost function.
3) The variable cost to manufacture an item is \$20 per item, and it costs a total of \$750 to produce 20 items. If x represents the number of items manufactured and y is the cost, write the cost function.	4) To manufacture 30 items, it costs \$2700, and to manufacture 50 items, it costs \$3200. If x represents the number of items manufactured and y the cost, write the cost function.
5) To manufacture 100 items, it costs \$32,000, and to manufacture 200 items, it costs \$40,000. If x is the number of items manufactured and y is the cost, write the cost function.	6) It costs \$1900 to manufacture 60 items, and the fixed costs are \$700. If x represents the number of items manufactured and y the cost, write the cost function.

SECTION 3.7 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

7) A person who weighs 150 pounds has 60 pounds of muscles; a person that weighs 180 pounds has 72 pounds of muscles. If x represents body weight and y is muscle weight, write an equation describing their relationship. Use this relationship to determine the muscle weight of a person that weighs 170 pounds.	8) A spring on a door stretches 6 inches if a force of 30 pounds is applied. It stretches 10 inches if a 50 pound force is applied. If x represents the number of inches stretched, and y is the force, write a linear equation describing the relationship. Use it to determine the amount of force required to stretch the spring 12 inches.
9). A male college student who is 64 inches tall weighs 110 pounds. Another student who is 74 inches tall weighs 180 pounds. Assuming the relationship between male students' heights (x), and weights (y) is linear, write a function to express weights in terms of heights, and use this function to predict the weight of a student who is 68 inches tall.	10) EZ Clean company has determined that if it spends \$30,000 on advertising, it can hope to sell 12,000 of its Minivacs a year, but if it spends \$50,000, it can sell 16,000. Write an equation that gives a relationship between the number of dollars spent on advertising (x) and the number of minivacs sold(y).
11) The freezing temperatures for water for Celsius and Fahrenheit scales are 0ºC and 32ºF. The boiling temperatures for water are 100 ºC and 212 ºF. Let C denote the temperature in Celsius and F in Fahrenheit. Write the conversion function from Celsius to Fahrenheit. Use the function to convert 25 ºC into ºF.	12) By reversing the coordinates in the previous problem, find a conversion function that converts Fahrenheit into Celsius, and use this conversion function to convert 72 ºF into an equivalent Celsius measure.

SECTION 3.7 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

13) California's population was 29.8 million in the year 1990, and 37.3 million in 2010. Assume that the population trend was and continues to be linear, write the population function. Use this function to predict the population in 2025. *Hint: Use 1990 as the base year (year 0); then 2010 and 2025 are years 20, and 35, respectively.)*	14) Use the population function for California in the previous problem to find the year in which the population will be 40 million people.

15) A college's enrollment was 13,200 students in the year 2000, and 15,000 students in 2015. Enrollment has followed a linear pattern. Write the function that models enrollment as a function of time. Use the function to find the college's enrollment in the year 2010. *Hint: Use year 2000 as the base year.*	16) If the college's enrollment continues to follow this pattern, in what year will the college have 16,000 students enrolled.
17) The cost of electricity in residential homes is a linear function of the amount of energy used. In Grove City, a home using 250 kilowatt hours (kwh) of electricity per month pays $55. A home using 600 kwh per month pays $118. Write the cost of electricity as a function of the amount used. Use the function to find the cost for a home using 400 kwh of electricity per month.	18) Find the level of electricity use that would correspond to a monthly cost of $100.

SECTION 3.7 PROBLEM SET: APPLICATIONS

In the following application problems, assume a linear relationship holds.

19) At ABC Co., sales revenue is $170,000 when it spends $5000 on advertising. Sales revenue is $254,000 when $12,000 is spent on advertising. a) Find a linear function for y = amount of sales revenue as a function of x = amount spent on advertising. b) Find revenue if $10,000 is spent on advertising. c) Find the amount that should be spent on advertising to achieve $200,000 in revenue.	20) For problem 19, explain the following: a. Explain what the slope of the line tells us about the effect on sales revenue of money spent on advertising. Be specific, explaining both the number and the sign of the slope in the context of this problem. b. Explain what the y intercept of the line tells us about the sales revenue in the context of this problem
21) Mugs Café sells 1000 cups of coffee per week if it does not advertise. For every $50 spent in advertising per week, it sells an additional 150 cups of coffee. a) Find a linear function that gives y = number of cups of coffee sold per week x = amount spent on advertising per week. b) How many cups of coffee does Mugs Café expect to sell if $100 per week is spent on advertising?	22) Party Sweets makes baked goods that can be ordered for special occasions. The price is $24 to order one dozen (12 cupcakes) and $9 for each additional 6 cupcakes. a. Find a linear function that gives the total price of a cupcake order as a function of the number of cupcakes ordered b. Find the price for an order of 5 dozen (60) cupcakes

3.8: More Applications

Learning Objectives

In this section, you will learn to:

1. Solve a linear system in two variables.
2. Find the equilibrium point when a demand and a supply equation are given.
3. Find the break-even point when the revenue and the cost functions are given.

Finding the Point of Intersection of Two Lines

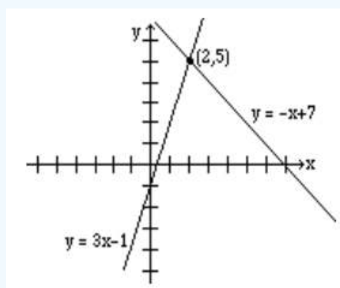
In this section, we will do application problems that involve the intersection of lines. Therefore, before we proceed any further, we will first learn how to find the intersection of two lines.

Example 3.8.1

Find the intersection of the line $y=3x-1$ and the line $y=-x+7$.

Solution

We graph both lines on the same axes, as shown below, and read the solution (2, 5).



Finding an intersection of two lines graphically is not always easy or practical; therefore, we will now learn to solve these problems algebraically.

At the point where two lines intersect, the x and y values for both lines are the same. So in order to find the intersection, we either let the x-values or the y-values equal.

If we were to solve the above example algebraically, it will be easier to let the y-values equal. Since $y=3x-1$ for the first line, and $y=-x+7$ for the second line, by letting the y-values equal, we get

$$\begin{aligned} 3x-1 &= -x+7 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

By substituting $x=2$ in any of the two equations, we obtain $y=5$.

Hence, the solution (2, 5).

A common algebraic method used to solve systems of equations is called the **elimination method**. The object is to eliminate one of the two variables by adding the left and right sides of the equations together. Once one variable is eliminated, we have an equation with only one variable for can be solved. Finally, by substituting the value of the variable that has been found in one of the original equations, we get the value of the other variable.

Example 3.8.2

Find the intersection of the lines $2x+y=7$ and $3x-y=3$ by the elimination method.

Solution

We add the left and right sides of the two equations.

$$\begin{array}{r} 2x+y=7 \\ 3x-y=3 \\ \hline 5x=10 \\ x=2 \end{array}$$

Now we substitute $x=2$ in any of the two equations and solve for y.

$$\begin{aligned} 2(2)+y &= 7 \\ y &= 3 \end{aligned}$$

Therefore, the solution is (2, 3).

Example 3.8.3

Solve the system of equations $x+2y=3$ and $2x+3y=4$ by the elimination method.

Solution

If we add the two equations, none of the variables are eliminated. But the variable x can be eliminated by multiplying the first equation by -2, and leaving the second equation unchanged.

$$\begin{array}{r} -2x-4y=-6 \\ 2x+3y=4 \\ \hline -y=-2 \\ y=2 \end{array}$$

Substituting $y=2$ in $x+2y=3$, we get

$$\begin{aligned} & x+2(2)=3 \\ & x=-1 \end{aligned} \tag{3.8.1}$$

Therefore, the solution is (-1, 2).

Example 3.8.4

Solve the system of equations $3x-4y=5$ and $4x-5y=6$.

Solution

This time, we multiply the first equation by - 4 and the second by 3 before adding. (The choice of numbers is not unique.)

$$\begin{array}{r} -12x+16y=-20 \\ 12x-15y=18 \\ \hline y=-2 \end{array}$$

By substituting y = - 2 in any one of the equations, we get x = -1.
Hence the solution is (-1, -2).

SUPPLY, DEMAND AND THE EQUILIBRIUM MARKET PRICE

In a free market economy the supply curve for a commodity is the number of items of a product that can be made available at different prices, and the demand curve is the number of items the consumer will buy at different prices.

As the price of a product increases, its demand decreases and supply increases. On the other hand, as the price decreases the demand increases and supply decreases. The **equilibrium price** is reached when the demand equals the supply.

Example $3.8.5$

The supply curve for a product is $y = 3.5x - 14$ and the demand curve for the same product is $y = -2.5x + 34$, where x is the price and y the number of items produced. Find the following.

a. How many items will be supplied at a price of $10?
b. How many items will be demanded at a price of $10?
c. Determine the equilibrium price.
d. How many items will be produced at the equilibrium price?

Solution

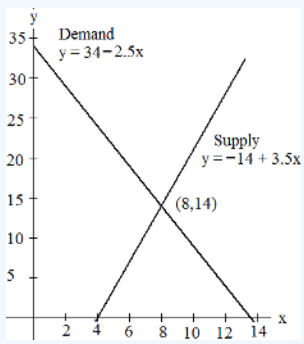
a) We substitute $x = 10$ in the supply equation, $y = 3.5x - 14$; the answer is $y = 3.5(10) - 14 = 21$.

b) We substitute $x = 10$ in the demand equation, $y = -2.5x + 34$; the answer is $y = -2.5(10) + 34 = 9$.

c) By letting the supply equal the demand, we get

$$\begin{aligned} 3.5x - 14 &= -25x + 34 \\ 6x &= 48 \\ x &= \$8 \end{aligned}$$

d) We substitute $x = 8$ in either the supply or the demand equation; we get $y = 14$.



The graph shows the intersection of the supply and the demand functions and their point of intersection, (8,14).

Interpretation: At equilibrium, the price is $8 per item, and 14 items are produced by suppliers and purchased by consumers.

The Break-Even Point

In a business, the profit is generated by selling products.

- If a company sells x number of items at a price P, then the **revenue R** is the price multiplied by number of items sold: $R = P \cdot x$.
- The **production costs C** are the sum of the variable costs and the fixed costs, and are often written as C = mx + b, where x is the number of items manufactured.
-
  - The slope m is the called marginal cost and represents the cost to produce one additional item or unit.
  - The variable cost, mx, depends on how much is being produced
  - The fixed cost b is constant; it does not change no matter how much is produced.
- **Profit** is equal to Revenue minus Cost: Profit = R − C

A company makes a profit if the revenue is greater than the cost. There is a loss if the cost is greater than the revenue. The point on the graph where the revenue equals the cost is called the break-even point. At the break-even point, profit is 0.

Example 3.8.6

If the revenue function of a product is $R = 5x$ and the cost function is $y = 3x + 12$, find the following.

a. If 4 items are produced, what will the revenue be?
b. What is the cost of producing 4 items?
c. How many items should be produced to break even?
d. What will be the revenue and the cost at the break-even point?

Solution

a) We substitute $x = 4$ in the revenue equation $R = 5x$, and the answer is $R = 20$.

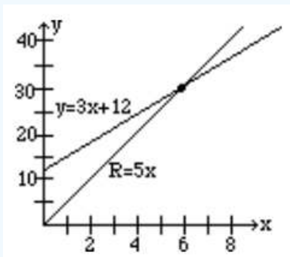
b) We substitute $x = 4$ in the cost equation $C = 3x + 12$, and the answer is $C = 24$.

c) By letting the revenue equal the cost, we get

$$\begin{aligned} 5x &= 3x + 12 \\ x &= 6 \end{aligned}$$

d) Substitute $x = 6$ in either the revenue or the cost equation: we get $R = C = 30$.

The graph below shows the intersection of the revenue and cost functions and their point of intersection, (6, 30).



3.8.1: More Applications (Exercises)

SECTION 3.8 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

1) Solve for x and y. $y = 3x + 4$ $y = 5x - 2$	2) Solve for x and y. $2x - 3y = 4$ $3x - 4y = 5$
3) The supply and demand curves for a product are: Supply $y = 2000x - 6500$ Demand $y = - 1000x + 28000$, where x is price and y is the number of items. At what price will supply equal demand and how many items will be produced at that price?	4) The supply and demand curves for a product are Supply $y = 300x - 18000$ and Demand $y = - 100x + 14000$, where x is price and y is the number of items. At what price will supply equal demand, and how many items will be produced at that price?
5) A car rental company offers two plans for one way rentals. Plan I charges \$36 per day and 17 cents per mile. Plan II charges \$24 per day and 25 cents per mile. a. If you were to drive 300 miles in a day, which plan is better? b. For what mileage are both rates equal?	

SECTION 3.8 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

6) A demand curve for a product is the number of items the consumer will buy at different prices. At a price of \$2 a store can sell 2400 of a particular type of toy truck. At a price of \$8 the store can sell 600 such trucks. If x represents the price of trucks and y the number of items sold, write an equation for the demand curve.	7) A supply curve for a product is the number of items that can be made available at different prices. A manufacturer of toy trucks can supply 2000 trucks if they are sold for \$8 each; it can supply only 400 trucks if they are sold for \$4 each. If x is the price and y the number of items, write an equation for the supply curve.
8) The equilibrium price is the price where the supply equals the demand. From the demand and supply curves obtained in the previous two problems, find the equilibrium price, and determine the number of items that can be sold at that price.	9) A break-even point is the intersection of the cost function and the revenue function, that is, where total cost equals revenue, and profit is zero. Mrs. Jones Cookies Store's cost and revenue, in dollars, for x number of cookies is given by $C = .05x + 3000$ and $R = .80x$. Find the number of cookies that must be sold to break even.

SECTION 3.8 PROBLEM SET: MORE APPLICATIONS

Solve the following problems.

10) A company's revenue and cost in dollars are given by $R = 225x$ and $C = 75x + 6000$, where x is the number of items. Find the number of items that must be produced to break-even.	11) A firm producing socks has a fixed cost of \$20,000 and variable cost of \$2 per pair of socks. Let x = the number of pairs of socks. Find the break-even point if the socks sell for \$4.50 per pair.

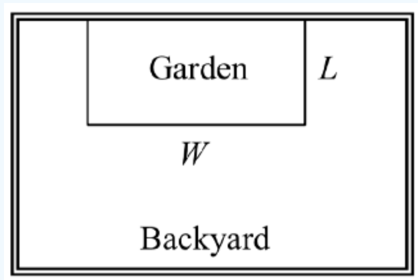
12) Whackemhard Sports is planning to introduce a new line of tennis rackets. The fixed costs for the new line are $25,000 and the variable cost of producing each racket is $60. x is the number of rackets; y is in dollars. If the racket sells for $80, how many rackets must be sold in order to break even?	13) It costs $1,200 to produce 50 pounds of a chemical and it costs $2,200 to produce 150 pounds. The chemical sells for $15 per pound x is the amount of chemical; y is in dollars. a. Find the cost function. b. What is the fixed cost? c. How many pounds must be sold to break even? d. Find the cost and revenue at the break-even point.

3.9: Quadratic Functions

In this section, we will explore the family of 2^{nd} degree polynomials, the quadratic functions. While they share many characteristics of polynomials in general, the calculations involved in working with quadratics is typically a little simpler, which makes them a good place to start our exploration of short run behavior. In addition, quadratics commonly arise from problems involving area and projectile motion, providing some interesting applications.

Example 3.9.1

A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose 3 sides, and will put the 4^{th} side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.



Solution

In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, W, to represent the side of fencing parallel to the 4^{th} side or backyard fence.

Since we know we only have 80 feet of fence available, we know that $L+W+L=80$, or more simply, $2L+W=80$. This allows us to represent the width, W, in terms of L:

$$W=80-2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A=LW=L(80-2L)$$

$$A(L)=80L-2L^2$$

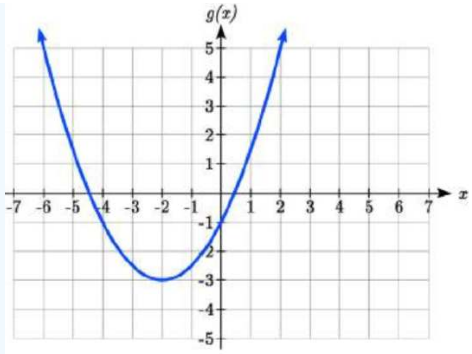
This formula represents the area of the fence in terms of the variable length L.

Short run Behavior: Vertex

We now explore the interesting features of the graphs of quadratics. In addition to intercepts, quadratics have an interesting feature where they change direction, called the **vertex**. You probably noticed that all quadratics are related to transformations of the basic quadratic function $f(x)=x^2$.

Example 3.9.2

Write an equation for the quadratic graphed below as a transformation of $f(x)=x^2$, then expand the formula and simplify terms to write the equation in standard polynomial form.



Solution

We can see the graph is the basic quadratic shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$. By plugging in a point that falls on the grid, such as (0,-1), we can solve for the stretch factor:

$$-1 = a(0+2)^2 - 3$$
$$2 = 4a$$
$$a = \frac{1}{2}$$

Written as a transformation, the equation for this formula is $g(x) = \frac{1}{2}(x+2)^2 - 3$. To write this in standard polynomial form, we can expand the formula and simplify terms:

$$g(x) = \frac{1}{2}(x+2)^2 - 3$$
$$g(x) = \frac{1}{2}(x+2)(x+2) - 3$$
$$g(x) = \frac{1}{2}(x^2 + 4x + 4) - 3$$
$$g(x) = \frac{1}{2}x^2 + 2x + 2 - 3$$
$$g(x) = \frac{1}{2}x^2 + 2x - 1$$

Notice that the horizontal and vertical shifts of the basic quadratic determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Exercise 3.9.1

A coordinate grid has been superimposed over the quadratic path of a basketball(From http://blog.mrmeyer.com/?p=4778, © Dan Meyer, CC-BY). Find an equation for the path of the ball. Does he make the basket?

Answer

The path passes through the origin with vertex at (-4, 7).

$$h(x) = -\frac{7}{16}(x+4)^2 + 7$$

To make the shot, $h(-7.5)$ would need to be about 4. $h(-7.5) \approx 1.64$; he doesn't make it.

forms of quadratic function

The **standard form** of a quadratic function is

$$f(x) = ax^2 + bx + c \tag{3.9.1}$$

The **transformation form** of a quadratic function is

$$f(x) = a(x-h)^2 + k \tag{3.9.2}$$

The **vertex** of the quadratic function is located at (h, k), where h and k are the numbers in the transformation form of the function. Because the vertex appears in the transformation form, it is often called the **vertex form**.

In the previous example, we saw that it is possible to rewrite a quadratic function given in transformation form and rewrite it in standard form by expanding the formula. It would be useful to reverse this process since the transformation form reveals the vertex.

Expanding out the general transformation form of a quadratic gives:

$$f(x) = a(x-h)^2 + k = a(x-h)(x-h) + k$$
$$f(x) = a(x^2 - 2xh + h^2) + k = ax^2 - 2ahx + ah^2 + k$$

This should be equal to the standard form of the quadratic:

$$ax^2 - 2ahx + ah^2 + k = ax^2 + bx + c$$

The second degree terms are already equal. For the linear terms to be equal, the coefficients must be equal:

$$-2ah = b \text{ ,so } h = -\frac{b}{2a}$$

This provides us a method to determine the horizontal shift of the quadratic from the standard form. We could likewise set the constant terms equal to find:

$$ah^2 + k = c \text{ ,so } k = c - ah^2 = c - a\left(-\frac{b}{2a}\right)^2 = c - a\frac{b^2}{4a^2} = c - \frac{b^2}{4a}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $k = f(h)$.

finding the vertex of a quadratic

For a quadratic given in standard form, the vertex (h, k) is located at:

$$h = -\frac{b}{2a}, k = f(h) = f\left(\frac{-b}{2a}\right) \tag{3.9.3}$$

Example 3.9.3

Find the vertex of the quadratic $f(x)=2x^2-6x+7$. Rewrite the quadratic into transformation form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$h=-\frac{b}{2a}=-\frac{-6}{2(2)}=\frac{6}{4}=\frac{3}{2}$$

The vertical coordinate of the vertex will be at

$$f(\frac{3}{2})=2(\frac{3}{2})^2-6(\frac{3}{2})+7=\frac{5}{2}$$

Rewriting into transformation form, the stretch factor will be the same as the a in the original quadratic. Using the vertex to determine the shifts,

$$f(x)=2(x-\frac{3}{2})^2+\frac{5}{2}$$

Exercise 3.9.2

Given the equation $g(x)=13+x^2-6x$ write the equation in standard form and then in transformation/vertex form.

Answer

$g(x)=x^2-6x+13$ in Standard form (Equation 3.9.1

Finding the vertex, $h=\dfrac{-(-6)}{2(1)}=3$.

$$k=g(3)=3^2-6(3)+13=4$$

$g(x)=(x-3)^2+4$ in Transformation form (Equation 3.9.2)

As an alternative to using a formula for finding the vertex, the equation can also be written into vertex form by **completing the square**. This process is most easily explained through example. In most cases, using the formula for finding the vertex will be quicker and easier than completing the square, but completing the square is a useful technique when faced with some other algebraic problems.

Example 3.9.4

Rewrite $f(x)=2x^2-12x+14$ into vertex form by completing the square.

Solution

We start by factoring the leading coefficient from the quadratic and linear terms.

$$2(x^2-6x)+14$$

Next, we are going to add something inside the parentheses so that the quadratic inside the parentheses becomes a perfect square. In other words, we are looking for values p and q so that $(x^2-6x+p)=(x-q)^2$.

Notice that if multiplied out on the right, the middle term would be -2q, so q must be half of the middle term on the left; $q=-3$. In that case, p must be $(-3)^2=9$.

$$(x^2-6x+9)=(x-3)^2$$

Now, we can't just add 9 into the expression – that would change the value of the expression. In fact, adding 9 inside the parentheses actually adds 18 to the expression, since the 2 outside the parentheses will distribute. To keep the expression balanced, we can subtract 18.

$$2(x^2-6x+9)+14-18$$

Simplifying, we are left with vertex form.

$$2(x-3)^2-4$$

In addition to enabling us to more easily graph a quadratic written in standard form, finding the vertex serves another important purpose – it allows us to determine the maximum or minimum value of the function, depending on which way the graph opens.

Example 3.9.5

Returning to our backyard farmer from the beginning of the section, what dimensions should she make her garden to maximize the enclosed area?

Solution

Earlier we determined the area she could enclose with 80 feet of fencing on three sides was given by the equation $A(L)=80L-2L^2$. Notice that quadratic has been vertically reflected, since the coefficient on the squared term is negative, so the graph will open downwards, and the vertex will be a maximum value for the area.

In finding the vertex, we take care since the equation is not written in standard polynomial form with decreasing powers. But we know that a is the coefficient on the squared term, so $a=-2$, $b=80$, and $c=0$.

Finding the vertex:

$$h=-\frac{80}{2(-2)}=20, k=A(20)=80(20)-2(20)^2=800$$

The maximum value of the function is an area of 800 square feet, which occurs when $L=20$ feet. When the shorter sides are 20 feet, that leaves 40 feet of fencing for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet, and the longer side parallel to the existing fence has length 40 feet.

Example 3.9.6

A local newspaper currently has 84,000 subscribers, at a quarterly charge of $30. Market research has suggested that if they raised the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the charge per subscription times the number of subscribers. We can introduce variables, C for charge per subscription and S for the number subscribers, giving us the equation

$$\text{Revenue}=CS$$

Since the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $S=84,000$ and $C=30$, and that if they raise the price to $32 they would lose 5,000 subscribers, giving a second pair of values, $C=32$ and $S=79,000$. From this we can find a linear equation relating the two quantities. Treating C as the input and S as the output, the equation will have form $S=mC+b$. The slope will be

$$m=\frac{79,000-84,000}{32-30}=\frac{-5,000}{2}=-2,500$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the vertical intercept

$$S=-2500C+b$$

Plug in the point $S=84,000$ and $C=30$

$$84,000=-2500(30)+b$$

Solve for b

$$b = 159,000$$

This gives us the linear equation $S = -2,500C + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = CS$$

Substituting the equation for S from above

$$\text{Revenue} = C(-2,500C + 159,000)$$

Expanding

$$\text{Revenue} = -2,500C^2 + 159,000C$$

We now have a quadratic equation for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex:

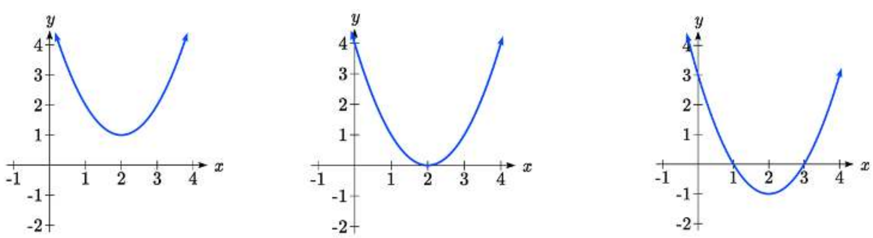
$$h = -\frac{159,000}{2(-2,500)} = 31.8$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we can evaluate the revenue equation:

$$\text{Maximum Revenue} = -2,500(31.8)^2 + 159,000(31.8) = \$2,528,100$$

Short run Behavior: Intercepts

As with any function, we can find the vertical intercepts of a quadratic by evaluating the function at an input of zero, and we can find the horizontal intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.



Zero horizontal intercepts | one horizontal intercept | two horizontal intercepts

Example 3.9.7

Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$

Solution

We can find the vertical intercept by evaluating the function at an input of zero:

$$f(0) = 3(0)^2 + 5(0) - 2 = -2$$

Vertical intercept at (0,-2)

For the horizontal intercepts, we solve for when the output will be zero

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution

$$0 = (3x - 1)(x + 2)$$

$$\begin{aligned} 0 &= 3x - 1 \\ x &= \frac{1}{3} \end{aligned} \quad \text{or} \quad \begin{aligned} 0 &= x + 2 \\ x &= -2 \end{aligned}$$

Horizontal intercepts at $\left(\frac{1}{3}, 0\right)$ and (-2,0)

Notice that in the standard form of a quadratic, the constant term *c* reveals the vertical intercept of the graph.

Example 3.9.8

Find the horizontal intercepts of the quadratic $f(x) = 2x^2 + 4x - 4$

Solution

Again we will solve for when the output will be zero

$$0 = 2x^2 + 4x - 4$$

Since the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic into transformation form.

$$h = -\frac{b}{2a} = -\frac{4}{2(2)} = -1 \quad k = f(-1) = 2(-1)^2 + 4(-1) - 4 = -6$$

$$f(x) = 2(x+1)^2 - 6$$

Now we can solve for when the output will be zero

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x + 1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has horizontal intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$

Exercise 3.9.3

In Try it Now problem 2 we found the standard & transformation form for the function $g(x) = 13 + x^2 - 6x$. Now find the Vertical & Horizontal intercepts (if any).

Answer

Vertical intercept at (0, 13), No horizontal intercepts since the vertex is above the x-axis and the graph opens upwards.

The process in the last example is done commonly enough that sometimes people find it easier to solve the problem once in general and remember the formula for the result, rather than repeating the process each time. Based on our previous work we showed that any quadratic in standard form can be written into transformation form as:

$$f(x) = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a} \tag{3.9.4}$$

Solving for the horizontal intercepts using this general equation gives:

$$0 = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

start to solve for x by moving the constants to the other side

$$\frac{b^2}{4a} - c = a(x + \frac{b}{2a})^2$$

divide both sides by a

$$\frac{b^2}{4a^2} - \frac{c}{a} = (x + \frac{b}{2a})^2$$

find a common denominator to combine fractions

$$\frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

combine the fractions on the left side of the equation

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

take the square root of both sides

$$\pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = x + \frac{b}{2a}$$

subtract $b/2a$ from both sides

$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = x$$

combining the fractions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice that this can yield two different answers for x

Quadratic Formula

For a quadratic function given in standard form $f(x) = ax^2 + bx + c$, the **quadratic formula** gives the horizontal intercepts of the graph of this function.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.9.5}$$

Example 3.9.9

A ball is thrown upwards from the top of a 40-foot-tall building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

What is the maximum height of the ball?

When does the ball hit the ground?

Solution

To find the maximum height of the ball, we would need to know the vertex of the quadratic.

$$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2}, k = H\left(\frac{5}{2}\right) = -16\left(\frac{5}{2}\right)^2 + 80\left(\frac{5}{2}\right) + 40 = 140$$

The ball reaches a maximum height of 140 feet after 2.5 seconds.

To find when the ball hits the ground, we need to determine when the height is zero – when $H(t) = 0$. While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:

$$t=\frac{-80 \pm \sqrt{80^2-4(-16)(40)}}{2(-16)}=\frac{-80 \pm \sqrt{8960}}{-32}$$

Since the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions:

$$t=\frac{-80-\sqrt{8960}}{-32} \approx 5.458 \text{ or } t=\frac{-80+\sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds.

Exercise 3.9.4

For these two equations determine if the vertex will be a maximum value or a minimum value.

a. $g(x)=-8x+x^2+7$
b. $g(x)=-3(3-x)^2+2$

Answer

a. Vertex is a minimum value, since $a>0$ and the graph opens upwards

b. Vertex is a maximum value, since $a<0$ and the graph opens downwards

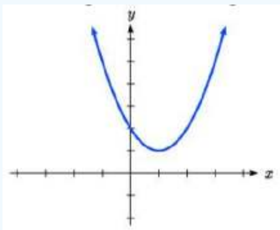
Important Topics of this Section

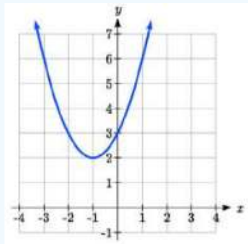
- Quadratic functions
- Standard form
- Transformation form/Vertex form
- Vertex as a maximum / Vertex as a minimum
- Short run behavior
- Vertex / Horizontal & Vertical intercepts
- Quadratic formula

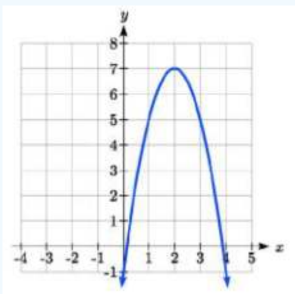
3.9E: Quadratic Functions (Exercises)

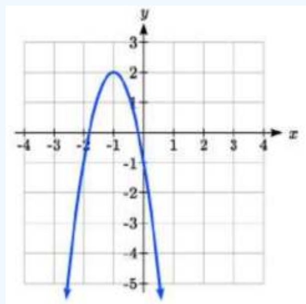
section 3.9 exercise

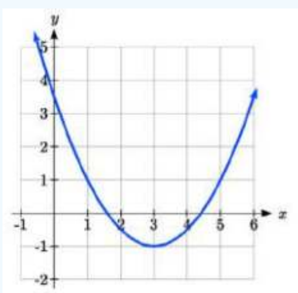
Write an equation for the quadratic function graphed.

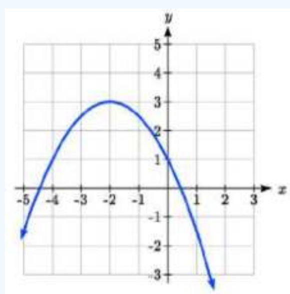
1. 

2. 

3. 

4. 

5. 

6. 

For each of the follow quadratic functions, find a) the vertex, b) the vertical intercept, and c) the horizontal intercepts.

7. $y(x) = 2x^2 + 10x + 12$

8. $z(p) = 3x^2 + 6x - 9$

9. $f(x) = 2x^2 - 10x + 4$

10. $g(x) = -2x^2 - 14x + 12$

11. $h(t) = -4t^2 + 6t - 1$

12. $k(t) = 2x^2 + 4x - 15$

Rewrite the quadratic function into vertex form.

13. $(x) = x^2 - 12x + 32$

14. $g(x) = x^2 + 2x - 3$

15. $h(x) = 2x^2 + 8x - 10$

16. $k(x) = 3x^2 - 6x - 9$

17. Find the values of b and c so $f(x) = -8x^2 + bx + c$ has vertex $2, -7)$

18. Find the values of b and c so $f(x) = 6x^2 + bx + c$ has vertex $(7, -9)$

Write an equation for a quadratic with the given features

19. x-intercepts (-3, 0) and (1, 0), and y intercept (0, 2)

20. x-intercepts (2, 0) and (-5, 0), and y intercept (0, 3)

21. x-intercepts (2, 0) and (5, 0), and y intercept (0, 6)

22. x-intercepts (1, 0) and (3, 0), and y intercept (0, 4)

23. Vertex at (4, 0), and y intercept (0, -4)

24. Vertex at (5, 6), and y intercept (0, -1)

25. Vertex at (-3, 2), and passing through (3, -2)

26. Vertex at (1, -3), and passing through (-2, 3)

27. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$.

a. From what height was the rocket launched?
b. How high above sea level does the rocket reach its peak?
c. Assuming the rocket will splash down in the ocean, at what time does splashdown occur?

28. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$.

a. From what height was the ball thrown?
b. How high above ground does the ball reach its peak?
c. When does the ball hit the ground?

29. The height of a ball thrown in the air is given by $h(x) = -\frac{1}{12}x^2 + 6x + 3$, where x is the horizontal distance in feet from the point at which the ball is thrown.

a. How high is the ball when it was thrown?
b. What is the maximum height of the ball?
c. How far from the thrower does the ball strike the ground?

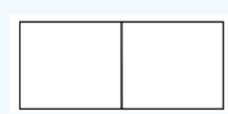
30. A javelin is thrown in the air. Its height is given by $h(x) = -\frac{1}{20}x^2 + 8x + 6$, where x is the horizontal distance in feet from the point at which the javelin is thrown.

a. How high is the javelin when it was thrown?
b. What is the maximum height of the javelin?
c. How far from the thrower does the javelin strike the ground?

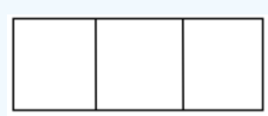
31. A box with a square base and no top is to be made from a square piece of cardboard by cutting 6 in. squares out of each corner and folding up the sides. The box needs to hold 1000 in^3. How big a piece of cardboard is needed?

32. A box with a square base and no top is to be made from a square piece of cardboard by cutting 4 in. squares out of each corner and folding up the sides. The box needs to hold 2700 in^3. How big a piece of cardboard is needed?

33. A farmer wishes to enclose two pens with fencing, as shown. If the farmer has 500 feet of fencing to work with, what dimensions will maximize the area enclosed?



34. A farmer wishes to enclose three pens with fencing, as shown. If the farmer has 700 feet of fencing to work with, what dimensions will maximize the area enclosed?



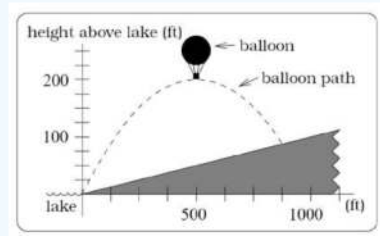
35. You have a wire that is 56 cm long. You wish to cut it into two pieces. One piece will be bent into the shape of a square. The other piece will be bent into the shape of a circle. Let A represent the total area enclosed by the square and the circle. What is the circumference of the circle when A is a minimum?

36. You have a wire that is 71 cm long. You wish to cut it into two pieces. One piece will be bent into the shape of a right triangle with legs of equal length. The other piece will be bent into the shape of a circle. Let A represent the total area enclosed by the triangle and the circle. What is the circumference of the circle when A is a minimum?

37. A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

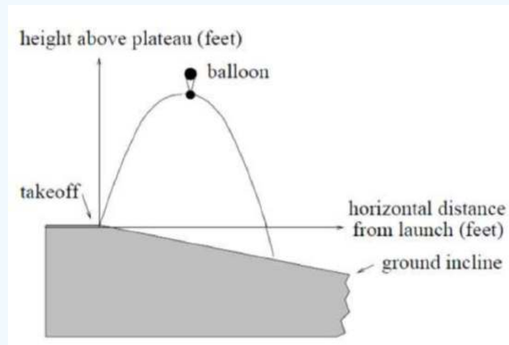
38. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

39. A hot air balloon takes off from the edge of a mountain lake. Impose a coordinate system as pictured and assume that the path of the balloon follows the graph of $f(x)=-\frac{2}{2500}x^2+\frac{4}{5}x$. The land rises at a constant incline from the lake at the rate of 2 vertical feet for each 20 horizontal feet. [UW]



a. What is the maximum height of the balloon above water level?
b. What is the maximum height of the balloon above ground level?
c. Where does the balloon land on the ground?
d. Where is the balloon 50 feet above the ground?

40. A hot air balloon takes off from the edge of a plateau. Impose a coordinate system as pictured below and assume that the path the balloon follows is the graph of the quadratic function $f(x)=-\frac{4}{2500}x^2+\frac{4}{5}x$. The land drops at a constant incline from the plateau at the rate of 1 vertical foot for each 5 horizontal feet. [UW]



a. What is the maximum height of the balloon above plateau level?
b. What is the maximum height of the balloon above ground level?
c. Where does the balloon land on the ground?
d. Where is the balloon 50 feet above the ground?

Answer

1. $f(x)=(x-2)^2-3$

3. $f(x)=-2(x-2)^2+7$

5. $f(x)=\frac{1}{2}(x-3)^2-1$

	Vertex	Vertical Intercept	Horizontal Intercepts
7.	(-2.5, -0.5)	(0, 12)	(-2, 0). (-3, 0)
9.	(2.5, -8.5)	(0, 4)	(0.438, 0). (4.562, 0)
11.	(0.75, 1.25)	(0, -1)	(0.191, 0). (1.309, 0)

13. $f(x)=(x-6)^2-4$

15. $f(x_{=}2(x+2)^2-18$

17. $b=32$ and $c=-39$

19. $f(x)--\frac{2}{3}(x+3)(x-1)$

21. $f(x)=\frac{3}{5}(x-2)(x-5)$

23. $f(x)=-\frac{1}{4}(x-4)^2$

25. $f(x)=-\frac{1}{9}(x+3)^2+2$

27a. 234m
b. 2909.561 ft
c. 47.735 seconds

29a. 3 ft
b. 111 ft
c. 72.497 ft

31. 24.91 in by 24.91 in

33. 125 ft by $83\frac{1}{3}ft$

35. 24.6344 cm

37. $10.70

3.10: Power Functions and Polynomial Functions

Learning Objectives

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table $3.10.1$.

Table 3.10.1

Year	2009	2010	2011	2012	2013
Bird Population	800	897	992	1,083	1,169

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$, where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.

Figure $3.10.1$: (credit: Jason Bay, Flickr)

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius r is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Definition: Power Function

A *power function* is a function that can be represented in the form

$$f(x) = kx^p \tag{3.10.1}$$

where k and p are real numbers, and k is known as the *coefficient*.

Q&A: Is $f(x) = 2^x$ a power function?

No. A power function contains a variable base raised to a fixed power (Equation 3.10.1). This function has a constant base raised to a variable power. This is called an exponential function, not a power function. This function will be discussed later.

Example 3.10.1: Identifying Power Functions

Which of the following functions are power functions?

$$\begin{array}{lr} f(x)=1 & \text{Constant function} \\ f(x)=x & \text{Identify function} \\ f(x)=x^2 & \text{Quadratic function} \\ f(x)=x^3 & \text{Cubic function} \\ f(x)=\dfrac{1}{x} & \text{Reciprocal function} \\ f(x)=\dfrac{1}{x^2} & \text{Reciprocal squared function} \\ f(x)=\sqrt{x} & \text{Square root function} \\ f(x)=\sqrt[3]{x} & \text{Cube root function} \end{array}$$

Solution

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x)=x^0$ and $f(x)=x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x)=x^2$ and $f(x)=x^3$.

The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x)=x^{-1}$ and $f(x)=x^{-2}$.

The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x)=x^{1/2}$ or $f(x)=x^{1/3}$.

Exercise 3.10.1

Which functions are power functions?

- $f(x)=2x^2 \cdot 4x^3$
- $g(x)=-x^5+5x^3-4x$
- $h(x)=\frac{2x^5-1}{3x^2+4}$

Answer

$f(x)$ is a power function because it can be written as $f(x)=8x^5$. The other functions are not power functions.

Identifying End Behavior of Power Functions

Figure 3.10.2 shows the graphs of $f(x)=x^2$, $g(x)=x^4$ and and $h(x)=x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

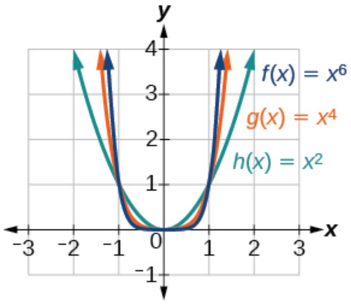


Figure $3.10.2$ Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty, \; f(x) \to \infty$$

Figure $3.10.3$ shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

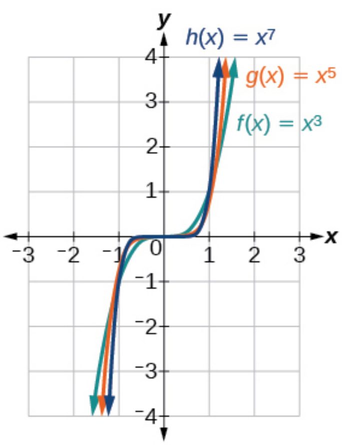


Figure $3.10.3$ Odd-power function

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure $3.10.2$ we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y-axis. In Figure $3.10.3$ we see that odd functions of the form $f(x) = x^n$, n odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\begin{array}{c} \text{as } x \to -\infty, \; f(x) \to -\infty \\ \text{as } x \to \infty, \; f(x) \to \infty \end{array}$$

The behavior of the graph of a function as the input values get very small $(x \to -\infty)$ and get very large $x \to \infty$ is referred to as the *end behavior* of the function. We can use words or symbols to describe end behavior.

Figure $3.10.4$ shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

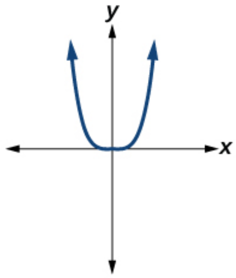
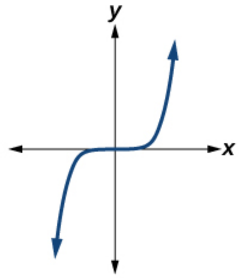
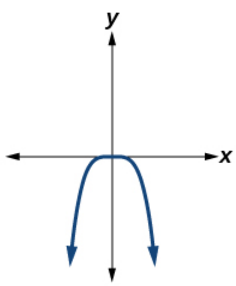
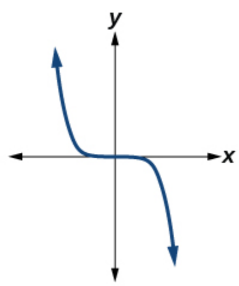
	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty,\ f(x) \to \infty$ and $x \to \infty,\ f(x) \to \infty$	 $x \to -\infty,\ f(x) \to -\infty$ and $x \to \infty,\ f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty,\ f(x) \to -\infty$ and $x \to \infty,\ f(x) \to -\infty$	 $x \to -\infty,\ f(x) \to \infty$ and $x \to \infty,\ f(x) \to -\infty$

Figure $3.10.4$

How To: Given a power function $f(x) = kx^n$ where n is a non-negative integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure $3.10.4$ to identify the end behavior.

Example $3.10.2$: Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x \to \infty$, $f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty$, $f(x) \to \infty$. We can graphically represent the function as shown in Figure $3.10.5$.

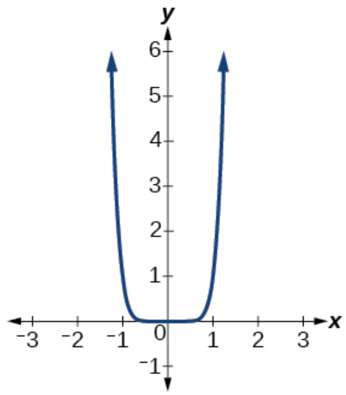


Figure 3.10.5 Graph of $f(x) = x^8$.

Example 3.10.3: Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. Figure 3.10.6 shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty$$
$$\text{as } x \to \infty,\ f(x) \to -\infty$$

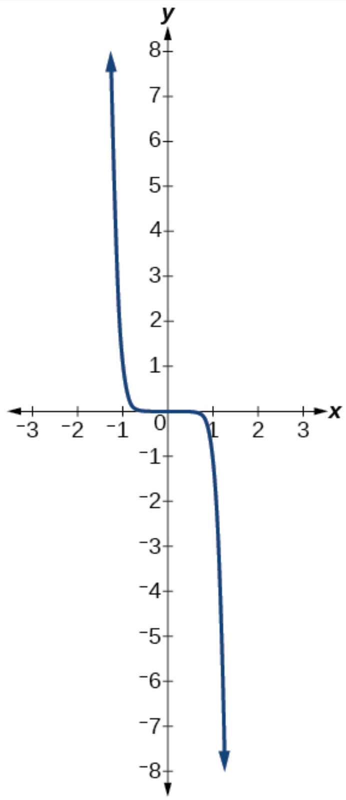


Figure 3.10.6 Graph of $f(x) = -x^9$.

Analysis

We can check our work by using the table feature on a graphing utility.

Table 3.10.2

x	$f(x)$

x	$f(x)$
-10	1,000,000,000
-5	1,953,125
0	0
5	-1,953,125
10	-1,000,000,000

We can see from Table $3.10.2$ that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).

Exercise 3.10.2

Describe in words and symbols the end behavior of $f(x)=-5x^4$.

Answer

As x approaches positive or negative infinity, $f(x)$ decreases without bound: as $x \rightarrow \pm\infty$, $f(x) \rightarrow -\infty$ because of the negative coefficient.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius r of the spill depends on the number of weeks w that have passed. This relationship is linear.

$$r(w)=24+8w$$

We can combine this with the formula for the area A of a circle.

$$A(r)=\pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24+8w) \\ &= \pi(24+8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w)=576\pi+384\pi w+64\pi w^2$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Definition: Polynomial Functions

Let n be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x)=a_nx^n+\ldots+a_2x^2+a_1x+a_0 \tag{3.10.2}$$

This is called the general form of a polynomial function. Each a_i is a coefficient and can be any real number. Each product a_ix^i is a *term of a polynomial function.*

Example 3.10.4: Identifying Polynomial Functions

Which of the following are polynomial functions?

- $f(x)=2x^3 \cdot 3x+4$
- $g(x)=-x(x^2-4)$
- $h(x)=5\sqrt{x}+2$

Solution

The first two functions are examples of polynomial functions because they can be written in the form of Equation 3.10.2, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x)=6x^4+4$.
- $g(x)$ can be written as $g(x)=-x^3+4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.

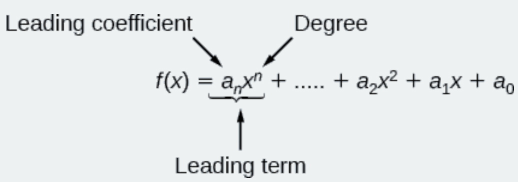


Figure 3.10.7

When a polynomial is written in this way, we say that it is in *general form*.

How To: Given a polynomial function, identify the degree and leading coefficient

1. Find the highest power of x to determine the degree function.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example 3.10.5: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x)=3+2x^2-4x^3$$

$$g(t)=5t^5-2t^3+7t$$

$$h(p)=6p-p^3-2$$

Solution

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, −4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, −1.

Exercise 3.10.3

Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

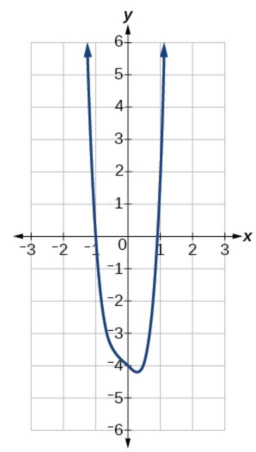
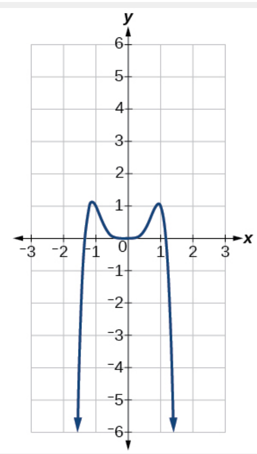
Answer

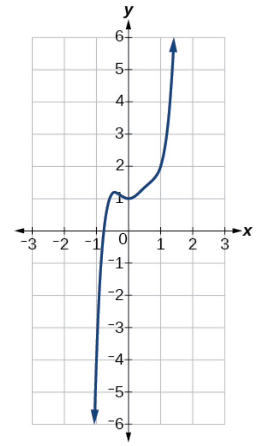
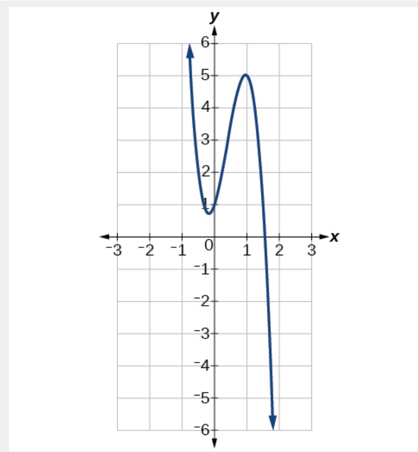
The degree is 6. The leading term is $-x^6$. The leading coefficient is -1.

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree (Table $3.10.3$).

Table 3.10.3

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x)=3x^5-4x^4+2x^2+1$	$3x^5$	
$f(x)=-6x^3+7x^2+3x+1$	$-6x^3$	

Example 3.10.6: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 3.10.8.

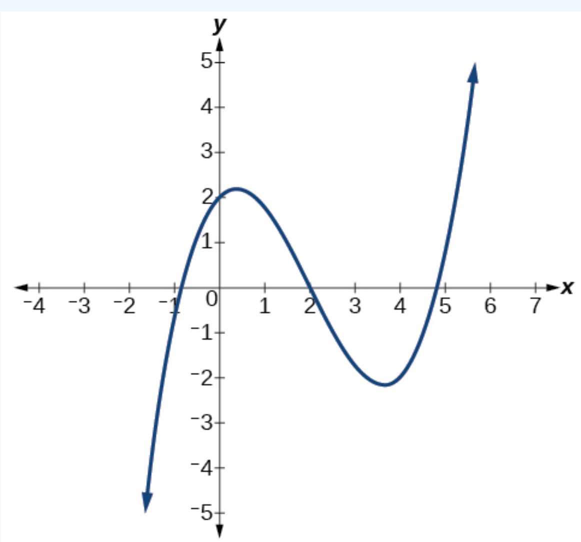


Figure 3.10.8

Solution

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \rightarrow \infty, \ f(x) \rightarrow \infty$$

$$\text{as } x \rightarrow -\infty, \ f(x) \rightarrow -\infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

Exercise 3.10.1

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 3.10.9.

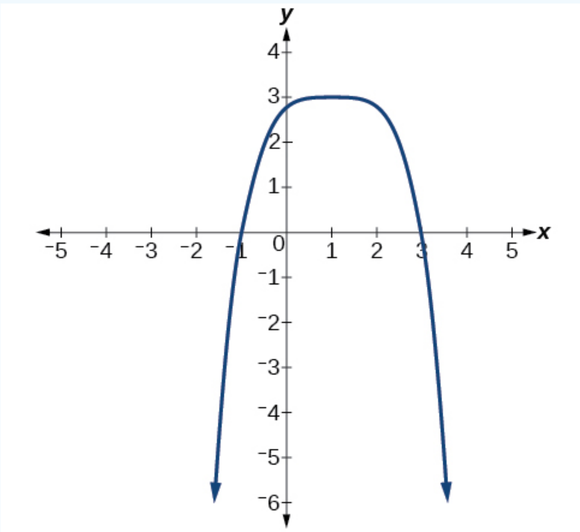


Figure 3.10.9

Answer

As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$; as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$. It has the shape of an even degree power function with a negative coefficient.

Example 3.10.7: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x \rightarrow -\infty, \ f(x) \rightarrow -\infty$$

$$\text{as } x \rightarrow \infty, \ f(x) \rightarrow -\infty$$

Exercise 3.10.7

Given the function $f(x)=0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Answer

The leading term is $0.2x^3$, so it is a degree 3 polynomial. As x approaches positive infinity, $f(x)$ increases without bound; as x approaches negative infinity, $f(x)$ decreases without bound.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the “middle” of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the y-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See Figure $3.10.10$

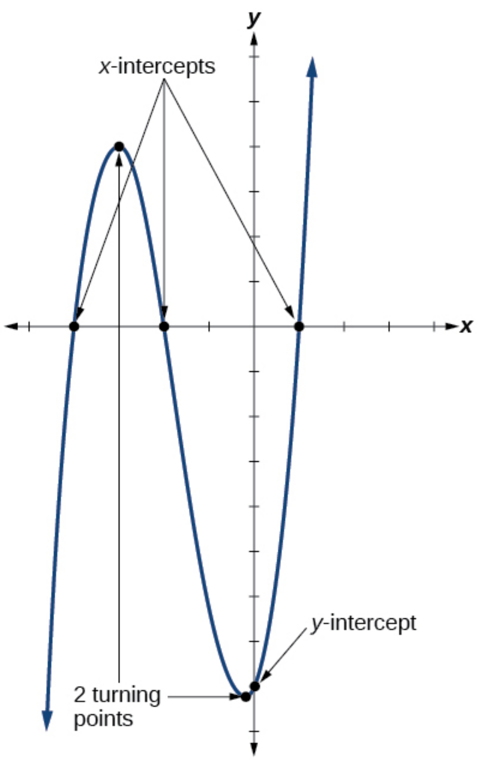


Figure $3.10.10$

Defintion: Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The y-intercept is the point at which the function has an input value of zero. The x-intercepts are the points at which the output value is zero.

How To: Given a polynomial function, determine the intercepts.

1. Determine the y-intercept by setting $x=0$ and finding the corresponding output value.
2. Determine the x-intercepts by solving for the input values that yield an output value of zero.

Example 3.10.8: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero, so substitute 0 for x.

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The y-intercept is $(0, 8)$.

The x-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{array}{rcccc} x-2=0 & \text{or} & x+1=0 & \text{or} & x-4=0 \\ x=2 & \text{or} & x=-1 & \text{or} & x=4 \end{array}$$

The x-intercepts are $(2, 0)$,$(-1, 0)$, and $(4, 0)$.

We can see these intercepts on the graph of the function shown in Figure $3.10.11$.

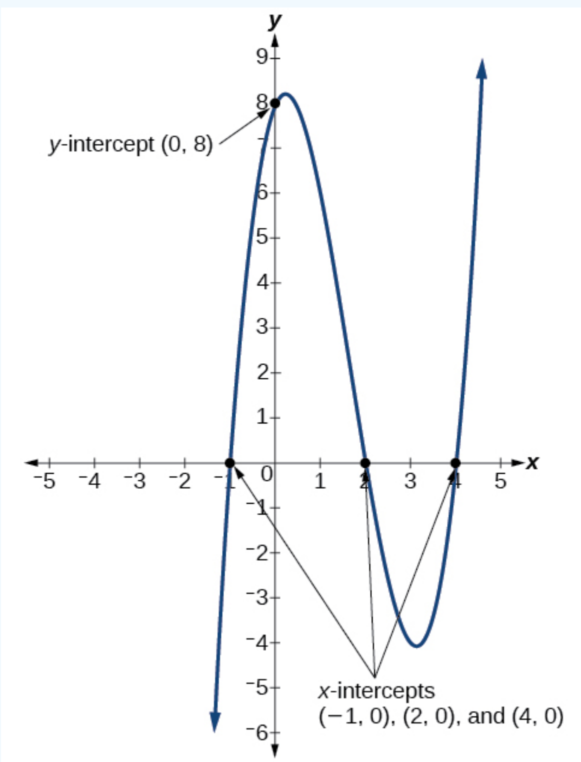


Figure $3.10.11$: Graph of $f(x) = (x-2)(x+1)(x-4)$.

Example 3.10.9: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^2 - 45 \\ &= -45 \end{aligned}$$

The y-intercept is $(0, -45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4 - 4x^2 - 45 \\ &= (x^2 - 9)(x^2 + 5) \\ &= (x - 3)(x + 3)(x^2 + 5) \end{aligned}$$

$$0 = (x - 3)(x + 3)(x^2 + 5)$$

$$\begin{array}{ccccc} x - 3 = 0 & \text{or} & x + 3 = 0 & \text{or} & x^2 + 5 = 0 \\ x = 3 & \text{or} & x = -3 & \text{or} & (\text{no real solution}) \end{array}$$

The x-intercepts are $(3, 0)$ and $(-3, 0)$.

We can see these intercepts on the graph of the function shown in Figure $3.10.12$ We can see that the function is even because $f(x) = f(-x)$.

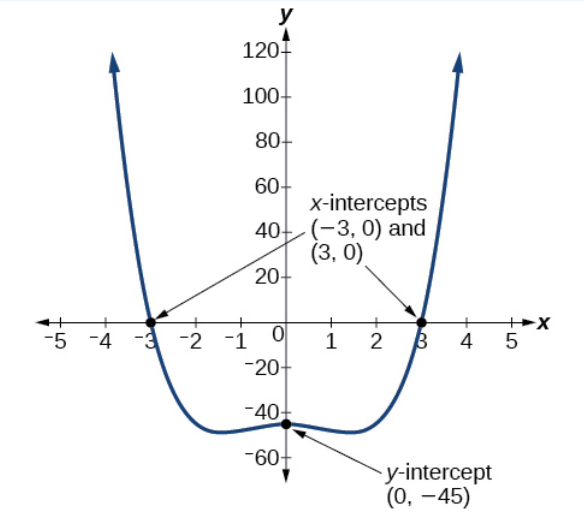


Figure $3.10.12$ Graph of $f(x) = x^4 - 4x^2 - 45$.

Exercise $3.10.5$

$\mathbf{3.10.5:}$ Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the y- and x-intercepts.

Solution

y-intercept $(0, 0)$; x-intercepts $(0, 0)$,$(-2, 0)$, and $(5, 0)$

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. A polynomial function of n^{th} degree is the product of n factors, so it will have at most n roots or zeros, or x-intercepts. The graph of the polynomial function of degree n must have at most $n-1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A smooth curve is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Intercepts and Turning Points of Polynomials

A polynomial of degree n will have, at most, n x-intercepts and $n - 1$ turning points.

Example 3.10.10: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

Solution

The polynomial has a degree of **10**, so there are at most n x-intercepts and at most $n-1$ turning points.

Exercise 3.10.6

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$

Answer

There are at most 12 x-intercepts and at most 11 turning points.

Example 3.10.11: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure $3.10.12$ based on its intercepts and turning points?

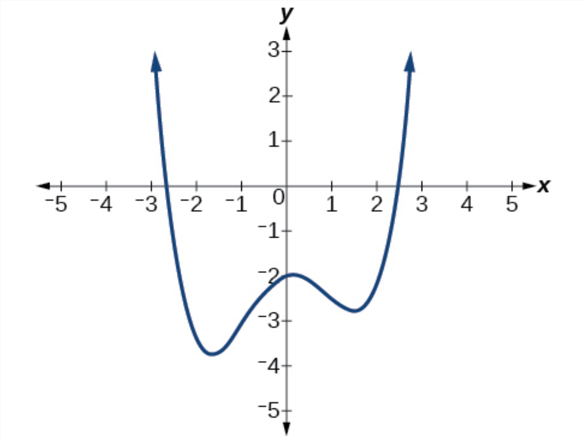


Figure $3.10.13$

Solution

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure $3.10.14$

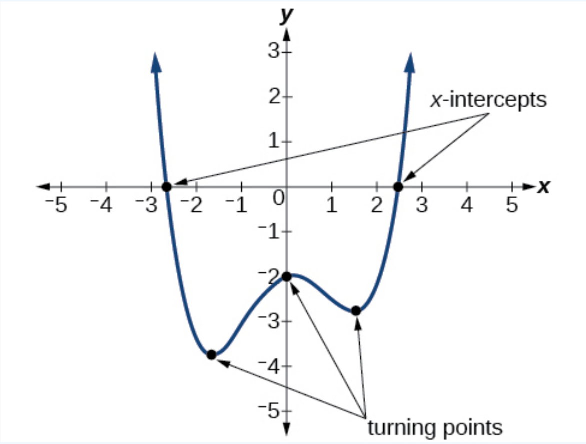


Figure $3.10.14$ Graph of an even-degree polynomial.

The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

Exercise 3.10.7

What can we conclude about the polynomial represented by the graph shown in Figure $3.10.15$ based on its intercepts and turning points?

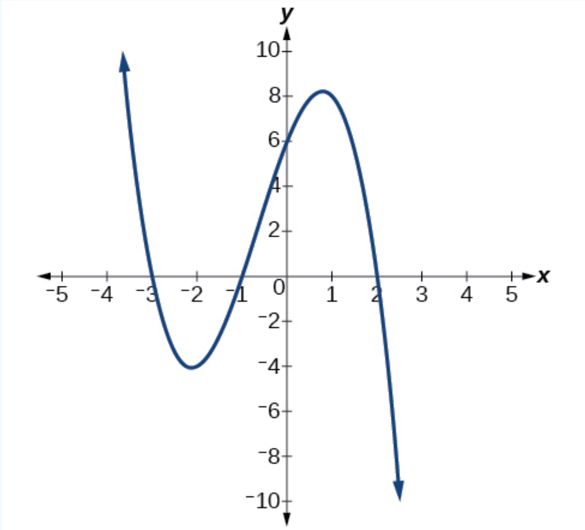


Figure $\mathbf{3.10.15}$.

Answer

Add texts here. Do not delete this text first.

Solution

The end behavior indicates an odd-degree polynomial function; there are 3 x-intercepts and 2 turning points, so the degree is odd and at least 3. Because of the end behavior, we know that the lead coefficient must be negative.

Example 3.10.12: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

Solution

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The y-intercept is $(0, 0)$.

The x-intercepts are found by determining the zeros of the function.

$$\begin{array}{lllll} 0 = -4x(x+3)(x-4) & & & & \\ x = 0 & \text{or} & x+3 = 0 & \text{or} & x-4 = 0 \\ x = 0 & \text{or} & x = -3 & \text{or} & x = 4 \end{array}$$

The x-intercepts are $(0, 0)$,$(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

Exercise 3.10.8

Given the function $f(x)=0.2(x-2)(x+1)(x-5)$, determine the local behavior.

Answer

The x-intercepts are $(2,0)$, $(-1,0)$, and $(5,0)$, the y-intercept is $(0,2)$, and the graph has at most 2 turning points.

Key Equations

- general form of a polynomial function: $f(x)=a_nx^n+a_{n-1}x^{n-1}...+a_2x^2+a_1x+a_0$

Key Concepts

- A power function is a variable base raised to a number power.
- The behavior of a graph as the input decreases without bound and increases without bound is called the end behavior.
- The end behavior depends on whether the power is even or odd.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function.
- A polynomial of degree n will have at most n x-intercepts and at most $n-1$ turning points.

Glossary

coefficient

a nonzero real number that is multiplied by a variable raised to an exponent (only the number factor is the coefficient)

continuous function

a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

degree

the highest power of the variable that occurs in a polynomial

end behavior

the behavior of the graph of a function as the input decreases without bound and increases without bound

leading coefficient

the coefficient of the leading term

leading term

the term containing the highest power of the variable

polynomial function

a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

power function

a function that can be represented in the form $f(x)=kx^p$ where k is a constant, the base is a variable, and the exponent, p, is a constant

smooth curve

a graph with no sharp corners

term of a polynomial function

any a_ix^i of a polynomial function in the form $f(x) = a_nx^n + a_{n-1}x^{n-1} \ldots + a_2x^2 + a_1x + a_0$

turning point

the location at which the graph of a function changes direction

---

3.11: Graphs of Polynomial Functions

Learning Objectives

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in Table $3.11.1$

Table 3.11.1

Year	2006	2007	2008	2009	2010	2011	2012	2013
Revenues	52.4	52.8	51.2	49.5	48.6	48.6	48.7	47.1

The revenue can be modeled by the polynomial function

$$R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332 \tag{3.11.1}$$

where R represents the revenue in millions of dollars and t represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing? These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure $3.11.1$ shows a graph that represents a **polynomial function** and a graph that represents a function that is not a polynomial.

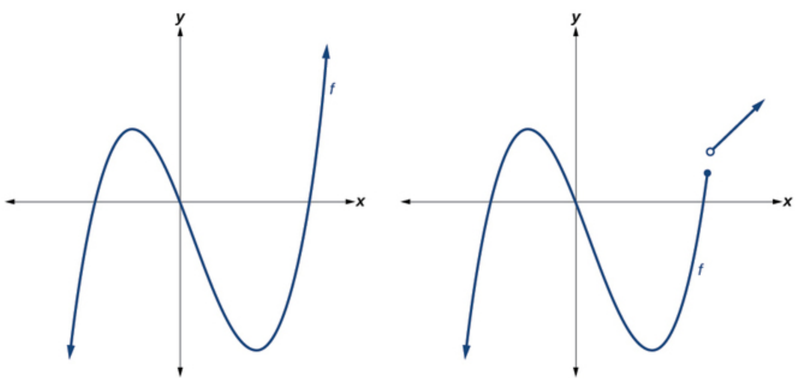


Figure $3.11.1$: Graph of $f(x) = x^3 - 0.01x$.

Example 3.11.1: Recognizing Polynomial Functions

Which of the graphs in Figure $3.11.2$ represents a polynomial function?

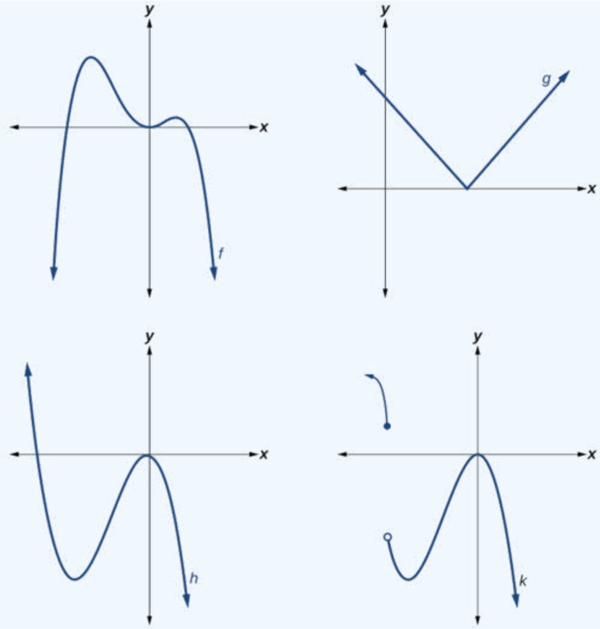


Figure $3.11.2$

Solution

- The graphs of f and h are graphs of polynomial functions. They are smooth and **continuous**.
- The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Q&A

Do all polynomial functions have as their domain all real numbers?

- Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x)=0$ are called **zeros** of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
The polynomial is given in factored form.
Technology is used to determine the intercepts.

HOwTO: Given a polynomial function f, find the x-intercepts by factoring

1. Set $f(x)=0$.
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

Example 3.11.2: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x)=x^6-3x^4+2x^2$.

Solution

We can attempt to factor this polynomial to find solutions for $f(x)=0$.

$$\begin{aligned} x^6-3x^4+2x^2 &= 0 && \text{Factor out the greatest common factor.} \\ x^2(x^4-3x^2+2) &= 0 && \text{Factor the trinomial.} \\ x^2(x^2-1)(x^2-2) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{array}{ccccc} x^2=0 & & (x^2-1)=0 & & (x^2-2)=0 \\ x^2=0 & \text{or} & x^2=1 & \text{or} & x^2=2 \\ x=0 & & x=\pm 1 & & x=\pm\sqrt{2} \end{array}$$

.

This gives us five x-intercepts: $(0,0)$, $(1,0)$, $(-1,0)$, $(\sqrt{2},0)$,and $(-\sqrt{2},0)$ (Figure $3.11.3$). We can see that this is an even function.

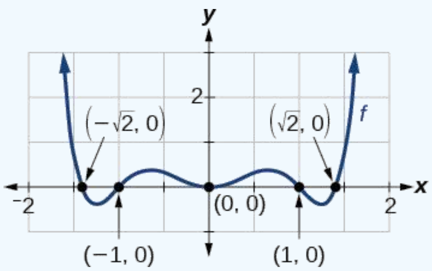


Figure $3.11.3$

Example 3.11.3: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x)=x^3-5x^2-x+5$.

Solution

Find solutions for $f(x)=0$ by factoring.

$$\begin{aligned} x^3-5x^2-x+5 &= 0 && \text{Factor by grouping.} \\ x^2(x-5)-(x-5) &= 0 && \text{Factor out the common factor.} \\ (x^2-1)(x-5) &= 0 && \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{array}{ccccc} x+1=0 & \text{or} & x-1=0 & \text{or} & x-5=0 \\ x=-1 & & x=1 & & x=5 \end{array}$$

There are three x-intercepts: $(-1,0)$, $(1,0)$, and $(5,0)$ (Figure $3.11.4$).

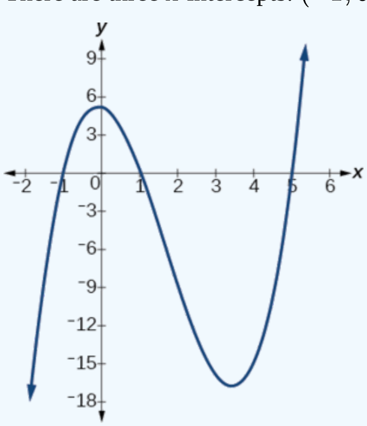


Figure $3.11.4$ Graph of $f(x)$.

Example 3.11.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x-2)^2(2x+3)$.

Solution

The y-intercept can be found by evaluating $g(0)$.

$$\begin{aligned} g(0) &= (0-2)^2(2(0)+3) \\ &= 12 \end{aligned}$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0 \tag{3.11.2}$$

$$\begin{aligned} (x-2)^2 &= 0 & \quad & (2x+3) &= 0 \\ x-2 &= 0 & \text{or} \quad & x &= -\frac{3}{2} \\ x &= 2 & & & \end{aligned}$$

So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

Analysis

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in Figure $3.11.5$.

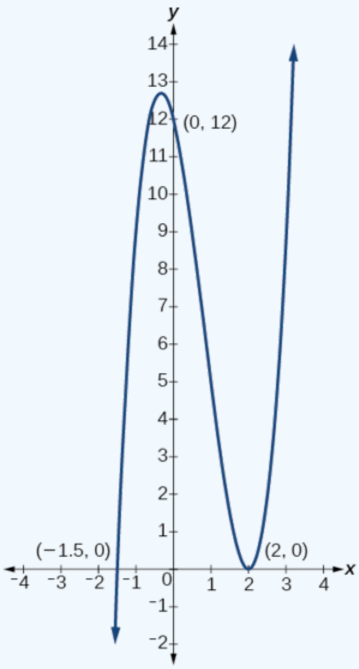


Figure $3.11.5$: Graph of $g(x)$.

Example 3.11.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

Solution

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure $3.11.6$, it appears that there are x-intercepts at $x = -3, -2,$ and 1.

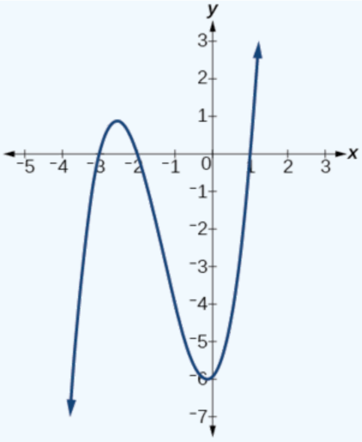


Figure $3.11.6$ Graph of $h(x)$.

We can check whether these are correct by substituting these values for x and verifying that

$$h(-3)=h(-2)=h(1)=0.$$

Since $h(x)=x^3+4x^2+x-6$, we have:

$$\begin{aligned}h(-3)&=(-3)^3+4(-3)^2+(-3)-6=-27+36-3-6=0\\h(-2)&=(-2)^3+4(-2)^2+(-2)-6=-8+16-2-6=0\\h(1)&=(1)^3+4(1)^2+(1)-6=1+4+1-6=0\end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$\begin{aligned}h(x)&=x^3+4x^2+x-6\\&=(x+3)(x+2)(x-1)\end{aligned}$$

Exercise $3.11.1$

Find the y-and x-intercepts of the function $f(x)=x^4-19x^2+30x$.

Answer

- y-intercept $(0,0)$;
- x-intercepts $(0,0)$, $(-5,0)$, $(2,0)$, and $(3,0)$

Identifying Zeros and Their Multiplicities

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off. Suppose, for example, we graph the function

$$f(x)=(x+3)(x-2)^2(x+1)^3. \tag{3.11.3}$$

Notice in Figure $3.11.7$ that the behavior of the function at each of the x-intercepts is different.

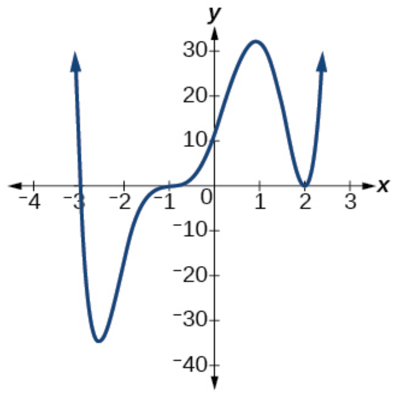


Figure $3.11.7$: Identifying the behavior of the graph at an x-intercept by examining the multiplicity of the zero.

The x-intercept −3 is the solution of equation $(x+3)=0$. The graph passes directly through thex-intercept at $x=-3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The x-intercept 2 is the repeated solution of equation $(x-2)^2=0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.

$$(x-2)^2=(x-2)(x-2) \tag{3.11.4}$$

The factor is repeated, that is, the factor $(x-2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x=2$, has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept −1 is the repeated solution of factor $(x+1)^3=0$.The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near the intercept as the toolkit function $f(x)=x^3$. We call this a triple zero, or a zero with multiplicity 3.

For **zeros** with even multiplicities, the graphs touch or are tangent to the x-axis. For zeros with odd multiplicities, the graphs cross or intersect the x-axis. See Figure $3.11.8$ for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

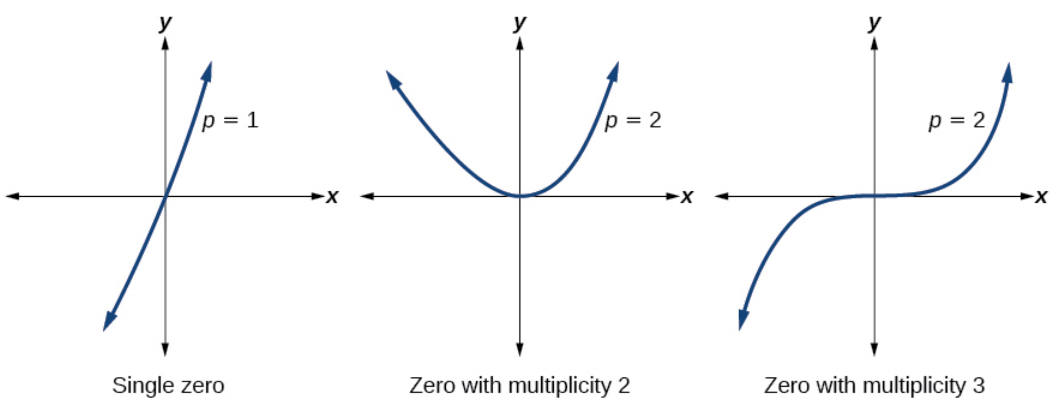


Figure $3.11.8$ Three graphs showing three different polynomial functions with multiplicity 1, 2, and 3.

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

Graphical Behavior of Polynomials at x-Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercepth is determined by the power p. We say that $x = h$ is a zero of **multiplicity** p.

The graph of a polynomial function will touch the x-axis at zeros with even multiplicities. The graph will cross the x-axis at zeros with odd multiplicities.

The sum of the multiplicities is the degree of the polynomial function.

HOWTO: Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities

1. If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is n.

Example 3.11.6: Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure $3.11.9$to identify the zeros of the function and their possible multiplicities.

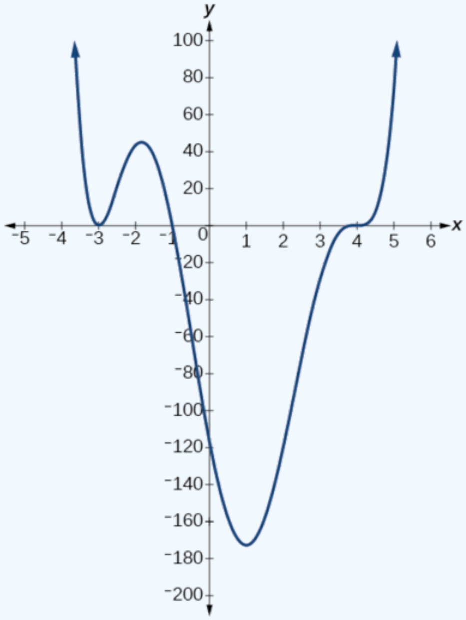


Figure $3.11.9$: Graph of a polynomial function with degree 5.

Solution

The polynomial function is of degree n. The sum of the multiplicities must be n.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of −3 has multiplicity 2.

The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 4$.The graph crosses the x-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3 and that the sum of the multiplicities is likely 6.

Exercise 3.11.2

Use the graph of the function of degree 5 in Figure $3.11.10$to identify the zeros of the function and their multiplicities.

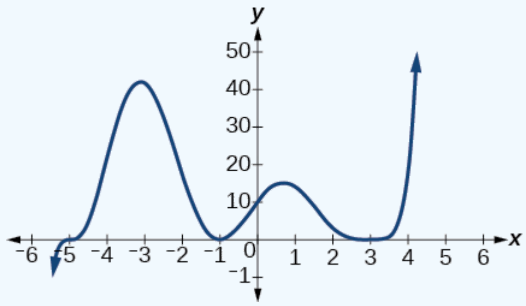


***Figure 3.11.10**: Graph of a polynomial function with degree 5.*

Answer

The graph has a zero of –5 with multiplicity 1, a zero of –1 with multiplicity 2, and a zero of 3 with even multiplicity.

Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{3.11.5}$$

will either ultimately rise or fall as x increases without bound and will either rise or fall as x decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure 3.11.11summarizes all four cases.

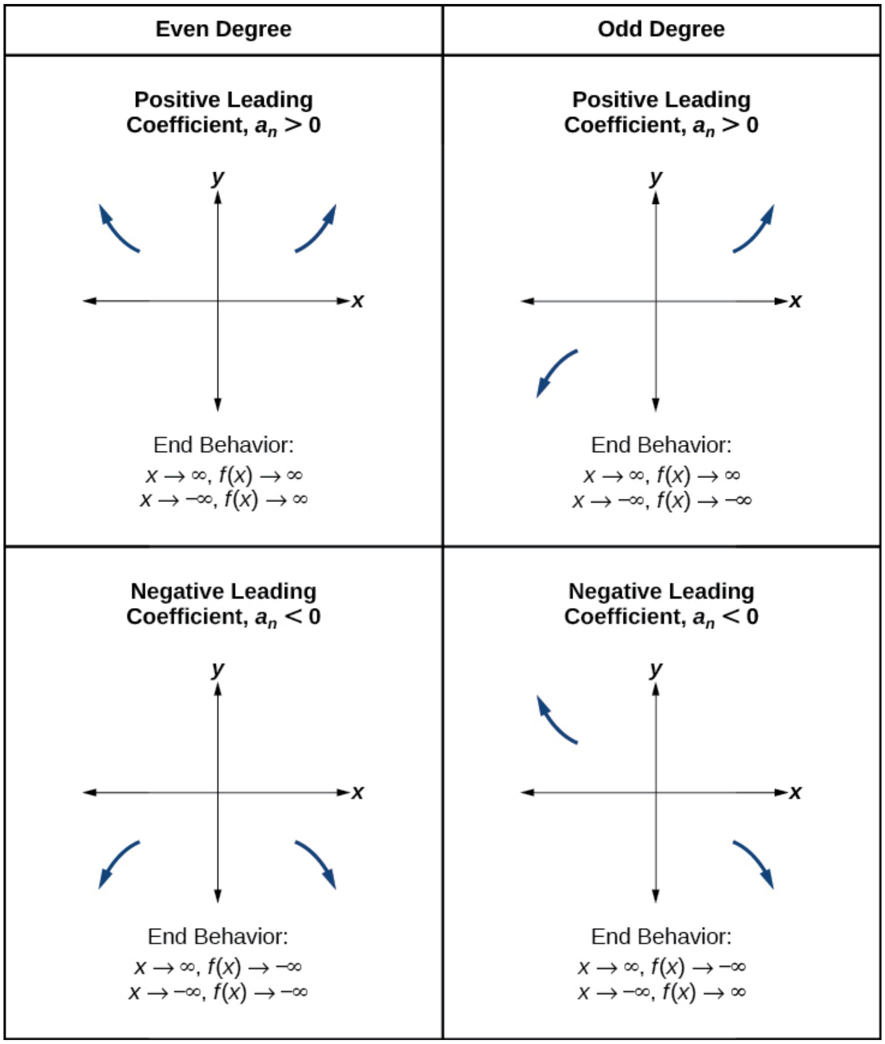


Figure $3.11.11$

Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure $3.11.12$ The graph has three turning points.

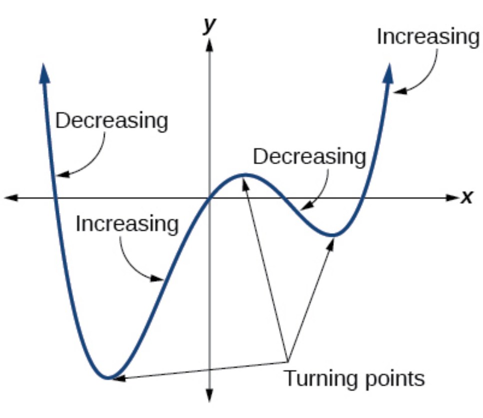


Figure $3.11.12$ Graph of $f(x) = x^4 - x^3 - 4x^2 + 4x$

This function f is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

> **Definition: Interpreting Turning Points**
>
> A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). A polynomial of degree n will have at most $n-1$ turning points.

> **Example 3.11.7: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function**
>
> Find the maximum number of turning points of each polynomial function.
>
> a. $f(x)=-x^3+4x^5-3x^2+1$
> b. $f(x)=-(x-1)^2(1+2x^2)$
>
> **Solution**
>
> a. $f(x)=-x^3+4x^5-3x^2+1$
>
> First, rewrite the polynomial function in descending order: $f(x)=4x^5-x^3-3x^2+1$
>
> Identify the degree of the polynomial function. This polynomial function is of degree 5.
>
> The maximum number of turning points is $5-1=4$.
>
> b. $f(x)=-(x-1)^2(1+2x^2)$
>
> First, identify the leading term of the polynomial function if the function were expanded.
>
> $$f(x) = -(x-1)^2\ (1+2x^2)$$
> $$a_n = -(x^2)\ (2x^2)\ {-2x^4}$$
>
> Then, identify the degree of the polynomial function. This polynomial function is of degree 4.
>
> The maximum number of turning points is $4-1=3$.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

> **Howto: Given a polynomial function, sketch the graph**
>
> 1. Find the intercepts.
> 2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x)=f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x)=-f(x)$.
> 3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
> 4. Determine the end behavior by examining the leading term.
> 5. Use the end behavior and the behavior at the intercepts to sketch a graph.
> 6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
> 7. Optionally, use technology to check the graph.

> **Example 3.11.8: Sketching the Graph of a Polynomial Function**
>
> Sketch a graph of $f(x)=-2(x+3)^2(x-5)$.
>
> **Solution**
>
> This graph has two x-intercepts. At $x=-3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x=5$,the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.
>
> The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The y-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure $3.11.13$

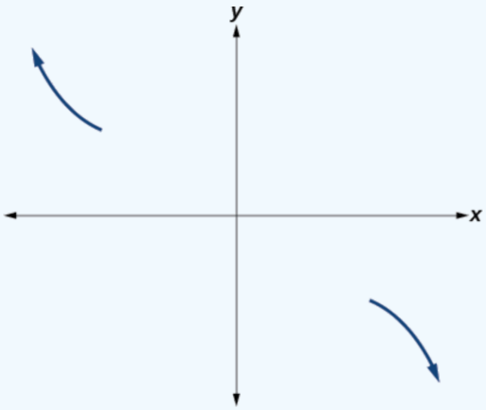


Figure $3.11.13$ Showing the distribution for the leading term.

To sketch this, we consider that:

- As $x \rightarrow -\infty$ the function $f(x) \rightarrow \infty$,so we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of thex-axis, so the function must start increasing.
- At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure $3.11.14$

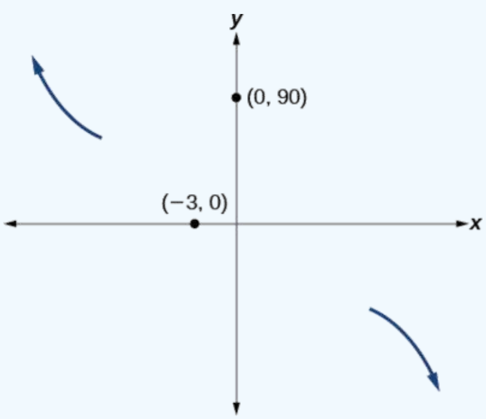


Figure $3.11.14$ Graph of the end behavior and intercepts, $(-3, 0)$ and $(0, 90)$, for the function $f(x) = -2(x+3)^2(x-5)$.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5, 0)$. See Figure $3.11.15$

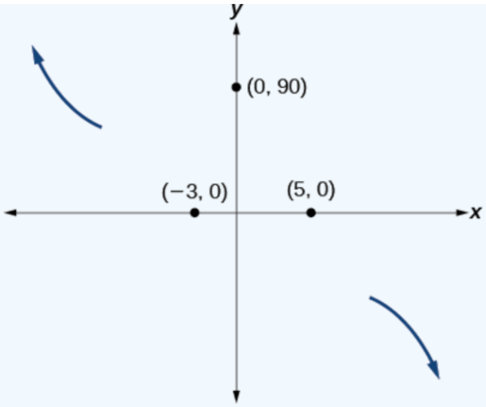


Figure $3.11.15$ Graph of the end behavior and intercepts, $(-3,0)$, $(0,90)$ and $(5,0)$, for the function $f(x)=-2(x+3)^2(x-5)$.

As $x \to \infty$ the function $f(x) \to -\infty$,

so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure $3.11.16$, and verify that the resulting graph looks like our sketch in Figure $3.11.15$.

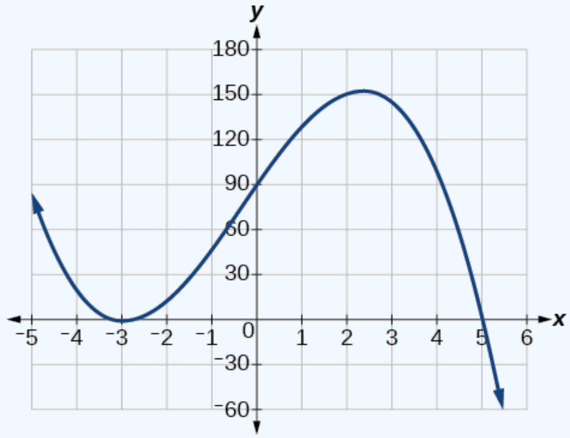


Figure* $3.11.16$*: *The complete graph of the polynomial function* $f(x)=-2(x+3)^2(x-5)$.

Exercise $3.11.8$

Sketch a graph of $f(x)=\dfrac{1}{4}x(x-1)^4(x+3)^3$.

Answer

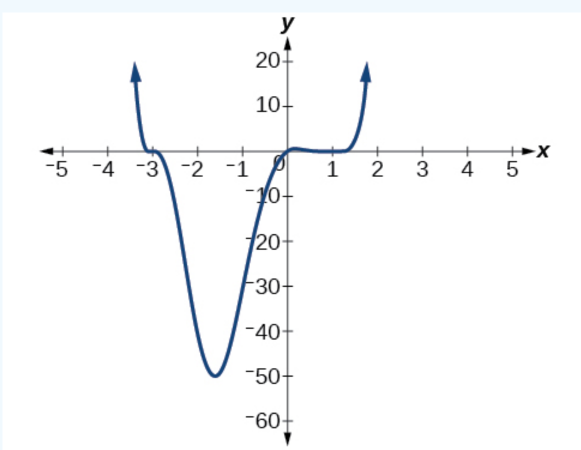


Figure $3.11.17$ Graph of $f(x)=\frac{1}{4}x(x-1)^4(x+3)^3$

Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f,if $a < b$ and $f(a) \neq f(b)$,then the function f takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function f at $x = a$ lies above the x-axis and another point at $x = b$ lies below thex-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. Call this point $(c, f(c))$.This means that we are assured there is a solution c where $f(c) = 0$.

In other words, the **Intermediate Value Theorem** tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. Figure $3.11.18$shows that there is a zero between a and b.

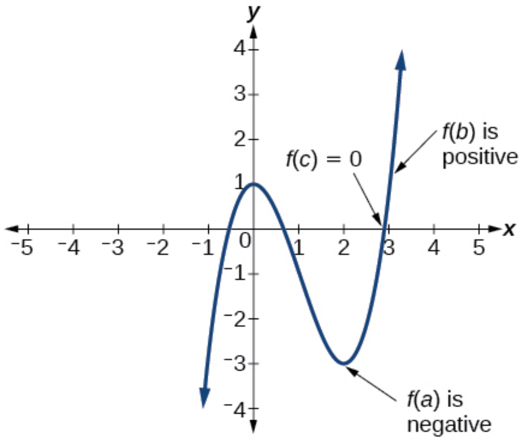


Figure $3.11.18$ Using the Intermediate Value Theorem to show there exists a zero.

Definition: Intermediate Value Theorem

Let f be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

Example 3.11.9: Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution

As a start, evaluate $f(x)$ at the integer values $x = 1,\ 2,\ 3,$ and 4 (Table $3.11.2$).

Table 3.11.2

x	1	2	3	4
$f(x)$	5	0	-3	2

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.

We have shown that there are at least two real zeros between $x = 1$ and $x = 4$.

Analysis

We can also see on the graph of the function in Figure $3.11.19$that there are two real zeros between $x = 1$ and $x = 4$.

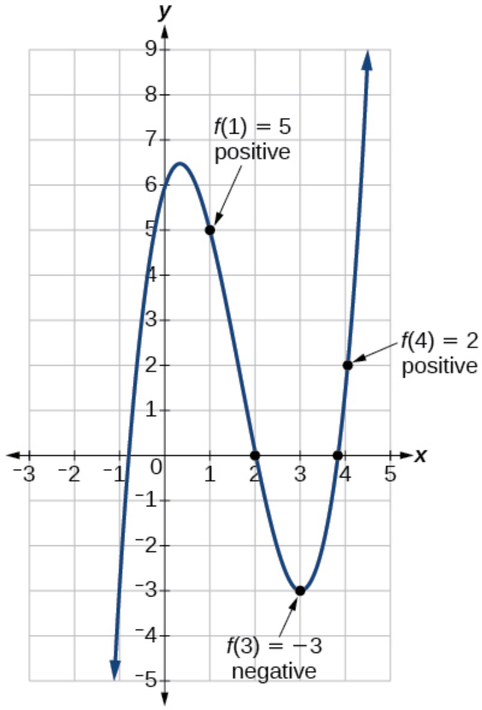


Figure $3.11.19$

Exercise $3.11.4$

Show that the function $f(x)=7x^5-9x^4-x^2$ has at least one real zero between $x=1$ and $x=2$.

Answer

Because f is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x=1$ and $x=2$.

Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a **polynomial function** written in factored form will have an x-intercept where each factor is equal to zero, we can form a function that will pass through a set of x-intercepts by introducing a corresponding set of factors.

Note: Factored Form of Polynomials

If a polynomial of lowest degree p has horizontal intercepts at $x=x_1,x_2,\ldots,x_n$, then the polynomial can be written in the factored form: $f(x)=a(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}$ where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the x-intercept.

How To: Given a graph of a polynomial function, write a formula for the function.

1. Identify the x-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the x-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

Example $3.11.10$: Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure $3.11.20$

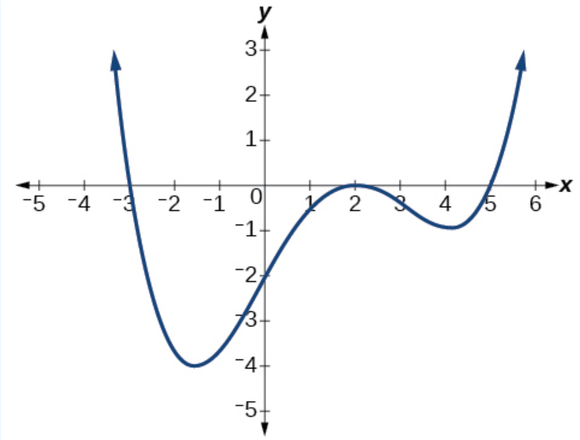


Figure 3.11.20

Solution

This graph has three x-intercepts: $x = -3,\ 2,$ and 5. The y-intercept is located at $(0, 2)$.At $x = -3$ and $x = 5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x) = a(x+3)(x-2)^2(x-5) \tag{3.11.6}$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, -2)$, to solve for a.

$$\begin{aligned} f(0) &= a(0+3)(0-2)^2(0-5) \\ -2 &= a(0+3)(0-2)^2(0-5) \\ -2 &= -60a \\ a &= \frac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$.

Exercise 3.11.5

Given the graph shown in Figure $3.11.21$, write a formula for the function shown.

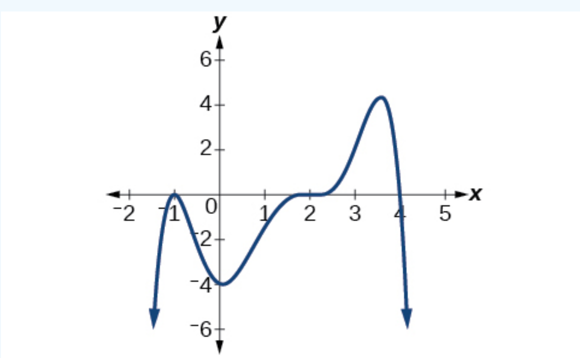


Figure 3.11.21

Answer

$f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

Note: Local and Global Extrema

A **local maximum** or **local minimum** at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$.If a function has a local maximum at a, then $f(a) \geq f(x)$for all x in an open interval around $x = a$. If a function has a local minimum at a, then $f(a) \leq f(x)$for all x in an open interval around $x = a$.

A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at a, then $f(a) \geq f(x)$ for all x. If a function has a global minimum at a, then $f(a) \leq f(x)$ for all x.

We can see the difference between local and global extrema in Figure $3.11.22$

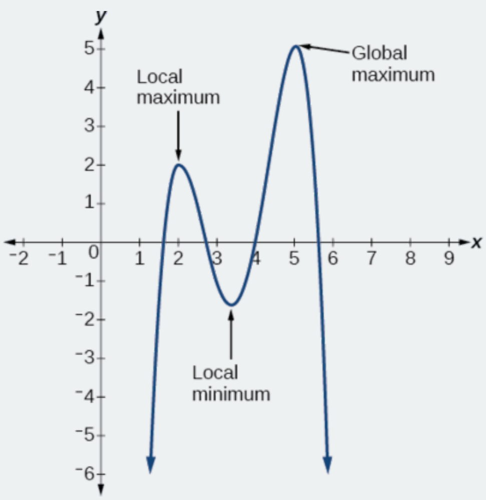


Figure $3.11.22$ Graph of an even-degree polynomial that denotes the local maximum and minimum and the global maximum.

Do all polynomial functions have a global minimum or maximum?

No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.

Example 3.11.11: Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

Solution

We will start this problem by drawing a picture like that in Figure $3.11.23$, labeling the width of the cut-out squares with a variable, w.

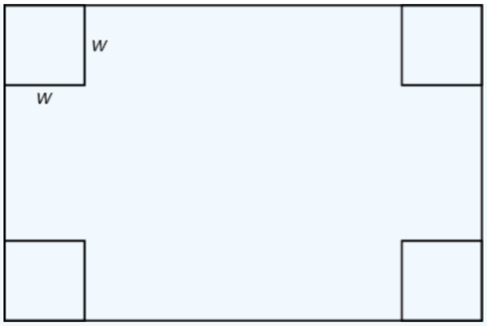


Figure $3.11.23$ Diagram of a rectangle with four squares at the corners.

Notice that after a square is cut out from each end, it leaves $a(14-2w)$ cm by $(20-2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume

$$\begin{aligned} V(w) &= (20-2w)(14-2w)w \\ &= 280w - 68w^2 + 4w^3 \end{aligned}$$

Notice, since the factors are $20-2w$, $14-2w$, and w, the three zeros are $10,\ 7,$ and $0,\ )respectively. Becauseaheightof\backslash(0$ cm is not reasonable, we consider the only the zeros 10 and 7. The shortest side is 14 and we are cutting off two squares, so values w may take on are greater than zero or less than 7. This means we will restrict the domain of this function to $0 < w < 7$.Using technology to sketch the graph of $V(w)$ on this reasonable domain, we get a graph like that in Figure $3.11.24$ We can use this graph to estimate the maximum value for the volume, restricted to values for w that are reasonable for this problem—values from 0 to 7.

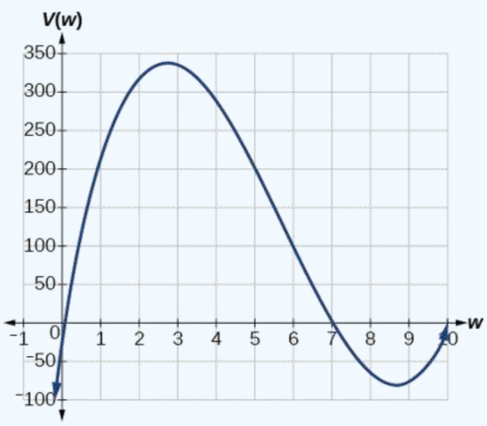


Figure $3.11.24$ Graph of $V(w) = (20-2w)(14-2w)w$

From this graph, we turn our focus to only the portion on the reasonable domain, $[0, 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce Figure $3.11.25$

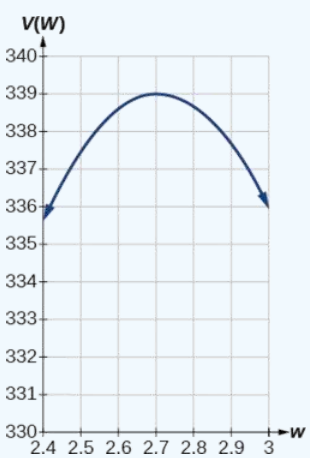


Figure $3.11.25$ Graph of $V(w) = (20-2w)(14-2w)w$.

From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

Exercise 3.11.1

Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

Answer

The minimum occurs at approximately the point $(0, -6.5)$,
and the maximum occurs at approximately the point $(3.5, 7)$.

Key Concepts

- Polynomial functions of degree 2 or more are smooth, continuous functions.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero.
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis.
- The multiplicity of a zero determines how the graph behaves at the x-intercepts.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n-1$ turning points.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points.
- Graphing a polynomial function helps to estimate local and global extremas.
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

Glossary

global maximum
highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all x.

global minimum
lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all x.

Intermediate Value Theorem
for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the functionf takes on every value between $f(a)$ and $f(b)$; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis

multiplicity
the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x-h)^p$, $x = h$ is a zero of multiplicity p.

3.12: Graphing Rational Functions

We've seen that the denominator of a rational function is never allowed to equal zero; division by zero is not defined. So, with rational functions, there are special values of the independent variable that are of particular importance. Now, it comes as no surprise that near values that make the denominator zero, rational functions exhibit special behavior, but here, we will also see that values that make the numerator zero sometimes create additional special behavior in rational functions.

We begin our discussion by focusing on the domain of a rational function.

The Domain of a Rational Function

When presented with a rational function of the form

$$f(x)=\frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}$$

the first thing we must do is identify the domain. Equivalently, we must identify the **restrictions**, values of the independent variable (usually x) that are **not** in the domain. To facilitate the search for restrictions, we should factor the denominator of the rational function (it won't hurt to factor the numerator at this time as well, as we will soon see). Once the domain is established and the restrictions are identified, here are the pertinent facts.

Behavior of a Rational Function at Its Restrictions

A rational function can only exhibit one of two behaviors at a restriction (a value of the independent variable that is not in the domain of the rational function).

1. The graph of the rational function will have a vertical asymptote at the restricted value.
2. The graph will exhibit a "hole" at the restricted value.

In the next two examples, we will examine each of these behaviors. In this first example, we see a restriction that leads to a vertical asymptote.

Example $3.12.1$

Sketch the graph of

$$f(x)=\frac{1}{x+2}$$

Solution

The first step is to identify the domain. Note that x = −2 makes the denominator of f(x) = 1/(x + 2) equal to zero. Division by zero is undefined. Hence, x = −2 is **not** in the domain of f; that is, x = −2 is a restriction. Equivalently, the domain of f is $\{x : x \neq -2\}$.

Now that we've identified the restriction, we can use the theory of Section 7.1 to shift the graph of y = 1/x two units to the left to create the graph of $f(x)=1/(x+2)$, as shown in Figure $3.12.1$.

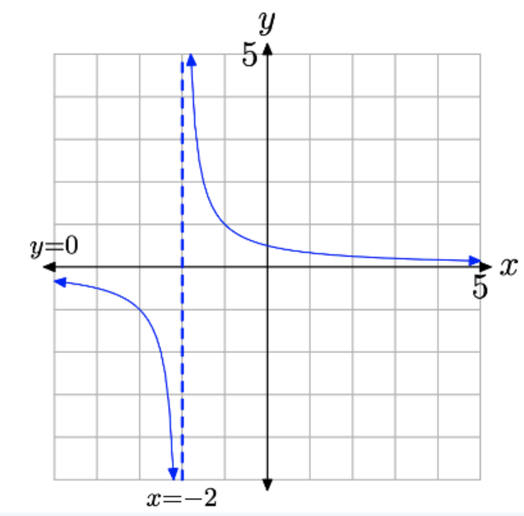


Figure 3.12.1 The function f(x) = 1/(x + 2) has a restriction at x = −2. The graph of f has a vertical asymptote with equation x = −2.

The function f(x) = 1/(x + 2) has a restriction at x = −2 and the graph of f exhibits a vertical asymptote having equation x = −2.

It is important to note that although the restricted value x = −2 makes the denominator of f(x) = 1/(x + 2) equal to zero, it does not make the numerator equal to zero. We'll soon have more to say about this observation.

Let's look at an example of a rational function that exhibits a "hole" at one of its restricted values.

Example 3.12.2

Sketch the graph of

$$f(x)=\frac{x-2}{x^2-4}$$

Solution

We highlight the first step.

Factor Numerators and Denominators

When working with rational functions, the first thing you should always do is factor both numerator and denominator of the rational function.

Following this advice, we factor both numerator and denominator of $f(x)=(x-2)/(x^2-4)$.

$$f(x)=\frac{x-2}{(x-2)(x+2)}$$

It is easier to spot the restrictions when the denominator of a rational function is in factored form. Clearly, x = −2 and x = 2 will both make the denominator of f(x) = (x−2)/((x−2)(x+ 2)) equal to zero. Hence, x = −2 and x = 2 are restrictions of the rational function f.

Now that the restrictions of the rational function f are established, we proceed to the second step.

Reduce to Lowest Terms

After you establish the restrictions of the rational function, the second thing you should do is reduce the rational function to lowest terms.

Following this advice, we cancel common factors and reduce the rational function f(x) = (x − 2)/((x − 2)(x + 2)) to lowest terms, obtaining a new function,

$$g(x)=\frac{1}{x+2}$$

The functions f(x) = (x − 2)/((x − 2)(x + 2)) and g(x) = 1/(x + 2) are not identical functions. They have different domains. The domain of f is $D_f=\{x: x\neq -2, 2\}$, but the domain of g is $D_g=\{x: x\neq -2\}$. Hence, the only difference between the two functions occurs at x = 2. The number 2 is in the domain of g, but not in the domain of f.

We know what the graph of the function g(x) = 1/(x + 2) looks like. We drew this graph in Example $3.12.1$ and we picture it anew in Figure $3.12.2$

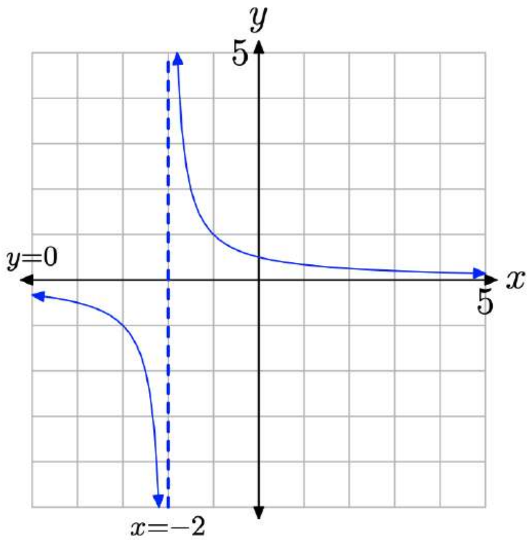


Figure $3.12.2$ The graph of g(x) = 1/(x + 2) exhibits a vertical asymptote at its restriction x = −2.

The difficulty we now face is the fact that we've been asked to draw the graph of f, not the graph of g. However, we know that the functions f and g agree at all values of x except x = 2. If we remove this value from the graph of g, then we will have the graph of f.

So, what point should we remove from the graph of g? We should remove the point that has an x-value equal to 2. Therefore, we evaluate the function g(x) = 1/(x + 2) at x = 2 and find

$$g(2)=\frac{1}{2+2}=\frac{1}{4}$$

Because g(2) = 1/4, we remove the point (2, 1/4) from the graph of g to produce the graph of f. The result is shown in Figure $3.12.3$

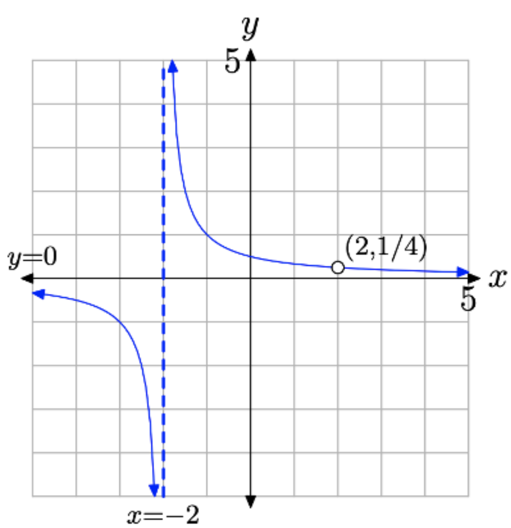


Figure $3.12.3$ The graph of f(x) = (x − 2)/((x − 2)(x + 2)) exhibits a vertical asymptote at its restriction x = −2 and a hole at its second restriction x = 2.

We pause to make an important observation. In Example $3.12.2$, we started with the function

$$f(x)=\frac{x-2}{(x-2)(x+2)}$$

which had restrictions at x = 2 and x = −2. After reducing, the function

$$g(x)=\frac{1}{x+2}$$

no longer had a restriction at x = 2. The function g had a single restriction at x = −2. The result, as seen in Figure $3.12.3$, was a vertical asymptote at the remaining restriction, and a hole at the restriction that "went away" due to cancellation. This leads us to the following procedure.

Asymptote or Hole?

To determine whether the graph of a rational function has a vertical asymptote or a hole at a restriction, proceed as follows:

1. Factor numerator and denominator of the original rational function f. Identify the restrictions of f.
2. Reduce the rational function to lowest terms, naming the new function g. Identify the restrictions of the function g.
3. Those restrictions of f that remain restrictions of the function g will introduce vertical asymptotes into the graph of f.
4. Those restrictions of f that are no longer restrictions of the function g will introduce "holes" into the graph of f. To determine the coordinates of the holes, substitute each restriction of f that is not a restriction of g into the function g to determine the y-value of the hole.

We now turn our attention to the zeros of a rational function.

The Zeros of a Rational Function

We've seen that division by zero is undefined. That is, if we have a fraction N/D, then D (the denominator) must not equal zero. Thus, 5/0, −15/0, and 0/0 are all undefined. On the other hand, in the fraction N/D, if N = 0 and $D\neq 0$, then the fraction is equal to zero. For example, 0/5, 0/(−15), and $0/\pi$ are all equal to zero.

Therefore, when working with an arbitrary rational function, such as

$$f(x)=\frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}$$

whatever value of x that will make the numerator zero without simultaneously making the denominator equal to zero will be a zero of the rational function f.

This discussion leads to the following procedure for identifying the zeros of a rational function.

Finding Zeros of Rational Functions

To determine the zeros of a rational function, proceed as follows.

1. Factor both numerator and denominator of the rational function f.
2. Identify the restrictions of the rational function f.
3. Identify the values of the independent variable (usually x) that make the numerator equal to zero.
4. The zeros of the rational function f will be those values of x that make the numerator zero but are not restrictions of the rational function f.

Let's look at an example.

Example $3.12.3$

Find the zeros of the rational function defined by

$$f(x)=\frac{x^2+3x+2}{x^2-2x-3}$$

Solution

Factor numerator and denominator of the rational function f.

$$f(x)=\frac{(x+1)(x+2)}{(x+1)(x-3)}$$

The values x = −1 and x = 3 make the denominator equal to zero and are restrictions.

Next, note that x = −1 and x = −2 both make the numerator equal to zero. However, x = −1 is also a restriction of the rational function f, so it will not be a zero of f. On the other hand, the value x = −2 is not a restriction and will be a zero of f.

Although we've correctly identified the zeros of f, it's instructive to check the values of x that make the numerator of f equal to zero. If we substitute x = −1 into original function defined by equation (6), we find that

$$f(-1)=\frac{(-1)^2+3(-1)+2}{(-1)^2-2(-1)-3}=\frac{0}{0}$$

is undefined. Hence, x = −1 is not a zero of the rational function f. The difficulty in this case is that x = −1 also makes the denominator equal to zero.

On the other hand, when we substitute x = −2 in the function defined by equation (6),

$$f(-2)=\frac{(-2)^2+3(-2)+2}{(-2)^2-2(-2)-3}=\frac{0}{5}=0$$

In this case, x = −2 makes the numerator equal to zero without making the denominator equal to zero. Hence, x = −2 is a zero of the rational function f.

It's important to note that you must work with the original rational function, and not its reduced form, when identifying the zeros of the rational function.

Example 3.12.4

Identify the zeros of the rational function

$$f(x)=\frac{x^2-6x+9}{x^2-9}$$

Solution

Factor both numerator and denominator.

$$f(x)=\frac{(x-3)^2}{(x+3)(x-3)}$$

Note that x = −3 and x = 3 are restrictions. Further, the only value of x that will make the numerator equal to zero is x = 3. However, this is also a restriction. Hence, the function f has no zeros.

The point to make here is what would happen if you work with the reduced form of the rational function in attempting to find its zeros. Cancelling like factors leads to a new function,

$$g(x)=\frac{x-3}{x+3}$$

Note that g has only one restriction, x = −3. Further, x = 3 makes the numerator of g equal to zero and is not a restriction. Hence, x = 3 is a zero of the function g, but it is not a zero of the function f.

This example demonstrates that we must identify the zeros of the rational function before we cancel common factors.

Drawing the Graph of a Rational Function

In this section we will use the zeros and asymptotes of the rational function to help draw the graph of a rational function. We will also investigate the end-behavior of rational functions. Let's begin with an example.

Example $3.12.5$

Sketch the graph of the rational function

$$f(x)=\frac{x+2}{x-3}$$

Solution

First, note that both numerator and denominator are already factored. The function has one restriction, x = 3. Next, note that x = −2 makes the numerator of equation (9) zero and is not a restriction. Hence, x = −2 is a zero of the function. Recall that a function is zero where its graph crosses the horizontal axis. Hence, the graph of f will cross the x-axis at (−2, 0), as shown in Figure $3.12.4$.

Note that the rational function (9) is already reduced to lowest terms. Hence, the restriction at x = 3 will place a vertical asymptote at x = 3, which is also shown in Figure $3.12.4$.

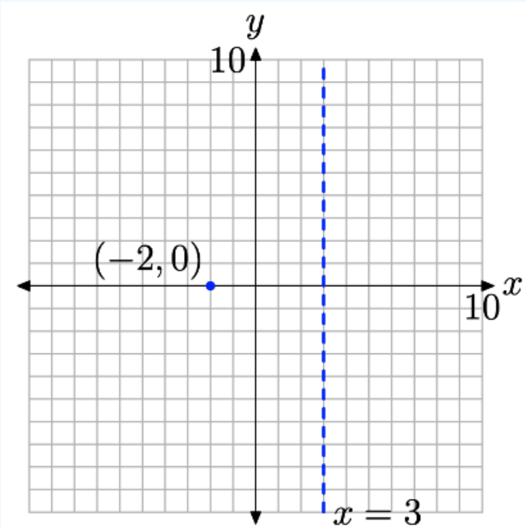
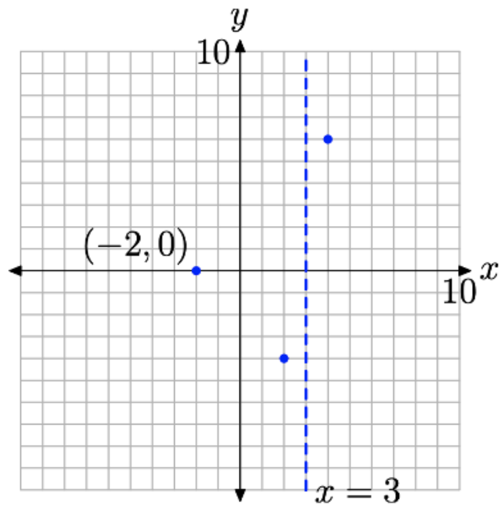


Figure $3.12.4$ Plot and label the x-intercept and vertical asymptote.

At this point, we know two things:

1. The graph will cross the x-axis at (−2, 0).
2. On each side of the vertical asymptote at x = 3, one of two things can happen. Either the graph will rise to positive infinity or the graph will fall to negative infinity.

To discover the behavior near the vertical asymptote, let’s plot one point on each side of the vertical asymptote, as shown in Figure $3.12.5$.



x	$y = (x+2)/(x-3)$
2	-4
4	6

Figure $3.12.5$ Additional points help determine the behavior near the vertical asymptote.

Consider the right side of the vertical asymptote and the plotted point (4, 6) through which our graph must pass. As the graph approaches the vertical asymptote at x = 3, only one of two things can happen. Either the graph rises to positive infinity or the graph falls to negative infinity. However, in order for the latter to happen, the graph must first pass through the point (4, 6), then cross the x-axis between x = 3 and x = 4 on its descent to minus infinity. But we already know that the only x-intercept is at the point (2, 0), so this cannot happen. Hence, on the right, the graph must pass through the point (4, 6), then rise to positive infinity, as shown in Figure $3.12.6$.

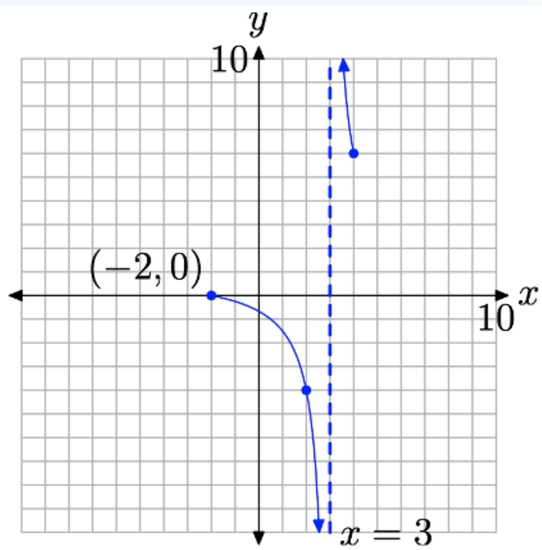


Figure $3.12.6$ Behavior near the vertical asymptote.

A similar argument holds on the left of the vertical asymptote at x = 3. The graph cannot pass through the point (2, −4) and rise to positive infinity as it approaches the vertical asymptote, because to do so would require that it cross the x-axis between x = 2 and x = 3. However, there is no x-intercept in this region available for this purpose. Hence, on the left, the graph must pass through the point (2, −4) and fall to negative infinity as it approaches the vertical asymptote at x = 3. This behavior is shown in Figure $3.12.6$.

Finally, what about the end-behavior of the rational function? What happens to the graph of the rational function as x increases without bound? What happens when x decreases without bound? One simple way to answer these questions is to use a table to investigate the behavior numerically. The graphing calculator facilitates this task.

First, enter your function as shown in Figure $3.12.7$(a), then press 2nd TBLSET to open the window shown in Figure $3.12.7$(b). For what we are about to do, all of the settings in this window are irrelevant, save one. Make sure you use the arrow keys to highlight ASK for the Indpnt (independent) variable and press ENTER to select this option. Finally, select 2nd TABLE, then enter the x-values 10, 100, 1000, and 10000, pressing ENTER after each one.

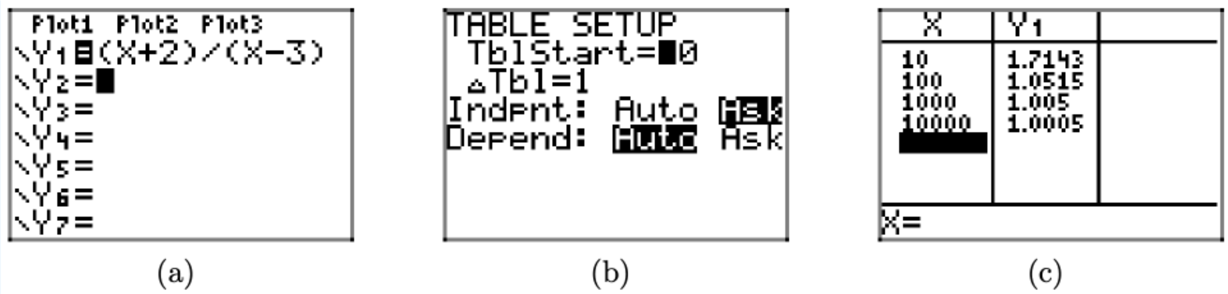


Figure 3.12.7 Using the table feature of the graphing calculator to investigate the end-behavior as x approaches positive infinity.

Note the resulting y-values in the second column of the table (the Y1 column) in Figure 3.12.7(c). As x is increasing without bound, the y-values are greater than 1, yet appear to be approaching the number 1. Therefore, as our graph moves to the extreme right, it must approach the horizontal asymptote at y = 1, as shown in Figure 3.12.9.

A similar effort predicts the end-behavior as x decreases without bound, as shown in the sequence of pictures in Figure 3.12.8. As x decreases without bound, the y-values are less than 1, but again approach the number 1, as shown in Figure 3.12.8(c).

Figure 3.12.8 Using the table feature of the graphing calculator to investigate the end-behavior as x approaches negative infinity.

The evidence in Figure 3.12.8(c) indicates that as our graph moves to the extreme left, it must approach the horizontal asymptote at y = 1, as shown in Figure 3.12.9.

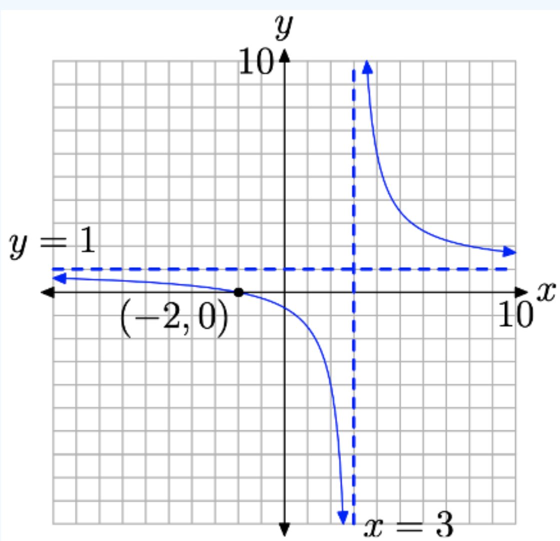


Figure 3.12.9 The graph approaches the horizontal asymptote y = 1 at the extreme right- and left-ends.

What kind of job will the graphing calculator do with the graph of this rational function? In Figure 3.12.10(a), we enter the function, adjust the window parameters as shown in Figure 3.12.10(b), then push the GRAPH button to produce the result in Figure 3.12.10(c).

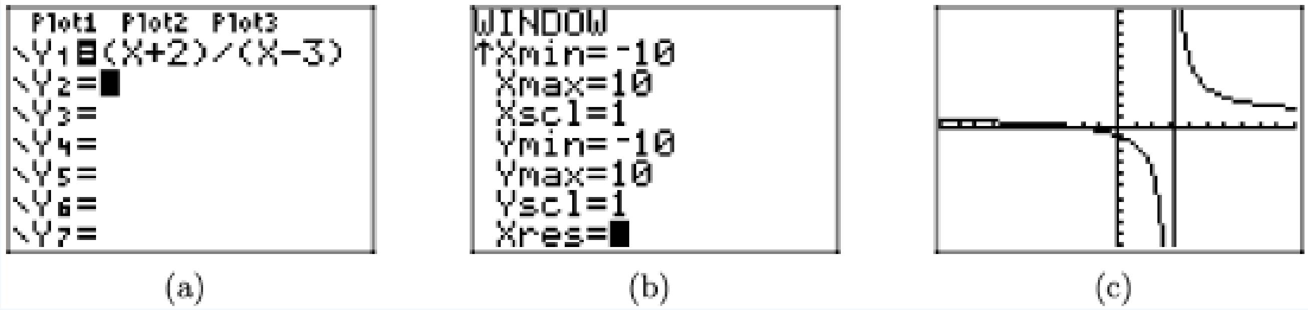


Figure 3.12.10 Drawing the graph of the rational function with the graphing calculator.

As was discussed in the first section, the graphing calculator manages the graphs of “continuous” functions extremely well, but has difficulty drawing graphs with discontinuities. In the case of the present rational function, the graph “jumps” from negative infinity to positive infinity across the vertical asymptote x = 3. The calculator knows only one thing: plot a point, then connect it to the previously plotted point with a line segment. Consequently, it does what it is told, and “connects” infinities when it shouldn’t.

However, if we have prepared in advance, identifying zeros and vertical asymptotes, then we can interpret what we see on the screen in Figure 3.12.10(c), and use that information to produce the correct graph that is shown in Figure 3.12.9. We can even add the horizontal asymptote to our graph, as shown in the sequence in Figure 3.12.11.

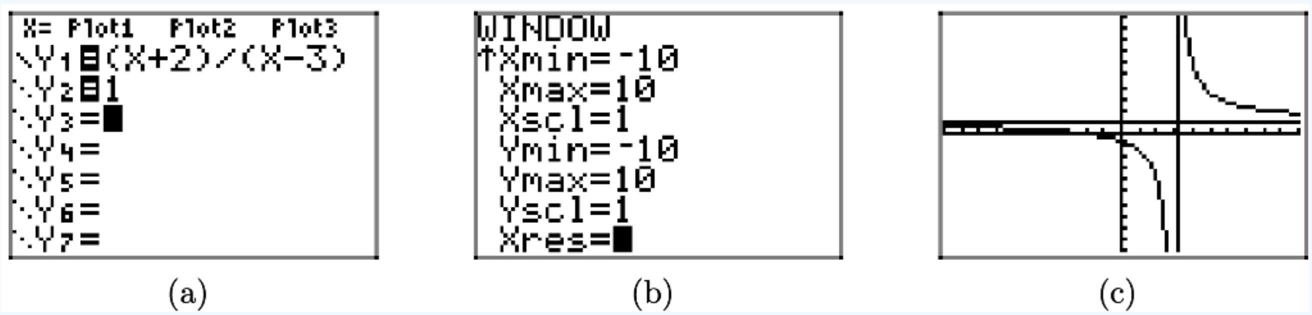


Figure 3.12.11 Adding a suspected horizontal asymptote.

This is an appropriate point to pause and summarize the steps required to draw the graph of a rational function.

Procedure for Graphing Rational Functions

Consider the rational function

$$f(x)=\frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}$$

. To draw the graph of this rational function, proceed as follows:

1. Factor the numerator and denominator of the rational function f.
2. Identify the domain of the rational function f by listing each restriction, values of the independent variable (usually x) that make the denominator equal to zero.
3. Identify the values of the independent variable that make the numerator of f equal to zero and are not restrictions. These are the zeros of f and they provide the x-coordinates of the x-intercepts of the graph of the rational function. Plot these intercepts on a coordinate system and label them with their coordinates.
4. Cancel common factors to reduce the rational function to lowest terms. − The restrictions of f that remain restrictions of this reduced form will place vertical asymptotes in the graph of f. Draw the vertical asymptotes on your coordinate system as dashed lines and label them with their equations. − The restrictions of f that are not restrictions of the reduced form will place “holes” in the graph of f. We’ll deal with the holes in step 8 of this procedure.
5. To determine the behavior near each vertical asymptote, calculate and plot one point on each side of each vertical asymptote.

6. To determine the end-behavior of the given rational function, use the table capability of your calculator to determine the limit of the function as x approaches positive and/or negative infinity (as we did in the sequences shown in Figure $3.12.7$ and Figure $3.12.8$). This determines the horizontal asymptote. Sketch the horizontal asymptote as a dashed line on your coordinate system and label it with its equation.
7. Draw the graph of the rational function.
8. If you determined that a restriction was a “hole,” use the restriction and the reduced form of the rational function to determine the y-value of the “hole.” Draw an open circle at this position to represent the “hole” and label the “hole” with its coordinates.
9. Finally, use your calculator to check the validity of your result.

Let’s look at another example.

Example $3.12.6$

Sketch the graph of the rational function

$$f(x)=\frac{x-2}{x^2-3x-4}$$

Solution

We will follow the outline presented in the Procedure for Graphing Rational Functions.

Step 1: First, factor both numerator and denominator.

$$f(x)=\frac{x-2}{(x+1)(x-4)}$$

Step 2: Thus, f has two restrictions, x = −1 and x = 4. That is, the domain of f is $D_f=\{s : x \neq -1, 4\}$.

Step 3: The numerator of equation (12) is zero at x = 2 and this value is not a restriction. Thus, 2 is a zero of f and (2, 0) is an x-intercept of the graph of f, as shown in Figure $3.12.12$

Step 4: Note that the rational function is already reduced to lowest terms (if it weren’t, we’d reduce at this point). Note that the restrictions x = −1 and x = 4 are still restrictions of the reduced form. Hence, these are the locations and equations of the vertical asymptotes, which are also shown in Figure $3.12.12$

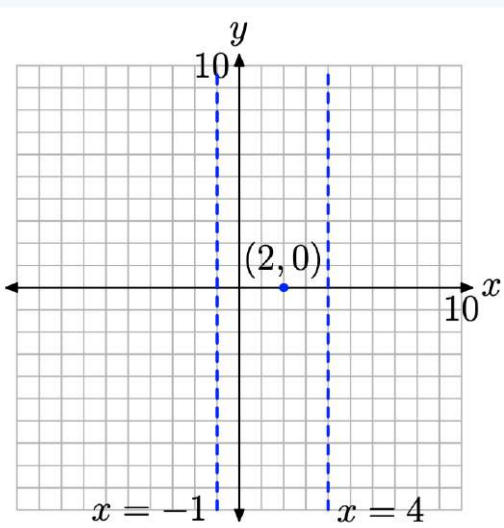
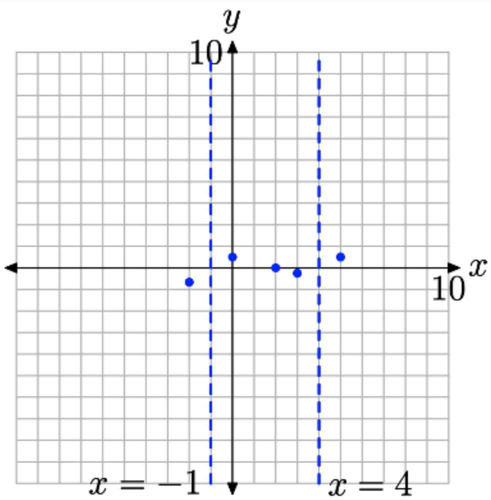


Figure $3.12.12$ Plot the x-intercepts and draw the vertical asymptotes.

All of the restrictions of the original function remain restrictions of the reduced form. Therefore, there will be no “holes” in the graph of f.

Step 5: Plot points to the immediate right and left of each asymptote, as shown in Figure $3.12.13$ These additional points completely determine the behavior of the graph near each vertical asymptote. For example, consider the point (5, 1/2) to the immediate right of the vertical asymptote x = 4 in Figure $3.12.13$ Because there is no x-intercept between x = 4 and x = 5, and the graph is already above the x-axis at the point (5, 1/2), the graph is forced to increase to positive infinity as it approaches the vertical asymptote x = 4. Similar comments are in order for the behavior on each side of each vertical asymptote.

Step 6: Use the table utility on your calculator to determine the end-behavior of the rational function as x decreases and/or increases without bound. To determine the end-behavior as x goes to infinity (increases without bound), enter the equation in your calculator, as shown in Figure $3.12.14$(a). Select 2nd TBLSET and highlight ASK for the independent variable. Select 2nd TABLE, then enter 10, 100, 1000, and 10000, as shown in Figure $3.12.14$(c).



x	$y = \dfrac{x-2}{(x+1)(x-4)}$
-2	$-2/3$
0	$1/2$
3	$-1/4$
5	$1/2$

Figure $3.12.13$ Additional points help determine the behavior near the vertical asymptote.

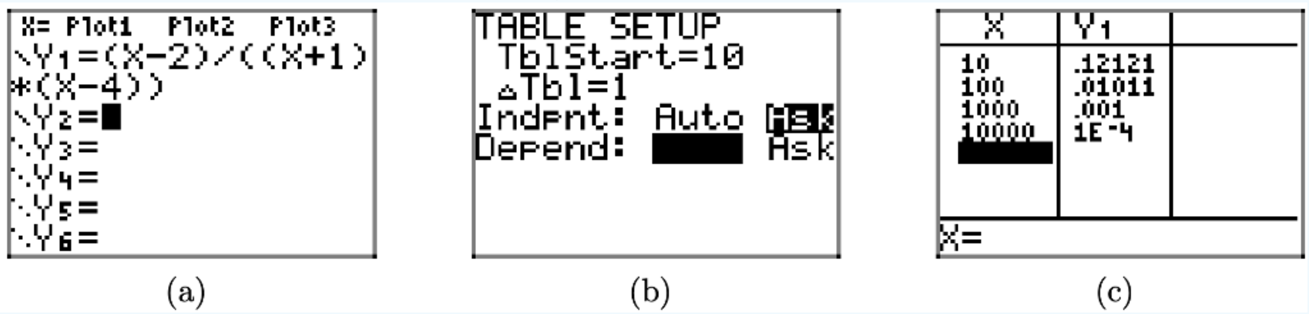


Figure $3.12.14$ Examining end-behavior as x approaches positive infinity

If you examine the y-values in Figure $3.12.14$(c), you see that they are heading towards zero (1e-4 means 1×10^{-4}, which equals 0.0001). This implies that the line y = 0 (the x-axis) is acting as a horizontal asymptote.

You can also determine the end-behavior as x approaches negative infinity (decreases without bound), as shown in the sequence in Figure $3.12.15$. The result in Figure $3.12.15$(c) provides clear evidence that the y-values approach zero as x goes to negative infinity. Again, this makes y = 0 a horizontal asymptote.

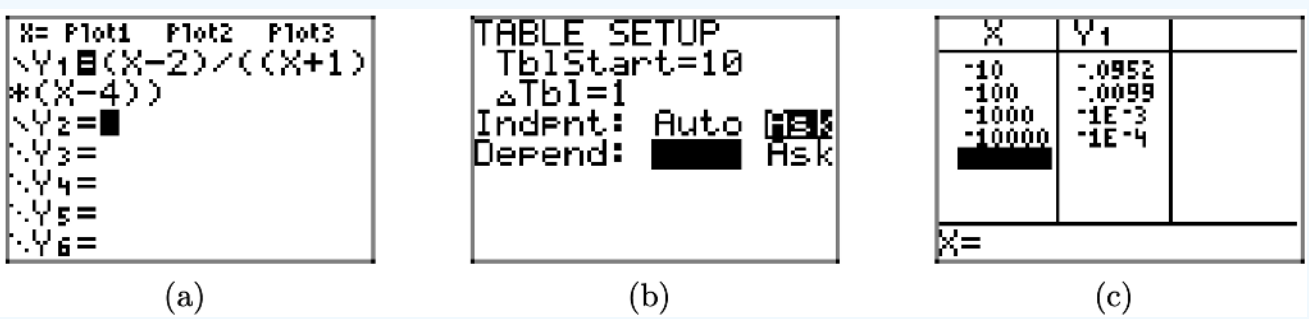


Figure $3.12.15$ Examining end-behavior as x approaches negative infinity

Add the horizontal asymptote y = 0 to the image in Figure $3.12.13$.

Step 7: We can use all the information gathered to date to draw the image shown in Figure $3.12.16$.

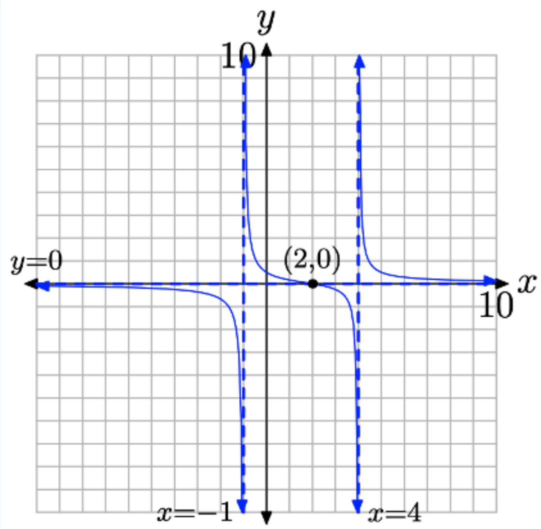


Figure 3.12.16 The completed graph runs up against vertical and horizontal asymptotes and crosses the x-axis at the zero of the function.

Step 8: As stated above, there are no “holes” in the graph of f.

Step 9: Use your graphing calculator to check the validity of your result. Note how the graphing calculator handles the graph of this rational function in the sequence in Figure 3.12.17. The image in Figure 3.12.17(c) is nowhere near the quality of the image we have in Figure 3.12.16, but there is enough there to intuit the actual graph if you prepare properly in advance (zeros, vertical asymptotes, end-behavior analysis, etc.).

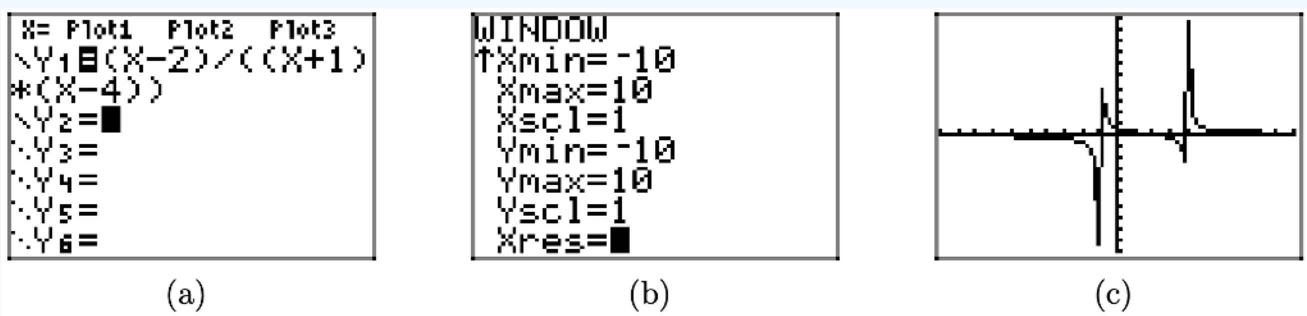


Figure 3.12.17 The user of the graphing calculator must decipher the image in the calculator’s view screen.

Exercise

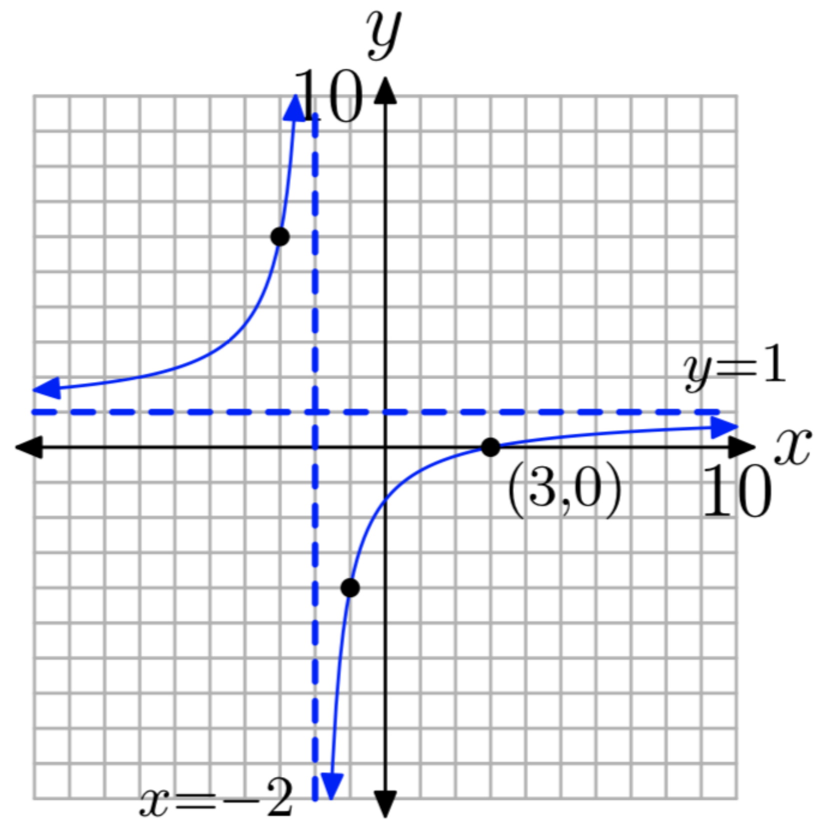
For rational functions **Exercises 1-20**, follow the Procedure for Graphing Rational Functions in the narrative, performing each of the following tasks.

1. Set up a coordinate system on graph paper. Label and scale each axis. Remember to draw all lines with a ruler.
2. Perform each of the nine steps listed in the Procedure for Graphing Rational Functions in the narrative.

Exercise 3.12.1

$f(x)=\frac{x-3}{x+2}$

Answer



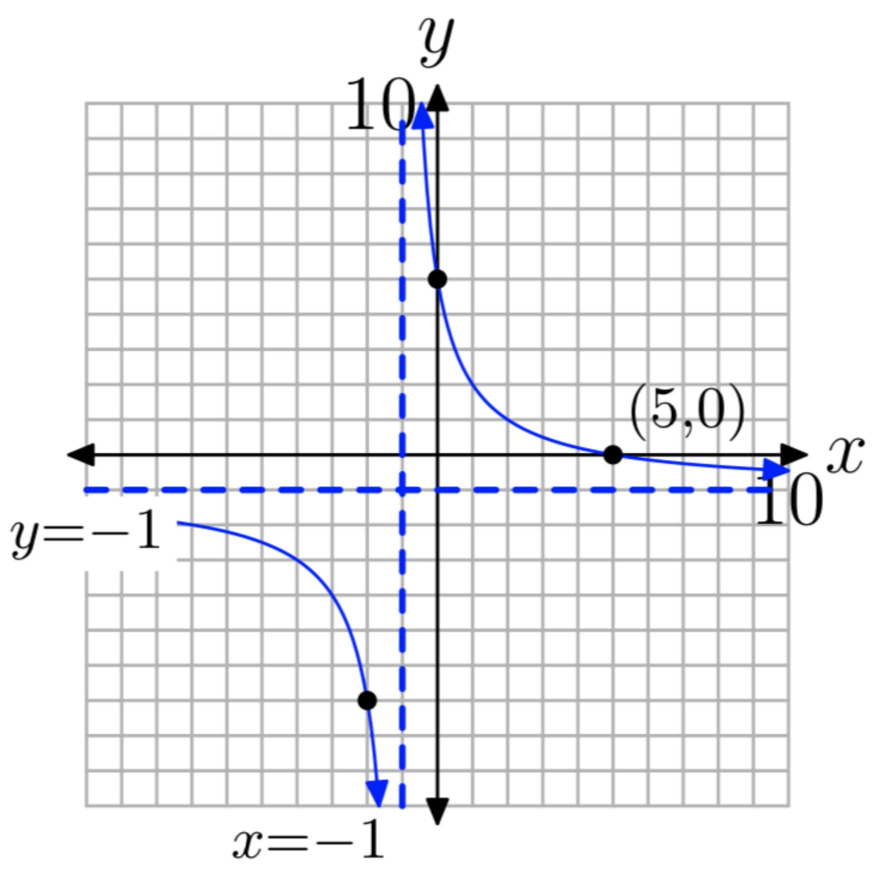
Exercise 3.12.2

$f(x) = \frac{x+2}{x-4}$

Exercise 3.12.3

$f(x) = \frac{5-x}{x+1}$

Answer



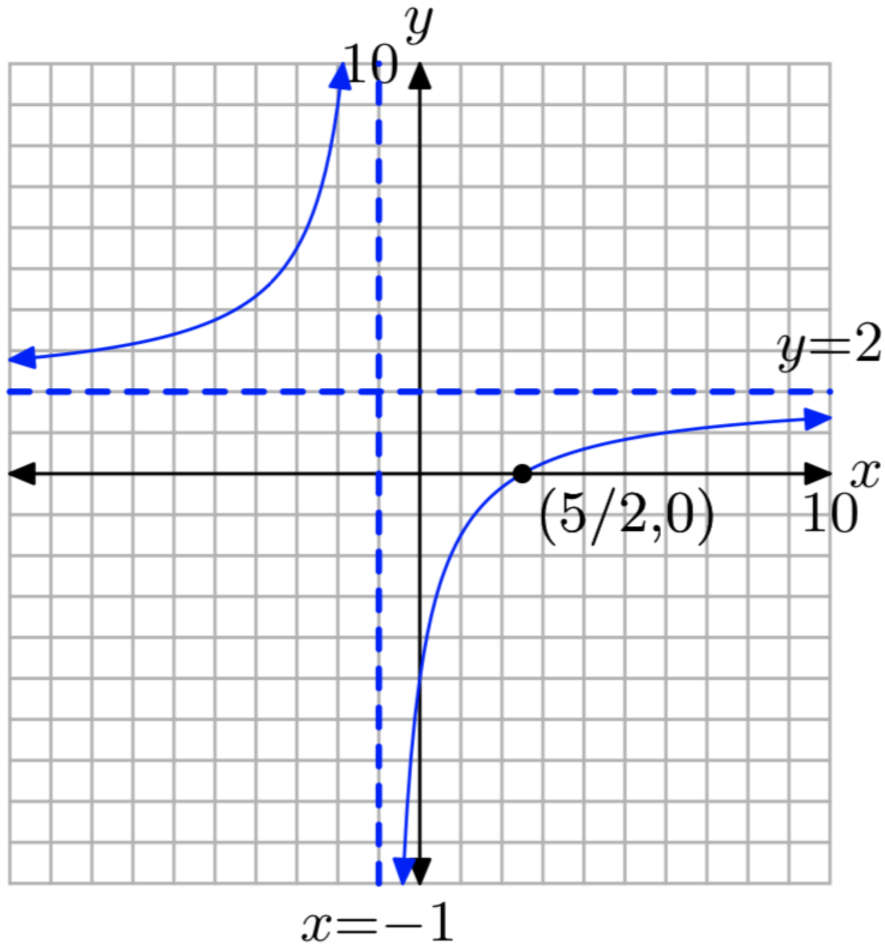
? Exercise 3.12.4

$f(x)=\frac{x+2}{4-x}$

? Exercise 3.12.5

$f(x)=\frac{2x-5}{x+1}$

Answer



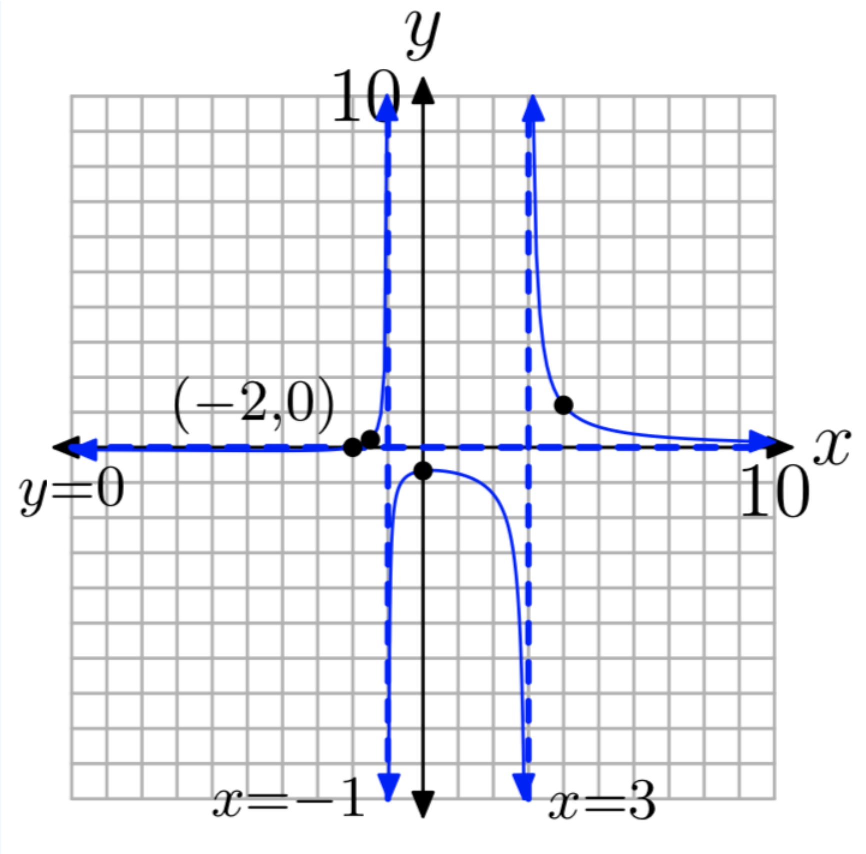
Exercise 3.12.6

$f(x)=\frac{2x+5}{3-x}$

Exercise 3.12.7

$f(x)=\frac{x+2}{x^{2}-2x-3}$

Answer



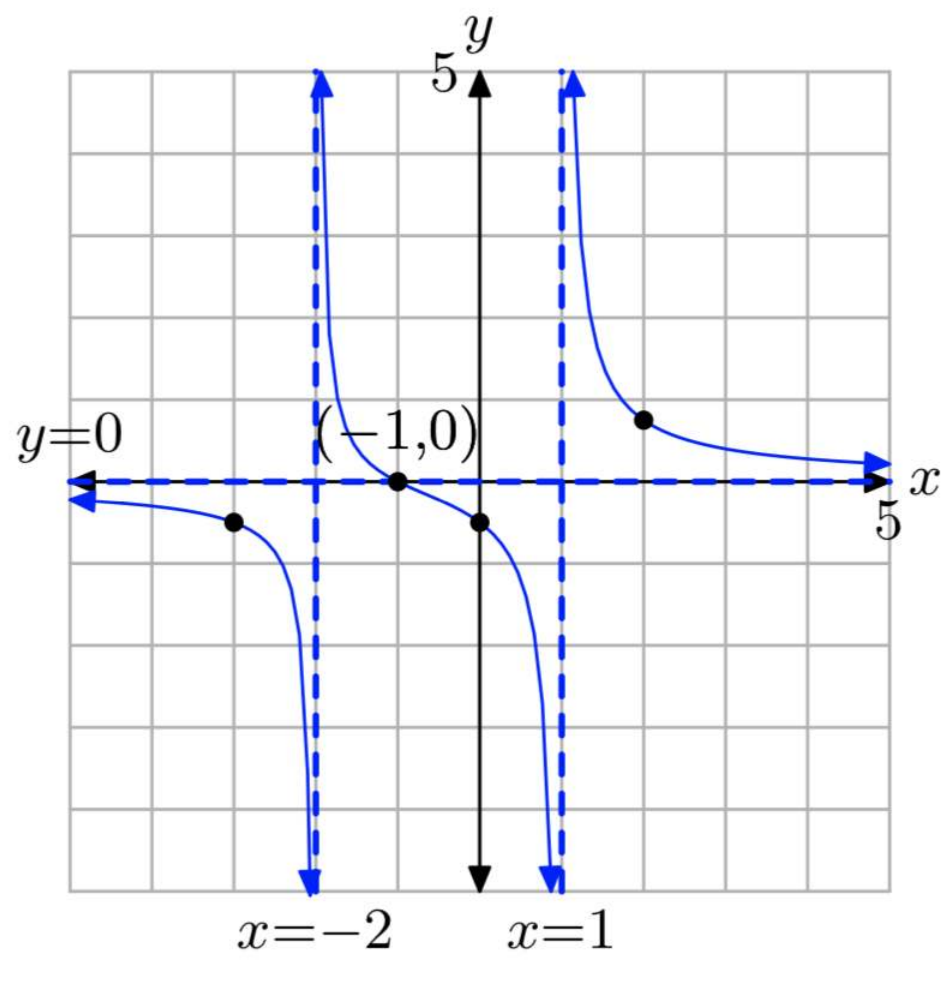
Exercise 3.12.8

$f(x) = \frac{x-3}{x^2-3x-4}$

Exercise 3.12.9

$f(x) = \frac{x+1}{x^2+x-2}$

Answer



Exercise 3.12.10

$f(x)=\frac{x-1}{x^2-x-2}$

Exercise 3.12.11

$f(x)=\frac{x^2-2x}{x^2+x-2}$

Answer

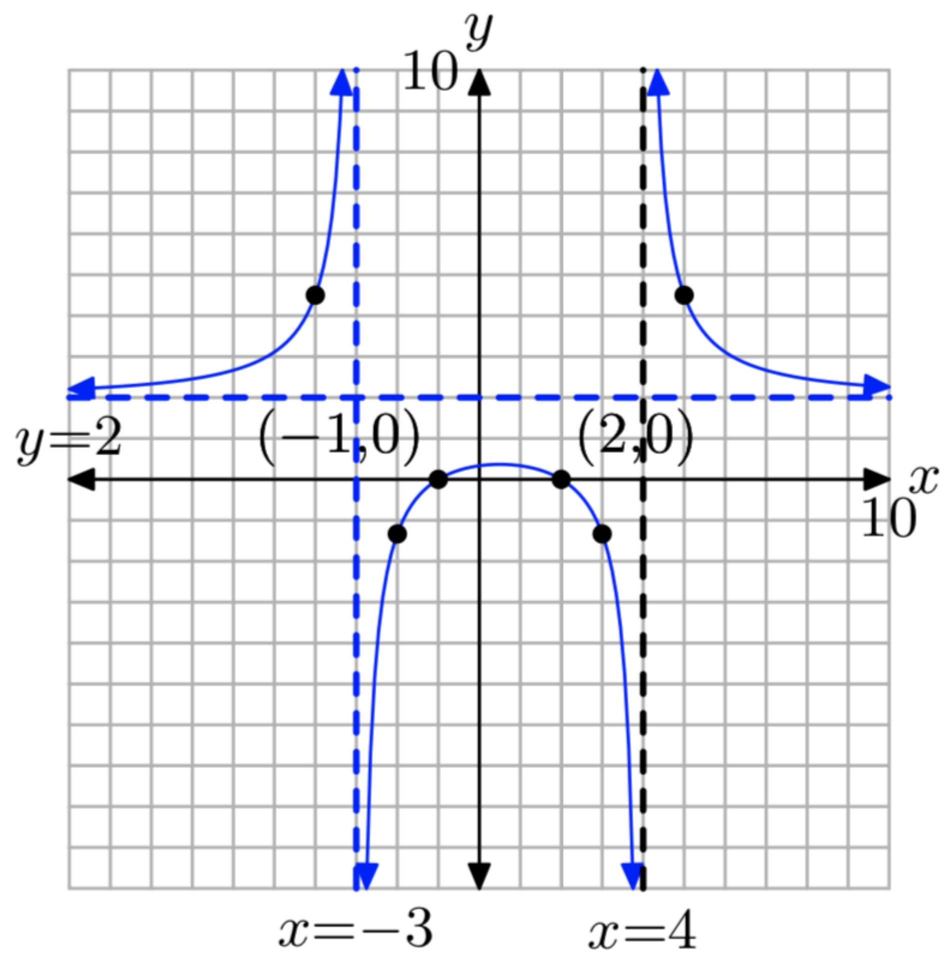
Exercise 3.12.12

$f(x)=\frac{x^2-2x}{x^2-2x-8}$

Exercise 3.12.13

$f(x)=\frac{2x^2-2x-4}{x^2-x-12}$

Answer



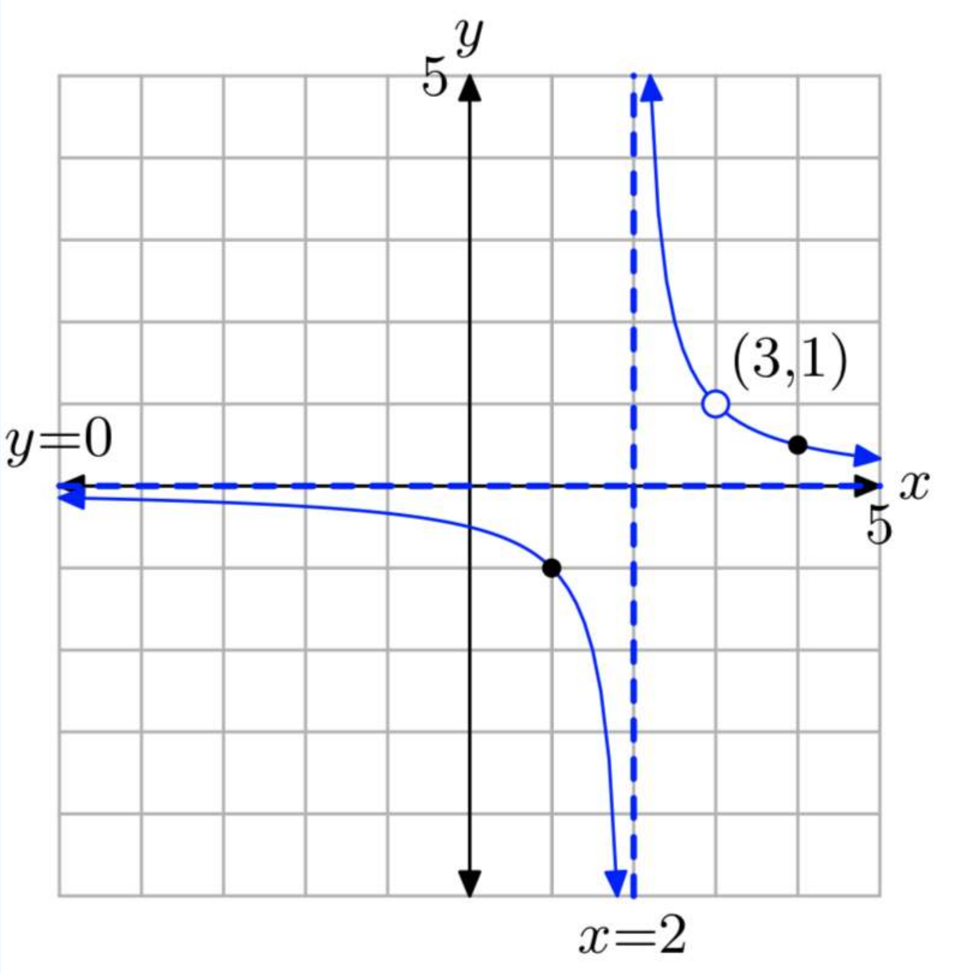
? Exercise 3.12.14

$f(x) = \frac{8x-2x^2}{x^2-x-6}$

? Exercise 3.12.15

$f(x) = \frac{x-3}{x^2-5x+6}$

Answer



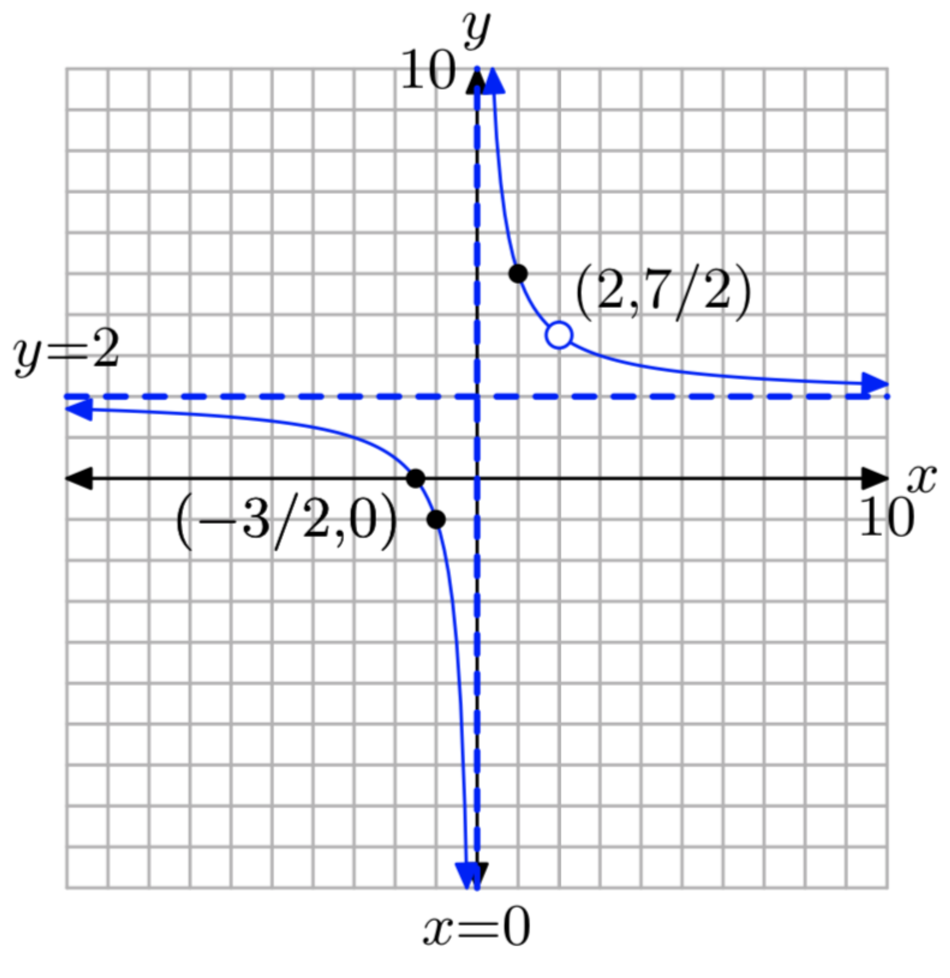
? Exercise 3.12.16

$f(x) = \frac{2x-4}{x^2-x-2}$

? Exercise 3.12.17

$f(x) = \frac{2x^2-x-6}{x^2-2x}$

Answer



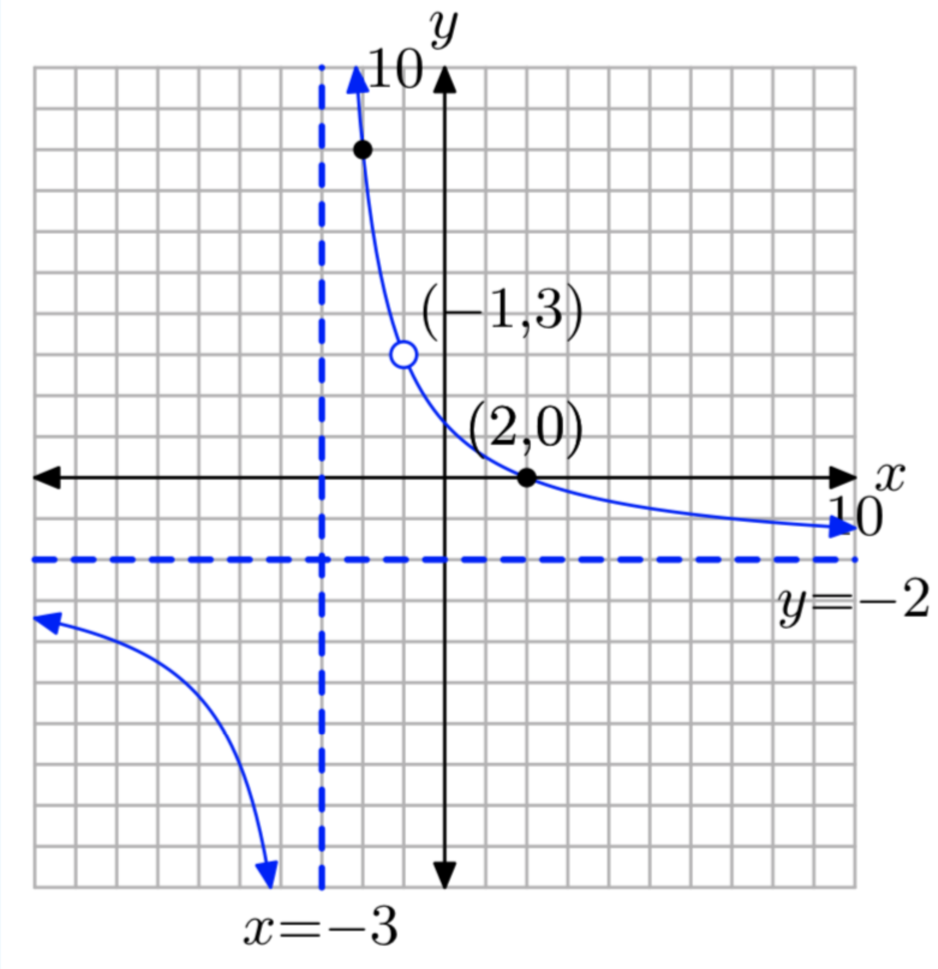
? Exercise 3.12.18

$f(x)=\frac{2x^2-x-6}{x^2-2x}$

? Exercise 3.12.19

$f(x)=\frac{4+2x-2x^2}{x^2+4x+3}$

Answer



Exercise 3.12.20

$f(x)=\frac{3x^2-6x-9}{1-x^2}$

In **Exercises 21-28**, find the coordinates of the x-intercept(s) of the graph of the given rational function.

Exercise 3.12.21

$f(x)=\frac{81-x^2}{x^2+10x+9}$

Answer

(9, 0)

Exercise 3.12.22

$f(x)=\frac{x-x^2}{x^2+5x-6}$

Exercise 3.12.23

$f(x)=\frac{x^2-x-12}{x^2+2x-3}$

Answer

(4, 0)

Exercise 3.12.24

$f(x)=\frac{x^2-81}{x^2-4x-45}$

Exercise 3.12.25

$f(x)=\frac{6x-18}{x^2-7x+12}$

Answer

no x-intercepts

Exercise 3.12.26

$f(x)=\frac{4x+36}{x^2+15x+54}$

Exercise 3.12.27

$f(x)=\frac{x^2-9x+14}{x^2-2x}$

Answer

(7, 0)

Exercise 3.12.28

$f(x)=\frac{x^2-5x-36}{x^2-9x+20}$

In **Exercises 29-36**, find the equations of all vertical asymptotes.

Exercise 3.12.29

$f(x)=\frac{x^2-7x}{x^2-2x}$

Answer

x = 2

Exercise 3.12.30

$f(x)=\frac{x^2+4x-45}{3x+27}$

? Exercise 3.12.31

$f(x)=\frac{x^2-6x+8}{x^2-16}$

Answer

x = −4

? Exercise 3.12.32

$f(x)=\frac{x^2-11x+18}{2x-x^2}$

? Exercise 3.12.33

$f(x)=\frac{x^2+x-12}{-4x+12}$

Answer

no vertical asymptotes

? Exercise 3.12.34

$f(x)=\frac{x^2-3x-54}{9x-x^2}$

? Exercise 3.12.35

$f(x)=\frac{16-x^2}{x^2+7x+12}$

Answer

x = −3

? Exercise 3.12.36

$f(x)=\frac{x^2-11x+30}{-8x+48}$

In **Exercises 37-42**, use a graphing calculator to determine the behavior of the given rational function as x approaches both positive and negative infinity by performing the following tasks:

1. Load the rational function into the Y=menu of your calculator.
2. Use the TABLE feature of your calculator to determine the value of f(x) for x = 10, 100, 1000, and 10000. Record these results on your homework in table form.
3. Use the TABLE feature of your calculator to determine the value of f(x) for x = −10, −100, −1000, and −10000. Record these results on your home- work in table form.
4. Use the results of your tabular exploration to determine the equation of the horizontal asymptote.

? Exercise 3.12.37

$f(x)=\frac{2x+3}{x-8}$

Answer

Horizontal asymptote at y = 2.

X	Y1	X	Y1
10	11.5	-10	0.944444
100	2.20652	-100	1.82407
1000	2.01915	-1000	1.98115
10000	2.0019	-10000	1.998

Exercise 3.12.38

$f(x) = \frac{4-3x}{x+2}$

Exercise 3.12.39

$f(x) = \frac{4-x^2}{x^2+4x+3}$

Answer

Horizontal asymptote at y = −1.

X	Y1	X	Y1
10	-0.671329	-10	-1.52381
100	-0.960877	-100	-1.04092
1000	-0.996009	-1000	-1.00401
10000	-0.9996	-10000	-1

Exercise 3.12.40

$f(x) = \frac{10-2x^2}{x^2-4}$

Exercise 3.12.41

$f(x) = \frac{x^2-2x-3}{2x^2-3x-2}$

Answer

Horizontal asymptote at $y = \frac{1}{2}$.

X	Y1	X	Y1
10	0.458333	-10	0.513158
100	0.49736	-100	0.502365
1000	0.499749	-1000	0.500249
10000	0.49997	-10000	0.50002

Exercise 3.12.42

$f(x) = \frac{2x^2-3x-5}{x^2-x-6}$

In **Exercises 43-48**, use a purely analytical method to determine the domain of the given rational function. Describe the domain using set-builder notation.

Exercise 3.12.43

$f(x) = \frac{x^2-5x-6}{-9x-9}$

Answer

Domain = {x: $x \neq -1$ }

Exercise 3.12.44

$f(x) = \frac{x^2+4x+3}{x^2-5x-6}$

Exercise 3.12.45

$f(x) = \frac{x^2+5x-24}{x^2-3x}$

Answer

Domain = {x: $x \neq 3, 0$}

Exercise 3.12.46

$f(x) = \frac{x^2-3x-4}{x^2-5x-6}$

Exercise 3.12.47

$f(x) = \frac{x^2-4x+3}{x-x^2}$

Answer

Domain = {x: $x \neq 0, 1$}

Exercise 3.12.48

$f(x) = \frac{x^2-4}{x^2-9x+14}$

CHAPTER OVERVIEW

4: Exponential and Logarithmic Functions

Learning Objectives

In this chapter, you will

1. Examine exponential and logarithmic functions and their properties
2. Identify exponential growth and decay functions and use them to model applications
3. Use the natural base e to represent an exponential functions
4. Use logarithmic functions to solve equations involving exponential functions
5. Use logarithmic and exponential functions to solve equations.

4.1: Prelude to Exponential and Logarithmic Functions including Exponential properties
4.2: Definition of Exponential Function, Exponential Growth and Decay Models
4.2.1: Exponential Growth and Decay Models (Exercises)
4.3: Graphs and Properties of Exponential Growth and Decay Functions/Use TI-84 calculator
4.3.1: Graphs and Properties of Exponential Growth and Decay Functions (Exercises)
4.4: Logarithms and Logarithmic Functions
4.4.1: Logarithms and Logarithmic Functions (Exercises)
4.5: Graphs and Properties of Logarithmic Functions
4.5.1: Graphs and Properties of Logarithmic Functions (Exercises)
4.6: Exponential and Logarithmic Equations
4.7: Chapter Review
4.E: Exponential and Logarithmic Functions (Exercises)

4.1: Prelude to Exponential and Logarithmic Functions including Exponential properties

In this chapter we examine exponential and logarithmic functions. We will need these functions in the next chapter, when examining financial calculations.

This chapter is a new addition to this textbook. The California Community Colleges Curriculum Course Descriptor for Finite Mathematics (C-ID; https://c-id.net/descriptors.html, http://www.ccccurriculum.net/articulation/) now requires coverage of exponential and logarithmic functions in a Finite Mathematics course that is part of an Associate Degree for Transfer.

Students enrolling in Finite Mathematics typically are required to complete an Intermediate Algebra course or equivalent, as a prerequisite, so students have already been exposed to much of the material in this chapter. However many students require a review of this material, which is the basis for financial calculations based on compound interest in the following chapter. In addition, review of this material is particularly important at colleges where Finite Mathematics serves as a prerequisite for Business Calculus.

This book assumes students have mastered working with exponents, and properties of exponents; it focuses on review of exponential and logarithmic functions with an eye toward skills needed to use exponential growth and decay models for financial calculations and other business applications, as well as subsequent use in a course on Business Calculus. For the most part, financial applications are not stressed in this new chapter, as financial calculations are the focus of the following chapter.

Definition: Properties of Exponential Functions

Adding Exponents

$a^n * a^m = a^{n+m}$

Example: $a^2 * a^4 = a^{2+4} = a^6$

Subtracting Exponents

$a^n/a^m = a^{n-m}$ provided a not equal to 0

$a^6/a^2 = a^{6-2} = a^4$

Multiplying Exponents

$(a^n)^m = a^{n*m}$

Zero Exponent

$a^0 = 1$

Negative Exponents

$a^{-n} = 1/a^n$

1 in Exponent

$a = a^1$

Distributing Exponents

$(ab)^n = (a^1b^1)^n = a^{1*n}\,b^{1*n} = a^n\,b^n$

$(a/b)^n = (a^1 / b^1)^n = a^{1*n} / b^{1*n} = a^n/b^n$

? Exercise 4.1.1

w^3w^5 = ?

Answer

$w^{3+5} = w^8$

? Exercise 4.1.1

y^6/y^2 = ?

Answer

$y^{6-2} = y^4$

? Exercise 4.1.1

$(z^3)^5$ = ?

Answer

$z^{3*5} = z^{15}$

? Exercise 4.1.1

2345^0 = ?

Answer

Any number raised to the 0 power is 1.

? Exercise 4.1.1

t^{-6} = ?

Answer

$t^{-6} = 1/t^6$

? Exercise 4.1.1

4^1 = ?

Answer

4

? Exercise 4.1.1

$(a^3b^2)4$ = ?

Answer

$a^{3*4}b^{2*4} = a^{12}b^8$

4.2: Definition of Exponential Function, Exponential Growth and Decay Models

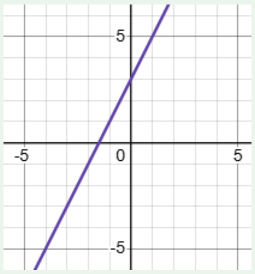
Learning Objectives

In this section, you will learn to

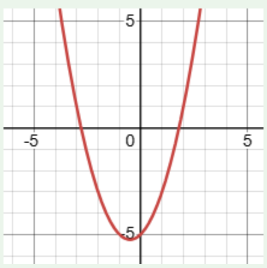
1. Define an exponential function
2. Recognize and model exponential growth and decay
3. Compare linear and exponential growth
4. Distinguish between exponential and power functions

Definition: Exponential Function

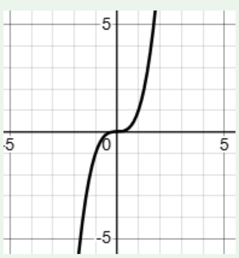
A line is of the form $y = ax + b$ **Doman:** $(-\infty, \infty)$ **Range**: $(-\infty, \infty)$



A quadratic is of the form $y = ax^2 + bx + c$ **Domain:** $(-\infty, \infty)$ **Range:** depends on the values of a, b, and c. If $a > 0$, the **Range is** $[-b/2a, \infty)$. If $a < 0$, **the Range is** $(-\infty, -b/2a]$



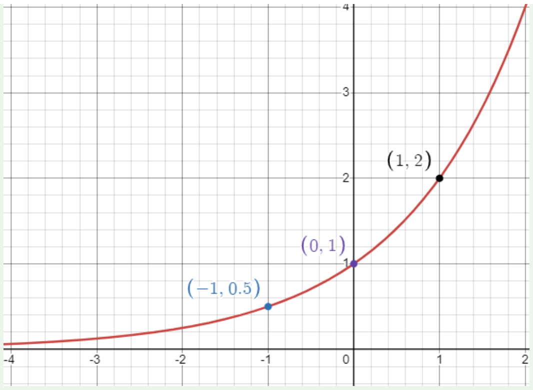
A cubic is of the form $y = ax^3 + bx^2 + cx + d$ **Domain:** $(-\infty, \infty)$ **Range:** $(-\infty, \infty)$



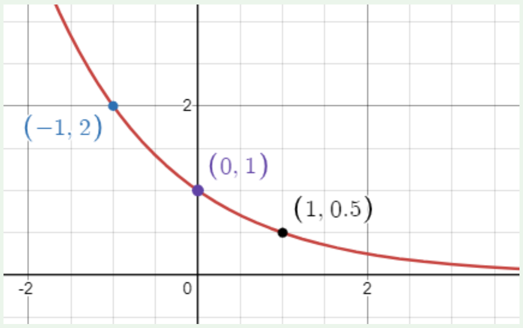
An exponential function is different from the function above because the x variable is in the exponent. The exponential function takes the following form.

$y = ab^x$. where $b > 0$. **Domain:** $(-\infty, \infty)$ **Range:** If $a > 0$, $y > 0$ or $(0, \infty)$.

If $b > 0$, the exponential function increases as x increases. For example $y = 2^x$. The leading coefficient $a = 1$.



If $0 < b < 1$, the exponential function decreases as x increases. For example $y = (1/2)^x$. The leading coefficient a = 1.



Comparing Exponential and Linear Growth

Consider two social media sites which are expanding the number of users they have:

- **Site A** has 10,000 users, and expands by **adding 1,500 new users each month**
- **Site B** has 10,000 users, and expands by **increasing the number of users by 10% each month.**

The number of users for **Site A can be modeled as linear growt**h. The number of users increases by a constant number, 1500, each month. If x = the number of months that have passed and y is the number of users, the number of users after x months is $y = 10000 + 1500x$. For **site B, the user base expands by a constant percent each month**, rather than by a constant number. **Growth that occurs at a constant percent each unit of time is called exponential growth.**

We can look at growth for each site to understand the difference. The table compares the number of users for each site for 12 months. The table shows the calculations for the first 4 months only, but uses the same calculation process to complete the rest of the 12 months.

Month	Users at Site A	Users at Site B
0	10000	10000
1	$\mathbf{10000 + 1500 = 11500}$	$10000 + 10\%\text{ of }10000$ $= 10000 + 0.10(10000)$ $= 10000(1.10) = 11000$
2	$11500 + 1500 = 13000$	$11000 + 10\%\text{ of }11000$ $= 11000 + 0.10(11000)$ $= 11000(1.10) = 12100$
3	$\mathbf{13000 + 1500 = 14500}$	$12100 + 10\%\text{ of }12100$ $= 12100 + 0.10(12100)$ $= 12100(1.10) = 13310$
4	$14500 + 1500 = 16000$	$13310 + 10\%\text{ of }13310$ $= 13310 + 0.10(13310)$ $= 13310(1.10) = 14641$

Month	Users at Site A	Users at Site B
5	**17500**	**16105**
6	19000	17716
7	**20500**	**19487**
8	22000	21436
9	**23500**	**23579**
10	25000	25937
11	**26500**	**28531**
12	28000	31384

For Site B, we can re-express the calculations to help us observe the patterns and develop a formula for the number of users after x months.

- Month 1: $y = 10000(1.1) = 11000$
- Month 2: $y = 11000(1.1) = 10000(1.1)(1.1) = \mathbf{10000(1.1)^2} = 12100$
- Month 3: $y = 12100(1.1) = 10000(1.1)^2(1.1) = \mathbf{10000(1.1)^3} = 13310$
- Month 4: $y = 13310(1.1) = 10000(1.1)^3(1.1) = \mathbf{10000(1.1)^4} = 14641$

By looking at the patterns in the calculations for months 2, 3, and 4, we can generalize the formula. After x months, the number of users y is given by the function $\mathbf{y = 10000(1.1)^x}$

Using Exponential Functions to Model Growth and Decay

In **exponential growth**, the value of the dependent variable y increases at a constant percentage rate as the value of the independent variable (x or t) increases. Examples of exponential growth functions include:

- the number of residents of a city or nation that grows at a constant percent rate.
- the amount of money in a bank account that earns interest if money is deposited at a single point in time and left in the bank to compound without any withdrawals.

In **exponential decay,** the value of the dependent variable y decreases at a constant percentage rate as the value of the independent variable (x or t) increases. Examples of exponential decay functions include:

- value of a car or equipment that depreciates at a constant percent rate over time
- the amount a drug that still remains in the body as time passes after it is ingested
- the amount of radioactive material remaining over time as a radioactive substance decays.

Exponential functions often model quantities as a function of time; thus we often use the letter t as the independent variable instead of x.

The table compares exponential growth and exponential decay functions:

Exponential Growth	Exponential Decay
Quantity grows by a constant percent per unit of time	**Quantity decreases by a constant percent per unit of time**
$\mathbf{y = ab^x}$ • a is a positive number representing the initial value of the function when $x = 0$ • b is a real number that is greater than 1: $b > 1$ • the growth rate r is a positive number, $r > 0$ where $b = 1 + r$ (so that $r = b - 1$)	$\mathbf{y = ab^x}$ • a is a positive number representing the initial value of the function when $x = 0$ • b is a real number that is between 0 and 1: $0 < b < 1$ • the decay rate r is a negative number, $r < 0$ where $b = 1 + r$ (so that $r = b - 1$)

In general, **the domain of exponential functions is the set of all real numbers.** The range of an **exponential growth or decay function is the set of all positive real numbers.**

In most applications, the independent variable, x or t, represents time. When the independent variable represents time, we may choose to restrict the domain so that independent variable can have only non-negative values in order for the application to make sense. If we restrict the domain, then the range is also restricted as well.

- For an exponential growth function $y = ab^x$ with $b > 1$ and $a > 0$, if we restrict the domain so that $x \geq 0$, then the range is $y \geq a$.
- For an exponential decay function $y = ab^x$ with $0 < b < 1$ and $a > 0$, if we restrict the domain so that $x \geq 0$, then the range is $0 < y \leq a$.

Example 4.2.1

Consider the growth models for social media sites A and B, where x = number of months since the site was started and y = number of users. The number of users for Site A follows the linear growth model:

$$y = 10000 + 1500x.$$

The number of users for Site B follows the exponential growth model:

$$y = 10000(1.1^x)$$

For each site, use the function to calculate the number of users at the end of the first year, to verify the values in the table. Then use the functions to predict the number of users after 30 months.

Solution

Since x is measured in months, then $x = 12$ at the end of one year.

Linear Growth Model:

When $x = 12$ months, then $y = 10000 + 1500(12) = 28,000$ users
When $x = 30$ months, then $y = 10000 + 1500(30) = 55,000$ users

Exponential Growth Model:

When $x = 12$ months, then $y = 10000(1.1^{12}) = 31,384$ users
When $x = 30$ months, then $y = 10000(1.1^{30}) = 174,494$ users

We see that as x, the number of months, gets larger, the exponential growth function grows large faster than the linear function (even though in Example 4.2.1 the linear function initially grew faster). This is an important characteristic of exponential growth: exponential growth functions always grow faster and larger in the long run than linear growth functions.

It is helpful to use function notation, writing $y = f(t) = ab^t$, to specify the value of t at which the function is evaluated.

Example 4.2.2

A forest has a population of 2000 squirrels that is increasing at the rate of 3% per year. Let t = number of years and $y = f(t) =$ number of squirrels at time t.

a. Find the exponential growth function that models the number of squirrels in the forest at the end of t years.
b. Use the function to find the number of squirrels after 5 years and after 10 years

Solution

a. The exponential growth function is $y = f(t) = ab^t$, where $a = 2000$ because the initial population is 2000 squirrels

The annual growth rate is 3% per year, stated in the problem. We will express this in decimal form as $r = 0.03$

Then $b = 1 + r = 1 + 0.03 = 1.03$

Answer: The exponential growth function is $y = f(t) = 2000(1.03^t)$

b. After 5 years, the squirrel population is $y = f(5) = 2000(1.03^5) \approx 2319$ squirrels

After 10 years, the squirrel population is $y=f(10)=2000(1.03^{10})\approx 2688$ squirrels

Example 4.2.3

A large lake has a population of 1000 frogs. Unfortunately the frog population is decreasing at the rate of 5% per year. Let t = number of years and y = $g(t)$ = the number of frogs in the lake at time t.

a. Find the exponential decay function that models the population of frogs.
b. Calculate the size of the frog population after 10 years.

Solution

a. The exponential decay function is $y=g(t)=ab^t$, where $a=1000$ because the initial population is 1000 frogs

The annual decay rate is 5% per year, stated in the problem. The words decrease and decay indicated that r is negative. We express this as $r=-0.05$ in decimal form.

Then, $b=1+r=1+(-0.05)=0.95$

Answer: The exponential decay function is: $y=g(t)=1000(0.95^t)$

b. After 10 years, the frog population is $y=g(10)=1000(0.95^{10})\approx 599$ frogs

Example 4.2.4

A population of bacteria is given by the function $y=f(t)=100(2^t)$, where t is time measured in hours and y is the number of bacteria in the population.

a. What is the initial population?
b. What happens to the population in the first hour?
c. How long does it take for the population to reach 800 bacteria?

Solution

a. The initial population is 100 bacteria. We know this because $a=100$ and because at time $t=0$, then $f(0)=100(2^0)=100(1)=100$

b. At the end of 1 hour, the population is $y=f(1)=100(2^1)=100(2)=200$ bacteria.
The population has doubled during the first hour.

c. We need to find the time t at which $f(t)=800$. Substitute 800 as the value of y:

$$\begin{aligned} y=f(t)&=100\left(2^t\right)\\ 800&=100\left(2^t\right) \end{aligned}$$

Divide both sides by 100 to isolate the exponential expression on the one side

$$8=1\left(2^t\right)$$

8 = 2^3, so it takes $t=3$ hours for the population to reach 800 bacteria.

Two important notes about Example $4.2.4$:

- In solving $8=2^t$, we “knew” that t is 3. But we usually can **not** know the value of the variable just by looking at the equation. Later we will use logarithms to solve equations that have the variable in the exponent.
- To solve $800=100(2^t)$, we divided both sides by 100 to isolate the exponential expression 2^t. We can not multiply 100 by 2. Even if we write it as $800=100(2)^t$, which is equivalent, we still can **not** multiply 100 by 2. The exponent applies **only** to the quantity immediately before it, so the exponent t applies only to the base of 2.

Comparing Linear, Exponential and Power Functions

To identify the type of function from its formula, we need to carefully note the position that the variable occupies in the formula.

A linear function can be written in the form $\mathbf{y=ax+b}$

As we studied in chapter 1, there are other forms in which linear equations can be written, but linear functions can all be rearranged to have form $y=mx+b$.

An exponential function has form $\mathbf{y=ab^x}$

The variable x is in the exponent. The base b is a positive number.

- If $b>1$, the function represents exponential growth.
- If $0<b<1$, the function represents exponential decay

A power function has form $\mathbf{y=cx^P}$

The variable x is in the base. The exponent p is a non-zero number.

We compare three functions:

- linear function $y=f(x)=2x$
- exponential function $y=g(x)=2^x$
- power function $y=h(x)=x^2$

$\mathbf{x}$	$\mathbf{y=f(x)=2x}$	$\mathbf{y=g(x)=2^x}$	$\mathbf{y=h(x)=x^2}$
0	0	1	0
1	2	2	1
2	4	4	4
3	6	8	9
4	8	16	16
5	10	32	25
6	12	64	36
10	20	1024	100
Type of function	Linear $y=mx+b$	Exponential $y=ab^x$	Power $y=cx^P$
How to recognize this type of function.	all terms are first degree; m is slope, b is the y intercept	base is a number $b>0$; the variable is in the exponent	variable is in the base; exponent is a number p $\neq 0$
	For equal intervals of change in x, y increases by a constant amount	For equal intervals of change in x, y increases by a constant ratio	

For the functions in the previous table: linear function $y=f(x)=2x$, exponential function $y=g(x)=2^x$, and power function $y=h(x)=x^2$, if we restrict the domain to $x\geq 0$ only, then all these functions are growth functions. When $x\geq 0$, the value of y increases as the value of x increases.

The exponential growth function grows large faster than the linear and power functions, as x gets large. This is always true of exponential growth functions, as x gets large enough.

Example 4.2.5

Classify the functions below as exponential, linear, or power functions.

a. $y=10x^3$
b. $y=1000-30x$
c. $y=1000\left(1.05^x\right)$
d. $y=500(0.75^x)$

e. $y=10\sqrt[3]{x}=x^{1/3}$
f. $y=5x-1$
g. $y=6/x^2=6x^{-2}$

Solution:

The exponential functions are

c. $y=1000\left(1.05^x\right)$ The variable is in the exponent; the base is the number $b=1.05$

d. $y=500(0.75^x)$\) The variable is in the exponent; the base is the number $b=0.75$

The linear functions are

b. $y=1000-30x$

f. $y=5x-1$

The power functions are

a. $y=10x^3$ The variable is the base; the exponent is a fixed number, $p=3$.

e. $y=10\sqrt[3]{x}=x^{1/3}$ The variable is the base; the exponent is a number, $p=1/3$.

g. $y=6/x^2=6x^{-2}$ The variable is the base; the exponent is a number, $p=-2$.

NATURAL BASE: *e*

The number *e* is often used as the base of an exponential function. *e* is called the natural base.

***e* is approximately 2.71828**

e is an irrational number with an infinite number of decimals; the decimal pattern never repeats.

Section 6.2 includes an example that shows how the value of *e* is developed and why this number is mathematically important. Students studying Finite Math should already be familiar with the number *e* from their prerequisite algebra classes.

When *e* is the base in an exponential growth or decay function, it is referred to as **continuous growth or continuous decay**. We will use *e* in chapter 6 in financial calculations when we examine interest that compounds continuously.

Any exponential function can be written in the form $\mathbf{y=ae^{kx}}$

k is called the continuous growth or decay rate.

- If $k>0$, the function represents exponential growth
- If $k<0$, the function represents exponential decay

a is the initial value

We can rewrite the function in the form $\mathbf{y=ab^x}$, where $\mathbf{b=e^k}$

In general, if we know one form of the equation, we can find the other forms. For now, we have not yet covered the skills to find k when we know b. After we learn about logarithms later in this chapter, we will find k using natural log: $k=\ln b$.

The table below summarizes the forms of exponential growth and decay functions.

	$y = ab^x$	**$y = a(1+r)^x$**	**$y = ae^{kx}$, $k \neq 0$**
Initial value	a>0	a>0	a>0
Relationship between b, r, k	b > 0	b=1+ r	$b = e^k$ and k = ln b
Growth	b > 1	r > 0	k > 0
Decay	0 < b < 1	r < 0	k < 0

The value of houses in a city are increasing at a continuous growth rate of 6% per year. For a house that currently costs $400,000:

a. Write the exponential growth function in the form $y = ae^{kx}$.
b. What would be the value of this house 4 years from now?
c. Rewrite the exponential growth function in the form $y = ab^x$.
d. Find and interpret r.

Solution

a. The initial value of the house is a = $400000

The problem states that the **continuous** growth rate is 6% per year, so k = 0.06

The growth function is : $y = 400000e^{0.06x}$

b. After 4 years, the value of the house is $y = 400000e^{0.06(4)}$ = $508,500.

c. To rewrite $y = 400000e^{0.06x}$ in the form $y = ab^x$, we use the fact that $b = e^k$.

$$\begin{aligned}\mathrm{b} &= e^{0.06}\\ \mathrm{b} &= 1.06183657 \approx 1.0618\\ \mathrm{y} &= 400000(1.0618)^{\mathrm{x}}\end{aligned}$$

d. To find r, we use the fact that $b = 1 + r$

$$\begin{aligned}\mathrm{b} &= 1.0618\\ 1 + \mathrm{r} &= 1.0618\\ \mathrm{r} &= 0.0618\end{aligned}$$

The value of the house is **increasing** at an **annual rate** of 6.18%.

Example 4.2.7

Suppose that the value of a certain model of new car decreases at a continuous decay rate of 8% per year. For a car that costs $20,000 when new:

a. Write the exponential decay function in the form $y = ae^{kx}$.
b. What would be the value of this car 5 years from now?
c. Rewrite the exponential decay function in the form $y = ab^x$.
d. Find and interpret r.

Solution

a. The initial value of the car is a = $20000

The problem states that the **continuous** decay rate is 8% per year, so k = -0.08

The growth function is : $y = 20000e^{-0.08x}$

b. After 5 years, the value of the car is $y = 20000e^{-0.08(5)}$ = $13,406.40.

c. To rewrite $y = 20000e^{-0.08x}$ in the form $y = ab^x$, we use the fact that $b = e^k$.

$$\begin{aligned}\mathrm{b} &= e^{-0.08}\\ \mathrm{b} &= 0.9231163464 \approx 0.9231\\ \mathrm{y} &= 20000(0.9231)^{\mathrm{x}}\end{aligned}$$

d. To find r, we use the fact that $b = 1 + r$

$$\begin{aligned}\mathrm{b} &= 0.9231\\ 1 + \mathrm{r} &= 0.9231\\ \mathrm{r} &= 0.9231 - 1 = -0.0769\end{aligned}$$

The value of the car is **decreasing** at an **annual rate** of 7.69%.

4.2.1: Exponential Growth and Decay Models (Exercises)

SECTION 4.2 PROBLEM SET: EXPONENTIAL GROWTH AND DECAY FUNCTIONS

Identify each as an exponential, linear, or power function.

1. $y=640\left(1.25^{x}\right)$	2. $y=640\left(x^{1.25}\right)$
3. $y=640(1.25x)$	4. $y=1.05x-2.5$
5. $y=90-(4/5)x$	6. $y=42\left(0.92^{x}\right)$
7. $y=37\left(x^{0.25}\right)$	8. $y=4(1/3)^{x}$

Indicate if the function represents exponential growth or exponential decay.

9. $y=127\mathrm{e}^{-0.35\mathrm{t}}$	10. $y=70\left(0.8^{t}\right)$
11. $y=453\left(1.2^{t}\right)$	12. $y=16e^{0.2t}$

In each of the following, y is an exponential function of t stated in the form $y=ae^{kt}$ where t represents time measured in years.. For each:

a. re-express each function in the form $y=ab^{t}$ (state the value of b accurate to 4 decimal places)
b. state the annual growth rate or annual decay rate as a percent, accurate to 2 decimal places

13. $y=127\mathrm{e}^{-0.35\mathrm{t}}$	14. $y=16e^{0.4t}$
15. $y=17250\mathrm{e}^{0.24\mathrm{t}}$	16. $y=4700\mathrm{e}^{-0.07t}$

Identify if the function represents exponential growth, exponential decay, linear growth, or linear decay. In each case write the function and find the value at the indicated time.

17. A house was purchased for \$350,000 in the year 2010. The value has been increasing by \$7,000 per year. Write the function and find the value of the house after 5 years.	18. A house was purchased for \$350,000 in the year 2010. The value has been increasing at the rate of 2% per year. Write the function and find the value of the house after 5 years.
19. A lab purchases new equipment for \$50,000. Its value depreciates over time. The value decreases at the rate of 6% annually. Write the function and find the value after 10 years.	20. A lab purchases new equipment for \$50,000. Its value depreciates over time. The value decreases by \$3000 annually. Write the function and find the value after 10 years
21. A population of bats in a cave has 200 bats. The population is increasing by 10 bats annually. Write the function. How many bats live in the cave after 7 years?	22. A population of bats in a cave has 200 bats. The population is increasing at the rate of 5% annually. Write the function. How many bats live in the cave after 7 years?
23. A population of a certain species of bird in a state park has 300 birds. The population is decreasing at the rate of 7% year. Write the function. How many birds are in the population after 6 years?	24. A population of a certain species of bird in a state park has 300 birds. The population decreases by 20 birds per year. Write the function. How many birds are in the population after 6 years?

In problems 25-28, the problem represents exponential growth or decay and states the CONTINUOUS growth rate or continuous decay rate. Write the exponential growth or decay function and find the value at the indicated time.

Hint: Use the form of the exponential function that is appropriate when the CONTINUOUS growth or decay rate is given.

25. A population of 400 microbes increases at the **continuous** growth rate of 26% per day. Write the function and find the number of microbes in the population at the end of 7 days.	26. The price of a machine needed by a production factory is $28,000. The business expects to replace the machine in 4 years. Due to inflation the price of the machine is increasing at the **continuous** rate of 3.5% per year. Write the function and find the value of the machine 4 years from now.
27. A population of an endangered species consists of 4000 animals of that species. The population is decreasing at the continuous rate of 12% per year. Write the function and find the size of the population at the end of 10 years.	28. A business buys a computer system for $12000. The value of the system is depreciating and decreases at the continuous rate of 20% per year. Write the function and find the value at the end of 3 years.

4.3: Graphs and Properties of Exponential Growth and Decay Functions/Use TI-84 calculator

Learning Objectives

In this section, you will:

1. Examine properties of exponential functions
2. Examine graphs of exponential functions
3. How to graph and evaluate an exponential function using the TI-84 calculator.

An exponential function can be written in forms

$$f(x)=ab^x=a(1+r)^x=ae^{kx}$$

where

- **a is the initial value** because $f(0)=a$. In the growth and decay models that we examine in this finite math textbook, $a>0$.
- **b is often called the growth factor**. We restrict b to be positive ($b>0$) because even roots of negative numbers are undefined. We want the function to be defined for all values of x, but b^x would be undefined for some values of x if $b<0$.
- **r is called the growth or decay rate**. In the formula for the functions, we use r in decimal form, but in the context of a problem we usually state r as a percent.
- **k is called the continuous growth rate or continuous decay rate.**

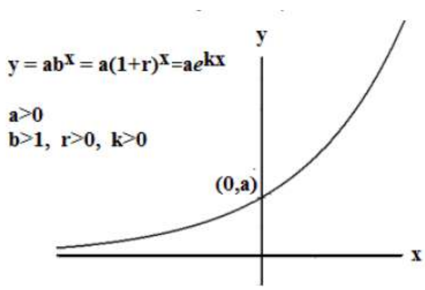
Properties of Exponential Growth Functions

The function $y=f(x)=ab^x$ represents growth if $b>1$ and $a>0$.

The growth rate r is positive when $b>1$. Because $b=1+r>1$, then $r=b-1>0$

The function $y=f(x)=ae^{kx}$ represents growth if $k>0$ and $a>0$.

The function is an increasing function; y increases as x increases.



- Domain: { all real numbers} ; all real numbers can be input to an exponential function
- Range: If $a>0$, the range is {positive real numbers} The graph is always above the x axis.
- Horizontal Asymptote: when $b>1$, the horizontal asymptote is the negative x axis, as x becomes large negative. Using mathematical notation: as x → −∞, then y → 0.
- The vertical intercept is the point $(0,a)$ on the y-axis. There is no horizontal intercept because the function does not cross the x-axis.

Properties of Exponential Decay Functions

The function $y=f(x)=ab^x$ function represents decay if $0<b<1$ and $a>0$.

The growth rate r is negative when $0<b<0$. Because $b=1+r<1$, then $r=b-1<0$.

The function $y=f(x)=ae^{kx}$ function represents decay if $k<0$ and $a>0$.

The function is a decreasing function; y decreases as x increases.

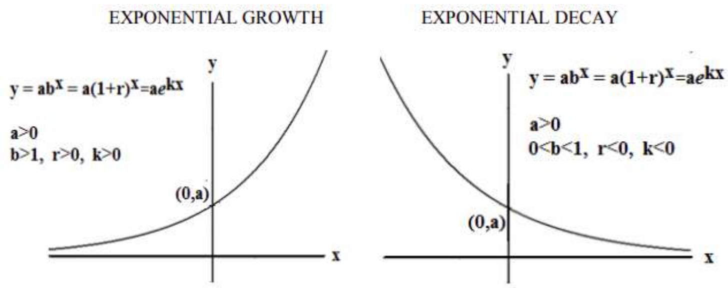
Domain: { all real numbers} ; all real numbers can be input to an exponential function

Range: If $a>0$, the range is { positive real numbers } The graph is always above the x axis.

Horizontal Asymptote: when $b < 1$, the horizontal asymptote is the positive x axis as x becomes large positive. Using mathematical notation: as x → ∞, then y → 0.

The vertical intercept is the point $(0, a)$ on the y-axis. There is no horizontal intercept because the function does not cross the x-axis.

The graphs for exponential growth and decay functions are displayed below for comparison.



How to find an exponential function from two points

Example 4.3.0

Suppose the following points define an exponential function that describes the exponential growth or decline of customers coming to a restaurant if the initial point is (0, 1). Find the exponential function.

a) (4, 625)

b) (-3, 343)

Solution

a) The general exponential function is y = ab^x.

- Substituting the point (0, 1), you get 1 = ab^0.
- We know that b^0 = 1; therefore, the exponential function is y = b^x.
- Next, substitute the point (4, 625) in the exponential function 625 = b^4.
- Take the 4th root of each side $(625)^{(1/4)}$ = b
- b = 5
- The exponential equation is y = 5^x.

b) The general exponential function is y = abx.

- Substituting the point (0, 1), you get 1 = ab^0.
- We know that b^0 = 1; therefore, the exponential function is y = b^x.
- Next, substitute the point (-3, 343) in the exponential function 343 = b^{-3}.
- A negative exponent indicates the b should be in the denominator 343 = $(1/b)^3$.
- Take the cubic root of both sides $(343)^{(1/3)}$ = (1/b).
- 7 = (1/b)
- Multiply both sides by b and solve for b. 7b = (1/b)(b) = 1. b = 1/7
- The exponential equation is y = $(1/7)^x$.

An Exponential Function is a One-to-One Function

Observe that in the graph of an exponential function, each y value on the graph occurs only once. Therefore, every y value in the range corresponds to only one x value. So, for any particular value of y, you can use the graph to see which value of x is the input to produce that y value as output. This property is called "**one-to-one**".

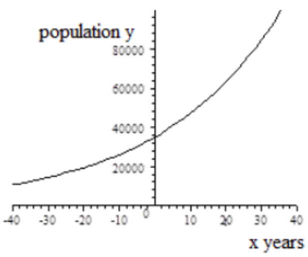
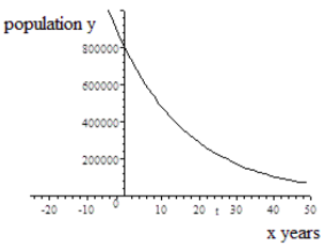
Because for each value of the output y, you can uniquely determine the value of the corresponding input x, thus every exponential function has an inverse function. The inverse function of an exponential function is a logarithmic function, which we will investigate in the next section.

Example 4.3.1

x years after the year 2015, the population of the city of Fulton is given by the function $y = f(x) = 35000(1.03^x)$. x years after the year 2015, the population of the city of Greenville is given by the function $y = g(x) = 80000(0.95^x)$. Compare the graphs of these functions.

Solution

The graphs below were created using computer graphing software. You can also graph these functions using a graphing calculator.

Population of Fulton	Population of Greenville
$y = f(x) = 35000\,(1.03^x)$	$y = g(x) = 80000\,(0.95^x)$
Fulton's population is increasing. $b = 1.03 > 1$ and $r = 0.03 > 0$ Exponential Growth	Greenville’s population is decreasing. $b = 0.95 < 1$ and $r = -0.05 < 0$ Exponential Decay
	
y-intercept: (0, 35000) The initial population in 2015 is 35000.	**y-intercept:** (0,80000) The initial population in 2015 is 80000.
Horizontal Asymptote: The negative x axis is the horizontal asymptote. y → 0 as x → − ∞	**Horizontal Asymptote:** The positive x axis is the horizontal asymptote. y → 0 as x → ∞

Domain: In general, the domain of both functions $y = f(x) = 35000(1.03^x)$ and $y = g(x) = 80000(0.95^x)$ is the set of all real numbers.

Range: The range of both functions is the set of positive real numbers. Both graphs always lie above the x-axis.

Domain and Range in context of this problem:

The functions represent population size as a function of time ***after*** the year 2015 . We restrict the domain in this context, using the “practical domain” as the set of all non-negative real numbers: $x \geq 0$. Then we would consider only the portion of the graph that lies in the first quadrant.

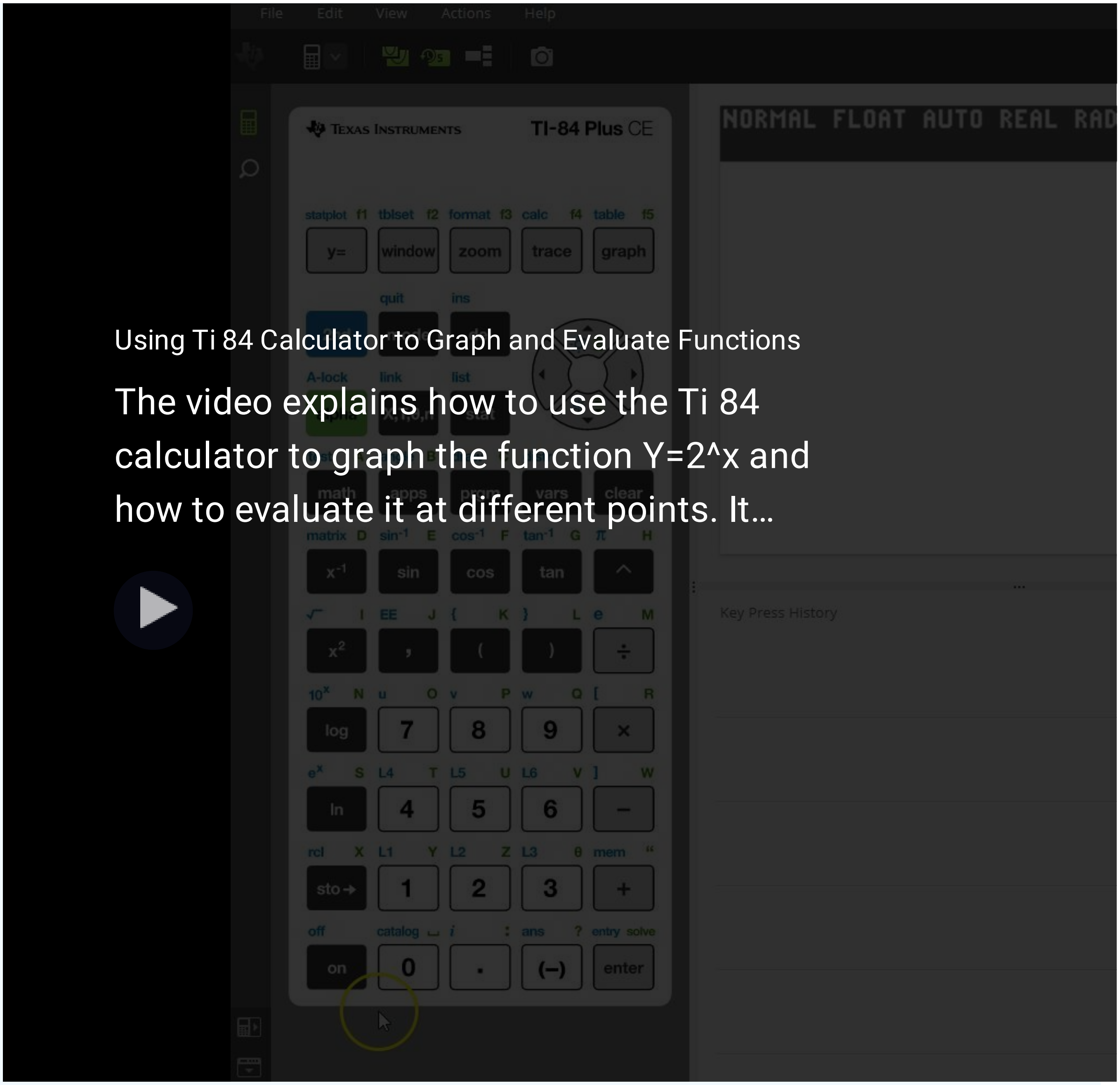
- If we restrict the domain to $x \geq 0$ for the growth function $y = f(x) = 35000(1.03^x)$, then the range for the population of Fulton is $y \geq 35,000$
- If we restrict the domain to $x \geq 0$ for the decay function $y = g(x) = 80000(0.95^x)$, then the range for the population of Greenville is $y \leq 80,000$.

How to graph an exponential function

Example 4.3.1

Graph y = 2^x using the TI-84 calculator. Evaluate the exponential function at x = 25.4 and x = 0.25.

Solution



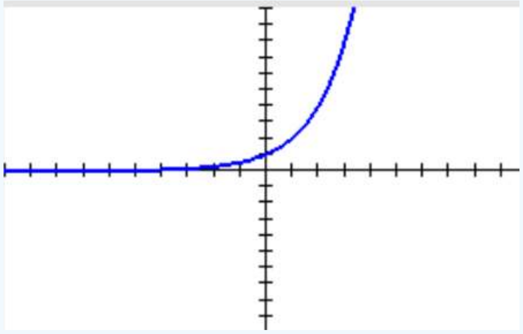
1) Click the y= button.

2) Hit the Clear button to erase any function entered.

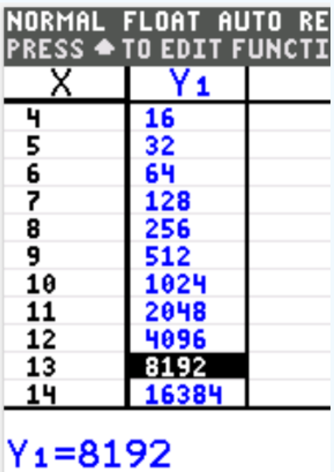
3) Enter 2 and hit the ^ button and hit the X button 

.

4) Hit the Graph button to view the graph of y = 2^x.



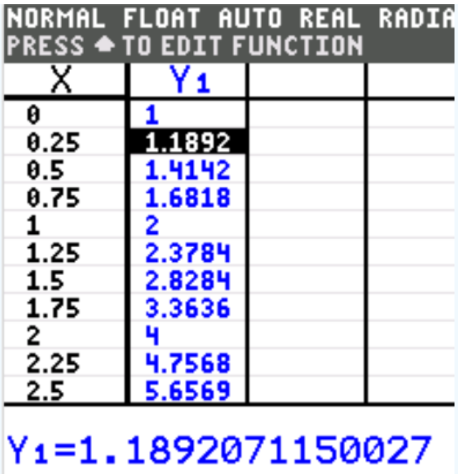
5) To evaluate $y = 2^x$, hit the 2nd button and the graph button. Scroll to the x value and move to the y column to find the answer (i.e. x = 13).



6) To set the X values to what you want, hit the 2nd button and the Window button 

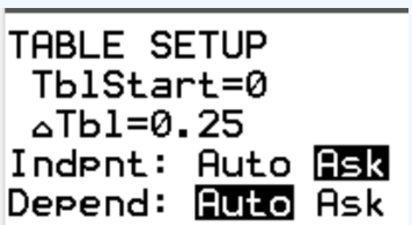
. Set the beginning value of 0, and the amount to increase the x value by 0.25 then hit the 2nd button then the Graph button.

Scroll down to x = 0.25 and move to the Y_1 value. The Y value is 1.1892071150027. Round it to the desired decimal places.

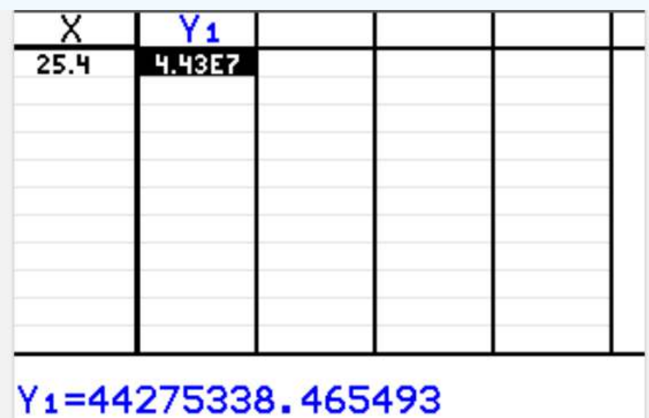


7) To set the table to ask you for a specific value. Hit the 2nd button and the Windows key 

. Scroll down to Indpnt and highlight Ask then hit the enter button.



Then hit the 2nd button and the graph button. Enter 25.4 and hit enter button. Move over to the Y_1 column to see the answer.



Round to the desired decimal places.

4.3.1: Graphs and Properties of Exponential Growth and Decay Functions (Exercises)

SECTION 4.3 PROBLEM SET: GRAPHS AND PROPERTIES OF EXPONENTIAL GROWTH AND DECAY FUNCTIONS

In questions 1-4, let t = time in years and y = the value at time t or y = the size of the population at time t. The domain is the set of non-negative values for t; $t \geq 0$, because y represents a physical quantity and negative values for time may not make sense. For each question:

a. Write the formula for the function in the form $y = ab^t$
b. Sketch the graph of the function and mark the coordinates of the y-intercept.

1. A house was purchased for \$350,000 in the year 2010. The value has been increasing at the rate of 2% per year.	2. A population of a certain species of bird in a state park has 300 birds. The population is decreasing at the rate of 7% year.
3. A lab buys equipment \$50,000. Its value depreciates over time. The value decreases at the rate of 6% annually.	4. A population of bats in a cave has 200 bats. The population is increasing at the rate of 5% annually.

In questions 5-8, let t = time in years and y = the value at time t or y = the size of the population at time t. The domain is the set of non-negative values for t; $t \geq 0$, because y represents a physical quantity and negative values for time may not make sense. For each question:

a. Write the formula for the function in the form $y = ae^{kt}$
b. Sketch the graph of the function and mark the coordinates of the y-intercept.

5. A population of 400 microbes increases at the **continuous** growth rate of 26% per day.	6. The price of a machine needed by a production factory is \$28,000. Due to inflation the price of the machine is increasing at the **continuous** rate of 3.5% per year.
7. A population of an endangered species consists of 4000 animals of that species. The population is decreasing at the **continuous** rate of 12% per year.	8. A business buys a computer system for \$12000. The value of the system is depreciating and decreases at the **continuous** rate of 20% per year.

For questions 9-12

a. Sketch a graph of exponential function.
b. List the coordinates of the y intercept.
c. State the equation of any asymptotes and state the whether the function approaches the asymptote as x $\rightarrow \infty$ or as x$\rightarrow -\infty$.
d. State the domain and range.

9. $y = 10\left(1.5^x\right)$	10. $y = 10\left(e^{1.2x}\right)$
11. $y = 32\left(0.75^x\right)$	12. $y = 200\left(e^{-0.5x}\right)$

4.4: Logarithms and Logarithmic Functions

Learning Objectives

In this section you will learn

1. the definition of logarithmic function as the inverse of the exponential function
2. to write equivalent logarithmic and exponential expressions
3. the definition of common log and natural log
4. properties of logs
5. to evaluate logs using the change of base formula

The Logarithm

Suppose that a population of 50 flies is expected to double every week, leading to a function of the form $f(x)=50(2)^{x}$, where x represents the number of weeks that have passed. When will this population reach 500?

Trying to solve this problem leads to

$$500=50(2)^{x}$$

Dividing both sides by 50 to isolate the exponential leads to

$$10=2^{x}.$$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$ above. We know that $2^3 = 8$ and $2^4 = 16$, so it is clear that x must be some value between 3 and 4 since $g(x) = 2^x$ is increasing. We could use technology to create a table of values or graph to better estimate the solution, but we would like to find an algebraic way to solve the equation.

We need an inverse operation to exponentiation in order to solve for the variable if the variable is in the exponent. As we learned in algebra class (prerequisite to this finite math course), the inverse function for an exponential function is a logarithmic function.

We also learned that an exponential function has an inverse function, because each output (y) value corresponds to only one input (x) value. The name given this property was "one-to-one".

Source: The material in this section of the textbook originates from David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, "Chapter 4: Exponential and Logarithmic Functions," licensed under a Creative Commons CC BY-SA 3.0 license. The material here is based on material contained in that textbook but has been modified by Roberta Bloom, as permitted under this license.

Logarithm

The logarithm (base b) function, written $\log_b(x)$, is the inverse of the exponential function (base b), b^x.

$$\mathbf{y=\log_b(x) \quad \text{is equivalent to} \quad b^y=x}$$

In general, the statement $b^{a}=c$ is equivalent to the statement $\log_{b}(c)=a$.

Note: The base b must be positive: $b>0$

Inverse Property of Logarithms

Since the logarithm and exponential are inverses, it follows that:

$$\log_b(b^x) \quad \text{and } b^{\log_b(x)}=x$$

Since log is a function, it is most correctly written as $\log_b(c)$, using parentheses to denote function evaluation, just as we would with *f(c)*. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_b c$.

Example 4.4.1

Write these exponential equations as logarithmic equations:

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-3} = \frac{1}{1000}$

Solution

a. $2^3 = 8$ can be written as a logarithmic equation as $\log_2(8) = 3$

b. $5^2 = 25$ can be written as a logarithmic equation as $\log_5(25) = 2$

c. $10^{-3} = \frac{1}{1000}$ can be written as a logarithmic equation as $\log_{10}\left(\frac{1}{1000}\right) = -3$

Example 4.4.2

Write these logarithmic equations as exponential equations:

a. $\log_6(\sqrt{6}) = \frac{1}{2}$
b. $\log_3(9) = 2$

Solution

a. $\log_6(\sqrt{6}) = \frac{1}{2}$ can be written as an exponential equation as $6^{\frac{1}{2}} = \sqrt{6}$
b. $\log_3(9) = 2$ can be written as an exponential equation as $3^2 = 9$

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

Example 4.4.3

Solve $\log_4(x) = 2$ for x.

Solution

By rewriting this expression as an exponential, $4^2 = x$, so $x = 16$

Example 4.4.4

Solve $2^x = 10$ for x.

Solution

By rewriting this expression as a logarithm, we get $x = \log_2(10)$

While this does define a solution, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful—often we really need a decimal approximation to the solution. Luckily, this is a task that calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases: base 10 and base e. Happily, this ends up not being a problem, as we'll see soon that we can use a "change of base" formula to evaluate logarithms for other bases.

Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written $\log(x)$ and sometimes like $\log_{10}(x)$. If the base is not indicated in the log function, then the base b used is $b = 10$.

The **natural log** is the logarithm with base e, and is typically written $\ln(x)$.

Note that for any other base b, other than 10, the base must be indicated in the notation $\log_b(x)$.

Example 4.4.5

Evaluate $\log(1000)$using the definition of the common log.

Solution

The table shows values of the common log

number	number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

To evaluate log(1000), we can say

$$x=\log(1000)$$

Then rewrite the equation in exponential form using the common log base of 10

$$10^x=1000$$

From this, we might recognize that 1000 is the cube of 10, so

$$x=3$$

Alternatively, we can use the inverse property of logs to write

$$\log_{10}(10^3)=3$$

Example 4.4.6

Evaluate $\log\left(\frac{1}{1,000,000}\right)$

Solution

To evaluate log(1/1,000,000), we can say

$$x=\log(1/1,000,000)=\log\left(1/10^6\right)=\log\left(10^{-6}\right)$$

Then rewrite the equation in exponential form: $10^x=10^{-6}$

Therefore $x=-6$

Alternatively, we can use the inverse property of logs to find the answer:

$$\log_{10}\left(10^{-6}\right)=-6$$

Example 4.4.7

Evaluate

a. $\ln e^5$
b. $\ln\sqrt{e}$

Solution

a. To evaluate $\ln e^5$, we can say

$$x = \ln e^5$$

Then rewrite into exponential form using the natural log base of e

$$e^x = e^5$$

Therefore $x = 5$.

Alternatively, we can use the inverse property of logs to write $\ln\left(e^5\right) = 5$.

b. To evaluate $\ln \sqrt{e}$, we recall that roots are represented by fractional exponents

$$\mathrm{x} = \ln \sqrt{e} = \ln(\sqrt{e}) = \ln\left(e^{1/2}\right)$$

Then rewrite into exponential form using the natural log base of e

$$\mathrm{e}^x = \mathrm{e}^{1/2}$$

Therefore $x = 1/2$

Alternatively, we can use the inverse property of logs to write

$$(\ln\left(\mathrm{e}^{1/2}\right) = 1/2$$

Example 4.4.8

Evaluate the following using your calculator or computer:

a. $\log 500$
b. $\ln 500$

Solution

a. Using the LOG key on the calculator to evaluate logarithms in base 10, we evaluate LOG(500)

Answer: $\log 500 \approx 2.69897$

b. Using the LN key on the calculator to evaluate natural logarithms, we evaluate LN(500)

Answer: $\ln 500 \approx 6.214608$

Exercise 4.4.1

Convert $8^x = 272$ to an logarithmic function.

Answer

$\log_8(272) = x$

Exercise 4.4.2

Convert from $\log_e(456) = x$ to an exponential function.

Answer

$e^x = 456$

Some Properties of Logarithms

We often need to evaluate logarithms using a base other than 10 or *e*. To find a way to utilize the common or natural logarithm functions to evaluate expressions like $\log_2(10)$, we need some additional properties.

> **Properties of logs: Exponential Property**
>
> $$\log_b(A^q) = q\log_b(A)$$

The exponent property allows us to find a method for changing the base of a logarithmic expression.

> **Properties of Logs: Change of Base**
>
> $$\log_b(A) = \frac{\log_c(A)}{\log_c(b)} \text{ for any bases } b, c > 0$$

To show why these properties are true, we offer proofs.

Proof of Exponent Property: $\log_b(A^q) = q\log_b(A)$

Since the logarithmic and exponential functions are inverses,

$$\log_b(A^q) = \mathrm{A}$$

So

$$A^q = \left(b^{\log_b A}\right)^q$$

Utilizing the exponential rule that states $(x^p)^q = x^{pq}$, we get

$$A^q = \left(b^{\log_b A}\right)^q = b^{q\log_b A}$$

Then

$$\log_b A^q = \log_b b^{q\log_b A}$$

Again utilizing the inverse property on the right side yields the result

$$\log_b A^q = q\log_b A$$

Proof of Change of Base Property: $\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$ for any bases b, $c > 0$

Let $\log_b(A) = x$.

Rewriting as an exponential gives $b^x = A$.

Taking the log base c of both sides of this equation gives $\log_c b^x = \log_c A$.

Now utilizing the exponent property for logs on the left side,

$$x\log_c b = \log_c A$$

Dividing, we obtain $x = \frac{\log_c(A)}{\log_c(b)}$ which is the change of base formula.

Evaluating Logarithms

With the change of base formula, $\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$ for any bases b, $c > 0$, we can finally find a decimal approximation to our question from the beginning of the section.

✔ Example 4.4.9

Solve $2^x = 10$ for x.

Solution

Rewrite exponential equation $2^x = 10$ as a logarithmic equation

$$x = \log_2(10)$$

Using the change of base formula, we can rewrite log base 2 as a logarithm of any other base. Since our calculators can evaluate natural log, we can choose to use the natural logarithm, which is the log base *e*:

Using our calculators to evaluate this, $\frac{\ln(10)}{\ln(2)} = \mathrm{LN}(10)/\mathrm{LN}(2) \approx 3.3219$

This finally allows us to answer our original question from the beginning of this section:
For the population of 50 flies that doubles every week, it will take approximately 3.32 weeks to grow to 500 flies.

✔ Example 4.4.10

Evaluate $\log_5(100)$ using the change of base formula.

Solution

We can rewrite this expression using any other base.

Method 1: We can use natural logarithm base *e* with the change of base formula

$$\log_5(100) = \frac{\ln(100)}{\ln(5)} = \mathrm{LN}(100)/\mathrm{LN}(5) \approx 2.861$$

Method 2: We can use common logarithm base 10 with the change of base formula,

$$\log_5(100) = \frac{\log(100)}{\log(5)} = \mathrm{LOG}(100)/\mathrm{LOG}(5) \approx 2.861$$

We summarize the relationship between exponential and logarithmic functions

Logarithms

The logarithm (base *b*) function, written $\log_b(x)$, is the inverse of the exponential function (base *b*), b^x.

$$\mathbf{y = \log_b(x) \quad \text{is equivalent to} \quad b^y = x}$$

In general, the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Note: The base b must be positive: b>0

Inverse Property of Logarithms

Since the logarithm and exponential are inverses, it follows that:

$$\log_b(b^x) \quad \text{and } b^{\log_b(x)} = x$$

Properties of Logs: Exponential Property: $\log_b(A^q) = q\log_b(A)$

Properties of Logs: Change of Base: $\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$ for any base $b, c > 0$

The inverse, exponential and change of base properties above will allow us to solve the equations that arise in problems we encounter in this textbook. For completeness, we state a few more properties of logarithms

Sum of Logs Property: $\log_b(A) + \log_b(C) = \log_b(AC)$

Difference of Logs Property: $\log_b(A)-\log_b(C)=\log_b\left(\frac{A}{C}\right)$

Logs of Reciprocals: $\log_b\left(\frac{1}{C}\right)=-\log_b(C)$

Reciprocal Bases: $\log_{1/b}C=-\log_b(C)$

Source: The material in this section of the textbook originates from David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, "Chapter 4: Exponential and Logarithmic Functions," licensed under a Creative Commons CC BY-SA 3.0 license. The material here is based on material contained in that textbook but has been modified by Roberta Bloom, as permitted under this license.

4.4.1: Logarithms and Logarithmic Functions (Exercises)

SECTION 4.4 PROBLEM SET: LOGARITHMS AND LOGARITHMIC FUNCTIONS

Rewrite each of these exponential expressions in logarithmic form:

1. $3^4 = 81$	2. $10^5 = 100,000$
3. $5^{-2} = 0.04$	4. $4^{-1} = 0.25$
5. $16^{1/4} = 2$	6. $9^{1/2} = 3$

Rewrite each of these logarithmic expressions in exponential form:

7. $\log_5 625 = 4$	8. $\log_2(1/32) = -5$
9. $\log_{11} 1331 = 3$	10. $\log_{10} 0.0001 = -4$
11. $\log_{64} 4 = 1/3$	12. $\ln\sqrt{e} = \frac{1}{2}$

If the expression is in exponential form, rewrite it in logarithmic form.

If the expression is in logarithmic form, rewrite it in exponential form.

13. $5^x = 15625$	14. $x = 9^3$
15. $\log_5 125 = x$	16. $\log_3 x = 5$
17. $\log_{10} y = 4$	18. $\mathrm{e}^{\mathrm{x}} = 10$
19. $\ln x = -1$	20. $e^5 = y$

For each equation, rewrite in exponential form and solve for x.

21. $\log_5(x) = 3$	22. $\log_2(x) = -2$
23. $\log_{10}(x) = -3$	24. $\log_3(x) = 6$
25. $\log_{25}(x) = 1/2$	26. $\log_{64}(x) = 1/3$

Evaluate without using your calculator.

27. $\ln\sqrt[3]{e}$	28. $\ln\frac{1}{e^2}$
29. $\ln \mathrm{e}^{10}$	30. $\log_{10}(10^e)$

For problems 31 – 38: Evaluate using your calculator. Use the change of base formula if needed

31. $\log 20$	32. $\ln 42$
33. $\ln 2.9$	34. $\log 0.5$
35. $\log_4 36$	36. $\log_7 100$
37. $\log_{105} 3.5$	38. $\log_{1.067} 2$

4.5: Graphs and Properties of Logarithmic Functions

Learning Objectives

In this section, you will:

1. examine properties of logarithmic functions
2. examine graphs of logarithmic functions
3. examine the relationship between graphs of exponential and logarithmic functions

Recall that the exponential function $f(x) = 2^x$ produces this table of values

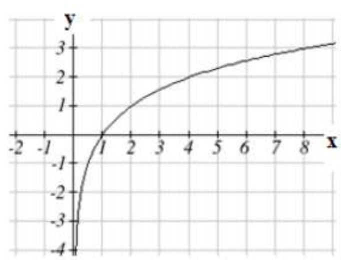
x	-3	-2	-1	0	1	2	3
$f(x)$	1/8	1/4	1/2	1	2	4	8

Since the logarithmic function is an inverse of the exponential, $g(x) = \log_2(x)$ produces the table of values

x	1/8	1/4	1/2	1	2	4	8
$g(x)$	-3	-2	-1	0	1	2	3

In this second table, notice that

1. As the input increases, the output increases.
2. As input increases, the output increases more slowly.
3. Since the exponential function only outputs positive values, the logarithm can only accept positive values as inputs, so the domain of the log function is $(0, \infty)$.
4. Since the exponential function can accept all real numbers as inputs, the logarithm can have any real number as output, so the range is all real numbers or $(-\infty, \infty)$.



Plotting the graph of $g(x) = \log_2(x)$ from the points in the table , notice that as the input values for x approach zero, the output of the function grows very large in the negative direction, indicating a vertical asymptote at $x = 0$.

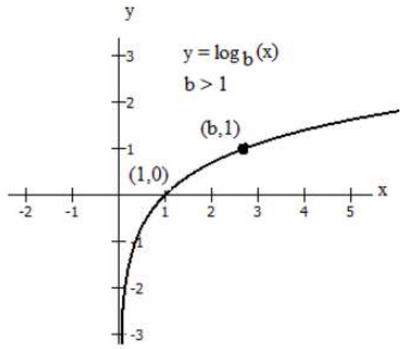
In symbolic notation we write

as $x \to 0^+$, $f(x) \to -\infty$

and as $x \to \infty$, $f(x) \to \infty$

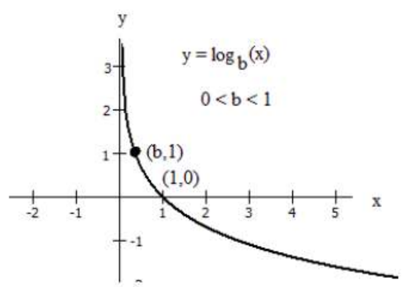
Source: The material in this section of the textbook originates from David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, "Chapter 4: Exponential and Logarithmic Functions," licensed under a Creative Commons CC BY-SA 3.0 license. The material here is based on material contained in that textbook but has been modified by Roberta Bloom, as permitted under this license.

Graphically, in the function $g(x) = \log_b(x)$, $b > 1$, we observe the following properties:



- The graph has a horizontal intercept at (1, 0)
- The line x = 0 (the y-axis) is a vertical asymptote; as $x \to 0^+$, $y \to -\infty$
- The graph is increasing if $b > 1$
- The domain of the function is $x > 0$, or (0, ∞)
- The range of the function is all real numbers, or $(-\infty, \infty)$

However if the base b is less than 1, 0 < b < 1, then the graph appears as below.
This follows from the log property of reciprocal bases : $\log_{1/b} C = -\log_b(C)$



- The graph has a horizontal intercept at (1, 0)
- The line x = 0 (the y-axis) is a vertical asymptote; as $x \to 0^+$, $y \to \infty$
- The graph is decreasing if 0 < b < 1
- The domain of the function is x > 0, or (0, ∞)
- The range of the function is all real numbers, or $(-\infty, \infty)$

When graphing a logarithmic function, it can be helpful to remember that the graph will pass through the points (1, 0) and (b, 1).

Finally, we compare the graphs of $y = b^x$ and $y = \log_b(x)$, shown below on the same axes.

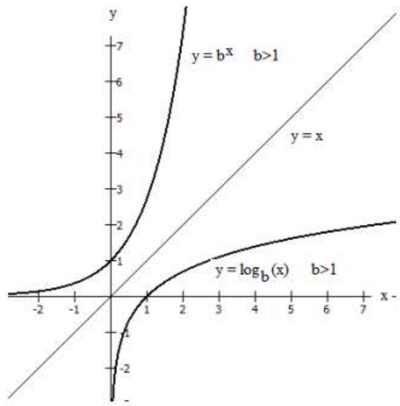
Because the functions are inverse functions of each other, for every specific ordered pair (h, k) on the graph of $y = b^x$, we find the point (k, h) with the coordinates reversed on the graph of $y = \log_b(x)$.

In other words, if the point with $x = h$ and $y = k$ is on the graph of $y = b^x$, then the point with $x = k$ and $y = h$ lies on the graph of $y = \log_b(x)$

The domain of $y = b^x$ is the range of $y = \log_b(x)$

The range of $y = b^x$ is the domain of $y = \log_b(x)$

For this reason, the graphs appear as reflections, or mirror images, of each other across the diagonal line $y = x$. This is a property of graphs of inverse functions that students should recall from their study of inverse functions in their prerequisite algebra class.



	$\mathbf{y = b^x}$**, with** $\mathbf{b > 1}$	$\mathbf{y = \log_b(x)}$**, with** $\mathbf{b > 1}$
Domain	all real numbers	all positive real numbers
Range	all positive real numbers	all real numbers
Intercepts	(0,1)	(1,0)
Asymptotes	Horizontal asymptote is the line y = 0 (the x-axis) As $x \to -\infty,\ y \to 0$	Vertical asymptote is the line x = 0 (the y axis) As $x \to 0^+,\ y \to -\infty$

Logarithmic Properties

Product Property: $y = \log_b(ac)$ = $\log_b(a)$ + $\log_b(c)$

Quotient Property: $y = \log_b \frac{a}{c}$ = $\log_b(a)$ - $\log_b(c)$

Power Property: $y = \log_b a^c$ = $c\log_b(a)$

Source: The material in this section of the textbook originates from David Lippman and Melonie Rasmussen, Open Text Bookstore, Precalculus: An Investigation of Functions, “Chapter 4: Exponential and Logarithmic Functions,” licensed under a Creative Commons CC BY-SA 3.0 license. The material here is based on material contained in that textbook but has been modified by Roberta Bloom, as permitted under this license.

4.5.1: Graphs and Properties of Logarithmic Functions (Exercises)

SECTION 4.5 PROBLEM SET: GRAPHS AND PROPERTIES OF LOGARITHMIC FUNCTIONS

Questions 1 – 3: For each of the following functions

a. Sketch a reasonably accurate graph showing the shape of the graph of the function
b. State the domain
c. State the range
d. State whether the graph has a vertical asymptote or a horizontal asymptote and write the equation of that asymptote
e. Does the graph have an x-intercept or a y-intercept asymptote? Write the coordinates of the x-intercept or the y-intercept.

1. $y=\ln x$ a. Sketch the graph below	b. domain: ________ c. range: ________ d. Is the asymptote horizontal or vertical? _________ Equation of the asymptote: ________ e. Coordinates of x-intercept or y-intercept: ________
2. $y=\log x$ a. Sketch the graph below	b. domain: ________ c. range: ________ d. Is the asymptote horizontal or vertical? _________ Equation of the asymptote: ________ e. Coordinates of x-intercept or y-intercept: ________
3. $y=\log_{0.8} x$ a. Sketch the graph below	b. domain: ________ c. range: ________ d. Is the asymptote horizontal or vertical? _________ Equation of the asymptote: ________ e. Coordinates of x-intercept or y-intercept: ________

Questions 4 - 5: For the pair of inverse functions $y=e^x$ and $y=\ln x$

a. Sketch a reasonably accurate graph showing the shape of the graph of the function
b. State the domain
c. State the range
d. State whether the graph has a vertical asymptote or a horizontal asymptote and write the equation of that asymptote
e. Does the graph have an x-intercept or a y-intercept asymptote? Write the coordinates of the xintercept or the y-intercept.

4. $y=e^x$ a. Sketch the graph below	b. domain: ________ c. range: ________ d. Is the asymptote horizontal or vertical? _________ Equation of the asymptote: ________ e. Coordinates of x-intercept or y-intercept: ________
5. $y=\ln x$ a. Sketch the graph below	b. domain: ________ c. range: ________ d. Is the asymptote horizontal or vertical? _________ Equation of the asymptote: ________ e. Coordinates of x-intercept or y-intercept: ________

Questions 6-11: Match the graph with the function.

Choose the function from the list below and write it on the line underneath the graph.

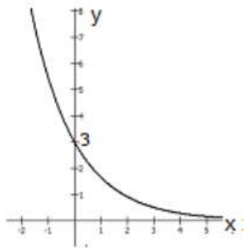
Hint: To match the function and the graph, identify these properties of the graph and function

- Is the function increasing decreasing?
- Examine the asymptote

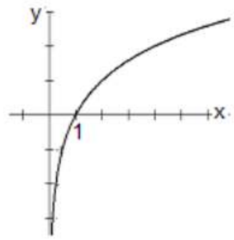
- Determine the x or y intercept

$$\mathrm{y}=3\left(2^{x}\right) \quad y=5\left(0.4^{x}\right) \quad y=\log_{2}(x) \quad y=\log_{1/2}(x) \quad y=3e^{-0.6x} \quad y=5e^{0.3x}$$

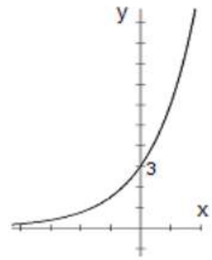
6. Function: ________



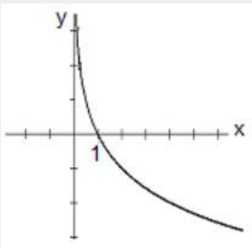
7. Function: ________



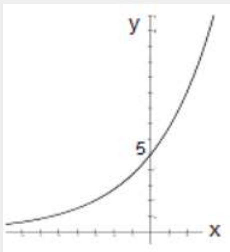
8. Function: ________



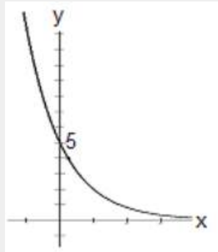
9. Function: ________



10. Function: ________



11. Function: ________



4.6: Exponential and Logarithmic Equations

Learning Objectives

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.

Figure 4.6.1: Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the “rabbit plague.” (credit: Richard Taylor, Flickr)

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers b, S, and T, where $b>0$, $b\neq 1$, $b^S=b^T$ if and only if $S=T$.

In other words, when an **exponential equation** has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7}=\dfrac{3^{2x}}{3}$. To solve for x, we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for x:

$$\begin{aligned}
3^{4x-7} &= \frac{3^{2x}}{3} && \\
3^{4x-7} &= \frac{3^{2x}}{3^1} && \text{Rewrite 3 as } 3^1 \\
3^{4x-7} &= 3^{2x-1} && \text{Use the division property of exponents} \\
4x-7 &= 2x-1 && \text{Apply the one-to-one property of exponents} \\
2x &= 6 && \text{Subtract 2x and add 7 to both sides} \\
x &= 3 && \text{Divide by 3}
\end{aligned}$$

USING THE ONE-TO-ONE PROPERTY OF EXPONENTIAL FUNCTIONS TO SOLVE EXPONENTIAL EQUATIONS

For any algebraic expressions S and T, and any positive real number $b\neq 1$,

$$b^S=b^T \text{ if and only if } S=T \tag{4.6.1}$$

How to: Given an exponential equation with the form $b^S=b^T$, where S and T are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S=b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S=T$, for the unknown.

Example 4.6.1: Solving an Exponential Equation with a Common Base

Solve $2^{x-1}=2^{2x-4}$.

Solution

$$\begin{aligned} 2^{x-1} &= 2^{2x-4} && \text{The common base is 2} \\ x-1 &= 2x-4 && \text{By the one-to-one property the exponents must be equal} \\ x &= 3 && \text{Solve for x} \end{aligned}$$

Exercise 4.6.1

Solve $5^{2x}=5^{3x+2}$.

Answer

$x=-2$

Rewrite Equations So All Powers Have the Same Base

Sometimes the *common base* for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256=4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x:

$$\begin{aligned} 256 &= 4^{x-5} && \\ 2^8 &= (2^2)^{x-5} && \text{Rewrite each side as a power with base 2} \\ 2^8 &= 2^{2x-10} && \text{Use the one-to-one property of exponents} \\ 8 &= 2x-10 && \text{Apply the one-to-one property of exponents} \\ 18 &= 2x && \text{Add 10 to both sides} \\ x &= 9 && \text{Divide by 2} \end{aligned}$$

How to: Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S=b^T$.
3. Use the one-to-one property to set the exponents equal.
4. Solve the resulting equation, $S=T$, for the unknown.

Example 4.6.2: Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2}=16^{x+1}$.

Solution

$$\begin{aligned} 8^{x+2} &= 16^{x+1} && \\ (2^3)^{x+2} &= (2^4)^{x+1} && \text{Write 8 and 16 as powers of 2} \\ 2^{3x+6} &= 2^{4x+4} && \text{To take a power of a power, multiply exponents} \\ 3x+6 &= 4x+4 && \text{Use the one-to-one property to set the exponents equal} \\ x &= 2 && \text{Solve for x} \end{aligned}$$

Exercise 4.6.2

Solve $5^{2x}=25^{3x+2}$.

Answer

$x=-1$

Example 4.6.3: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x}=\sqrt{2}$.

Solution

$$\begin{aligned} 2^{5x} &= 2^{\frac{1}{2}} && \text{Write the square root of 2 as a power of 2} \\ 5x &= \frac{1}{2} && \text{Use the one-to-one property} \\ x &= \frac{1}{10} && \text{Solve for x} \end{aligned}$$

Exercise 4.6.3

Solve $5^x=\sqrt{5}$.

Answer

$x=\frac{1}{2}$

Q&A: Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?

No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

Example 4.6.4: Solving an Equation with Positive and Negative Powers

Solve $3^{x+1}=-2$.

Solution

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Analysis

Figure 4.6.2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

Figure 4.6.2

Exercise 4.6.4

Solve $2^x=-100$.

Answer

The equation has no solution.

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a)=\log(b)$ is equivalent to $a=b$, we may apply logarithms with the same base on both sides of an exponential equation.

How to: Given an exponential equation in which a common base cannot be found, solve for the unknown

1. Apply the logarithm of both sides of the equation.
   - If one of the terms in the equation has base 10, use the common logarithm.
   - If none of the terms in the equation has base 10, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

Example 4.6.5: Solving an Equation Containing Powers of Different Bases

Solve $5^{x+2}=4^x$.

Solution

$$\begin{aligned} 5^{x+2} &= 4^x && \text{There is no easy way to get the powers to have the same base} \\ \ln 5^{x+2} &= \ln 4^x && \text{Take ln of both sides} \\ (x+2)\ln 5 &= x\ln 4 && \text{Use laws of logs} \\ x\ln 5+2\ln 5 &= x\ln 4 && \text{Use the distributive law} \\ x\ln 5-x\ln 4 &= -2\ln 5 && \text{Get terms containing x on one side, terms without x on the other} \\ x(\ln 5-\ln 4) &= -2\ln 5 && \text{On the left hand side, factor out an x} \\ x\ln\left(\frac{5}{4}\right) &= \ln\left(\frac{1}{25}\right) && \text{Use the laws of logs} \\ x &= \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)} && \text{Divide by the coefficient of x} \end{aligned}$$

Exercise 4.6.5

Solve $2^x=3^{x+1}$.

Answer

$$x=\frac{\ln 3}{\ln\left(\frac{2}{3}\right)}$$

Q&A: Is there any way to solve $2^x=3^x$?

Yes. The solution is 0.

Equations Containing e

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

How to: Given an equation of the form $y=Ae^{kt}$, solve for t.

1. Divide both sides of the equation by A.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by k.

Example 4.6.6: Solve an Equation of the Form $y = Ae^{kt}$

Solve $100 = 20e^{2t}$.

Solution

$$\begin{aligned} 100 &= 20e^{2t} && \\ 5 &= e^{2t} && \text{Divide by the coefficient of the power} \\ \ln 5 &= 2t && \text{Take ln of both sides. Use the fact that } ln(x) \text{ and } e^x \text{ are inverse functions} \\ t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of t} \end{aligned}$$

Analysis

Using laws of logs, we can also write this answer in the form $t = \ln \sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

Exercise 4.6.6

Solve $3e^{0.5t} = 11$.

Answer

$$t = 2\ln\left(\frac{11}{3}\right) \text{ or } \ln\left(\frac{11}{3}\right)^2$$

Q&A: Does every equation of the form $y = Ae^{kt}$ have a solution?

No. There is a solution when $k \neq 0$,and when y and A are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2 = -3e^t$.

Example 4.6.7: Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x} + 5 = 12$.

Solution

$$\begin{aligned} 4e^{2x} + 5 &= 12 && \\ 4e^{2x} &= 7 && \text{Combine like terms} \\ e^{2x} &= \frac{7}{4} && \text{Divide by the coefficient of the power} \\ 2x &= \ln\left(\frac{7}{4}\right) && \text{Take ln of both sides} \\ x &= \frac{1}{2}\ln\left(\frac{7}{4}\right) && \text{Solve for x} \end{aligned}$$

Exercise 4.6.7

Solve $3 + e^{2t} = 7e^{2t}$.

Answer

$$t = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2)$$

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

✓ Example 4.6.8: Solving Exponential Functions in Quadratic Form

Solve $e^{2x} - e^{x} = 56$.

Solution

$$\begin{aligned} e^{2x} - e^{x} &= 56 && \\ e^{2x} - e^{x} - 56 &= 0 && \text{Get one side of the equation equal to zero} \\ (e^{x} + 7)(e^{x} - 8) &= 0 && \text{Factor by the FOIL method} \\ e^{x} + 7 &= 0 && \text{or} \\ e^{x} - 8 &= 0 && \text{If a product is zero, then one factor must be zero} \\ e^{x} &= -7 && \text{or} \\ e^{x} &= 8 && \text{Isolate the exponentials} \\ e^{x} &= 8 && \text{Reject the equation in which the power equals a negative number} \\ x &= \ln 8 && \text{Solve the equation in which the power equals a positive number} \end{aligned}$$

Analysis

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^{x} = -7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

? Exercise 4.6.8

Solve $e^{2x} = e^{x} + 2$.

Answer

$x = \ln 2$

Q&A: Does every logarithmic equation have a solution?

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^{y} = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$$\begin{aligned} \log_2(2) + \log_2(3x - 5) &= 3 && \\ \log_2(2(3x - 5)) &= 3 && \text{Apply the product rule of logarithms} \\ \log_2(6x - 10) &= 3 && \text{Distribute} \\ 2^{3} &= 6x - 10 && \text{Apply the definition of a logarithm} \\ 8 &= 6x - 10 && \text{Calculate } 2^{3} \\ 18 &= 6x && \text{Add 10 to both sides} \\ x &= 3 && \text{Divide by 6} \end{aligned}$$

USING THE DEFINITION OF A LOGARITHM TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expression S and real numbers b and c,where $b>0$, $b\neq 1$,

$$\log_b(S)=c \text{ if and only if } b^c=S \tag{4.6.2}$$

Example 4.6.9: Using Algebra to Solve a Logarithmic Equation

Solve $2\ln x+3=7$.

Solution

$$\begin{aligned} 2\ln x+3 &= 7 \\ 2\ln x &= 4 && \text{Subtract 3} \\ \ln x &= 2 && \text{Divide by 2} \\ x &= e^2 && \text{Rewrite in exponential form} \end{aligned}$$

Exercise 4.6.9

Solve $6+\ln x=10$.

Answer

$x=e^4$

Example 4.6.10: Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x)=7$.

Solution

$$\begin{aligned} 2\ln(6x) &= 7 \\ \ln(6x) &= \frac{7}{2} && \text{Divide by 2} \\ 6x &= e^{\left(\frac{7}{2}\right)} && \text{Use the definition of } \ln \\ x &= \frac{1}{6}e^{\left(\frac{7}{2}\right)} && \text{Divide by 6} \end{aligned}$$

Exercise 4.6.10

Solve $2\ln(x+1)=10$.

Answer

$x=e^5-1$

Example 4.6.11: Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x=3$.

Solution

$$\begin{aligned} \ln x &= 3 \\ x &= e^3 && \text{Use the definition of the natural logarithm} \end{aligned}$$

Figure $4.6.3$ represents the graph of the equation. On the graph, the *x*-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3\approx 20$. A calculator gives a better approximation: $e^3\approx 20.0855$.

Figure 4.6.3: The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately $(20.0855, 3)$.

? Exercise 4.6.11

Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

Answer

$x \approx 9.97$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for x,and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x:

$$\begin{aligned} \log(3x-2) - \log(2) &= \log(x+4) \\ \log\left(\frac{3x-2}{2}\right) &= \log(x+4) && \text{Apply the quotient rule of logarithms} \\ \frac{3x-2}{2} &= x+4 && \text{Apply the one to one property of a logarithm} \\ 3x-2 &= 2x+8 && \text{Multiply both sides of the equation by 2} \\ x &= 10 && \text{Subtract 2x and add 2} \end{aligned}$$

To check the result, substitute $x = 10$ into $\log(3x-2) - \log(2) = \log(x+4)$.

$$\begin{aligned} \log(3(10)-2) - \log(2) &= \log((10)+4) \\ \log(28) - \log(2) &= \log(14) \\ \log\left(\frac{28}{2}\right) &= \log(14) && \text{The solution checks} \end{aligned}$$

USING THE ONE-TO-ONE PROPERTY OF LOGARITHMS TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expressions S and T and any positive real number b, where $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{4.6.3}$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

How to: Given an equation containing logarithms, solve it using the one-to-one property

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S = \log_b T$.
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Example 4.6.12: Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x+3)$.

Solution

$$\begin{aligned} \ln(x^2) &= \ln(2x+3) \\ x^2 &= 2x+3 && \text{Use the one-to-one property of the logarithm} \\ x^2-2x-3 &= 0 && \text{Get zero on one side before factoring} \\ (x-3)(x+1) &= 0 && \text{Factor using FOIL} \\ x-3 &= 0 && \text{or } x+1=0 \text{ If a product is zero, one of the factors must be zero} \\ x=3 && \text{or} \\ x &= -11 && \text{Solve for x} \end{aligned}$$

Analysis

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

Exercise 4.6.12

Solve $\ln(x^2) = \ln 1$.

Answer

$x = 1$ or $x = -1$

Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its **half-life**. Table $4.6.1$ lists the half-life for several of the more common radioactive substances.

Table 4.6.1

Substance	***Use***	***Half-life***
gallium-67	*nuclear medicine*	*80 hours*
cobalt-60	*manufacturing*	*5.3 years*
technetium-99m	*nuclear medicine*	*6 hours*
americium-241	*construction*	*432 years*
carbon-14	*archeological dating*	*5,715 years*
uranium-235	*atomic power*	*703,800,000 years*

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$A(t) = A_0 e^{\frac{\ln(0.5)}{T}t} \tag{4.6.4}$$

$$A(t) = A_0 e^{\frac{\ln(0.5)t}{T}} \tag{4.6.5}$$

$$A(t) = A_0 (e^{\ln(0.5)})^{\frac{t}{T}} \tag{4.6.6}$$

$$A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}} \tag{4.6.7}$$

where

- A_0 is the amount initially present
- T is the half-life of the substance
- t is the time period over which the substance is studied
- y is the amount of the substance present after time t

Example 4.6.13: Using the Formula for Radioactive Decay to Find the Quantity of a Substance

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

Solution

$$\begin{aligned} y &= 1000e^{\frac{\ln(0.5)}{703,800,000}t} \\ 900 &= 1000e^{\frac{\ln(0.5)}{703,800,000}t} && \text{After } 10\% \text{ decays, 900 grams are left} \\ 0.9 &= e^{\frac{\ln(0.5)}{703,800,000}t} && \text{Divide by 1000} \\ \ln(0.9) &= \ln\left(e^{\frac{\ln(0.5)}{703,800,000}t}\right) && \text{Take ln of both sides} \\ \ln(0.9) &= \frac{\ln(0.5)}{703,800,000}t && \ln(e^M) = M \\ t &= 703,800,000 \times \frac{\ln(0.9)}{\ln(0.5)} && \text{years Solve for t} \\ t &\approx 106,979,777 && \text{years} \end{aligned}$$

Analysis

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

Exercise 4.6.13

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

Answer

$t = 703,800,000 \times \frac{\ln(0.8)}{\ln(0.5)}$ years ≈ 226,572,993 years.

Media

Access these online resources for additional instruction and practice with exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Exponential Equations with Logarithms

Key Equations

One-to-one property for exponential functions	For any algebraic expressions S and T and any positive real number b, where $b^S = b^T$ if and only if $S = T$.
Definition of a logarithm	For any algebraic expression S and positive real numbers b and c, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$.
One-to-one property for logarithmic functions	For any algebraic expressions S and T and any positive real number b, where $b \neq 1$, $\log_b S = \log_b T$ if and only if $S = T$.

Key Concepts

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown. See Example 4.6.1.
- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See Example 4.6.2, Example 4.6.3, and Example 4.6.4.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See Example 4.6.5.
- We can solve exponential equations with base e,by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See Example 4.6.6 and Example 4.6.7.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions. See Example 4.6.8.
- When given an equation of the form $\log_b(S) = c$, where S is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See Example 4.6.9 and Example 4.6.10.
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the *x*-value of the intersecting point. See Example 4.6.11.
- When given an equation of the form $\log_b S = \log_b T$, where S and T are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See Example 4.6.12.
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See Example 4.6.13.

4.7: Chapter Review

1. The value of a new boat depreciates after it is purchased. The value of the boat 7 years after it was purchased is $25,000 and its value has been decreasing at the rate of 8.2% per year.
   a. Find the initial value of the boat when it was purchased.
   b. How many years after it was purchased will the boat's value be $20,000?
   c. What was its value 3 years after the boat was purchased?
2. Tony invested $40,000 in 2010; unfortunately his investment has been losing value at the rate of 2.7% per year.
   a. Write the function that gives the value of the investment as a function of time t in years after 2010.
   b. Find the value of the investment in 2020, if its value continues to decrease at this rate.
   c. In what year will the investment be worth half its original value?
3. Rosa invested $25,000 in 2005; its value has been increasing at the rate of 6.4% annually.
   a. Write the function that gives the value of the investment as a function of time t in years after 2005.
   b. Find the value of the investment in 2025.
4. The population of a city is increasing at the rate of 3.2% per year, since the year 2000. Its population in 2015 was 235,000 people.
   a. Find the population of the city in the year 2000.
   b. In what year with the population be 250, 000 if it continues to grow at this rate.
   c. What was the population of this city in the year 2008?
5. The population of an endangered species has only 5000 animals now. Its population has been decreasing at the rate of 12% per year.
   a. If the population continues to decrease at this rate, how many animals will be in this population 4 years from now.
   b. In what year will there be only 2000 animals remaining in this population?
6. 300 mg of a medication is administered to a patient. After 5 hours, only 80 mg remains in the bloodstream.
   a. Using an exponential decay model, find the hourly decay rate.
   b. How many hours after the 300 mg dose of medication was administered was there 125 mg in the bloodstream
   c. How much medication remains in the bloodstream after 8 hours?
7. If $y=240b^t$ and $y=600$ when $t=6$ years, find the annual growth rate. State your answer as a percent.
8. If the function is given in the form $y=ae^{kt}$, rewrite it in the form $y=ab^t$.
   If the function is given in the form $y=ab^t$, rewrite it in the form $y=ae^{kt}$.
   a. $y=375000\left(1.125^t\right)$
   b. $y=5400e^{0.127t}$
   c. $y=230e^{-0.62t}$
   d. $y=3600\left(0.42^t\right)$

4.E: Exponential and Logarithmic Functions (Exercises)

4.1: Exponential Functions

When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term exponential growth has a very specific meaning. In this section, we will take a look at exponential functions, which model this kind of rapid growth.

Verbal

1) Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

Answer:

Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

2) Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

3)The Oxford Dictionary defines the word nominal as a value that is "stated or expressed but not necessarily corresponding exactly to the real value."Develop a reasonable argument for why the term nominal rate is used to describe the annual percentage rate of an investment account that compounds interest.

Answer:

When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of nominal.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

4) The average annual population increase of a pack of wolves is 25.

5) A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.

Answer:

exponential; the population decreases by a proportional rate.

6) The value of a coin collection has increased by 3.25%annually over the last 20 years.

7) For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.

Answer:

not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function.

8) The height of a projectile at time t is represented by the function $h(t)=-4.9t^2+18t+40$

For the following exercises, consider this scenario: For each year$t$, the population of a forest of trees is represented by the function$A(t)=115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t)=82(1.029)^t$. (Round answers to the nearest whole number.)

9) Which forest's population is growing at a faster rate?

Answer:

The forest represented by the function $B(t)=82(1.029)^t$.

10) Which forest had a greater number of trees initially? By how many?

11) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?

Answer:

After $t=20$ years, forest A will have 43 more trees than forest B.

12) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?

13) Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

Answer:

Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

14) $y=300(1-t)^5$

15) $y=220(1.06)^x$

Answer:

exponential growth; The growth factor, 1.06 is greater than 1.

16) $y=16.5(1.025)^{\frac{1}{x}}$

17) $y=11,701(0.97)^t$

Answer:

exponential decay; The decay factor, 0.97, is between 0 and 1.

For the following exercises, find the formula for an exponential function that passes through the two points given.

18) $(0,6)$ and $(3,750)$

19) $(0,2000)$ and $(2,20)$

Answer:

$f(x)=2000(0.1)^x$

20) $\left(-1,\frac{3}{2}\right)$ and $(3,24)$

21) $(-2,6)$ and $(3,1)$

Answer:

$$f(x)=\left(\frac{1}{6}\right)^{-\frac{3}{5}}\left(\frac{1}{6}\right)^{\frac{x}{5}}\approx 2.93(0.699)^x$$

22) $(3,1)$ and $(5,4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

23)

x	1	2	3	4
$f(x)$	70	40	10	-20

Answer:

Linear

24)

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

25)

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

Answer:

Neither

26)

x	1	2	3	4
$f(x)$	10	20	40	80

27)

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

Answer:

Linear

For the following exercises, use the compound interest formula, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

28) After a certain number of years, the value of an investment account is represented by the equation $10,250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the value of the account?

29) What was the initial deposit made to the account in the previous exercise?

Answer:

$\$10,250$

30) How many years had the account from the previous exercise been accumulating interest?

31) An account is opened with an initial deposit of $\$6,500$ and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?

Answer:

$\$13,268.58$

32) How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

33) Solve the compound interest formula for the principal, P.

Answer:

$P = A(t) \cdot \left(1 + \frac{r}{n}\right)^{-nt}$

34) Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)

35) How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?

Answer:

$\$4,572.56$

36) Use properties of rational exponents to solve the compound interest formula for the interest rate, r.

37) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $\$9,000$ and was worth $\$13,373.53$ after 10 years.

Answer:

4%

38) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $\$5,500$, and was worth $\$38,455$ after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

39) $y = 3742(e)^{0.75t}$

Answer:

continuous growth; the growth rate is greater than 0.

40) $y = 150(e)^{\frac{3.25}{t}}$

41) $y = 2.25(e)^{-2t}$

Answer:

continuous decay; the growth rate is less than 0.

42) Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?

43) How much less would the account from Exercise 42 be worth after 30 years if it were compounded monthly instead?

Answer:

$\$669.42$

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

44) $f(x) = 2(5)^x$ for $f(-3)$

45) $f(x) = -4^{2x+3}$ for $f(-1)$

Answer:

$f(-1) = -4$

46) $f(x) = e^x$, for $f(3)$

47) $f(x) = -2e^{x-1}$, for $f(-1)$

Answer:

$f(-1) \approx -0.2707$

48) $f(x)=2.7(4)^{-x+1}+1.5$, for $f(-2)$

49) $f(x)=1.2e^{2x}-0.3$, for $f(3)$

Answer:

$f(3)\approx 483.8146$

50) $f(x)=-\frac{3}{2}(3)^{-x}+\frac{3}{2}$, for $f(2)$

Technology

For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.

51) $(0,3)$ and $(3,375)$

Answer:

$y=3\cdot 5^x$

52) $(3,222.62)$and $(10,77.456)$

53) $(20,29.495)$and $(150,730.89)$

Answer:

$y\approx 18\cdot 1.025^x$

54) $(5,2.909)$and $(13,0.005)$

55) ((11,310.035)\) and $(25,356.3652)$

Answer:

$y\approx 0.2\cdot 1.95^x$

Extensions

56) The annual percentage yield (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $APY=\left(1+\frac{r}{12}\right)^{12}-1$

57) Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.

Answer:

$$\begin{aligned}APY&=\frac{A(t)-a}{a}\\&=\frac{a\left(1+\frac{r}{365}\right)^{365(1)}-a}{a}\\&=\frac{a\left[\left(1+\frac{r}{365}\right)^{365}-1\right]}{a}\\&=\left(1+\frac{r}{365}\right)^{365}-1\end{aligned};\ I(n)=\left(1+\frac{r}{n}\right)^{n}-1$$

58) Recall that an exponential function is any equation written in the form $f(x)=a\cdot b^x$ such that a and b are positive numbers and $b\neq 1$. Any positive number b can be written as $b=e^n$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number e as a base.

59) In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b>1$, the exponential decay function can be written as $f(x)=a\cdot\left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b=e^n$, to show that an exponential decay function takes the form $f(x)=a\cdot(e)^{-nx}$ for some positive number n.

Answer:

Let f be the exponential decay function $f(x)=a\cdot\left(\frac{1}{b}\right)^x$ such that $b>1$. Then for some number $n>0$,

$$\begin{aligned}f(x)&=a\cdot\left(\frac{1}{b}\right)^x\\&=a\left(b^{-1}\right)^x\\&=a\left((e^n)^{-1}\right)^x\\&=a\left(e^{-n}\right)^x\\&=a(e)^{-nx}\end{aligned}$$

60) The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t)=a(e)^{rt}$, where a is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time t can be calculated with the formula $I(t)=e^{rt}-1$

Real-World Applications

61) The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were $23,900$ fox counted in the area. What is the fox population predicted to be in the year 2020?

Answer:

$47,622$ fox

62) A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?

63) In the year 1985, a house was valued at $\$110,000$. By the year 2005, the value had appreciated to $\$145,000$. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

Answer:

1.39%; $\$155,368.09$

64) A car was valued at $\$38,000$ in the year 2007. By 2013, the value had depreciated to $\$11,000$. If the car's value continues to drop by the same percentage, what will it be worth by 2017?

65) Jamal wants to save $\$54,000$ for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?

Answer:

$\$35,838.76$

66) Kyoko has $\$10,000$ that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $\$15,000$ by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (Hint: solve the compound interest formula for the interest rate.)

67) Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $\$13,500$. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

Answer:

$\$82,247.78$; $\$449.75$

68) An investment account with an annual interest rate of 7% was opened with an initial deposit of $\$4,000$ Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

4.2: Graphs of Exponential Functions

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool.

Verbal

1) What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?

Answer:

An asymptote is a line that the graph of a function approaches, as x either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

2) What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

Algebraic

3) The graph of $f(x)=3^x$ is reflected about the y-axis and stretched vertically by a factor of 4. What is the equation of the new function, $g(x)$. State its y-intercept, domain, and range.

Answer:

$g(x)=4(3)^{-x}$; y-intercept: $(0,4)$; Domain: all real numbers; Range: all real numbers greater than 0.

4) The graph of $f(x)=\left(\frac{1}{2}\right)^{-x}$ is reflected about the y-axis and compressed vertically by a factor of $\left(\frac{1}{5}\right)$. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

5) The graph of $f(x)=10^x$ is reflected about the x-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

Answer:

$g(x)=-10^x+7$; y-intercept: $(0,6)$; Domain: all real numbers; Range: all real numbers less than 7.

6) The graph of $f(x)=(1.68)^x$ is shifted right 3 units, stretched vertically by a factor of 2,reflected about the x-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its y-intercept (to the nearest thousandth), domain, and range.

7) The graph of $f(x)=2\left(\frac{1}{4}\right)^{x-20}$ is shifted left 2 units, stretched vertically by a factor of 4,reflected about the x-axis, and then shifted downward 4 units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

Answer:

$g(x)=2\left(\frac{1}{4}\right)^x$; y-intercept: $(0,2)$; Domain: all real numbers; Range: all real numbers greater than 0.

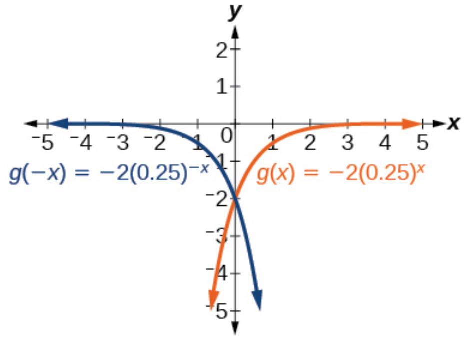
Graphical

For the following exercises, graph the function and its reflection about the y-axis on the same axes, and give the y-intercept.

8) $f(x)=3\left(\frac{1}{2}\right)^x$

9) $g(x)=-2(0.25)^x$

Answer:



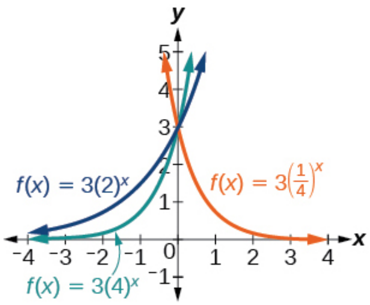
y-intercept: $(0, -2)$

10) $h(x) = 6(1.75)^{-x}$

For the following exercises, graph each set of functions on the same axes.

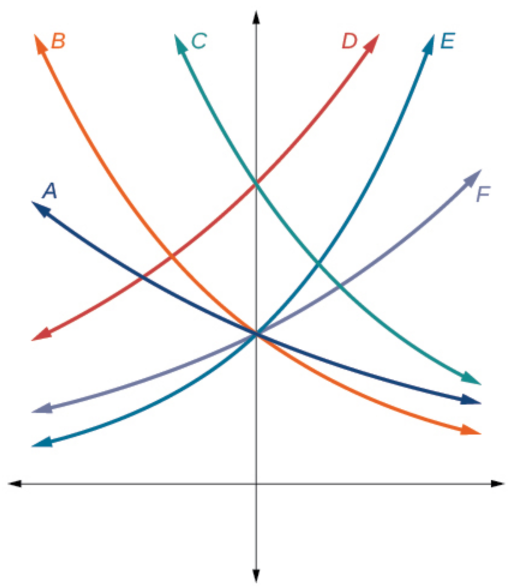
11) $f(x) = 3\left(\frac{1}{4}\right)^x, g(x) = 3(2)^x, h(x) = 3(4)^x$

Answer:



12) $f(x) = \frac{1}{4}(3)^x, g(x) = 2(3)^x, h(x) = 4(3)^x$

For the following exercises, match each function with one of the graphs in Figure below.



13) $f(x) = 2(0.69)^x$

Answer:

B

14) $f(x) = 2(1.28)^x$

15) $f(x) = 2(0.81)^x$

Answer:

A

16) $f(x) = 4(1.28)^x$

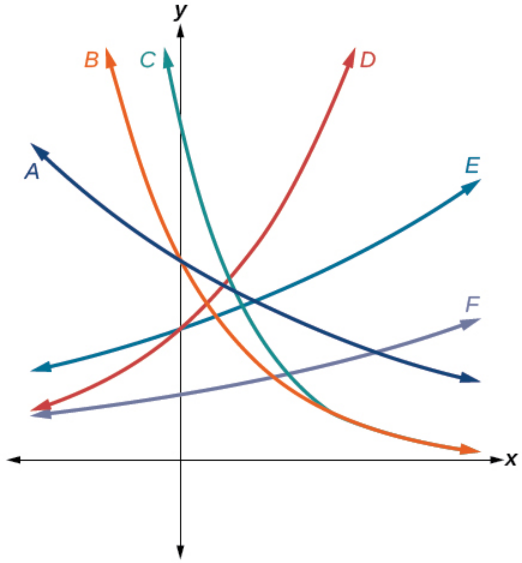
17) $f(x) = 2(1.59)^x$

Answer:

E

18) $f(x) = 4(0.69)^x$

For the following exercises, use the graphs shown in Figure below. All have the form $f(x) = ab^x$.



19) Which graph has the largest value for b?

Answer:

D

20) Which graph has the smallest value for b?

21) Which graph has the largest value for a?

Answer:

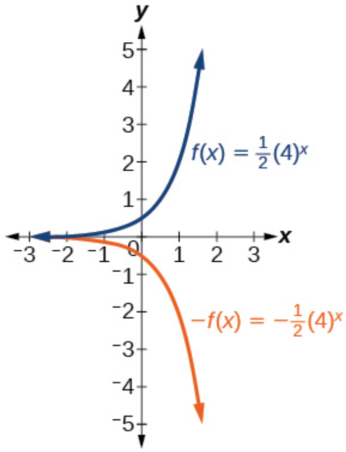
C

22) Which graph has the smallest value for a?

For the following exercises, graph the function and its reflection about the x-axis on the same axes.

23) $f(x) = \frac{1}{2}(4)^x$

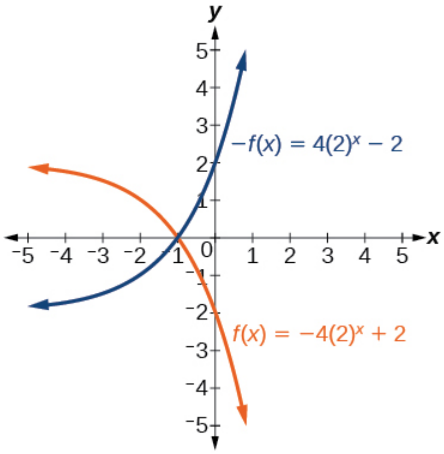
Answer:



24) $f(x)=3(0.75)^{x}-1$

25) $f(x)=-4(2)^{x}+2$

Answer:

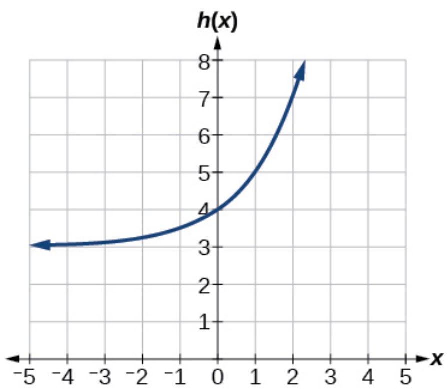


For the following exercises, graph the transformation of $f(x)=2^{x}$.Give the horizontal asymptote, the domain, and the range.

26) $f(x)=2^{-x}$

27) $h(x)=2^{x}+3$

Answer:



Horizontal asymptote: $h(x) = 3$; Domain: all real numbers; Range: all real numbers strictly greater than 3.

28) $f(x) = 2^{x-2}$

For the following exercises, describe the end behavior of the graphs of the functions.

29) $f(x) = -5(4)^x - 1$

Answer:

As $x \to \infty, f(x) \to -\infty$

As $x \to -\infty, f(x) \to -1$

30) $f(x) = 3\left(\frac{1}{2}\right)^x - 2$

31) $f(x) = 3(4)^{-x} + 2$

Answer:

As $x \to \infty, f(x) \to 2$

As $x \to -\infty, f(x) \to \infty$

For the following exercises, start with the graph of $f(x) = 4^x$.Then write a function that results from the given transformation.

32) Shift $f(x)$ 4 units upward

33) Shift $f(x)$ 3 units downward

Answer:

$f(x) = 4^x - 3$

34) Shift $f(x)$ 2 units left

35) Shift $f(x)$ 5 units right

Answer:

$f(x) = 4^{x-5}$

36) Reflect $f(x)$ about the x-axis

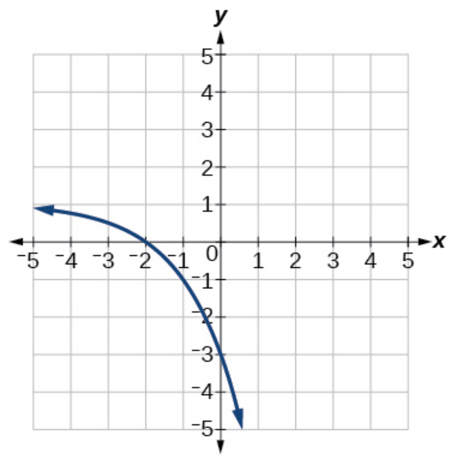
37) Reflect $f(x)$ about the y-axis

Answer:

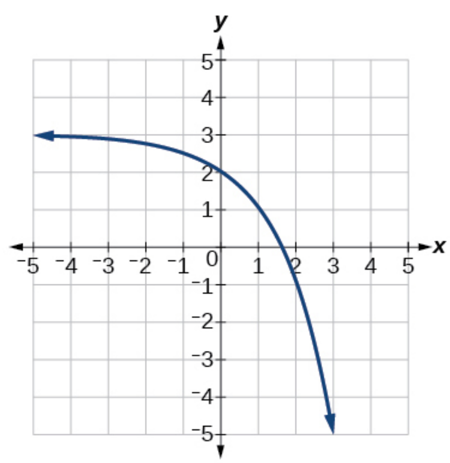
$f(x) = 4^{-x}$

For the following exercises, each graph is a transformation of $f(x) = 2^x$.Write an equation describing the transformation.

38)



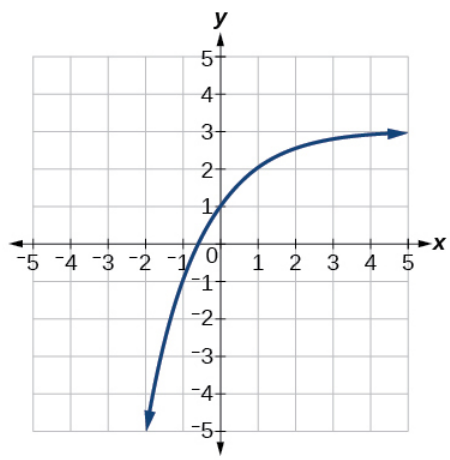
39)



Answer:

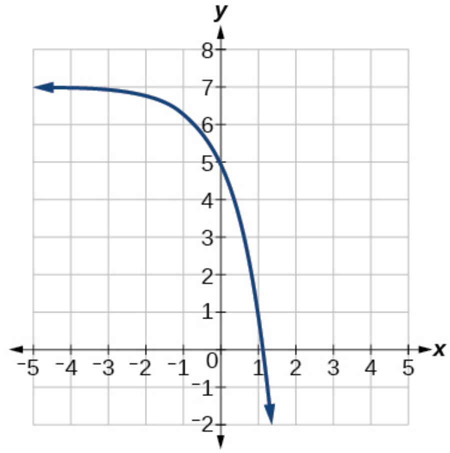
$y = -2^x + 3$

40)



For the following exercises, find an exponential equation for the graph.

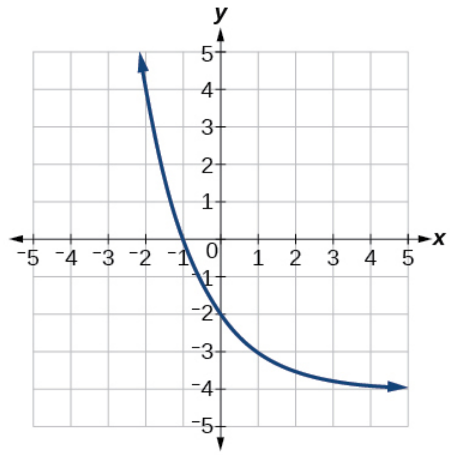
41)



Answer:

$$y = -2(3)^x + 7$$

42)



Numeric

For the following exercises, evaluate the exponential functions for the indicated value of x.

43) $g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.

Answer:

$$g(6) = 800 + \frac{1}{3} \approx 800.3333$$

44) $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.

45) $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^x + 6$ for $h(-7)$.

Answer:

$$h(-7) = -58$$

Technology

For the following exercises, use a graphing calculator to approximate the solutions of the equation. Round to the nearest thousandth.

46) $-50 = \left(\frac{1}{2}\right)^{-x}$

47) $116 = \left(\frac{1}{4}\right)\left(\frac{1}{8}\right)^{x}$

Answer:

$x \approx -2.953$

48) $12 = 2(3)^{x} + 1$

49) $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$

Answer:

$x \approx -0.222$

50) $-30 = -4(2)^{x+2} + 2$

Extensions

51) Explore and discuss the graphs of $F(x) = (b)^{x}$ and $G(x) = \left(\frac{1}{b}\right)^{x}$. Then make a conjecture about the relationship between the graphs of the functions b^{x} and $\left(\frac{1}{b}\right)^{x}$ for any real number $b > 0$.

Answer:

The graph of $G(x) = \left(\frac{1}{b}\right)^{x}$ is the refelction about the y-axis of the graph of $F(x) = (b)^{x}$; For any real number $b > 0$ and function $f(x) = (b)^{x}$, the graph of $\left(\frac{1}{b}\right)^{x}$ is the the reflection about the y-axis, $F(-x)$.

52) Prove the conjecture made in the previous exercise.

53) Explore and discuss the graphs of $f(x) = 4^{x}$, $g(x) = 4^{x-2}$, and $h(x) = \left(\frac{1}{16}\right)4^{x}$. Then make a conjecture about the relationship between the graphs of the functions b^{x} and $\left(\frac{1}{b^{n}}\right)b^{x}$ for any real number n and real number $b > 0$.

Answer:

The graphs of $g(x)$ and $h(x)$ are the same and are a horizontal shift to the right of the graph of $f(x)$; For any real number n, real number $b > 0$, and function $f(x) = b^{x}$, the graph of $\left(\frac{1}{b^{n}}\right)b^{x}$ is the horizontal shift $f(x-n)$.

54) Prove the conjecture made in the previous exercise.

4.3: Logarithmic Functions

The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.

Verbal

1) What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^{y} = x$ and $\log_{b} x = y$ for $b > 0, b \neq 1$.

Answer

A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, y, in the expression b^{y} can also be written as the logarithm, $\log_{b} x = y$,and the value of x is the result of raising b to the power of y.

2) How is the logarithmic function $f(x) = \log_{b} x$ related to the exponential function $g(x) = b^{x}$? What is the result of composing these two functions?

3) How can the logarithmic equation $\log_{b} x = y$ be solved for x using the properties of exponents?

Answer

Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$, and then properties of exponents can be applied to solve for x.

4) Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

5) Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Answer

The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_e(x)$,the notation used is $\ln(x)$.

Algebraic

For the following exercises, rewrite each equation in exponential form.

6) $\log_4(q) = m$

7) $\log_a(b) = c$

Answer

$a^c = b$

8) $\log_{16}(y) = x$

9) $\log_x(64) = y$

Answer

$x^y = 64$

10) $\log_y(x) = -11$

11) $\log_{15}(a) = b$

Answer

$15^b = a$

12) $\log_y(137) = x$

13) $\log_{13}(142) = a$

Answer

$13^a = 142$

14) $\log(v) = t$

15) $\ln(w) = n$

Answer

$e^n = w$

For the following exercises, rewrite each equation in logarithmic form.

16) $4^x = y$

17) $c^d = k$

Answer

$\log_c(k) = d$

18) $m^{-7} = n$

19) $19^x = y$

Answer

$\log_{19}(y) = x$

20) $x^{-\frac{10}{13}} = y$

21) $n^4 = 103$

Answer

$\log_n(103) = 4$

22) $\left(\frac{7}{5}\right)^m = n$

23) $y^x = \frac{39}{100}$

Answer

$\log_y\left(\frac{39}{100}\right) = x$

24) $10^a = b$

25) $e^k = h$

Answer

$\ln(w) = n$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

26) $\log_3(x) = 2$

27) $\log_2(x) = -3$

Answer

$x = 2^{-3} = \frac{1}{8}$

28) $\log_5(x) = 2$

29) $\log_3(x) = 3$

Answer

$x = 3^3 = 27$

30) $\log_2(x) = 6$

31) $\log_9(x) = \frac{1}{2}$

Answer

$x = 9^{\frac{1}{2}} = 3$

32) $\log_{18}(x) = 2$

33) $\log_6(x) = -3$

Answer

$$x = 6^{-3} = \frac{1}{216}$$

34) $\log(x) = 3$

35) $\ln(x) = 2$

Answer

$$x = e^2$$

For the following exercises, use the definition of common and natural logarithms to simplify.

36) $\log(100^8)$

37) $10^{\log(32)}$

Answer

$$32$$

38) $2\log(.0001)$

39) $e^{\ln(1.06)}$

Answer

$$1.06$$

40) $\ln(e^{-5.03})$

41) $e^{\ln(10.125)} + 4$

Answer

$$14.125$$

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

42) $\log_3\left(\frac{1}{27}\right)$

43) $\log_6(\sqrt{6})$

Answer

$$\frac{1}{2}$$

44) $\log_2\left(\frac{1}{8}\right) + 4$

45) $6\log_8(4)$

Answer

$$4$$

For the following exercises, evaluate the common logarithmic expression without using a calculator.

46) $\log(10,000)$

47) $\log(0.001)$

Answer

$$-3$$

48) $\log(1)+7$

49) $2\log(100^{-3})$

Answer

-12

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

50) $\ln\left(e^{\frac{1}{3}}\right)$

51) $\ln(1)$

Answer

0

52) $\ln\left(e^{-0.225}\right)-3$

53) $25\ln\left(e^{\frac{2}{5}}\right)$

Answer

10

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

54) $\log(0.04)$

55) $\ln(15)$

Answer

2.708

56) $\ln\left(\frac{4}{5}\right)$

57) $\log(\sqrt{2})$

Answer

0.151

58) $\ln(\sqrt{2})$

Extensions

59) Is $x=0$ in the domain of the function $f(x)=\log x$? If so, what is the value of the function when $x=0$? Verify the result.

Answer

No, the function has no defined value for $x=0$. To verify, suppose $x=0$ is in the domain of the function $f(x)=\log(x)$. Then there is some number n such that $n=\log(0)$. Rewriting as an exponential equation gives: $10^n=0$, which is impossible since no such real number n exists. Therefore, $x=0$ is not the domain of the function $f(x)=\log(x)$.

60) Is $f(x)=0$ in the range of the function $f(x)=\log(x)$? If so, for what value of x? Verify the result.

61) Is there a number x such that $\ln x=2$? If so, what is that number? Verify the result.

Answer

Yes. Suppose there exists a real number x such that $\ln x=2$. Rewriting as an exponential equation gives $x=e^2$, which is a real number. To verify, let $x=e^2$. Then, by definition, $\ln(x)=\ln\left(e^2\right)=2$.

62) Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)}=-1$ log3(27)log4(164)=−1? log3(27)log4(164)=−1? Verify the result.

63) Is the following true: $\frac{\ln(e^{1.725})}{\ln(1)}=1.725$ Verify the result.

Answer

No; $\ln(1)=0$, so $\frac{\ln(e^{1.725})}{\ln(1)}=1.725$ is undefined.

Real-World Applications

64) The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI=\log_2\left(\frac{f^2}{t}\right)$, where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?

65) Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?

Answer

2

66) The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula $\log\left(\frac{I_1}{I_2}\right)=M_1-M_2$ where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0. How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

4.4: Graphs of Logarithmic Functions

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Verbal

1) The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

Answer

Since the functions are inverses, their graphs are mirror images about the line $y-x$. So for every point (a,b) on the graph of a logarithmic function, there is a corresponding point (b,a) on the graph of its inverse exponential function.

2) What type(s) of translation(s), if any, affect the range of a logarithmic function?

3) What type(s) of translation(s), if any, affect the domain of a logarithmic function?

Answer

Shifting the function right or left and reflecting the function about the y-axis will affect its domain.

4) Consider the general logarithmic function $f(x)=\log_b(x)$. Why can't x be zero?

5) Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

Answer

No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

Algebraic

For the following exercises, state the domain and range of the function.

6) $f(x) = \log_3(x+4)$

7) $h(x) = \ln\left(\frac{1}{2} - x\right)$

Answer

Domain: $\left(-\infty, \frac{1}{2}\right)$; Range: $(-\infty, \infty)$

8) $g(x) = \log_5(2x+9) - 2$

9) $h(x) = \ln(4x+17) - 5$

Answer

Domain: $\left(-\frac{17}{4}, \infty\right)$; Range: $(-\infty, \infty)$

10) $f(x) = \log_2(12 - 3x) - 3$

For the following exercises, state the domain and the vertical asymptote of the function.

11) $f(x) = \log_b(x-5)$

Answer

Domain: $(5, \infty)$; Vertical asymptote: $x = 5$

12) $g(x) = \ln(3 - x)$

13) $f(x) = \log(3x+1)$

Answer

Domain: $\left(-\frac{1}{3}, \infty\right)$; Vertical asymptote: $x = -\frac{1}{3}$

14) $f(x) = 3\log(-x) + 2$

15) $g(x) = -\ln(3x+9) - 7$

Answer

Domain: $(-3, \infty)$; Vertical asymptote: $x = -3$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

16) $f(x) = \ln(2 - x)$

17) $f(x) = \log\left(x - \frac{3}{7}\right)$

Answer

Domain: $\left(\frac{3}{7}, \infty\right)$

Vertical asymptote: $x = \frac{3}{7}$

End behavior: as $x \to \left(\frac{3}{7}\right)^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

18) $h(x) = -\log(3x - 4) + 3$

19) $g(x) = \ln(2x+6) - 5$

Answer

Domain: $(-3, \infty)$

Vertical asymptote: $x = -3$

End behavior: as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

20) $f(x) = \log_3(15 - 5x) + 6$

For the following exercises, state the domain, range, and x- and y-intercepts, if they exist. If they do not exist, write DNE.

21) $h(x) = \log_4(x - 1) + 1$

Answer

Domain: $(1, \infty)$

Range: $-\infty, \infty$

Vertical asymptote: $x = 1$

x-intercept: $\left(\frac{5}{4}, 0\right)$

y-intercept: DNE

22) $f(x) = \log(5x + 10) + 3$

23) $g(x) = \ln(-x) - 2$

Answer

Domain: $(-\infty, 0)$

Range: $-\infty, \infty$

Vertical asymptote: $x = 0$

x-intercept: $\left(-e^2, 0\right)$

y-intercept: DNE

24) $f(x) = \log_2(x + 2) - 5$

25) $h(x) = 3\ln(x) - 9$

Answer

Domain: $(0, \infty)$

Range: $-\infty, \infty$

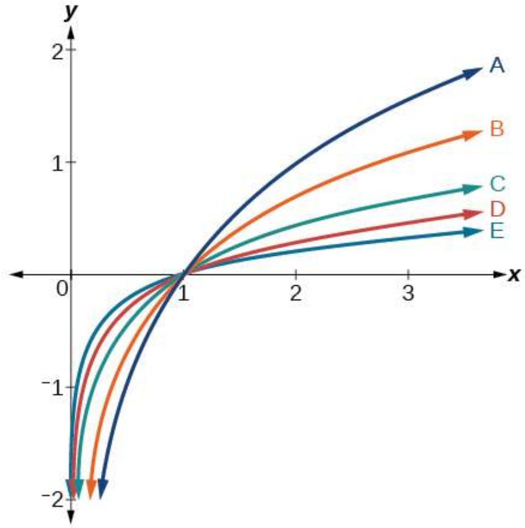
Vertical asymptote: $x = 0$

x-intercept: $\left(e^3, 0\right)$

y-intercept: DNE

Graphical

For the following exercises, match each function in Figure below with the letter corresponding to its graph.



26) $d(x) = \log(x)$

27) $f(x) = \ln(x)$

Answer

B

28) $g(x) = \log_2(x)$

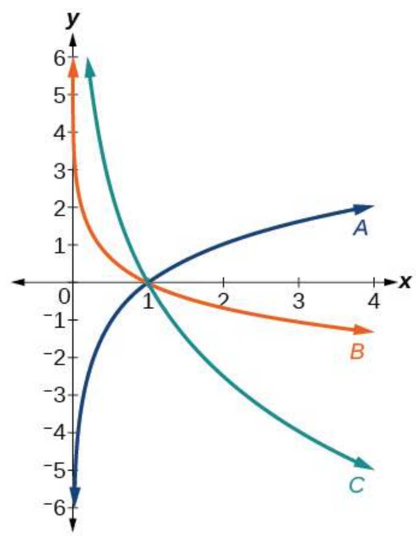
29) $h(x) = \log_5(x)$

Answer

C

30) $j(x) = \log_{25}(x)$

For the following exercises, match each function in Figure with the letter corresponding to its graph.



31) $f(x) = \log_{\frac{1}{3}}(x)$

Answer

B

32) $g(x) = \log_2(x)$

33) $h(x) = \log_{\frac{3}{4}}(x)$

Answer

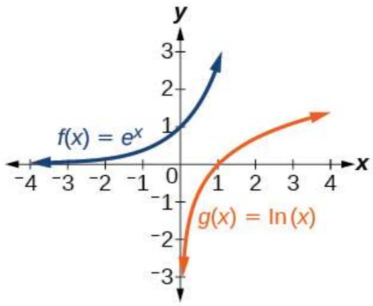
C

For the following exercises, sketch the graphs of each pair of functions on the same axis.

34) $f(x) = \log(x)$ and $g(x) = 10^x$

35) $f(x) = e^x$ and $g(x) = \ln(x)$

Answer



For the following exercises, match each function in Figure with the letter corresponding to its graph.

36) $f(x) = \log_4(-x+2)$

37) $g(x) = -\log_4(x+2)$

Answer

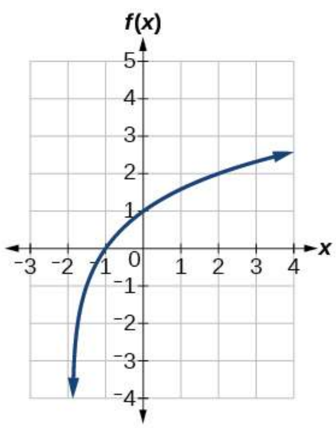
C

38) $h(x) = \log_4(x+2)$

For the following exercises, sketch the graph of the indicated function.

39) $f(x) = \log_2(x+2)$

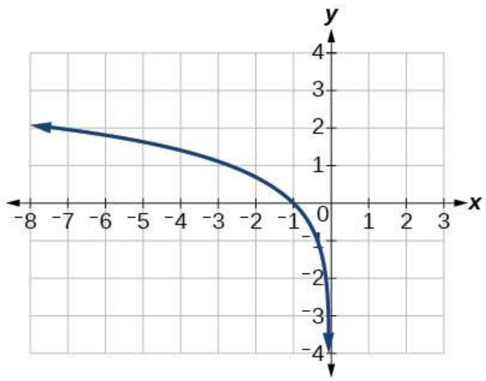
Answer



40) $f(x) = 2\log(x)$

41) $f(x) = \ln(-x)$

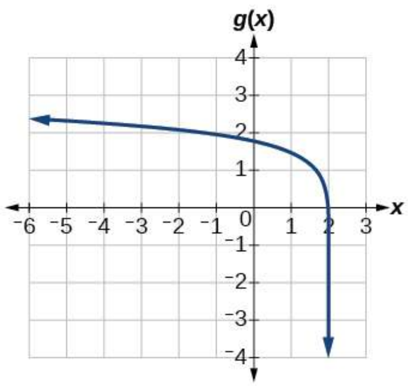
Answer



42) $g(x) = \log(4x + 16) + 4$

43) $g(x) = \log(6 - 3x) + 1$

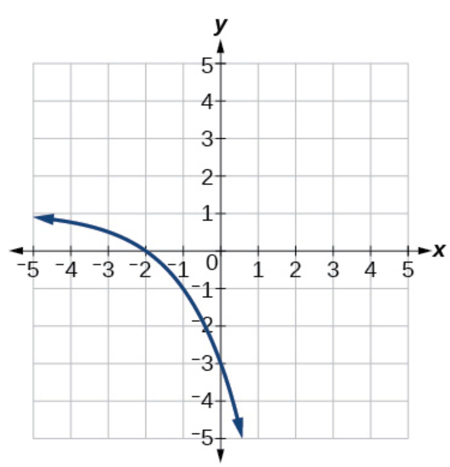
Answer



44) $h(x) = -\frac{1}{2}\log(x + 1) - 3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

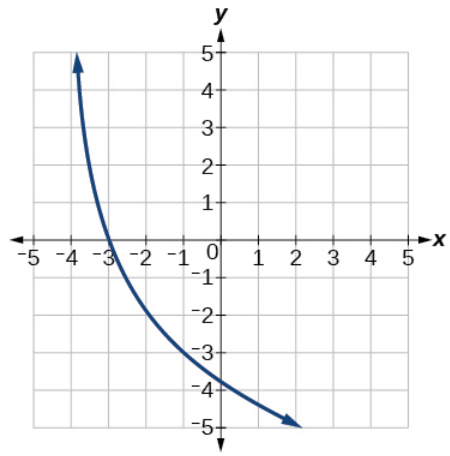
45) Use $y = \log_2(x)$ as the parent function.



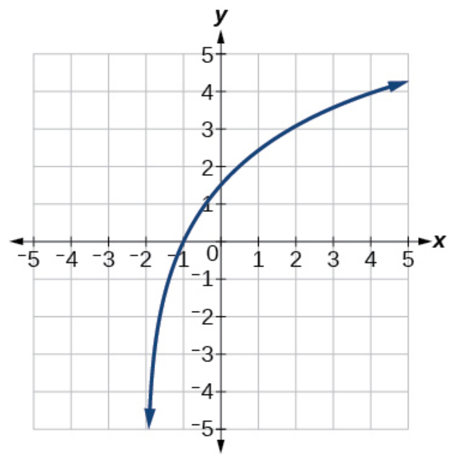
Answer

$f(x) = \log_2(-(x - 1))$

46) Use $f(x)=\log_3(x)$ as the parent function.



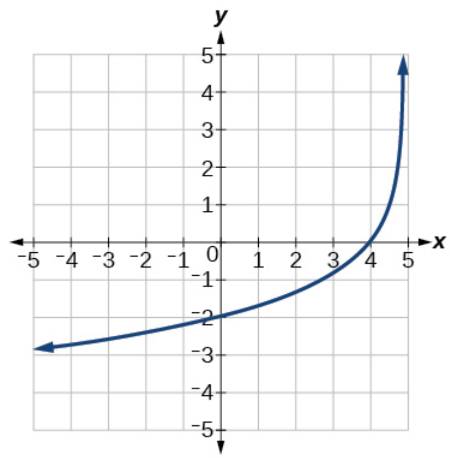
47) Use $f(x)=\log_4(x)$ as the parent function.



Answer

$f(x)=3\log_4(x+2)$

48) Use $f(x)=\log_5(x)$ as the parent function.



Technology

For the following exercises, use a graphing calculator to find approximate solutions to each equation.

49) $\log(x-1)+2=\ln(x-1)+2$

Answer

$x=2$

50) $\log(2x-3)+2=-\log(2x-3)+5$

51) $\ln(x-2)+2=-\ln(x+1)$

Answer

$x\approx 2.303$

52) $2\ln(5x+1)=\frac{1}{2}\ln(-5x)+1$

53) $\frac{1}{3}\log(1-x)=\log(x+1)+\frac{1}{3}$

Answer

$x\approx -0.472$

Extensions

54) Let b be any positive real number such that $b\neq 1$. What must $\log_b 1$be equal to? Verify the result.

55) Explore and discuss the graphs of $f(x)=\log_{\frac{1}{2}}(x)$ and $g(x)=-\log_2(x)$. Make a conjecture based on the result.

Answer

The graphs of $f(x)=\log_{\frac{1}{2}}(x)$ and $g(x)=-\log_2(x)$ appear to be the same;

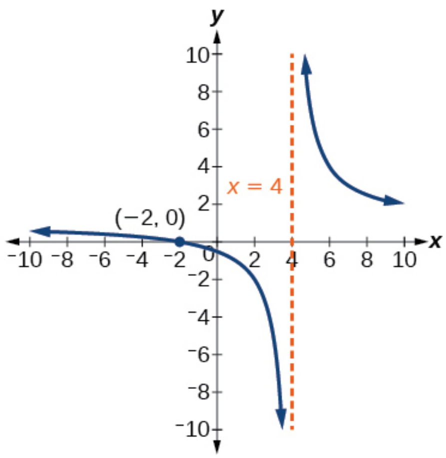
Conjecture: for any positive base $b\neq 1$, $\log_b(x)=\log_{\frac{1}{b}}(x)$

56) Prove the conjecture made in the previous exercise.

57) What is the domain of the function $f(x)=\ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.

Answer

Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4}>0$. From the graph of the function $f(x)=\frac{x+2}{x-4}$, note that the graph lies above the x-axis on the interval $(-\infty,-2)$ and again to the right of the vertical asymptote, that is $(4,\infty)$. Therefore, the domain is $(-\infty,-2)\cup(4,\infty)$.



58) Use properties of exponents to find the x-intercepts of the function $f(x)=\log\left(x^2+4x+4\right)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.

4.5: Logarithmic Properties

Recall that the logarithmic and exponential functions “undo” each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here.

Verbal

1) How does the power rule for logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?

Answer

Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right)=\frac{1}{n}\log_b(x)$.

2) What does the change-of-base formula do? Why is it useful when using a calculator?

Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

3) $\log_b(7x\cdot 2y)$

Answer

$\log_b(2)+\log_b(7)+\log_b(x)+\log_b(y)$

4) $\ln(3ab\cdot 5c)$

5) $\log_b\left(\frac{13}{17}\right)$

Answer

$\log_b(13)-\log_b(17)$

6) $\log_4\left(\frac{\frac{x}{z}}{w}\right)$

7) $\ln\left(\frac{1}{4^k}\right)$

Answer

$-k\ln(4)$

8) $\log_2(y^x)$

For the following exercises, condense to a single logarithm if possible.

9) $\ln(7)+\ln(x)+\ln(y)$

Answer

$\ln(7xy)$

10) $\log_3(2)+\log_3(a)+\log_3(11)+\log_3(b)$

11) $\log_b(28)-\log_b(7)$

Answer

$\log_b(4)$

12) $\ln(a)-\ln(d)-\ln(c)$

13) $-\log_b\left(\frac{1}{7}\right)$

Answer

$\log_b(7)$

14) $\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

15) $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

Answer

$15\log(x)+13\log(y)-19\log(z)$

16) $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$

17) $\log\left(\sqrt{x^3y^{-4}}\right)$

Answer

$\frac{3}{2}\log(x)-2\log(y)$

18) $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

19) $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$

Answer

$\frac{8}{3}\log(x)+\frac{14}{3}\log(y)$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

20) $\log\left(2x^4\right)+\log\left(3x^5\right)$

21) $\ln\left(6x^9\right)-\ln\left(3x^2\right)$

Answer

$\ln(2x^7)$

22) $2\log(x)+3\log(x+1)$

23) $\log(x)-\frac{1}{2}\log(y)+3\log(z)$

Answer

$\log\left(\frac{xz^3}{\sqrt{y}}\right)$

24) $4\log_7(c)+\frac{\log_7(a)}{3}+\frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

25) $\log_7(15)$ to base e

Answer

$\log_7(15)=\frac{\ln(15)}{\ln(7)}$

For the following exercises, suppose $\log_5(6)=a$ and $\log_5(11)=b$.Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b.Show the steps for solving.

27) $\log_{11}(5)$

Answer

$\log_{11}(5)=\frac{\log_5(5)}{\log_5(11)}=\frac{1}{b}$

28) $\log_6(55)$

29) $\log_{11}\left(\frac{6}{11}\right)$

Answer

$\log_{11}\left(\frac{6}{11}\right)=\frac{\log_{11}\left(\frac{6}{11}\right)}{\log_5(11)}=\frac{\log_5(6)-\log_5(11)}{\log_5(11)}=\frac{a-b}{b}=\frac{a}{b}-1$

Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

30) $\log_3\left(\frac{1}{9}\right)-3\log_3(3)$

31) $6\log_8(2)+\frac{\log_8(64)}{3\log_8(4)}$

Answer

3

32) $2\log_9(3)-4\log_9(3)+\log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

33) $\log_3(22)$

Answer

2.81359

34) $\log_8(65)$

35) $\log_6(5.38)$

Answer

0.93913

36) $\log_4\left(\frac{15}{2}\right)$

37) $\log_{\frac{1}{2}}(4.7)$

Answer

-2.23266

Extensions

38) Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2$. Show the steps for solving.

39) Use the quotient rule for logarithms to find all x values such that $\log_6(x+2)-\log_6(x-3)=1$. Show the steps for solving.

Answer

Rewriting as an exponential equation and solving for x:

$$\begin{aligned} 6^1 &= \frac{x+2}{x-3} \\ 0 &= \frac{x+2}{x-3}-6 \\ 0 &= \frac{x+2}{x-3}-\frac{6(x-3)}{(x-3)} \\ 0 &= \frac{x+2-6x+18}{x-3} \\ 0 &= \frac{x-4}{x-3} \\ x &= 4 \end{aligned}$$

Checking, we find that $\log_6(4+2)-\log_6(4-3)=\log_6(6)-\log_6(1)$ is defined, so $x=4$

40) Can the power property of logarithms be derived from the power property of exponents using the equation $b^x=m$? If not, explain why. If so, show the derivation.

41) Prove that $\log_b(n)=\frac{1}{\log_b(n)}$ for any positive integers $b>1$ and $n>1$.

Answer

Let b and n be positive integers greater than 1. Then, by the change-of-base formula, $\log_b(n)=\frac{\log_n(n)}{\log_n(b)}=\frac{1}{\log_n(b)}$

42) Does $\log_{81}(2401)=\log_3(7)$? Verify the claim algebraically.

4.6: Exponential and Logarithmic Equations

Uncontrolled population growth can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Verbal

1) How can an exponential equation be solved?

Answer

Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

2) When does an extraneous solution occur? How can an extraneous solution be recognized?

3) When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

Answer

The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

Algebraic

For the following exercises, use like bases to solve the exponential equation.

4) $4^{-3v-2}=4^{-v}$

5) $64\cdot 4^{3x}=16$

Answer

$x=-\frac{1}{3}$

6) $3^{2x+1}\cdot 3^{x}=243$

7) $2^{-3n}\cdot\frac{1}{4}=2^{n+2}$

Answer

$n=-1$

8) $625\cdot 5^{3x+3}=125$

9) $\frac{36^{3b}}{36^{2b}}=216^{2-b}$

Answer

$b=\frac{6}{5}$

10) $\left(\frac{1}{64}\right)^{3n}\cdot 8=2^{6}$

For the following exercises, use logarithms to solve.

11) $9^{x-10}=1$

Answer

$x=10$

12) $2e^{6x}=13$

13) $e^{r+10}-10=-42$

Answer

No solution

14) $2 \cdot 10^{9a} = 29$

15) $-8 \cdot 10^{p+7} - 7 = -24$

Answer

$p = \log\left(\frac{17}{8}\right) - 7$

16) $7e^{3n-5} + 5 = -89$

17) $e^{-3k} + 6 = 44$

Answer

$k = -\frac{\ln(38)}{3}$

18) $-5e^{9x-8} - 8 = -62$

19) $-6e^{9x+8} + 2 = -74$

Answer

$x = \frac{\frac{\ln(38)}{3} - 8}{9}$

20) $2^{x+1} = 5^{2x-1}$

21) $e^{2x} - e^{x} - 132 = 0$

Answer

$x = \ln 12$

22) $7e^{8x+8} - 5 = -95$

23) $10e^{8x+3} + 2 = 8$

Answer

$x = \frac{\frac{\ln(3)}{5} - 3}{8}$

24) $4e^{3x+3} - 7 = 53$

25) $8e^{-5x-2} - 4 = -90$

Answer

No solution

26) $3^{2x+1} = 7^{x-2}$

27) $e^{2x} - e^{x} - 6 = 0$

Answer

$x = \ln 3$

28) $3e^{3-3x} + 6 = -31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

29) $\log\left(\frac{1}{100}\right) = -2$

Answer

$$10^{-2} = \frac{1}{100}$$

30) $\log_{324}(18) = \frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

31) $5\log_7 n = 10$

Answer

$$n = 49$$

32) $-8\log_9 x = 16$

33) $4 + \log_2(9k) = 2$

Answer

$$k = \frac{1}{36}$$

34) $2\log(8n+4) + 6 = 10$

35) $10 - 4\ln(9-8x) = 6$

Answer

$$x = \frac{9-e}{8}$$

For the following exercises, use the one-to-one property of logarithms to solve.

36) $\ln(10-3x) = \ln(-4x)$

37) $\log_{13}(5n-2) = \log_{13}(8-5n)$

Answer

$$n = 1$$

38) $\log(x+3) - \log(x) = \log(74)$

39) $\ln(-3x) = \ln(x^2 - 6x)$

Answer

No solution

40) $\log_4(6-m) = \log_4(3m)$

41) $\ln(x-2) - \ln(x) = \ln(54)$

Answer

No solution

42) $\log_9(2n^2 - 14n) = \log_9(-45 + n^2)$

43) $\ln(x^2 - 10) + \ln(9) = \ln(10)$

Answer

$$x = \pm\frac{10}{3}$$

For the following exercises, solve each equation for x.

44) $\log(x+12) = \log(x) + \log(12)$

45) $\ln(x)+\ln(x-3)=\ln(7x)$

Answer

$x=10$

46) $\log_2(7x+6)=3$

47) $\ln(7)+\ln(2-4x^2)=\ln(14)$

Answer

$x=0$

48) $\log_8(x+6)-\log_8(x)=\log_8(58)$

49) $\ln(3)-\ln(3-3x)=\ln(4)$

Answer

$x=\dfrac{3}{4}$

50) $\log_3(3x)-\log_3(6)=\log_3(77)$

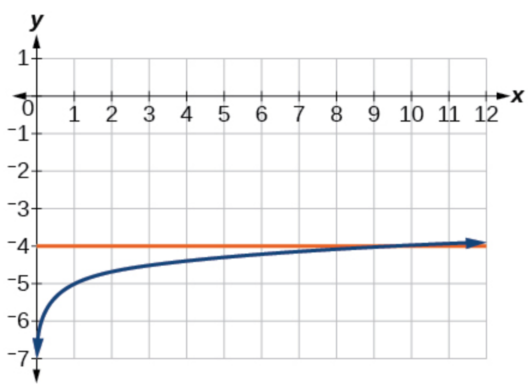
Graphical

For the following exercises, solve the equation for x, if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.

51) $\log_9(x)-5=-4$

Answer

$x=9$

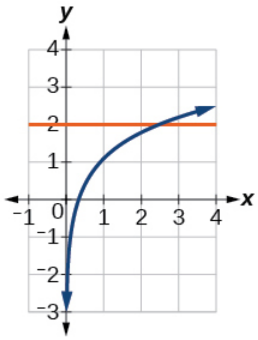


52) $\log_3(x)+3=2$

53) $\ln(3x)=2$

Answer

$x=\dfrac{e^2}{3}\approx 2.5$

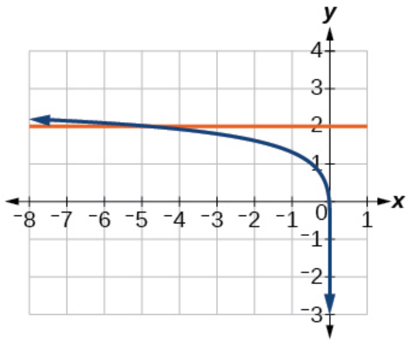


54) $\ln(x-5)=1$

55) $\log(4)+\log(-5x)=2$

Answer

$x=-5$

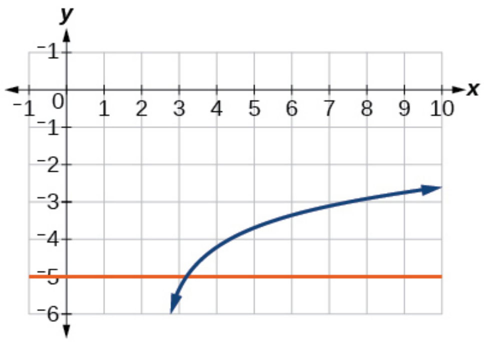


56) $-7+\log_3(4-x)=-6$

57) $\ln(4x-10)-6=-5$

Answer

$x=\dfrac{e+10}{4}\approx 3.2$

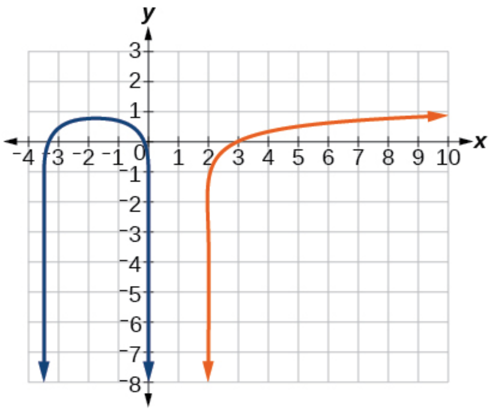


58) $\log(4-2x)=\log(-4x)$

59) $\log_{11}(-2x^2-7x)=\log_{11}(x-2)$

Answer

No solution

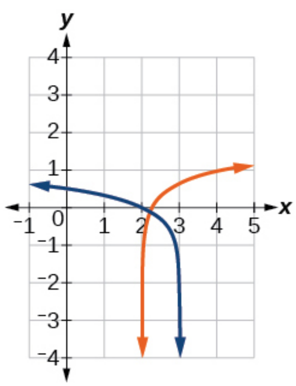


60) $\ln(2x+9)=\ln(-5x)$

61) $\log_9(3-x)=\log_9(4x-8)$

Answer

$$x=\frac{11}{5}\approx 2.2$$

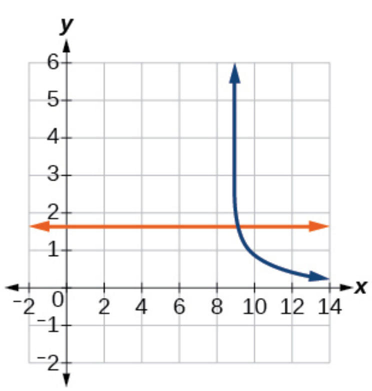


62) $\log(x^2+13)=\log(7x+3)$

63) $\dfrac{3}{\log_2(10)}-\log(x-9)=\log(44)$

Answer

$$x=\frac{101}{11}\approx 9.2$$



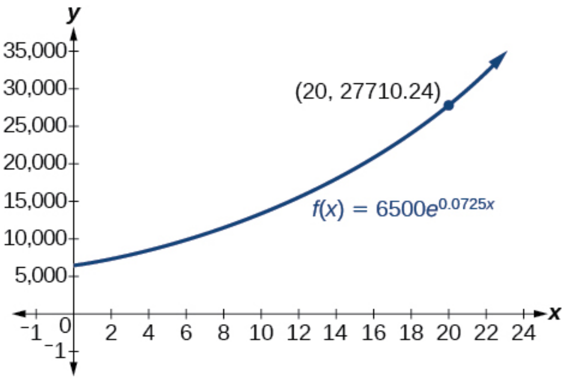
64) $\ln(x)-\ln(x+3)=\ln(6)$

For the following exercises, solve for the indicated value, and graph the situation showing the solution point.

65) An account with an initial deposit of $\$6,500$ earns 7.25% annual interest, compounded continuously. How much will the account be worth after 20 years?

Answer

about $\$27,710.24$

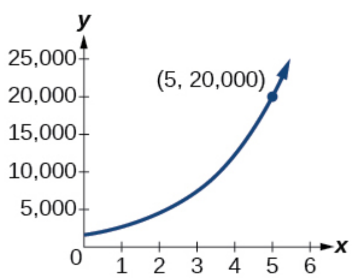


66) The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8 \cdot 3 \cdot 10^2$ watts per square meter?

67) The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach $20,000$?

Answer

about 5 years



Technology

For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to 3 decimal places.

68) $1000(1.03)^t = 5000$ using the common log.

69) $e^{5x} = 17$ using the natural log.

Answer

$$\frac{\ln(17)}{5} \approx 0.567$$

70) $3(1.04)^{3t} = 8$ using the common log

71) $3^{4x-5} = 38$ using the common log

Answer

$$x = \frac{\log(38+5\log(3))}{4\log(3)} \approx 2.078$$

72) $50e^{-0.12t} = 10$ using the natural log

For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.

73) $7e^{3x-5} + 7.9 = 47$

Answer

$x \approx 2.2401$

74) $\ln(3) + \ln(4.4x + 6.8) = 2$

75) $\log(-0.7x - 9) = 1 + 5\log(5)$

Answer

$x \approx -44655.7143$

76) Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch? (Hint: there are 5280 feet in a mile)

77) The magnitude M of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules $E_0 = 10^{4.4}$ and is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

Answer

about 5.83

Extensions

78) Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b x} = x$.

79) Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Answer

$$t = \ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$$

80) Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.

81) Newton's Law of Cooling states that the temperature T of an object at any time t can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$, where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and k is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Answer

$$t = \ln\left(\left(\frac{T-T_s}{T_0-T_s}\right)^{-\frac{1}{k}}\right)$$

4.7: Exponential and Logarithmic Models

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton’s Law of Cooling.

Verbal

1) With what kind of exponential model would *half-life* be associated? What role does half-life play in these models?

Answer

Half-life is a measure of decay and is thus associated with exponential decay models. The half-life of a substance or quantity is the amount of time it takes for half of the initial amount of that substance or quantity to decay.

2) What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.

3) With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?

Answer

Doubling time is a measure of growth and is thus associated with exponential growth models. The doubling time of a substance or quantity is the amount of time it takes for the initial amount of that substance or quantity to double in size.

4) Define Newton’s Law of Cooling. Then name at least three real-world situations where Newton’s Law of Cooling would be applied.

5) What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

Answer

An order of magnitude is the nearest power of ten by which a quantity exponentially grows. It is also an approximate position on a logarithmic scale; Sample response: Orders of magnitude are useful when making comparisons between numbers that differ by a great amount. For example, the mass of Saturn is 95 times greater than the mass of Earth. This is the same as saying that the mass of Saturn is about 10^2 times, or 2 orders of magnitude greater, than the mass of Earth.

Numeric

6) The temperature of an object in degrees Fahrenheit after t minutes is represented by the equation $T(t)=68e^{-0.0174t}+72$. To the nearest degree, what is the temperature of the object after one and a half hours?

For the following exercises, use the logistic growth model $f(x)=\dfrac{150}{1+8e^{-2x}}$

7) Find and interpret $f(0)$. Round to the nearest tenth.

Answer

$f(0)\approx 16.7$; The amount initially present is about 16.7units.

8) Find and interpret $f(4)$. Round to the nearest tenth.

9) Find the carrying capacity.

Answer

150

10) Graph the model.

11) Determine whether the data from the table could best be represented as a function that is linear, exponential, or logarithmic. Then write a formula for a model that represents the data.

x	$f(x)$
–2	0.694

x	$f(x)$
–1	0.833
0	1
1	1.2
2	1.44
3	1.728
4	2.074
5	2.488

Answer

exponential; $f(x) = 1.2^x$

12) Rewrite $f(x) = 1.68(0.65)^x$ as an exponential equation with base e to five significant digits.

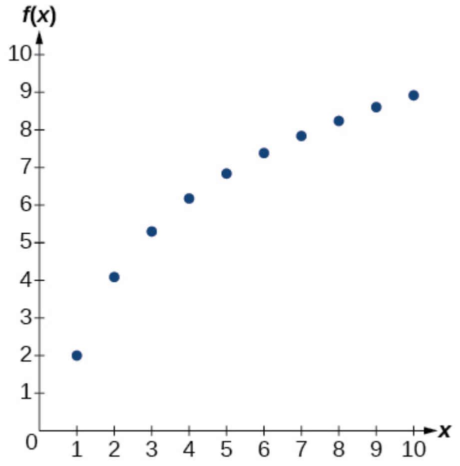
Technology

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table could represent a function that is linear, exponential, or logarithmic.

13)

x	$f(x)$
1	2
2	4.079
3	5.296
4	6.159
5	6.828
6	7.375
7	7.838
8	8.238
9	8.592
10	8.908

Answer



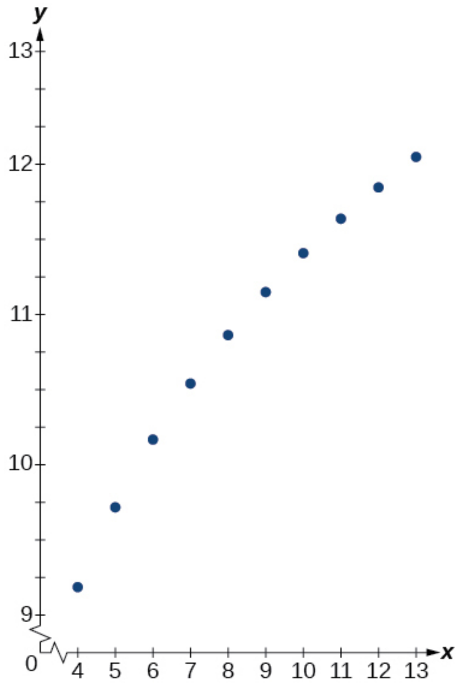
14)

x	$f(x)$
1	2.4
2	2.88
3	3.456
4	4.147
5	4.977
6	5.972
7	7.166
8	8.6
9	10.32
10	12.383

15)

x	$f(x)$
4	9.429
5	9.972
6	10.415
7	10.79
8	11.115
9	11.401
10	11.657
11	11.889
12	12.101
13	12.295

Answer



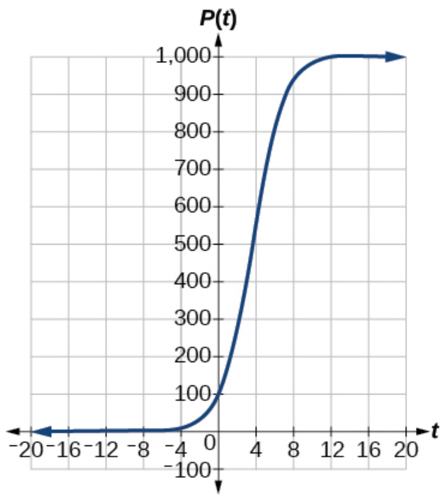
16)

x	$f(x)$
1.25	5.75
2.25	8.75
3.56	12.68
4.2	14.6
5.65	18.95
6.75	22.25
7.25	23.75
8.6	27.8
9.25	29.75
10.5	33.5

For the following exercises, use a graphing calculator and this scenario: the population of a fish farm in t years is modeled by the equation $P(t)=\dfrac{1000}{1+9e^{-0.6t}}$

17) Graph the function.

Answer



18) What is the initial population of fish?

19) To the nearest tenth, what is the doubling time for the fish population?

Answer

about 1.4 years

20) To the nearest whole number, what will the fish population be after 2 years?

21) To the nearest tenth, how long will it take for the population to reach 900?

Answer

about 7.3 years

22) What is the carrying capacity for the fish population? Justify your answer using the graph of P.

Extensions

23) A substance has a half-life of 2.045 minutes. If the initial amount of the substance was 132.8 grams, how many half-lives will have passed before the substance decays to 8.3 grams? What is the total time of decay?

Answer

4 half-lives; 8.18 minutes

24) The formula for an increasing population is given by $P(t) = P_0 e^{rt}$ where P_0 is the initial population and $r > 0$. Derive a general formula for the time t it takes for the population to increase by a factor of M.

25) Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. Show each step for solving this equation algebraically for the seismic moment S.

Answer

$$\begin{aligned} M &= \frac{2}{3}\log\left(\frac{S}{S_0}\right) \\ \log\left(\frac{S}{S_0}\right) &= \frac{3}{2}M \\ \frac{S}{S_0} &= 10^{\frac{3M}{2}} \\ S &= S_0 10^{\frac{3M}{2}} \end{aligned}$$

26) What is the y-intercept of the logistic growth model $y = \dfrac{c}{1 + ae^{-rx}}$? Show the steps for calculation. What does this point tell us about the population?

27) Prove that $b^x = e^{x \ln(b)}$ for positive $b \neq 1$.

Answer

Let $y = b^x$ for some non-negative real number b such that $b \neq 1$. Then,

$$\begin{aligned} \ln(y) &= \ln(b^x) \\ \ln(y) &= x \ln(b) \\ e^{\ln(y)} &= e^{x \ln(b)} \\ y &= e^{x \ln(b)} \end{aligned}$$

Real-World Applications

For the following exercises, use this scenario: A doctor prescribes 125 milligrams of a therapeutic drug that decays by about 30% each hour.

28) To the nearest hour, what is the half-life of the drug?

29) Write an exponential model representing the amount of the drug remaining in the patient's system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 3 hours. Round to the nearest milligram.

Answer

$A = 125e^{(-0.3567t)}$; $A \approx 43$ mg

30) Using the model found in the previous exercise, find $f(10$ and interpret the result. Round to the nearest hundredth.

For the following exercises, use this scenario: A tumor is injected with 0.5 grams of Iodine-125, which has a decay rate of 1.15% per day.

31) To the nearest day, how long will it take for half of the Iodine-125 to decay?

Answer

about 60 days

32) Write an exponential model representing the amount of Iodine-125 remaining in the tumor after t days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after 60 days. Round to the nearest tenth of a gram.

33) A scientist begins with 250 grams of a radioactive substance. After 250 minutes, the sample has decayed to 32 grams. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?

Answer

$f(t) = 250e^{(-0.00914t)}$; half-life: about 76 minutes

34) The half-life of Radium-226 is 1590 years. What is the annual decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

35) The half-life of Erbium-165 is 10.4 hours. What is the hourly decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

Answer

$r \approx -0.0667$, So the hourly decay rate is about 6.67%

36) A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is 5730 years.)

37) A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was 1350 bacteria. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest whole number, what is the population size after 3 hours?

Answer

$f(t)=1350e^{(0.03466t)}$; after 3 hours: $P(180)\approx 691,200$

For the following exercises, use this scenario: A biologist recorded a count of 360 bacteria present in a culture after 5 minutes and 1000 bacteria present after 20 minutes.

38) To the nearest whole number, what was the initial population in the culture?

39) Rounding to six significant digits, write an exponential equation representing this situation. To the nearest minute, how long did it take the population to double?

Answer

$f(t)=256e^{(0.068110t)}$; doubling time: about 10 minutes

For the following exercises, use this scenario: A pot of boiling soup with an internal temperature of 100° Fahrenheit was taken off the stove to cool in a 69° F room. After fifteen minutes, the internal temperature of the soup was 95° F.

40) Use Newton's Law of Cooling to write a formula that models this situation.

41) To the nearest minute, how long will it take the soup to cool to 80° F?

Answer

about 88 minutes

42) To the nearest degree, what will the temperature be after 2 and a half hours?

For the following exercises, use this scenario: A turkey is taken out of the oven with an internal temperature of 165° F and is allowed to cool in a 75° F room. After half an hour, the internal temperature of the turkey is 145° F.

43) Write a formula that models this situation.

Answer

$T(t)=90e^{(-0.008377t)}+75$, where t is in minutes.

44) To the nearest degree, what will the temperature be after 50 minutes?

45) To the nearest minute, how long will it take the turkey to cool to 110° F?

Answer

about 113 minutes

For the following exercises, find the value of the number shown on each logarithmic scale. Round all answers to the nearest thousandth.

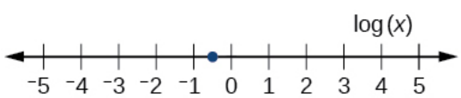


Figure_04_07_207.jpg" src="/@api/deki/files/13777/CNX_PreCalc_Figure_04_07_207.jpg" />

Answer

$\log(x)=1.5; x\approx 31.623$

48) Plot each set of approximate values of intensity of sounds on a logarithmic scale: Whisper: $10^{-10}\frac{W}{m^2}$, Vacuum: $10^{-4}\frac{W}{m^2}$, Jet: $10^{2}\frac{W}{m^2}$

49) Recall the formula for calculating the magnitude of an earthquake, $M=\frac{2}{3}\log\left(\frac{S}{S_0}\right)$. One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake. Round to the nearest hundredth.

Answer

MMS magnitude: 5.82

For the following exercises, use this scenario: The equation $N(t)=\frac{500}{1+49e^{-0.7t}}$ models the number of people in a town who have heard a rumor after t days.

50) How many people started the rumor?

51) To the nearest whole number, how many people will have heard the rumor after 3 days?

Answer

$N(3)\approx 71$

52) As t t t increases without bound, what value does $N(t)$ approach? Interpret your answer.

For the following exercise, choose the correct answer choice.

53) A doctor and injects a patient with 13 milligrams of radioactive dye that decays exponentially. After 12 minutes, there are 4.75 milligrams of dye remaining in the patient’s system. Which is an appropriate model for this situation?

a. $f(t)=13(0.0805)^t$
b. $f(t)=13e^{0.9195t}$
c. $f(t)=13e^{(-0.0839t)}$
d. $f(t)=\frac{4.75}{1+13e^{-0.83925t}}$

Answer

c

4.8: Fitting Exponential Models to Data

We will concentrate on three types of regression models in this section: exponential, logarithmic, and logistic. Having already worked with each of these functions gives us an advantage. Knowing their formal definitions, the behavior of their graphs, and some of their real-world applications gives us the opportunity to deepen our understanding. As each regression model is presented, key features and definitions of its associated function are included for review.

Verbal

1) What situations are best modeled by a logistic equation? Give an example, and state a case for why the example is a good fit.

Answer

Logistic models are best used for situations that have limited values. For example, populations cannot grow indefinitely since resources such as food, water, and space are limited, so a logistic model best describes populations.

2) What is a carrying capacity? What kind of model has a carrying capacity built into its formula? Why does this make sense?

3) What is regression analysis? Describe the process of performing regression analysis on a graphing utility.

Answer

Regression analysis is the process of finding an equation that best fits a given set of data points. To perform a regression analysis on a graphing utility, first list the given points using the STAT then EDIT menu. Next graph the scatter plot using the STAT PLOT feature. The shape of the data points on the scatter graph can help determine which regression feature to use. Once this is determined, select the appropriate regression analysis command from the STAT then CALC menu.

4) What might a scatterplot of data points look like if it were best described by a logarithmic model?

5) What does the y-intercept on the graph of a logistic equation correspond to for a population modeled by that equation?

Answer

The y-intercept on the graph of a logistic equation corresponds to the initial population for the population model.

Graphical

For the following exercises, match the given function of best fit with the appropriate scatterplot in Figure (a) through Figure (e). Answer using the letter beneath the matching graph.

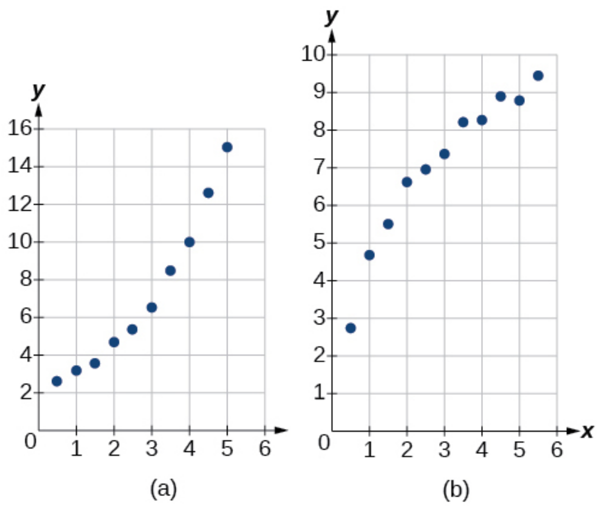
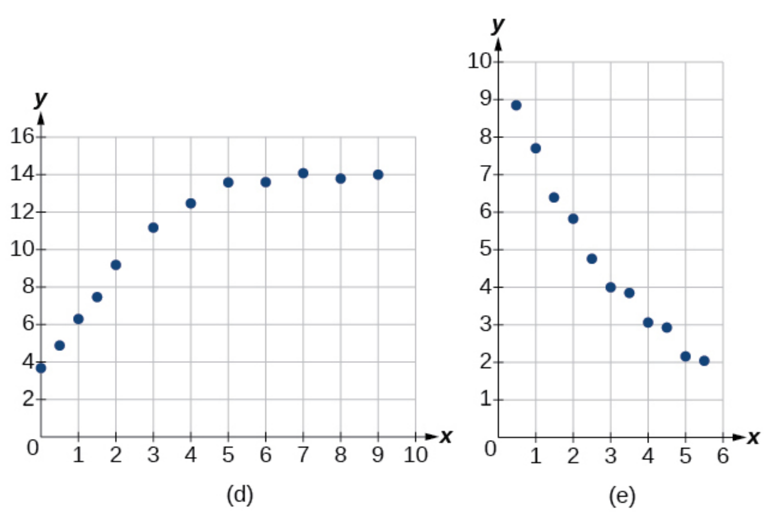


Figure_04_08_203.jpg" src="/@api/deki/files/13803/CNX_PreCalc_Figure_04_08_203.jpg" />



6) $y = 10.209e^{-0.294x}$

7) $y = 5.598 - 1.912\ln(x)$

Answer

c

8) $y = 2.104(1.479)^x$

9) $y = 4.607 + 2.733\ln(x)$

Answer

b

10) $y = \dfrac{14.005}{1 + 2.79e^{-0.812x}}$

Numeric

11) To the nearest whole number, what is the initial value of a population modeled by the logistic equation $P(t) = \dfrac{175}{1 + 6.995e^{-0.68t}}$? What is the carrying capacity?

Answer

$P(0) = 22;\ 175$

12) Rewrite the exponential model $A(t) = 1550(1.085)^x$ as an equivalent model with base e. Express the exponent to four significant digits.

13) A logarithmic model is given by the equation $h(p) = 67.682 - 5.792\ln(p)$. To the nearest hundredth, for what value of p does $h(p) = 62$?

Answer

$p \approx 2.67$

14) A logistic model is given by the equation $P(t) = \dfrac{90}{1 + 5e^{-0.42t}}$. To the nearest hundredth, for what value of t does $P(t) = 45$?

15) What is the y-intercept on the graph of the logistic model given in the previous exercise?

Answer

y-intercept: $(0, 15)$

Technology

For the following exercises, use this scenario: The population P of a koi pond over x months is modeled by the function $P(x) = \dfrac{68}{1 + 16e^{-0.28x}}$.

16) Graph the population model to show the population over a span of 3 years.

17) What was the initial population of koi?

Answer

\(4)\ koi

18) How many koi will the pond have after one and a half years?

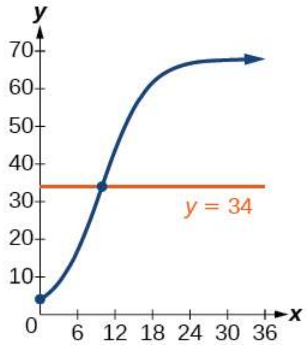
19) How many months will it take before there are 20 koi in the pond?

Answer

about 6.8 months

20) Use the intersect feature to approximate the number of months it will take before the population of the pond reaches half its carrying capacity.

Answer



For the following exercises, use this scenario: The population P of an endangered species habitat for wolves is modeled by the function $P(x) = \dfrac{558}{1 + 54.8e^{-0.462x}}$, where x is given in years.

21) Graph the population model to show the population over a span of 10 years.

22) What was the initial population of wolves transported to the habitat?

Answer

\(10)\ wolves

23) How many wolves will the habitat have after 3 years?

24) How many years will it take before there are 100 wolves in the habitat?

Answer

about 5.4 years

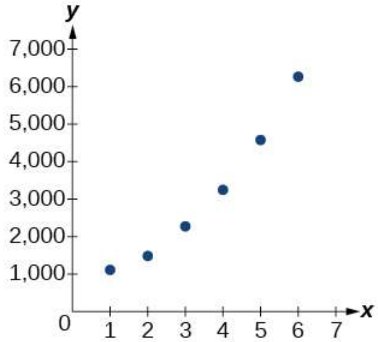
25) Use the intersect feature to approximate the number of years it will take before the population of the habitat reaches half its carrying capacity.

For the following exercises, refer to Table below.

x	$f(x)$
1	1125
2	1495
3	2310
4	3294
5	4650
6	6361

26) Use a graphing calculator to create a scatter diagram of the data.

Answer



27) Use the regression feature to find an exponential function that best fits the data in the table.

28) Write the exponential function as an exponential equation with base e.

Answer

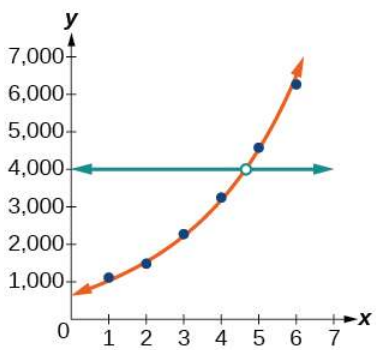
$f(x) = 776.682e^{0.3549x}$

29) Graph the exponential equation on the scatter diagram.

30) Use the intersect feature to find the value of x for which $f(x) = 4000$.

Answer

When $f(x) = 4000$, $x \approx 4.6$



For the following exercises, refer to Table below.

x	$f(x)$
1	555
2	383
3	307
4	210
5	158
6	122

31) Use a graphing calculator to create a scatter diagram of the data.

32) Use the regression feature to find an exponential function that best fits the data in the table.

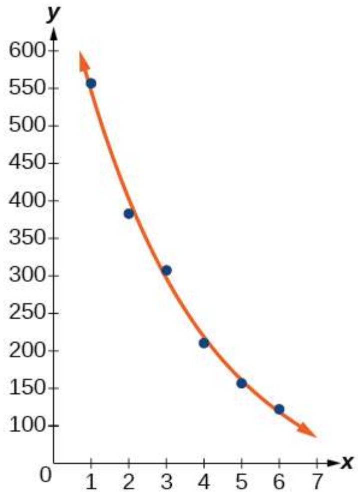
Answer

$f(x) = 731.92(0.738)^x$

33) Write the exponential function as an exponential equation with base e.

34) Graph the exponential equation on the scatter diagram.

Answer



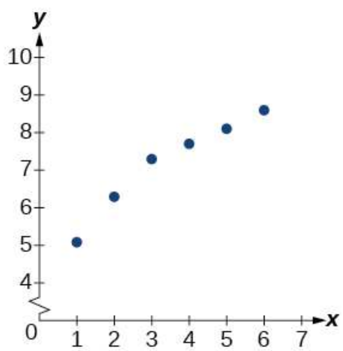
35) Use the intersect feature to find the value of x for which $f(x)=250$.

For the following exercises, refer to Table below.

x	$f(x)$
1	5.1
2	6.3
3	7.3
4	7.7
5	8.1
6	8.6

36) Use a graphing calculator to create a scatter diagram of the data.

Answer



37) Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y=a+b\ln(x)$ that best fits the data in the table.

38) Use the logarithmic function to find the value of the function when $x=10$.

Answer

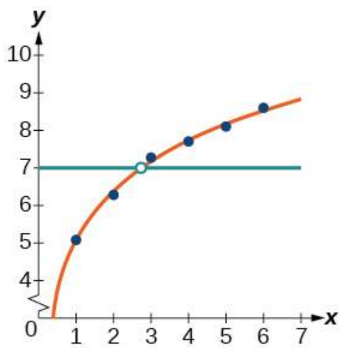
$f(10) \approx 9.5$

39) Graph the logarithmic equation on the scatter diagram.

40) Use the intersect feature to find the value of x for which $f(x) = 7$.

Answer

When $f(x) = 7$, $x \approx 2.7$



For the following exercises, refer to Table below.

x	$f(x)$
1	7.5
2	6
3	5.2
4	4.3
5	3.9
6	3.4
7	3.1
8	2.9

41) Use a graphing calculator to create a scatter diagram of the data.

42) Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

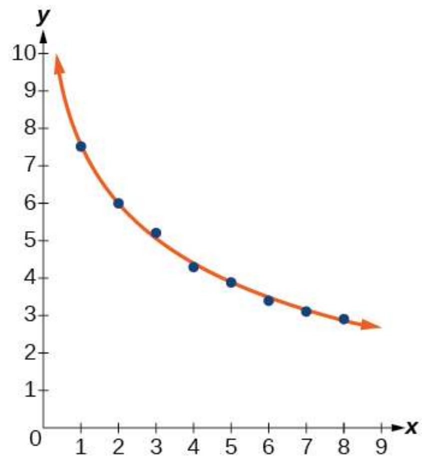
Answer

$f(x) = 7.544 - 2.268\ln(x)$

43) Use the logarithmic function to find the value of the function when $x = 10$.

44) Graph the logarithmic equation on the scatter diagram.

Answer



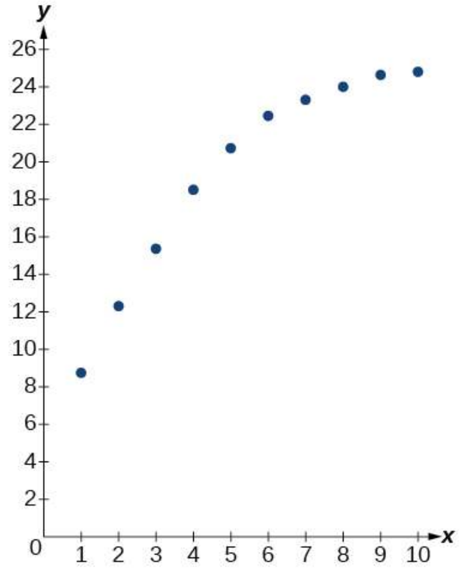
45) Use the intersect feature to find the value of x for which $f(x) = 8$.

For the following exercises, refer to Table below.

x	$f(x)$
1	8.7
2	12.3
3	15.4
4	18.5
5	20.7
6	22.5
7	23.3
8	24
9	24.6
10	24.8

46) Use a graphing calculator to create a scatter diagram of the data.

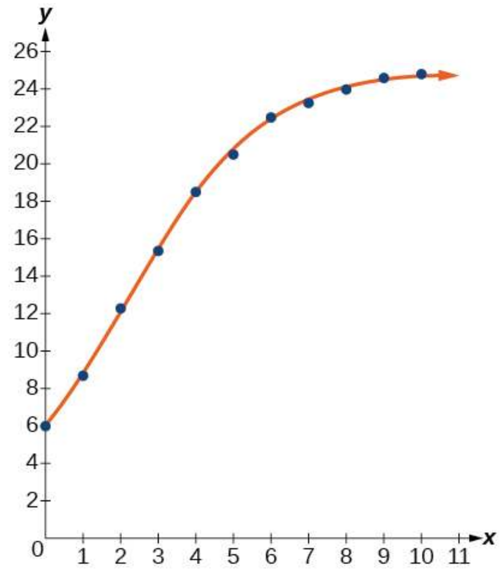
Answer



47) Use the LOGISTIC regression option to find a logistic growth model of the form $y = \frac{c}{1+ae^{-bx}}$ that best fits the data in the table.

48) Graph the logistic equation on the scatter diagram.

Answer

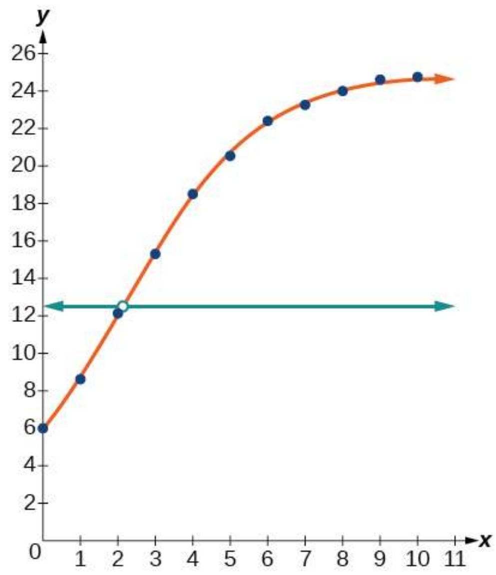


49) To the nearest whole number, what is the predicted carrying capacity of the model?

50) Use the intersect feature to find the value of x for which the model reaches half its carrying capacity.

Answer

When $f(x) = 12.5$, $x \approx 2.1$



For the following exercises, refer to Table below.

x	$f(x)$
0	12
2	28.6
4	52.8
5	70.3
7	99.9
8	112.5
10	125.8
11	127.9
15	135.1
17	135.9

51) Use a graphing calculator to create a scatter diagram of the data.

52) Use the LOGISTIC regression option to find a logistic growth model of the form $y=\dfrac{c}{1+ae^{-bx}}$ that best fits the data in the table.

Answer

$$y=\frac{136.068}{1+10.324e^{-0.480x}}$$

53) Graph the logistic equation on the scatter diagram.

54) To the nearest whole number, what is the predicted carrying capacity of the model?

Answer

about 136

55) Use the intersect feature to find the value of x for which the model reaches half its carrying capacity.

Extensions

56) Recall that the general form of a logistic equation for a population is given by $P(t)=\frac{c}{1+ae^{-bt}}$, such that the initial population at time $t=0$ is $P(0)=P_0$. Show algebraically that $\frac{c-P(t)}{P(t)}=\frac{c-P_0}{P_0}e^{-bt}$.

Answer

Working with the left side of the equation, we see that it can be rewritten as ae^{-bt} :

$$\begin{aligned}\frac{c-P(t)}{P(t)} &= \frac{c-\frac{c}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}}\\ &= \frac{\frac{c(1+ae^{-bt})-c}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}}\\ &= \frac{\frac{c(1+ae^{-bt}-1)}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}}\\ &= 1+ae^{-bt}-1\\ &= ae^{-bt}\end{aligned}$$

$$\begin{aligned}P_0 &= \frac{c}{1+ae^{-b(0)}}\\ &= \frac{c}{1+a}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{c-P_0}{P_0}e^{-bt} &= \frac{c-\frac{c}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= \frac{\frac{c(1+a)-c}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= \frac{\frac{c(1+a-1)}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= (1+a-1)e^{-bt}\\ &= ae^{-bt}\end{aligned}$$

Thus,

$$\frac{c-P(t)}{P(t)}=\frac{c-P_0}{P_0}e^{-bt}$$

57) Use a graphing utility to find an exponential regression formula $f(x)$ and a logarithmic regression formula $g(x)$ for the points $(1.5, 1.5)$ and $(8.5, 8.5$. Round all numbers to 6 decimal places. Graph the points and both formulas along with the line $y=x$ on the same axis. Make a conjecture about the relationship of the regression formulas.

58) Verify the conjecture made in the previous exercise. Round all numbers to six decimal places when necessary.

Answer

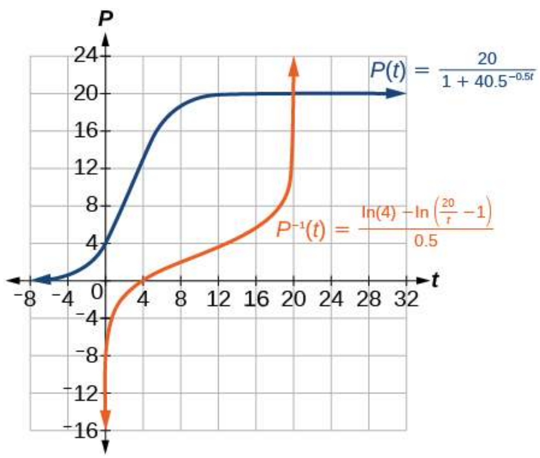
First rewrite the exponential with base e: $f(x)=1.034341e^{0.247800x}$. Then test to verify that $f(g(x))=x$,taking rounding error into consideration:

$$
\begin{aligned}
g(f(x)) &= 4.035510 \ln\left(1.034341 e^{0.247800x}\right) - 0.136259 \\
&= 4.03551\left(\ln(1.034341) + \ln\left(e^{0.2478x}\right)\right) - 0.136259 \\
&= 4.03551(\ln(1.034341) + 0.2478x) - 0.136259 \\
&= 0.136257 + 0.999999x - 0.136259 \\
&= -0.000002 + 0.999999x \\
&\approx 0 + x \\
&= x
\end{aligned}
$$

59) Find the inverse function $f^{-1}(x)$ for the logistic function $f(x) = \dfrac{c}{1+ae^{-bx}}$. Show all steps.

60) Use the result from the previous exercise to graph the logistic model $P(t) = \dfrac{20}{1+4e^{-0.5t}}$ along with its inverse on the same axis. What are the intercepts and asymptotes of each function?

Answer



The graph of $P(t)$ has a y-intercept at $(0,4)$ and horizontal asymptotes at $y=0$ and $y=20$. The graph of $P^{-1}(t)$ has an x-intercept at $(4,0)$ and vertical asymptotes at $x=0$ and $x=20$.

Contributors and Attributions

- Jay Abramson (Arizona State University) with contributing authors. Textbook content produced by OpenStax College is licensed under a Creative Commons Attribution License 4.0 license. Download for free at https://openstax.org/details/books/precalculus.

CHAPTER OVERVIEW

5: Introduction to Calculus

Calculus is the broad area of mathematics dealing with such topics as instantaneous rates of change, areas under curves, and sequences and series. Underlying all of these topics is the concept of a limit, which consists of analyzing the behavior of a function at points ever closer to a particular point, but without ever actually reaching that point. Calculus has two basic applications: differential calculus and integral calculus.

5.1: Prelude to Calculus

5.2: Finding Limits - Numerical and Graphical Approaches

5.3: Finding Limits - Properties of Limits

5.4: Continuity

5.5: Average rate of Change and Derivatives

5.E: Introduction to Calculus (Exercises)

1.1: 5.6 Derivatives of Polynomials

5.R: Introduction to Calculus (Review)

5.1: Prelude to Calculus

The eight-time world champion and winner of six Olympic gold medals in sprinting, Usain Bolt has truly earned his nickname as the “fastest man on Earth.” Also known as the “lightning bolt,” he set the track on fire by running at a top speed of 27.79 mph—the fastest time ever recorded by a human runner.

Like the fastest land animal, a cheetah, Bolt does not run at his top speed at every instant. How then, do we approximate his speed at any given instant? We will find the answer to this and many related questions in this chapter.

5.2: Finding Limits - Numerical and Graphical Approaches

Intuitively, we know what a **limit** is. A car can go only so fast and no faster. A trash can might hold 33 gallons and no more. It is natural for measured amounts to have limits. What, for instance, is the limit to the height of a woman? The tallest woman on record was Jinlian Zeng from China, who was 8 ft 1 in.[1] Is this the limit of the height to which women can grow? Perhaps not, but there is likely a limit that we might describe in inches if we were able to determine what it was.

To put it mathematically, the function whose input is a woman and whose output is a measured height in inches has a limit. In this section, we will examine numerical and graphical approaches to identifying limits.

Definition: Limits

A limit describe the value that a function approaches as x approaches a specific value.

Limit notation

$$\lim_{x \to a} f(x) = L \tag{5.2.1}$$

The function does not have to exist at the x value of the limit.

The function does not exist at x = -6 (open circle).

f(-6) = DNE (Does not exist)

The limit exist since the red line approaches 3 as x approaches -6 from the left side (-).

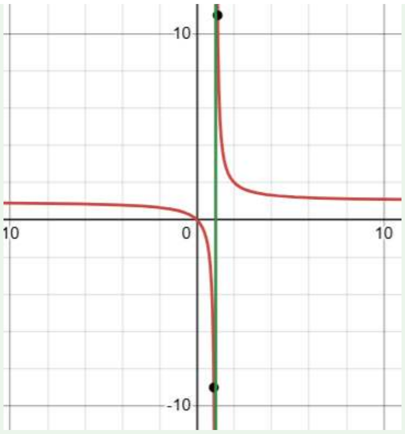
$$\lim_{x \to -6^-} f(x) = 3 \tag{5.2.2}$$

The blue line approaches 3 as x approaches -6 from the right side (+).

$$\lim_{x \to -6^+} f(x) = 3 \tag{5.2.3}$$

The limit exist since the limit from the left is equal to the limit on the right.

$$\lim_{x \to -6} f(x) = 3 \tag{5.2.4}$$



f(1) = DNE (Does not exist)

The limit exist since the red line approaches

$$-\infty \tag{5.2.5}$$

as x approaches -6 from the left side (-).

$$\lim_{x \to 1^-} f(x) = -\infty \tag{5.2.6}$$

The blue line approaches

$$\infty \tag{5.2.7}$$

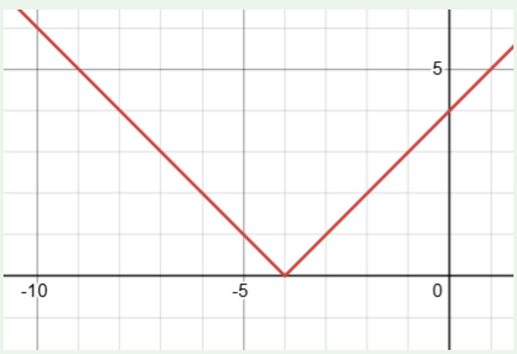
as x approaches -6 from the right side (+).

$$\lim_{x \to 1^+} f(x) = \infty \tag{5.2.8}$$

The limit does not exist since the limit from the left is not equal to the limit on the right.

$$\lim_{x \to 1} f(x) = DNE \tag{5.2.9}$$

The function can exist at the x value.



The function exist at x = -4.

f(-4) = 0

The limit exist since the red line on the left hand side approaches 0 as x is approach -4 (-).

$$\lim_{x \to -4^-} f(x) = 0 \tag{5.2.10}$$

The function exist at x = -4. The limit exist since the red line on the right hand side approaches 0 as x is approaching -4 (+).

$$\lim_{x\to -4^+} f(x) = 0 \tag{5.2.11}$$

The limit exist since the limit from the left is equal to the limit on the right.

$$\lim_{x\to -4} f(x) = 0 \tag{5.2.12}$$

Understanding Limit Notation

We have seen how a **sequence** can have a limit, a value that the sequence of terms moves toward as the nu mber of terms increases. For example, the terms of the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$$

gets closer and closer to 0. A sequence is one type of function, but functions that are not sequences can also have limits. We can describe the behavior of the function as the input values get close to a specific value. If the limit of a function $f(x) = L$, then as the input x gets closer and closer to a, the output *y*-coordinate gets closer and closer to L. We say that the output "approaches" L.

Figure $5.2.1$ provides a visual representation of the mathematical concept of limit. As the input value x approaches a, the output value $f(x)$ approaches L.

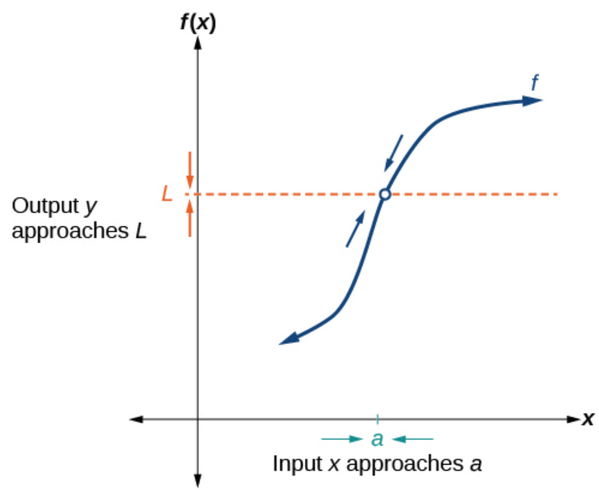


Figure $5.2.1$: The output (*y*--coordinate) approaches L as the input (*x*-coordinate) approaches a.

We write the equation of a limit as

$$\lim_{x\to a} f(x) = L.$$

This notation indicates that as x approaches a both from the left of $x = a$ and the right of $x = a$, the output value approaches L.

Consider the function

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}. \tag{5.2.13}$$

We can factor the function in Equation 5.2.13 as shown.

$$\begin{array}{ll} f(x) = \dfrac{\cancel{(x-7)}(x+1)}{\cancel{x-7}} & \text{Cancel like factors in numerator and denominator.} \\ f(x) = x+1, x \neq 7 & \text{Simplify.} \end{array}$$

Notice that x cannot be 7, or we would be dividing by 0, so 7 is not in the domain of the function in Equation 5.2.13. To avoid changing the function when we simplify, we set the same condition, $x \neq 7$, for the simplified function. We can represent the function graphically as shown in Figure $5.2.2$.

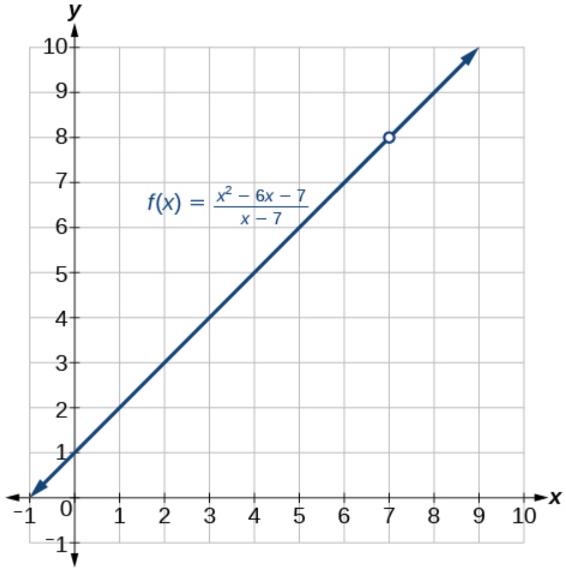


Figure $5.2.2$: Because 7 is not allowed as an input, there is no point at $x = 7$.

What happens at $x = 7$ is completely different from what happens at points close to $x = 7$ on either side. The notation

$$\lim_{x \to 7} f(x) = 8$$

indicates that as the input x approaches 7 from either the left or the right, the output approaches 8. The output can get as close to 8 as we like if the input is sufficiently near 7.

What happens at $x = 7$? When $x = 7$, there is no corresponding output. We write this as

$$f(7) \text{does not exist.}$$

This notation indicates that 7 is not in the domain of the function. We had already indicated this when we wrote the function as

$$f(x) = x + 1, x \neq 7.$$

Notice that the limit of a function can exist even when $f(x)$ is not defined at $x = a$. Much of our subsequent work will be determining limits of functions as x nears a, even though the output at $x = a$ does not exist.

THE LIMIT OF A FUNCTION

A quantity L is the limit of a function $f(x)$ as x approaches a if, as the input values of x approach a (but do not equal a),the corresponding output values of $f(x)$ get closer to L. Note that the value of the limit is not affected by the output value of $f(x)$ at a. Both a and L must be real numbers. We write it as

$$\lim_{x \to a} f(x) = L$$

Example $5.2.1$: Understanding the Limit of a Function

For the following limit, define a, $f(x)$, and L.

$$\lim_{x \to 2} (3x + 5) = 11$$

Solution

First, we recognize the notation of a limit. If the limit exists, as x approaches a, we write

$$\lim_{x \to a} f(x) = L.$$

We are given

$$\lim_{x \to 2} (3x + 5) = 11.$$

This means that $a = 2, f(x) = 3x + 5,$ and $L = 11.$

Analysis

Recall that $y = 3x + 5$ is a line with no breaks. As the input values approach 2, the output values will get close to 11. This may be phrased with the equation $\lim_{x \to 2} (3x + 5) = 11,$ which means that as x nears 2 (but is not exactly 2), the output of the function $f(x) = 3x + 5$ gets as close as we want to $3(2) + 5,$ or 11, which is the limit L, as we take values of x sufficiently near 2 but not at $x = 2.$

Exercise 5.2.1

For the following limit, define a, $f(x)$, and L.

$$\lim_{x \to 5} (2x^2 - 4) = 46$$

Solution

$a = 5, f(x) = 2x^2 - 4,$ and $L = 46.$

Understanding Left-Hand Limits and Right-Hand Limits

We can approach the input of a function from either side of a value—from the left or the right. Figure $5.2.3$ shows the values of

$$f(x) = x + 1, x \neq 7$$

as described earlier and depicted in Figure $5.2.3$.

	Values of x approach 7 from the left ($x < 7$)			$x = 7$	Values of x approach 7 from the right ($x > 7$)		
x	6.9	6.99	6.999	7	7.001	7.01	7.1
$f(x)$	7.9	7.99	7.999	Undefined	8.001	8.01	8.1

Values of output approach the limit, 8 $\longrightarrow$ $\longleftarrow$ Values of output approach the limit, 8

Figure $5.2.3$ are $6.9, 6.99,$ and 6.999. The corresponding outputs are $7.9, 7.99,$ and 7.999. These values are getting closer to 8. The limit of values of $f(x)$ as x approaches from the left is known as the left-hand limit. For this function, 8 is the left-hand limit of the function $f(x) = x + 1, x \neq 7$ as x approaches 7.

Values described as "from the right" are greater than the input value 7 and would therefore appear to the right of the value on a number line. The input values that approach 7 from the right in Figure $5.2.3$ are $7.1, 7.01,$and $7.001.$ The corresponding outputs are $8.1, 8.01,$and $8.001.$ These values are getting closer to 8. The limit of values of $f(x)$ as x approaches from the right is known as the right-hand limit. For this function, 8 is also the right-hand limit of the function $f(x) = x + 1, x \neq 7$ as x approaches 7.

Figure $5.2.3$ shows that we can get the output of the function within a distance of 0.1 from 8 by using an input within a distance of 0.1 from 7. In other words, we need an input x within the interval $6.9 < x < 7.1$ to produce an output value of $f(x)$ within the interval $7.9 < f(x) < 8.1.$

We also see that we can get output values of $f(x)$ successively closer to 8 by selecting input values closer to 7. In fact, we can obtain output values within any specified interval if we choose appropriate input values.

Figure $5.2.4$ provides a visual representation of the left- and right-hand limits of the function. From the graph of $f(x)$, we observe the output can get infinitesimally close to $L = 8$ as x approaches 7 from the left and as x approaches 7 from the right.

To indicate the left-hand limit, we write

$$\lim_{x \to 7^-} f(x) = 8.$$

To indicate the right-hand limit, we write

$$\lim_{x \to 7^+} f(x) = 8.$$

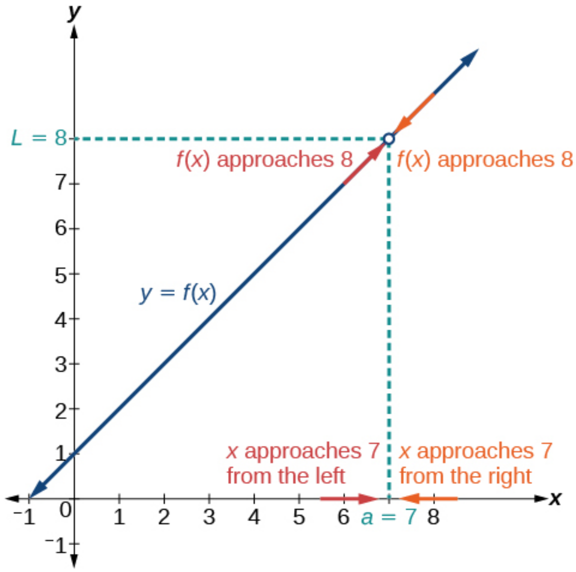


Figure 5.2.4: The left- and right-hand limits are the same for this function.

LEFT- AND RIGHT-HAND LIMITS

The left-hand limit of a function $f(x)$ as x approaches a from the left is equal to L, denoted by

$$\lim_{x \to a^-} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a such that $x < a$ and $x \neq a$.

The right-hand limit of a function $f(x)$, as x approaches a from the right, is equal to L, denoted by

$$\lim_{x \to a^+} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a but greater than a. Both a and L are real numbers.

Understanding Two-Sided Limits

In the previous example, the left-hand limit and right-hand limit as x approaches a are equal. If the left- and right-hand limits are equal, we say that the function $f(x)$ has a two-sided limit as x approaches a. More commonly, we simply refer to a **two-sided limit** as a limit. If the left-hand limit does not equal the right-hand limit, or if one of them does not exist, we say the limit does not exist.

A

The limit of a function $f(x)$, as x approaches a, is equal to L, that is,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

In other words, the left-hand limit of a function $f(x)$ as x approaches a is equal to the right-hand limit of the same function as x approaches a. If such a limit exists, we refer to the limit as a two-sided limit. Otherwise we say the limit does not exist.

Finding a Limit Using a Graph

To visually determine if a limit exists as x approaches a, we observe the graph of the function when x is very near to $x = a$. In Figure $5.2.5$ we observe the behavior of the graph on both sides of a.

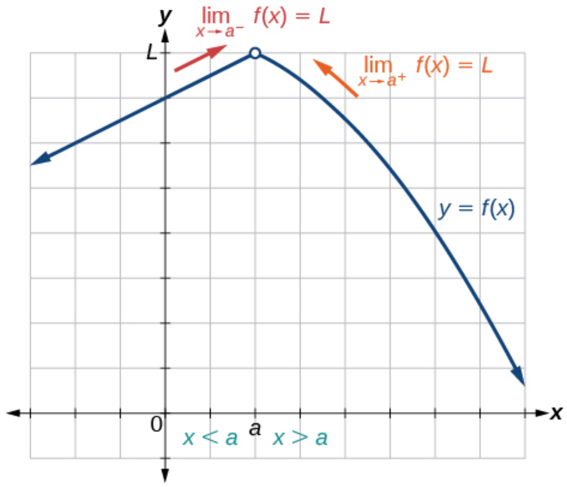


Figure 5.2.5

To determine if a left-hand limit exists, we observe the branch of the graph to the left of $x = a$, but near $x = a$. This is where $x < a$. We see that the outputs are getting close to some real number L so there is a left-hand limit.

To determine if a right-hand limit exists, observe the branch of the graph to the right of $x = a$, but near $x = a$. This is where $x > a$. We see that the outputs are getting close to some real number L, so there is a right-hand limit.

If the left-hand limit and the right-hand limit are the same, as they are in Figure $5.2.5$, then we know that the function has a two-sided limit. Normally, when we refer to a "limit," we mean a two-sided limit, unless we call it a one-sided limit.

Finally, we can look for an output value for the function $f(x)$ when the input value x is equal to a. The coordinate pair of the point would be $(a, f(a))$. If such a point exists, then $f(a)$ has a value. If the point does not exist, as in Figure $5.2.5$, then we say that $f(a)$ does not exist.

HOW TO: Given a function $f(x)$, use a graph to find the limits and a function value as x approaches a.

1. Examine the graph to determine whether a left-hand limit exists.
2. Examine the graph to determine whether a right-hand limit exists.
3. If the two one-sided limits exist and are equal, then there is a two-sided limit—what we normally call a "limit."
4. If there is a point at $x = a$, then $f(a)$ is the corresponding function value.

Example 5.2.2: Finding a Limit Using a Graph

Determine the following limits and function value for the function f shown in Figure $5.2.6$.

a. $\lim_{x\to 2^-} f(x)$
b. $\lim_{x\to 2^+} f(x)$
c. $\lim_{x\to 2} f(x)$
d. $f(2)$

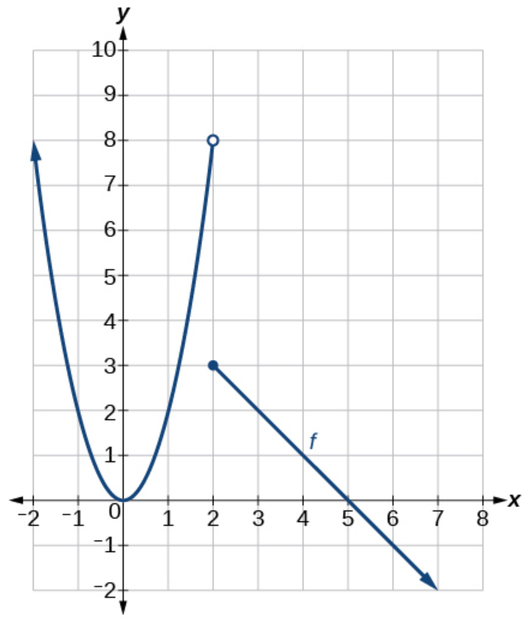


Figure 5.2.6

Determine the following limits and function value for the function f f shown in Figure.

i. $\lim_{x\to 2^-} f(x)$
ii. $\lim_{x\to 2^+} f(x)$
iii. $\lim_{x\to 2} f(x)$
iv. $f(2)$

a. Looking at Figure:

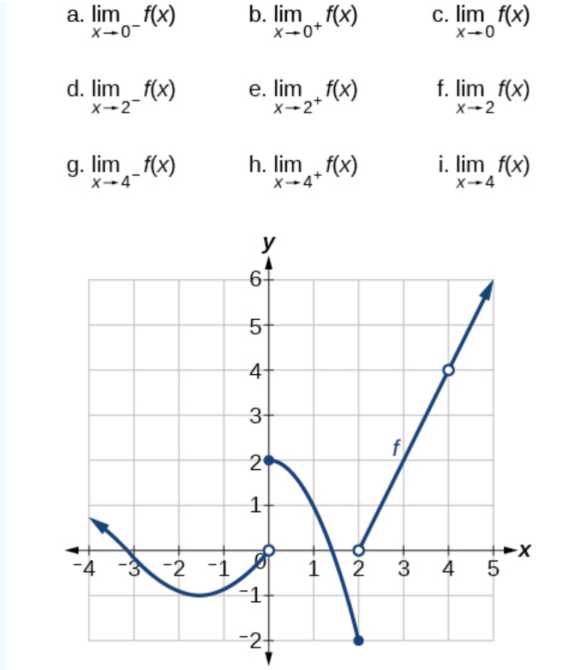
i. $\lim_{x\to 2^-} f(x)=8$; when $x<2$,but infinitesimally close to 2, the output values get close to $y=8$.
ii. $\lim_{x\to 2^+} f(x)=3$; when $x>2$,but infinitesimally close to 2, the output values approach $y=3$.
iii. $\lim_{x\to 2} f(x)$ does not exist because $\lim_{x\to 2^-} f(x)\neq \lim_{x\to 2^+} f(x)$; the left and right-hand limits are not equal.
iv. $f(2)=3$ because the graph of the function f passes through the point $(2,f(2))$ or $(2,3)$.

b. Looking at Figure:

i. $\lim_{x\to 2^-} f(x)=8$; when $x<2$ but infinitesimally close to 2, the output values approach $y=8$.
ii. $\lim_{x\to 2^+} f(x)=8$; when $x>2$ but infinitesimally close to 2, the output values approach $y=8$.
iii. $\lim_{x\to 2} f(x)=8$; because $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=8$; the left and right-hand limits are equal.
iv. $f(2)=4$ because the graph of the function f passes through the point $(2,f(2))$ or $(2,4)$.

Exercise 5.2.2:

Using the graph of the function y=f(x) y=f(x) shown in Figure, estimate the following limits.



Solution

a. 0; b. 2; c. does not exist; d.−2; −2; e. 0; f. does not exist; g. 4; h. 4; i. 4

Finding a Limit Using a Table

Creating a table is a way to determine limits using numeric information. We create a table of values in which the input values of x approach a from both sides. Then we determine if the output values get closer and closer to some real value, the limit L.

Let's consider an example using the following function:

$$\lim_{x \to 5} \left(\frac{x^3 - 125}{x - 5} \right)$$

To create the table, we evaluate the function at values close to $x = 5$. We use some input values less than 5 and some values greater than 5 as in Figure. The table values show that when $x > 5$ but nearing 5, the corresponding output gets close to 75. When $x > 5$ but nearing 5, the corresponding output also gets close to 75.

x	4.9	4.99	4.999	5	5.001	5.01	5.1
f(x)	73.51	74.8501	74.985001	Undefined	75.015001	75.1501	76.51

$\lim_{x \to 5^-} f(x) = 75$ $\lim_{x \to 5^+} f(x) = 75$

Because

$$\lim_{x \to 5^-} f(x) = 75 = \lim_{x \to 5^+} f(x),$$

then

$$\lim_{x \to 5} f(x) = 75.$$

Remember that $f(5)$ does not exist.

How to: Given a function f,use a table to find the limit as x approaches a and the value of $f(a)$, if it exists.

1. Choose several input values that approach a a from both the left and right. Record them in a table.
2. Evaluate the function at each input value. Record them in the table.
3. Determine if the table values indicate a left-hand limit and a right-hand limit.
4. If the left-hand and right-hand limits exist and are equal, there is a two-sided limit.
5. Replace x with a to find the value of $f(a)$.

Example 5.2.3: Finding a Limit Using a Table

Numerically estimate the limit of the following expression by setting up a table of values on both sides of the limit.

$$\lim_{x\to 0}\left(\frac{5\sin(x)}{3x}\right)$$

Solution

We can estimate the value of a limit, if it exists, by evaluating the function at values near $x=0$. We cannot find a function value for $x=0$ directly because the result would have a denominator equal to 0, and thus would be undefined.

$$f(x)=\frac{5\sin(x)}{3x}$$

We create Figure by choosing several input values close to $x=0$, with half of them less than $x=0$ and half of them greater than $x=0$. Note that we need to be sure we are using radian mode. We evaluate the function at each input value to complete the table.

The table values indicate that when $x<0$ but approaching 0, the corresponding output nears $\frac{5}{3}$.

When $x>0$ but approaching 0, the corresponding output also nears $\frac{5}{3}$.

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	1.66389	1.666639	1.666666	Undefined	1.666666	1.666639	1.66389

$$\lim_{x\to 0^-} f(x)=\frac{5}{3} \qquad \lim_{x\to 0^+} f(x)=\frac{5}{3}$$

Because

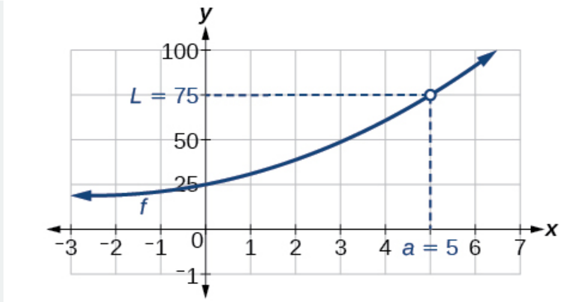
$$\lim_{x\to 0^-} f(x)=\frac{5}{3}=\lim_{x\to 0^+} f(x),$$

then

$$\lim_{x\to 0} f(x)=\frac{5}{3}.$$

Q & A: Is it possible to check our answer using a graphing utility?

Yes. We previously used a table to find a limit of 75 for the function $f(x)=\frac{x^3-125}{x-5}$ *as* x *approaches 5. To check, we graph the function on a viewing window as shown in Figure. A graphical check shows both branches of the graph of the function get close to the output 75 as* x *nears 5. Furthermore, we can use the 'trace' feature of a graphing calculator. By appraoching* $x=5$ *we may numerically observe the corresponding outputs getting close to 75.*



Exercise 5.2.3:

Numerically estimate the limit of the following function by making a table:

$$\lim_{x \to 0} \left(\frac{20 \sin(x)}{4x} \right)$$

Solution

$$\lim_{x \to 0} \left(\frac{20 \sin(x)}{4x} \right) = 5$$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	4.9916708	4.9999167	4.9999992	Error	4.9999992	4.9999167	4.9916708

$\lim_{x \to 0^-} \left(\frac{20 \sin(x)}{4x} \right) \longrightarrow 5$ $\qquad$ $5 \longleftarrow \lim_{x \to 0^+} \left(\frac{20 \sin(x)}{4x} \right)$

Q & A

Is one method for determining a limit better than the other?

No. Both methods have advantages. Graphing allows for quick inspection. Tables can be used when graphical utilities aren't available, and they can be calculated to a higher precision than could be seen with an unaided eye inspecting a graph.

Example 5.2.4: Using a Graphing Utility to Determine a Limit

With the use of a graphing utility, if possible, determine the left- and right-hand limits of the following function as x approaches 0. If the function has a limit as x approaches 0, state it. If not, discuss why there is no limit.

$$f(x) = 3 \sin(\frac{\pi}{x})$$

Solution

We can use a graphing utility to investigate the behavior of the graph close to $x = 0$. Centering around $x = 0$, we choose two viewing windows such that the second one is zoomed in closer to $x = 0$ than the first one. The result would resemble Figure for $[-2, 2]$ by $[-3, 3]$.

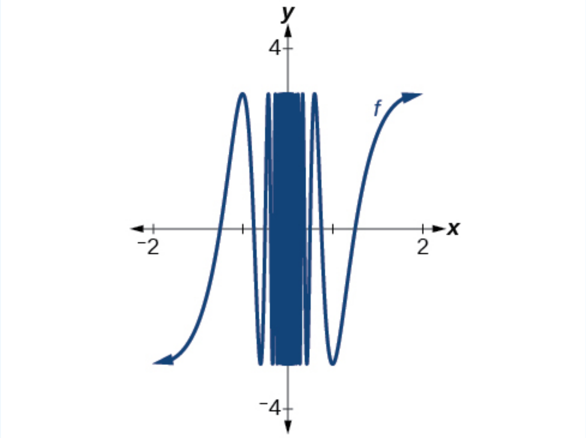


Figure_12_01_012">Figure for$[-0.1, 0.1]$ by $[-3, 3]$.

Even closer to zero, we are even less able to distinguish any limits.

The closer we get to 0, the greater the swings in the output values are. That is not the behavior of a function with either a left-hand limit or a right-hand limit. And if there is no left-hand limit or right-hand limit, there certainly is no limit to the function $f(x)$ as x approaches 0.

We write

$$\lim_{x \to 0^-} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

$$\lim_{x \to 0^+} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

$$\lim_{x \to 0} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

Exercise 5.2.4:

Numerically estimate the following limit: $\lim_{x \to 0}(\sin(\frac{2}{x}))$.

Solution

does not exist

Access these online resources for additional instruction and practice with finding limits.

- Introduction to Limits
- Formal Definition of a Limit

Key Concepts

- A function has a limit if the output values approach some value L as the input values approach some quantity a. a. See Example.
- A shorthand notation is used to describe the limit of a function according to the form $\lim_{x \to a} f(x) = L$, which indicates that as x approaches a, both from the left of $x = a$ and the right of $x = a$, the output value gets close to L.
- A function has a left-hand limit if $f(x)$ approaches L as x approaches a a where $x < a$. A function has a right-hand limit if $f(x)$ approaches L as x approaches a where $x > a$.
- A two-sided limit exists if the left-hand limit and the right-hand limit of a function are the same. A function is said to have a limit if it has a two-sided limit.
- A graph provides a visual method of determining the limit of a function.
- If the function has a limit as x approaches a, the branches of the graph will approach the same $y-$ coordinate near $x = a$ from the left and the right. See Example.

- A table can be used to determine if a function has a limit. The table should show input values that approach a from both directions so that the resulting output values can be evaluated. If the output values approach some number, the function has a limit. See Example.
- A graphing utility can also be used to find a limit. See Example.

Footnotes

- 1 https://en.Wikipedia.org/wiki/Human_height and http://en.Wikipedia.org/wiki/List_of_tallest_people

Glossary

left-hand limit

the limit of values of $f(x)$ as x approaches from a the left, denoted

$$\lim_{x \to a^-} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a a such that $x < a$ and $x \neq a.$. Both a and L are real numbers.

limit

when it exists, the value, L,that the output of a function $f(x)$ approaches as the input x gets closer and closer to a but does not equal a. The value of the output, $f(x)$, can get as close to L as we choose to make it by using input values of x sufficiently near to $x = a$,but not necessarily at $x = a$. Both a and L are real numbers, and L is denoted

$$\lim_{x \to a} f(x) = L.$$

right-hand limit

the limit of values of $f(x)$ as x approaches a from the right, denoted

$$\lim_{x \to a^+} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a where $x > a,$ and $x \neq a.$ Both a and L are real numbers.

two-sided limit

the limit of a function

$$f(x),$$

as x approaches a,is equal to L,that is,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

5.3: Finding Limits - Properties of Limits

Consider the *rational function*

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

The function can be factored as follows:

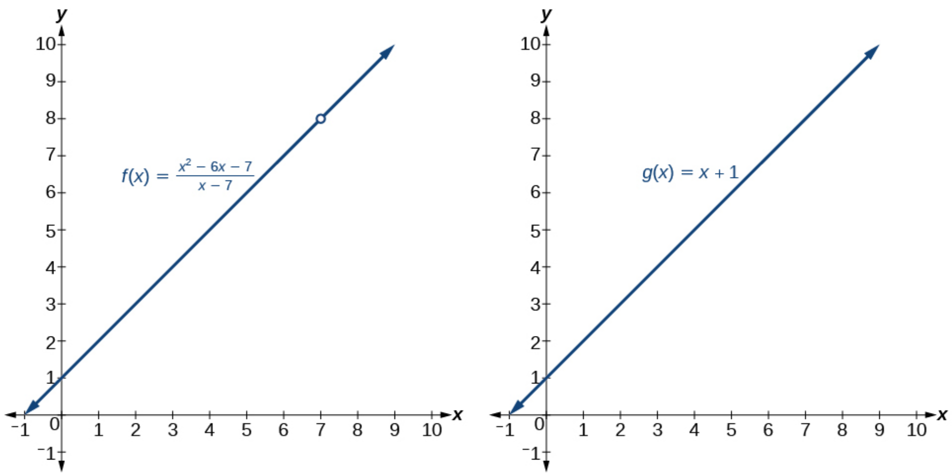
$$f(x) = \frac{\cancel{(x-7)}(x+1)}{\cancel{x-7}}$$

which gives us

$$f(x) = x + 1, x \neq 7.$$

Does this mean the function $f(x)$ is the same as the function $g(x) = x + 1$?

The answer is no. Function $f(x)$ does not have $x = 7$ in its domain, but $g(x)$ does. Graphically, we observe there is a hole in the graph of $f(x)$ at $x = 7$, as shown in Figure and no such hole in the graph of $g(x)$, as shown in Figure.



(left) The graph of function f contains a break at $x = 7$ and is therefore not continuous at $x = 7$. (Right)The graph of function g is continuous.

So, do these two different functions also have different limits as x approaches 7? Not necessarily. Remember, in determining a limit of a function as x approaches a, what matters is whether the output approaches a real number as we get close to $x = a$. The existence of a limit does not depend on what happens when x equals a.

Look again at Figure and Figure. Notice that in both graphs, as x approaches 7, the output values approach 8. This means

$$\lim_{x \to 7} f(x) = \lim_{x \to 7} g(x).$$

Remember that when determining a limit, the concern is what occurs near $x = a$, not at $x = a$. In this section, we will use a variety of methods, such as rewriting functions by factoring, to evaluate the limit. These methods will give us formal verification for what we formerly accomplished by intuition.

Finding the Limit of a Sum, a Difference, and a Product

Graphing a function or exploring a table of values to determine a limit can be cumbersome and time-consuming. When possible, it is more efficient to use the **properties of limits**, which is a collection of theorems for finding limits.

Knowing the properties of limits allows us to compute limits directly. We can add, subtract, multiply, and divide the limits of functions as if we were performing the operations on the functions themselves to find the limit of the result. Similarly, we can find the limit of a function raised to a power by raising the limit to that power. We can also find the limit of the root of a function by taking the root of the limit. Using these operations on limits, we can find the limits of more complex functions by finding the limits of their simpler component functions.

Properties of Limits

Let $a, k, A,$ and B represent real numbers, and f and g be functions, such that $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$. For limits that exist and are finite, the properties of limits are summarized in Table

Constant, k	$\lim_{x \to a} k = k$
Constant times a function	$\lim_{x \to a} [k \cdot f(x)] = k \lim_{x \to a} f(x) = kA$
Sum of functions	$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = A + B$

Difference of functions	$\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)=A-B$
Product of functions	$\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)=A\cdot B$
Quotient of functions	$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{A}{B}, B\neq 0$
Function raised to an exponent	$\lim_{x\to a}[f(x)]^n=[\lim_{x\to\infty}f(x)]^n=A^n$, where n is a positive integer
*n*th root of a function, where n is a positive integer	$\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a}[f(x)]}=\sqrt[n]{A}$
Polynomial function	$\lim_{x\to a}p(x)=p(a)$

Example 5.3.1 : Given Limit of a Function

Given $\lim_{x\to 3}g(x)=3$ and $\lim_{x\to 3}h(x)=5$. Evaluate the following.

a) $\lim_{x\to 3}5g(x)$

b) $\lim_{x\to 3}(g(x)+f(x))$

Solution

a) $\lim_{x\to 3}5g(x)=$ = 5(3) = 15

b) $\lim_{x\to 3}(g(x)+f(x))$ = $\lim_{x\to 3}g(x)$ + $\lim_{x\to 3}h(x)$ = 3 + 5 = 8

Example 5.3.1: Evaluating the Limit of a Function Algebraically

Evaluate

$$\lim_{x\to 3}(2x+5).$$

Solution

$$\begin{aligned}
\lim_{x\to 3}(2x+5) &= \lim_{x\to 3}(2x)+\lim_{x\to 3}(5) && \text{Sum of functions property} \quad (5.3.1)\\
&= 2\lim_{x\to 3}(x)+\lim_{x\to 3}(5) && \text{Constant times a function property} \quad (5.3.2)\\
&= 2(3)+5 && \text{Evaluate} \quad (5.3.3)\\
&= 11 && (5.3.4)
\end{aligned}$$

Exercise 5.3.1:

Evaluate the following limit:

$$\lim_{x\to -12}(-2x+2).$$

Solution

26

Finding the Limit of a Polynomial

Not all functions or their limits involve simple addition, subtraction, or multiplication. Some may include polynomials. Recall that a polynomial is an expression consisting of the sum of two or more terms, each of which consists of a constant and a variable raised to a nonnegative integral power. To find the limit of a polynomial function, we can find the limits of the individual terms of the function, and then add them together. Also, the limit of a polynomial function as x approaches a is equivalent to simply evaluating the function for a.

how to: Given a function containing a polynomial, find its limit

1. Use the properties of limits to break up the polynomial into individual terms.
2. Find the limits of the individual terms.
3. Add the limits together.
4. Alternatively, evaluate the function for a.

Example 5.3.1: Evaluating the Limit of a Function Algebraically

Evaluate

$$\lim_{x\to 3}(5x^2).$$

Solution

$$\begin{align} \lim_{x\to 3}(5x^2) &= 5\lim_{x\to 3}(x^2) && \text{Constant times a function property} \tag{5.3.5}\\ &= 5(3^2) && \text{Function raised to an exponent property} \tag{5.3.6}\\ &= 45 \tag{5.3.7}\end{align}$$

Exercise 5.3.1:

Evaluate

$$\lim_{x\to 4}(x^3-5).$$

Solution

59

Example 5.3.2: Evaluating the Limit of a Polynomial Algebraically

Evaluate

$$\lim_{x\to 5}(2x^3-3x+1).$$

Solution

$$\begin{align} \lim_{x\to 5}(2x^3-3x+1) &= \lim_{x\to 5}(2x^3)-\lim_{x\to 5}(3x)+\lim_{x\to 5}(1) && \text{Sum of functions} \tag{5.3.8}\\ &= 2\lim_{x\to 5}(x^3)-3\lim_{x\to 5}(x)+\lim_{x\to 5}(1) && \text{Constant times a function} \tag{5.3.9}\\ &= 2(5^3)-3(5)+1 && \text{Function raised to an exponent} \tag{5.3.10}\\ &= 236 && \text{Evaluate} \tag{5.3.11}\end{align}$$

Exercise 5.3.2:

Evaluate the following limit:

$$\lim_{x\to -1}(x^4-4x^3+5).$$

Solution

10

Finding the Limit of a Power or a Root

When a limit includes a power or a root, we need another property to help us evaluate it. The square of the limit of a function equals the limit of the square of the function; the same goes for higher powers. Likewise, the square root of the limit of a function equals the limit of the square root of the function; the same holds true for higher roots.

Example 5.3.3: Evaluating a Limit of a Power

Evaluate

$$\lim_{x\to 2}(3x+1)^5.$$

Solution

We will take the limit of the function as x approaches 2 and raise the result to the 5^{th} power.

$$\begin{align} \lim_{x\to 2}(3x+1)^5 &= (\lim_{x\to 2}(3x+1))^5 \tag{5.3.12}\\ &= (3(2)+1)^5 \tag{5.3.13}\\ &= 7^5 \tag{5.3.14}\\ &= 16,807 \tag{5.3.15}\end{align}$$

Exercise $5.3.3$:

Evaluate the following limit: $\lim_{x \to -4} (10x + 36)^3$.

Solution

−64

Q & A: If we can't directly apply the properties of a limit, for example in $\lim_{x \to 2} (\frac{x^2+6x+8}{x-2})$, can we still determine the limit of the function as x approaches a?

Yes. Some functions may be algebraically rearranged so that one can evaluate the limit of a simplified equivalent form of the function.

Finding the Limit of a Quotient

Finding the limit of a function expressed as a quotient can be more complicated. We often need to rewrite the function algebraically before applying the properties of a limit. If the denominator evaluates to 0 when we apply the properties of a limit directly, we must rewrite the quotient in a different form. One approach is to write the quotient in factored form and simplify.

Example $5.3.4$: Evaluating the Limit of a Quotient by Factoring

Evaluate

$$\lim_{x \to 2} (\frac{x^2 - 6x + 8}{x - 2}).$$

Solution

Factor where possible, and simplify.

$$\begin{aligned}
\lim_{x \to 2} (\frac{x^2 - 6x + 8}{x - 2}) &= \lim_{x \to 2} (\frac{(x-2)(x-4)}{x-2}) && \text{Factor the numerator.} \quad (5.3.16) \\
&= \lim_{x \to 2} (\frac{\cancel{(x-2)}(x-4)}{\cancel{x-2}}) && \text{Cancel the common factors.} \quad (5.3.17) \\
&= \lim_{x \to 2} (x-4) && \text{Evaluate.} \quad (5.3.18) \\
&= 2 - 4 = -2 && (5.3.19)
\end{aligned}$$

Analysis

When the limit of a rational function cannot be evaluated directly, factored forms of the numerator and denominator may simplify to a result that can be evaluated.

Notice, the function

$$f(x) = \frac{x^2 - 6x + 8}{x - 2}$$

is equivalent to the function

$$f(x) = x - 4, x \neq 2.$$

Notice that the limit exists even though the function is not defined at $x = 2$.

Exercise $5.3.4$

Evaluate the following limit:

$$\lim_{x \to 7} \left(\frac{x^2 - 11x + 28}{7 - x} \right).$$

Solution

-3

Example $5.3.5$: Evaluating the Limit of a Quotient by Finding the LCD

Evaluate

$$\lim_{x \to 5} \left(\frac{\frac{1}{x} - \frac{1}{5}}{x - 5} \right).$$

Solution

Find the LCD for the denominators of the two terms in the numerator, and convert both fractions to have the LCD as their denominator.

Analysis

When determining the limit of a rational function that has terms added or subtracted in either the numerator or denominator, the first step is to find the common denominator of the added or subtracted terms; then, convert both terms to have that denominator, or simplify the rational function by multiplying numerator and denominator by the least common denominator. Then check to see if the resulting numerator and denominator have any common factors.

Exercise 5.3.5:

Evaluate

$$\lim_{x\to -5}\left(\frac{\frac{1}{5}+\frac{1}{x}}{10+2x}\right).$$

Solution

$-\frac{1}{50}$

how to: Given a limit of a function containing a root, use a conjugate to evaluate

1. If the quotient as given is not in indeterminate $\left(\frac{0}{0}\right)$ form, evaluate directly.
2. Otherwise, rewrite the sum (or difference) of two quotients as a single quotient, using the **least common denominator (LCD).**
3. If the numerator includes a root, rationalize the numerator; multiply the numerator and denominator by the **conjugate** of the numerator. Recall that $a\pm\sqrt{b}$ are conjugates.
4. Simplify.
5. Evaluate the resulting limit.

Example 5.3.6: Evaluating a Limit Containing a Root Using a Conjugate

Evaluate

$$\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right).$$

Solution

$$\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right)=\lim_{x\to 0}\left(\frac{(\sqrt{25-x}-5)}{x}\cdot\frac{(\sqrt{25-x}+5)}{(\sqrt{25-x}+5)}\right) \quad \text{Multiply numerator and denominator by the conjugate.} \tag{5.3.20}$$

$$=\lim_{x\to 0}\left(\frac{(25-x)-25}{x(\sqrt{25-x}+5)}\right) \quad \text{Multiply: } (\sqrt{25-x}-5)\cdot(\sqrt{25-x}+5)=(25-x)-25. \tag{5.3.21}$$

$$=\lim_{x\to 0}\left(\frac{-\cancel{x}}{\cancel{x}(25-x+5)}\right) \quad \text{Combine like terms.} \tag{5.3.22}$$

$$=\lim_{x\to 0}\left(\frac{-\cancel{x}}{\cancel{x}(\sqrt{25-x}+5)}\right) \quad \text{Simplify } \frac{-x}{x}=-1. \tag{5.3.23}$$

$$=\frac{-1}{\sqrt{25-0}+5} \quad \text{Evaluate.} \tag{5.3.24}$$

$$=\frac{-1}{5+5}=-\frac{1}{10} \tag{5.3.25}$$

Analysis

When determining a **limit** of a function with a root as one of two terms where we cannot evaluate directly, think about multiplying the numerator and denominator by the conjugate of the terms.

Exercise 5.3.6

Evaluate the following limit: $\lim_{h\to 0}\left(\frac{\sqrt{16-h}-4}{h}\right)$.

Solution

$-\frac{1}{8}$

Example 5.3.7: Evaluating the Limit of a Quotient of a Function by Factoring

Evaluate

$$\lim_{x\to 4}\left(\frac{4-x}{\sqrt{x}-2}\right).$$

Solution

$$\begin{aligned}
\lim_{x\to 4}\left(\frac{4-x}{\sqrt{x}-2}\right) &= \lim_{x\to 4}\left(\frac{(2+\sqrt{x})(2-x)}{\sqrt{x}-2}\right) && \text{Factor.} && (5.3.26)\\
&= \lim_{x\to 4}\left(\frac{(2+\sqrt{x})(\cancel{2-\sqrt{x}})}{-\cancel{(2-\sqrt{x})}}\right) && \text{Factor } -1 \text{ out of the denominator. Simplify.} && (5.3.27)\\
&= \lim_{x\to 4} -(2+x) && \text{Evaluate.} && (5.3.28)\\
&= -(2+\sqrt{4}) && && (5.3.29)\\
&= -4 && && (5.3.30)
\end{aligned}$$

Analysis

Multiplying by a conjugate would expand the numerator; look instead for factors in the numerator. Four is a perfect square so that the numerator is in the form

$$a^2-b^2$$

and may be factored as

$$(a+b)(a-b).$$

Exercise 5.3.7

Evaluate the following limit:

$$\lim_{x\to 3}\left(\frac{x-3}{\sqrt{x}-\sqrt{3}}\right).$$

Solution

$2\sqrt{3}$

how to: Given a quotient with absolute values, evaluate its limit

1. Try factoring or finding the LCD.
2. If the **limit** cannot be found, choose several values close to and on either side of the input where the function is undefined.
3. Use the numeric evidence to estimate the limits on both sides.

Example 5.3.8: Evaluating the Limit of a Quotient with Absolute Values

Evaluate

$$\lim_{x\to 7}\frac{|x-7|}{x-7}.$$

Solution

The function is undefined at $x=7$, so we will try values close to 7 from the left and the right.

Left-hand limit:

$$\frac{|6.9-7|}{6.9-7}=\frac{|6.99-7|}{6.99-7}=\frac{|6.999-7|}{6.999-7}=-1$$

Right-hand limit:

$$\frac{|7.1-7|}{7.1-7}=\frac{|7.01-7|}{7.01-7}=\frac{|7.001-7|}{7.001-7}=1$$

Since the left- and right-hand limits are not equal, there is no limit.

Exercise 5.3.8

Evaluate

$$\lim_{x\to 6^+}\frac{6-x}{|x-6|}.$$

Solution

Key Concepts

- The properties of limits can be used to perform operations on the limits of functions rather than the functions themselves. See Example.
- The limit of a polynomial function can be found by finding the sum of the limits of the individual terms. See Example and Example.
- The limit of a function that has been raised to a power equals the same power of the limit of the function. Another method is direct substitution. See Example.
- The limit of the root of a function equals the corresponding root of the limit of the function.
- One way to find the limit of a function expressed as a quotient is to write the quotient in factored form and simplify. See Example.
- Another method of finding the limit of a complex fraction is to find the LCD. See Example.
- A limit containing a function containing a root may be evaluated using a conjugate. See Example.
- The limits of some functions expressed as quotients can be found by factoring. See Example.
- One way to evaluate the limit of a quotient containing absolute values is by using numeric evidence. Setting it up piecewise can also be useful. See Example.

Glossary

properties of limits

a collection of theorems for finding limits of functions by performing mathematical operations on the limits

5.4: Continuity

Arizona is known for its dry heat. On a particular day, the temperature might rise as high as $118^{\circ}F$ and drop down only to a brisk $95^{\circ}F$. Figure $5.4.1$ shows the function T, where the output of $T(x)$ is the temperature in Fahrenheit degrees and the input x is the time of day, using a 24-hour clock on a particular summer day.

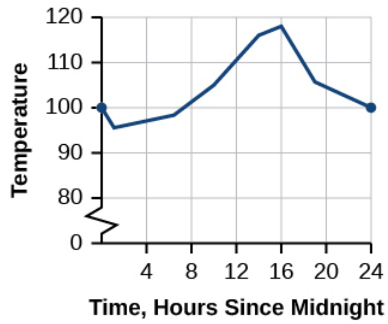


Figure $5.4.1$*: Temperature as a function of time forms a continuous function.*

When we analyze this graph, we notice a specific characteristic. There are no breaks in the graph. We could trace the graph without picking up our pencil. This single observation tells us a great deal about the function. In this section, we will investigate functions with and without breaks.

Determining Whether a Function Is Continuous at a Number

Let's consider a specific example of temperature in terms of date and location, such as June 27, 2013, in Phoenix, AZ. The graph in Figure $5.4.1$ indicates that, at 2 a.m., the temperature was $96^{\circ}F$. By 2 p.m. the temperature had risen to $116^{\circ}F$, and by 4 p.m. it was $118^{\circ}F$. Sometime between 2 a.m. and 4 p.m., the temperature outside must have been exactly $110.5^{\circ}F$. In fact, any temperature between $96^{\circ}F$ and $118^{\circ}F$ occurred at some point that day. This means all real numbers in the output between $96^{\circ}F$ and $118^{\circ}F$ are generated at some point by the function according to the intermediate value theorem,

Look again at Figure $5.4.1$. There are no breaks in the function's graph for this 24-hour period. At no point did the temperature cease to exist, nor was there a point at which the temperature jumped instantaneously by several degrees. A function that has no holes or breaks in its graph is known as a continuous function. Temperature as a function of time is an example of a continuous function.

If temperature represents a continuous function, what kind of function would not be continuous? Consider an example of dollars expressed as a function of hours of parking. Let's create the function D, where $D(x)$ is the output representing cost in dollars for parking x number of hours (Figure $5.4.2$).

Suppose a parking garage charges $4.00 per hour or fraction of an hour, with a $25 per day maximum charge. Park for two hours and five minutes and the charge is $12. Park an additional hour and the charge is $16. We can never be charged $13, $14, or $15. There are real numbers between 12 and 16 that the function never outputs. There are breaks in the function's graph for this 24-hour period, points at which the price of parking jumps instantaneously by several dollars.

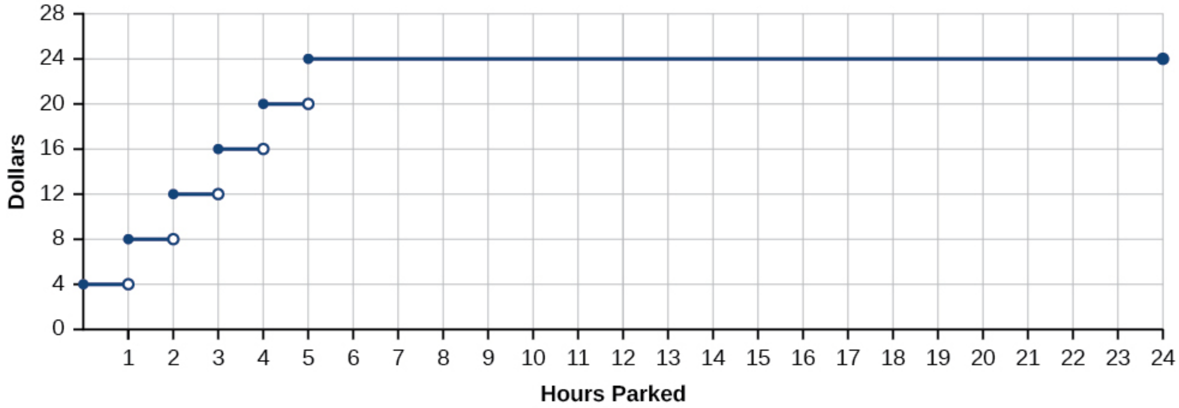


Figure 5.4.2: Parking-garage charges form a discontinuous function.

A function that remains level for an interval and then jumps instantaneously to a higher value is called a **stepwise** function. This function is an example.

A function that has any hole or break in its graph is known as a **discontinuous function**. A stepwise function, such as parking-garage charges as a function of hours parked, is an example of a discontinuous function.

So how can we decide if a function is continuous at a particular number? We can check three different conditions. Let's use the function $y = f(x)$ represented in Figure as an example.

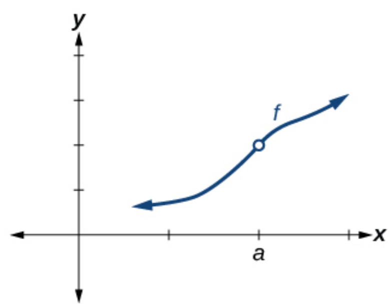


Figure_12_03_004">Figure.

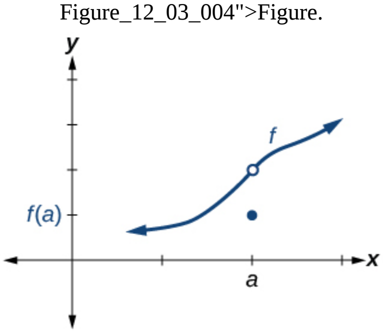


Figure_12_03_003">Figure approaches $x = a$ from the left and right, the same *y*-coordinate is approached. Therefore, Condition 2 is satisfied. However, there could still be a hole in the graph at $x = a$.

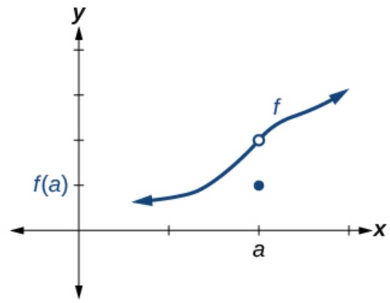
Condition 3 According to Condition 3, the corresponding y y coordinate at $x = a$ fills in the hole in the graph of f. This is written $\lim_{x \to a} f(x) = f(a)$.

Satisfying all three conditions means that the function is continuous. All three conditions are satisfied for the function represented in Figure so the function is continuous as $x = a$.

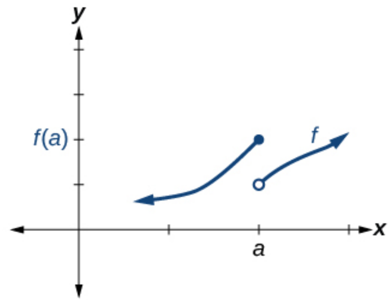
All three conditions are satisfied. The function is continuous at $x = a$.

Figure through Figure provide several examples of graphs of functions that are not continuous at $x = a$ and the condition or conditions that fail.

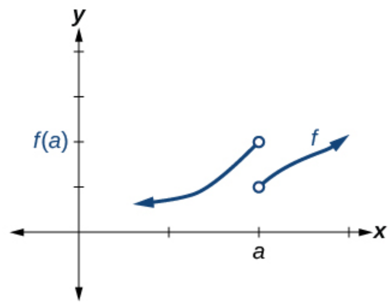
Condition 2 is satisfied. Conditions 1 and 3 both fail.



Conditions 1 and 2 are both satisfied. Condition 3 fails.



Condition 1 is satisfied. Conditions 2 and 3 fail.



Conditions 1, 2, and 3 all fail.

Definition of continuity

A function $f(x)$ is **continuous** at $x = a$ provided all three of the following conditions hold true:

- Condition 1: $f(a)$ exists.
- Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
- Condition 3: $\lim_{x \to a} f(x) = f(a)$

If a function $f(x)$ is not continuous at $x = a$,the function is **discontinuous** at $x = a$.

Identifying a Jump Discontinuity

Discontinuity can occur in different ways. We saw in the previous section that a function could have a left-hand limit and a right-hand limit even if they are not equal. If the left- and right-hand limits exist but are different, the graph “jumps” at $x = a$. The function is said to have a jump discontinuity.

As an example, look at the graph of the function $y = f(x)$ in Figure. Notice as x approaches a how the output approaches different values from the left and from the right.

Graph of a function with a jump discontinuity.

JUMP DISCONTINUITY

A function $f(x)$ has a jump discontinuity at $x=a$ if the left- and right-hand limits both exist but are not equal: $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

Identifying Removable Discontinuity

Some functions have a discontinuity, but it is possible to redefine the function at that point to make it continuous. This type of function is said to have a removable discontinuity. Let's look at the function $y=f(x)$ represented by the graph in Figure. The function has a limit. However, there is a hole at $x=a$. The hole can be filled by extending the domain to include the input $x=a$ and defining the corresponding output of the function at that value as the limit of the function at $x=a$.

Graph of function f with a removable discontinuity at $x=a$.

removable discontinuity

A function f(x) f(x) has a removable discontinuity at $x=a$ if the limit, $\lim_{x \to a} f(x)$, exists, but either

1. $f(a)$ does not exist *or*
2. $f(a)$, the value of the function at $x=a$ does not equal the limit, $f(a) \neq \lim_{x \to a} f(x)$.

Example 5.4.1: Identifying Discontinuities

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x)=\frac{x^2-2x-15}{x-5}$

b.
$$g(x)=\begin{cases} x+1, & x<2 \\ -x, & x \geq 2 \end{cases} \tag{5.4.1}$$

---

a. Notice that the function is defined everywhere except at $x=5$.

Thus, $f(5)$ does not exist, Condition 2 is not satisfied. Since Condition 1 is satisfied, the limit as x approaches 5 is 8, and Condition 2 is not satisfied.This means there is a removable discontinuity at $x=5$.

b. Condition 2 is satisfied because $g(2)=-2$.

Notice that the function is a **piecewise function**, and for each piece, the function is defined everywhere on its domain. Let's examine Condition 1 by determining the left- and right-hand limits as x approaches 2.

Left-hand limit: $\lim_{x \to 2^-} (x+1)=2+1=3$. The left-hand limit exists.

Right-hand limit: $\lim_{x \to 2^+} (-x)=-2$. The right-hand limit exists. But

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x). \tag{5.4.2}$$

So, $\lim_{x \to 2} f(x)$ does not exist, and Condition 2 fails: There is no removable discontinuity. However, since both left- and right-hand limits exist but are not equal, the conditions are satisfied for a jump discontinuity at $x=2$.

Exercise 5.4.1:

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x)=\frac{x^2-6x}{x-6}$

b. $g(x)=\begin{cases} \sqrt{x}, & 0 \leq x<4 \\ 2x, & x \geq 4 \end{cases}$

---

a. removable discontinuity at $x = 6$;
b. jump discontinuity at $x = 4$

Recognizing Continuous and Discontinuous Real-Number Functions

Many of the functions we have encountered in earlier chapters are continuous everywhere. They never have a hole in them, and they never jump from one value to the next. For all of these functions, the limit of $f(x)$ as x approaches a a is the same as the value of $f(x)$ when $x = a$. So $\lim_{x \to a} f(x) = f(a)$. There are some functions that are continuous everywhere and some that are only continuous where they are defined on their domain because they are not defined for all real numbers.

EXAMPLES OF CONTINUOUS FUNCTIONS

The following functions are continuous everywhere:

Polynomial functions	Ex: $f(x) = x^4 - 9x^2$
Exponential functions	Ex: $f(x) = 4^{x+2} - 5$
Sine functions	Ex: $f(x) = \sin(2x) - 4$
Cosine functions	Ex: $f(x) = -\cos(x + \frac{\pi}{3})$

The following functions are continuous everywhere they are defined on their domain:

Logarithmic functions	Ex: $f(x) = 2\ln(x), x > 0$
Tangent functions	Ex: $f(x) = \tan(x) + 2, x \neq \frac{\pi}{2} + k\pi, k$ is an integer
Rational functions	Ex: $f(x) = \frac{x^2 - 25}{x - 7}, x \neq 7$

how to:Given a function $f(x)$, determine if the function is continuous at $x = a$.

1. Check Condition 1: $f(a)$ exists.
2. Check Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
3. Check Condition 3: $\lim_{x \to a} f(x) = f(a)$.
4. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions is not satisfied, the function is not continuous at $x = a$.

Example 5.4.2: Determining Whether a Piecewise Function is Continuous at a Given Number

Determine whether the function $f(x) = \begin{cases} 4x, & x \leq 3 \\ 8 + x, & x > 3 \end{cases}$ is continuous at

a. $x = 3$
b. $x = \frac{8}{3}$

To determine if the function f is continuous at $x = a$, we will determine if the three conditions of continuity are satisfied at $x = a$.

a. Condition 1: Does $f(a)$ exist?

$$f(3) = 4(3) = 12 \tag{5.4.3}$$
$$\Rightarrow \text{Condition 1 is satisfied.} \tag{5.4.4}$$

Condition 2: Does $\lim_{x \to 3} f(x)$ exist?

To the left of $x=3$, $f(x)=4x$; to the right of $x=3$, $f(x)=8+x$. We need to evaluate the left- and right-hand limits as x approaches 1.

- Left-hand limit: $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-} 4(3)=12$
- Right-hand limit: $\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}(8+x)=8+3=11$

Because $\lim\limits_{x\to 1^-} f(x)\neq \lim\limits_{x\to 1^+} f(x)$, $\lim\limits_{x\to 1} f(x)$ does not exist.

$$\Rightarrow \text{Condition 2 fails.} \tag{5.4.5}$$

There is no need to proceed further. Condition 2 fails at $x=3$. If any of the conditions of continuity are not satisfied at $x=3$, the function $f(x)$ is not continuous at $x=3$.

b. $x=\frac{8}{3}$

Condition 1: Does $f(\frac{8}{3})$ exist?

$$f(\frac{8}{3})=4(\frac{8}{3})=\frac{32}{3} \tag{5.4.6}$$

$$\Rightarrow \text{Condition 1 is satisfied.} \tag{5.4.7}$$

Condition 2: Does $\lim\limits_{x\to \frac{8}{3}} f(x)$ exist?

To the left of $x=\frac{8}{3}$, $f(x)=4x$; to the right of $x=\frac{8}{3}$, $f(x)=8+x$. We need to evaluate the left- and right-hand limits as x approaches $\frac{8}{3}$.

- Left-hand limit: $\lim\limits_{x\to \frac{8}{3}^-} f(x)=\lim\limits_{x\to \frac{8}{3}^-} 4(\frac{8}{3})=\frac{32}{3}$
- Right-hand limit: $\lim\limits_{x\to \frac{8}{3}^+} f(x)=\lim\limits_{x\to \frac{8}{3}^+}(8+x)=8+\frac{8}{3}=\frac{32}{3}$

Because $\lim\limits_{x\to \frac{8}{3}} f(x)$ exists,

$$\Rightarrow \text{Condition 2 is satisfied.} \tag{5.4.8}$$

Condition 3: Is $f(\frac{8}{3})=\lim\limits_{x\to \frac{8}{3}} f(x)$?

$$f(\frac{32}{3})=\frac{32}{3}=\lim_{x\to \frac{8}{3}} f(x) \tag{5.4.9}$$

$$\Rightarrow \text{Condition 3 is satisfied.} \tag{5.4.10}$$

Because all three conditions of continuity are satisfied at $x=\frac{8}{3}$, the function $f(x)$ is continuous at $x=\frac{8}{3}$.

Exercise 5.4.2:

Determine whether the function $f(x)=\begin{cases}\frac{1}{x}, & x\leq 2\\ 9x-11.5, & x>2\end{cases}$ is continuous at $x=2$.

yes

Example 5.4.3: Determining Whether a Rational Function is Continuous at a Given Number

Determine whether the function $f(x)=\frac{x^2-25}{x-5}$ is continuous at $x=5$.

To determine if the function f is continuous at $x=5$, we will determine if the three conditions of continuity are satisfied at $x=5$.

Condition 1:

$$f(5) \text{ does not exist.} \tag{5.4.11}$$
$$\Rightarrow \text{Condition 1 fails.} \tag{5.4.12}$$

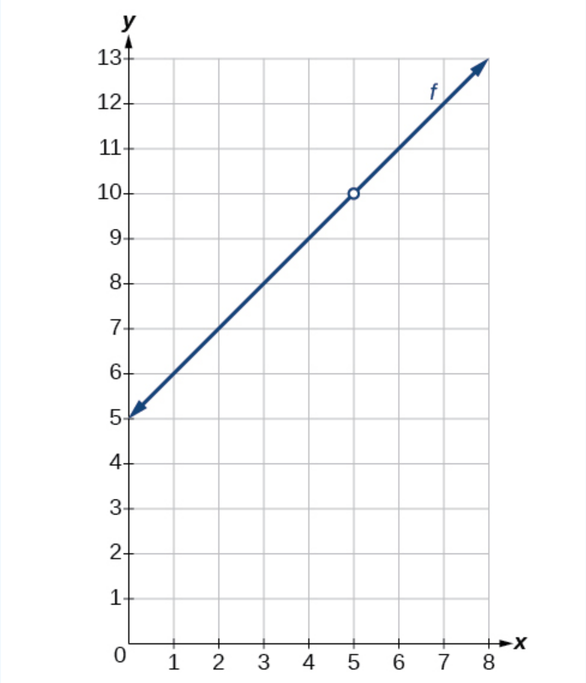
There is no need to proceed further. Condition 2 fails at $x=5$. If any of the conditions of continuity are not satisfied at $x=5$,the function f f is not continuous at $x=5$.

Analysis

See Figure. Notice that for Condition 2 we have

$$\lim_{x\to 5}\frac{x^2-25}{x-5} = \lim_{x\to 3}\frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}} \tag{5.4.13}$$
$$= \lim_{x\to 5}(x+5) \tag{5.4.14}$$
$$= 5+5 = 10 \tag{5.4.15}$$
$$\Rightarrow \text{Condition 2 is satisfied.} \tag{5.4.16}$$

At x=5, x=5,there exists a removable discontinuity. See Figure.



Exercise 5.4.3:

Determine whether the function $f(x)=\frac{9-x^2}{x^2-3x}$ is continuous at $x=3$. If not, state the type of discontinuity.

No, the function is not continuous at $x=3$. There exists a removable discontinuity at $x=3$.

Determining the Input Values for Which a Function Is Discontinuous

Now that we can identify continuous functions, jump discontinuities, and removable discontinuities, we will look at more complex functions to find discontinuities. Here, we will analyze a piecewise function to determine if any real numbers exist where the function is not continuous. A **piecewise function** may have discontinuities at the boundary points of the function as well as within the functions that make it up.

To determine the real numbers for which a piecewise function composed of **polynomial** functions is not continuous, recall that polynomial functions themselves are continuous on the set of real numbers. Any discontinuity would be at the boundary points. So we need to explore the three conditions of continuity at the boundary points of the piecewise function.

how to: Given a piecewise function, determine whether it is continuous at the boundary points

1. For each boundary point a of the piecewise function, determine the left- and right-hand limits as x approaches a, as well as the function value at a.
2. Check each condition for each value to determine if all three conditions are satisfied.
3. Determine whether each value satisfies condition 1: $f(a)$ exists.
4. Determine whether each value satisfies condition 2: $\lim\limits_{x\to a} f(x)$ exists.
5. Determine whether each value satisfies condition 3: $\lim\limits_{x\to a} f(x) = f(a)$.
6. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions fails, the function is not continuous at $x = a$.

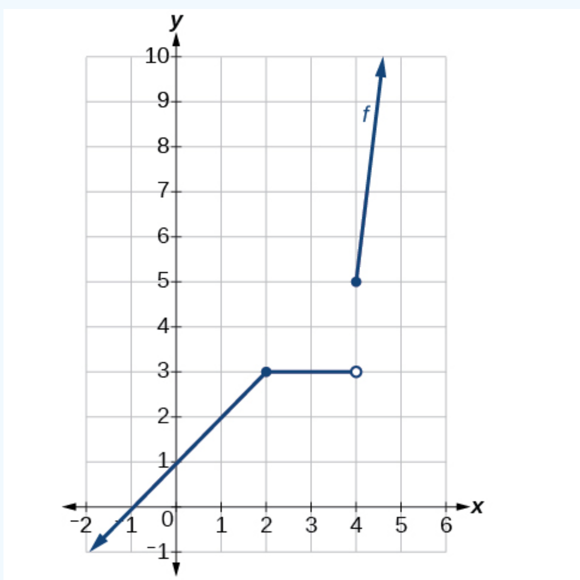
Example 5.4.4: Determining the Input Values for Which a Piecewise Function Is Discontinuous

Determine whether the function f f is discontinuous for any real numbers.

$$fx = \begin{cases} x+1, & x<2 \\ 3, & 2\le x<4 \\ x^2-11, & x\ge 4 \end{cases} \tag{5.4.17}$$

Analysis

See Figure. At $x = 4$,there exists a jump discontinuity. Notice that the function is continuous at $x = 2$.



Graph is continuous at $x = 2$ but shows a jump discontinuity at $x = 4$.

Exercise 5.4.4:

Determine where the function $f(x) = \begin{cases} \frac{\pi x}{4}, & x<2 \\ \frac{\pi}{x}, & 2\le x\le 6 \\ 2\pi x, & x>6 \end{cases}$ is discontinuous.

$x=6$

Determining Whether a Function Is Continuous

To determine whether a **piecewise function** is continuous or discontinuous, in addition to checking the boundary points, we must also check whether each of the functions that make up the piecewise function is continuous.

how to: Given a piecewise function, determine whether it is continuous.

1. Determine whether each component function of the piecewise function is continuous. If there are discontinuities, do they occur within the domain where that component function is applied?
2. For each boundary point $x=a$ of the piecewise function, determine if each of the three conditions hold.

Example 5.4.5: Determining Whether a Piecewise Function Is Continuous

Determine whether the function below is continuous. If it is not, state the location and type of each discontinuity.

$$fx=\begin{cases}\sin(x), & x<0\\ x^3, & x>0\end{cases} \tag{5.4.18}$$

The two functions composing this piecewise function are $f(x)=\sin(x)$ on $x<0$ and $f(x)=x^3$ on $x>0$. The sine function and all polynomial functions are continuous everywhere. Any discontinuities would be at the boundary point,

At $x=0$,let us check the three conditions of continuity.

Condition 1:

$$f(0) \text{ does not exist.} \tag{5.4.19}$$

$$\Rightarrow \text{Condition 1 fails.} \tag{5.4.20}$$

Because all three conditions are not satisfied at $x=0$, the function $f(x)$ is discontinuous at $x=0$.

Analysis

See Figure. There exists a removable discontinuity at $x=0$; $\lim_{x\to 0} f(x)=0$, thus the limit exists and is finite, but $f(a)$ does not exist.

Function has removable discontinuity at 0.

Media

Access these online resources for additional instruction and practice with continuity.

- Continuity at a Point
- Continuity at a Point: Concept Check

Key Concepts

- A continuous function can be represented by a graph without holes or breaks.
- A function whose graph has holes is a discontinuous function.
- A function is continuous at a particular number if three conditions are met:
  - Condition 1: $f(a)$ exists.
  - Condition 2: $\lim_{x\to a} f(x)$ exists at $x=a$.
  - Condition 3: $\lim_{x\to a} f(x)=f(a)$.
- A function has a jump discontinuity if the left- and right-hand limits are different, causing the graph to "jump."
- A function has a removable discontinuity if it can be redefined at its discontinuous point to make it continuous. See Example.
- Some functions, such as polynomial functions, are continuous everywhere. Other functions, such as logarithmic functions, are continuous on their domain. See Example and Example.

- For a piecewise function to be continuous each piece must be continuous on its part of the domain and the function as a whole must be continuous at the boundaries. See Example and Example.

Glossary

continuous function
: a function that has no holes or breaks in its graph

discontinuous function
: a function that is not continuous at $x = a$

jump discontinuity
: a point of discontinuity in a function $f(x)$ at $x = a$ where both the left and right-hand limits exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

removable discontinuity
: a point of discontinuity in a function $f(x)$ where the function is discontinuous, but can be redefined to make it continuous

---

5.5: Average rate of Change and Derivatives

The average teen in the United States opens a refrigerator door an estimated 25 times per day. Supposedly, this average is up from 10 years ago when the average teenager opened a refrigerator door 20 times per day[1]. It is estimated that a television is on in a home 6.75 hours per day, whereas parents spend an estimated 5.5 minutes per day having a meaningful conversation with their children. These averages, too, are not the same as they were 10 years ago, when the television was on an estimated 6 hours per day in the typical household, and parents spent 12 minutes per day in meaningful conversation with their kids. What do these scenarios have in common? The functions representing them have changed over time. In this section, we will consider methods of computing such changes over time.

Finding the Average Rate of Change of a Function

The functions describing the examples above involve a change over time. Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

A **tangent line** to a curve is a line that intersects the curve at only a single point but does not cross it there. (The tangent line may intersect the curve at another point away from the point of interest.) If we zoom in on a curve at that point, the curve appears linear, and the **slope of the curve** at that point is close to the slope of the tangent line at that point.

Figure $5.5.1$ represents the function $f(x)=x^3-4x$. We can see the slope at various points along the curve.

- slope at $x=-2$ is 8
- slope at $x=-1$ is –1
- slope at $x=2$ is 8

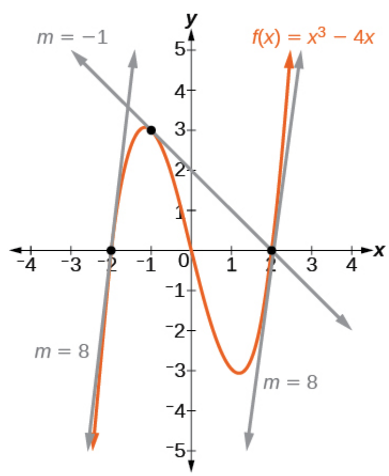


Figure $5.5.1$: Graph showing tangents to curve at –2, –1, and 2.

Let's imagine a point on the curve of function f at $x=a$ as shown in Figure $5.5.1$. The coordinates of the point are $(a, f(a))$. Connect this point with a second point on the curve a little to the right of $x=a$, with an *x*-value increased by some small real number h. The coordinates of this second point are $(a+h, f(a+h))$ for some positive-value h.

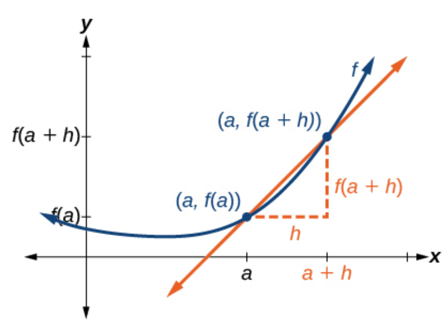


Figure $5.5.2$: Connecting point a with a point just beyond allows us to measure a slope close to that of a tangent line at $x=a$.

We can calculate the slope of the line connecting the two points $(a, f(a))$ and $(a+h, f(a+h))$, called a **secant line**, by applying the slope formula,

$$\text{slope} = \frac{\text{change in y}}{\text{change in x}} \tag{5.5.1}$$

We use the notation m_{sec} to represent the slope of the secant line connecting two points.

$$m_{sec} = \frac{f(a+h)-f(a)}{(a+h)-(a)} \tag{5.5.2}$$

$$= \frac{f(a+h)-f(a)}{\cancel{a}+h-\cancel{a}} \tag{5.5.3}$$

The slope m_{sec} equals the *average rate of change* between two points $(a, f(a))$ and $(a+h, f(a+h))$.

$$m_{sec} = \frac{f(a+h)-f(a)}{h} \tag{5.5.4}$$

the AVERAGE RATE OF CHANGE BETWEEN TWO POINTS ON A CURVE

The **average rate of change** (AROC) between two points $(a, f(a))$ and $(a+h, f(a+h))$ on the curve of f is the slope of the line connecting the two points and is given by

$$\text{AROC} = \frac{f(a+h)-f(a)}{h} \tag{5.5.5}$$

Example 5.5.1: Finding the Average Rate of Change

Find the average rate of change connecting the points $(2, -6)$ and $(-1, 5)$.

Solution

We know the average rate of change connecting two points may be given by

$$\text{AROC} = \frac{f(a+h)-f(a)}{h} \tag{5.5.6}$$

If one point is $(2, -6)$, or $(2, f(2))$, then $f(2) = -6$.

The value h is the displacement from 2 to -1, which equals $-1-2 = -3$.

For the other point, $f(a+h)$ is the *y*-coordinate at $a+h$, which is $2+(-3)$ or -1, so $f(a+h) = f(-1) = 5$.

$$\text{AROC} = \frac{f(a+h)-f(a)}{h} \tag{5.5.7}$$

$$= \frac{5-(-6)}{-3} \tag{5.5.8}$$

$$= \frac{11}{-3} \tag{5.5.9}$$

$$= -\frac{11}{3} \tag{5.5.10}$$

Exercise 5.5.1

Find the average rate of change connecting the points $(-5, 1.5)$ and $(-2.5, 9)$

Solution

3

Understanding the Instantaneous Rate of Change

Now that we can find the average rate of change, suppose we make h in Figure $5.5.2$ smaller and smaller. Then $a+h$ will approach a as h gets smaller, getting closer and closer to 0. Likewise, the second point $(a+h, f(a+h))$ will approach the first point, $(a, f(a))$. As a consequence, the connecting line between the two points, called the secant line, will get closer and closer to being a tangent to the function at $x = a$, and the slope of the secant line will get closer and closer to the slope of the tangent at $x = a$ (Figure $5.5.3$).

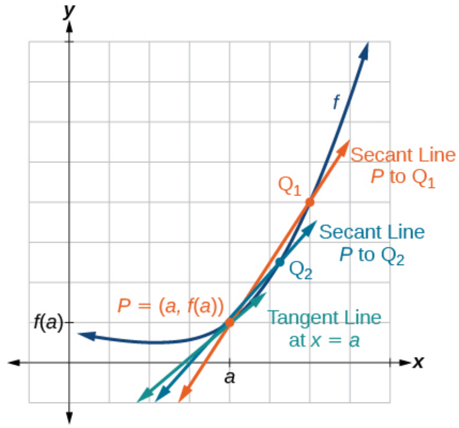


Figure 5.5.3: The connecting line between two points moves closer to being a tangent line at $x=a$.

Because we are looking for the slope of the tangent at $x=a$, we can think of the measure of the slope of the curve of a function f at a given point as the rate of change at a particular instant. We call this slope the **instantaneous rate of change**, or the **derivative** of the function at $x=a$. Both can be found by finding the limit of the slope of a line connecting the point at $x=a$ with a second point infinitesimally close along the curve. For a function f both the instantaneous rate of change of the function and the derivative of the function at $x=a$ are written as $f'(a)$, and we can define them as a two-sided limit that has the same value whether approached from the left or the right.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \tag{5.5.11}$$

The expression by which the limit is found is known as the **difference quotient**.

DEFINITION OF INSTANTANEOUS RATE OF CHANGE AND DERIVATIVE

The derivative, or instantaneous rate of change, of a function f at $x=a$, is given by

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \tag{5.5.12}$$

The expression $\frac{f(a+h)-f(a)}{h}$ is called the difference quotient.

We use the difference quotient to evaluate the limit of the rate of change of the function as h approaches 0.

Derivatives: Interpretations and Notation

The derivative of a function can be interpreted in different ways. It can be observed as the behavior of a graph of the function or calculated as a numerical rate of change of the function.

- The derivative of a function $f(x)$ at a point $x=a$ is the slope of the tangent line to the curve $f(x)$ at $x=a$. The derivative of $f(x)$ at $x=a$ is written $f'(a)$.
- The derivative $f'(a)$ measures how the curve changes at the point $(a,f(a))$.
- The derivative $f'(a)$ may be thought of as the instantaneous rate of change of the function $f(x)$ at $x=a$.
- If a function measures distance as a function of time, then the derivative measures the instantaneous **velocity** at time $t=a$.

NOTATIONS FOR THE DERIVATIVE

The equation of the derivative of a function $f(x)$ is written as $y'=f'(x)$, where $y=f(x)$. The notation $f'(x)$ is read as "f prime of x." Alternate notations for the derivative include the following:

$$f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\frac{d}{dx}f(x)=Df(x) \tag{5.5.13}$$

The expression $f'(x)$ is now a function of x; this function gives the slope of the curve $y=f(x)$ at any value of x. The derivative of a function $f(x)$ at a point $x=a$ is denoted $f'(a)$.

how to: Given a function f, find the derivative by applying the definition of the derivative.

1. Calculate $f(a+h)$.
2. Calculate $f(a)$.
3. Substitute and simplify $\frac{f(a+h)-f(a)}{h}$.
4. Evaluate the limit if it exists: $f'(a)=\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

Example 5.5.1: Finding the Derivative of a Polynomial Function

Find the derivative of the function $f(x)=x^2-3x+5$ at $x=a$.

Solution

We have:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad \text{Definition of a derivative} \tag{5.5.14}$$

Substitute $f(a+h)=(a+h)^2-3(a+h)+5$ and $f(a)=a^2-3a+5$.

$$\begin{aligned}
f'(a) &= \lim_{h\to 0}\frac{(a+h)(a+h)-3(a+h)+5-(a^2-3a+5)}{h} && (5.5.15)\\
&= \lim_{h\to 0}\frac{a^2+2ah+h^2-3a-3h+5-a^2+3a-5}{h} && \text{Evaluate to remove parentheses.} \quad (5.5.16)\\
&= \lim_{h\to 0}\frac{\cancel{a^2}+2ah+h^2-\cancel{3a}-3h+\cancel{5}-\cancel{a^2}+\cancel{3a}-\cancel{5}}{h} && \text{Simplify.} \quad (5.5.17)\\
&= \lim_{h\to 0}\frac{2ah+h^2-3h}{h} && \text{Factor out an } h. \quad (5.5.18)\\
&= 2a+0-3 && \text{Evaluate the limit.} \quad (5.5.19)\\
&= 2a-3 && (5.5.20)
\end{aligned}$$

Exercise 5.5.1

Find the derivative of the function $f(x)=3x^2+7x$ at $x=a$

Solution

$f'(a)=6a+7$

Definition: Derivative of a Polynomial

We **do not** have to use the limit to get a derivative of a polynomial.

$f(x)=ax^n$

The derivative is as follows

$f'(x)=nax^{n-1}$

Repeat this process for each term in the polynomial.

Example 5.5.1

Find the derivative of the function $f(x)=3x^2+7x$ at $x=a$

Solution

The derivative of the function $f'(x)=2*3x^1+1*7x^0$.

$x^1=x$

$x^0=1$

$f'(x)=2*3x+1*7x$.

at x = a

$f'(a)=2*3a+1*7=6a+7$.

Finding Derivatives of Functions with Roots

To find derivatives of functions with roots, we use the methods we have learned to find limits of functions with roots, including multiplying by a conjugate.

Example 5.5.1: Finding the Derivative of a Function with a Root

Find the derivative of the function $f(x)=4\sqrt{x}$ at $x=36$.

We have $f(x)=4x^{1/2}$ the derivative is

$f'(x)=(1/2)4x^{1/2-1}$.
$f'(x)=(1/2)4x^{-1/2}$
$f'(x)=2x^{-1/2}$.

x = 36

$f'(36)=2*36^{-1/2}$
$f'(36)=2/36^{1/2}$
\(f'(36) = 2/6 = 1/3.

Exercise 5.5.1:

Find the derivative of the function $f(x)=9\sqrt{x}$ at $x=9$.

$\frac{3}{2}$

Finding Instantaneous Rates of Change

Many applications of the derivative involve determining the rate of change at a given instant of a function with the independent variable time—which is why the term *instantaneous* is used. Consider the height of a ball tossed upward with an initial velocity of 64 feet per second, given by $s(t)=-16t^2+64t+6$, where t is measured in seconds and $s(t)$ is measured in feet. We know the path is that of a parabola. The derivative will tell us how the height is changing at any given point in time. The height of the ball is shown in Figure as a function of time. In physics, we call this the "*s*-*t* graph."

Example 5.5.1: Finding the Instantaneous Rate of Change

Using the function above, $s(t)=-16t^2+64t+6$,what is the instantaneous velocity of the ball at 1 second and 3 seconds into its flight?

The velocity at $t=1$ and $t=3$ is the instantaneous rate of change of distance per time, or velocity. Notice that the initial height is 6 feet. To find the instantaneous velocity, we find the **derivative** and evaluate it at $t=1$ and $t=3$:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \tag{5.5.21}$$

$$=\lim_{h\to 0}\frac{-16(t+h)^2+64(t+h)+6-(-16t^2+64t+6)}{h} \quad \text{Substitute } s(t+h) \text{ and } s(t). \tag{5.5.22}$$

$$=\lim_{h\to 0}\frac{-16t^2-32ht-h^2+64t+64h+6+16t^2-64t-6}{h} \quad \text{Distribute} \tag{5.5.23}$$

$$=\lim_{h\to 0}\frac{-32ht-h^2+64h}{h} \quad \text{Simplify.} \tag{5.5.24}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(-32t-h+64)}{\cancel{h}} \quad \text{Factor the numerator.} \tag{5.5.25}$$

$$=\lim_{h\to 0}-32t-h+64 \quad \text{Cancel out the common factor} h. \tag{5.5.26}$$

$$s'(t)=-32t+64 \quad \text{Evaluate the limit by letting} h=0. \tag{5.5.27}$$

For any value of t, $s'(t)$ tells us the velocity at that value of t.

Evaluate $t=1$ and $t=3$.

$$s'(1)=-32(1)+64=32 \tag{5.5.28}$$

$$s'(3)=-32(3)+64=-32 \tag{5.5.29}$$

The velocity of the ball after 1 second is 32 feet per second, as it is on the way up.

The velocity of the ball after 3 seconds is −32 feet per second, as it is on the way down.

Exercise 5.5.1:

The position of the ball is given by $s(t) = -16t^2 + 64t + 6$. What is its velocity 2 seconds into flight?

0

Using Graphs to Find Instantaneous Rates of Change

We can estimate an instantaneous rate of change at $x = a$ by observing the slope of the curve of the function $f(x)$ at $x = a$. We do this by drawing a line tangent to the function at $x = a$ and finding its slope.

how to: Given a graph of a function $f(x)$, find the instantaneous rate of change of the function at $x = a$.

1. Locate $x = a$ on the graph of the function $f(x)$.
2. Draw a tangent line, a line that goes through $x = a$ at a and at no other point in that section of the curve. Extend the line far enough to calculate its slope as

$$\frac{\text{change in } y}{\text{change in } x}. \tag{5.5.30}$$

Example 5.5.1: Estimating the Derivative at a Point on the Graph of a Function

From the graph of the function $y = f(x)$ presented in Figure, estimate each of the following:

$f(0)$; $f(2)$; $f'(0)$; $f'(2)$

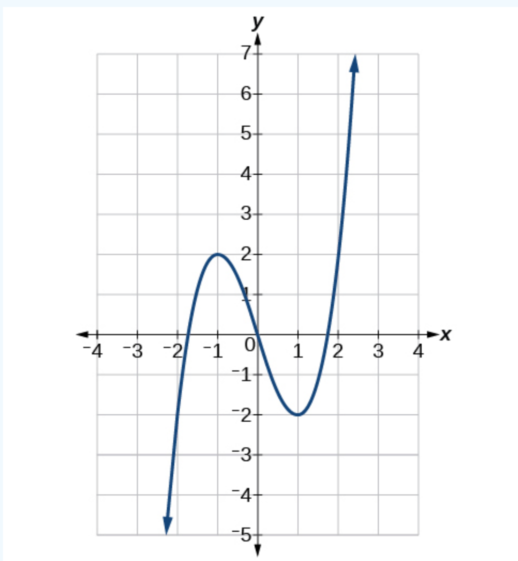
To find the functional value, $f(a)$, find the y-coordinate at $x = a$.

To find the **derivative** at $x = a$, $f'(a)$, draw a tangent line at $x = a$, and estimate the slope of that tangent line. See Figure.

- $f(0)$ is the y-coordinate at $x = 0$. The point has coordinates $(0, 1)$, thus $f(0) = 1$.
- $f(2)$ is the y-coordinate at $x = 2$. The point has coordinates $(2, 1)$, thus $f(2) = 1$.
- $f'(0)$ is found by estimating the slope of the tangent line to the curve at $x = 0$. The tangent line to the curve at $x = 0$ appears horizontal. Horizontal lines have a slope of 0, thus $f'(0) = 0$.
- $f'(2)$ is found by estimating the slope of the tangent line to the curve at $x = 2$. Observe the path of the tangent line to the curve at $x = 2$. As the x value moves one unit to the right, the y value moves up four units to another point on the line. Thus, the slope is 4, so $f'(2) = 4$.
-

Exercise 5.5.1:

Using the graph of the function $f(x) = x^3 - 3x$ shown in Figure, estimate: $f(1)$, $f'(1)$, $f(0)$and $f'(0)$.



−2,−2,0, 0, −3

Using Instantaneous Rates of Change to Solve Real-World Problems

Another way to interpret an instantaneous rate of change at $x = a$ is to observe the function in a real-world context. The unit for the derivative of a function $f(x)$ is

$$\frac{\text{output units}}{\text{input unit}} \tag{5.5.31}$$

Such a unit shows by how many units the output changes for each one-unit change of input. The instantaneous rate of change at a given instant shows the same thing: the units of change of output per one-unit change of input.

One example of an instantaneous rate of change is a marginal cost. For example, suppose the production cost for a company to produce x items is given by $C(x)$, in thousands of dollars. The derivative function tells us how the cost is changing for any value of x in the domain of the function. In other words, $C'(x)$ is interpreted as a **marginal cost**, the additional cost in thousands of dollars of producing one more item when x items have been produced. For example, $C'(11)$ is the approximate additional cost in thousands of dollars of producing the 12$^{\text{th}}$ item after 11 items have been produced. $C'(11) = 2.50$ means that when 11 items have been produced, producing the 12$^{\text{th}}$ item would increase the total cost by approximately $2,500.00.

Example 5.5.1: Finding a Marginal Cost

The cost in dollars of producing x laptop computers in dollars is $f(x) = x^2 - 100x$. At the point where 200 computers have been produced, what is the approximate cost of producing the 201$^{\text{st}}$unit?

If $f(x) = x^2 - 100x$ describes the cost of producing x computers, $f'(x)$ will describe the marginal cost. We need to find the derivative. For purposes of calculating the derivative, we can use the following functions:

$$f(a+b) = (x+h)^2 - 100(x+h) \tag{5.5.32}$$
$$f(a) = a^2 - 100a \tag{5.5.33}$$

$$f'(x) = \frac{f(a+h) - f(a)}{h} \quad \text{Formula for a derivative} \tag{5.5.34}$$
$$= \frac{(x+h)^2 - 100(x+h) - (x^2 - 100x)}{h} \tag{5.5.35}$$
$$\text{Substitute } f(a+h) \text{ and } f(a). \tag{5.5.36}$$
$$= \frac{x^2 + 2xh + h^2 - 100x - 100h - x^2 + 100x}{h} \quad \text{Multiply polynomials, distribute.} \tag{5.5.37}$$
$$= \text{2xh+h\^{}2−100h}h \quad \text{Collect like terms.} \tag{5.5.38}$$
$$= \frac{\cancel{h}(2x + h - 100)}{\cancel{h}} \quad \text{Factor and cancel like terms.} \tag{5.5.39}$$
$$= 2x + h - 100 \quad \text{Simplify.} \tag{5.5.40}$$
$$= 2x - 100 \quad \text{Evaluate when } h = 0. \tag{5.5.41}$$
$$f'(x) = 2x - 100 \quad \text{Formula for marginal cost} \tag{5.5.42}$$
$$f'(200) = 2(200) - 100 = 300 \quad \text{Evaluate for 200 units.} \tag{5.5.43}$$

The marginal cost of producing the 201$^{\text{st}}$ unit will be approximately $300.

Example 5.5.1:Interpreting a Derivative in Context

A car leaves an intersection. The distance it travels in miles is given by the function $f(t)$, where t represents hours. Explain the following notations:

$f(0) = 0 f'(1) = 60 f(1) = 70 f(2.5) = 150$

First we need to evaluate the function $f(t)$ and the derivative of the function $f'(t)$, and distinguish between the two. When we evaluate the function $f(t)$, we are finding the distance the car has traveled in t hours. When we evaluate the derivative f′(t), f′(t), we are finding the speed of the car after t hours.

a. $f(0) = 0$ means that in zero hours, the car has traveled zero miles.
b. $f'(1) = 60$ means that one hour into the trip, the car is traveling 60 miles per hour.
c. $f(1) = 70$ means that one hour into the trip, the car has traveled 70 miles. At some point during the first hour, then, the car must have been traveling faster than it was at the 1-hour mark.
d. $f(2.5) = 150$ means that two hours and thirty minutes into the trip, the car has traveled 150 miles.

Exercise 5.5.1

A runner runs along a straight east-west road. The function $f(t)$ gives how many feet eastward of her starting point she is after t seconds. Interpret each of the following as it relates to the runner.

$f(0)=0$; $f(10)=150$; $f'(10)=15$; $f'(20)=-10$; $f(40)=-100$

---

a. After zero seconds, she has traveled 0 feet.
b. After 10 seconds, she has traveled 150 feet east.
c. After 10 seconds, she is moving eastward at a rate of 15 ft/sec.
d. After 20 seconds, she is moving westward at a rate of 10 ft/sec.
e. After 40 seconds, she is 100 feet westward of her starting point.

Finding Points Where a Function's Derivative Does Not Exist

To understand where a function's derivative does not exist, we need to recall what normally happens when a function $f(x)$ has a derivative at $x=a$. Suppose we use a graphing utility to zoom in on $x=a$. If the function $f(x)$ is differentiable, that is, if it is a function that can be differentiated, then the closer one zooms in, the more closely the graph approaches a straight line. This characteristic is called *linearity*.

Look at the graph in Figure. The closer we zoom in on the point, the more linear the curve appears.

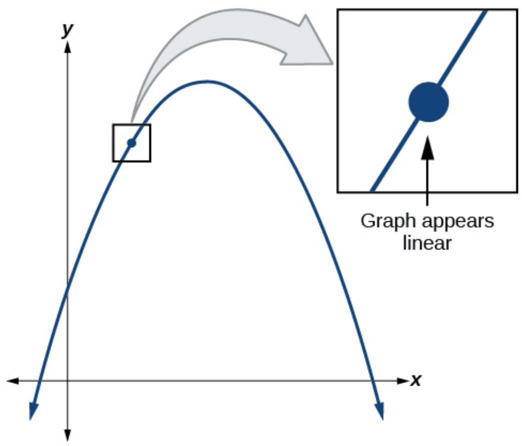


Figure_12_04_009">Figure, the graph does not approach a straight line. No matter how close we zoom in, the graph maintains its sharp corner.

Graph of the function $f(x)=|x|$,with *x*-axis from –0.1 to 0.1 and *y*-axis from –0.1 to 0.1.

What are the characteristics of a graph that is not differentiable at a point? Here are some examples in which function $f(x)$ is not differentiable at $x=a$.

In Figure, we see the graph of

$$f(x)=\begin{cases} x^2, & x\le 2 \\ 8-x, & x>2. \end{cases} \tag{5.5.44}$$

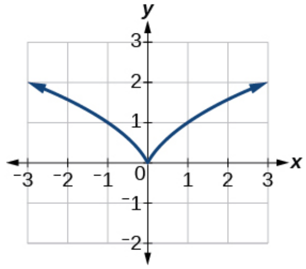
Notice that, as x approaches 2 from the left, the left-hand limit may be observed to be 4, while as x approaches 2 from the right, the right-hand limit may be observed to be 6. We see that it has a discontinuity at $x=2$.

The graph of $f(x)$ has a discontinuity at $x=2$.

In Figure, we see the graph of $f(x)=|x|$. We see that the graph has a corner point at $x=0$.

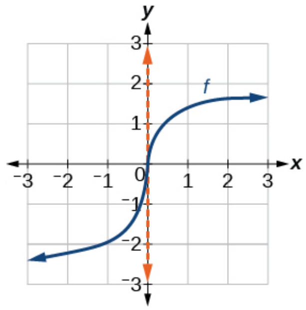
The graph of $f(x)=|x|$ *has a corner point at* $x=0$.

In Figure, we see that the graph of $f(x)=x^{\frac{2}{3}}$ has a cusp at $x=0$. A cusp has a unique feature. Moving away from the cusp, both the left-hand and right-hand limits approach either infinity or negative infinity. Notice the tangent lines as x approaches 0 from both the left and the right appear to get increasingly steeper, but one has a negative slope, the other has a positive slope.



The graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$.

In Figure, we see that the graph of $f(x) = x^{frac13}$ has a vertical tangent at $x = 0$. Recall that vertical tangents are vertical lines, so where a vertical tangent exists, the slope of the line is undefined. This is why the derivative, which measures the slope, does not exist there.



The graph of $f(x) = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$.

differentiability

A function $f(x)$ is differentiable at $x = a$ if the derivative exists at $x = a$,which means that $f'(a)$ exists.

There are four cases for which a function $f(x)$ is not differentiable at a point $x = a$.

1. When there is a discontinuity at $x = a$.
2. When there is a corner point at $x = a$.
3. When there is a cusp at $x = a$.
4. Any other time when there is a vertical tangent at $x = a$.

Example 5.5.1: Determining Where a Function Is Continuous and Differentiable from a Graph

Using Figure, determine where the function is

a. continuous
b. discontinuous
c. differentiable
d. not differentiable

At the points where the graph is discontinuous or not differentiable, state why.

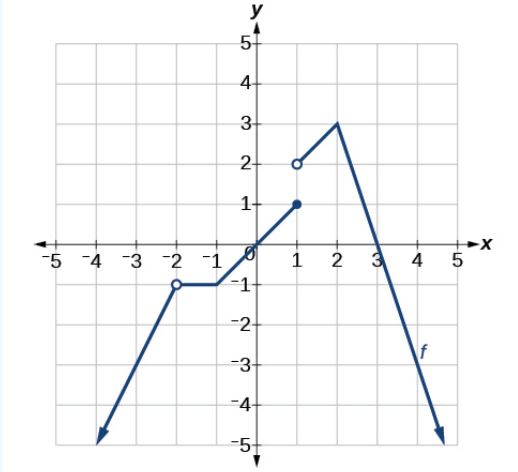


Figure_12_04_016">Figure.

Three intervals where the function is continuous

The graph of is differentiable on $(-\infty,-2)\cup(-2,-1)\cup(-1,1)\cup(1,2)\cup(2,\infty)$. The graph of $f(x)$ is not differentiable at $x=-2$ because it is a point of discontinuity, at $x=-1$ because of a sharp corner, at $x=1$ because it is a point of discontinuity, and at $x=2$ because of a sharp corner. See Figure.

Five intervals where the function is differentiable

Exercise 5.5.1:

Determine where the function $y=f(x)$ shown in Figure is continuous and differentiable from the graph.

The graph of f is continuous on $(-\infty,1)\cup(1,3)\cup(3,\infty)$. The graph of f f is discontinuous at $x=1$ and $x=3$. The graph of f is differentiable on $(-\infty,1)\cup(1,3)\cup(3,\infty)$. The graph of f is not differentiable at $x=1$ and $x=3$.

Finding an Equation of a Line Tangent to the Graph of a Function

The equation of a tangent line to a curve of the function $f(x)$ at $x=a$ is derived from the point-slope form of a line, $y=m(x-x_1)+y_1$. The slope of the line is the slope of the curve at $x=a$ and is therefore equal to $f'(a)$, the derivative of $f(x)$ at $x=a$. The coordinate pair of the point on the line at $x=a$ is $(a,f(a))$.

If we substitute into the point-slope form, we have

The equation of the tangent line is

$$y=f'(a)(x-a)+f(a) \tag{5.5.45}$$

F

The equation of a line tangent to the curve of a function f at a point $x=a$ is

$$y=f'(a)(x-a)+f(a) \tag{5.5.46}$$

how to: Given a function f, find the equation of a line tangent to the function at $x=a$.

1. Find the derivative of $f(x)$ at $x=a$ using $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.
2. Evaluate the function at $x=a$. This is $f(a)$.
3. Substitute $(a,f(a))$ and $f'(a)$ into $y=f'(a)(x-a)+f(a)$.
4. Write the equation of the tangent line in the form $y=mx+b$.

Example 5.5.1: Finding the Equation of a Line Tangent to a Function at a Point

Find the equation of a line tangent to the curve $f(x)=x^2-4x$ at $x=3$.

Using:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{5.5.47}$$

Substitute $f(a+h) = (a+h)^2 - 4(a+h)$ and $f(a) = a^2 - 4a$.

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{(a+h)(a+h) - 4(a+h) - (a2 - 4a)}{h} && && (5.5.48)\\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 4a - 4h - a^2 + 4a}{h} && \text{Remove parentheses.} && (5.5.49)\\
&= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{4a} - 4h - \cancel{a^2} + \cancel{4a}}{h} && \text{Combine like terms.} && (5.5.50)\\
&= \lim_{h \to 0} \frac{2ah + h^2 - 4h}{h} && && (5.5.51)\\
&= \lim_{h \to 0} \frac{\cancel{h}(2a + h - 4)}{h} && \text{Factor out } h. && (5.5.52)\\
&= 2a + 0 - 4 && && (5.5.53)\\
f'(a) &= 2a - 4 && \text{Evaluate the limit.} && (5.5.54)\\
f'(3) &= 2(3) - 4 = 2 && && (5.5.55)
\end{aligned}$$

Equation of tangent line at $x = 3$:

$$\begin{aligned}
y &= f'(a)(x-a) + f(a) && (5.5.56)\\
y &= f'(3)(x-3) + f(3) && (5.5.57)\\
y &= 2(x-3) + (-3) && (5.5.58)\\
y &= 2x - 9 && (5.5.59)
\end{aligned}$$

Analysis

We can use a graphing utility to graph the function and the tangent line. In so doing, we can observe the point of tangency at $x = 3$ as shown in Figure.

Graph confirms the point of tangency at $x = 3$.

Exercise 5.5.1:

Find the equation of a tangent line to the curve of the function $f(x) = 5x^2 - x + 4$ at $x = 2$.

$y = 19x - 16$

Finding the Instantaneous Speed of a Particle

If a function measures position versus time, the derivative measures displacement versus time, or the speed of the object. A change in speed or direction relative to a change in time is known as ***velocity***. The velocity at a given instant is known as **instantaneous velocity**.

In trying to find the speed or velocity of an object at a given instant, we seem to encounter a contradiction. We normally define speed as the distance traveled divided by the elapsed time. But in an instant, no distance is traveled, and no time elapses. How will we divide zero by zero? The use of a derivative solves this problem. A derivative allows us to say that even while the object's velocity is constantly changing, it has a certain velocity at a given instant. That means that if the object traveled at that exact velocity for a unit of time, it would travel the specified distance.

INSTANTANEOUS VELOCITY

Let the function $s(t)$ represent the position of an object at time t. The **instantaneous velocity** or velocity of the object at time $t = a$ is given by

$$s'(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} \tag{5.5.60}$$

Example 5.5.1: Finding the Instantaneous Velocity

A ball is tossed upward from a height of 200 feet with an initial velocity of 36 ft/sec. If the height of the ball in feet after t seconds is given by $s(t) = -16t^2 + 36t + 200$, find the instantaneous velocity of the ball at $t = 2$.

First, we must find the derivative $s'(t)$. Then we evaluate the derivative at $t = 2$, using $s(a+h) = -16(a+h)^2 + 36(a+h) + 200$ and $s(a) = -16a^2 + 36a + 200$.

$$\begin{aligned}
s'(a) &= \lim_{h\to 0}\frac{s(a+h)-s(a)}{h} && (5.5.61)\\
&= \lim_{h\to 0}\frac{-16(a+h)^2+36(a+h)+200-(-16a^2+36a+200)}{h} && (5.5.62)\\
&= \lim_{h\to 0}\frac{-16(a^2+2ah+h^2)+36(a+h)+200-(-16a^2+36a+200)}{h} && (5.5.63)\\
&= \lim_{h\to 0}\frac{-16a^2-32ah-16h^2+36a+36h+200+16a^2-36a-200}{h} && (5.5.64)\\
&= \lim_{h\to 0}\frac{-\cancel{16a^2}-32ah-16h^2+\cancel{36a}+36h+\cancel{200}+\cancel{16a^2}-\cancel{36a}-\cancel{200}}{h} && (5.5.65)\\
&= \lim_{h\to 0}\frac{-32ah-16h^2+36h}{h} && (5.5.66)\\
&= \lim_{h\to 0}\frac{\cancel{h}(-32a-16h+36)}{\cancel{h}} && (5.5.67)\\
&= \lim_{h\to 0}(-32a-16h+36) && (5.5.68)\\
&= -32a-16\cdot 0+36 && (5.5.69)\\
s'(a) &= -32a+36 && (5.5.70)\\
s'(2) &= -32(2)+36 && (5.5.71)\\
&= -28 && (5.5.72)
\end{aligned}$$

Analysis

This result means that at time $t=2$ seconds, the ball is dropping at a rate of 28 ft/sec.

Exercise 5.5.1:

A fireworks rocket is shot upward out of a pit 12 ft below the ground at a velocity of 60 ft/sec. Its height in feet after t seconds is given by $s=-16t^2+60t-12$. What is its instantaneous velocity after 4 seconds?

–68 ft/sec, it is dropping back to Earth at a rate of 68 ft/s.

Access these online resources for additional instruction and practice with derivatives.

- Estimate the Derivative
- Estimate the Derivative Ex. 4

Visit this website for additional practice questions from Learningpod.

Key Equations

average rate of change	$\text{AROC}=\frac{f(a+h)-f(a)}{h}$
derivative of a function	$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Key Concepts

- The slope of the secant line connecting two points is the average rate of change of the function between those points. See Example.
- The derivative, or instantaneous rate of change, is a measure of the slope of the curve of a function at a given point, or the slope of the line tangent to the curve at that point. See Example, Example, and Example.
- The difference quotient is the quotient in the formula for the instantaneous rate of change:

  $\frac{f(a+h)-f(a)}{h}$
- Instantaneous rates of change can be used to find solutions to many real-world problems. See Example.
- The instantaneous rate of change can be found by observing the slope of a function at a point on a graph by drawing a line tangent to the function at that point. See Example.
- Instantaneous rates of change can be interpreted to describe real-world situations. See Example and Example.
- Some functions are not differentiable at a point or points. See Example.
- The point-slope form of a line can be used to find the equation of a line tangent to the curve of a function. See Example.
- Velocity is a change in position relative to time. Instantaneous velocity describes the velocity of an object at a given instant. Average velocity describes the velocity maintained over an interval of time.

- Using the derivative makes it possible to calculate instantaneous velocity even though there is no elapsed time. See Example.

Section Exercises

Verbal

How is the slope of a linear function similar to the derivative?

The slope of a linear function stays the same. The derivative of a general function varies according to x. Both the slope of a line and the derivative at a point measure the rate of change of the function.

What is the difference between the average rate of change of a function on the interval $[x, x+h]$ and the derivative of the function at x?

A car traveled 110 miles during the time period from 2:00 P.M. to 4:00 P.M. What was the car's average velocity? At exactly 2:30 P.M., the speed of the car registered exactly 62 miles per hour. What is another name for the speed of the car at 2:30 P.M.? Why does this speed differ from the average velocity?

Average velocity is 55 miles per hour. The instantaneous velocity at 2:30 p.m. is 62 miles per hour. The instantaneous velocity measures the velocity of the car at an instant of time whereas the average velocity gives the velocity of the car over an interval.

Explain the concept of the slope of a curve at point x.

Suppose water is flowing into a tank at an average rate of 45 gallons per minute. Translate this statement into the language of mathematics.

The average rate of change of the amount of water in the tank is 45 gallons per minute. If $f(x)$ is the function giving the amount of water in the tank at any time t,then the average rate of change of $f(x)$ between $t=a$ and $t=b$ is $f(a)+45(b-a)$.

Algebraic

For the following exercises, use the definition of derivative $\lim\limits_{hto0} \frac{f(x+h)-f(x)}{h}$ to calculate the derivative of each function.

$f(x)=3x-4$

$f(x)=-2x+1$

$f'(x)=-2$

$f(x)=x^2-2x+1$

\(f(x)=2x^2+x−3

$f'(x)=4x+1$

$f(x)=2x^2+5$

$f(x)=\frac{-1}{x-2}$

$f'(x)=\frac{1}{(x-2)^2}$

$f(x)=\frac{2+x}{1-x}$

$f(x)=\frac{5-2x}{3+2x}$

$\frac{-16(}{3+2x)^2}$

$f(x)=\sqrt{1+3x}$

$f(x)=3x^3-x^2+2x+5$

$f'(x)=9x^2-2x+2$

$f(x)=5$

$f(x) = 5\pi$

$f'(x) = 0$

For the following exercises, find the average rate of change between the two points.

$(-2, 0)$ and $(-4, 5)$

$(4, -3)$ and $(-2, -1)$

$-\frac{1}{3}$

$(0, 5)$ and $(6, 5)$

$(7, -2)$ and $(7, 10)$

---

undefined

For the following polynomial functions, find the derivatives.

$f(x) = x^3 + 1$

$f(x) = -3x^2 - 7x = 6$

$f'(x) = -6x - 7$

$f(x) = 7x^2$

$f(x) = 3x^3 + 2x^2 + x - 26$

$f'(x) = 9x^2 + 4x + 1$

For the following functions, find the equation of the tangent line to the curve at the given point x on the curve.

$f(x) = 2x^2 - 3x \quad x = 3$

$f(x) = x^3 + 1 \quad x = 2$

$y = 12x - 15$

$f(x) = \sqrt{x} \quad x = 9$

For the following exercise, find k such that the given line is tangent to the graph of the function.

$f(x) = x^2 - kx, \quad y = 4x - 9$

$k = -10$ or $k = 2$

Graphical

For the following exercises, consider the graph of the function f and determine where the function is continuous/discontinuous and differentiable/not differentiable.

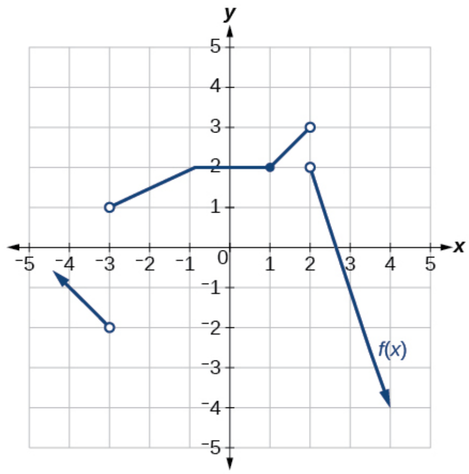


Figure_12_04_202.jpg" style="width: 487px; height: 457px;" width="487px" height="457px" src="/@api/deki/files/26500/CNX_Precalc_Figure_12_04_202.jpg" />

Discontinuous at $x = -2$ and $x = 0$. Not differentiable at –2, 0, 2.

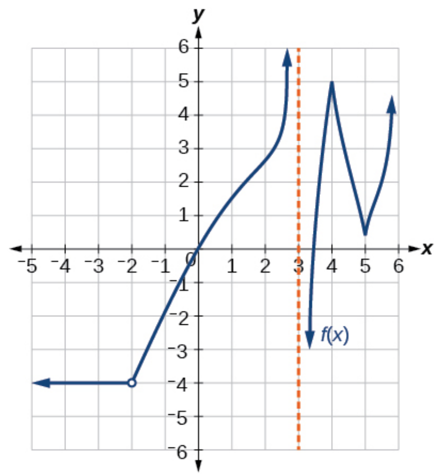
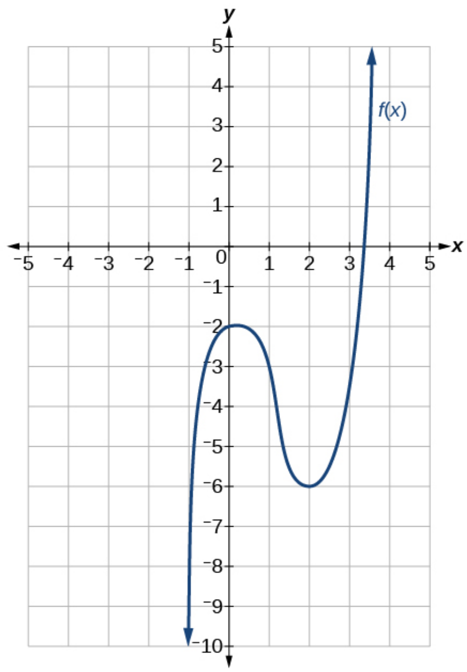


Figure_12_04_204.jpg" style="width: 402px; height: 374px;" width="402px" height="374px" src="/@api/deki/files/26493/CNX_Precalc_Figure_12_04_204.jpg" />

Discontinuous at $x = 5$. Not differentiable at -4, –2, 0, 1, 3, 4, 5.

For the following exercises, use Figure to estimate either the function at a given value of x or the derivative at a given value of x, as indicated.



$f(-1)$

$f(0)$

$f(0) = -2$

$f(1)$

$f(2)$

\(f(2)=−6f(2)=−6

$f(3)$

$f'(-1)$

$f'(-1) = 9$

$f'(0)$

$f'(1)$

$f'(1) = -3$

$f'(2)$

$f'(3)$

$f'(3) = 9$

Sketch the function based on the information below:

$f'(x) = 2x, f(2) = 4$

Technology

Numerically evaluate the derivative. Explore the behavior of the graph of $f(x) = x^2$ around $x = 1$ by graphing the function on the following domains: $[0.9, 1.1]$, $[0.99, 1.01]$, $[0.999, 1.001]$, and $[0.9999, 1.0001]$. We can use the feature on our calculator that automatically sets Ymin and Ymax to the Xmin and Xmax values we preset. (On some of the commonly used graphing calculators, this feature may be called ZOOM FIT or

ZOOM AUTO). By examining the corresponding range values for this viewing window, approximate how the curve changes at $x=1$, that is, approximate the derivative at $x=1$.

Answers vary. The slope of the tangent line near $x=1$ is 2.

Real-World Applications

For the following exercises, explain the notation in words. The volume $f(t)$ of a tank of gasoline, in gallons, t minutes after noon.

$f(0)=600$

$f'(30)=-20$

At 12:30 p.m., the rate of change of the number of gallons in the tank is –20 gallons per minute. That is, the tank is losing 20 gallons per minute.

$f(30)=0$

$f'(200)=30$

At 200 minutes after noon, the volume of gallons in the tank is changing at the rate of 30 gallons per minute.

$f(240)=500$

For the following exercises, explain the functions in words. The height, s, of a projectile after t seconds is given by $s(t)=-16t^2+80t$.

$s(2)=96$

The height of the projectile after 2 seconds is 96 feet.

$s'(2)=16$

$s(3)=96$

The height of the projectile at $t=3$ seconds is 96 feet.

$s'(3)=-16$

$s(0)=0, s(5)=0.$

The height of the projectile is zero at $t=0$ and again at $t=5$. In other words, the projectile starts on the ground and falls to earth again after 5 seconds.

For the following exercises, the volume V of a sphere with respect to its radius r is given by $V=\frac{4}{3}\pi r^3$.

Find the average rate of change of V as r changes from 1 cm to 2 cm.

Find the instantaneous rate of change of V when r=3 cm. r=3 cm.

36π

For the following exercises, the revenue generated by selling x items is given by $R(x)=2x^2+10x$.

Find the average change of the revenue function as x changes from $x=10$ to $x=20$.

Find $R'(10)$ and interpret.

$50.00 per unit, which is the instantaneous rate of change of revenue when exactly 10 units are sold.

Find $R'(15)$ and interpret. Compare $R'(15)$ to $R'(10)$, and explain the difference.

For the following exercises, the cost of producing x cellphones is described by the function $C(x)=x^2-4x+1000$.

Find the average rate of change in the total cost as x changes from $x=10$ to $x=15$.

$21 per unit

Find the approximate marginal cost, when 15 cellphones have been produced, of producing the 16th cellphone.

Find the approximate marginal cost, when 20 cellphones have been produced, of producing the 21st cellphone.

$36

Extension

For the following exercises, use the definition for the derivative at a point $x = a$, $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$, to find the derivative of the functions.

$f(x) = \frac{1}{x^2}$

$f(x) = 5x^2 - x + 4$

$f'(x) = 10a - 1$

$f(x) = -x^2 + 4x + 7$

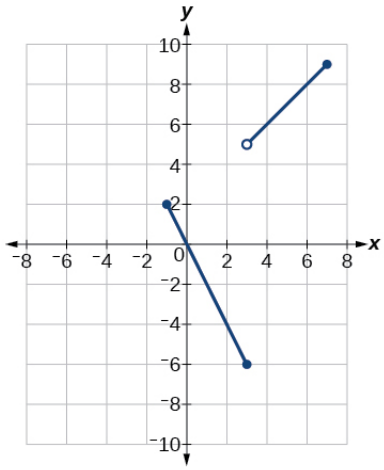
$f(x) = \frac{-4}{3-x^2}$

$\frac{4}{(3-x)^2}$

Chapter Review Exercises

Finding Limits: A Numerical and Graphical Approach

For the following exercises, use Figure.



$\lim\limits_{x \to -1^+} f(x)$

2

$\lim\limits_{x \to -1^-} f(x)$

$\lim\limits_{x \to -1} f(x)$

---

does not exist

$\lim\limits_{x \to 3} f(x)$

At what values of x is the function discontinuous? What condition of continuity is violated?

Discontinuous at $x = -1$ ($\lim\limits_{x \to a} f(x)$ does not exist), $x = 3$ (jump discontinuity),and $x = 7$ ((\lim \limits_{x \to a} f(x)\) does not exist).

Using Table, estimate $\lim_{x\to 0} f(x)$.

x	$F(x)$
−0.1	2.875
−0.01	2.92
−0.001	2.998
0	Undefined
0.001	2.9987
0.01	2.865
0.1	2.78145
0.15	2.678

3

For the following exercises, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has limit as x approaches a, state it. If not, discuss why there is no limit.

$f(x)=\begin{cases} |x|-1, & \text{if } x\neq 1 \\ x^3, \text{if } x=1 \end{cases}\ a=1$

$f(x)=\begin{cases} \frac{1}{x+1}, & \text{if } x=-2 \\ (x+1)^2, & \text{if } x\neq -2 \end{cases}\ a=-2$

$\lim_{x\to -2} f(x)=1$

$f(x)=\begin{cases} \sqrt{x+3} & \text{if } x<1 \\ -\sqrt[3]{x} & \text{if } x>1 \end{cases}\ a=1$

Finding Limits: Properties of Limits

For the following exercises, find the limits if $\lim_{x\to c} f(x)=-3$ and $\lim_{x\to c} g(x)=5$.

$\lim_{x\to c}(f(x)+g(x))$

2

$\lim_{x\to c} \frac{f(x)}{g(x)}$

$\lim_{xtoc}(f(x)\cdot g(x))$

-15

$\lim_{x\to 0^+} f(x), f(x)=\begin{cases} 3x^2+2x+1 & x>0 \\ 5x+3 & x<0 \end{cases}$

$\lim_{x\to 0^-} f(x), f(x)=\begin{cases} 3x^2+2x+1 & x>0 \\ 5x+3 & x<0 \end{cases}$

3

$\lim_{x\to 3^+}(3x-[\![x]\!])$

For the following exercises, evaluate the limits using algebraic techniques.

$\lim_{h\to 0}(\frac{(h+6)^2-36}{h})$

12

$\lim\limits_{x\to 25}(\frac{x^2-625}{\sqrt{x}-5)}$

$\lim\limits_{x\to 1}(\frac{-x^2-9x}{x})$

-10

$\lim\limits_{x\to 4}\frac{7-\sqrt{12x+1}}{x-4}$

$\lim\limits_{x\to -3}(\frac{\frac{1}{3}+\frac{1}{x}}{3+x})$

$-\frac{1}{9}$

Continuity

For the following exercises, use numerical evidence to determine whether the limit exists at $x=a$. If not, describe the behavior of the graph of the function at $x=a$.

$f(x)=\frac{-2}{x-4};a=4$

$f(x)=\frac{-2}{(x-4)^2};a=4$

At $x=4$,the function has a vertical asymptote.

$f(x)=\frac{-x}{x^2-x-6};a=3$

$f(x)=\frac{6x^2+23x+20}{4x^2-25};a=-\frac{5}{2}$

---

removable discontinuity at $a=-\frac{5}{2}$

$f(x)=\frac{\sqrt{x}-3}{9-x};a=9$

For the following exercises, determine where the given function $f(x)$ is continuous. Where it is not continuous, state which conditions fail, and classify any discontinuities.

$f(x)=x^2-2x-15$

---

continuous on $(-\infty,\infty)$

$f(x)=\frac{x^2-2x-15}{x-5}$

$f(x)=\frac{x^2-2x}{x^2-4x+4}$

---

removable discontinuity at $x=2. f(2)$ is not defined, but limits exist.

$f(x)=\frac{x^3-125}{2x^2-12x+10}$

$f(x)=\frac{x^2-\frac{1}{x}}{2-x}$

---

discontinuity at $x=0$ and $x=2$. Both $f(0)$ and $f(2)$ are not defined.

$f(x)=\frac{x+2}{x^2-3x-10}$

$f(x)=\frac{x+2}{x^3+8}$

---

removable discontinuity at $x=-2. f(-2)$ is not defined.

[Derivatives](#)

For the following exercises, find the average rate of change $\frac{f(x+h)-f(x)}{h}$.

$f(x)=3x+2$

$f(x)=5$

0

$f(x)=\frac{1}{x+1}$

$f(x)=\ln(x)$

$\frac{\ln(x+h)-\ln(x)}{h}$

$f(x)=e^{2x}$

For the following exercises, find the derivative of the function.

$f(x)=4x-6$

$=4$

$f(x)=5x^2-3x$

Find the equation of the tangent line to the graph of $f(x)$ at the indicated x value.

$f(x)=-x^3+4x; x=2.$

$y=-8x+16$

For the following exercises, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

$f(x)=\frac{x}{|x|}$

Given that the volume of a right circular cone is $V=\frac{1}{3}\pi r^2 h$ and that a given cone has a fixed height of 9 cm and variable radius length, find the instantaneous rate of change of volume with respect to radius length when the radius is 2 cm. Give an exact answer in terms of π

12π

Practice Test

For the following exercises, use the graph of f in [Figure](#).

$f(1)$

3

$\lim\limits_{x\to -1^+} f(x)$

$\lim\limits_{x\to -1^-} f(x)$

0

$\lim\limits_{x\to -1} f(x)$

$\lim\limits_{x\to -2} f(x)$

-1

At what values of x is f discontinuous? What property of continuity is violated?

For the following exercises, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has a limit as x approaches a,state it. If not, discuss why there is no limit

$f(x)=\begin{cases} \frac{1}{x}-3, & \text{if } x\le 2 \\ x^3+1, & \text{if } x>2 \end{cases}\ a=2$

$\lim_{x\to 2^-} f(x)=-\frac{5}{2}a$ and $\lim_{x\to 2^+} f(x)=9$ Thus, the limit of the function as x approaches 2 does not exist.

$f(x)=\begin{cases} x^3+1, & \text{if } x<1 \\ 3x^2-1, & \text{if } x=1 \\ -\sqrt{x+3}+4, & \text{if } x>1 \end{cases}\ a=1$

For the following exercises, evaluate each limit using algebraic techniques.

$\lim_{x\to -5}(\frac{\frac{1}{5}+\frac{1}{x}}{10+2x})$

$-\frac{1}{50}$

$\lim_{h\to 0}(\frac{\sqrt{h^2+25}-5}{h^2})$

$\lim_{h\to 0}(\frac{1}{h}-\frac{1}{h^2+h})$

1

For the following exercises, determine whether or not the given function f is continuous. If it is continuous, show why. If it is not continuous, state which conditions fail.

$f(x)=\sqrt{x^2-4}$

$f(x)=\frac{x^3-4x^2-9x+36}{x^3-3x^2+2x-6}$

---

removable discontinuity at $x=3$

For the following exercises, use the definition of a derivative to find the derivative of the given function at $x=a$.

$f(x)=\frac{3}{5+2x}$

$f(x)=\frac{3}{\sqrt{x}}$

$f'(x)=-\frac{3}{2a^{\frac{3}{2}}}$

$f(x)=2x^2+9x$

discontinuous at –2,0, not differentiable at –2,0, 2.

For the following exercises, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

$f(x)=|x-2|-|x+2|$

$f(x)=\frac{2}{1+e^{\frac{2}{x}}}$

---

not differentiable at $x=0$ (no limit)

For the following exercises, explain the notation in words when the height of a projectile in feet, s, is a function of time t t in seconds after launch and is given by the function $s(t)$.

$s(0)$

$s(2)$

---

the height of the projectile at $t=2$ seconds

$s'(2)$

$\frac{s(2)-s(1)}{2-1}$

---

the average velocity from $t=1$ to $t=2$

$s(t)=0$

For the following exercises, use technology to evaluate the limit.

$\lim_{x\to 0}\frac{\sin(x)}{3x}$

$\frac{1}{3}$

$\lim_{x\to 0}\frac{\tan^2(x)}{2x}$

$\lim_{x\to 0}\frac{\sin(x)(1-\cos(x))}{2x^2}$

0

Evaluate the limit by hand.

$\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases}4x-7 & x\neq 1\\ x^2-4 & x=1\end{cases}$

At what value(s) of x is the function below discontinuous?

$f(x)=\begin{cases}4x-7 & x\neq 1\\ x^2-4 & x=1\end{cases}$

For the following exercises, consider the function whose graph appears in Figure.

Find the average rate of change of the function from $x=1$ to $x=3$.

2

Find all values of x at which $f'(x)=0$.

$x=1$

Find all values of x at which $f'(x)$ does not exist.

Find an equation of the tangent line to the graph of f the indicated point: $f(x)=3x^2-2x-6, x=-2$

$y=-14x-18$

For the following exercises, use the function $f(x)=x(1-x)^{\frac{2}{5}}$.

Graph the function $f(x)=x(1-x)^{\frac{2}{5}}$ by entering $f(x)=x((1-x)^2)^{\frac{1}{5}}$ and then by entering $f(x)=x((1-x)^{\frac{1}{5}})^2$.

Explore the behavior of the graph of $f(x)$ around $x=1$ by graphing the function on the following domains, [0.9, 1.1], [0.99, 1.01], [0.999, 1.001], and [0.9999, 1.0001]. Use this information to determine whether the function appears to be differentiable at $x=1$.

The graph is not differentiable at $x=1$ (cusp).

For the following exercises, find the derivative of each of the functions using the definition: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$f(x)=2x-8$

$f(x)=4x^2-7$

$f'(x)=8x$

$f(x)=x-\frac{1}{2}x^2$

$f(x) = \frac{1}{x+2}$

$f'(x) = -\frac{1}{(2+x)^2}$

$f(x) = \frac{3}{x-1}$

$f(x) = -x^3 + 1$

$f\,'(x) = -3^x 2$

$f(x) = x^2 + x^3$

$f(x) = \sqrt{x-1}$

$f'(x) = \frac{1}{2\sqrt{x-1}}$

Footnotes

- 1 www.csun.edu/science/health/d...tv&health.html Source provided.

Glossary

average rate of change

the slope of the line connecting the two points $(a, f(a))$ and $(a+h, f(a+h))$ on the curve of $f(x)$; it is given by

$$\text{AROC} = \frac{f(a+h) - f(a)}{h}. \tag{5.5.73}$$

derivative

the slope of a function at a given point; denoted $f\,'(a)$,at a point $x = a$ it is $f\,'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$,providing the limit exists.

differentiable

a function $f(x)$ for which the derivative exists at $x = a$. In other words, if f′(a) f′(a) exists.

instantaneous rate of change

the slope of a function at a given point; at $x = a$ it is given by $f\,'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

instantaneous velocity

the change in speed or direction at a given instant; a function $s(t)$ represents the position of an object at time t,and the instantaneous velocity or velocity of the object at time $t = a$ is given by $s\,'(a) = \lim_{h \to 0} \frac{s(a+h)-s(a)}{h}$.

secant line

a line that intersects two points on a curve

tangent line

a line that intersects a curve at a single point

5.E: Introduction to Calculus (Exercises)

5.1: Finding Limits - Numerical and Graphical Approaches

In this section, we will examine numerical and graphical approaches to identifying limits.

Verbal

1) Explain the difference between a value at $x = a$ and the limit as x approaches a.

Answer

The value of the function, the output, at $x = a$ is $f(a)$. When the $\lim_{x \to a} f(x)$ is taken, the values of x get infinitely close to a but never equal a. As the values of x approach a from the left and right, the limit is the value that the function is approaching.

2) Explain why we say a function does not have a limit as x approaches a if, as x approaches a, the left-hand limit is not equal to the right-hand limit.

Graphical

For the exercises 3-14, estimate the functional values and the limits from the graph of the function f provided in the Figure below.

3) $\lim_{x \to -2^-} f(x)$

Answer

-4

4) $\lim_{x \to -2^+} f(x)$

5) $\lim_{x \to -2 f(x)}$

Answer

-4

6) $f(-2)$

7) $\lim_{x \to -1^- f(x)}$

Answer

2

8) $\lim_{x \to 1^+} f(x)$

9) $\lim_{x \to 1} f(x)$

Answer

does not exist

10) $f(1)$

11) $\lim_{x \to 4^-} f(x)$

Answer

4

12) $\lim_{x \to 4^+} f(x)$

13) $\lim_{x \to 4} f(x)$

Answer

does not exist

14) $f(4)$

For the exercises 15-21, draw the graph of a function from the functional values and limits provided.

15) $\lim_{x \to 0^-} f(x) = 2,\ \lim_{x \to 0^+} f(x) = -3, \lim_{x \to 2} f(x) = 2, f(0) = 4, f(2) = -1, f(-3)$ does not exist.

Answer

Answers will vary.

16) $\lim_{x \to 2^-} f(x) = 0,\ \lim_{x \to 2^+} = -2, \lim_{x \to 0} f(x) = 3, f(2) = 5, f(0)$

Answer

Answers will vary.

17) $\lim_{x \to 2^-} f(x) = 2,\ \lim_{x \to 2^+} f(x) = -3, \lim_{x \to 0} f(x) = 5, f(0) = 1, f(1) = 0$

Answer

Answers will vary.

18) $\lim_{x \to 3^-} f(x) = 0,\ \lim_{x \to 3^+} f(x) = 5, \lim_{x \to 5} f(x) = 0, f(5) = 4, f(3)$ does not exist.

Answer

Answers will vary.

19) $\lim_{x \to 4} f(x) = 6,\ \lim_{x \to 6^+} f(x) = -1, \lim_{x \to 0} f(x) = 5, f(4) = 6, f(2) = 6$

Answer

Answers will vary.

20) $\lim_{x \to -3} f(x) = 2,\ \lim_{x \to 1^+} f(x) = -2, \lim_{x \to 3} f(x) = -4, f(-3) = 0, f(0) = 0$

Answer

Answers will vary.

21) $\lim_{x \to \pi} f(x) = \pi^2,\ \lim_{x \to -\pi} f(x) = \frac{\pi}{2},\ \lim_{x \to 1^-} f(x) = 0, f(\pi) = \sqrt{2}, f(0)$ does not exist.

Answer

Answers will vary.

For the exercises 22-26, use a graphing calculator to determine the limit to 5 decimal places as x approaches 0.

22) $f(x) = (1 + x)^{\frac{1}{x}}$

23) $g(x) = (1 + x)^{\frac{2}{x}}$

Answer

7.38906

24) $h(x)=(1+x)^{\frac{3}{x}}$

25) $i(x)=(1+x)^{\frac{4}{x}}$

Answer

54.59815

26) $j(x)=(1+x)^{\frac{5}{x}}$

27) Based on the pattern you observed in the exercises above, make a conjecture as to the limit of $f(x)=(1+x)^{\frac{6}{x}}, g(x)=(1+x)^{\frac{7}{x}},$ and $h(x)=(1+x)^{\frac{n}{x}}$.

Answer

$e^6 \approx 403.428794, e^7 \approx 1096.633158, e^n$

For the exercises 28-29, use a graphing utility to find graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has a limit as x approaches a,state it. If not, discuss why there is no limit.

28) $(x)=\begin{cases} |x|-1, & \text{if } x \neq 1 \\ x^3, & \text{if } x=1 \end{cases} \; a=1$

29) $(x)=\begin{cases} \frac{1}{x+1}, & \text{if } x=-2 \\ (x+1)^2, & \text{if } x \neq -2 \end{cases} \; a=-2$

Answer

$\lim_{x \to -2} f(x)=1$

Numeric

For the exercises 30-38, use numerical evidence to determine whether the limit exists at $x=a$. If not, describe the behavior of the graph of the function near $x=a$. Round answers to two decimal places.

30) $f(x)=\frac{x^2-4x}{16-x^2}; a=4$

31) $f(x)=\frac{x^2-x-6}{x^2-9}; a=3$

Answer

$\lim_{x \to 3}\left(\frac{x^2-x-6}{x^2-9}\right)=\frac{5}{6} \approx 0.83$

32) $f(x)=\frac{x^2-6x-7}{x^2-7x}; a=7$

33) $f(x)=\frac{x^2-1}{x^2-3x+2}; a=1$

Answer

$\lim_{x \to 1}\left(\frac{x^2-1}{x^2-3x+2}\right)=-2.00$

34) $f(x)=\frac{1-x^2}{x^2-3x+2}; a=1$

35) $f(x)=\frac{10-10x^2}{x^2-3x+2}; a=1$

Answer

$$\lim_{x\to 1}\left(\frac{10-10x^2}{x^2-3x+2}\right)=20.00$$

36) $f(x)=\dfrac{x}{6x^2-5x-6}; a=\dfrac{3}{2}$

37) $f(x)=\dfrac{x}{4x^2+4x+1}; a=-\dfrac{1}{2}$

Answer

$\lim_{x\to\frac{-1}{2}}\left(\dfrac{x}{4x^2+4x+1}\right)$ does not exist. Function values decrease without bound as x approaches -0.5 from either left or right.

38) $f(x)=\frac{2}{x-4}; a=4$

For the exercises 39-41, use a calculator to estimate the limit by preparing a table of values. If there is no limit, describe the behavior of the function as x approaches the given value.

39) $\lim_{x\to 0}\dfrac{7\tan x}{3x}$

Answer

$$\lim_{x\to 0}\frac{7\tan x}{3x}=\frac{7}{3}$$

40) $\lim_{x\to 4}\dfrac{x^2}{x-4}$

Answer

41) $\lim_{x\to 0}\dfrac{2\sin x}{4\tan x}$

Answer

$$\lim_{x\to 0}\frac{2\sin x}{4\tan x}=\frac{1}{2}$$

For the exercises 42-49, use a graphing utility to find numerical or graphical evidence to determine the left and right-hand limits of the function given as x approaches a. If the function has a limit as x approaches a, state it. If not, discuss why there is no limit.

42) $\lim_{x\to 0}e^{e^{\frac{1}{x}}}$

43) $\lim_{x\to 0}e^{e^{-\frac{1}{x^2}}}$

Answer

$$\lim_{x\to 0}e^{e^{-\frac{1}{x^2}}}=1.0$$

44) $\lim_{x\to 0}\dfrac{|x|}{x}$

45) $\lim_{x \to -1} \frac{|x+1|}{x+1}$

Answer

$\lim_{x \to -1^-} \frac{|x+1|}{x+1} = \frac{-(x+1)}{(x+1)} = -1$ and $\lim_{x \to -1^+} \frac{|x+1|}{x+1} = \frac{(x+1)}{(x+1)} = 1$; since the right-hand limit does not equal the left-hand limit, $\lim_{x \to -1} \frac{|x+1|}{x+1}$ does not exist.

46) $\lim_{x \to 5} \frac{|x-5|}{5-x}$

47) $\lim_{x \to -1} \frac{1}{(x+1)^2}$

Answer

$\lim_{x \to -1} \frac{1}{(x+1)^2}$ does not exist. The function increases without bound as x approaches -1 from either side.

48) $\lim_{x \to 1} \frac{1}{(x-1)^3}$

49) $\lim_{x \to 0} \frac{5}{1-e^{\frac{2}{x}}}$

Answer

$\lim_{x \to 0} \frac{5}{1-e^{\frac{2}{x}}}$ does not exist. Function values approach 5 from the left and approach 0 from the right.

50) Use numerical and graphical evidence to compare and contrast the limits of two functions whose formulas appear similar: $f(x) = \left|\frac{1-x}{x}\right|$ and $g(x) = \left|\frac{1+x}{x}\right|$ as x approaches 0. Use a graphing utility, if possible, to determine the left- and right-hand limits of the functions $f(x)$ and $g(x)$ as x approaches 0. If the functions have a limit as x approaches 0, state it. If not, discuss why there is no limit.

Extensions

51) According to the Theory of Relativity, the mass m m of a particle depends on its velocity v. That is

$$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$$

where m_o is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass, m, as v approaches c^-.

Answer

Through examination of the postulates and an understanding of relativistic physics, as $v \to c, m \to \infty$. Take this one step further to the solution,

$$\lim_{v \to c^-} m = \lim_{v \to c^-} \frac{m_o}{\sqrt{1-(v^2/c^2)}} = \infty$$

52) Allow the speed of light, c, to be equal to 1.0. If the mass, m, is 1, what occurs to m as $v \to c$? Using the values listed in the Table below, make a conjecture as to what the mass is as v approaches 1.00.

v	m
0.5	1.15
0.9	2.29

v	m
0.95	3.20
0.99	7.09
0.999	22.36
0.99999	223.61

5.2: Finding Limits - Properties of Limits

Graphing a function or exploring a table of values to determine a limit can be cumbersome and time-consuming. When possible, it is more efficient to use the properties of limits, which is a collection of theorems for finding limits. Knowing the properties of limits allows us to compute limits directly.

Verbal

1) Give an example of a type of function f whose limit, as x approaches a, is $f(a)$.

Answer

If f is a polynomial function, the limit of a polynomial function as x approaches a will always be $f(a)$.

2) When direct substitution is used to evaluate the limit of a rational function as x approaches a and the result is $f(a)=\frac{0}{0}$,does this mean that the limit of f does not exist?

3) What does it mean to say the limit of $f(x)$, as x approaches c, is undefined?

Answer

It could mean either (1) the values of the function increase or decrease without bound as x approaches c, or (2) the left and right-hand limits are not equal.

Algebraic

For the exercises 4-30, evaluate the limits algebraically.

4) $\lim_{x\to 0}(3)$

5) $\lim_{x\to 2}\left(\frac{-5x}{x^2-1}\right)$

Answer

$\frac{-10}{3}$

6) $\lim_{x\to 2}\left(\frac{x^2-5x+6}{x+2}\right)$

7) $\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$

Answer

6

8) $\lim_{x\to -1}\left(\frac{x^2-2x-3}{x+1}\right)$

9) $\lim_{x\to \frac{3}{2}}\left(\frac{6x^2-17x+12}{2x-3}\right)$

Answer

$\frac{1}{2}$

10) $\lim_{x \to -\frac{7}{2}} \left(\frac{8x^2 + 18x - 35}{2x + 7} \right)$

11) $\lim_{x \to 3} \left(\frac{x^2 - 9}{x - 5x + 6} \right)$

Answer

6

12) $\lim_{x \to -3} \left(\frac{-7x^4 - 21x^3}{-12x^4 + 108x^2} \right)$

13) $\lim_{x \to 3} \left(\frac{x^2 + 2x - 3}{x - 3} \right)$

Answer

does not exist

14) $\lim_{h \to 0} \left(\frac{(3+h)^3 - 27}{h} \right)$

15) $\lim_{h \to 0} \left(\frac{(2-h)^3 - 8}{h} \right)$

Answer

-12

16) $\lim_{h \to 0} \left(\frac{(h+3)^2 - 9}{h} \right)$

17) $\lim_{h \to 0} \left(\frac{\sqrt{5-h} - \sqrt{5}}{h} \right)$

Answer

$-\frac{\sqrt{5}}{10}$

18) $\lim_{x \to 0} \left(\frac{\sqrt{3-x} - \sqrt{3}}{x} \right)$

19) $\lim_{x \to 9} \left(\frac{x^2 - 81}{3 - x} \right)$

Answer

-108

20) $\lim_{x \to 1} \left(\frac{\sqrt{x} - x^2}{1 - \sqrt{x}} \right)$

21) $\lim_{x \to 0} \left(\frac{x}{\sqrt{1+2x} - 1} \right)$

Answer

1

22) $\lim_{x\to\frac{1}{2}}\left(\frac{x^2-\frac{1}{4}}{2x-1}\right)$

23) $\lim_{x\to 4}\left(\frac{x^3-64}{x^2-16}\right)$

Answer

6

24) $\lim_{x\to 2^-}\left(\frac{|x-2|}{x-2}\right)$

25) $\lim_{x\to 2^+}\left(\frac{|x-2|}{x-2}\right)$

Answer

1

26) $\lim_{x\to 2}\left(\frac{|x-2|}{x-2}\right)$

27) $\lim_{x\to 4^-}\left(\frac{|x-4|}{4-x}\right)$

Answer

1

28) $\lim_{x\to 4^+}\left(\frac{|x-4|}{4-x}\right)$

29) $\lim_{x\to 4}\left(\frac{|x-4|}{4-x}\right)$

Answer

does not exist

30) $\lim_{x\to 2}\left(\frac{-8+6x-x^2}{x-2}\right)$

For the exercises 31-33, use the given information to evaluate the limits: $\lim_{x\to c}f(x)=3, \lim_{x\to c}g(x)=5$

31) $\lim_{x\to c}[2f(x)+\sqrt{g(x)}]$

Answer

$6+\sqrt{5}$

32) $\lim_{x\to c}[3f(x)+\sqrt{g(x)}]$

33) $\lim_{x\to c}\frac{f(x)}{g(x)}$

Answer

$\frac{3}{5}$

For the exercises 34-43, evaluate the following limits.

34) $\lim\limits_{x\to 2}\cos(\pi x)$

35) $\lim\limits_{x\to 2}\sin(\pi x)$

Answer

0

36) $\lim\limits_{x\to 2}\sin\left(\frac{\pi}{x}\right)$

37) $f(x)=\begin{cases}2x^2+2x+1, & x\le 0\\ x-3, & x>0;\end{cases}\ \lim\limits_{x\to 0^+} f(x)$

Answer

-3

38) $f(x)=\begin{cases}2x^2+2x+1, & x\le 0\\ x-3, & x>0;\end{cases}\ \lim\limits_{x\to 0^-} f(x)$

39) $f(x)=\begin{cases}2x^2+2x+1, & x\le 0\\ x-3, & x>0;\end{cases}\ \lim\limits_{x\to 0} f(x)$

Answer

does not exist; right-hand limit is not the same as the left-hand limit.

40) $\lim\limits_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}$

41) $\lim\limits_{x\to 2^+}(2x-[\![x]\!])$

Answer

2

42) $\lim\limits_{x\to 2}\frac{\sqrt{x+7}-3}{x^2-x-2}$

43) $\lim\limits_{x\to 3^+}\frac{x^2}{x^2-9}$

Answer

Limit does not exist; limit approaches infinity.

For the exercises 44-53, find the average rate of change $\frac{f(x+h)-f(x)}{h}$.

44) $f(x)=x+1$

45) $f(x)=2x^2-1$

Answer

$4x+2h$

46) $f(x)=x^2+3x+4$

47) $f(x)=x^2+4x-100$

Answer

$2x+h+4$

48) $f(x)=3x^2+1$

49) $f(x)=\cos(x)$

Answer

$$\frac{\cos(x+h)-\cos(x)}{h}$$

50) $f(x)=2x^3-4x$

51) $f(x)=\frac{1}{x}$

Answer

$$\frac{-1}{x(x+h)}$$

52) $f(x)=\frac{1}{x^2}$

53) $f(x)=\sqrt{x}$

Answer

$$\frac{-1}{\sqrt{x+h}+\sqrt{x}}$$

Graphical

54) Find an equation that could be represented by the Figure below.

Figure below.

Answer

$$f(x)=\frac{x^2+5x+6}{x+3}$$

For the exercises 56-57, refer to the Figure below.

56) What is the right-hand limit of the function as x approaches 0?

57) What is the left-hand limit of the function as x approaches 0?

Answer

does not exist

Real-World Applications

58) The position function $s(t)=-16t^2+144t$ gives the position of a projectile as a function of time. Find the average velocity (average rate of change) on the interval $[1,2]$.

59) The height of a projectile is given by $s(t)=-64t^2+192t$ Find the average rate of change of the height from $t=1$ second to $t=1.5$ seconds.

Answer

52

60) The amount of money in an account after t years compounded continuously at 4.25% interest is given by the formula $A=A_0e^{0.0425t}$,where A_0 is the initial amount invested. Find the average rate of change of the balance of the account from $t=1$ year to $t=2$ years if the initial amount invested is $\$1,000.00$.

5.3: Continuity

A function that remains level for an interval and then jumps instantaneously to a higher value is called a stepwise function. This function is an example. A function that has any hole or break in its graph is known as a discontinuous function. A stepwise function, such as parking-garage charges as a function of hours parked, is an example of a discontinuous function. We can check three different conditions to decide if a function is continuous at a particular number.

Verbal

1) State in your own words what it means for a function f to be continuous at $x=c$.

Answer

Informally, if a function is continuous at $x=c$, then there is no break in the graph of the function at $f(c)$, and $f(c)$ is defined.

2) State in your own words what it means for a function to be continuous on the interval (a,b).

Algebraic

For the exercises 3-22, determine why the function f is discontinuous at a given point a on the graph. State which condition fails.

3) $f(x)=\ln|x+3|, a=-3$

Answer

discontinuous at $a=-3$; $f(-3)$ does not exist

4) $f(x)=\ln|5x-2|, a=\frac{2}{5}$

5) $f(x)=\frac{x^2-16}{x+4}, a=-4$

Answer

removable discontinuity at $a=-4$; $f(-4)$ is not defined

6) $f(x)=\frac{x^2-16x}{x}, a=0$

7) $f(x)=\begin{cases} x, & x\neq 3 \\ 2x, & x=3 \end{cases}\ a=3$

Answer

Discontinuous at $a=3$; $\lim\limits_{x\to 3} f(x)=3$, but $f(3)=6$, which is not equal to the limit.

8) $f(x) = \begin{cases} 5, & x \neq 0 \\ 3, & x = 0 \end{cases} \; a = 0$

9) $f(x) = \begin{cases} \dfrac{1}{2-x}, & x \neq 2 \\ 3, & x = 2 \end{cases} \; a = 2$

Answer

$\lim\limits_{x \to 2} f(x)$ does not exist.

10) $f(x) = \begin{cases} \dfrac{1}{x+6}, & x = -6 \\ x^2, & x \neq -6 \end{cases} \; a = -6$

11) $f(x) = \begin{cases} 3+x, & x < 1 \\ x, & x = 1 \\ x^2, & x > 1 \end{cases} \; a = 1$

Answer

$\lim\limits_{x \to 1^-} f(x) = 4$; $\lim\limits_{x \to 1^+} f(x) = 1$. Therefore, $\lim\limits_{x \to 1} f(x)$ does not exist.

12) $f(x) = \begin{cases} 3-x, & x < 1 \\ x, & x = 1 \\ 2x^2, & x > 1 \end{cases} \; a = 1$

13) $f(x) = \begin{cases} 3+2x, & x < 1 \\ x, & x = 1 \\ -x^2, & x > 1 \end{cases} \; a = 1$

Answer

$\lim\limits_{x \to 1^-} f(x) = 5 \neq \lim\limits_{x \to 1^+} f(x) = -1$. Thus $\lim\limits_{x \to 1} f(x)$ does not exist.

14) $f(x) = \begin{cases} x^2, & x < -2 \\ 2x+1, & x = -2 \\ x^3, & x > -2 \end{cases} \; a = -2$

15) $f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x < -3 \\ x-9, & x = -3 \\ \dfrac{1}{x}, & x > -3 \end{cases} \; a = -3$

Answer

$\lim\limits_{xto-3^+} f(x) = -\dfrac{1}{3}$

Therefore, $\lim\limits_{x \to -3} f(x)$ does not exist.

16) $f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x < -3 \\ x-9, & x = -3 \\ -6, & x > -3 \end{cases} \; a = 3$

17) $f(x) = \dfrac{x^2-4}{x-2}, a = 2$

Answer

$f(2)$ is not defined.

18) $f(x)=\frac{25-x^2}{x^2-10x+25}, a=5$

19) $f(x)=\frac{x^3-9x}{x^2+11x+24}, a=-3$

Answer

$f(-3)$ is not defined.

20) $f(x)=\frac{x^3-27}{x^2-3x}, a=3$

21) $f(x)=\frac{x}{|x|}, a=0$

Answer

$f(0)$ is not defined.

22) $f(x)=\frac{2|x+2|}{x+2}, a=-2$

For the exercises 23-35, determine whether or not the given function f is continuous everywhere. If it is continuous everywhere it is defined, state for what range it is continuous. If it is discontinuous, state where it is discontinuous.

23) $f(x)=x^3-2x-15$

Answer

Continuous on $(-\infty,\infty)$

24) $f(x)=\frac{x^2-2x-15}{x-5}$

25) $f(x)=2\cdot 3^{x+4}$

Answer

Continuous on $(-\infty,\infty)$

26) $f(x)=-\sin(3x)$

27) $f(x)=\frac{|x-2|}{x^2-2x}$

Answer

Discontinuous at $x=0$ and$x=2$

28) $f(x)=\tan(x)+2$

29) $f(x)=2x+\frac{5}{x}$

Answer

Discontinuous at $x=0$

30) $f(x)=\log_2(x)$

31) $f(x)=\ln x^2$

Answer

Continuous on $(0,\infty)$

32) $f(x) = e^{2x}$

33) $f(x) = \sqrt{x-4}$

Answer

Continuous on $[4, \infty)$

34) $f(x) = \sec(x) - 3$

35) $f(x) = x^2 + \sin(x)$

Answer

Continuous on $(-\infty, \infty)$.

36) Determine the values of b and c such that the following function is continuous on the entire real number line.

\[f(x)= \begin{cases}x+1, && 1

Graphical

For the exercises 37-39, refer to the Figure below. Each square represents one square unit. For each value of a, determine which of the three conditions of continuity are satisfied at $x = a$ and which are not.

37) $x = -3$

Answer

1, but not 2 or 3

38) $x = 2$

39) $x = 4$

Answer

1 and 2, but not 3

For the exercises 40-43, use a graphing utility to graph the function $f(x) = \sin\left(\frac{12\pi}{x}\right)$ as in Figure. Set the x-axis a short distance before and after 0 to illustrate the point of discontinuity.

40) Which conditions for continuity fail at the point of discontinuity?

41) Evaluate $f(0)$.

Answer

$f(0)$ is undefined.

42) Solve for x if $f(x) = 0$.

43) What is the domain of $f(x)$?

Answer

$(-\infty, 0) \cup (0, \infty)$

For the exercises 44-45, consider the function shown in the Figure below.

44) At what x-coordinates is the function discontinuous?

45) What condition of continuity is violated at these points?

Answer

At $x = -1$, the limit does not exist. At $x = 1$, $f(1)$ does not exist.

At $x=2$, there appears to be a vertical asymptote, and the limit does not exist.

46) Consider the function shown in the Figure below. At what x-coordinates is the function discontinuous? What condition(s) of continuity were violated?

47) Construct a function that passes through the origin with a constant slope of 1, with removable discontinuities at $x=-7$ and $x=1$.

Answer

$\dfrac{x^3+6x^2-7x}{(x+7)(x-1)}$

48) The function $f(x)=\dfrac{x^3-1}{x-1}$ is graphed in the Figure below. It appears to be continuous on the interval $[-3,3]$, but there is an x-value on that interval at which the function is discontinuous. Determine the value of x at which the function is discontinuous, and explain the pitfall of utilizing technology when considering continuity of a function by examining its graph.

49) Find the limit $\lim_{x\to 1} f(x)$ and determine if the following function is continuous at $x=1$:

$$fx=\begin{cases} x^2+4 & x\neq 1 \\ 2 & x=1 \end{cases}$$

Answer

The function is discontinuous at $x=1$ because the limit as x approaches 1 is 5 and $f(1)=2$.

50) The graph of $f(x)=\dfrac{\sin(2x)}{x}$ is shown in the Figure below. Is the function $f(x)$ continuous at $x=0$? Why or why not?

5.4: Derivatives

Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

Verbal

1) How is the slope of a linear function similar to the derivative?

Answer

The slope of a linear function stays the same. The derivative of a general function varies according to x. Both the slope of a line and the derivative at a point measure the rate of change of the function.

2) What is the difference between the average rate of change of a function on the interval $[x,x+h]$ and the derivative of the function at x?

3) A car traveled 110 miles during the time period from 2:00 P.M. to 4:00 P.M. What was the car's average velocity? At exactly 2:30 P.M., the speed of the car registered exactly 62 miles per hour. What is another name for the speed of the car at 2:30 P.M.? Why does this speed differ from the average velocity?

Answer

Average velocity is 55 miles per hour. The instantaneous velocity at 2:30 p.m. is 62 miles per hour. The instantaneous velocity measures the velocity of the car at an instant of time whereas the average velocity gives the velocity of the car over an interval.

4) Explain the concept of the slope of a curve at point x.

5) Suppose water is flowing into a tank at an average rate of 45 gallons per minute. Translate this statement into the language of mathematics.

Answer

The average rate of change of the amount of water in the tank is 45 gallons per minute. If $f(x)$ is the function giving the amount of water in the tank at any time t, then the average rate of change of $f(x)$ between $t=a$ and $t=b$ is $f(a)+45(b-a)$.

Algebraic

For the exercises 6-17, use the definition of derivative $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to calculate the derivative of each function.

6) $f(x)=3x-4$

7) $f(x)=-2x+1$

Answer

$f'(x)=-2$

8) $f(x)=x^2-2x+1$

9) $f(x)=2x^2+x-3$

Answer

$f'(x)=4x+1$

10) $f(x)=2x^2+5$

11) $f(x)=\frac{-1}{x-2}$

Answer

$f'(x)=\frac{1}{(x-2)^2}$

12) $f(x)=\frac{2+x}{1-x}$

13) $f(x)=\frac{5-2x}{3+2x}$

Answer

$\frac{-16}{(3+2x)^2}$

14) $f(x)=\sqrt{1+3x}$

15) $f(x)=3x^3-x^2+2x+5$

Answer

$f'(x)=9x^2-2x+2$

16) $f(x)=5$

17) $f(x)=5\pi$

Answer

$f'(x)=0$

For the exercises 18-21, find the average rate of change between the two points.

18) $(-2,0)$ and $(-4,5)$

19) $(4,-3)$ and $(-2,-1)$

Answer

$$-\frac{1}{3}$$

20) $(0, 5)$ and $(6, 5)$

21) $(7, -2)$ and $(7, 10)$

Answer

undefined

For the polynomial functions 22-25, find the derivatives.

22) $f(x) = x^3 + 1$

23) $f(x) = -3x^2 - 7x = 6$

Answer

$f'(x) = -6x - 7$

24) $f(x) = 7x^2$

25) $f(x) = 3x^3 + 2x^2 + x - 26$

Answer

$f'(x) = 9x^2 + 4x + 1$

For the functions 26-28, find the equation of the tangent line to the curve at the given point x on the curve.

26) $f(x) = 2x^2 - 3x \quad x = 3$

27) $f(x) = x^2 + 1 \quad x = 2$

Answer

$y = 12x - 15$

28) $f(x) = \sqrt{x} \quad x = 9$

29) For the following exercise, find k such that the given line is tangent to the graph of the function.

$$f(x) = x^2 - kx \quad y = 4x - 9$$

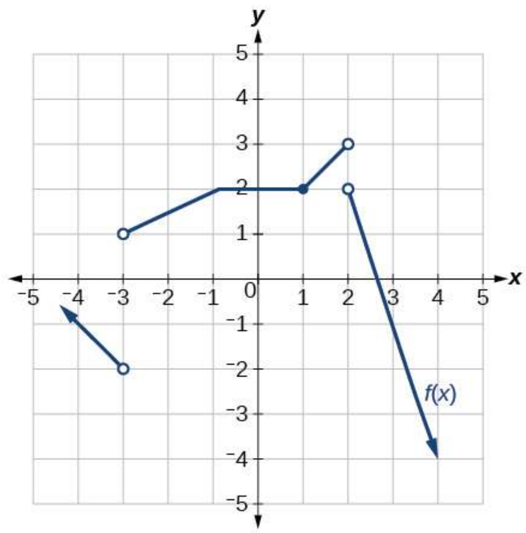
Answer

$k = -10$ or $k = 2$

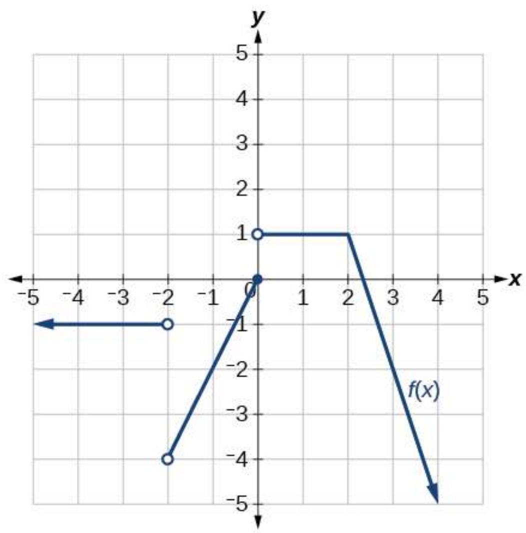
Graphical

For the exercises 30-33, consider the graph of the function f and determine where the function is continuous/discontinuous and differentiable/not differentiable.

30)



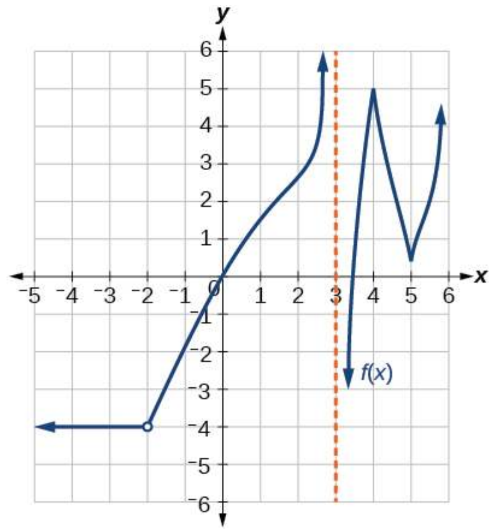
31)



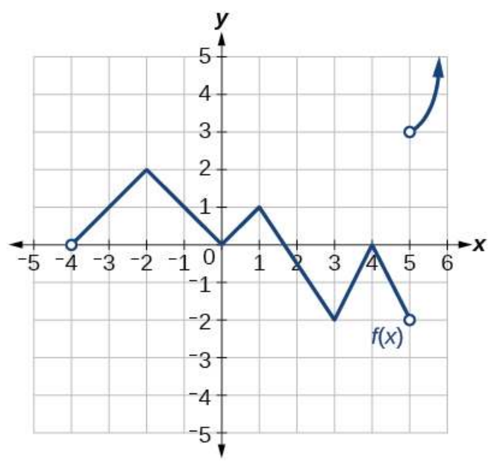
Answer

Discontinuous at $x = -2$ and $x = 0$. Not differentiable at $-2, 0, 2$.

32)



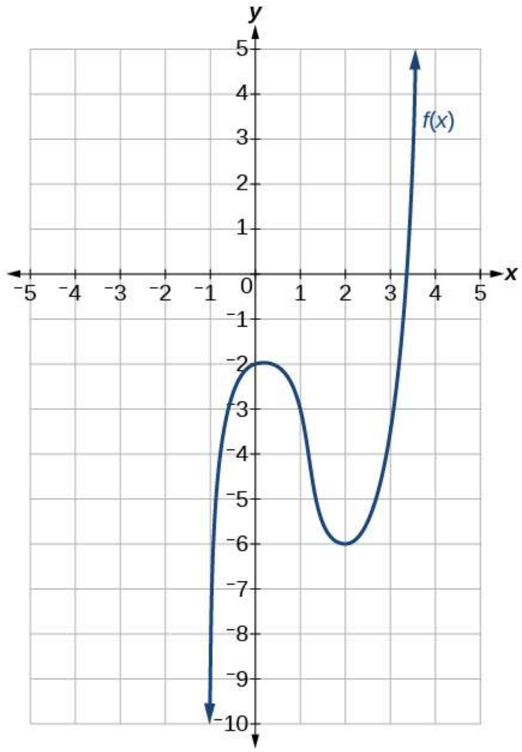
33)



Answer

Discontinuous at $x = 5$. Not differentiable at $-4, -2, 0, 1, 3, 4, 5$

For the exercises 34-43, use the Figure below to estimate either the function at a given value of x or the derivative at a given value of x, as indicated.



34) $f(-1)$

35) $f(0)$

Answer

$f(0) = -2$

36) $f(1)$

37) $f(2)$

Answer

$f(2) = -6$

38) $f(3)$

39) $f'(-1)$

Answer

$f'(-1) = 9$

40) $f'(0)$

41) $f'(1)$

Answer

$f'(1) = -3$

42) $f'(2)$

43) $f'(3)$

Answer

$f'(3) = 9$

44) Sketch the function based on the information below:

$$f'(x) = 2x, f(2) = 4$$

Technology

45) Numerically evaluate the derivative. Explore the behavior of the graph of $f(x) = x^2$ around $x = 1$ by graphing the function on the following domains: $[0.9, 1.1]$, $[0.99, 1.01]$, $[0.999, 1.001]$, $[0.9999, 1.0001]$. We can use the feature on our calculator that automatically sets Ymin and Ymax to the Xmin and Xmax values we preset. (On some of the commonly used graphing calculators, this feature may be called ZOOM FIT or ZOOM AUTO). By examining the corresponding range values for this viewing window, approximate how the curve changes at $x = 1$, that is, approximate the derivative at $x = 1$.

Answer

Answers vary. The slope of the tangent line near $x = 1$ is 2.

Real-World Applications

For the exercises 46-50, explain the notation in words. The volume $f(t)$ of a tank of gasoline, in gallons, t minutes after noon.

46) $f(0) = 600$

47) $f'(30) = -20$

Answer

At 12:30 p.m., the rate of change of the number of gallons in the tank is -20 gallons per minute. That is, the tank is losing 20 gallons per minute.

48) $f(30) = 0$

49) $f'(200) = 30$

Answer

At 200 minutes after noon, the volume of gallons in the tank is changing at the rate of 30 gallons per minute.

50) $f(240) = 500$

For the exercises 51-55, explain the functions in words. The height, s, of a projectile after t seconds is given by $s(t) = -16t^2 + 80t$.

51) $s(2) = 96$

Answer

The height of the projectile after 2 seconds is 96 feet.

52) $s'(2) = 16$

53) $s(3) = 96$

Answer

The height of the projectile at $t = 3$ seconds is 96 feet.

54) $s'(3) = -16$

55) $s(0) = 0, s(5) = 0$

Answer

The height of the projectile is zero at $t=0$ and again at $t=5$. In other words, the projectile starts on the ground and falls to earth again after 5 seconds.

For the exercises 56-57, the volume V of a sphere with respect to its radius r is given by $V=\frac{4}{3}\pi r^3$.

56) Find the average rate of change of V as r changes from 1 cm to 2 cm.

57) Find the instantaneous rate of change of V when $r=3$ cm.

Answer

36π

For the exercises 58-60, the revenue generated by selling x items is given by $R(x)=2x^2+10x$.

58) Find the average change of the revenue function as x changes from $x=10$ to $x=20$.

59) Find $R'(10)$ and interpret.

Answer

$\$50.00$per unit, which is the instantaneous rate of change of revenue when exactly 10 units are sold.

60) Find $R'(15)$ and interpret. Compare $R'(15)$ to $R'(10)$, and explain the difference.

For the exercises 61-63, the cost of producing x cellphones is described by the function $C(x)=x^2-4x+1000$.

61) Find the average rate of change in the total cost as x changes from $x=10$ to $x=15$.

Answer

$\$21$ per unit

62) Find the approximate marginal cost, when 15 cellphones have been produced, of producing the 16^{th} cellphone.

63) Find the approximate marginal cost, when 20 cellphones have been produced, of producing the 21^{st} cellphone.

Answer

$\$36$

Extension

For the exercises 64-67, use the definition for the derivative at a point $x=a$, $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, to find the derivative of the functions.

64) $f(x)=\frac{1}{x^2}$

65) $f(x)=5x^2-x+4$

Answer

$f'(x)=10a-1$

66) $f(x)=-x^2+4x+7$

67) $f(x)=\frac{-4}{3-x^2}$

Answer

$\frac{4}{(3-x)^2}$

Contributors and Attributions

- Template:OpenStaxPreCalc

1.1: 5.6 Derivatives of Polynomials

Definition: Derivative of Polynomials

Given the function $f(x) = 6x^2 + 3x + 6$, find the derivative. When the function is a polynomial, you can find the derivative by doing the following for each term.

$f'(x) = 2(6)x^{2-1} + (1)(3)x^0 + (0)6x^{0-1} = 12x^1 + 3x^0 + 0 = 12x + 3$

1) Bring the exponent down and multiply it by the coefficient.

2) Subtract one from the exponent.

3) Simplify.

4) The derivative of a constant is 0.

Example 1.1.5.6.1

Find the derivative of the following polynomial.

$f(x) = 4x^3 - 5x^2 + 10x - 5$

Solution

$f'(x) = 3(4)x^{3-1} - (2)(5)x^{2-1} + (1)(10)x^{1-1} - 0$

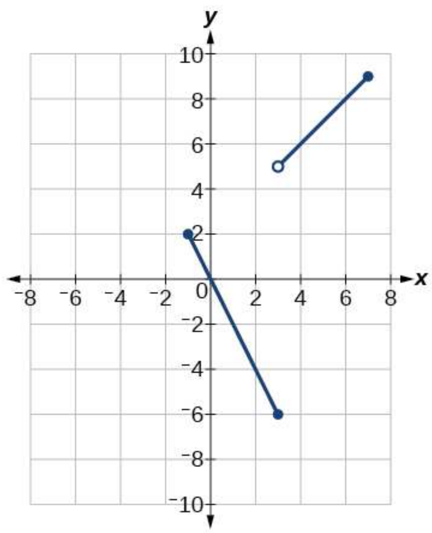
$f'(x) = 12x^2 - 10x^1 + 10x^0 - 0$

$f'(x) = 12x^2 - 10x + 10$

5.R: Introduction to Calculus (Review)

5.1: Finding Limits - Numerical and Graphical Approaches

For the exercises 1-6, use the Figure below.



1) $\lim_{x\to -1^+} f(x)$

Answer

2

2) $\lim_{x\to -1^-} f(x)$

3) $\lim_{x\to -1} f(x)$

Answer

does not exist

4) $\lim_{x\to 3} f(x)$

5) At what values of x is the function discontinuous? What condition of continuity is violated?

Answer

Discontinuous at $x=-1\left(\lim_{x\to a} f(x) \text{ does not exist}\right)$, $x=3$ (jump discontinuity) , and $x=7\left(\lim_{x\to a} f(x) \text{ does not exist}\right)$.

6) Using the Table below, estimate $\lim_{x\to 0} f(x)$.

x	$F(x)$
−0.1	2.875
−0.01	2.92
−0.001	2.998
0	Undefined
0.001	2.9987

x	$F(x)$
0.01	2.865
0.1	2.78145
0.15	2.678

Answer

3

For the exercises 7-9, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as x approaches a. If the function has limit as x approaches a, state it. If not, discuss why there is no limit.

7) $f(x)=\begin{cases} |x|-1 & \text{if } x\neq 1 \\ x^3 & \text{if } x=1 \end{cases}\quad a=1$

8) $f(x)=\begin{cases} \dfrac{1}{x+1} & \text{if } x=-2 \\ (x+1)^2 & \text{if } x\neq -2 \end{cases}\quad a=-2$

Answer

$\lim\limits_{x\to -2} f(x)=1$

9) $f(x)=\begin{cases} \sqrt{x+3} & \text{if } x<1 \\ -\sqrt[3]{x} & \text{if } x>1 \end{cases}\quad a=1$

5.2: Finding Limits - Properties of Limits

For the exercises 1-6, find the limits if $\lim\limits_{x\to c} f(x)=-3$ and $\lim\limits_{x\to c} g(x)=5$.

1) $\lim\limits_{x\to c}(f(x)+g(x))$

Answer

2

2) $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$

3) $\lim\limits_{x\to c}(f(x)\cdot g(x))$

Answer

-15

4) $\lim\limits_{x\to 0^+} f(x), f(x)=\begin{cases} 3x^2+2x+1 & x>0 \\ 5x+3 & x<0 \end{cases}$

5) $\lim\limits_{x\to 0^-} f(x), f(x)=\begin{cases} 3x^2+2x+1 & x>0 \\ 5x+3 & x<0 \end{cases}$

Answer

3

6) $\lim\limits_{x\to 3^+}(3x-[\![x]\!])$

For the exercises 7-11, evaluate the limits using algebraic techniques.

7) $\lim_{h\to 0}\left(\frac{(h+6)^2-36}{h}\right)$

Answer

12

8) $\lim_{x\to 25}\left(\frac{x^2-625}{\sqrt{x}-5}\right)$

9) $\lim_{x\to 1}\left(\frac{-x^2-9x}{x}\right)$

Answer

-10

10) $\lim_{x\to 4}\left(\frac{7-\sqrt{12x+1}}{x-4}\right)$

11) $\lim_{x\to 3}\left(\frac{\frac{1}{3}+\frac{1}{x}}{3+x}\right)$

Answer

$-\frac{1}{9}$

5.3: Continuity

For the exercises 1-5, use numerical evidence to determine whether the limit exists at $x=a$. If not, describe the behavior of the graph of the function at $x=a$.

1) $f(x)=\frac{-2}{x-4};\ a=4$

2) $f(x)=\frac{-2}{(x-4)^2};\ a=4$

Answer

At $x=4$, the function has a vertical asymptote.

3) $f(x)=\frac{-x}{x^2-x-6};\ a=3$

4) $f(x)=\frac{6x^2+23x+20}{4x^2-25};\ a=-\frac{5}{2}$

Answer

removable discontinuity at $a=-\frac{5}{2}$

5) $f(x)=\frac{\sqrt{x}-3}{9-x};\ a=9$

For the exercises 6-12, determine where the given function $f(x)$ is continuous. Where it is not continuous, state which conditions fail, and classify any discontinuities.

6) $f(x)=x^2-2x-15$

Answer

continuous on $(-\infty,\infty)$

7) $f(x)=\dfrac{x^2-2x-15}{x-5}$

8) $f(x)=\dfrac{x^2-2x}{x^2-4x+4}$

Answer

removable discontinuity at $x=2$. $f(2)$ is not defined, but limits exist.

9) $f(x)=\dfrac{x^3-125}{2x^2-12x+10}$

10) $f(x)=\dfrac{x^2-\frac{1}{x}}{2-x}$

Answer

discontinuity at $x=0$ and $x=2$. Both $f(0)$ and $f(2)$ are not defined.

11) $f(x)=\dfrac{x+2}{x^2-3x-10}$

12) $f(x)=\dfrac{x+2}{x^3+8}$

Answer

removable discontinuity at $x=-2$. $f(-2)$ is not defined.

5.4: Derivatives

For the exercises 1-5, find the average rate of change $f(x)=\dfrac{f(x+h)-f(x)}{h}$.

1) $f(x)=3x+2$

2) $f(x)=5$

Answer

0

3) $f(x)=\dfrac{1}{x+1}$

4) $f(x)=\ln(x)$

Answer

$$f(x)=\frac{\ln(x+h)-\ln(x)}{h}$$

5) $f(x)=e^{2x}$

For the exercises 6-7, find the derivative of the function.

6) $f(x)=4x-6$

Answer

4

7) $f(x)=5x^2-3x$

8) Find the equation of the tangent line to the graph of $f(x)$ at the indicated x value.

$$f(x)=-x^3+4x;\ x=2$$

Answer

$y=-8x+16$

9) For the following exercise, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

$$f(x)=\frac{x}{|x|}$$

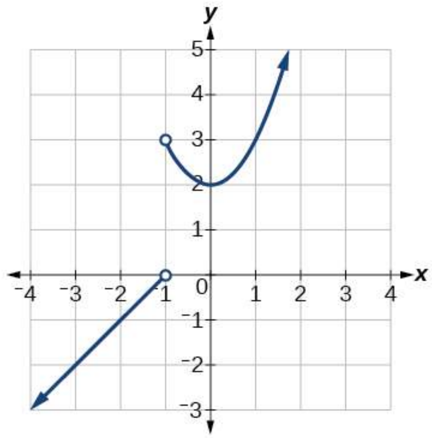
10) Given that the volume of a right circular cone is $V=\frac{1}{3}\pi r^2 h$ and that a given cone has a fixed height of 9 cm and variable radius length, find the instantaneous rate of change of volume with respect to radius length when the radius is 2 cm. Give an exact answer in terms of π.

Answer

12π

Practice Test

For the exercises 1-6, use the graph of f in the Figure below.



1) $f(1)$

Answer

3

2) $\lim\limits_{x\to -1^+} f(x)$

3) $\lim\limits_{x\to -1^-} f(x)$

Answer

0

4) $\lim\limits_{x\to -1} f(x)$

5) $\lim\limits_{x\to -2} f(x)$

Answer

-1

6) At what values of x is f discontinuous? What property of continuity is violated?

7) $f(x)=\begin{cases}\frac{1}{3}-3 & \text{if } x\le 2\\ x^3+1 & \text{if } x>2\end{cases}$ $a=2$

Answer

$$\lim_{x\to 2^-} f(x)=-\frac{5}{2}a \text{ and } \lim_{x\to 2^+} f(x)=9$$

Thus, the limit of the function as x approaches 2 does not exist.

8) $f(x)=\begin{cases}x^3+1 & \text{if } x<1\\ 3x^2-1 & \text{if } x=1\\ -\sqrt{x+3}+4 & \text{if } x>1\end{cases}$ $a=1$

For the exercises 9-11, evaluate each limit using algebraic techniques.

9) $\lim_{x\to -5}\left(\frac{\frac{1}{5}+\frac{1}{x}}{10+2x}\right)$

Answer

$$-\frac{1}{50}$$

10) $\lim_{h\to 0}\left(\frac{\sqrt{h^2+25}-5}{h^2}\right)$

11) $\lim_{h\to 0}\left(\frac{1}{h}-\frac{1}{h^2+h}\right)$

Answer

1

For the exercises 12-13, determine whether or not the given function f is continuous. If it is continuous, show why. If it is not continuous, state which conditions fail.

12) $f(x)=\sqrt{x^2-4}$

13) $f(x)=\frac{x^3-4x^2-9x+36}{x^3-3x^2+2x-6}$

Answer

removable discontinuity at $x=3$

For the exercises 14-16, use the definition of a derivative to find the derivative of the given function at $x=a$.

14) $f(x)=\frac{3}{5+2x}$

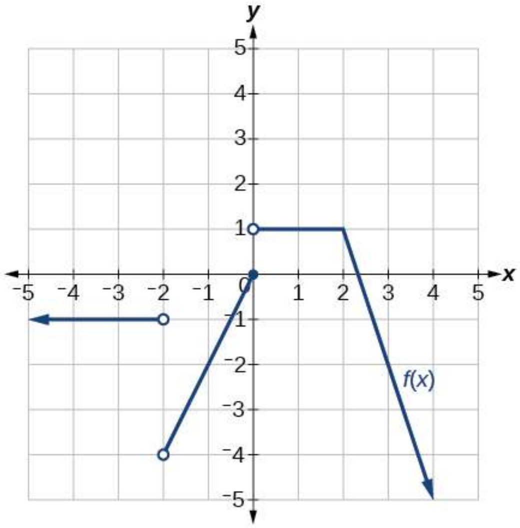
15) $f(x)=\frac{3}{\sqrt{x}}$

Answer

$$f'(x)=-\frac{3}{2a^{\frac{3}{2}}}$$

16) $f(x)=2x^2+9x$

17) For the graph in the Figure below, determine where the function is continuous/discontinuous and differentiable/not differentiable.



Answer

discontinuous at $-2, 0$, not differentiable at $-2, 0, 2$.

For the exercises 18-19, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

18) $f(x) = |x-2| - |x+2|$

19) $f(x) = \dfrac{2}{1+e^{\frac{2}{x}}}$

Answer

not differentiable at $x = 0$ (no limit)

For the exercises 20-24, explain the notation in words when the height of a projectile in feet, s, is a function of time t in seconds after launch and is given by the function $s(t)$.

20) $s(0)$

21) $s(2)$

Answer

the height of the projectile at $t = 2$ seconds

22) $s'(2)$

23) $\dfrac{s(2)-s(1)}{2-1}$

Answer

the average velocity from $t = 1$ to $t = 2$

24) $s(t) = 0$

For the exercises 25-28, use technology to evaluate the limit.

25) $\lim\limits_{x\to 0} \dfrac{\sin(x)}{3x}$

Answer

$\frac{1}{3}$

26) $\lim_{x\to 0} \frac{\tan^2(x)}{2x}$

27) $\lim_{x\to 0} \frac{\sin(x)(1-\cos(x))}{2x^2}$

Answer

0

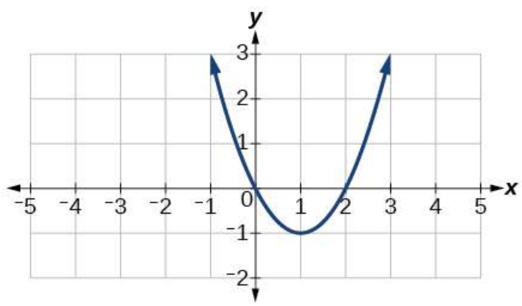
28) Evaluate the limit by hand.

$$\lim_{x\to 1} f(x), \text{ where } f(x) = \begin{cases} 4x-7 & x \neq 1 \\ x^2-4 & x=1 \end{cases}$$

At what value(s) of x is the function below discontinuous?

$$f(x) = \begin{cases} 4x-7 & x \neq 1 \\ x^2-4 & x=1 \end{cases}$$

For the exercises 29-32, consider the function whose graph appears in Figure.



29) Find the average rate of change of the function from $x = 1$ to $x = 3$.

Answer

2

30) Find all values of x at which $f'(x) = 0$.

Answer

$x = 1$

31) Find all values of x at which $f'(x)$ does not exist.

32) Find an equation of the tangent line to the graph of f the indicated point: $f(x) = 3x^2 - 2x - 6, \; x = -2$

Answer

$y = -14x - 18$

For the exercises 33-34, use the function $f(x) = x(1-x)^{\frac{2}{5}}$

33) Graph the function $f(x) = x(1-x)^{\frac{2}{5}}$ by entering $f(x) = x\left((1-x)^2\right)^{\frac{1}{5}}$ and then by entering $f(x) = x\left((1-x)^{\frac{1}{5}}\right)^2$.

34) Explore the behavior of the graph of $f(x)$ around $x = 1$ by graphing the function on the following domains, $[0.9, 1.1]$, $[0.99, 1.01]$, $[0.999, 1.001]$, and $[0.9999, 1.0001]$. Use this information to determine whether the function appears to be differentiable at $x = 1$.

Answer

The graph is not differentiable at $x=1$ (cusp).

For the exercises 35-42, find the derivative of each of the functions using the definition: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

35) $f(x)=2x-8$

36) $f(x)=4x^2-7$

Answer

$f'(x)=8x$

37) $f(x)=x-\frac{1}{2}x^2$

38) $f(x)=\frac{1}{x+2}$

Answer

$$f'(x)=-\frac{1}{(2+x)^2}$$

39) $f(x)=\frac{3}{x-1}$

40) $f(x)=-x^3+1$

Answer

$f'(x)=-3x^2$

41) $f(x)=x^2+x^3$

42) $f(x)=\sqrt{x-1}$

Answer

$$f'(x)=-\frac{1}{2\sqrt{x-1}}$$

Contributors and Attributions

- Template:OpenStaxPreCalc

CHAPTER OVERVIEW

6: Mathematics of Finance

Learning Objectives

In this chapter, you will learn to:

1. Solve financial problems that involve simple interest.
2. Solve problems involving compound interest.
3. Find the future value of an annuity, and the amount of payments to a sinking fund.
4. Find the future value of an annuity, and an installment payment on a loan.

6.1: Simple and Compound Interest
  6.1.1: Simple and Compound Interest (Exercises)
6.2: Annuities and Sinking Funds
  6.2.1: Annuities and Sinking Funds (Exercises)
6.3: Present Value of an Annuity and Installment Payment
  6.3.1: Present Value of an Annuity and Installment Payment (Exercises)
6.4: Miscellaneous Application Problems
  6.4.1: Miscellaneous Application Problems (Exercises)
6.5: Classification of Finance Problems
  6.5.1: Classification of Finance Problems (Exercises)
6.6: Chapter Review

6.1: Simple and Compound Interest

Learning Objectives

In this section, you will learn to:

1. Find simple interest.
2. Find the present value for simple interest.
3. Find discounts and proceeds for simple interest.
4. Find the future value of a lump-sum using compound interest.
5. Find the present value of a lump-sum using compound interest.
6. Find the effective interest rate using compound interest.

Simple Interest

It costs to borrow money. The rent one pays for the use of money is called the **interest**. The amount of money that is being borrowed or loaned is called the **principle** or **present value**. Simple interest is paid only on the original amount borrowed. When the money is loaned out, the person who borrows the money generally pays a fixed rate of interest on the principle for the time period he keeps the money. Although the interest rate is often specified for a year, it may be specified for a week, a month, or a quarter, etc. The credit card companies often list their charges as monthly rates, sometimes it is as high as 2.75% a month.

Definition: Simple Interest

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$I = P \cdot r \cdot t$$

Definition: Accumulated Value

The total amount FV, also called the **accumulated value** or the future value, is given by

$$\begin{aligned} FV &= P + I \\ &= P + Prt \end{aligned}$$

or

$$FV = P(1 + rt) \tag{6.1.1}$$

where interest rate r is expressed in decimals.

Example 6.1.1

Ursula borrows \$600 for 5 months at a simple interest rate of 15% per year. Find the interest, and the total amount she is obligated to pay?

Solution

The interest is computed by multiplying the principle with the interest rate and the time.

$$\begin{aligned} \mathrm{I} &= \mathrm{Prt} \\ &= \$600(0.15)\frac{5}{12} \\ &= \$37.50 \end{aligned}$$

The total amount is

$$\begin{aligned} \mathrm{FV} &= \mathrm{P}+\mathrm{I} \\ &= \$600+\$37.50 \\ &= \$637.50 \end{aligned}$$

Incidentally, the total amount can be computed directly via Equation 6.1.1 as

$$\begin{aligned} FV &= P(1+rt) \\ &= \$600[1+(0.15)(5/12)] \\ &= \$600(1+0.0625) \\ &= \$637.50 \end{aligned}$$

Example 6.1.2

Jose deposited $2500 in an account that pays 6% simple interest. How much money will he have at the end of 3 years?

Solution

The total amount or the future value is given by Equation 6.1.1.

$$\begin{aligned} FV &= P(1+rt) \\ &= \$2500[1+(.06)(3)] \\ \mathrm{FV} &= \$2950 \end{aligned}$$

Example 6.1.3

Darnel owes a total of $3060 which includes 12% interest for the three years he borrowed the money. How much did he originally borrow?

Solution

This time we are asked to compute the principle P via Equation 6.1.1.

$$\begin{aligned} \$3060 &= \mathrm{P}[1+(0.12)(3)] \\ \$3060 &= \mathrm{P}(1.36) \\ \frac{\$3060}{1.36} &= \mathrm{P} \\ \$2250 &= \mathrm{P} \quad \text{Darnel originally borrowed } \$2250. \end{aligned}$$

Example 6.1.4

A Visa credit card company charges a 1.5% finance charge for each month on the unpaid balance. If Martha owed $2350 and has not paid her bill for three months, how much does she owe now?

Solution

Before we attempt the problem, the reader should note that in this problem the rate of finance charge is given per month and not per year.

The total amount Martha owes is the previous unpaid balance plus the finance charge.

$$FV=\$2350+\$2350(.015)(3)=\$2350+\$105.75=\$2455.75$$

Alternatively, again, we can compute the amount directly by using formula $FV=P(1+rt)$

$$FV=\$2350[1+(.015)(3)]=\$2350(1.045)=\$2455.75$$

Discount and Proceeds

Banks often deduct the simple interest from the loan amount at the time that the loan is made. When this happens, we say the loan has been **discounted.** The interest that is deducted is called the **discount**, and the actual amount that is given to the borrower is called the **proceeds**. The amount the borrower is obligated to repay is called the **maturity value.**

Discount and Proceeds

If an amount M is borrowed for a time t at a discount rate of r per year, then the discount D is

$$D = M \cdot r \cdot t$$

The proceeds P, the actual amount the borrower gets, is given by

$$\begin{aligned} P &= M - D \\ &= M - Mrt \end{aligned}$$

or

$$P = M(1 - rt)$$

where interest rate r is expressed in decimals.

Example 6.1.5

Francisco borrows $1200 for 10 months at a simple interest rate of 15% per year. Determine the discount and the proceeds.

Solution

The discount D is the interest on the loan that the bank deducts from the loan amount.

$$\begin{aligned} &\mathrm{D} = \mathrm{Mrt} \\ &\mathrm{D} = \$1200(0.15)\left(\tfrac{10}{12}\right) = \$150 \end{aligned}$$

Therefore, the bank deducts $150 from the maturity value of $1200, and gives Francisco $1050. Francisco is obligated to repay the bank $1200.

In this case, the discount D = $150, and the proceeds

$$P = \$1200 - \$150 = \$1050.$$

Example 6.1.6

If Francisco wants to receive $1200 for 10 months at a simple interest rate of 15% per year, what amount of loan should he apply for?

Solution

In this problem, we are given the proceeds P and are being asked to find the maturity value M.

We have P = $1200, $r = 0.15$, t = 10/12 . We need to find M.

We know $P = M - D$

but also $D = Mrt$

therefore

$$\begin{aligned} \mathrm{P} &= \mathrm{M} - \mathrm{Mrt} \\ &= \mathrm{M}(1 - \mathrm{rt}) \\ \$1200 &= \mathrm{M}\left[1 - (0.15)\left(\frac{10}{12}\right)\right] \end{aligned}$$

We need to solve for M.

$$\begin{aligned} \$1200 &= \mathrm{M}(1-0.125) \\ \$1200 &= \mathrm{M}(0.875) \\ \frac{\$1200}{0.875} &= \mathrm{M} \\ \$1371.43 &= \mathrm{M} \end{aligned}$$

Therefore, Francisco should ask for a loan for \$1371.43.

The bank will discount \$171.43 and Francisco will receive \$1200.

Simple Interest Summary

Below is a summary of the formulas we developed for calculations involving simple interest:

Simple interest

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$I = P \cdot r \cdot t$$

The total amount A, also called the accumulated value or the future value, is given by

$$FV = P + I = P + Prt$$

or

$$FV = P(1+rt)$$

where interest rate r is expressed in decimals.

Discount and Proceeds

If an amount M is borrowed for a time t at a discount rate of r per year, then the discount D is

$$D = M \cdot r \cdot t$$

The proceeds P, the actual amount the borrower gets, is given by

$$\begin{gathered} P = M - D \\ P = M - Mrt \end{gathered}$$

or

$$P = M(1-rt)$$

where interest rate r is expressed in decimals.

At the end of the loan's term, the borrower repays the entire maturity amount M.

Compound Interest

In the last section, we examined problems involving simple interest. Simple interest is generally charged when the lending period is short and often less than a year. When the money is loaned or borrowed for a longer time period, if the interest is paid (or charged) not only on the principle, but also on the past interest, then we say the interest is **compounded.**

Suppose we deposit \$200 in an account that pays 8% interest. At the end of one year, we will have \$200 + \$200(.08) = \$200(1 + .08) = \$216.

Now, suppose we put this amount, \$216, in the same account. After another year, we will have \$216 + \$216(.08) = \$216(1 + .08) = \$233.28.

So, an initial deposit of \$200 has accumulated to \$233.28 in two years. Further note that had it been simple interest, this amount would have accumulated to only \$232. The reason the amount is slightly higher is because the interest (\$16) we earned the first year, was put back into the account. And this \$16 amount itself earned for one year interest of \$16(.08) = \$1.28, thus resulting in the increase. So, we have earned interest on the principle as well as on the past interest, and that is why we call it compound interest.

Now suppose we leave this amount, \$233.28, in the bank for another year, the final amount will be \$233.28 + \$233.28(.08) = \$233.28(1 + .08) = \$251.94.

Now, let us look at the mathematical part of this problem to devise an easier way to solve these problems.

After one year, we had \$200(1 + .08) = \$216

After two years, we had \$216(1 + .08)

But \$216 = \$200(1 + .08), therefore, the above expression becomes

$$\$200(1+.08)(1+.08)=\$200(1+.08)^2=\$233.28$$

After three years, we get

$$\$233.28(1+.08)=\$200(1+.08)(1+.08)(1+.08)$$

which can be written as

$$\$200(1+.08)^3=\$251.94$$

Suppose we are asked to find the total amount at the end of 5 years, we will get

$$200(1+.08)^5=\$293.87$$

We summarize as follows:

The original amount	\$200	= \$200
The amount after one year	\$200(1 + .08)	= \$216
The amount after two years	$\$200(1 + .08)^2$	= \$233.28
The amount after three years	$\$200(1 + .08)^3$	= \$251.94
The amount after five years	$\$200(1 + .08)^5$	= \$293.87
The amount after t years	$\$200(1 + .08)^t$	

COMPOUNDING PERIODS

Banks often compound interest more than one time a year. Consider a bank that pays 8% interest but compounds it four times a year, or quarterly. This means that every quarter, the bank will pay an interest equal to one-fourth of 8%, or 2%.

Now if we deposit \$200 in the bank, after one quarter we will have $\$200\left(1+\frac{.08}{4}\right)$ or \$204.

After two quarters, we will have $\$200\left(1+\frac{.08}{4}\right)^2$ or \$208.08.

After one year, we will have $\$200\left(1+\frac{.08}{4}\right)^4$ or \$216.49.

After three years, we will have $\$200\left(1+\frac{.08}{4}\right)^{12}$ or \$253.65, etc.

The original amount	\$200	= \$200
The amount after one quarter	$\$200\left(1+\frac{.08}{4}\right)$	= \$204

The amount after two quarters	$\$200\left(1+\frac{.08}{4}\right)^{2}$	= $208.08
The amount after one year	$\$200\left(1+\frac{.08}{4}\right)^{4}$	= $216.49
The amount after two years	$\$200\left(1+\frac{.08}{4}\right)^{8}$	= $234.31
The amount after three years	$\$200\left(1+\frac{.08}{4}\right)^{12}$	= $253.65
The amount after five years	$\$200\left(1+\frac{.08}{4}\right)^{20}$	= $297.19
The amount after t years	$\$200\left(1+\frac{.08}{4}\right)^{4t}$	

Therefore, if we invest a lump-sum amount of P dollars at an interest rate r, compounded n times a year, then after t years the final amount is given by

$$FV = P\left(1+\frac{r}{m}\right)^{mt}$$

The following examples use the compound interest formula $FV = P\left(1+\frac{r}{m}\right)^{mt}$

Example 6.1.7

If $3500 is invested at 9% compounded monthly, what will the future value be in four years?

Solution

Clearly an interest of .09/12 is paid every month for four years. The interest is compounded $4 \times 12 = 48$ times over the four-year period. We get

$$\mathrm{FV} = \$3500\left(1+\frac{.09}{12}\right)^{48} = \$3500(1.0075)^{48} = \$5009.92$$

$3500 invested at 9% compounded monthly will accumulate to $5009.92 in four years.

Example 6.1.8

How much should be invested in an account paying 9% compounded daily for it to accumulate to $5,000 in five years?

Solution

We know the future value, but need to find the principle.

$$\begin{aligned}\$5000 &= P\left(1+\tfrac{.09}{365}\right)^{365\times 5}\\ \$5000 &= P(1.568225)\\ \$3188.32 &= P\end{aligned}$$

$3188,32 invested into an account paying 9% compounded daily will accumulate to $5,000 in five years.

Example 6.1.9

If $4,000 is invested at 4% compounded annually, how long will it take to accumulate to $6,000?

Solution

$m=1$ because annual compounding means compounding only once per year. The formula simplifies to $FV=(1+r)^{t}$ when $m=1$.

$$\begin{aligned} \$6000 &= 4000(1+.04)^t \\ \frac{6000}{4000} &= 1.04^t \\ 1.5 &= 1.04^t \end{aligned}$$

We use logarithms to solve for the value of t because the variable t is in the exponent.

$$t = \log_{1.04}(1.5)$$

Using the change of base formula we can solve for t:

$$t = \frac{\ln(1.5)}{\ln(1.04)} = 10.33 \text{ years}$$

It takes 10.33 years for $4000 to accumulate to $6000 if invested at 4% interest, compounded annually

Example 6.1.10

If $5,000 is invested now for 6 years what interest rate compounded quarterly is needed to obtain an accumulated value of $8000.

Solution

We have $m = 4$ for quarterly compounding.

$$\begin{aligned} \$8000 &= \$5000\left(1+\frac{r}{4}\right)^{4\times 6} \\ \frac{\$8000}{\$5000} &= \left(1+\frac{r}{4}\right)^{24} \\ 1.6 &= \left(1+\frac{r}{4}\right)^{24} \end{aligned}$$

We use roots to solve for t because the variable r is in the base, whereas the exponent is a known number.

$$\sqrt[24]{1.6} = 1 + \frac{\mathrm{r}}{4}$$

Many calculators have a built in “nth root” key or function. In the TI-84 calculator, this is found in the Math menu. Roots can also be calculated as fractional exponents; if necessary, the previous step can be rewritten as

$$1.6^{1/24} = 1 + \frac{\mathrm{r}}{4}$$

Evaluating the left side of the equation gives

$$\begin{aligned} &1.0197765 = 1 + \tfrac{\mathrm{r}}{4} \\ &0.0197765 = \tfrac{\mathrm{r}}{4} \\ &\mathrm{r} = 4(0.0197765) = 0.0791 \end{aligned}$$

An interest rate of (0.0791 * 100 = 7.91) 7.91% is needed in order for $5000 invested now to accumulate to $8000 at the end of 6 years, with interest compounded quarterly.

Effective Interest Rate

Banks are required to state their interest rate in terms of an **“effective yield”** ” or **“effective interest rate”**, for comparison purposes. The effective rate is also called the Annual Percentage Yield (APY) or Annual Percentage Rate (APR).

The effective rate is the interest rate compounded annually would be equivalent to the stated rate and compounding periods. The next example shows how to calculate the effective rate.

To examine several investments to see which has the best rate, we find and compare the effective rate for each investment.

Example $6.1.11$ illustrates how to calculate the effective rate.

Example 6.1.11

If Bank A pays 7.2% interest compounded monthly, what is the effective interest rate?
If Bank B pays 7.25% interest compounded semiannually, what is the effective interest rate? Which bank pays more interest?

Solution

Bank A: Suppose we deposit $1 in this bank and leave it for a year, we will get

$$1\left(1+\frac{0.072}{12}\right)^{12}=1.0744$$
$$\mathrm{r_{EFF}}=1.0744-1=0.0744$$

We earned interest of $1.0744 - $1.00 = $.0744 on an investment of $1.

The effective interest rate is 7.44%, often referred to as the APY or APR.

Bank B: The effective rate is calculated as

$$\mathbf{r}_{\mathrm{EFF}}=1\left(1+\frac{0.072}{2}\right)^{2}-1=.0738$$

The effective interest rate is 7.38%.

Bank A pays slightly higher interest, with an effective rate of 7.44%, compared to Bank B with effective rate 7.38%.

Continuous Compounding

Interest can be compounded yearly, semiannually, quarterly, monthly, and daily. Using the same calculation methods, we could compound every hour, every minute, and even every second. As the compounding period gets shorter and shorter, we move toward the concept of continuous compounding.

But what do we mean when we say the interest is compounded continuously, and how do we compute such amounts? When interest is compounded "infinitely many times", we say that the interest is **compounded continuously**. Our next objective is to derive a formula to model continuous compounding.

Suppose we put $1 in an account that pays 100% interest. If the interest is compounded once a year, the total amount after one year will be $\$1(1+1)=\$2$.

- If the interest is compounded semiannually, in one year we will have $\$1(1+1/2)^2=\$2.25$
- If the interest is compounded quarterly, in one year we will have $\$1(1+1/4)^4=\$2.44$
- If the interest is compounded monthly, in one year we will have $\$1(1+1/12)^{12}=\$2.61$
- If the interest is compounded daily, in one year we will have $\$1(1+1/365)^{365}=\$2.71$

We show the results as follows:

Frequency of compounding	Formula	Total amount
Annually	$\$1(1+1)$	$2
Semiannually	$\$1(1+1/2)^2$	$2.25
Quarterly	$\$1(1+1/4)^4=\$2.44$	$2.44140625
Monthly	$\$1(1+1/12)^{12}$	$2.61303529
Daily	$\$1(1+1/365)^{365}$	$2.71456748
Hourly	$\$1(1+1/8760)^{8760}$	$2.71812699
Every minute	$\$1(1+1/525600)^{525600}$	$2.71827922
Every Second	$\$1(1+1/31536000)^{31536000}$	$2.71828247
Continuously	$\$1(2.718281828\ldots)$	$2.718281828...

We have noticed that the $1 we invested does not grow without bound. It starts to stabilize to an irrational number 2.718281828... given the name "e" after the great mathematician Euler.

In mathematics, we say that as m becomes infinitely large the expression equals $\left(1+\frac{1}{m}\right)^{m}$ e.

Therefore, it is natural that the number e play a part in continuous compounding.
It can be shown that as m becomes infinitely large the expression $\left(1+\frac{r}{m}\right)^{mt}=e^{rt}$

Therefore, it follows that if we invest $\$P$ at an interest rate r per year, compounded continuously, after t years the final amount will be given by

$$FV=P\cdot e^{rt}$$

Example \(\PageIndex{12\)

$3500 is invested at 9% compounded continuously. Find the future value in four years.

Solution

Using the formula for the continuous compounding, we get $FV=Pe^{rt}$.

$$\begin{aligned}FV&=\$3500e^{0.09\times 4}\\FV&=\$3500e^{0.36}\\FV&=\$5016.65\end{aligned}$$

Example 6.1.13

If an amount is invested at 7% compounded continuously, what is the effective interest rate?

Solution

If we deposit $1 in the bank at 7% compounded continuously for one year, and subtract that $1 from the final amount, we get the effective interest rate in decimals.

$$\begin{aligned}\mathrm{r_{EFF}}&=1\mathrm{e}^{0.07}-1\\\mathrm{r_{EFF}}&=1.0725-1\\\mathrm{r_{EFF}}&=.0725\text{ or }7.25\%\end{aligned}$$

Example 6.1.14

If an amount is invested at 7% compounded continuously, how long will it take to double?

We offer two solutions.

Solution 1 uses logarithms to calculate the exact answer, so it is preferred. We already used this method in Example 6.1.9 to solve for time needed for an investment to accumulate to a specified future value.

Solution 2 provides an estimated solution that is applicable only to doubling time, but not to other multiples. Students should find out from their instructor if there is a preference as to which solution method is to be used for doubling time problems.

Solution: Solution 1: **Calculating the answer exactly:** $Pe^{0.07t}=FV$.

We don’t know the initial value of the priniciple but we do know that the accumulated value is double (twice) the principle.

$$\mathrm{P}_{e}^{0.07t}=2\mathrm{P}$$

We divide both sides by P

$$e^{.07t}=2$$

Using natural logarithm:

$$\begin{aligned}.07\mathrm{t}&=\ln(2)\\\mathrm{t}&=\ln(2)/.07=9.9\text{ years}\end{aligned}$$

It takes 9.9 years for money to double if invested at 7% continuous interest.

Solution 2: **Estimating the answer using the Law of 72:**

The Law of 72 is a useful tool for estimating the time needed for an investment to double in value.

The Rule of 72

The **rule of 72** is a simple and often very useful mathematical shortcut that can help you estimate the impact **of** any interest or growth rate and can be used in situations ranging from financial calculations to projections **of** population growth. The formula for the **rule of 72** is expressed as the unknown (the required amount **of** time to double a value) calculated by taking the number **72** and dividing it by the known interest rate or growth rate. When using this formula, it is important to note that the rate should be expressed as a whole integer, not as a percentage. So, as a result, we have

Years for an Amount to Double=72/(Interest or Growth Rate)

This formula can be extremely practical when working with financial estimates or projections and for understanding how compound interest can have a dramatic effect on an original amount or monetary balance.

Following are just a few examples of how the rule **of 72** can help you solve problems very quickly and very easily, often enabling you to solve them "in your head," without the need for a calculator or spreadsheet.

Let's say you are interested in knowing how long it will take your savings account balance to double. If your account earns an interest rate of 9%, your money will take 72/9 or 8, years to double. However, if you are earning only 6% on this same investment, your money will take 72/6, or 12, years to double.

Now let's say you have a specific future purchasing need and you know that you will need to double your money in five years. In this case, you would be required to invest it at an interest rate of 72/5, or 14.4%. Through these sample examples, it is easy to see how relatively small changes in a growth or interest rate can have significant impact on the time required for a balance to double in size.

The number of years required to double money ≈ 72 ÷ interest rate(%) = years.

With technology available to do calculations using logarithms, we would use the Law of 72 only for quick estimates of doubling times. Using the Law of 72 as an estimate works only for doubling times, but not other multiples, so it's not a replacement for knowing how to find exact solutions.

However, the Law of 72 can be useful to help quickly estimate many "doubling time" problems mentally, which can be useful in compound interest applications as well as other applications involving exponential growth.

Example 6.1.15

a. At the peak growth rate in the 1960's the world's population had a doubling time of 35 years. At that time, approximately what was the growth rate?
b. As of 2015, the world population's annual growth rate was approximately 1.14%. Based on that rate, find the approximate doubling time.

Solution

a. According to the law of 72,

$$\text{doubling time} = 35 \approx 72 \div r$$

$$r \approx 2.057 \text{ expressed as a percent}$$

Therefore, the world population was growing at an approximate rate of 2.057% in the 1960's.

b.. According to the law of 72,

$$\text{doubling time } t \approx 72 \div r = 72 \div 1.14 \approx 63.157 \text{ years}$$

If the world population were to continue to grow at the annual growth rate of 1.14% , it would take approximately 64 years for the population to double.

COMPOUND INTEREST SUMMARY

Below is a summary of the formulas we developed for calculations involving compound interest:

COMPOUND INTEREST $\mathbf{n}$ times per year

1. If an amount P is invested for t years at an interest rate r per year, compounded m times a year, then the future value is given by

$$FV = P\left(1+\frac{r}{m}\right)^{mt}$$

$\mathbf{P}$ is called the principle and is also called the present value.

2. If a bank pays an interest rate r per year, compounded m times a year, then the effective interest rate is given by

$$\mathbf{r}_{\mathrm{EFF}} = \left(1+\frac{r}{m}\right)^{m} - 1$$

CONTINUOUSLY COMPOUNDED INTEREST

3. If an amount P is invested for t years at an interest rate r per year, compounded continuously, then the future value is given by

$$\mathrm{FV} = \mathrm{P}e^{rt}$$

4. If a bank pays an interest rate r per year, compounded n times a year, then the effective interest rate is given by

$$\mathrm{r}_{\mathrm{EFF}} = e^{\mathbf{r}} - 1$$

5. The Law of 72 states that

The number of years to double money is approximately 72 ÷ interest rate

6.1.1: Simple and Compound Interest (Exercises)

SECTION 6.1 PROBLEM SET: SIMPLE INTEREST AND DISCOUNT

Do the following simple interest problems.

1) If an amount of $2,000 is borrowed at a simple interest rate of 10% for 3 years, how much is the interest?	2) You borrow $4,500 for six months at a simple interest rate of 8%. How much is the interest?
3) John borrows $2400 for 3 years at 9% simple interest. How much will he owe at the end of 3 years?	4) Jessica takes a loan of $800 for 4 months at 12% simple interest. How much does she owe at the end of the 4-month period?
5) If an amount of $2,160, which includes a 10% simple interest for 2 years, is paid back, how much was borrowed 2 years earlier?	6) Jamie just paid off a loan of $2,544, the principal and simple interest. If he took out the loan six months ago at 12% simple interest, what was the amount borrowed?
7) Shanti charged $800 on her charge card and did not make a payment for six months. If there is a monthly charge of 1.5%, how much does she owe?	8) A credit card company charges 18% interest on the unpaid balance. If you owed $2000 three months ago and have been delinquent since, how much do you owe?

SECTION 6.1 PROBLEM SET: SIMPLE INTEREST AND DISCOUNT

Do the following simple interest problems.

9) An amount of $2000 is borrowed for 3 years. At the end of the three years, $2660 is paid back. What was the simple interest rate?	10) Nancy borrowed $1,800 and paid back $1,920, four months later. What was the simple interest rate?
11) Jose agrees to pay $2,000 in one year at an interest rate of 12%. The bank subtracts the discount of 12% of $2,000, and gives the rest to Jose. Find the amount of the discount and the proceeds to Jose.	12) Tasha signs a note for a discounted loan agreeing to pay $1200 in 8 months at an 18% discount rate. Determine the amount of the discount and the proceeds to her.
13) An amount of $8,000 is borrowed at a discount rate of 12%, find the proceeds if the length of the loan is 7 months.	14) An amount of $4,000 is borrowed at a discount rate of 10%, find the proceeds if the length of the loan is 180 days.
15) Derek needs $2400 new equipment for his shop. He can borrow this money at a discount rate of 14% for a year. Find the amount of the loan he should ask for so that his proceeds are $2400.	16) Mary owes Jim $750, and wants to repay him. Mary decides to borrow the amount from her bank at a discount rate of 16%. If she borrows the money for 10 months, find the amount of the loan she should ask for so that her proceeds are $750?

SECTION 6.1 PROBLEM SET: COMPOUND INTEREST

Do the following compound interest problems involving a lump-sum amount.

17) What will the final amount be in 4 years if $8,000 is invested at 9.2% compounded monthly.?	18) How much should be invested at 10.3% for it to amount to $10,000 in 6 years?
19) Lydia's aunt Rose left her $5,000. Lydia spent $1,000 on her wardrobe and deposited the rest in an account that pays 6.9% compounded daily. How much money will she have in 5 years?	20) Thuy needs $1,850 in eight months for her college tuition. How much money should she deposit lump sum in an account paying 8.2% compounded monthly to achieve that goal?
21) Bank A pays 5% compounded daily, while Bank B pays 5.12% compounded monthly. Which bank pays more? Explain.	22) EZ Photo Company needs five copying machines in 2 1/2 years for a total cost of $15,000. How much money should be deposited now to pay for these machines, if the interest rate is 8% compounded semiannually?

23) Jon's grandfather planned to give him \$12,000 in 10 years. Jon has convinced his grandfather to pay him \$6,000 now, instead. If Jon invests this \$6,000 at 7.5% compounded continuously, how much money will he have in 10 years?	24) What will be the price of a \$20,000 car in 5 years if the inflation rate is 6%?

SECTION 6.1 PROBLEM SET: COMPOUND INTEREST

Do the following compound interest problems.

25) At an interest rate of 8% compounded continuously, how many years will it take to double your money?	26) If an investment earns 10% compounded continuously, in how many years will it triple?
27) The City Library ordered a new computer system costing \$158,000; it will be delivered in 6 months, and the full amount will be due 30 days after delivery. How much must be deposited today into an account paying 7.5% compounded monthly to have \$158,000 in 7 months?	28) Mr. and Mrs. Tran is expecting a baby girl in a few days. They want to put away money for her college education now. How much money should they deposit in an account paying 10.2% so they will have \$100,000 in 18 years to pay for their daughter's educational expenses?
29) Find the effective interest rate for an account paying 7.2% compounded quarterly.	30) If a bank pays 5.75% compounded monthly, what is the effective interest rate?
31) The population of the African nation of Cameroon was 12 million people in the year 2015; it has been growing at a rate of 2.5% per year. If the population continues to grows at a rate, what will the population be in 2030? (http://databank.worldbank.org/data on 4/26/2016)	32) According to the Law of 72, if an amount grows at an annual rate of 1%, then it doubles every seventy-two years. Suppose a bank pays 5% interest, how long will it take for you to double your money? How about at 15%?

6.2: Annuities and Sinking Funds

Learning Objectives

In this section, you will learn to:

1. Find the future value of an annuity.
2. Find the amount of payments to a sinking fund.

Ordinary Annuity

In the first two sections of this chapter, we examined problems where money was deposited in a lump sum in an account and left there for the entire period. Now, we will solve problems where timely payments are made in an account. When a sequence of payments of some fixed amount is made in an account at equal intervals, we call that an **annuity**. And this is the subject of this section.

To develop a formula to find the value of an annuity, we will need to recall the formula for the sum of a geometric series. A geometric series is of the form:

$$a+ax+ax^2+ax^3+\ldots+ax^n. \tag{6.2.1}$$

In a geometric series, each subsequent term is obtained by multiplying the preceding term by a number, called the **common ratio**. A geometric series is completely determined by knowing its first term, the common ratio, and the number of terms.

The first term of the series in Equation 6.2.1 is a, the common ratio is x, and the number of terms is $n+1$. The following are some examples of geometric series.

$$3+6+12+24+48$$

This above series has first term $a=3$ and common ratio $x=2$

$$2+6+18+54+162$$

This above series has first term $a=2$ and common ratio $x=3$

$$37+3.7+0.37+0.037+0.0037$$

This above series has first term $a=35$ and common ratio $x=0.1$

In your algebra class, you developed a formula for finding the sum of a geometric series. You probably used r as the symbol for the ratio, but we are using x because r is the symbol we have been using for the interest rate. The formula for the sum of the terms of a geometric series with first term a, common ratio x and last term ax^n is:

$$\frac{a\left(x^{n+1}-1\right)}{x-1}$$

We will use this formula to find the value of an annuity. Consider the following example.

Example 6.2.1

If at the end of each month a deposit of $500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

Solution

There are 60 deposits made in this account. The first payment stays in the account for 59 months, the second payment for 58 months, the third for 57 months, and so on.

- The first payment of $500 will accumulate to an amount of $500(1 + 0.08/12)^{59}.
- The second payment of $500 will accumulate to an amount of $500(1 + 0.08/12)^{58}.
- The third payment will accumulate to $500(1 + 0.08/12)^{57}.
- The fourth payment will accumulate to $500(1 + 0.08/12)^{56}.

And so on . . .

Finally the next to last (59th) payment will accumulate to $\$500(1+0.08/12)^1$.

The last payment is taken out at the same time it is made and will not earn any interest.

To find the total amount in five years, we need to add the accumulated value of these sixty payments.

In other words, we must find the sum of the following series.

$$\$500(1+0.08/12)^{59}+\$500(1+0.08/12)^{58}+\$500(1+0.08/12)^{57}+\ldots+\$500$$

Written backward, we have

$$\$500+\$500(1+0.08/12)+\$500(1+0.08/12)^2+\ldots+\$500(1+0.08/12)^{59}$$

This is a geometric series with $a=\$500$, $r=(1+0.08/12$, and $n=59$. The sum is

$$\begin{aligned}\text{sum} &= \frac{\$500\left[(1+0.08/12)^{60}-1\right]}{0.08/12}\\ &= \$500(73.47686)\\ &= \$36,738.43\end{aligned}$$

When the payments are made at the end of each period rather than the beginning, we call it an **ordinary annuity.**

Future Value of an Ordinary Annuity

If a payment of PMT dollars is made in account m times a year at an interest r, then the final amount FV after t years is

$$\mathbf{FV}=\frac{\mathbf{PMT}\left[(\mathbf{1}+\mathbf{r}/\mathbf{m})^{\mathbf{mt}}-\mathbf{1}\right]}{\mathbf{r}/\mathbf{m}}$$

The future value is also called the accumulated value

Example 6.2.2

Tanya deposits \$300 at the end of each quarter in her savings account. If the account earns 5.75% compounded quarterly, how much money will she have in 4 years?

Solution

The future value of this annuity can be found using the above formula.

$$\begin{aligned}FV &= \frac{\$300\left[(1+.0575/4)^{16}-1\right]}{0.0575/4}\\ &= \$300(17.8463)\\ &= \$5353.89\end{aligned}$$

If Tanya deposits \$300 into a savings account earning 5.75% compounded quarterly for four years, then at the end of 4 years, she will have \$5,353.89

Example 6.2.3

Robert needs \$5,000 in three years. How much should he deposit monthly in an account that pays 8% compounded monthly to achieve his goal?

Solution

If Robert saves PMT dollars per month, after three years, he will have

$$\frac{PMT\left[(1+.08/12)^{36}-1\right]}{0.08/12}$$

But we'd like this amount to be $5,000. Therefore,

$$\begin{aligned}\frac{\mathrm{PMT}\left[(1+.08/12)^{36}-1\right]}{.08/12} &= \$5000\\ \mathrm{PMT}(40.5356) &= \$5000\\ \mathrm{PMT} &= \frac{5000}{40.5356}\\ &= \$123.35\end{aligned}$$

Robert needs to deposit $123.35 at the end of each month for three years into an account paying 8% compounded monthly to have $5,000 at the end of 5 years.

Sinking Fund

When a business deposits money at regular intervals into an account to save for a future purchase of equipment, the savings fund is referred to as a "**sinking fund**." Calculating the sinking fund deposit uses the same method as the previous problem.

Example 6.2.4

A business needs $450,000 in five years. How much should be deposited each quarter in a sinking fund that earns 9% compounded quarterly to have this amount in five years?

Solution

Again, suppose that PMT dollars are deposited each quarter in the sinking fund. After five years, the future value of the fund should be $450,000. This suggests the following relationship:

$$\begin{aligned}\frac{\mathrm{PMT}\left[(1+0.09/4)^{20}-1\right]}{0.09/4} &= \$450,000\\ \mathrm{PMT}(24.9115) &= 450,000\\ \mathrm{PMT} &= \frac{450000}{24.9115}\\ &= \$18,063.93\end{aligned}$$

The business needs to deposit $18,063.93 at the end of each quarter for five years into a sinking fund earning interest of 9% compounded quarterly to have $450,000 at the end of 5 years.

Annuity Due

If the payment is made at the beginning of each period rather than at the end, we call it an **annuity due**. The formula for the annuity due can be derived similarly. Reconsider Example 1, with the change that the deposits are made at the beginning of each month.

Example 6.2.5

If at the beginning of each month, a deposit of $500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

Solution

There are 60 deposits made in this account. The first payment stays in the account for 60 months, the second payment for 59 months, the third for 58 months, and so on.

- The first payment of $500 will accumulate to an amount of $\$500(1+0.08/12)^{60}$.
- The second payment of $500 will accumulate to an amount of $\$500(1+.08/12)^{59}$.

- The third payment will accumulate to $\$500(1+0.08/12)^{58}$.

And so on . . .

The last payment is in the account for a month and accumulates to $\$500(1+0.08/12)$

To find the total amount in five years, we need to find the sum of the series:

$$\$500(1+0.08/12)^{60}+\$500(1+0.08/12)^{59}+\$500(1+0.08/12)^{58}+\ldots+\$500(1+0.08/12)$$

Written backward, we have

$$\$500(1+0.08/12)+\$500(1+0.08/12)^{2}+\ldots+\$500(1+0.08/12)^{60}$$

If we add $500 to this series and later subtract that $500, the value will not change. We get

$$\mathbf{\$500}+\$500(1+0.08/12)+\$500(1+0.08/12)^{2}+\ldots+\$500(1+0.08/12)^{60}-\mathbf{\$500}$$

Except for the last term, we have a geometric series with a = $500, r = (1 + .08/12), and n = 60. Therefore, the sum is

$$\begin{aligned} \mathrm{A} &= \frac{\$500\left[(1+0.08/12)^{61}-1\right]}{0.08/12}-\$500 \\ &=\$500(74.9667)-\$500 \\ &=\$37483.35-\$500 \\ &=\$36983.35 \end{aligned}$$

So, in the case of an annuity due, to find the future value, we increase the number of periods n by one and subtract one payment.

Future Value of an "Annuity Due"

$$\mathrm{FV}=\frac{\mathrm{PMT}\left[(1+\mathrm{r}/\mathrm{m})^{\mathrm{mt}+1}-1\right]}{\mathrm{r}/\mathrm{m}}-\mathrm{PMT}$$

Summary

Finally, the author wishes that the students learn the concepts so they will not have to memorize every formula. It is for this reason formulas are kept at a minimum. But before we conclude this section, we will once again mention one single equation that will help us find the future value and the sinking fund payment.

If a payment of PMT dollars is made in an account m times a year at an interest r, then the future value FV after t years is

$$\mathbf{FV}=\frac{\mathbf{PMT}\left[(\mathbf{1}+\mathbf{r}/\mathbf{m})^{\mathbf{mt}}-\mathbf{1}\right]}{\mathbf{r}/\mathbf{m}}$$

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

6.2.1: Annuities and Sinking Funds (Exercises)

SECTION 6.2 PROBLEM SET: ANNUITIES AND SINKING FUNDS

Each of the following problems involves an annuity - a sequence of payments.

1) Find the future value of an annuity of $200 per month for 5 years at 6% compounded monthly.	2) How much money should be deposited at the end of each month in an account paying 7.5% for it to amount to $10,000 in 5 years?
3) At the end of each month Rita deposits $300 in an account that pays 5%. What will the final amount be in 4 years?	4) Mr. Chang wants to retire in 10 years and can save $650 every three months. If the interest rate is 7.8%, how much will he have (a) at the end of 5 years? (b) at the end of 10 years?
5) A firm needs to replace most of its machinery in five years at a cost of $500,000. The company wishes to create a sinking fund to have this money available in five years. How much should the quarterly deposits be if the fund earns 8%?	6) Mrs. Brown needs $5,000 in three years. If the interest rate is 9%, how much should she save at the end of each month to have that amount in three years?
7) A company has a $120,000 note due in 4 years. How much should be deposited at the end of each quarter in a sinking fund to pay off the note in four years if the interest rate is 8%?	8) You are now 20 years of age and decide to save $100 at the end of each month until you are 65. If the interest rate is 9.2%, how much money will you have when you are 65?
9) Is it better to receive $400 at the beginning of each month for six years, or a lump sum of $25,000 today if the interest rate is 7%? Explain.	10) To save money for a vacation, Jill decided to save $125 at the beginning of each month for the next 8 months. If the interest rate is 7%, how much money will she have at the end of 8 months?
11) Mrs. Gill puts $2200 at the end of each year in her IRA account which earns 9% per year. How much total money will she have in this account after 20 years?	12) If the inflation rate stays at 6% per year for the next five years, how much will the price be of a $15,000 car in five years? How much must you save at the end of each month at an interest rate of 7.3% to buy that car in 5 years?

6.3: Present Value of an Annuity and Installment Payment

Learning Objectives

In this section, you will learn to:

1. Find the present value of an annuity.
2. Find the amount of installment payment on a loan.

PRESENT VALUE OF AN ANNUITY

In Section 6.2, we learned to find the future value of a series of payments. We will now learn to amortize a loan (determine the periodic payment in exchange for a loan taken), and to find the present value of an annuity.

The **present value** of an annuity is the amount of money we would need now in order to be able to make the payments of a specific amount in the future. In other words, the present value is the value needed to fund a future stream of payments, such as a monthly retirement payment.

We break this down step by step to understand the concept of the present value of an annuity. After that, the examples provide a more efficient way to do the calculations by working with concepts and calculations we have already explored in Sections 6.1 and 6.2.

Suppose Carlos owns a small business and employs an assistant manager to help him run the business. Assume it is January 1 now. Carlos plans to pay his assistant manager a \$1000 bonus at the end of this year and another \$1000 bonus at the end of the following year. Carlos' business had good profits this year so he wants to put the money for his assistant's future bonuses into a savings account now. The money he puts in now will earn interest at the rate of 4% per year compounded annually while in the savings account.

How much money should Carlos put into the savings account now to withdraw \$1000 one year from now and another \$1000 two years from now?

At first, this sounds like a sinking fund. But it is different. In a sinking fund, we put money into the fund with periodic payments to save to accumulate to a specified lump sum that is the future value at the end of a specified time period.

In this case, we want to put a lump sum into the savings account now, so that lump sum is our principle or present value, PV. Then, we want to withdraw that amount as a series of period payments; in this case, the withdrawals are an annuity with \$1000 payments at the end of years 1 and 2.

We need to determine the amount we need in the account now, the present value, to be able to withdraw the periodic payments later.

We use the compound interest formula from Section 6.1 with $r = 0.04$ and $m = 1$ for annual compounding to determine the present value of each payment of \$1000.

Consider the first payment of \$1000 at the end of year 1. Let P_1 be its present value for the first payment.

$$\$1000 = P_1(1.04)^1 \text{ so } P_1 = \$\,961.54$$

Now consider the second payment of \$1000 at the end of year 2. Let P_2 is the present value for the second payment.

$$\$1000 = P_2(1.04)^2 \text{ so } P_2 = \$924.56$$

To make the \$1000 payments at the specified times in the future, the amount that Carlos needs to deposit now is the present value $PV = P_1 + P_2 = \$961.54 + \$924.56 = \$1886.10$

The calculation above was useful to illustrate the meaning of the present value of an annuity.

But, it is not an efficient way to calculate the present value. If we were to have a large number of annuity payments, the step-by-step calculation would be long and tedious.

Example $6.3.1$ investigates and develops an efficient way to calculate the present value of an annuity, by relating the future (accumulated) value of an annuity and its present value.

Example $6.3.1$

Suppose you have won a lottery that pays $1,000 per month for the next 20 years. But, you prefer to have the entire amount now. If the interest rate is 8%, how much will you accept?

Solution

This classic present value problem needs our complete attention because the rationalization we use to solve this problem will be used again in the following problems.

Consider, for argument purposes, that two people Mr. Cash, and Mr. Credit have won the same lottery of $1,000 per month for the next 20 years. Mr. Credit is happy with his $1,000 monthly payment, but Mr. Cash wants to have the entire amount now.

Our job is to determine how much Mr. Cash should get. We reason as follows:

If Mr. Cash accepts P dollars, then the PV dollars deposited at 8% for 20 years should yield the same amount as the $1,000 monthly payments for 20 years. In other words, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like the future values to be equal.

Since Mr. Cash is receiving a lump sum of x dollars, its future value is given by the lump sum formula we studied in Section 6.1, and it is

$$\mathrm{PV}=\mathrm{P}(1+.08/12)^{240}$$

Since Mr. Credit is receiving a sequence of payments, or an annuity, of $1,000 per month, its future value is given by the annuity formula we learned in Section 6.1. This value is

$$\mathrm{FV}=\frac{\$1000\left[(1+.08/12)^{240}-1\right]}{.08/12}$$

The only way Mr. Cash will agree to the amount he receives is if these two future values are equal. So, we set them equal and solve for the unknown.

$$\begin{aligned}
&\mathrm{PV}(1+.08/12)^{240}=\frac{\$1000\left[(1+.08/12)^{240}-1\right]}{.08/12}\\
&\mathrm{PV}(4.9268)=\$1000(589.02041)\\
&\mathrm{PV}(4.9268)=\$589020.41\\
&\mathrm{PV}=\$119,554.36
\end{aligned}$$

The present value of an ordinary annuity of $1,000 each month for 20 years at 8% is $119,554.36

The reader should also note that if Mr. Cash takes his lump sum of PV = $119,554.36 and invests it at 8% compounded monthly, he will have an accumulated value of FV=$589,020.41 in 20 years.

INSTALLMENT PAYMENT ON A LOAN

If a person or business needs to buy or pay for something now (a car, a home, college tuition, equipment for a business) but does not have the money, they can borrow the money as a loan.

They receive the loan amount called the principle (or present value) now and are obligated to pay back the principle in the future over a stated amount of time (term of the loan), as regular periodic payments with interest.

Example $6.3.2$ examines how to calculate the loan payment, using reasoning similar to Example $6.3.1$.

Example 6.3.2

Find the monthly payment for a car costing $15,000 if the loan is amortized over five years at an interest rate of 9%.

Solution

Again, consider the following scenario:

Two people, Mr. Cash and Mr. Credit, go to buy the same car that costs $15,000. Mr. Cash pays cash and drives away, but Mr. Credit wants to make monthly payments for five years.

Our job is to determine the amount of the monthly payment. We reason as follows:

If Mr. Credit pays PMT dollars per month, then the PMT dollar payment deposited each month at 9% for 5 years should yield the same amount as the $15,000 lump sum deposited for 5 years.

Again, we are comparing the future values for both Mr. Cash and Mr. Credit, and we would like them to be the same.

Since Mr. Cash is paying a lump sum of $15,000, its future value is given by the lump sum formula, and it is

$$\$15,000(1+.09/12)^{60}$$

Mr. Credit wishes to make a sequence of payments, or an annuity, of PMT dollars per month, and its future value is given by the annuity formula, and this value is

$$\frac{\text{PMT}\left[(1+.09/12)^{60}-1\right]}{.09/12}$$

We set the two future amounts equal and solve for the unknown.

$$\begin{aligned}
\$15,000(1+.09/12)^{60} &= \frac{PMT\left[(1+.09/12)^{60}-1\right]}{.09/12} \\
\$15,000(1.5657) &= PMT(75.4241) \\
\$311.38 &= PMT
\end{aligned}$$

Therefore, the monthly payment needed to repay the loan is $311.38 for five years.

SECTION 6.3 SUMMARY

We summarize the method used in examples 6.3.1 and 6.3.2 below.

The Equation to Find the Present Value of an Annuity,

Or the Installment Payment for a Loan

If a payment of PMT dollars is made in an account m times a year at an interest r, then the present value PV of the annuity after t years is

$$\mathbf{PV(1+r/m)^{mt}} = \frac{\mathbf{PMT\left[(1+r/m)^{mt}-1\right]}}{\mathbf{r/m}}$$

When used for a loan, the amount PV is the loan amount, and PMT is the periodic payment needed to repay the loan over a term of t years with m payments per year.

If the present value or loan amount is needed, solve for PV

If the periodic payment is needed, solve for PMT.

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.

Finally, we note that many finite mathematics and finance books develop the formula for the present value of an annuity differently.

Instead of using the formula:

$$\mathrm{PV}(1+\mathrm{r}/\mathrm{m})^{\mathrm{mt}}=\frac{\mathrm{PMT}\left[(1+\mathrm{r}/\mathrm{m})^{\mathrm{mt}}-1\right]}{\mathrm{r}/\mathrm{m}} \tag{6.3.1}$$

and solving for the present value PV after substituting the numerical values for the other items in the formula, many textbooks first solve the formula for PV in order to develop a new formula for the present value. Then the numerical information can be substituted into the present value formula and evaluated, without needing to solve algebraically for PV.

Alternate Method to find Present Value of an Annuity

Starting with formula 6.3.1: $\mathrm{PV}(1+\mathrm{r}/\mathrm{m})^{\mathrm{mt}}=\frac{\mathrm{PMT}\left[(1+\mathrm{r}/\mathrm{m})^{\mathrm{mt}}-1\right]}{\mathrm{r}/\mathrm{m}}$

Divide both sides by $(1+r/m)^{mt}$ to isolate PV, and simplify

$$PV=\frac{PMT\left[(1+r/m)^{mt}-1\right]}{r/m}\cdot\frac{1}{(1+r/m)^{mt}}$$

$$PV=\frac{PMT\left[1-(1+r/m)^{-mt}\right]}{r/m} \tag{6.3.2}$$

The authors of this book believe that it is easier to use formula 6.3.1 at the top of this page and solve for PV or PMT as needed. In this approach, there are fewer formulas to understand, and many students find it easier to learn.

However, some people prefer formula 6.3.2, and it is mathematically correct to use that method. Note that if you choose to use formula 6.3.2, you need to be careful with the negative exponents in the formula. And if you needed to find the periodic payment, you would still need to do the algebra to solve for the value of m.

It would be a good idea to check with your instructor to see if he or she has a preference. In fact, you can usually tell your instructor's preference by noting how he or she explains and demonstrates these types of problems in class.

6.3.1: Present Value of an Annuity and Installment Payment (Exercises)

SECTION 6.3 PROBLEM SET: PRESENT VALUE OF AN ANNUITY AND INSTALLMENT PAYMENT

For the following problems, show all work.

1) Shawn has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum today. If the current interest rate is 8.2%, how much money can he hope to get?	2) Sonya bought a car for $15,000. Find the monthly payment if the loan is to be amortized over 5 years at a rate of 10.1%.
3) You determine that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for a car if the interest rate is 9% and you want to repay the loan in 5 years?	4) Compute the monthly payment for a house loan of $200,000 to be financed over 30 years at an interest rate of 10%.
5) If the $200,000 loan in the previous problem is financed over 15 years rather than 30 years at 10%, what will the monthly payment be?	6) Friendly Auto offers Jennifer a car for $2000 down and $300 per month for 5 years. Jason wants to buy the same car but wants to pay cash. How much must Jason pay if the interest rate is 9.4%?
7) The Gomez family bought a house for $450,000. They paid 20% down and amortized the rest at 5.2% over a 30-year period. Find their monthly payment.	8) Mr. and Mrs. Wong purchased their new house for $350,000. They made a down payment of 15%, and amortized the rest over 30 years. If the interest rate is 5.8%, find their monthly payment.
9) A firm needs a piece of machinery that has a useful life of 5 years. It has an option of leasing it for $10,000 a year, or buying it for $40,000 cash. If the interest rate is 10%, which choice is better?	10) Jackie wants to buy a $19,000 car, but she can afford to pay only $300 per month for 5 years. If the interest rate is 6%, how much does she need to put down?
11) Vijay's tuition at college for the next year is $32,000. His parents have decided to pay the tuition by making nine monthly payments. If the interest rate is 6%, what is the monthly payment?	12) Glen borrowed $10,000 for his college education at 8% compounded quarterly. Three years later, after graduating and finding a job, he decided to start paying off his loan. If the loan is amortized over five years at 9%, find his monthly payment for the next five years.

6.4: Miscellaneous Application Problems

Learning Objectives

In this section, you will learn to apply to concepts for compound interest for savings and annuities to:

1. Find the outstanding balance, partway through the term of a loan, of the future payments still remaining on the loan.
2. Perform financial calculations in situations involving several stages of savings and/or annuities.
3. Find the fair market value of a bond.
4. Construct an amortization schedule for a loan.

We have already developed the tools to solve most finance problems. Now we use these tools to solve some application problems.

OUTSTANDING BALANCE ON A LOAN

One of the most common problems deals with finding the balance owed at a given time during the life of a loan. Suppose a person buys a house and amortizes the loan over 30 years, but decides to sell the house a few years later. At the time of the sale, he is obligated to pay off his lender, therefore, he needs to know the balance he owes. Since most long term loans are paid off prematurely, we are often confronted with this problem.

To find the outstanding balance of a loan at a specified time, we need to find the present value P of all future payments that have not yet been paid. In this case t does not represent the entire term of the loan. Instead:

- t represents the time that still remains on the loan
- nt represents the total number of future payments.

Example $6.4.1$

Mr. Jackson bought his house in 1995, and financed the loan for 30 years at an interest rate of 7.8%. His monthly payment was $1260. In 2015, Mr. Jackson decides to pay off the loan. Find the balance of the loan he still owes.

Solution

The reader should note that the original amount of the loan is not mentioned in the problem. That is because we don't need to know that to find the balance.

The original loan was for 30 years. 20 years have past so there are years still remaining. 12(10) = 120 payments still remain to be paid on this loan.

As for the bank or lender is concerned, Mr. Jackson is obligated to pay $1260 each month for 10 more years; he still owes a total of 120 payments. But since Mr. Jackson wants to pay it all off now, we need to find the present value P at the time of repayment of the remaining 10 years of payments of $1260 each month. Using the formula we get for the present value of an annuity, we get

$$\begin{aligned}
\mathrm{P}(1+.078/12)^{120} &= \frac{\$1260\left[(1+.078/12)^{120}-1)\right]}{(.078/12)} \\
\mathrm{P}(2.17597) &= \$227957.85 \\
\mathrm{P} &= \$104761.48
\end{aligned}$$

to find the outstanding balance of a loan

If a loan has a payment of m dollars made n times a year at an interest r, then the outstanding value of the loan when there are t years still remaining on the loan is given by P:

$$\mathbf{P(1+r/n)^{nt}} = \frac{\mathbf{m\left[(1+r/n)^{nt}-1\right|}}{\mathbf{r/n}}$$

IMPORTANT: Note that t is not the original term of the loan but instead t is the amount of time still remaining in the future nt is the number of payments still remaining in the future

> If the problem does not directly state the amount of time still remaining in the term of the loan, then it must be calculated BEFORE using the above formula as t = original term of loan - time already passed since the start date of the loan.

Note that there are other methods to find the outstanding balance on a loan, but the method illustrated above is the easiest.

One alternate method would be to use an amortization schedule, as illustrated toward the end of this section. An amortization schedule shows the payments, interest, and outstanding balance step by step after each loan payment. An amortization schedule is tedious to calculate by hand but can be easily constructed using spreadsheet software.

Another way to find the outstanding balance, that we will not illustrate here, is to find the difference A - B, where

A = the original loan amount (principal) accumulated to the date on which we want to find the outstanding balance (using compound interest formula)

B = the accumulated value of all payments that have been made as of the date on which we want to find the outstanding balance (using formula for accumulated value of an annuity)

In this case we would need do a compound interest calculation and an annuity calculation; we then need to find the difference between them. Three calculations are needed instead of one.

It is a mathematically acceptable way to calculate the outstanding balance. However, **it is very strongly recommended that students use the method explained in box above** and illustrated in Example $6.4.1$, as it is much simpler.

PROBLEMS INVOLVING MULTIPLE STAGES OF SAVINGS AND/OR ANNUITIES

Consider the following situations:

a. Suppose a baby, Aisha, is born and her grandparents invest \$5000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to need to finish college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?
b. Aisha graduates college and starts a job. She saves \$1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

These problems appear complicated. But each can be broken down into two smaller problems involving compound interest on savings or involving annuities. Often the problem involves a savings period followed by an annuity period. ; the accumulated value from first part of the problem may become a present value in the second part. Read each problem carefully to determine what is needed.

✓ Example $6.4.2$

Suppose a baby, Aisha, is born and her grandparents invest \$8000 in a college fund. The money remains invested for 18 years until Aisha enters college, and then is withdrawn in equal semiannual payments over the 4 years that Aisha expects to attend college. The college investment fund earns 5% interest compounded semiannually. How much money can Aisha withdraw from the account every six months while she is in college?

Solution

Part 1: Accumulation of College Savings: Find the accumulated value at the end of 18 years of a sum of \$8000 invested at 5% compounded semiannually.

$$A = \$8000(1+.05/2)^{(2\times 18)} = \$8000(1.025)^{36} = \$8000(2.432535)$$
$$A = \$19460.28$$

Part 2: Seminannual annuity payout from savings to put toward college expenses. Find the amount of the semiannual payout for four years using the accumulated savings from part 1 of the problem with an interest rate of 5% compounded semiannually.

A= \$19460.28 in Part 1 is the accumulated value at the end of the savings period. This becomes the present value P=\$19460.28 when calculating the semiannual payments in Part 2.

$$\begin{gathered}\$19460.28\left(1+\frac{.05}{2}\right)^{2\times4}=\frac{m\left[\left(1+\frac{.05}{2}\right)^{2\times4}-1\right]}{(.05/2)}\\ \$23710.46=m(8.73612)\\ m=\$2714.07\end{gathered}$$

Aisha will be able to withdraw \$2714.07 semiannually for her college expenses.

Example 6.4.3

Aisha graduates college and starts a job. She saves \$1000 each quarter, depositing it into a retirement savings account. Suppose that Aisha saves for 30 years and then retires. At retirement she wants to withdraw money as an annuity that pays a constant amount every month for 25 years. During the savings phase, the retirement account earns 6% interest compounded quarterly. During the annuity payout phase, the retirement account earns 4.8% interest compounded monthly. Calculate Aisha's monthly retirement annuity payout.

Solution

Part 1: Accumulation of Retirement Savings: Find the accumulated value at the end of 30 years of \$1000 deposited at the end of each quarter into a retirement savings account earning 6% interest compounded quarterly.

$$\begin{aligned}&A=\frac{\$1000\left[(1+.06/4)^{4\times30}-1\right]}{(.06/4)}\\ &A=\$331288.19\end{aligned}$$

Part 2: Monthly retirement annuity payout: Find the amount of the monthly annuity payments for 25 years using the accumulated savings from part 1 of the problem with an interest rate of 4.8% compounded monthly.

A= \$331288.19 in Part 1 is the accumulated value at the end of the savings period. This amount will become the present value P =\$331288.19 when calculating the monthly retirement annuity payments in Part 2.

$$\begin{aligned}&\$331288.19(1+.048/12)^{12\times25}=\frac{m\left[(1+.048/12)^{12\times25}-1\right]}{(.048/12)}\\ &\$1097285.90=m(578.04483)\\ &m=\$1898.27\end{aligned}$$

Aisha will have a monthly retirement annuity income of \$1898.27 when she retires.

FAIR MARKET VALUE OF A BOND

Whenever a business, and for that matter the U. S. government, needs to raise money it does it by selling bonds. A **bond** is a certificate of promise that states the terms of the agreement. Usually the business sells bonds for the **face amount** of \$1,000 each for a stated **term**, a period of time ending at a specified **maturity** date.

The person who buys the bond, the **bondholder**, pays \$1,000 to buy the bond.

The bondholder is promised two things: First that he will get his \$1,000 back at the maturity date, and second that he will receive a fixed amount of interest every six months.

As the market interest rates change, the price of the bond starts to fluctuate. The bonds are bought and sold in the market at their **fair market value**.

The interest rate a bond pays is fixed, but if the market interest rate goes up, the value of the bond drops since the money invested in the bond could earn more if invested elsewhere. When the value of the bond drops, we say it is trading at a **discount**.

On the other hand, if the market interest rate drops, the value of the bond goes up since the bond now yields a higher return than the market interest rate, and we say it is trading at a **premium**.

Example 6.4.4

The Orange Computer Company needs to raise money to expand. It issues a 10-year \$1,000 bond that pays \$30 every six months. If the current market interest rate is 7%, what is the fair market value of the bond?

Solution

The bond certificate promises us two things - An amount of \$1,000 to be paid in 10 years, and a semi-annual payment of \$30 for ten years. Therefore, to find the fair market value of the bond, we need to find the present value of the lump sum of \$1,000 we are to receive in 10 years, as well as, the present value of the \$30 semi-annual payments for the 10 years.

We will let P_1 = the present value of the face amount of \$1,000

$$P_1(1+.07/2)^{20} = \$1,000$$

Since the interest is paid twice a year, the interest is compounded twice a year and nt = 2(10)=20

$$\begin{aligned} P_1(1.9898) &= \$1,000 \\ P_1 &= \$502.56 \end{aligned}$$

We will let P_2 = the present value of the \$30 semi-annual payments is

$$\begin{aligned} \mathrm{P}_2(1+.07/2)^{20} &= \frac{\$30\left[(1+.07/2)^{20}-1\right]}{(.07/2)} \\ \mathrm{P}_2(1.9898) &= 848.39 \\ \mathrm{P}_2 &= \$426.37 \end{aligned}$$

The present value of the lump-sum \$1,000 = \$502.56

The present value of the \$30 semi-annual payments = \$426.37

The fair market value of the bond is P = P_{1+} P_2 = \$502.56 + \$426.37 = \$928.93

Note that because the market interest rate of 7% is higher than the bond's implied interest rate of 6% implied by the semiannual payments, the bond is selling at a discount; its fair market value of \$928.93 is less than its face value of \$1000.

Example 6.4.5

A state issues a 15 year \$1000 bond that pays \$25 every six months. If the current market interest rate is 4%, what is the fair market value of the bond?

Solution

The bond certificate promises two things - an amount of \$1,000 to be paid in 15 years, and semi-annual payments of \$25 for 15 years. To find the fair market value of the bond, we find the present value of the \$1,000 face value we are to receive in 15 years and add it to the present value of the \$25 semi-annual payments for the 15 years. In this example, $nt = 2(15) = 30$.

We will let P_1 = the present value of the lump-sum \$1,000

$$\begin{aligned} \mathrm{P}_1(1+.04/2)^{30} &= \$1,000 \\ \mathrm{P}_1 &= \$552.07 \end{aligned}$$

We will let P_2 = the present value of the \$25 semi-annual payments is

$$\begin{aligned} \mathrm{P}_2(1+.04/2)^{30} &= \frac{\$25\left[(1+.04/2)^{30}-1\right]}{(.04/2)} \\ \mathrm{P}_2(1.18114) &= \$1014.20 \\ \mathrm{P}_2 &= \$559.90 \end{aligned}$$

The present value of the lump-sum \$1,000 = \$552.07

The present value of the \$30 semi-annual payments = \$559.90

Therefore, the fair market value of the bond is

$$P = P_1 + P_2 = \$552.07 + \$559.90 = \$1111.97$$

Because the market interest rate of 4% is lower than the interest rate of 5% implied by the semiannual payments, the bond is selling at a premium: the fair market value of $1,111.97 is more than the face value of $1,000.

To summarize:

to find the Fair Market Value of a Bond

Find the present value of the face amount $\mathbf{A}$ that is payable at the maturity date:

$$\mathbf{A} = \mathbf{P}_1(\mathbf{1} + \mathbf{r}/\mathbf{n})^{\mathbf{nt}}; \text{ solve to find } \mathrm{P}_1$$

Find the present value of the semiannually payments of $\$m$ over the term of the bond:

$$\mathbf{P}_2(\mathbf{1} + \mathbf{r}/\mathbf{n})^{\mathbf{nt}} = \frac{\mathbf{m}\left[(\mathbf{1} + \mathbf{r}/\mathbf{n})^{\mathbf{nt}} - \mathbf{1}\right]}{\mathbf{r}/\mathbf{n}} \quad ; \text{ solve to find } \mathbf{P}_2$$

The fair market value (or present value or price or current value) of the bond is the sum of the present values calculated above:

$$\mathrm{P} = \mathbf{P}_1 + \mathbf{P}_2$$

AMORTIZATION SCHEDULE FOR A LOAN

An amortization schedule is a table that lists all payments on a loan, splits them into the portion devoted to interest and the portion that is applied to repay principal, and calculates the outstanding balance on the loan after each payment is made.

Example 6.4.6

An amount of $500 is borrowed for 6 months at a rate of 12%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment contributing toward reducing the debt, and the outstanding balance.

Solution

The reader can verify that the monthly payment is $86.27.

The first month, the outstanding balance is $500, and therefore, the monthly interest on the outstanding balance is

(outstanding balance)(the monthly interest rate) = ($500)(.12/12) = $5

This means, the first month, out of the $86.27 payment, $5 goes toward the interest and the remaining $81.27 toward the balance leaving a new balance of $500 - $81.27 = $418.73.

Similarly, the second month, the outstanding balance is $418.73, and the monthly interest on the outstanding balance is ($418.73)(.12/12) = $4.19. Again, out of the $86.27 payment, $4.19 goes toward the interest and the remaining $82.08 toward the balance leaving a new balance of $418.73 - $82.08 = $336.65. The process continues in the table below.

Payment #	Payment	Interest	Debt Payment	Balance
1	$86.27	$5	$81.27	$418.73
2	$86.27	$4.19	$82.08	$336.65
3	$86.27	$3.37	$82.90	$253.75
4	$86.27	$2.54	$83.73	$170.02
5	$86.27	$1.70	$84.57	$85.45
6	$86.27	$0.85	$85.42	$0.03

Note that the last balance of 3 cents is due to error in rounding off.

An amortization schedule is usually lengthy and tedious to calculate by hand. For example, an amortization schedule for a 30 year mortgage loan with monthly payments would have (12)(30)=360 rows of calculations in the amortization schedule table. A car loan with 5 years of monthly payments would have 12(5)=60 rows of calculations in the amortization schedule table. However it would be straightforward to use a spreadsheet application on a computer to do these repetitive calculations by inputting and copying formulas for the calculations into the cells.

Most of the other applications in this section's problem set are reasonably straightforward, and can be solved by taking a little extra care in interpreting them. And remember, there is often more than one way to solve a problem.

6.4.1: Miscellaneous Application Problems (Exercises)

SECTION 6.4 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

For problems 1 - 4, assume a $200,000 house loan is amortized over 30 years at an interest rate of 5.4%.

1) Find the monthly payment.	2) Find the balance owed after 20 years.
3) Find the balance of the loan after 100 payments.	4) Find the monthly payment if the original loan were amortized over 15 years.

6.4 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

5) Mr. Patel wants to pay off his car loan. The monthly payment for his car is $365, and he has 16 payments left. If the loan was financed at 6.5%, how much does he owe?	6) An amount of $2000 is borrowed for a year at a rate of 7%. Make an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment going toward reducing the debt, and the balance.
7) Fourteen months after Dan bought his new car he lost his job. His car was repossessed by his lender after he made only 14 monthly payments of $376 each. If the loan was financed over a 4-year period at an interest rate of 6.3%, how much did the car cost the lender? In other words, how much did Dan still owe on the car?	8) You have a choice of either receiving $5,000 at the end of each year for the next 5 years or receiving $3000 per year for the next 10 years. If the current interest rate is 9%, which is better?
9) Mr. Smith is planning to retire in 25 years and would like to have $250,000 then. What monthly payment made at the end of each month to an account that pays 6.5% will achieve his objective?	10) Assume Mr. Smith has reached retirement and has $250,000 in an account which is earning 6.5%. He would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment.
11) Mrs. Garcia is planning to retire in 20 years. She starts to save for retirement by depositing $2000 each quarter into a retirement investment account that earns 6% interest compounded quarterly. Find the accumulated value of her retirement savings at the end of 20 years.	12) Assume Mrs. Garcia has reached retirement and has accumulated the amount found in question 13 in a retirement savings account. She would now like to make equal monthly withdrawals for the next 15 years to completely deplete this account. Find the withdrawal payment. Assume the account now pays 5.4% compounded monthly.

SECTION 6.4 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

13) A ten-year $1,000 bond pays $35 every six months. If the current interest rate is 8.2%, find the fair market value of the bond. Hint: You must do the following. a) Find the present value of $1000. b) Find the present value of the $35 payments. c) The fair market value of the bond = a + b	14) Find the fair market value of the ten-year $1,000 bond that pays $35 every six months, if the current interest rate has dropped to 6%. Hint: You must do the following. a) Find the present value of $1000. b) Find the present value of the $35 payments. c) The fair market value of the bond = a + b
15) A twenty-year $1,000 bond pays $30 every six months. If the current interest rate is 4.2%, find the fair market value of the bond. Hint: You must do the following. a) Find the present value of $1000. b) Find the present value of the $30 payments. c) The fair market value of the bond = a + b	16) Find the fair market value of the twenty-year $1,000 bond that pays $30 every six months, if the current interest rate has increased to 7.5%.

SECTION 6.4 PROBLEM SET: MISCELLANEOUS APPLICATION PROBLEMS

17) Mr. and Mrs. Nguyen deposit $10,000 into a college investment account when their new baby grandchild is born. The account earns 6.25% interest compounded quarterly. a) When their grandchild reaches the age of 18, what is the accumulated value of the college investment account? b) The Nguyen's grandchild has just reached the age of 18 and started college. If she is to withdraw the money in the college savings account n equal monthly payments over the next 4 years, how much money will be withdrawn each month?	18) Mr. Singh is 38 and plans to retire at age 65. He opens a retirement savings account. a) Mr. Singh wants to save enough money to accumulate $500,000 by the time he retires. The retirement investment account pays 7% interest compounded monthly. How much does he need to deposit each month to achieve this goal? b) Mr. Singh has now reached at 65 and retires. How much money can he withdraw each month for 25 years if the retirement investment account now pays 5.2% interest, compounded monthly?

6.5: Classification of Finance Problems

Learning Objectives

In this section, you will review the concepts of chapter 6 to:

1. Re-examine the types of financial problems and classify them.
2. Re-examine the vocabulary words used in describing financial calculations

We'd like to remind the reader that the hardest part of solving a finance problem is determining the category it falls into. So in this section, we will emphasize the classification of problems rather than finding the actual solution.

We suggest that the student read each problem carefully and look for the word or words that may give clues to the kind of problem that is presented. For instance, students often fail to distinguish a lump-sum problem from an annuity. Since the payments are made each period, an annuity problem contains words such as each, every, per etc.. One should also be aware that in the case of a lump-sum, only a single deposit is made, while in an annuity numerous deposits are made at equal spaced time intervals. To help interpret the vocabulary used in the problems, we include a glossary at the end of this section.

Students often confuse the present value with the future value. For example, if a car costs $15,000, then this is its present value. Surely, you cannot convince the dealer to accept $15,000 in some future time, say, in five years. Recall how we found the installment payment for that car. We assumed that two people, Mr. Cash and Mr. Credit, were buying two identical cars both costing $15, 000 each. To settle the argument that both people should pay exactly the same amount, we put Mr. Cash's cash of $15,000 in the bank as a lump-sum and Mr. Credit's monthly payments of x dollars each as an annuity. Then we make sure that the future values of these two accounts are equal. As you remember, at an interest rate of 9%

the future value of Mr. Cash's lump-sum was $\$15,000(1+.09/12)^{60}$, and

the future value of Mr. Credit's annuity was $\frac{x\left[(1+.09/12)^{60}-1\right]}{.09/12}$.

To solve the problem, we set the two expressions equal and solve for m.

The present value of an annuity is found in exactly the same way. For example, suppose Mr. Credit is told that he can buy a particular car for $311.38 a month for five years, and Mr. Cash wants to know how much he needs to pay. We are finding the present value of the annuity of $311.38 per month, which is the same as finding the price of the car. This time our unknown quantity is the price of the car. Now suppose the price of the car is P, then

the future value of Mr. Cash's lump-sum is $\mathrm{P}(1+.09/12)^{60}$, and

the future value of Mr. Credit's annuity is $\frac{\$311.38\left[(1+.09/12)^{60}-1\right]}{.09/12}$.

Setting them equal we get,

$$\begin{aligned} P(1+.09/12)^{60} &= \frac{\$311.38\left[(1+.09/12)^{60}-1\right]}{.09/12} \\ P(1.5657) &= (\$311.38)(75.4241) \\ P(1.5657) &= \$23,485.57 \\ P &= \$15,000.04 \end{aligned}$$

CLASSIFICATION OF PROBLEMS AND EQUATIONS FOR SolutionS

We now list six problems that form a basis for all finance problems.
Further, we classify these problems and give an equation for the solution.

Example 6.5.1

If $2,000 is invested at 7% compounded quarterly, what will the final amount be in 5 years?

Classification: **Future (accumulated) Value of a Lump-sum** or FV of a lump-sum.

Equation:

$$\mathrm{FV}=\mathrm{A}=\$2000(1+.07/4)^{20}$$

Example 6.5.2

How much should be invested at 8% compounded yearly, for the final amount to be $5,000 in five years?

Classification: **Present Value of a Lump-sum** or PV of a lump-sum.

Equation:

$$\mathrm{PV}(1+.08)^{5}=\$5,000$$

Example 6.5.3

If $200 is invested *each* month at 8.5% compounded monthly, what will the final amount be in 4 years?

Classification: **Future (accumulated) Value of an Annuity** or FV of an annuity.

Equation:

$$\mathrm{FV}=\mathrm{A}=\frac{\$200\left[(1+.085/12)^{48}-1\right]}{.085/12}$$

Example 6.5.4

How much should be invested *each* month at 9% for it to accumulate to $8,000 in three years?

Classification: **Sinking Fund Payment**

Equation:

$$\frac{m\left[(1+.09/12)^{36}-1\right]}{.09/12}=\$8,000$$

Example 6.5.5

Keith has won a lottery paying him $2,000 *per* month for the next 10 years. He'd rather have the entire sum now. If the interest rate is 7.6%, how much should he receive?

Classification: **Present Value of an Annuity** or PV of an annuity.

Equation:

$$\mathrm{PV}(1+.076/12)^{120}=\frac{\$2000\left[(1+.076/12)^{120}-1\right]}{.076/12}$$

Example 6.5.6

Mr. A has just donated $25,000 to his alma mater. Mr. B would like to donate an equivalent amount, but would like to pay by monthly payments over a five year period. If the interest rate is 8.2%, determine the size of the monthly payment?

Classification: **Installment Payment**.

Equation:

$$\frac{m\left[(1+.082/12)^{60}-1\right]}{.082/12}=\$25,000(1+.082/12)^{60}$$

GLOSSARY: VOCABULARY AND SYMBOLS USED IN FINANCIAL CALCULATIONS

As we've seen in these examples, it's important to read the problems carefully to correctly identify the situation. It is essential to understand to vocabulary for financial problems. Many of the vocabulary words used are listed in the glossary below for easy reference.

t	Term	Time period for a loan or investment. In this book t is represented in years and should be converted into years when it is stated in months or other units.
P	Principal	Principal is the amount of money borrowed in a loan. If a sum of money is invested for a period of time, the sum invested at the start is the Principal.
P	Present Value	Value of money at the beginning of the time period.
A	Accumulated Value Future Value	Value of money at the end of the time period
D	Discount	In loans involving simple interest, a discount occurs if the interest is deducted from the loan amount at the beginning of the loan period, rather than being repaid at the end of the loan period.
m	Periodic Payment	The amount of a constant periodic payment that occurs at regular intervals during the time period under consideration (examples: periodic payments made to repay a loan, regular periodic payments into a bank account as savings, regular periodic payment to a retired person as an annuity,)
n	Number of payment periods and compounding periods per year	In this book, when we consider periodic payments, we will always have the compounding period be the same as the payment period. In general the compounding and payment periods do not have to be the same, but the calculations are more complicated if they are different. If the periods differ, formulas for the calculations can be found in finance textbooks or various online resources. Calculations can easily be done using technology such as an online financial calculator, or financial functions in a spreadsheet, or a financial pocket calculator.

nt	Number of periods	nt = (number of periods per year) $\times$(number of years) nt gives the total number of payment and compounding periods In some situations we will calculate nt as the multiplication shown above. In other situations the problem may state nt, such as a problem describing an investment of 18 months duration compounded monthly. In this example: nt = 18 months and n = 12; then t = 1.5 years but t is not stated explicitly in the problem. The TI-84+ calculators built in TVM solver uses $N = nt$.
r	Annual interest rate Nominal rate	The stated annual interest rate. This is stated as a percent but converted to decimal form when using financial calculation formulas. If a bank account pays 3% interest compounded quarterly, then 3% is the nominal rate, and it is included in the financial formulas as r = 0.03
r/n	Interest rate per compounding period	If a bank account pays 3% interest compounded quarterly, then r/n = 0.03/4 = 0. 075, corresponding to a rate of 0.75% per quarter. Some Finite Math books use the symbol i to represent r/n

r_{EFF}	Effective Rate Effective Annual Interest Rate APY Annual Percentage Yield APR Annual Percentage Rate	The effective rate is the interest rate compounded annually that would give the same interest rate as the compounded rate stated for the investment. The effective rate provides a uniform way for investors or borrowers to compare different interest rates with different compounding periods.
I	Interest	Money paid by a borrower for the use of money borrowed as a loan. Money earned over time when depositing money into a savings account, certificate of deposit, or money market account. When a person deposits money in a bank account, the person depositing the funds is essentially temporarily lending the money to the bank and the bank pays interest to the depositor.

	Sinking Fund	A fund set up by making payments over a period of time into a savings or investment account in order to save to fund a future purchase. Businesses use sinking funds to save for a future purchase of equipment at the end of the savings period by making periodic installment payments into a sinking fund.
	Annuity	An annuity is a stream of periodic payments. In this book it refers to a stream of constant periodic payments made at the end of each compounding period for a specific amount of time. In common use the term annuity generally refers to a constant stream of periodic payments received by a person as retirement income, such as from a pension. Annuity payments in general may be made at the end of each payment period (ordinary annuity) or at the start of each period (annuity due). The compounding periods and payment periods do not need to be equal, but in this textbook we only consider situations when these periods are equal.
	Lump Sum	A single sum of money paid or deposited at one time, rather than being spread out over time. An example is lottery winnings if the recipient chooses to receive a single "lump sum" one-time payment, instead of periodic payments over a period of time or as. Use of the word lump sum indicates that this is a one time transaction and is not a stream of periodic payments.
	Loan	An amount of money that is borrowed with the understanding that the borrower needs to repay the loan to the lender in the future by the end of a period of time that is called the term of the loan. The repayment is most often accomplished through periodic payments until the loan has completely repaid over the term of the loan. However there are also loans that can be repaid as a single sum at the end of the term of the loan, with interest paid either periodically over the term or in a lump sum at the end of the loan or as a discount at the start of the loan.

6.5.1: Classification of Finance Problems (Exercises)

SECTION 6.5 PROBLEM SET: CLASSIFICATION OF FINANCE PROBLEMS

Let the letters A, B, C, D, E and F be represented as follows:

$$\begin{array}{lll} A = FV \text{ of a lump-sum} & C = FV \text{ of an annuity} & E = \text{ Installment payment} \\ B = PV \text{ of a lump-sum} & D = \text{ sinking fund payment} & F = PV \text{ of an annuity} \end{array}$$

Classify each by writing the appropriate letter in the box, and write an equation for solution.

1) What monthly deposits made to an account paying 9% will grow to $10,000 in 4 years?

2) An amount of $4000 is invested at 6% compounded daily. What will the final amount be in 5 years?

3) David has won a lottery paying him $10,000 per month for the next 20 years. He'd rather have the whole amount in one lump sum now. If the current interest rate is 7%, how much money can he hope to get?

4) Each month Linda deposits $250 in an account that pays 9%. How much money will she have in 4 years?

5) Find the monthly payment for a $15,000 car if the loan is amortized over 4 years at a rate of 10%.

6) What lump-sum deposited in an account paying 7% compounded daily will grow to $10,000 in 5 years?

7) What amount of quarterly payments will amount to $250,000 in 5 years at a rate of 8%?

8) The Chang family bought their house 25 years ago. They had their loan financed for 30 years at an interest rate of 11% resulting in a payment of $1350 a month. Find the balance of the loan.

A 10-year $1000 bond pays $35 every six months. If the current interest rate is 8%, in order to find the fair market value of the bond, we need to find the following.

9) The present value of $1000.

10) The present value of the $35 per six month payments.

SECTION 6.5 PROBLEM SET: CLASSIFICATION OF FINANCE PROBLEMS

$$\begin{array}{lll} A = FV \text{ of a lump-sum} & C = FV \text{ of an annuity} & E = \text{ Installment payment} \\ B = PV \text{ of a lump-sum} & D = \text{ sinking fund payment} & F = PV \text{ of an annuity} \end{array}$$

11) What lump-sum deposit made today is equal to 33 monthly deposits of $500 if the interest rate is 8%?

12) What monthly deposits made to an account paying 10% will accumulated to $10,000 in six years?

13) A department store charges a finance charge of 1.5% per month on the outstanding balance.
If Ned charged $400 three months ago and has not paid his bill, how much does he owe?

14) What will the value of $300 monthly deposits be in 10 years if the account pays 12% compounded monthly?

15) What lump-sum deposited at 6% compounded daily will grow to $2000 in three years?

16) A company buys an apartment complex for $5,000,000 and amortizes the loan over 10 years.
What is the yearly payment if the interest rate is 14%?

17) In 2002, a house in Rock City cost $300,000. Real estate in Rock City has been increasing in value at the annual rate of 5.3%..
Find the price of that house in 2016.

18) You determine that you can afford to pay $400 per month for a car. What is the maximum price you can pay for a car if the interest rate is 11% and you want to repay the loan in 4 years?

19) A business needs $350,000 in 5 years. How much lump-sum should be put aside in an account that pays 9% so that five years from now the company will have $350,000?

20) A person wishes to have $500,000 in a pension fund 20 years from now. How much should he deposit each month in an account paying 9% compounded monthly?

6.6: Chapter Review

SECTION 6.6 PROBLEM SET: CHAPTER REVIEW

1. Manuel borrows $800 for 6 months at 18% simple interest. How much does he owe at the end of 6 months?
2. The population of a city is 65,000 and expects to grow at a rate of 2.3% per year for the next 10 years. What will the population of this city be in 10 years?
3. The Gill family is buying a $250,000 house with a 10% down payment. If the loan is financed over a 30 year period at an interest rate of 4.8%, what is the monthly payment?
4. Find the monthly payment for the house in the above problem if the loan was amortized over 15 years.
5. You look at your budget and decide that you can afford $250 per month for a car. What is the maximum amount you can afford to pay for the car if the interest rate is 8.6% and you want to finance the loan over 5 years?
6. Mr. Nakahama bought his house in the year 1998. He had his loan financed for 30 years at an interest rate of 6.2% resulting in a monthly payment of $1500. In 2015, 17 years later, he paid off the balance of the loan. How much did he pay?
7. Lisa buys a car for $16,500, and receives $2400 for her old car as a trade-in value. Find the monthly payment for the balance if the loan is amortized over 5 years at 8.5%.
8. A car is sold for $3000 cash down and $400 per month for the next 4 years. Find the cash value of the car today if the money is worth 8.3% compounded monthly.
9. An amount of $2300 is borrowed for 7 months at a simple interest rate of 16%. Find the discount and the proceeds.
10. Marcus has won a lottery paying him $5000 per month for the next 25 years. He'd rather have the whole amount in one lump sum today. If the current interest rate is 7.3%, how much money can he hope to get?
11. In the year 2000, an average house in Star City cost $250,000. If the average annual inflation rate for the past years has been about 4.7%, what was the price of that house in 2015?
12. Find the 'fair market' value of a ten-year $1000 bond which pays $30 every six months if the current interest rate is 7%. What if the current interest rate is 5%?
13. A Visa credit card company has a finance charge of 1.5% per month (18% per year) on the outstanding balance. John owed $3200 and has been delinquent for 5 months. How much total does he owe, now?
14. You want to purchase a home for $200,000 with a 30-year mortgage at 9.24% interest. Find
    a. the monthly payment
    b. the balance owed after 20 years.
15. When Jose bought his car, he amortized his loan over 6 years at a rate of 9.2%, and his monthly payment came out to be $350 per month. He has been making these payments for the past 40 months and now wants to pay off the remaining balance. How much does he owe?
16. A lottery pays $10,000 per month for the next 20 years. If the interest rate is 7.8%, find both its present and future values.
17. A corporation estimates it will need $300,000 in 8 years to replace its existing machinery. How much should it deposit each quarter in a sinking fund earning 8.4% compounded quarterly to meet this obligation?
18. Our national debt in 1992 was about $4 trillion. If the annual interest rate was 7% then, what was the daily interest on the national debt?
19. A business must raise $400,000 in 10 years. What should be the size of the owners' monthly payments to a sinking fund paying 6.5% compounded monthly?
20. The population of a city of 80,000 is growing at a rate of 3.2% per year. What will the population be at the end of 10 years?
21. A sum of $5000 is deposited in a bank today. What will the final amount be in 20 months if the bank pays 9% and the interest is compounded monthly?
22. A manufacturing company buys a machine for $500 cash and $50 per month for the next 3 years. Find the cash value of the machine today if the money is worth 6.2% compounded monthly.
23. The United States paid about 4 cents an acre for the Louisiana Purchase in 1803. Suppose the value of this property grew at a rate of 5.5% annually. What would an acre be worth in the year 2000?
24. What amount should be invested per month at 9.1% compounded monthly so that it will become $5000 in 17 months?
25. A machine costs $8000 and has a life of 5 years. It can be leased for $160 per month for 5 years with a cash down payment of $750. The current interest rate is 8.3%. Is it cheaper to lease or to buy?
26. If inflation holds at 5.2% per year for 5 years, what will be the cost in 5 years of a car that costs $16,000 today? How much will you need to deposit each quarter in a sinking fund earning 8.7% per year to purchase the new car in 5 years?

27. City Bank pays an interest rate of 6%, while Western Bank pays 5.8% compounded continuously. Which one is a better deal?
28. Ali has inherited $20,000 and is planning to invest this amount at 7.9% interest. At the same time he wishes to make equal monthly withdrawals to use up the entire sum in 5 years. How much can he withdraw each month?
29. Jason has a choice of receiving $300 per month for the next 5 years or $500 per month for the next 3 years. Which one is worth more if the current interest rate is 7.7%?
30. If a bank pays 6.8% compounded continuously, how long will it take to double your money?
31. A mutual fund claims a growth rate of 8.3% per year. If $500 per month is invested, what will the final amount be in 15 years?
32. Mr. Vasquez has been given two choices for his compensation. He can have $20,000 cash plus $500 per month for 10 years, or he can receive $12,000 cash plus $1000 per month for 5 years. If the interest rate is 8%, which is the better offer?
33. How much should Mr. Shackley deposit in a trust account so that his daughter can withdraw $400 per month for 4 years if the interest rate is 8%?
34. Mr. Albers borrowed $425,000 from the bank for his new house at an interest rate of 4.7%. He will make equal monthly payments for the next 30 years. How much money will he end up paying the bank over the life of the loan, and how much is the interest?
35. Mr. Tong puts away $500 per month for 10 years in an account that earns 9.3%. After 10 years, he decides to withdraw $1,000 per month. If the interest rate stays the same, how long will it take Mr. Tong to deplete the account?
36. An amount of $5000 is borrowed for 15 months at an interest rate of 9%. Find the monthly payment and construct an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the amount of payment contributing towards debt, and the outstanding debt.

7: Matrices

Learning Objectives

In this chapter, you will learn to:

1. Do matrix operations.
2. Solve linear systems using the Gauss-Jordan method.
3. Solve linear systems using the matrix inverse method.
4. Do application problems.

- ## 7.1: Introduction to Matrices

  A matrix is a 2-dimensional array of numbers arranged in rows and columns. Matrices provide a method of organizing, storing, and working with mathematical information. Matrices have an abundance of applications and use in the real world. Matrices provide a useful tool for working with models based on systems of linear equations. We'll use matrices in sections 7.2, 7.3, and 7.4 to solve systems of linear equations with several variables in this chapter.

  7.1.1: Introduction to Matrices (Exercises)

- ## 7.2: Systems of Linear Equations and the Gauss-Jordan Method

  In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the augmented matrix, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a row operation.

  7.2.1: Systems of Linear Equations and the Gauss-Jordan Method (Exercises)

- ## 7.3: Systems of Linear Equations – Special Cases

  7.3.1: Systems of Linear Equations – Special Cases (Exercises)

- ## 7.4: Inverse Matrices

  In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems.

  7.4.1: Inverse Matrices (Exercises)

- ## 7.5: Application of Matrices in Cryptography

  In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems.

  7.5.1: Application of Matrices in Cryptography (Exercises)

- ## 7.6: Applications – Leontief Models

  In this section we will examine an application of matrices to model economic systems.

  7.6.1: Applications – Leontief Models (Exercises)

-

7.7: Chapter Review

Thumbnail: (via Wikipedia)

7.1: Introduction to Matrices

Learning Objectives

In this section, you will learn to:

1. Add and subtract matrices.
2. Multiply a matrix by a scalar.
3. Multiply two matrices.

A matrix is a 2-dimensional array of numbers arranged in rows and columns. Matrices provide a method of organizing, storing, and working with mathematical information. Matrices have an abundance of applications and use in the real world. Matrices provide a useful tool for working with models based on systems of linear equations. We'll use matrices in sections 7.2, 7.3, and 7.4 to solve systems of linear equations with several variables in this chapter.

Matrices are used in encryption, which we will explore in section 7.5, and in economic modeling, explored in section 7.6. We use matrices again in Chapter 9, in optimization problems such as maximizing profit/revenue or minimizing cost. Matrices are used in business for scheduling, routing transportation and shipments, and managing inventory.

Just about any application that collects and manages data can apply matrices. The use of matrices has grown as the availability of data in many areas of life and business has increased. They are important tools for organizing data and solving problems in all fields of science (physics, chemistry, biology, genetics, meteorology, and economics). In computer science, matrix mathematics lies behind the animation of images in movies and video games.

Computer science analyzes diagrams of networks to understand how things are connected, such as relationships between people on a social website, relationships between results in line searches, and how people link from one website to another. The mathematics to work with network diagrams comprise the field of "graph theory"; it relies on matrices to organize the information in the graphs that diagram connections and associations in a network. For example, if someone uses Facebook, Linked In, or other social media sites, these sites use network graphs and matrices to organize their relationships with other users.

Introduction to Matrices

A matrix is a rectangular array of numbers. Matrices are useful in organizing and manipulating large amounts of data. In order to understand matrices, we will look at the following example.

Example 7.1.1

Fine Furniture Company makes chairs and tables at its San Jose, Hayward, and Oakland factories. The total production, in hundreds, from the three factories for the years 2014 and 2015 is listed in the table below.

	2014		2015	
	CHAIRS	TABLES	CHAIRS	TABLES
SAN JOSE	30	18	36	20
HAYWARD	20	12	24	18
OAKLAND	16	10	20	12

a. Represent the production for the years 2014 and 2015 as the matrices A and B.
b. Find the difference in sales between the years 2014 and 2015.
c. The company predicts that in the year 2020, the production at these factories will be double that of the year 2014. What will the production be for the year 2020?

Solution

a) The matrices are as follows:

$$A = \begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} 36 & 20 \\ 24 & 18 \\ 20 & 12 \end{bmatrix}$$

b) We are looking for the matrix $B - A$. When two matrices have the same number of rows and columns, the matrices can be added or subtracted entry by entry. Therefore, we get

$$\mathrm{B} - \mathrm{A} = \begin{bmatrix} 36-30 & 20-18 \\ 24-20 & 18-12 \\ 20-16 & 12-10 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 4 & 6 \\ 4 & 2 \end{bmatrix}$$

c) We would like a matrix that is twice the matrix of 2014, i.e., $2A$.

Whenever a matrix is multiplied by a number, each entry is multiplied by the number.

$$2\mathrm{A} = 2\begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} 60 & 36 \\ 40 & 24 \\ 32 & 20 \end{bmatrix}$$

Before we go any further, we need to familiarize ourselves with some terms that are associated with matrices. The numbers in a matrix are called the **entries** or the **elements** of a matrix.

Whenever we talk about a matrix, we need to know the **size** or the **dimension** of the matrix. The dimension of a matrix is the number of rows and columns it has. When we say a matrix is a “3 by 4 matrix”, we are saying that it has 3 rows and 4 columns. The rows are always mentioned first and the columns second. This means that a 3×4 matrix does not have the same dimension as a 4×3 matrix.

$$A = \begin{bmatrix} 1 & 4 & -2 & 0 \\ 3 & -1 & 7 & 9 \\ 6 & 2 & 0 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 9 & 8 \\ -3 & 0 & 1 \\ 6 & 5 & -2 \\ -4 & 7 & 8 \end{bmatrix}$$

Matrix A has dimensions 3×4 and matrix B has dimensions 4×3.

A matrix that has the same number of rows as columns is called a **square matrix**. A matrix with all entries zero is called a **zero matrix.** A square matrix with 1's along the main diagonal and zeros everywhere else, is called an **identity matrix**. When a matrix (size r x c) is multiplied by an identity matrix(size c x c), the matrix remains the same (size r x c).

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix I is a 3×3 identity matrix

A matrix with only one row is called a **row matrix,** and a matrix with only one column is called a **column matrix**. Two matrices are **equal** if they have the same size and the corresponding entries are equal.

The following operations can be performed on matrices: addition, subtraction, scalar multiplication, and matrix multiplication. The topics are defined in the sections below.

Matrix Addition and Subtraction

If two matrices are the same size, they can be added or subtracted. The operations are performed on corresponding entries.

Example 7.1.2

Given the matrices A, B, C and D, below

$$A=\begin{bmatrix}1&2&4\\2&3&1\\5&0&3\end{bmatrix}\quad B=\begin{bmatrix}2&-1&3\\2&4&2\\3&6&1\end{bmatrix}\quad C=\begin{bmatrix}4\\2\\3\end{bmatrix}\quad D=\begin{bmatrix}-2\\-3\\4\end{bmatrix}$$

Find, if possible.

a. $A+B$
b. $C-D$
c. $A+D$.

Solution

As we mentioned earlier, matrix addition and subtraction involve performing these operations entry by entry.

a) We add each element of A to the corresponding entry of B.

$$A+B=\begin{bmatrix}1+2&2+-1&4+3\\2+2&3+4&1+2\\5+3&0+6&3+1\end{bmatrix}$$

$$A+B=\begin{bmatrix}3&1&7\\4&7&3\\8&6&4\end{bmatrix}$$

b) Just like the problem above, we perform the subtraction entry by entry.

$$\mathrm{C}-\mathrm{D}=\begin{bmatrix}4-(-2)\\2-(-3)\\3-4\end{bmatrix}$$

$$\mathrm{C}-\mathrm{D}=\begin{bmatrix}6\\5\\-1\end{bmatrix}$$

c) The sum $A+D$ cannot be found because the two matrices have different sizes.

Note: Two matrices can *only* be added or subtracted if they have the same dimension.

Multiplying a Matrix by a Scalar

If a matrix is multiplied by a scalar, each entry is multiplied by that scalar. We can consider scalar multiplication as multiplying a number and a matrix to obtain a new matrix as the product.

Example 7.1.3

Given the matrix A and C in the example above, find $2A$ and $-3C$.

Solution

To find $2A$, we multiply each entry of matrix A by 2, and to find $-3C$, we multiply each entry of C by -3. The results are given below.

a) We multiply each entry of A by 2.

$$2A=\begin{bmatrix}2\cdot1&2\cdot2&2\cdot4\\2\cdot2&2\cdot3&2\cdot1\\2\cdot5&2\cdot0&2\cdot3\end{bmatrix}$$

$$2A=\begin{bmatrix}2&4&8\\4&6&2\\10&0&6\end{bmatrix}$$

b) We multiply each entry of C by -3.

$$-3C=\begin{bmatrix}-3\cdot4\\-3\cdot2\\-3\cdot3\end{bmatrix}$$

$$-3C=\begin{bmatrix}-12\\-6\\-9\end{bmatrix}$$

Multiplication of Two Matrices

Multiplying two matrices is not as easy as the addition, subtraction, or scalar multiplication of matrices. Because of its wide use in application problems, it is important that we learn it well. Therefore, we will try to learn the process in a step-by-step manner. We first begin by finding a product of a row matrix and a column matrix.

Example 7.1.4

Find the product AB, given

$$A=\begin{bmatrix}2&3&4\end{bmatrix}$$

and

$$B=\begin{bmatrix}a\\b\\c\end{bmatrix}.$$

Solution

The product is a 1×1 matrix whose entry is obtained by multiplying the corresponding entries and then forming the sum.

$$\begin{aligned}AB&=\begin{bmatrix}2&3&4\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}\\&=[2a+3b+4c]\end{aligned}$$

Note that AB is a 1×1 matrix, and its only entry is $2a+3b+4c$.

Example 7.1.5

Find the product AB, given

$$A=\begin{bmatrix}2&3&4\end{bmatrix}$$

and

$$B=\begin{bmatrix}5\\6\\7\end{bmatrix}$$

Solution

Again, we multiply the corresponding entries and add.

$$\begin{aligned} \mathrm{AB} &= \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 \end{bmatrix} \\ &= \begin{bmatrix} 10 + 18 + 28 \end{bmatrix} \\ &= \begin{bmatrix} 56 \end{bmatrix} \end{aligned}$$

Note: In order for a product of a row matrix and a column matrix to exist, the number of entries in the row matrix must be the same as the number of entries in the column matrix.

Example 7.1.6

Find the product AB, given

$$A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}.$$

Solution

We know how to multiply a row matrix by a column matrix. To find the product AB, in this example, we will multiply the row matrix A to both the first and second columns of matrix B, resulting in a 1×2 matrix.

$$\mathrm{AB} = \begin{bmatrix} 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 & 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 \end{bmatrix} = \begin{bmatrix} 56 & 38 \end{bmatrix}$$

We multiplied a 1×3 matrix by a matrix whose size is 3×2. So unlike addition and subtraction, it is possible to multiply two matrices with different dimensions, if the number of entries in the columns of the first matrix is the same as the number of entries in the rows of the second matrix.

Example 7.1.7

Find the product AB, given:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

and

$$\mathrm{B} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$$

Solution

This time we are multiplying two rows of the matrix A with two columns of the matrix B. Since the number of entries in each row of A is the same as the number of entries in each column of B, the product is possible. We do exactly what we did in the last example. The only difference is that the matrix A has one more row.

We multiply the first row of the matrix A with the two columns of B, one at a time, and then repeat the process with the second row of A. We get

$$AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 2\cdot5+3\cdot6+4\cdot7 & 2\cdot3+3\cdot4+4\cdot5 \\ 1\cdot5+2\cdot6+3\cdot7 & 1\cdot3+2\cdot4+3\cdot5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 56 & 38 \\ 38 & 26 \end{bmatrix}$$

Example 7.1.8

Find, if possible:

a. EF
b. FE
c. FH
d. GH
e. HG

$$E = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 4 & 1 \end{bmatrix} \quad H = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Solution

a) To find EF, we multiply the first row $\begin{bmatrix} 1 & 2 \end{bmatrix}$

of E with the columns $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ of the matrix F, and then repeat the process by multiplying the other two rows of E with these columns of F. The result is as follows:

$$EF = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot2+2\cdot3 & 1\cdot-1+2\cdot2 \\ 4\cdot2+2\cdot3 & 4\cdot-1+2\cdot2 \\ 3\cdot2+1\cdot3 & 3\cdot-1+1\cdot2 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 14 & 0 \\ 9 & -1 \end{bmatrix}$$

b) Product FE is not possible because the matrix F has two entries in each row, while the matrix E has three entries in each column. In other words, the matrix F has two columns, while the matrix E has three rows.

c)

$$FH = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\cdot-3+-1\cdot-1 \\ 3\cdot-3+2\cdot-1 \end{bmatrix} = \begin{bmatrix} -5 \\ -11 \end{bmatrix}$$

d)

$$GH = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4\cdot-3+1\cdot-1 \\ -1 \end{bmatrix} = \begin{bmatrix} -13 \end{bmatrix}$$

e)

$$HG = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \begin{bmatrix} 4 & 1 \end{bmatrix} = \begin{bmatrix} -3\cdot4 & -3\cdot1 \\ -1\cdot4 & -1\cdot1 \end{bmatrix} = \begin{bmatrix} -12 & -3 \\ -4 & -1 \end{bmatrix}$$

We summarize some important properties of matrix multiplication that we observed in the previous examples.

In order for product $\mathbf{AB}$ to exist:

- **the number of columns of $\mathbf{A}$ must equal the number of rows of $\mathbf{B}$**
- **if matrix $\mathbf{A}$ has dimension $\mathbf{m \times n}$ and matrix $\mathbf{B}$ has dimension $\mathbf{n \times p}$, then the product $\mathbf{AB}$ will be a matrix with dimension $\mathbf{m \times p}$.**

Matrix multiplication is not commutative: if both matrix products $\mathbf{AB}$ and $\mathbf{BA}$ exist, most of the time $\mathbf{AB}$ will not equal $\mathbf{BA}$.

✔ Example 7.1.9

Given matrices R, S, and T below, find $2RS-3ST$.

$$R=\begin{bmatrix}1&0&2\\2&1&5\\2&3&1\end{bmatrix}\quad S=\begin{bmatrix}0&-1&2\\3&1&0\\4&2&1\end{bmatrix}\quad T=\begin{bmatrix}-2&3&0\\-3&2&2\\-1&1&0\end{bmatrix}$$

Solution

We multiply the matrices R and S.

$$\mathrm{RS}=\begin{bmatrix}8&3&4\\23&9&9\\13&3&5\end{bmatrix}$$

$$2\mathrm{RS}=2\begin{bmatrix}8&3&4\\23&9&9\\13&3&5\end{bmatrix}=\begin{bmatrix}16&6&8\\46&18&18\\26&6&10\end{bmatrix}$$

$$\mathrm{ST}=\begin{bmatrix}1&0&-2\\-9&11&2\\-15&17&4\end{bmatrix}$$

$$3\mathrm{ST}=3\begin{bmatrix}1&0&-2\\-9&11&2\\-15&17&4\end{bmatrix}=\begin{bmatrix}3&0&-6\\-27&33&6\\-45&51&12\end{bmatrix}$$

Thus

$$2\mathrm{RS}-3\mathrm{ST}=\begin{bmatrix}16&6&8\\46&18&18\\26&6&10\end{bmatrix}-\begin{bmatrix}3&0&-6\\-27&33&6\\-45&51&12\end{bmatrix}=\begin{bmatrix}13&6&14\\73&-15&12\\71&-45&-2\end{bmatrix}$$

✔ Example 7.1.10

Find F^2 given matrix

$$\mathrm{F}=\begin{bmatrix}2&-1\\3&2\end{bmatrix}$$

Solution

F^2 is found by multiplying matrix F by itself, using matrix multiplication.

$$\mathrm{F}^2=\begin{bmatrix}2&-1\\3&2\end{bmatrix}\begin{bmatrix}2&-1\\3&2\end{bmatrix}=\begin{bmatrix}2\cdot2+(-1)\cdot3&2\cdot(-1)+(-1)\cdot2\\3\cdot2+2\cdot3&3\cdot(-1)+2\cdot2\end{bmatrix}=\begin{bmatrix}1&-4\\12&1\end{bmatrix}$$

Note that F^2 is not found by squaring each entry of matrix F. The process of raising a matrix to a power, such as finding F^2, is only possible if the matrix is a **square matrix**.

USING MATRICES TO REPRESENT A SYSTEM OF LINEAR EQUATIONS

In this chapter, we will be using matrices to solve linear systems. In section 7.4, we will be asked to express linear systems as the **matrix equation $\mathbf{AX}=\mathbf{B}$**, where A, X, and B are matrices.

- Matrix A is called the **coefficient matrix.**
- Matrix X is a matrix with 1 column that contains the variables.
- Matrix B is a matrix with 1 column that contains the constants.

> ### ✓ Example 7.1.11
>
> Verify that the system of two linear equations with two unknowns:
>
> $$\begin{aligned} ax+by&=h \\ cx+dy&=k \end{aligned} \tag{7.1.1}$$
>
> can be written as $AX=B$, where
>
> $$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad X=\begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B=\begin{bmatrix} h \\ k \end{bmatrix}$$
>
> **Solution**
>
> If we multiply the matrices A and X, we get
>
> $$AX=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$
>
> If $AX=B$ then
>
> $$\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}=\begin{bmatrix} h \\ k \end{bmatrix}$$
>
> If two matrices are equal, then their corresponding entries are equal. It follows that
>
> $$\begin{aligned} ax+by&=h \\ cx+dy&=k \end{aligned} \tag{7.1.2}$$

> ### ✓ Example 7.1.12
>
> Express the following system as a matrix equation in the form $AX=B$.
>
> $$\begin{aligned} 2x+3y-4z&=5 \\ 3x+4y-5z&=6 \\ 5x\quad\;\; -6z&=7 \end{aligned} \tag{7.1.3}$$
>
> **Solution**
>
> This system of equations can be expressed in the form $AX=B$ as shown below.
>
> $$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

7.1.1: Introduction to Matrices (Exercises)

A vendor sells hot dogs and corn dogs at three different locations. His total sales(in hundreds) for January and February from the three locations are given in the table below.

	JANUARY			FEBRUARY	
	HOT DOGS	CORN DOGS		HOT DOGS	CORN DOGS
PLACE I	10	8		8	7
PLACE II	8	6		6	7
PLACE III	6	4		6	5

Represent these tables as 3×2 matrices J and F, and answer problems 1 - 5.

1) Determine total sales for the two months, that is, find $J+F$.	2) Find the difference in sales, $J-F$.
3) If hot dogs sell for \$3 and corn dogs for \$2, find the revenue from the sale of hot dogs and corn dogs. *Hint: Let* P *be a* 2×1 *matrix. Find* $(J+F)P$.	4) If March sales will be up from February by 10%, 15%, and 20% at Place I, Place II, and Place III, respectively, find the expected number of hot dogs and corn dogs to be sold in March. *Hint: Let* R *be a* 1×3 *matrix with entries 1.10, 1.15, and 1.20. Find* $M=RF$.
5) Hots dogs sell for \$3 and corn dogs sell for \$2. Using matrix M that predicts the number of hot dogs and corn dogs expected to be sold in March from problem (4), find the 1×1 matrix that predicts total revenue in March. *Hint: Use* 2×1 *price matrix* P *from problem (3) and find* MP.	

Determine the sums and products in problems 6-13. Given the matrices A, B, C, and D as follows:

$$\mathrm{A}=\begin{bmatrix}3 & 6 & 1\\0 & 1 & 3\\2 & 4 & 1\end{bmatrix} \quad \mathrm{B}=\begin{bmatrix}1 & -1 & 2\\1 & 4 & 2\\3 & 1 & 1\end{bmatrix} \quad \mathrm{C}=\begin{bmatrix}1\\2\\3\end{bmatrix} \quad \mathrm{D}=\begin{bmatrix}2 & 3 & 2\end{bmatrix}$$

6) $3A-2B$	7) AB
8) BA	9) $AB+BA$
10) A^2	11) $2BC$
12) $2CD+3AB$	13) A^2B

14) Let $E=\begin{bmatrix}m & n\\p & q\end{bmatrix}$ and $\mathrm{F}=\begin{bmatrix}\mathrm{a} & \mathrm{b}\\\mathrm{c} & \mathrm{d}\end{bmatrix}$, find EF.	15) Let $E=\begin{bmatrix}m & n\\p & q\end{bmatrix}$ and $\mathrm{F}=\begin{bmatrix}\mathrm{a} & \mathrm{b}\\\mathrm{c} & \mathrm{d}\end{bmatrix}$, find FE.
16) Let $G=\begin{bmatrix}3 & 6 & 1\\0 & 1 & 3\\2 & 4 & 1\end{bmatrix}$ and $H=\begin{bmatrix}x\\y\\z\end{bmatrix}$, find GH.	17) Let $G=\begin{bmatrix}3 & 6 & 1\\0 & 1 & 3\\2 & 4 & 1\end{bmatrix}$ and $H=\begin{bmatrix}x\\y\\z\end{bmatrix}$. Explain why the product HG does not exist.

Express the following systems as $AX=B$, where A, X, and B are matrices.

18) $$\begin{aligned}4x-5y&=6\\5x-6y&=7\end{aligned} \tag{7.1.1.1}$$	19) $$\begin{aligned}x-2y+2z&=3\\x-3y+4z&=7\\x-2y-3z&=-12\end{aligned} \tag{7.1.1.2}$$

20)	21)
$$\begin{aligned} 2x+3z&=17\\ 3x-2y&=10\\ 5y+2z&=11 \end{aligned} \tag{7.1.1.3}$$	$$\begin{array}{ccccccccl} x & + & 2y & + & 3z & + & 2w & = & 14\\ x & - & 2y & - & z & & & = & -5\\ & & y & - & 2z & + & 4w & = & 9\\ x & & & + & 3z & + & 3w & = & 15 \end{array} \tag{7.1.1.4}$$

7.2: Systems of Linear Equations and the Gauss-Jordan Method

Learning Objectives

In this section you will learn to

1. Represent a system of linear equations as an augmented matrix
2. Solve the system using elementary row operations.

In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method. The process begins by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the **augmented matrix**, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a **row operation**.

Example 7.2.1

Write the following system as an augmented matrix.

$$\begin{array}{l} 2x+3y-4z=5 \\ 3x+4y-5z=-6 \\ 4x+5y-6z=7 \end{array}$$

Solution

We express the above information in matrix form. Since a system is entirely determined by its coefficient matrix and by its matrix of constant terms, the augmented matrix will include only the coefficient matrix and the constant matrix. So the augmented matrix we get is as follows:

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 3 & 4 & -5 & -6 \\ 4 & 5 & -6 & 7 \end{array}\right]$$

In the last section, we expressed the system of equations as $AX=B$, where A represented the coefficient matrix, and B the matrix of constant terms. As an augmented matrix, we write the matrix as $\left[\begin{array}{c|c} A & B \end{array}\right]$. It is clear that all of the information is maintained in this matrix form, and only the letters x, y and z are missing. A student may choose to write x, y and z on top of the first three columns to help ease the transition.

Example 7.2.2

For the following augmented matrix, write the system of equations it represents.

$$\left[\begin{array}{ccccc} 1 & 3 & -5 & | & 2 \\ 2 & 0 & -3 & | & -5 \\ 3 & 2 & -3 & | & -1 \end{array}\right]$$

Solution

The system is readily obtained as below.

$$\begin{array}{l} x+3y-5z=2 \\ 2x-3z=-5 \\ 3x+2y-3z=-1 \end{array}$$

Once a system is expressed as an augmented matrix, the Gauss-Jordan method reduces the system into a series of equivalent systems by using the row operations. This row reduction continues until the system is expressed in what is called the **reduced row**

echelon form. The reduced row echelon form of the coefficient matrix has 1's along the main diagonal and zeros elsewhere. The solution is readily obtained from this form.

The method is not much different form the algebraic operations we employed in the elimination method in the first chapter. The basic difference is that it is algorithmic in nature, and, therefore, can easily be programmed on a computer.

We will next solve a system of two equations with two unknowns, using the elimination method, and then show that the method is analogous to the Gauss-Jordan method.

Example 7.2.3

Solve the following system by the elimination method.

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

Solution

We multiply the first equation by – 3, and add it to the second equation.

$$\begin{array}{r} -3x-9y=-21 \\ \underline{3x+4y=11} \\ -5y=-10 \end{array}$$

By doing this we transformed our original system into an equivalent system:

$$\begin{aligned} x+3y &= 7 \\ -5y &= -10 \end{aligned}$$

We divide the second equation by – 5, and we get the next equivalent system.

$$\begin{aligned} x+3y &= 7 \\ y &= 2 \end{aligned}$$

Now we multiply the second equation by – 3 and add to the first, we get

$$\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

Example 7.2.4

Solve the following system from Example 3 by the Gauss-Jordan method, and show the similarities in both methods by writing the equations next to the matrices.

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned} \tag{7.2.1}$$

Solution

The augmented matrix for the system is as follows.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right] \quad \left[\begin{array}{c} x+3y=7 \\ 3x+4y=11 \end{array}\right]$$

We multiply the first row by – 3, and add to the second row.

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right] \quad \left[\begin{array}{ll} x+3y & =7 \\ -5y & =-10 \end{array}\right]$$

We divide the second row by – 5, we get,

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{rl} x+3y & =7 \\ y & =2 \end{array}\right]$$

Finally, we multiply the second row by – 3 and add to the first row, and we get,

$$\begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \end{bmatrix} \quad \begin{bmatrix} x=1 \\ y=2 \end{bmatrix}$$

Now we list the three row operations the Gauss-Jordan method employs.

Row Operations

1. Any two rows in the augmented matrix may be interchanged.
2. Any row may be multiplied by a non-zero constant.
3. A constant multiple of a row may be added to another row.

One can easily see that these three row operation may make the system look different, but they do not change the solution of the system.

The first row operation states that if any two rows of a system are interchanged, the new system obtained has the same solution as the old one. Let us look at an example in two equations with two unknowns. Consider the system

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

We interchange the rows, and we get,

$$\begin{aligned} 3x+4y &= 11 \\ x+3y &= 7 \end{aligned}$$

Clearly, this system has the same solution as the one above.

The second operation states that if a row is multiplied by any non-zero constant, the new system obtained has the same solution as the old one. Consider the above system again,

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

We multiply the first row by –3, we get,

$$\begin{aligned} -3x-9y &= -21 \\ 3x+4y &= 11 \end{aligned}$$

Again, it is obvious that this new system has the same solution as the original.

The third row operation states that any constant multiple of one row added to another preserves the solution. Consider our system,

$$\begin{aligned} x+3y &= 7 \\ 3x+4y &= 11 \end{aligned}$$

If we multiply the first row by –3, and add it to the second row, we get,

$$\begin{aligned} x+3y &= 7 \\ -5y &= -10 \end{aligned}$$

And once again, the same solution is maintained.

Now that we understand how the three row operations work, it is time to introduce the Gauss-Jordan method to solve systems of linear equations. As mentioned earlier, the Gauss-Jordan method starts out with an augmented matrix, and by a series of row operations ends up with a matrix that is in the **reduced row echelon form**.

A matrix is in the **reduced row echelon form** if the first nonzero entry in each row is a 1, and the columns containing these 1's have all other entries as zeros. The reduced row echelon form also requires that the leading entry in each row be to the right of the leading entry in the row above it, and the rows containing all zeros be moved down to the bottom. We state the Gauss-Jordan method as follows.

Gauss-Jordan Method

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to get a 1 as the entry in the first row and first column.
4. Use row operations to make all other entries as zeros in column one.
5. Interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.
6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.

The final matrix is called the reduced row-echelon form.

Example 7.2.5

Solve the following system by the Gauss-Jordan method.

$$\begin{aligned} 2x+y+2z &= 10 \\ x+2y+z &= 8 \\ 3x+y-z &= 2 \end{aligned}$$

Solution

We write the augmented matrix.

$$\left[\begin{array}{rrrrr} 2 & 1 & 2 & | & 10 \\ 1 & 2 & 1 & | & 8 \\ 3 & 1 & -1 & | & 2 \end{array}\right]$$

We want a 1 in row one, column one. This can be obtained by dividing the first row by 2, or interchanging the second row with the first. Interchanging the rows is a better choice because that way we avoid fractions.

$$\left[\begin{array}{rrrrr} 1 & 2 & 1 & | & 8 \\ 2 & 1 & 2 & | & 10 \\ 3 & 1 & -1 & | & 2 \end{array}\right] \quad \text{we interchanged row 1 (R1) and row 2 (R2)}$$

We need to make all other entries zeros in column 1. To make the entry (2) a zero in row 2, column 1, we multiply row 1 by - 2 and add it to the second row. We get,

$$\left[\begin{array}{rrrrr} 1 & 2 & 1 & | & 8 \\ 0 & -3 & 0 & | & -6 \\ 3 & 1 & -1 & | & 2 \end{array}\right] \quad -2R1+R2$$

To make the entry (3) a zero in row 3, column 1, we multiply row 1 by - 3 and add it to the third row. We get,

$$\left[\begin{array}{rrrrr} 1 & 2 & 1 & | & 8 \\ 0 & -3 & 0 & | & -6 \\ 0 & -5 & -4 & | & -22 \end{array}\right] \quad -3R1+R3$$

So far we have made a 1 in the left corner and all other entries zeros in that column. Now we move to the next diagonal entry, row 2, column 2. We need to make this entry(–3) a 1 and make all other entries in this column zeros. To make row 2, column 2 entry a 1, we divide the entire second row by –3.

$$\left[\begin{array}{rrrrr} 1 & 2 & 1 & | & 8 \\ 0 & 1 & 0 & | & 2 \\ 0 & -5 & -4 & | & -22 \end{array}\right] \quad \text{R2} \div (-3)$$

Next, we make all other entries zeros in the second column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \quad -2R2+R1 \text{ and } 5R2+R3$$

We make the last diagonal entry a 1, by dividing row 3 by – 4.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R3 \div (-4)$$

Finally, we make all other entries zeros in column 3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad -\mathrm{R3}+\mathrm{R1}$$

Clearly, the solution reads $x=1$, $y=2$, and $z=3$.

Before we leave this section, we mention some terms we may need in Chapter 9.

The process of obtaining a 1 in a location, and then making all other entries zeros in that column, is called **pivoting**.

The number that is made a 1 is called the **pivot element,** and the row that contains the pivot element is called the **pivot row**.

We often multiply the pivot row by a number and add it to another row to obtain a zero in the latter. The row to which a multiple of pivot row is added is called the **target row**.

7.2.1: Systems of Linear Equations and the Gauss-Jordan Method (Exercises)

SECTION 7.2 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS

Solve the following by the Gauss-Jordan Method. Show all work.

1) $$\begin{aligned} x+3y&=1\\ 2x-5y&=13\end{aligned}$$	2) $$\begin{aligned} x-y-z&=-1\\ x-3y+2z&=7\\ 2x-y+z&=3\end{aligned}$$
3) $$\begin{aligned} x+2y+3z&=9\\ 3x+4y+z&=5\\ 2x-y+2z&=11\end{aligned}$$	4) $$\begin{aligned} x+2y&=0\\ y+z&=3\\ x+3z&=14\end{aligned}$$

SECTION 7.2 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS

Solve the following by the Gauss-Jordan Method. Show all work.

5) Two apples and four bananas cost \$2.00 and three apples and five bananas cost \$2.70. Find the price of each.	6) A bowl of corn flakes, a cup of milk, and an egg provide 16 grams of protein. A cup of milk and two eggs provide 21 grams of protein. Two bowls of corn flakes with two cups of milk provide 16 grams of protein. How much protein is provided by one unit of food?
7) $$\begin{aligned} x+2y&=10\\ y+z&=5\\ z+w&=3\\ x+w&=5\end{aligned}$$	8) $$\begin{aligned} x+w&=6\\ 2x+y+w&=16\\ x-2z&=0\\ z+w&=5\end{aligned}$$

7.3: Systems of Linear Equations – Special Cases

Learning Objectives

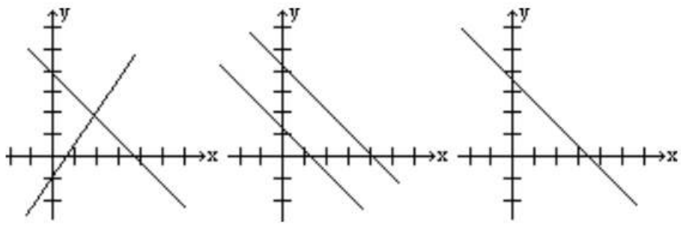
In this section, you will learn to:

1. Determine the linear systems that have no solution.
2. Solve the linear systems that have infinitely many solutions.

If we consider the intersection of two lines in a plane, three things can happen.

1. The lines intersect in exactly one point. This is called an **independent system**.
2. The lines are parallel, so they do not intersect. This is called an **inconsistent system**.
3. The lines coincide; they intersect at infinitely many points. This is a **dependent system**.

The figures below show all three cases.



Every system of equations has either one solution, no solution, or infinitely many solutions.

In the last section, we used the Gauss-Jordan method to solve systems that had exactly one solution. In this section, we will determine the systems that have no solution, and solve the systems that have infinitely many solutions.

Example 7.3.1

Solve the following system of equations:

$$\begin{aligned} x+y&=7 \\ x+y&=9 \end{aligned}$$

Solution

Let us use the Gauss-Jordan method to solve this system. The augmented matrix is

$$\left[\begin{array}{cccc} 1 & 1 & \mid & 7 \\ 1 & 1 & \mid & 9 \end{array}\right] \quad \left[\begin{array}{l} x+y=7 \\ x+y=9 \end{array}\right]$$

If we multiply the first row by - 1 and add to the second row, we get

$$\left[\begin{array}{cccc} 1 & 1 & \mid & 7 \\ 0 & 0 & \mid & 2 \end{array}\right] \quad \left[\begin{array}{l} x+y=7 \\ 0x+0y=2 \end{array}\right]$$

Since 0 cannot equal 2, the last equation cannot be true for any choices of x and y.

Alternatively, it is clear that the two lines are parallel; therefore, they do not intersect.

In the examples that follow, we are going to start using a calculator to row reduce the augmented matrix, in order to focus on understanding the answer rather than focusing on the process of carrying out the row operations.

Example 7.3.2

Solve the following system of equations.

$$\begin{aligned}2x+3y-4z&=7\\3x+4y-2z&=9\\5x+7y-6z&=20\end{aligned}$$

Solution

We enter the following augmented matrix in the calculator.

$$\begin{bmatrix}2&3&-4&|&7\\3&4&-2&|&9\\5&7&-6&|&20\end{bmatrix}$$

Now by pressing the key to obtain the reduced row-echelon(rref) form, we get

$$\begin{bmatrix}1&0&10&|&0\\0&1&-8&|&0\\0&0&0&|&1\end{bmatrix}$$

The last row states that $0x+0y+0z=1$. But the left side of the equation is equal to 0. So this last row states 0 = 1, which is a contradiction, a false statement.

This bottom row indicates that the system is inconsistent; therefore, there is no solution.

✔ Example 7.3.3

Solve the following system of equations.

$$\begin{aligned}x+y&=7\\x+y&=7\end{aligned}$$

Solution

The problem clearly asks for the intersection of two lines that are the same; that is, the lines coincide. This means the lines intersect at an infinite number of points.

A few intersection points are listed as follows: (3, 4), (5, 2), (-1, 8), (-6, 13) etc. However, when a system has an infinite number of solutions, the solution is often expressed in the parametric form. This can be accomplished by assigning an arbitrary constant, t, to one of the variables, and then solving for the remaining variables. Therefore, if we let $y=t$, then $x=7-t$. Or we can say all ordered pairs of the form $(7-t,t)$ satisfy the given system of equations.

Alternatively, while solving the Gauss-Jordan method, we will get the reduced row-echelon(rref) form given below.

$$\begin{bmatrix}1&1&|&7\\0&0&|&0\end{bmatrix}$$

The row of all zeros, can simply be ignored. This row says $0x+0y=0$; it provides no further information about the values of x and y that solve this system.

This leaves us with only one equation but two variables. And whenever there are more variables than the equations, the solution must be expressed as a parametric solution in terms of an arbitrary constant, as above.

Parametric Solution: $\mathbf{x=7-t, y=t}$.

✔ Example 7.3.4

Solve the following system of equations.

$$\begin{aligned}x+y+z&=2\\2x+y-z&=3\\3x+2y&=5\end{aligned}$$

Solution

The augmented matrix and the reduced row-echelon form are given below.

$$\begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 2 & 1 & -1 & | & 3 \\ 3 & 2 & 0 & | & 5 \end{bmatrix} \text{Augmented Matrix for this system}$$

$$\begin{bmatrix} 1 & 0 & -2 & | & 1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \text{Reduced Row Echelon Form}$$

Since the last equation dropped out, we are left with two equations and three variables. This means the system has infinite number of solutions. We express those solutions in the parametric form by letting the last variable z equal the parameter t.

The first equation reads $x - 2z = 1$, therefore, $x = 1 + 2z$.

The second equation reads $y + 3z = 1$, therefore, $y = 1 - 3z$.

And now if we let $z = t$, the parametric solution is expressed as follows:

Parametric Solution: $\mathbf{x = 1 + 2t, \quad y = 1 - 3t, \quad z = t}$ **.**

The reader should note that particular solutions, or specific solutions, to the system can be obtained by assigning values to the parameter t. For example:

- if we let $t = 2$, we have the solution $x = 5, y = -5, z = 2 : (5, -5, 2)$
- if we let $t = 0$, we have the solution $x = 1, y = 1, z = 0 : (1, 1, 0)$.

Example 7.3.5

Solve the following system of equations.

$$\begin{aligned} x + 2y - 3z &= 5 \\ 2x + 4y - 6z &= 10 \\ 3x + 6y - 9z &= 15 \end{aligned}$$

Solution

The reduced row-echelon form is given below.

$$\begin{bmatrix} 1 & 2 & -3 & | & 5 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This time the last two equations drop out. We are left with one equation and three variables. Again, there are an infinite number of solutions. But this time the answer must be expressed in terms of two arbitrary constants.
If we let $z = t$ and let $y = s$, the first equation $x + 2y - 3z = 5$ results in $x = 5 - 2s + 3t$.

We rewrite the **parametric solution:** $\mathbf{x = 5 - 2s + 3t, \quad y = s, \quad z = t}$ **.**

We summarize our discussion in the following table.

1. If any row of the reduced row-echelon(rref) form of the matrix gives a false statement such as 0 = 1, the system is inconsistent and has no solution.
2. If the reduced row echelon(rref) form has fewer equations than the variables and the system is consistent, then the system has an infinite number of solutions. Remember the rows that contain all zeros are dropped.
   a. If a system has an infinite number of solutions, the solution must be expressed in the parametric form.
   b. The number of arbitrary parameters equals the number of variables minus the number of equations.

7.3.1: Systems of Linear Equations – Special Cases (Exercises)

SECTION 7.3 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS - SPECIAL CASES

Solve the following inconsistent or dependent systems by using the Gauss-Jordan method.

1. $\begin{aligned} 2x+6y&=8 \\ x+3y&=4 \end{aligned}$	2. The sum of digits of a two digit number is 9. The sum of the number and the number obtained by interchanging the digits is 99. Find the number.
3. $\begin{aligned} 2x-y&=10 \\ -4x+2y&=15 \end{aligned}$	4. $\begin{aligned} x+y+z&=6 \\ 3x+2y+z&=14 \\ 4x+3y+2z&=20 \end{aligned}$
5. $\begin{aligned} x+2y-4z&=1 \\ 2x-3y+8z&=9 \end{aligned}$	6. Jessica has a collection of 15 coins consisting of nickels, dimes and quarters. If the total worth of the coins is $1.80, how many are there of each? Find all three solutions.

SECTION 7.3 PROBLEM SET: SYSTEMS OF LINEAR EQUATIONS - SPECIAL CASES

Solve the following inconsistent or dependent systems by using the Gauss-Jordan method.

7. A company is analyzing sales reports for three products: products X, Y, Z. One report shows that a combined total of 20,000 of items X, Y, and Z were sold. Another report shows that the sum of the number of item Z sold and twice the number of item X sold equals 10,000. Also item X has 5,000 more items sold than item Y. Are these reports consistent?	8. $\begin{aligned} x+y+2z&=0 \\ x+2y+z&=0 \\ 2x+3y+3z&=0 \end{aligned}$
9. Find three solutions to the following system of equations. $\begin{aligned} x+2y+z&=12 \\ y&=3 \end{aligned}$	10. $\begin{aligned} x+2y&=5 \\ 2x+4y&=k \end{aligned}$ For what values of k does this system of equations have a. No solution? b. Infinitely many solutions?
11. $x+3y-z=5$	12. Why is it not possible for a linear system to have exactly two solutions? Explain geometrically.

7.4: Inverse Matrices

Learning Objectives

In this section, you will learn to:

1. Find the inverse of a matrix, if it exists.
2. Use inverses to solve linear systems.

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems.

Definition of an Inverse: An $n \times n$ matrix has an inverse if there exists a matrix B such that $AB = BA = I_n$, where I_n is an $n \times n$ identity matrix. The inverse of a matrix A, if it exists, is denoted by the symbol A^{-1}.

Example 7.4.1

Given matrices A and B below, verify that they are inverses.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

Solution

The matrices are inverses if the product AB and BA both equal the identity matrix of dimension 2×2: I_2,

$$\mathrm{AB} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2$$

and

$$\mathrm{BA} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2$$

Clearly that is the case; therefore, the matrices A and B are inverses of each other.

Example 7.4.2

Find the inverse of the matrix $\mathrm{A} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

Solution

Suppose A has an inverse, and it is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $AB = I_2$: $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

After multiplying the two matrices on the left side, we get

$$\begin{bmatrix} 3a + c & 3b + d \\ 5a + 2c & 5b + 2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries, we get four equations with four unknowns:

$$\begin{array}{ll} 3a + c = 1 & 3b + d = 0 \\ 5a + 2c = 0 & 5b + 2d = 1 \end{array}$$

Solving this system, we get: $a = 2 \quad b = -1 \quad c = -5 \quad d = 3$

Therefore, the inverse of the matrix A is $B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

In this problem, finding the inverse of matrix A amounted to solving the system of equations:

$$\begin{array}{ll} 3a + c = 1 & 3b + d = 0 \\ 5a + 2c = 0 & 5b + 2d = 1 \end{array}$$

Actually, it can be written as two systems, one with variables a and c, and the other with b and d. The augmented matrices for both are given below.

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 5 & 2 & 0 \end{array}\right] \text{ and } \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 5 & 2 & 1 \end{array}\right]$$

As we look at the two augmented matrices, we notice that the coefficient matrix for both the matrices is the same. This implies the row operations of the Gauss-Jordan method will also be the same. A great deal of work can be saved if the two right hand columns are grouped together to form one augmented matrix as below.

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right]$$

And solving this system, we get

The matrix on the right side of the vertical line is the A^{-1} matrix.

What you just witnessed is no coincidence. This is the method that is often employed in finding the inverse of a matrix. We list the steps, as follows:

The Method for Finding the Inverse of a Matrix

1. Write the augmented matrix $[A|I_n]$.
2. Write the augmented matrix in step 1 in reduced row echelon form(rref).
3. If the reduced row echelon form (rref) in 2 is $[I_n|B]$, then B is the inverse of A.
4. If the left side of the row reduced echelon form (rref) is not an identity matrix, the inverse does not exist.

Example 7.4.3

Given the matrix A below, find its inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution

We write the augmented matrix as follows.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

We will reduce this matrix using the Gauss-Jordan method.

Multiplying the first row by -2 and adding it to the second row, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

If we swap the second and third rows, we get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Divide the second row by -2. The result is

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 5 & -2 & -2 & 1 & 0 \end{array}\right]$$

Let us do two operations here. 1) Add the second row to first, 2) Add -5 times the second row to the third. And we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & -2 & 1 & 5/2 \end{array}\right]$$

Multiplying the third row by 2 results in

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Multiply the third row by 1/2 and add it to the second.
Also, multiply the third row by -1/2 and add it to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Therefore, the inverse of matrix A is $\mathrm{A}^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

One should verify the result by multiplying the two matrices to see if the product does, indeed, equal the identity matrix.

Now that we know how to find the inverse of a matrix, we will use inverses to solve systems of equations. The method is analogous to solving a simple equation like the one below.

$$\frac{2}{3}x = 4$$

Example 7.4.4

Solve the following equation: $\frac{2}{3}x = 4$

Solution

To solve the above equation, we multiply both sides of the equation by the multiplicative inverse of $\frac{2}{3}$ which happens to be $\frac{3}{2}$. We get

$$\frac{3}{2} \cdot \frac{2}{3}x = 4 \cdot \frac{3}{2}$$
$$x = 6$$

We use the Example $7.4.4$ as an analogy to show how linear systems of the form $AX = B$ are solved.

To solve a linear system, we first write the system in the matrix equation $AX = B$, where A is the coefficient matrix, X the matrix of variables, and B the matrix of constant terms.
We then multiply both sides of this equation by the multiplicative inverse of the matrix A.

Consider the following example.

Example 7.4.5

Solve the following system

$$\begin{aligned} 3x+y&=3 \\ 5x+2y&=4 \end{aligned}$$

Solution

To solve the above equation, first we express the system as

$$AX=B$$

where A is the coefficient matrix, and B is the matrix of constant terms. We get

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

To solve this system, we multiply both sides of the matrix equation $AX=B$ by A^{-1}. Matrix multiplication is not commutative, so we need to multiply by A^{-1} on the left on both sides of the equation.

Matrix A is the same matrix A whose inverse we found in Example $7.4.2$, so $\mathrm{A}^{-1}=\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Multiplying both sides by A^{-1}, we get

$$\begin{gathered} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 2 \\ -3 \end{bmatrix} \end{gathered}$$

Therefore, $x=2$, and $y=-3$.

Example 7.4.6

Solve the following system:

$$\begin{aligned} x-y+z&=6 \\ 2x+3y&=1 \\ -2y+z&=5 \end{aligned}$$

Solution

To solve the above equation, we write the system in matrix form $AX=B$ as follows:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

To solve this system, we need inverse of A. From Example $7.4.3$, $\mathrm{A}^{-1}=\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

Multiplying both sides of the matrix equation $AX=B$ on the left by A^{-1}, we get

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

After multiplying the matrices, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

We remind the reader that not every system of equations can be solved by the matrix inverse method. Although the Gauss-Jordan method works for every situation, the matrix inverse method works only in cases where the inverse of the square matrix exists. In such cases the system has a unique solution.

The Method for Finding the Inverse of a Matrix

1. Write the augmented matrix $[\mathrm{A}|\mathrm{I_n}]$.
2. Write the augmented matrix in step 1 in reduced row echelon form(rref) .
3. If the reduced row echelon form(rref) in 2 is $[\mathrm{I_n}|\mathrm{B}]$, then B is the inverse of A.
4. If the left side of the row reduced echelon form(rref) is not an identity matrix, the inverse does not exist.

The Method for Solving a System of Equations When a Unique Solution Exists

1. Express the system in the matrix equation $AX = B$.

2. To solve the equation $AX = B$, we multiply on both sides by A^{-1}.

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B \text{ where } I \text{ is the identity matrix}$$

7.4.1: Inverse Matrices (Exercises)

SECTION 7.4 PROBLEM SET: INVERSE MATRICES

In problems 1- 2, verify that the given matrices are inverses of each other.

1. $\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$	2. $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 3 & -4 & 1 \\ 2 & -4 & 1 \\ 3 & -5 & 1 \end{bmatrix}$

In problems 3- 6, find the inverse of each matrix by the row-reduction method.

3. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$	4. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

SECTION 7.4 PROBLEM SET: INVERSE MATRICES

In problems 5 - 6, find the inverse of each matrix by the row-reduction method.

5. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$	6. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Problems 7 -10: Express the system as $AX = B$; then solve using matrix inverses found in problems 3 - 6.

7. $\begin{array}{l} 3x - 5y = 2 \\ -x + 2y = 0 \end{array}$	8. $\begin{array}{r} x + \quad 2z = 8 \\ y + 4z = 8 \\ z = 3 \end{array}$

SECTION 7.4 PROBLEM SET: INVERSE MATRICES

Problems 9 -10: Express the system as $AX = B$; then solve using matrix inverses found in problems 3 - 6.

9. $\begin{array}{rl} x + y - z & = 2 \\ x + z & = 7 \\ 2x + y + z & = 13 \end{array}$	10. $\begin{array}{l} x + y + z = 2 \\ 3x + y = 7 \\ x + y + 2z = 3 \end{array}$
11. Why is it necessary that a matrix be a square matrix for its inverse to exist? Explain by relating the matrix to a system of equations.	12. Suppose we are solving a system $AX = B$ by the matrix inverse method, but discover A has no inverse. How else can we solve this system? What can be said about the solutions of this system?

7.5: Application of Matrices in Cryptography

Learning Objectives

In this section, we will learn to find the inverse of a matrix, if it exists. Later, we will use matrix inverses to solve linear systems. In this section you will learn to

1. encode a message using matrix multiplication.
2. decode a coded message using the matrix inverse and matrix multiplication

Encryption dates back approximately 4000 years. Historical accounts indicate that the Chinese, Egyptians, Indian, and Greek encrypted messages in some way for various purposes. One famous encryption scheme is called the Caesar cipher, also called a substitution cipher, used by Julius Caesar, involved shifting letters in the alphabet, such as replacing A by C, B by D, C by E, etc, to encode a message. Substitution ciphers are too simple in design to be considered secure today.

In the middle ages, European nations began to use encryption. A variety of encryption methods were used in the US from the Revolutionary War, through the Civil War, and on into to modern times.

Applications of mathematical theory and methods to encryption became widespread in military usage in the 20$^{\text{th}}$ century. The military would encode messages before sending and the recipient would decode the message, in order to send information about military operations in a manner that kept the information safe if the message was intercepted. In World War II, encryption played an important role, as both Allied and Axis powers sent encrypted messages and devoted significant resources to strengthening their own encryption while also trying to break the opposition's encryption.

In this section we will examine a method of encryption that uses matrix multiplication and matrix inverses. This method, known as the Hill Algorithm, was created by Lester Hill, a mathematics professor who taught at several US colleges and also was involved with military encryption. The Hill algorithm marks the introduction of modern mathematical theory and methods to the field of cryptography.

These days, the Hill Algorithm is not considered a secure encryption method; it is relatively easy to break with modern technology. However, in 1929 when it was developed, modern computing technology did not exist. This method, which we can handle easily with today's technology, was too cumbersome to use with hand calculations. Hill devised a mechanical encryption machine to help with the mathematics; his machine relied on gears and levers, but never gained widespread use. Hill's method was considered sophisticated and powerful in its time and is one of many methods influencing techniques in use today. Other encryption methods at that time also utilized special coding machines. Alan Turing, a computer scientist pioneer in the field of artificial intelligence, invented a machine that was able to decrypt messages encrypted by the German Enigma machine, helping to turn the tide of World War II.

With the advent of the computer age and internet communication, the use of encryption has become widespread in communication and in keeping private data secure; it is no longer limited to military uses. Modern encryption methods are more complicated, often combining several steps or methods to encrypt data to keep it more secure and harder to break. Some modern methods make use of matrices as part of the encryption and decryption process; other fields of mathematics such as number theory play a large role in modern cryptography.

To use matrices in encoding and decoding secret messages, our procedure is as follows.

We first convert the secret message into a string of numbers by arbitrarily assigning a number to each letter of the message. Next we convert this string of numbers into a new set of numbers by multiplying the string by a square matrix of our choice that has an inverse. This new set of numbers represents the coded message.

To decode the message, we take the string of coded numbers and multiply it by the inverse of the matrix to get the original string of numbers. Finally, by associating the numbers with their corresponding letters, we obtain the original message.

In this section, we will use the correspondence shown below where letters A to Z correspond to the numbers 1 to 26, a space is represented by the number 27, and punctuation is ignored.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Example 7.5.1

Use matrix $A=\begin{bmatrix}1&2\\1&3\end{bmatrix}$ to encode the message: ATTACK NOW!

Solution

We divide the letters of the message into groups of two.

$$\text{AT} \quad \text{TA} \quad \text{CK} \quad \text{-N} \quad \text{OW}$$

We assign the numbers to these letters from the above table, and convert each pair of numbers into 2×1 matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix}\text{A}\\\text{T}\end{bmatrix}=\begin{bmatrix}1\\20\end{bmatrix} \quad \begin{bmatrix}\text{T}\\\text{A}\end{bmatrix}=\begin{bmatrix}20\\1\end{bmatrix} \quad \begin{bmatrix}\text{C}\\\text{K}\end{bmatrix}=\begin{bmatrix}3\\11\end{bmatrix}$$

$$\begin{bmatrix}-\\\text{N}\end{bmatrix}=\begin{bmatrix}27\\14\end{bmatrix} \quad \begin{bmatrix}\text{O}\\\text{W}\end{bmatrix}=\begin{bmatrix}15\\23\end{bmatrix}$$

So at this stage, our message expressed as 2×1 matrices is as follows.

$$\begin{bmatrix}1\\20\end{bmatrix}\begin{bmatrix}20\\1\end{bmatrix}\begin{bmatrix}3\\11\end{bmatrix}\begin{bmatrix}27\\14\end{bmatrix}\begin{bmatrix}15\\23\end{bmatrix} \quad \textbf{(I)}$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A. For example, the product of A with our first matrix is: $\begin{bmatrix}1&2\\1&3\end{bmatrix}\begin{bmatrix}1\\20\end{bmatrix}=\begin{bmatrix}41\\61\end{bmatrix}$

And the product of A with our second matrix is: $\begin{bmatrix}1&2\\1&3\end{bmatrix}\begin{bmatrix}20\\1\end{bmatrix}=\begin{bmatrix}22\\23\end{bmatrix}$

Multiplying each matrix in $\textbf{(I)}$ by matrix A, in turn, gives the desired coded message:

$$\begin{bmatrix}41\\61\end{bmatrix}\begin{bmatrix}22\\23\end{bmatrix}\begin{bmatrix}25\\36\end{bmatrix}\begin{bmatrix}55\\69\end{bmatrix}\begin{bmatrix}61\\84\end{bmatrix}$$

Example 7.5.2

Decode the following message that was encoded using matrix $A=\begin{bmatrix}1&2\\1&3\end{bmatrix}$.

$$\begin{bmatrix}21\\26\end{bmatrix}\begin{bmatrix}37\\53\end{bmatrix}\begin{bmatrix}45\\54\end{bmatrix}\begin{bmatrix}74\\101\end{bmatrix}\begin{bmatrix}53\\69\end{bmatrix} \quad \textbf{(II)}$$

Solution

Since this message was encoded by multiplying by the matrix A in Example 7.5.1, we decode this message by first multiplying each matrix, on the left, by the inverse of matrix A given below.

$$A^{-1}=\begin{bmatrix}3&-2\\-1&1\end{bmatrix}$$

For example: $\begin{bmatrix}3&-2\\-1&1\end{bmatrix}\begin{bmatrix}21\\26\end{bmatrix}=\begin{bmatrix}11\\5\end{bmatrix}$

By multiplying each of the matrices in $\textbf{(II)}$ by the matrix A^{-1}, we get the following.

$$\begin{bmatrix} 11 \\ 5 \end{bmatrix} \begin{bmatrix} 5 \\ 16 \end{bmatrix} \begin{bmatrix} 27 \\ 9 \end{bmatrix} \begin{bmatrix} 20 \\ 27 \end{bmatrix} \begin{bmatrix} 21 \\ 16 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix} \mathrm{K} \\ \mathrm{E} \end{bmatrix} \begin{bmatrix} \mathrm{E} \\ \mathrm{P} \end{bmatrix} \begin{bmatrix} - \\ \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathrm{T} \\ - \end{bmatrix} \begin{bmatrix} \mathrm{U} \\ \mathrm{P} \end{bmatrix}$$

And the message reads: KEEP IT UP.

Now suppose we wanted to use a 3×3 matrix to encode a message, then instead of dividing the letters into groups of two, we would divide them into groups of three.

Example 7.5.3

Using the matrix $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, encode the message: ATTACK NOW!

Solution

We divide the letters of the message into groups of three.

ATT ACK -NO W- -

Note that since the single letter "W" was left over on the end, we added two spaces to make it into a triplet.

Now we assign the numbers their corresponding letters from the table, and convert each triplet of numbers into 3×1 matrices. We get

$$\begin{bmatrix} \mathrm{A} \\ \mathrm{T} \\ \mathrm{T} \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} \begin{bmatrix} \mathrm{A} \\ \mathrm{C} \\ \mathrm{K} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} - \\ \mathrm{N} \\ \mathrm{O} \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix} \begin{bmatrix} \mathrm{W} \\ - \\ - \end{bmatrix} = \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix}$$

So far we have,

$$\begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix} \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix} \quad \textbf{(III)}$$

We multiply, on the left, each matrix of our message by the matrix B. For example,

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}$$

By multiplying each of the matrices in $\textbf{(III)}$ by the matrix B, we get the desired coded message as follows:

$$\begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix} \begin{bmatrix} -7 \\ 12 \\ 16 \end{bmatrix} \begin{bmatrix} 26 \\ 42 \\ 83 \end{bmatrix} \begin{bmatrix} 23 \\ 50 \\ 100 \end{bmatrix}$$

If we need to decode this message, we simply multiply the coded message by B^{-1}, and associate the numbers with the corresponding letters of the alphabet.

In Example $7.5.4$ we will demonstrate how to use matrix B^{-1} to decode an encrypted message.

Example 7.5.4

Decode the following message that was encoded using matrix $B=\begin{bmatrix}1 & 1 & -1\\ 1 & 0 & 1\\ 2 & 1 & 1\end{bmatrix}$.

$$\begin{bmatrix}11\\20\\43\end{bmatrix}\begin{bmatrix}25\\10\\41\end{bmatrix}\begin{bmatrix}22\\14\\41\end{bmatrix} \quad \textbf{(IV)}$$

Solution

Since this message was encoded by multiplying by the matrix B. We first determine inverse of B.

$$\mathrm{B}^{-1}=\begin{bmatrix}1 & 2 & -1\\ -1 & -3 & 2\\ -1 & -1 & 1\end{bmatrix}$$

To decode the message, we multiply each matrix, on the left, by B^{-1}. For example,

$$\begin{bmatrix}1 & 2 & -1\\ -1 & -3 & 2\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}11\\20\\43\end{bmatrix}=\begin{bmatrix}8\\15\\12\end{bmatrix}$$

Multiplying each of the matrices in $\textbf{(IV)}$ by the matrix B^{-1} gives the following.

$$\begin{bmatrix}8\\15\\12\end{bmatrix}\begin{bmatrix}4\\27\\6\end{bmatrix}\begin{bmatrix}9\\18\\5\end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain

$$\begin{bmatrix}\mathrm{H}\\\mathrm{O}\\\mathrm{L}\end{bmatrix}\begin{bmatrix}\mathrm{D}\\-\\\mathrm{F}\end{bmatrix}\begin{bmatrix}\mathrm{I}\\\mathrm{R}\\\mathrm{E}\end{bmatrix} \quad \text{The message reads: HOLD FIRE}$$

The message reads: HOLD FIRE.

We summarize:

TO ENCODE A MESSAGE

1. Divide the letters of the message into groups of two or three.

2. Convert each group into a string of numbers by assigning a number to each letter of the message. Remember to assign letters to blank spaces.

3. Convert each group of numbers into column matrices.

3. Convert these column matrices into a new set of column matrices by multiplying them with a compatible square matrix of your choice that has an inverse. This new set of numbers or matrices represents the coded message.

TO DECODE A MESSAGE

1. Take the string of coded numbers and multiply it by the inverse of the matrix that was used to encode the message.

2. Associate the numbers with their corresponding letters.

7.5.1: Application of Matrices in Cryptography (Exercises)

SECTION 7.5 PROBLEM SET: APPLICATION OF MATRICES IN CRYPTOGRAPHY

In problems 1 - 8, the letters A to Z correspond to the numbers 1 to 26, as shown below, and a space is represented by the number 27.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

In problems 1 - 2, use the matrix A, given below, to encode the given messages.

$$A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$$

In problems 3 - 4, decode the messages that were encoded using matrix A.
Make sure to consider the spaces between words, but ignore all punctuation. Add a final space if necessary.

1. Encode the message: WATCH OUT!	2. Encode the message: HELP IS ON THE WAY.
3. Decode the following message: 64 23 102 41 82 32 97 35 71 28 69 32	4. Decode the following message: 105 40 117 48 39 19 69 32 72 27 37 15 114 47

SECTION 7.5 PROBLEM SET: APPLICATION OF MATRICES IN CRYPTOGRAPHY

In problems 5 - 6, use the matrix B, given below, to encode the given messages.

$$\mathrm{B}=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 2\\ 1 & 0 & -1\end{bmatrix}$$

In problems 7 - 8, decode the messages that were encoded using matrix B.

Make sure to consider the spaces between words, but ignore all punctuation. Add a final space if necessary.

5. Encode the message using matrix B: LUCK IS ON YOUR SIDE.	6. Encode the message using matrix B: MAY THE FORCE BE WITH YOU.
7. Decode the following message that was encoded using matrix B: 8 23 7 4 47 - 2 15 102 -12 20 58 15 27 80 18 12 74 -7	8. Decode the following message that was encoded using matrix B: 12 69 - 3 11 53 9 5 46 -10 18 95 - 9 25 107 4 27 76 22 1 72 -26

7.6: Applications – Leontief Models

Learning Objectives

In this section we will examine an application of matrices to model economic systems.

In the 1930's, Wassily Leontief used matrices to model economic systems. His models, often referred to as the input-output models, divide the economy into sectors where each sector produces goods and services not only for itself but also for other sectors. These sectors are dependent on each other and the total input always equals the total output. In 1973, he won the Nobel Prize in Economics for his work in this field. In this section we look at both the closed and the open models that he developed.

The Closed Model

As an example of the closed model, we look at a very simple economy, where there are only three sectors: food, shelter, and clothing.

Example 7.6.1

We assume that in a village there is a farmer, carpenter, and a tailor, who provide the three essential goods: food, shelter, and clothing. Suppose the farmer himself consumes 40% of the food he produces, and gives 40% to the carpenter, and 20% to the tailor. Thirty percent of the carpenter's production is consumed by himself, 40% by the farmer, and 30% by the carpenter. Fifty percent of the tailor's production is used by himself, 30% by the farmer, and 20% by the tailor. Write the matrix that describes this closed model.

Solution

The table below describes the above information.

	Proportion produced by the farmer	Proportion produced by the carpenter	Proportion produced by the tailor
The proportion used by the farmer	.40	.40	.30
The proportion used by the carpenter	.40	.30	.20
The proportion used by the tailor	.20	.30	.50

In a matrix form it can be written as follows.

$$A=\begin{bmatrix} .40 & .40 & .30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix}$$

This matrix is called the **input-output matrix.** It is important that we read the matrix correctly. For example the entry A_{23}, the entry in row 2 and column 3, represents the following.

A_{23} = 20% of the tailor's production is used by the carpenter.

A_{33} = 50% of the tailor's production is used by the tailor.

Example 7.6.2

In Example $7.6.1$ above, how much should each person get for his efforts?

Solution

We choose the following variables.

x = Farmer's pay y = Carpenter's pay z = Tailor's pay

As we said earlier, in this model input must equal output. That is, the amount paid by each equals the amount received by each.

Let us say the farmer gets paid x dollars. Let us now look at the farmer's expenses. The farmer uses up 40% of his own production, that is, of the x dollars he gets paid, he pays himself .40x dollars, he pays .40y dollars to the carpenter, and .30z to the tailor. Since the expenses equal the wages, we get the following equation.

$$x = .40x + .40y + .30z$$

In the same manner, we get

$$\begin{aligned} y &= .40x + .30y + .20z \\ z &= .20x + .30y + .50z \end{aligned}$$

The above system can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .40 & .40 & 30 \\ .40 & .30 & .20 \\ .20 & .30 & .50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This system is often referred to as $X = AX$

Simplification results in the system of equations $(I - A)X = 0$

$$\begin{aligned} .60x - .40y - .30z &= 0 \\ -.40x + .70y - .20z &= 0 \\ -.20x - .30y + .50z &= 0 \end{aligned}$$

Solving for x, y, and z using the Gauss-Jordan method, we get

$$x = \frac{29}{26}t \quad y = \frac{12}{13}t \quad \text{ and } z = t$$

Since we are only trying to determine the proportions of the pay, we can choose t to be any value. Suppose we let \(t\(= $2600, then we get

$$x = \$2900 \quad y = \$2400 \quad \text{ and } z = \$2600$$

Note: The use of a graphing calculator or computer application in solving the systems of linear matrix equations in these problems is strongly recommended.

The Open Model

The open model is more realistic, as it deals with the economy where sectors of the economy not only satisfy each other's' needs, but they also satisfy some outside demands. In this case, the outside demands are put on by the consumer. But the basic assumption is still the same; that is, whatever is produced is consumed.

Let us again look at a very simple scenario. Suppose the economy consists of three people, the farmer F, the carpenter C, and the tailor T. A part of the farmer's production is used by all three, and the rest is used by the consumer. In the same manner, a part of the carpenter's and the tailor's production is used by all three, and rest is used by the consumer.

Let us assume that whatever the farmer produces, 20% is used by him, 15% by the carpenter, 10% by the tailor, and the consumer uses the other 40 billion dollars worth of the food. Ten percent of the carpenter's production is used by him, 25% by the farmer, 5% by the tailor, and 50 billion dollars worth by the consumer. Fifteen percent of the clothing is used by the tailor, 10% by the farmer, 5% by the carpenter, and the remaining 60 billion dollars worth by the consumer. We write the internal consumption in the following table, and express the demand as the matrix D.

	F produces	C produces	T produces
F uses	.20	.25	.10
C uses	.15	.10	.05

T uses	.10	.05	.15

The consumer demand for each industry in billions of dollars is given below.

$$\mathrm{D}=\begin{bmatrix}40\\50\\60\end{bmatrix}$$

Example 7.6.3

In the example above, what should be, in billions of dollars, the required output by each industry to meet the demand given by the matrix D?

Solution

We choose the following variables.

x = Farmer's output

y = Carpenter's output

z = Tailor's output

In the closed model, our equation was $X=AX$, that is, the total input equals the total output. This time our equation is similar with the exception of the demand by the consumer.

So our equation for the open model should be $X=AX+D$, where D represents the demand matrix.

We express it as follows:

$$X=AX+D$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}.20&.25&.10\\.15&.10&.05\\.10&.05&.15\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}40\\50\\60\end{bmatrix}$$

To solve this system, we write it as

$$\begin{aligned}&X=AX+D\\&(I-A)X=D \quad \text{where I is a 3 by 3 identity matrix}\\&X=(I-A)^{-1}D\end{aligned}$$

$$\mathrm{I}-\mathrm{A}=\begin{bmatrix}.80&-.25&-.10\\-.15&.90&-.05\\-.10&-.05&.85\end{bmatrix}$$

$$(\mathrm{I}-\mathrm{A})^{-1}=\begin{bmatrix}1.3445&.3835&.1807\\.2336&1.1814&.097\\.1719&.1146&1.2034\end{bmatrix}$$

$$X=\begin{bmatrix}1.3445&.3835&.1807\\.2336&1.1814&.097\\.1719&.1146&1.2034\end{bmatrix}\begin{bmatrix}40\\50\\60\end{bmatrix}$$

$$X=\begin{bmatrix}83.7999\\74.2341\\84.8138\end{bmatrix}$$

The three industries must produce the following amount of goods in billions of dollars.

Farmer = \$83.7999 Carpenter = \$74.2341 Tailor = \$84.813

We will do one more problem like the one above, except this time we give the amount of internal and external consumption in dollars and ask for the proportion of the amounts consumed by each of the industries. In other words, we ask for the matrix A.

Example 7.6.4

Suppose an economy consists of three industries F, C, and T. Each of the industries produces for internal consumption among themselves, as well as for external demand by the consumer. The table shows the use of each industry's production ,in dollars.

	F	C	T	Demand	Total
F	40	50	60	100	250
C	30	40	40	110	220
T	20	30	30	120	200

The first row says that of the \$250 dollars worth of production by the industry F, \$40 is used by F, \$50 is used by C, \$60 is used by T, and the remainder of \$100 is used by the consumer. The other rows are described in a similar manner.

Once again, the total input equals the total output. Find the proportion of the amounts consumed by each of the industries. In other words, find the matrix A.

Solution

We are being asked to determine the following:

How much of the production of each of the three industries, F, C, and T is required to produce one unit of F? In the same way, how much of the production of each of the three industries, F, C, and T is required to produce one unit of C? And finally, how much of the production of each of the three industries, F, C, and T is required to produce one unit of T?

Since we are looking for proportions, we need to divide the production of each industry by the total production for each industry.

We analyze as follows:

To produce 250 units of F, we need to use 40 units of F, 30 units of C, and 20 units of T.

Therefore, to produce 1 unit of F, we need to use 40/250 units of F, 30/250 units of C, and 20/250 units of T.

To produce 220 units of C, we need to use 50 units of F, 40 units of C, and 30 units of T.

Therefore, to produce 1 unit of C, we need to use 50/220 units of F, 40/220 units of C, and 30/220 units of T.

To produce 200 units of T, we need to use 60 units of F, 40 units of C, and 30 units of T.

Therefore, to produce 1 unit of T, we need to use 60/200 units of F, 40/200 units of C, and 30/200 units of T.

We obtain the following matrix.

$$\mathrm{A}=\begin{bmatrix}40/250 & 50/220 & 60/200\\ 30/250 & 40/220 & 40/200\\ 20/250 & 30/220 & 30/200\end{bmatrix}=\begin{bmatrix}.1600 & .2273 & .3000\\ .1200 & .1818 & .2000\\ .0800 & .1364 & .1500\end{bmatrix}$$

Clearly $AX+D=X$

$$\begin{bmatrix}40/250 & 50/220 & 60/200\\ 30/250 & 40/220 & 40/200\\ 20/250 & 30/220 & 30/200\end{bmatrix}\begin{bmatrix}250\\ 220\\ 200\end{bmatrix}+\begin{bmatrix}100\\ 110\\ 120\end{bmatrix}=\begin{bmatrix}250\\ 220\\ 200\end{bmatrix}$$

We summarize as follows:

LEONTIEF'S CLOSED MODEL

1. All consumption is within the industries. There is no external demand.
2. Input = Output

3. $X = AX$ or $(I - A)X = 0$

LEONTIEF'S OPEN MODEL

1. In addition to internal consumption, there is an outside demand by the consumer.
2. Input = Output
3. $X = AX + D$ or $X = (I - A)^{-1}D$

7.6.1: Applications – Leontief Models (Exercises)

SECTION 7.6 PROBLEM SET: APPLICATIONS - LEONTIEF MODELS

1) Solve the following homogeneous system.

$$\begin{aligned} x+y+z &= 0 \\ 3x+2y+z &= 0 \\ 4x+3y+2z &= 0 \end{aligned}$$

2) Solve the following homogeneous system.

$$\begin{aligned} x-y-z &= 0 \\ x-3y+2z &= 0 \\ 2x-4y+z &= 0 \end{aligned}$$

3) Chris and Ed decide to help each other by doing repairs on each others houses. Chris is a carpenter, and Ed is an electrician. Chris does carpentry work on his house as well as on Ed's house. Similarly, Ed does electrical repairs on his house and on Chris' house. When they are all finished they realize that Chris spent 60% of his time on his own house, and 40% of his time on Ed's house. On the other hand Ed spent half of his time on his house and half on Chris's house. If they originally agreed that each should get about a $1000 for their work, how much money should each get for their work?

4) Chris, Ed, and Paul decide to help each other by doing repairs on each others houses. Chris is a carpenter, Ed is an electrician, and Paul is a plumber. Each does work on his own house as well as on the others houses. When they are all finished they realize that Chris spent 30% of his time on his own house, 40% of his time on Ed's house, and 30% on Paul's house. Ed spent half of his time on his own house, 30% on Chris' house, and remaining on Paul's house. Paul spent 40% of the time on his own house, 40% on Chris' house, and 20% on Ed's house. If they originally agreed that each should get about a $1000 for their work, how much money should each get for their work?

SECTION 7.6 PROBLEM SET: APPLICATIONS - LEONTIEF MODELS

5) Given the internal consumption matrix A, and the external demand matrix D as follows.

$$A=\begin{bmatrix} .30 & .20 & .10 \\ .20 & .10 & .30 \\ .10 & .20 & .30 \end{bmatrix} \quad D=\begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix}$$

Solve the system using the open model: $X=AX+D$ or $X=(I-A)^{-1}D$

6) Given the internal consumption matrix A, and the external demand matrix D as follows.

$$A=\begin{bmatrix} .05 & .10 & .10 \\ .10 & .15 & .05 \\ .05 & .20 & .20 \end{bmatrix} \quad D=\begin{bmatrix} 50 \\ 100 \\ 80 \end{bmatrix}$$

Solve the system using the open model: $X=AX+D$ or $X=(I-A)^{-1}D$

7) An economy has two industries, farming and building. For every $1 of food produced, the farmer uses $.20 and the builder uses $.15. For every $1 worth of building, the builder uses $.25 and the farmer uses $.20. If the external demand for food is $100,000, and for building $200,000, what should be the total production for each industry in dollars?

SECTION 7.6 PROBLEM SET: APPLICATIONS - LEONTIEF MODELS

8) An economy has three industries, farming, building, and clothing. For every $1 of food produced, the farmer uses $.20, the builder uses $.15, and the tailor $.05. For every $1 worth of building, the builder uses $.25, the farmer uses $.20, and the tailor $.10. For every $1 worth of clothing, the tailor uses $.10, the builder uses $.20, the farmer uses $.15. If the external demand for food is $100 million, for building $200 million, and for clothing $300 million, what should be the total production for each in dollars?

9) Suppose an economy consists of three industries F, C, and T. The following table gives information about the internal use of each industry's production and external demand in dollars.

	F	C	T	Demand	Total
F	30	10	20	40	100
C	20	30	20	50	120
T	10	10	30	60	110

Find the proportion of the amounts consumed by each of the industries; that is, find the matrix A.

10) If in problem 9, the consumer demand for F, C, and T becomes 60, 80, and 100, respectively, find the total output and the internal use by each industry to meet that demand.

7.7: Chapter Review

SECTION 7.7 PROBLEM SET: CHAPTER REVIEW

1. To reinforce her diet, Mrs. Tam bought a bottle containing 30 tablets of Supplement A and a bottle containing 50 tablets of Supplement B. Each tablet of supplement A contains 1000 mg of calcium, 400 mg of magnesium, and 15 mg of zinc, and each tablet of supplement B contains 800 mg of calcium, 500 mg of magnesium, and 20 mg of zinc.}
   a. Represent the amount of calcium, magnesium and zinc in each tablet as a 2×3 matrix.
   b. Represent the number of tablets in each bottle as a row matrix.
   c. Use matrix multiplication to find the total amount of calcium, magnesium, and zinc in both bottles.

2. Let matrix $A=\begin{bmatrix}1 & -1 & 3\\ 3 & -2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}3 & 3 & -1\\ 1 & 4 & -3\end{bmatrix}$. Find the following.
   a. $\frac{1}{2}(A+B)$
   b. $3A=2B$

3. Let matrix $\mathrm{C}=\begin{bmatrix}1 & 1 & -1\\ 2 & 1 & 1\\ 1 & 0 & 1\end{bmatrix}$ and $D=\begin{bmatrix}2 & -3 & -1\\ 3 & -1 & -2\\ 3 & -3 & -2\end{bmatrix}$. Find the following.
   a. $2(C-D)$
   b. $C-3D$

4. Let matrix $E=\begin{bmatrix}1 & -1\\ 2 & 3\\ 1 & 2\end{bmatrix}$ and $\mathrm{F}=\begin{bmatrix}2 & 1 & -1\\ 1 & 2 & -3\end{bmatrix}$. Find the following.
   a. $2EF$
   b. $3FE$

5. Let matrix $G=\begin{bmatrix}1 & -1 & 3\\ 3 & 2 & 1\end{bmatrix}$ and $H=\begin{bmatrix}a & b\\ c & d\\ e & f\end{bmatrix}$. Find the following.
   a. $2GH$
   b. HG

6. Solve the following systems using the Gauss-Jordan Method.
   a. $$\begin{aligned} x+3y-2z&=7\\ 2x+7y-5z&=16\\ x+5y-3z&=10\end{aligned}$$
   b. $$\begin{aligned} 2x-4y+4z&=2\\ 2x+y+9z&=17\\ 3x-2y+2z&=7\end{aligned}$$

7. An apple, a banana and three oranges or two apples, two bananas, and an orange, or four bananas and two oranges cost $2. Find the price of each.

8. Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then determine one particular solution.
   a. $$\begin{aligned} x+y+z&=6\\ 2x-3y+2z&=12\\ 3x-2y+3z&=18\end{aligned}$$
   b. $$\begin{aligned} x+y+3z&=4\\ x+z&=1\\ 2x-y&=2\end{aligned}$$

9. Elise has a collection of 12 coins consisting of nickels, dimes and quarters. If the total worth of the coins is $1.80, how many are there of each? Find all possible solutions.

SECTION 7.7 PROBLEM SET: CHAPTER REVIEW

10. Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then find a particular solution.

a.
$$\begin{aligned}2x+y-2z&=0\\2x+2y-3z&=0\\6x+4y-7z&=0\end{aligned}$$

b.
$$\begin{aligned}3x+4y-3z&=5\\2x+3y-z&=4\\x+2y+z&=1\end{aligned}$$

11. Solve the following systems. If a system has an infinite number of solutions, first express the solution in parametric form, and then provide one particular solution.

a.
$$\begin{aligned}2x+y-2z&=0\\2x+2y-3z&=0\\6x+4y-7z&=0\end{aligned}$$

b.
$$\begin{aligned}3x+4y-3z&=5\\2x+3y-z&=4\\x+2y+z&=1\end{aligned}$$

12. Find the inverse of the following matrices:

a.
$$\begin{bmatrix}2&3\\3&5\end{bmatrix}$$

b.
$$\begin{bmatrix}1&1&1\\1&2&1\\2&3&1\end{bmatrix}$$

13. Solve the following systems using the matrix inverse method.

a.
$$\begin{aligned}2x+3y+z&=12\\x+2y+z&=9\\x+y+z&=5\end{aligned}$$

b.
$$\begin{aligned}x+2y-3z+w&=\\x-z&=4\\x-2y+z=0&\\y-2z+w&=-11=0\end{aligned}$$

14. Use matrix A to encode the following messages. The space between the letters is represented by the number 27, and all punctuation is ignored.

$$A=\begin{bmatrix}1&2&0\\1&2&1\\0&1&0\end{bmatrix}$$

a. TAKE IT AND RUN
b. GET OUT QUICK

15. Decode the following messages that were encoded using matrix A in the above problem.

a. 44, 71, 15, 18, 27, 1, 68, 82, 27, 69, 76, 27, 19, 33, 9
b. 37, 64, 15, 36, 54, 15, 67, 75, 20, 59, 66, 27, 39, 43, 12

16. Chris, Bob, and Matt decide to help each other study during the final exams. Chris's favorite subject is chemistry, Bob loves biology, and Matt knows his math. Each studies his own subject as well as helps the others learn their subjects. After the finals, they realize that Chris spent 40% of his time studying his own subject chemistry, 30% of his time helping Bob learn chemistry, and 30% of the time helping Matt learn chemistry. Bob spent 30% of his time studying his own subject biology, 30% of his time helping Chris learn biology, and 40% of the time helping Matt learn biology. Matt spent 20% of his time studying his own subject math, 40% of his time helping Chris learn math, and 40% of the time helping Bob learn math. If they originally agreed that each should work about 33 hours, how long did each work?
17. As in the previous problem, Chris, Bob, and Matt decide to not only help each other study during the final exams, but also tutor others to make a little money. Chris spends 30% of his time studying chemistry, 15% of his time helping Bob with chemistry, and 25% helping Matt with chemistry. Bob spends 25% of his time studying biology, 15% helping Chris with biology, and 30% helping Matt. Similarly, Matt spends 20% of his time on his own math, 20% helping Chris, and 20% helping Bob. If they spend respectively, 12, 12, and 10 hours tutoring others, how many total hours are they going to end up working?

CHAPTER OVERVIEW

8: Linear Programming - A Geometric Approach

Learning Objectives

In this chapter, you will learn to:

1. Graph a linear inequality.
2. Solve linear programming problems that maximize the objective function.
3. Solve linear programming problems that minimize the objective function.

- 8.1: Graph Linear Inequalities in Two Variables
  - 8.1E: Exercises
- 8.2: Maximization Applications
  - 8.2.1: Maximization Applications (Exercises)
- 8.3: Minimization Applications
  - 8.3.1: Minimization Applications (Exercises)
- 8.4: Chapter Review

8.1: Graph Linear Inequalities in Two Variables

SUMMARY

By the end of this section, you will be able to:

- Verify solutions to an inequality in two variables.
- Recognize the relation between the solutions of an inequality and its graph.
- Graph linear inequalities in two variables
- Solve applications using linear inequalities in two variables

Before you get started, take this readiness quiz.

1. Graph $x > 2$ on a number line.
   If you missed this problem, review [link].
2. Solve: $4x + 3 > 23$.
   If you missed this problem, review [link].
3. Translate: $8 < x > 3$.
   If you missed this problem, review [link].

Verify Solutions to an Inequality in Two Variables

Previously we learned to solve inequalities with only one variable. We will now learn about inequalities containing two variables. In particular we will look at **linear inequalities** in two variables which are very similar to linear equations in two variables.

Linear inequalities in two variables have many applications. If you ran a business, for example, you would want your revenue to be greater than your costs—so that your business made a profit.

LINEAR INEQUALITY

A **linear inequality** is an inequality that can be written in one of the following forms:

$$Ax + By > C \qquad Ax + By \geq C \qquad Ax + By < C \qquad Ax + By \leq C$$

Where *A* and *B* are not both zero.

Recall that an inequality with one variable had many solutions. For example, the solution to the inequality x>3x>3 is any number greater than 3. We showed this on the number line by shading in the number line to the right of 3, and putting an open parenthesis at 3. See Figure.

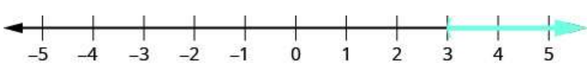


Figure 8.1.1

Similarly, linear inequalities in two variables have many solutions. Any ordered pair (x,y)(x,y) that makes an inequality true when we substitute in the values is a **solution to a linear inequality**.

SOLUTION TO A LINEAR INEQUALITY

An ordered pair (x, y) is a **solution to a linear inequality** if the inequality is true when we substitute the values of *x* and *y*.

Example 8.1.1

Determine whether each ordered pair is a solution to the inequality y>x+4:y>x+4:

ⓐ (0,0)(0,0) ⓑ (1,6)(1,6) ⓒ (2,6)(2,6) ⓓ (−5,−15)(−5,−15) ⓔ (−8,12)(−8,12)

Answer

ⓐ

$(0, 0)$	$y > x + 4$
Substitute 0 for *x* and 0 for *y*.	$0 \overset{?}{>} 0 + 4$
Simplify.	$0 \not> 4$
	So, $(0, 0)$ is not a solution to $y > x + 4$.

ⓑ

$(1, 6)$	$y > x + 4$
Substitute 1 for *x* and 6 for *y*.	$6 \overset{?}{>} 1 + 4$
Simplify.	$6 > 5$
	So, $(1, 6)$ is a solution to $y > x + 4$.

ⓒ

$(2, 6)$	$y > x + 4$
Substitute 2 for *x* and 6 for *y*.	$6 \overset{?}{>} 2 + 4$
Simplify.	$6 \not> 6$
	So, $(2, 6)$ is not a solution to $y > x + 4$.

ⓓ

$(-5, -15)$	$y > x + 4$
Substitute −5 for *x* and −15 for *y*.	$-15 \overset{?}{>} -5 + 4$
Simplify.	$-15 \not> -1$
	So, $(-5, -15)$ is not a solution to $y > x + 4$.

ⓔ

$(-8, 12)$	$y > x + 4$
Substitute −8 for *x* and 12 for *y*.	$12 \overset{?}{>} -8 + 4$
Simplify.	$12 > -4$
	So, $(-8, 12)$ is a solution to $y > x + 4$.

Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

ⓐ $(0, 0)$ ⓑ $(4, 9)$ ⓒ $(-2, 1)$ ⓓ $(-5, -3)$ ⓔ $(5, 1)$

Answer

ⓐ yes ⓑ yes ⓒ yes ⓓ yes ⓔ no

? Example 8.1.3

Determine whether each ordered pair is a solution to the inequality $y < x + 1$:

ⓐ $(0, 0)$ ⓑ $(8, 6)$ ⓒ $(-2, -1)$ ⓓ $(3, 4)$ ⓔ $(-1, -4)$

Answer

ⓐ yes ⓑ yes ⓒ no ⓓ no ⓔ yes

Recognize the Relation Between the Solutions of an Inequality and its Graph

Now, we will look at how the solutions of an inequality relate to its graph.

Let's think about the number line in shown previously again. The point $x = 3$ separated that number line into two parts. On one side of 3 are all the numbers less than 3. On the other side of 3 all the numbers are greater than 3. See Figure.

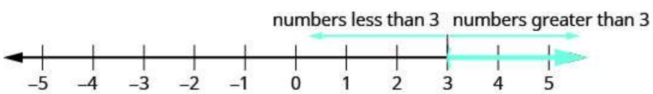


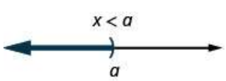
Figure 8.1.2The solution to $x > 3$ is the shaded part of the number line to the right of $x = 3$.

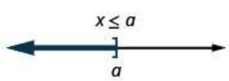
Similarly, the line $y = x + 4$ separates the plane into two regions. On one side of the line are points with $y < x + 4$. On the other side of the line are the points with $y > x + 4$. We call the line $y = x + 4$ a **boundary line**.

BOUNDARY LINE

The line with equation $Ax + By = C$ is the **boundary line** that separates the region where $Ax + By > C$ from the region where $Ax + By < C$.

For an inequality in one variable, the endpoint is shown with a parenthesis or a bracket depending on whether or not *a* is included in the solution:





Similarly, for an inequality in two variables, the boundary line is shown with a solid or dashed line to show whether or not it the line is included in the solution.

$Ax + By < C$	$Ax + By \leq C$
$Ax + By > C$	$Ax + By \geq C$
Boundary line is $Ax + By = C$.	Boundary line is $Ax + By = C$.
Boundary line is not included in solution.	Boundary line is included in solution.
Boundary line is dashed.	**Boundary line is solid.**

Now, let's take a look at what we found in Example. We'll start by graphing the line $y = x + 4$, and then we'll plot the five points we tested, as shown in the graph. See Figure.

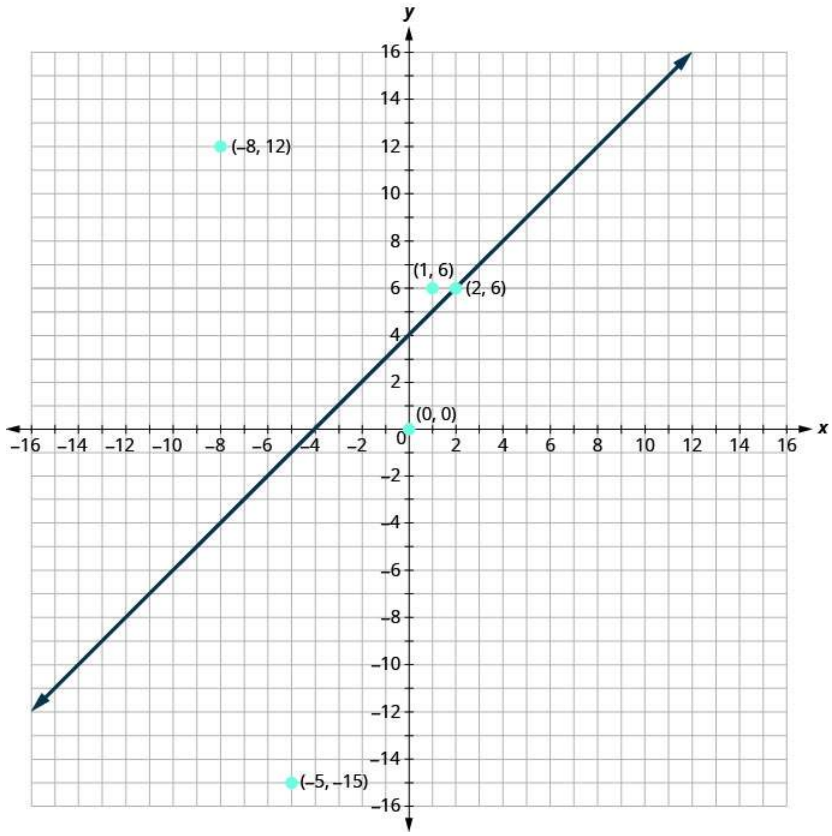


Figure 8.1.3

In Example we found that some of the points were solutions to the inequality $y > x+4$ and some were not.

Which of the points we plotted are solutions to the inequality $y > x+4$?

The points $(1,6)$ and $(-8,12)$ are solutions to the inequality $y > x+4$. Notice that they are both on the same side of the boundary line $y = x+4$.

The two points $(0,0)$ and $(-5,-15)$ are on the other side of the boundary line $y = x+4$, and they are not solutions to the inequality $y > x+4$. For those two points, $y < x+4$.

What about the point $(2,6)$? Because $6 = 2+4$, the point is a solution to the equation $y = x+4$, but not a solution to the inequality $y > x+4$. So the point $(2,6)$ is on the boundary line.

Let's take another point above the boundary line and test whether or not it is a solution to the inequality $y > x+4$. The point $(0,10)$clearly looks to above the boundary line, doesn't it? Is it a solution to the inequality?

$$\begin{array}{rcl} y & > & x+4 \\ 10 & \overset{?}{>} & 0+4 \\ 10 & > & 4 \end{array}$$

So, $(0,10)$ is a solution to $y > x+4$.

Any point you choose above the boundary line is a solution to the inequality $y > x+4$. All points above the boundary line are solutions.

Similarly, all points below the boundary line, the side with $(0,0)$ and $(-5,-15)$, are not solutions to $y > x+4$, as shown in Figure.

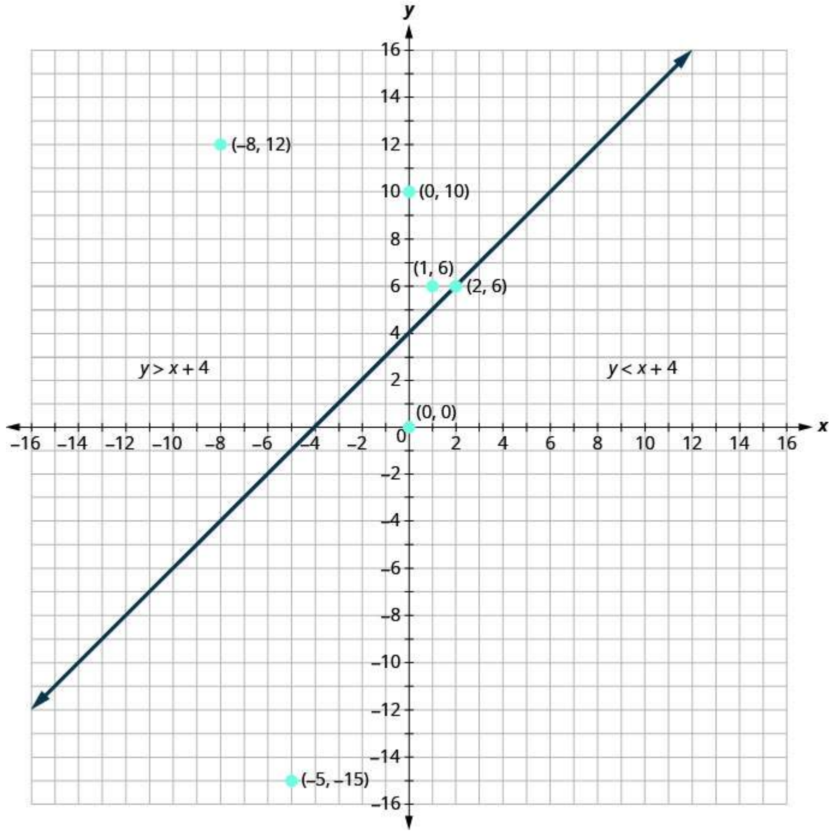
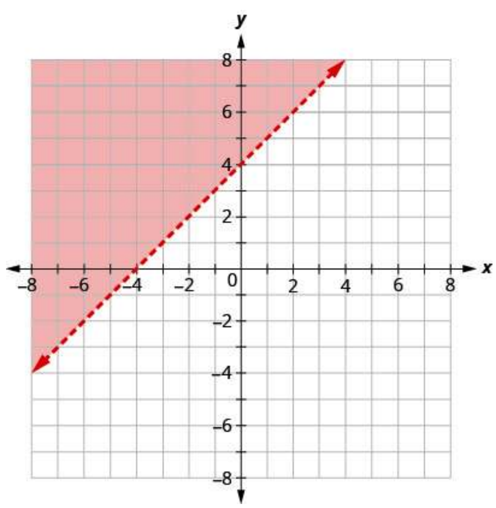


Figure $8.1.4$

The graph of the inequality $y > x + 4$ is shown in below.

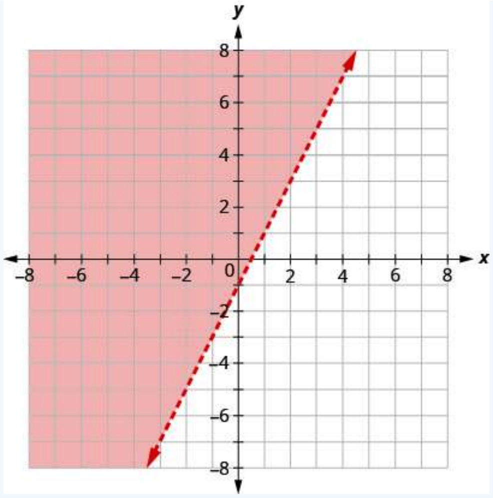
The line $y = x + 4$ divides the plane into two regions. The shaded side shows the solutions to the inequality $y > x + 4$.

The points on the boundary line, those where $y = x + 4$, are not solutions to the inequality $y > x + 4$, so the line itself is not part of the solution. We show that by making the line dashed, not solid.



Example $8.1.4$

The boundary line shown in this graph is $y = 2x - 1$. Write the inequality shown by the graph.



Answer

The line $y=2x-1$ is the boundary line. On one side of the line are the points with $y>2x-1$ and on the other side of the line are the points with $y<2x-1$.

Let's test the point $(0,0)$ and see which inequality describes its position relative to the boundary line.

At $(0,0)$, which inequality is true: $y>2x-1$ or $y<2x-1$?

$$\begin{array}{ll} y>2x-1 & y<2x-1 \\ 0 \stackrel{?}{>} 2\cdot 0-1 & 0 \stackrel{?}{<} 2\cdot 0-1 \\ 0>-1 \text{ True} & 0<-1 \text{ False} \end{array}$$

Since, $y>2x-1$ is true, the side of the line with $(0,0)$, is the solution. The shaded region shows the solution of the inequality $y>2x-1$.

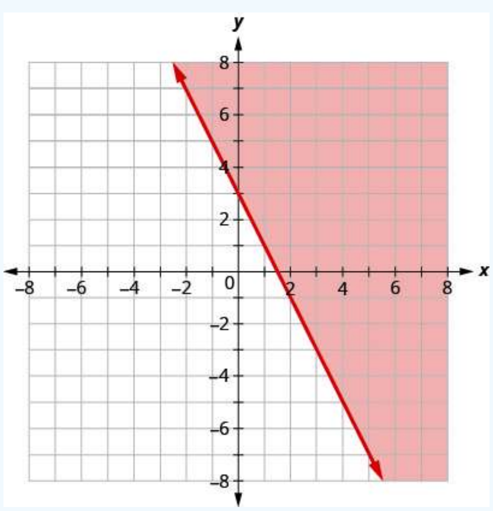
Since the boundary line is graphed with a solid line, the inequality includes the equal sign.

The graph shows the inequality $y\geq 2x-1$.

We could use any point as a test point, provided it is not on the line. Why did we choose $(0,0)$? Because it's the easiest to evaluate. You may want to pick a point on the other side of the boundary line and check that $y<2x-1$.

Example 8.1.5

Write the inequality shown by the graph with the boundary line $y=-2x+3$.

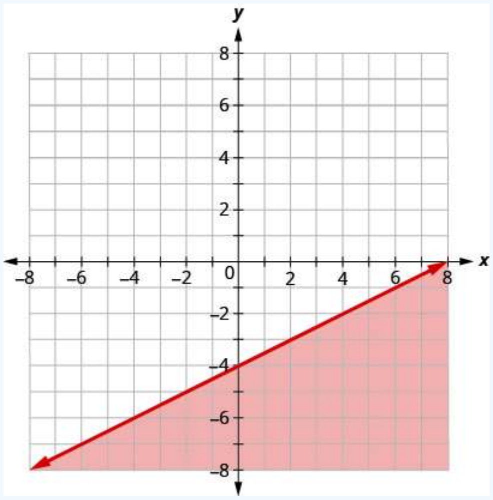


Answer

$y \geq -2x+3$

? Example 8.1.6

Write the inequality shown by the graph with the boundary line $y=\frac{1}{2}x-4$.

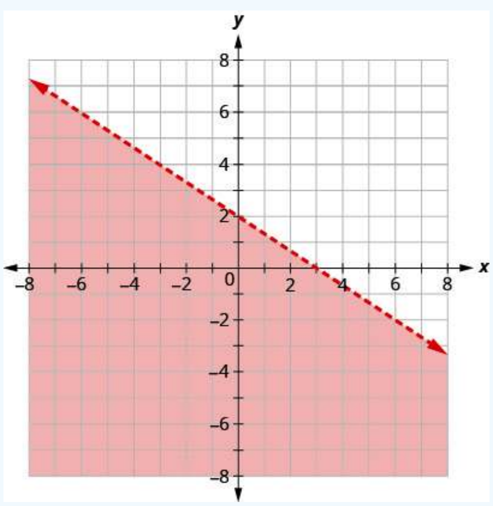


Answer

$y \leq \frac{1}{2}x-4$

? Example 8.1.7

The boundary line shown in this graph is $2x+3y=6$. Write the inequality shown by the graph.



Answer

The line $2x+3y=6$ is the boundary line. On one side of the line are the points with $2x+3y>6$ and on the other side of the line are the points with $2x+3y<6$.

Let's test the point $(0,0)$ and see which inequality describes its side of the boundary line.

At $(0,0)$, which inequality is true: $2x+3y>6$ or $2x+3y<6$?

$$\begin{array}{ll} 2x+3y>6 & 2x+3y<6 \\ 2(0)+3(0)\overset{?}{>}6 & 2(0)+3(0)\overset{?}{<}6 \\ 0>6 \text{ False} & 0<6 \text{ True} \end{array}$$

So the side with $(0,0)$ is the side where $2x+3y<6$.

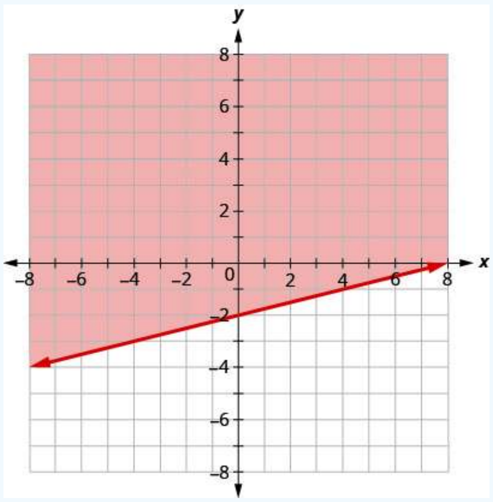
(You may want to pick a point on the other side of the boundary line and check that $2x+3y>6$.)

Since the boundary line is graphed as a dashed line, the inequality does not include an equal sign.

The shaded region shows the solution to the inequality $2x+3y<6$.

Example 8.1.8

Write the inequality shown by the shaded region in the graph with the boundary line $x-4y=8$.

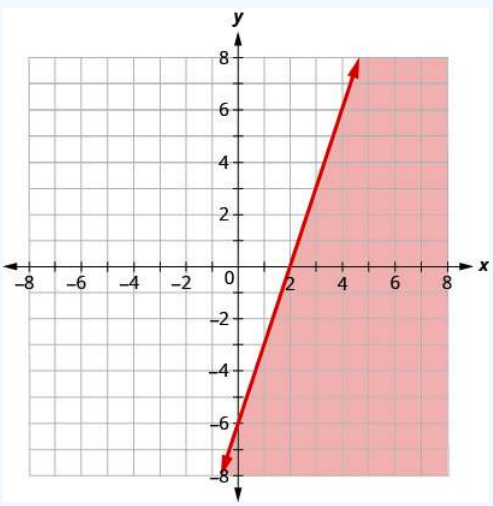


Answer

$x-4y\leq 8$

Example 8.1.9

Write the inequality shown by the shaded region in the graph with the boundary line $3x-y=6$.



Answer

$3x - y \geq 6$

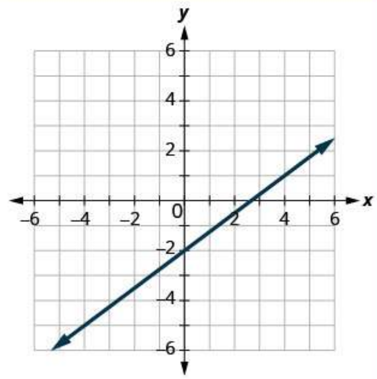
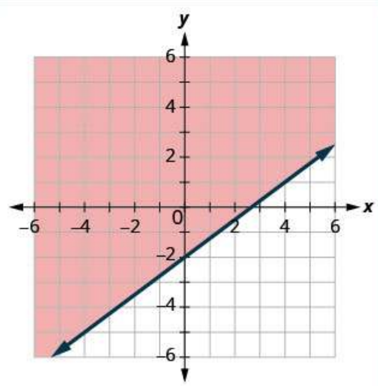
Graph Linear Inequalities in Two Variables

Now that we know what the graph of a linear inequality looks like and how it relates to a boundary equation we can use this knowledge to graph a given linear inequality.

❓ Example 8.1.10: How to Graph a Linear Equation in Two Variables

Graph the linear inequality $y \geq \frac{3}{4}x - 2$.

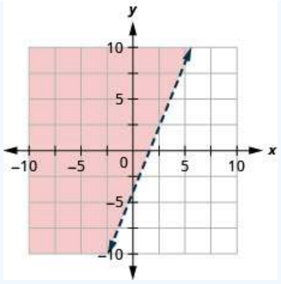
Answer

Step 1. Identify and graph the boundary line. • If the inequality is $\leq$ or $\geq$, the boundary line is solid. • If the inequality is $<$ or $>$, the boundary line is dashed.	Replace the inequality sign with an equal sign to find the boundary line. Graph the boundary line $y = \frac{3}{4}x - 2$. The inequality sign is $\geq$, so we draw a solid line.	
Step 2. Test a point that is not on the boundary line. Is it a solution of the inequality?	We'll test (0, 0). Is it a solution of the inequality?	At (0, 0), is $y \geq \frac{3}{4}x - 2$? $0 \overset{?}{\geq} \frac{3}{4}(0) - 2$ $0 \geq -2$ So, (0, 0) is a solution.
Step 3. Shade in one side of the boundary line. • If the test point is a solution, shade in the side that includes the point. • If the test point is not a solution, shade in the opposite side.	The test point (0, 0) is a solution to $y \geq \frac{3}{4}x - 2$. So we shade in that side.	 All points in the shaded region and on the boundary line represent the solutions to $y \geq \frac{3}{4}x - 2$.

❓ Example 8.1.11

Graph the linear inequality $y > \frac{5}{2}x - 4$.

Answer

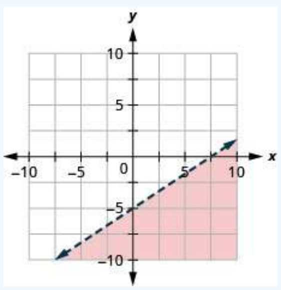


All points in the shaded region and on the boundary line, represent the solutions to $y > \frac{5}{2}x - 4$.

? Example 8.1.12

Graph the linear inequality $y < \frac{2}{3}x - 5$.

Answer



All points in the shaded region, but not those on the boundary line, represent the solutions to $y < \frac{2}{3}x - 5$.

The steps we take to graph a linear inequality are summarized here.

GRAPH A LINEAR INEQUALITY IN TWO VARIABLES.

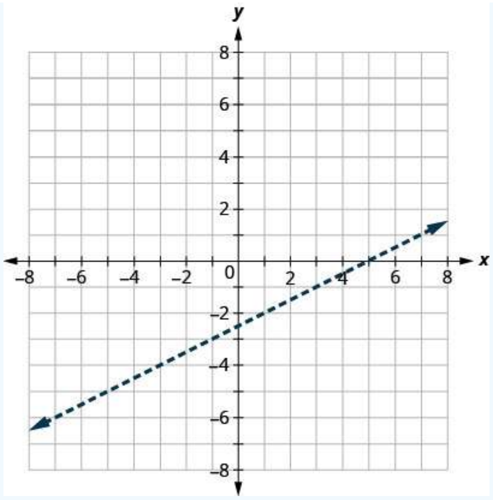
1. Identify and graph the boundary line.
   - If the inequality is \leq or\geq ,\leq or\geq , the boundary line is solid.
   - If the inequality is <or>,<or>, the boundary line is dashed.
2. Test a point that is not on the boundary line. Is it a solution of the inequality?
3. Shade in one side of the boundary line.
   - If the test point is a solution, shade in the side that includes the point.
   - If the test point is not a solution, shade in the opposite side.

? Example 8.1.13

Graph the linear inequality $x - 2y < 5$.

Answer

First, we graph the boundary line $x - 2y = 5$. The inequality is $<$ so we draw a dashed line.

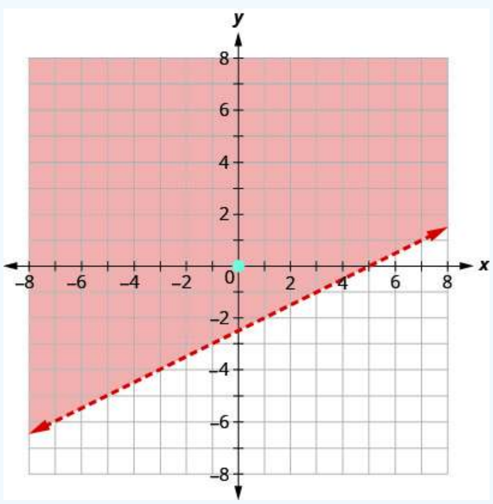


Then, we test a point. We'll use $(0,0)$ again because it is easy to evaluate and it is not on the boundary line.

Is $(0,0)$ a solution of $x-2y<5$?

$$0-2(0)\overset{?}{<}5$$
$$0-0\overset{?}{<}5$$
$$0<5$$

The point $(0,0)$ is a solution of $x-2y<5$, so we shade in that side of the boundary line.

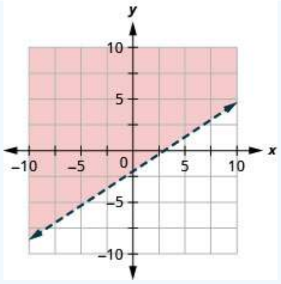


All points in the shaded region, but not those on the boundary line, represent the solutions to $x-2y<5$.

? Example 8.1.14

Graph the linear inequality: $2x-3y<6$.

Answer

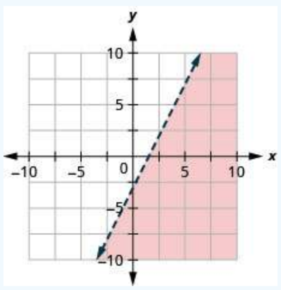


All points in the shaded region, but not those on the boundary line, represent the solutions to $2x - 3y < 6$.

Example 8.1.15

Graph the linear inequality: $2x - y > 3$.

Answer



All points in the shaded region, but not those on the boundary line, represent the solutions to $2x - y > 3$.

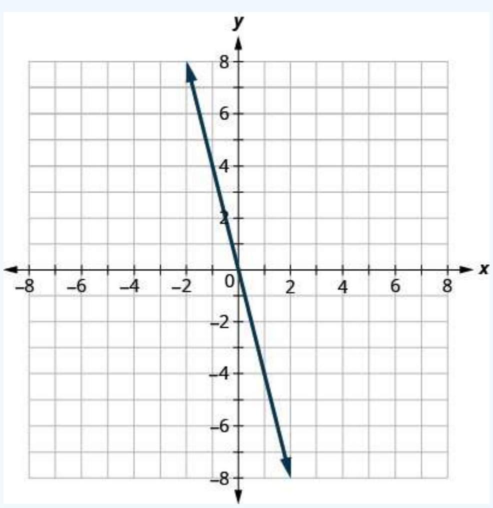
What if the boundary line goes through the origin? Then, we won't be able to use $(0, 0)$ as a test point. No problem—we'll just choose some other point that is not on the boundary line.

Example 8.1.16

Graph the linear inequality: $y \leq -4x$.

Answer

First, we graph the boundary line $y = -4x$. It is in slope–intercept form, with $m = -4$ and $b = 0$. The inequality is $\leq$ so we draw a solid line.



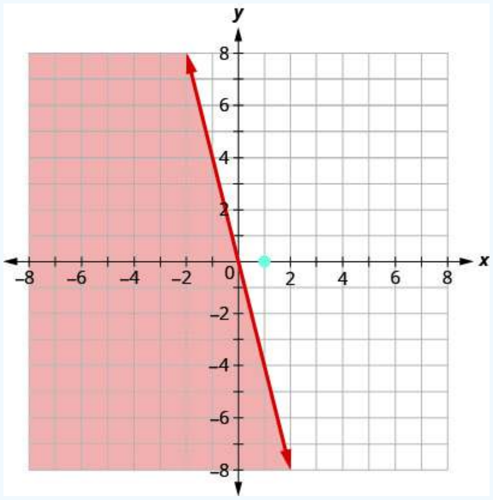
Now we need a test point. We can see that the point (1,0)(1,0) is not on the boundary line.

Is $(1, 0)$ a solution of $y \leq -4x$?

$$0 \overset{?}{\leq} -4(1)$$

$$0 \nleq -4$$

The point $(1, 0)$ is not a solution to $y \leq -4x$, so we shade in the opposite side of the boundary line.

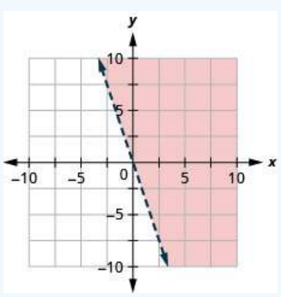


All points in the shaded region and on the boundary line represent the solutions to $y \leq -4x$.

? Example 8.1.17

Graph the linear inequality: $y > -3x$.

Answer

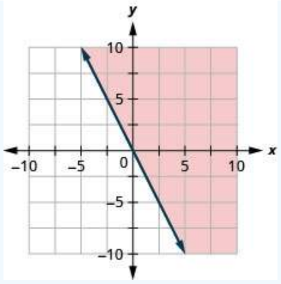


All points in the shaded region, but not those on the boundary line, represent the solutions to $y > -3x$.

? Example 8.1.18

Graph the linear inequality: $y \geq -2x$.

Answer



All points in the shaded region and on the boundary line, represent the solutions to $y \geq -2x$.

Some linear inequalities have only one variable. They may have an *x* but no *y*, or a *y* but no *x*. In these cases, the boundary line will be either a vertical or a horizontal line.

Recall that:

$$\begin{array}{ll} x=a & \text{vertical line} \\ y=b & \text{horizontal line} \end{array}$$

Example 8.1.19

Graph the linear inequality: $y>3$.

Answer

First, we graph the boundary line $y=3$. It is a horizontal line. The inequality is $>$ so we draw a dashed line.

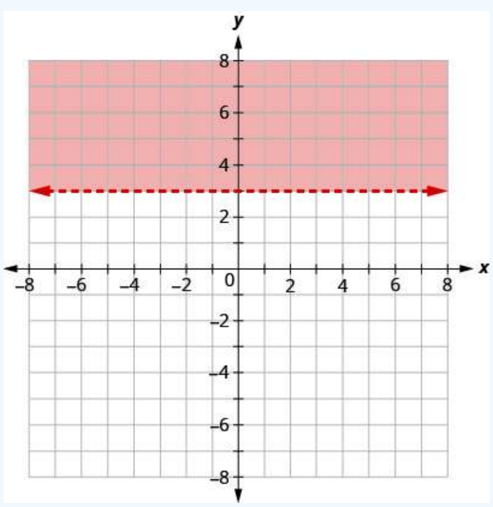
We test the point $(0,0)$.

$$y>3$$

$$0\slashed{>}3$$

So, $(0,0)$ is not a solution to $y>3$.

So we shade the side that does not include $(0,0)$ as shown in this graph.

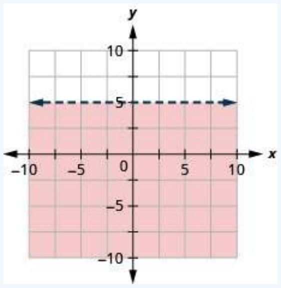


All points in the shaded region, but not those on the boundary line, represent the solutions to $y>3$.

Example 8.1.20

Graph the linear inequality: $y < 5$.

Answer

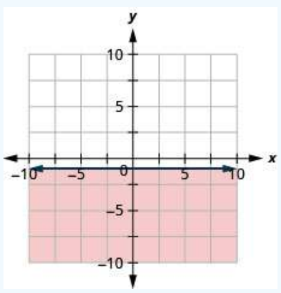


All points in the shaded region, but not those on the boundary line, represent the solutions to $y < 5$.

Example 8.1.21

Graph the linear inequality: $y \leq -1$.

Answer



All points in the shaded region and on the boundary line represent the solutions to $y \leq -1$.

Solve Applications using Linear Inequalities in Two Variables

Many fields use linear inequalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how thay might be used.

Example 8.1.22

Hilaria works two part time jobs in order to earn enough money to meet her obligations of at least $240 a week. Her job in food service pays $10 an hour and her tutoring job on campus pays $15 an hour. How many hours does Hilaria need to work at each job to earn at least $240?

ⓐ Let xx be the number of hours she works at the job in food service and let *y* be the number of hours she works tutoring. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hilaria.

Answer

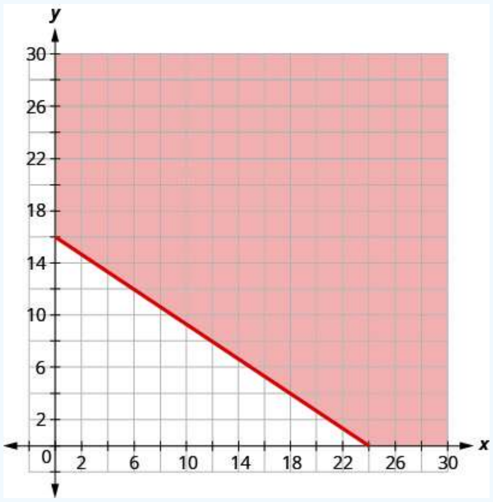
ⓐ We let *x* be the number of hours she works at the job in food service and let *y* be the number of hours she works tutoring.

She earns $10 per hour at the job in food service and $15 an hour tutoring. At each job, the number of hours multiplied by the hourly wage will gives the amount earned at that job.

Amount earned at the food service job	plus	the amount earned tutoring	is at least $240
$10x$	$+$	$15y$	≥ 240

ⓑ To graph the inequality, we put it in slope–intercept form.

$$10x+15y \geq 240 \tag{8.1.1}$$
$$15y \geq -10x+240 \tag{8.1.2}$$
$$y \geq -\frac{2}{3}x+16 \tag{8.1.3}$$



ⓒ From the graph, we see that the ordered pairs $(15,10)$, $(0,16)$, $(24,0)$ represent three of infinitely many solutions. Check the values in the inequality.

(15, 10)	(0, 16)	(24, 0)
$10x+15y \geq 240$	$10x+15y \geq 240$	$10x+15y \geq 240$
$10(15)+15(10) \overset{?}{\geq} 240$	$10(0)+15(16) \overset{?}{\geq} 240$	$10(24)+15(0) \overset{?}{\geq} 240$
$300 \geq 240$ True	$240 \geq 240$ True	$240 \geq 240$ True

For Hilaria, it means that to earn at least $240, she can work 15 hours tutoring and 10 hours at her fast-food job, earn all her money tutoring for 16 hours, or earn all her money while working 24 hours at the job in food service.

❓ Example $8.1.23$

Hugh works two part time jobs. One at a grocery store that pays $10 an hour and the other is babysitting for $13 hour. Between the two jobs, Hugh wants to earn at least $260 a week. How many hours does Hugh need to work at each job to earn at least $260?

ⓐ Let *x* be the number of hours he works at the grocery store and let *y* be the number of hours he works babysitting. Write an inequality that would model this situation.

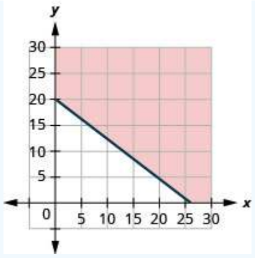
ⓑ Graph the inequality.

ⓒ Find three ordered pairs (*x*, *y*) that would be solutions to the inequality. Then, explain what that means for Hugh.

Answer

ⓐ $10x+13y \geq 260$

ⓑ



ⓒ Answers will vary..

❓ Example 8.1.24

Veronica works two part time jobs in order to earn enough money to meet her obligations of at least $280 a week. Her job at the day spa pays $10 an hour and her administrative assistant job on campus pays $17.50 an hour. How many hours does Veronica need to work at each job to earn at least $280?

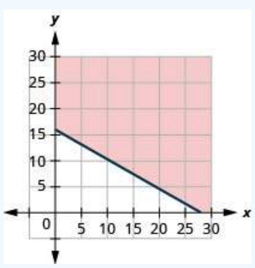
ⓐ Let *x* be the number of hours she works at the day spa and let *y* be the number of hours she works as administrative assistant. Write an inequality that would model this situation.

ⓑ Graph the inequality.

ⓒ Find three ordered pairs (*x*, *y*) that would be solutions to the inequality. Then, explain what that means for Veronica

Answer

ⓐ $10x + 17.5y \geq 280$
ⓑ



ⓒ Answers will vary.

Access this online resource for additional instruction and practice with graphing linear inequalities in two variables.

- [Graphing Linear Inequalities in Two Variables]

Key Concepts

- **How to graph a linear inequality in two variables.**
  1. Identify and graph the boundary line.
     If the inequality is $\leq$ or $\geq$, the boundary line is solid.
     If the inequality is $<$ or $>$, the boundary line is dashed.
  2. Test a point that is not on the boundary line. Is it a solution of the inequality?
  3. Shade in one side of the boundary line.
     If the test point is a solution, shade in the side that includes the point.
     If the test point is not a solution, shade in the opposite side.

Glossary

boundary line

The line with equation $Ax+By=C$ is the boundary line that separates the region where $Ax+By>C$ from the region where $Ax+By<C$.

linear inequality

A linear inequality is an inequality that can be written in one of the following forms: $Ax+By>C$, $Ax+By\geq C$, $Ax+By<C$, or $Ax+By\leq C$, where *A* and *B* are not both zero.

solution to a linear inequality

An ordered pair (x,y) is a solution to a linear inequality if the inequality is true when we substitute the values of *x* and *y*.

8.1E: Exercises

Practice Makes Perfect

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.

1. Determine whether each ordered pair is a solution to the inequality $y > x - 1$:

a. $(0, 1)$
b. $(-4, -1)$
c. $(4, 2)$
d. $(3, 0)$
e. $(-2, -3)$

Answer

a. yes b. yes c. no d. no e. no

2. Determine whether each ordered pair is a solution to the inequality $y > x - 3$:

a. $(0, 0)$
b. $(2, 1)$
c. $(-1, -5)$
d. $(-6, -3)$
e. $(1, 0)$

3. Determine whether each ordered pair is a solution to the inequality $y < 3x + 2$:

a. $(0, 3)$
b. $(-3, -2)$
c. $(-2, 0)$
d. $(0, 0)$
e. $(-1, 4)$

Answer

a. no b. no c. yes d. yes e. no

4. Determine whether each ordered pair is a solution to the inequality $y < -2x + 5$:

a. $(-3, 0)(-3, 0)b.$ \($(1, 6)$
c. $(-6, -2)$
d. $(0, 1)$
e. $(5, -4)$

5. Determine whether each ordered pair is a solution to the inequality $3x - 4y > 4$:

a. $(5, 1)$
b. $(-2, 6)$
c. $(3, 2)$
d. $(10, -5)$
e. $(0, 0)$

Answer

a. yes b. no c. no d. no e. no

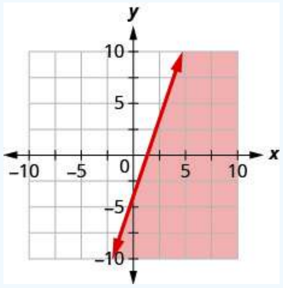
6. Determine whether each ordered pair is a solution to the inequality $2x+3y>2$:

a. $(1,1)$
b. $(4,-3)$
c. $(0,0)$
d. $(-8,12)$
e. $(3,0)$

Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

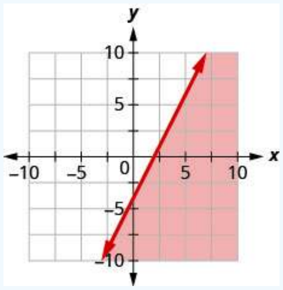
7. Write the inequality shown by the graph with the boundary line $y=3x-4$.



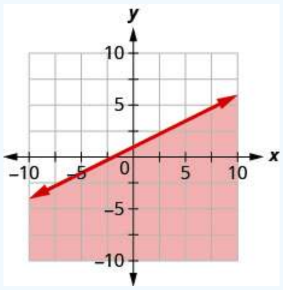
Answer

$y \leq 3x-4$

8. Write the inequality shown by the graph with the boundary line $y=2x-4$.



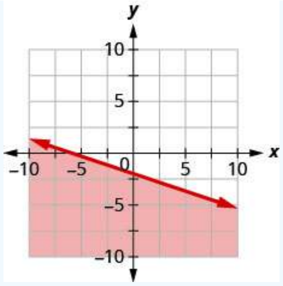
9. Write the inequality shown by the graph with the boundary line $y=-\frac{1}{2}x+1$.



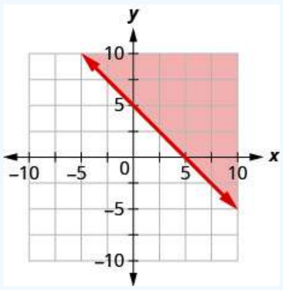
Answer

$y \leq -\frac{1}{2}x+1$

10. Write the inequality shown by the graph with the boundary line $y=-\frac{1}{3}x-2$.



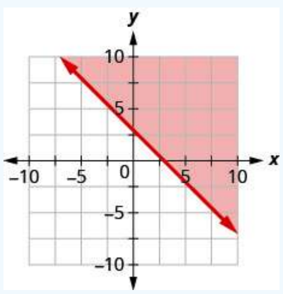
11. Write the inequality shown by the shaded region in the graph with the boundary line $x+y=5$.



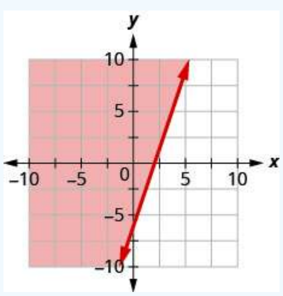
Answer

$x+y \geq 5$

12. Write the inequality shown by the shaded region in the graph with the boundary line $x+y=3$.



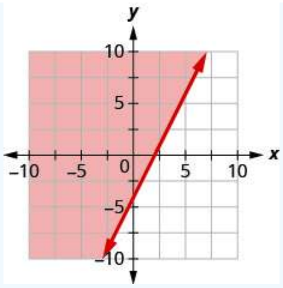
13. Write the inequality shown by the shaded region in the graph with the boundary line $3x-y=6$.



Answer

$3x-y \leq 6$

14. Write the inequality shown by the shaded region in the graph with the boundary line $2x-y=4$.

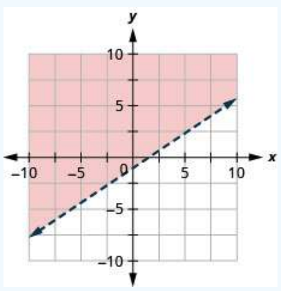


Graph Linear Inequalities in Two Variables

In the following exercises, graph each linear inequality.

15. Graph the linear inequality: $y > \frac{2}{3}x - 1$.

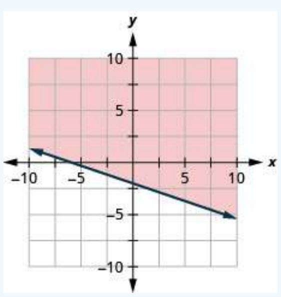
Answer



16. Graph the linear inequality: $y < \frac{3}{5}x + 2$.

17. Graph the linear inequality: $y \leq -\frac{1}{2}x + 4$.

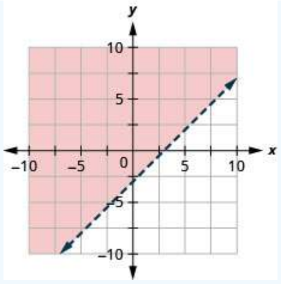
Answer



18. Graph the linear inequality: $y \geq -\frac{1}{3}x - 2$.

19. Graph the linear inequality: $x - y \leq 3$.

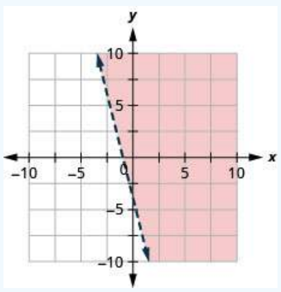
Answer



20. Graph the linear inequality: $x - y \geq -2$.

21. Graph the linear inequality: $4x + y > -4$.

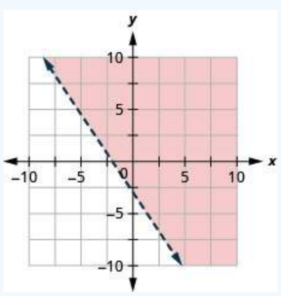
Answer



22. Graph the linear inequality: $x + 5y < -5$.

23. Graph the linear inequality: $3x + 2y \geq -6$.

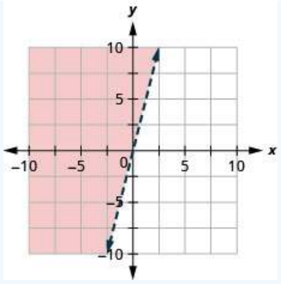
Answer



24. Graph the linear inequality: $4x + 2y \geq -8$.

25. Graph the linear inequality: $y > 4x$.

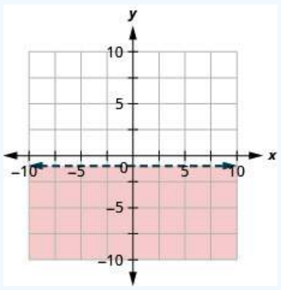
Answer



26. Graph the linear inequality: $y \leq -3x$.

27. Graph the linear inequality: $y < -10$.

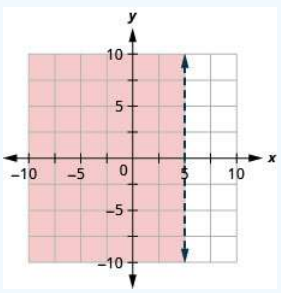
Answer



28. Graph the linear inequality: $y \geq 2$.

29. Graph the linear inequality: $x \leq 5$.

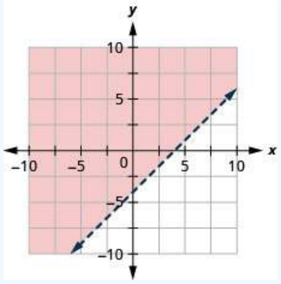
Answer



30. Graph the linear inequality: $x \geq 0$.

31. Graph the linear inequality: $x - y < 4$.

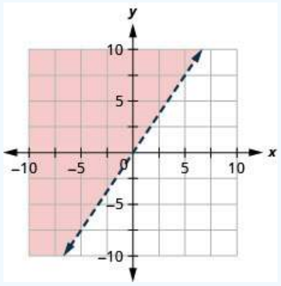
Answer



32. Graph the linear inequality: $x - y < -3$.

33. Graph the linear inequality: $y \geq \frac{3}{2}x$.

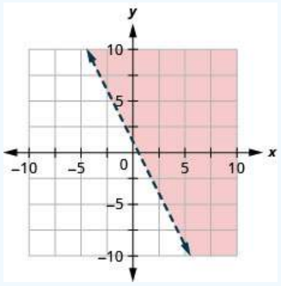
Answer



34. Graph the linear inequality: $y \leq \frac{5}{4}x$.

35. Graph the linear inequality: $y > -2x + 1$.

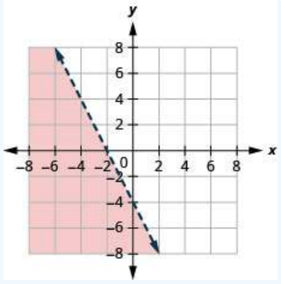
Answer



36. Graph the linear inequality: $y < -3x - 4$.

37. Graph the linear inequality: $2x + y \geq -4$.

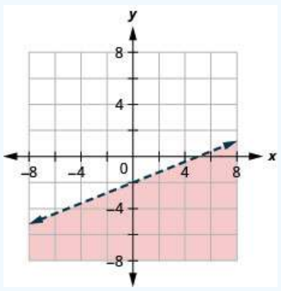
Answer



38. Graph the linear inequality: $x+2y \leq -2$.

39. Graph the linear inequality: $2x-5y > 10$.

Answer



40. Graph the linear inequality: $4x-3y > 12$.

Solve Applications using Linear Inequalities in Two Variables

41. Harrison works two part time jobs. One at a gas station that pays \$11 an hour and the other is IT troubleshooting for \$16.50\$16.50an hour. Between the two jobs, Harrison wants to earn at least \$330 a week. How many hours does Harrison need to work at each job to earn at least \$330?

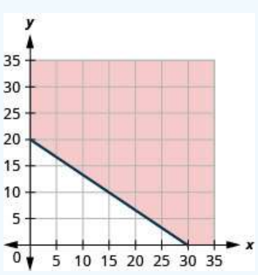
a. Let *x* be the number of hours he works at the gas station and let *y* be the number of (hours he works troubleshooting. Write an inequality that would model this situation.

b. Graph the inequality.

c. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Harrison.

Answer

a. $11x+16.5y \geq 330$
b.



c. Answers will vary.

42. Elena needs to earn at least \$450 a week during her summer break to pay for college. She works two jobs. One as a swimming instructor that pays \$9 an hour and the other as an intern in a genetics lab for \$22.50 per hour. How many hours does Elena need to work at each job to earn at least \$450 per week?

a. Let x be the number of hours she works teaching swimming and let y be the number of hours she works as an intern. Write an inequality that would model this situation.

b. Graph the inequality.

c. Find three ordered pairs (x,y)(x,y) that would be solutions to the inequality. Then, explain what that means for Elena.

43. The doctor tells Laura she needs to exercise enough to burn 500 calories each day. She prefers to either run or bike and burns 15 calories per minute while running and 10 calories a minute while biking.

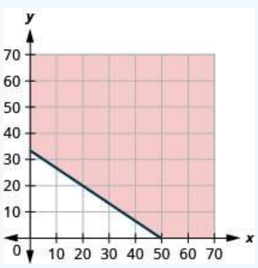
a. If x is the number of minutes that Laura runs and y is the number minutes she bikes, find the inequality that models the situation.

b. Graph the inequality.

c. List three solutions to the inequality. What options do the solutions provide Laura?

Answer

a. $15x + 10y \geq 500$
b.



c. Answers will vary.

44. Armando's workouts consist of kickboxing and swimming. While kickboxing, he burns 10 calories per minute and he burns 7 calories a minute while swimming. He wants to burn 600 calories each day.

a. If x is the number of minutes that Armando will kickbox and y is the number minutes he will swim, find the inequality that will help Armando create a workout for today.

b. Graph the inequality.

c. List three solutions to the inequality. What options do the solutions provide Armando?

Writing Exercises

45. Lester thinks that the solution of any inequality with a >> sign is the region above the line and the solution of any inequality with a << sign is the region below the line. Is Lester correct? Explain why or why not.

Answer

Answers will vary.

46. Explain why, in some graphs of linear inequalities, the boundary line is solid but in other graphs it is dashed.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
verify solutions to an inequality in two variables.			
recognize the relation between the solutions of an inequality and its graph.			
graph linear inequalities.			

b. On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

8.2: Maximization Applications

Learning Objectives

In this section, you will learn to:

1. Recognize the typical form of a linear programming problem
2. Formulate maximization linear programming problems
3. Graph feasibility regions for maximization of linear programming problems
4. Determine optimal solutions for the maximization of linear programming problems.

Linear Programming - Maximization

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. The conditions or constraints often take the form of inequalities. In this section, we will begin to formulate, analyze, and solve such problems, at a simple level, to understand the many components of such a problem.

A typical **linear programming** problem consists of finding an extreme value of a linear function subject to certain constraints. We are either trying to maximize or minimize the value of this linear function, such as to maximize profit or revenue, or to minimize cost. That is why these linear programming problems are classified as **maximization** or **minimization problems**, or just **optimization problems**. The function we are trying to optimize is called an **objective function**, and the conditions that must be satisfied are called **constraints**.

A typical example is to maximize profit from producing several products, subject to limitations on materials or resources needed for producing these items; the problem requires us to determine the amount of each item produced. Another type of problem involves scheduling; we need to determine how much time to devote to each of several activities in order to maximize income from (or minimize the cost of) these activities, subject to limitations on time and other resources available for each activity.

In this chapter, we will work with problems that involve only two variables, and therefore, can be solved by graphing. In the next chapter, we'll learn an algorithm to find a solution numerically. That will provide us with a tool to solve problems with more than two variables. At that time, with a little more knowledge about linear programming, we'll also explore the many ways these techniques are used in business and wide variety of other fields.

We begin by solving a maximization problem.

Example $8.2.1$

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation.

If Nikki makes \$40 an hour at Job I, and \$30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

We start by choosing our variables.

- Let x = The number of hours per week Niki will work at Job I.
- Let y = The number of hours per week Niki will work at Job II.

Now we write the objective function. Since Niki gets paid \$40 an hour at Job I, and \$30 an hour at Job II, her total income I is given by the following equation.

$$I = 40x + 30y$$

Our next task is to find the constraints. The second sentence in the problem states, "She never wants to work more than a total of 12 hours a week." This translates into the following constraint:

$$x + y \leq 12$$

The third sentence states, "For every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation." The translation follows.

$$2x + y \leq 16$$

The fact that x and y can never be negative is represented by the following two constraints:

$$x \geq 0, \text{ and } y \geq 0.$$

Well, good news! We have formulated the problem. We restate it as

$$\begin{array}{ll} \textbf{Maximize} & \mathrm{I = 40x + 30y} \\ \textbf{Subject to:} & \mathrm{x + y \leq 12} \\ & \mathrm{2x + y \leq 16} \\ & \mathrm{x \geq 0; y \geq 0} \end{array} \tag{8.2.1}$$

In order to solve the problem, we graph the constraints and shade the region that satisfies **all** the inequality constraints.

Any appropriate method can be used to graph the lines for the constraints. However often the easiest method is to graph the line by plotting the x-intercept and y-intercept.

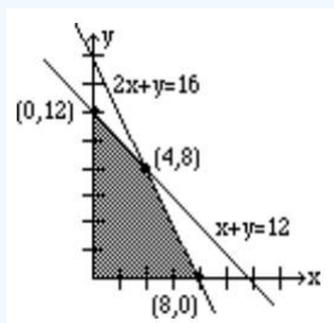
The line for a constraint will divide the plane into two region, one of which satisfies the inequality part of the constraint. A test point is used to determine which portion of the plane to shade to satisfy the inequality. Any point on the plane that is not on the line can be used as a test point.

- If the test point satisfies the inequality, then the region of the plane that satisfies the inequality is the region that contains the test point.
- If the test point does not satisfy the inequality, then the region that satisfies the inequality lies on the opposite side of the line from the test point.

In the graph below, after the lines representing the constraints were graphed using an appropriate method from Chapter 1, the point (0,0) was used as a test point to determine that

- (0,0) satisfies the constraint $x + y \leq 12$ because $0 + 0 < 12$
- (0,0) satisfies the constraint $2x + y \leq 16$ because $2(0) + 0 < 16$

Therefore, in this example, we shade the region that is below and to the left of **both** constraint lines, but also above the x axis and to the right of the y axis, in order to further satisfy the constraints $x \geq 0$ and $y \geq 0$.



The shaded region where all conditions are satisfied is called the **feasibility region** or the feasibility polygon.

The **Fundamental Theorem of Linear Programming** states that the maximum (or minimum) value of the objective function always takes place at the vertices of the feasibility region.

Therefore, we will identify all the vertices (corner points) of the feasibility region. We call these points **critical points.** They are listed as (0, 0), (0, 12), (4, 8), (8, 0). To maximize Niki's income, we will substitute these points in the objective function to see which point gives us the highest income per week. We list the results below.

Critical Points	Income

Critical Points	Income
(0, 0)	40(0) + 30(0) = $0
(0, 12)	40(0) + 30(12) = $360
(4, 8)	40(4) + 30(8) = $400
(8, 0)	40(8) + 30(0) = $320

Clearly, the point (4, 8) gives the most profit: $400.

Therefore, we conclude that Niki should work 4 hours at Job I, and 8 hours at Job II.

Example $8.2.2$

A factory manufactures two types of gadgets, regular and premium. Each gadget requires the use of two operations, assembly and finishing, and there are at most 12 hours available for each operation. A regular gadget requires 1 hour of assembly and 2 hours of finishing, while a premium gadget needs 2 hours of assembly and 1 hour of finishing. Due to other restrictions, the company can make at most 7 gadgets a day. If a profit of $20 is realized for each regular gadget and $30 for a premium gadget, how many of each should be manufactured to maximize profit?

Solution

We choose our variables.

- Let x = The number of regular gadgets manufactured each day.
- and y = The number of premium gadgets manufactured each day.

The objective function is

$$P=20x+30y$$

We now write the constraints. The fourth sentence states that the company can make at most 7 gadgets a day. This translates as

$$x+y\leq 7$$

Since the regular gadget requires one hour of assembly and the premium gadget requires two hours of assembly, and there are at most 12 hours available for this operation, we get

$$x+2y\leq 12$$

Similarly, the regular gadget requires two hours of finishing and the premium gadget one hour. Again, there are at most 12 hours available for finishing. This gives us the following constraint.

$$2x+y\leq 12$$

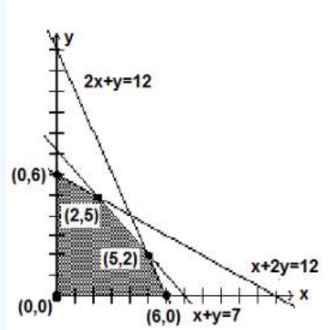
The fact that x and y can never be negative is represented by the following two constraints:

$$x\geq 0,\text{ and } y\geq 0.$$

We have formulated the problem as follows:

$$\begin{array}{ll} \textbf{Maximize} & \mathrm{P}=20\mathrm{x}+30\mathrm{y} \\ \textbf{Subject to:} & \mathrm{x}+\mathrm{y}\leq 7 \\ & \mathrm{x}+2\mathrm{y}\leq 12 \\ & 2\mathrm{x}+\mathrm{y}\leq 12 \\ & \mathrm{x}\geq 0;\mathrm{y}\geq 0 \end{array}$$

In order to solve the problem, we next graph the constraints and feasibility region.



Again, we have shaded the feasibility region, where all constraints are satisfied.

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify all the critical points. They are listed as (0, 0), (0, 6), (2, 5), (5, 2), and (6, 0). To maximize profit, we will substitute these points in the objective function to see which point gives us the maximum profit each day. The results are listed below.

Critical Point	Income
(0, 0)	20(0) + 30(0) = $0
(0, 6)	20(0) + 30(6) = $180
(2, 5)	20(2) + 30(5) = $190
(5, 2)	20(5) + 30(2) = $160
(6,0)	20(6) + 30(0) = $120

The point (2, 5) gives the most profit, and that profit is $190.
Therefore, we conclude that we should manufacture 2 regular gadgets and 5 premium gadgets daily to obtain the maximum profit of $190.

So far we have focused on “**standard maximization problems**” in which

1. The objective function is to be maximized
2. All constraints are of the form $ax + by \leq c$
3. All variables are constrained to by non-negative $(x \geq 0, y \geq 0)$

We will next consider an example where that is not the case. Our next problem is said to have “**mixed constraints**”, since some of the inequality constraints are of the form $ax + by \leq c$ and some are of the form $ax + by \geq c$. The non-negativity constraints are still an important requirement in any linear program.

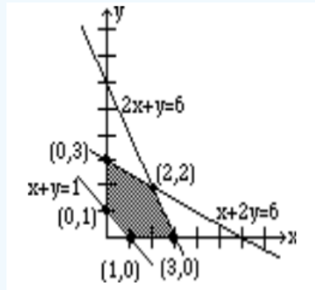
Example 8.2.3

Solve the following maximization problem graphically.

$$\begin{array}{ll} \textbf{Maximize} & \mathrm{P = 10x + 15y} \\ \textbf{Subject to:} & \mathrm{x + y \geq 1} \\ & \mathrm{x + 2y \leq 6} \\ & \mathrm{2x + y \leq 6} \\ & \mathrm{x \geq 0; y \geq 0} \end{array}$$

Solution

The graph is shown below.



The five critical points are listed in the above figure. The reader should observe that the first constraint $x+y \geq 1$ requires that the feasibility region must be bounded below by the line $x+y=1$; the test point (0,0) does not satisfy $x+y \geq 1$, so we shade the region on the opposite side of the line from test point (0,0).

Critical point	Income
(1, 0)	10(1) + 15(0) = $10
(3, 0)	10(3) + 15(0) = $30
(2, 2)	10(2) + 15(2) = $50
(0, 3)	10(0) + 15(3) = $45
(0,1)	10(0) + 15(1) = $15

Clearly, the point (2, 2) maximizes the objective function to a maximum value of 50.

It is important to observe that that if the point (0,0) lies on the line for a constraint, then (0,0) could not be used as a test point. We would need to select any other point we want that does not lie on the line to use as a test point in that situation.

Finally, we address an important question. Is it possible to determine the point that gives the maximum value without calculating the value at each critical point?

The answer is yes.

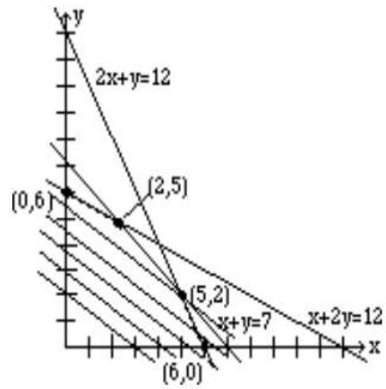
For example $8.2.2$, we substituted the points (0, 0), (0, 6), (2, 5), (5, 2), and (6, 0), in the objective function $P=20x+30y$, and we got the values $0, $180, $190, $160, $120, respectively.

Sometimes that is not the most efficient way of finding the optimum solution. Instead we could find the optimal value by also graphing the objective function.

To determine the largest P, we graph $P=20x+30y$ for any value P of our choice. Let us say, we choose $P=60$. We graph $20x+30y=60$.

Now we move the line parallel to itself, that is, keeping the same slope at all times. Since we are moving the line parallel to itself, the slope is kept the same, and the only thing that is changing is the P. As we move away from the origin, the value of P increases. The largest possible value of P is realized when the line touches the last corner point of the feasibility region.

The figure below shows the movements of the line, and the optimum solution is achieved at the point (2, 5). In maximization problems, as the line is being moved away from the origin, this optimum point is the farthest critical point.



We summarize:

The Maximization Linear Programming Problems

1. Write the objective function.
2. Write the constraints.
    1. For the standard maximization linear programming problems, constraints are of the form: $ax + by \leq c$
    2. Since the variables are non-negative, we include the constraints: $x \geq 0$; $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the maximum value.
    a. This is done by finding the value of the objective function at each corner point.
    b. This can also be done by moving the line associated with the objective function.

8.2.1: Maximization Applications (Exercises)

For the following maximization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

2) Mr. Tran has $24,000 to invest, some in bonds and the rest in stocks. He has decided that the money invested in bonds must be at least twice as much as that in stocks. But the money invested in bonds must not be greater than $18,000. If the bonds earn 6%, and the stocks earn 8%, how much money should he invest in each to maximize profit?

3) A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most 25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair and $30 for a table, how many units of each should be manufactured to maximize revenue?

4) The Silly Nut Company makes two mixtures of nuts: Mixture A and Mixture B. A pound of Mixture A contains 12 oz of peanuts, 3 oz of almonds and 1 oz of cashews and sells for $4. A pound of Mixture B contains 12 oz of peanuts, 2 oz of almonds and 2 oz of cashews and sells for $5. The company has 1080 lb. of peanuts, 240 lb. of almonds, 160 lb. of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit?
(Hint: Use consistent units. Work the entire problem in pounds by converting all values given in ounces into fractions of pounds).

5)

$$\begin{array}{ll}\text{Maximize:} & Z=4x+10y \\ \text{Subject to:} & x+y\le 5 \\ & 2x+y\le 8 \\ & x+2y\le 8 \\ & x\ge 0, y\ge 0\end{array}$$

6) This maximization linear programming problem is not in "standard" form. It has mixed constraints, some involving ≤ inequalities and some involving ≥ inequalities. However with careful graphing, we can solve this using the techniques we have learned in this section.

$$\begin{array}{ll}\text{Maximize:} & Z=5x+7y \\ \text{Subject to:} & x+y\le 30 \\ & 2x+y\le 50 \\ & 4x+3y\ge 60 \\ & 2x\ge y \\ & x\ge 0, y\ge 0\end{array}$$

8.3: Minimization Applications

Learning Objectives

In this section, you will learn to:

1. Formulate minimization linear programming problems
2. Graph feasibility regions for maximization linear programming problems
3. Determine optimal solutions for maximization linear programming problems.

Linear Programming - Minimization

Minimization linear programming problems are solved in much the same way as the maximization problems.

For the **standard minimization linear program**, the constraints are of the form $ax + by \geq c$, as opposed to the form $ax + by \leq c$ for the standard maximization problem. As a result, the feasible solution extends indefinitely to the upper right of the first quadrant, and is unbounded. But that is not a concern, since in order to minimize the objective function, the line associated with the objective function is moved towards the origin, and the critical point that minimizes the function is closest to the origin.

However, one should be aware that in the case of an unbounded feasibility region, the possibility of no optimal solution exists.

Example 8.3.1

At a university, Professor Symons wishes to employ two people, John and Mary, to grade papers for his classes. John is a graduate student and can grade 20 papers per hour; John earns \$15 per hour for grading papers. Mary is an post-doctoral associate and can grade 30 papers per hour; Mary earns \$25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment.
If Prof. Symons has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?

Solution

We choose the variables as follows:

Let x = The number of hours per week John is employed.

and y = The number of hours per week Mary is employed.

The objective function is

$$C = 15x + 25y$$

The fact that each must work at least one hour each week results in the following two constraints:

$$x \geq 1$$
$$y \geq 1$$

Since John can grade 20 papers per hour and Mary 30 papers per hour, and there are at least 110 papers to be graded per week, we get

$$20x + 30y \geq 110$$

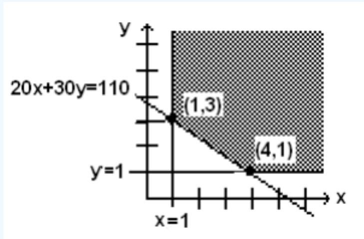
The fact that x and y are non-negative, we get

$$x \geq 0, \text{ and } y \geq 0.$$

The problem has been formulated as follows.

$$\begin{array}{ll} \textbf{Minimize} & C = 15x + 25y \\ \textbf{Subject to:} & x \geq 1 \\ & y \geq 1 \\ & 20x + 30y \geq 110 \\ & x \geq 0; y \geq 0 \end{array}$$

To solve the problem, we graph the constraints as follows:



Again, we have shaded the feasibility region, where all constraints are satisfied.

If we used test point (0,0) that does not lie on any of the constraints, we observe that (0, 0) **does not** satisfy any of the constraints $x \geq 1$, $y \geq 1$, $20x + 30y \geq 110$. Thus all the shading for the feasibility region lies on the opposite side of the constraint lines from the point (0,0).

Alternatively we could use test point (4,6), which also does not lie on any of the constraint lines. We'd find that (4,6) **does** satisfy all of the inequality constraints. Consequently all the shading for the feasibility region lies on the same side of the constraint lines as the point (4,6).

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify the two critical points, (1, 3) and (4, 1). To minimize cost, we will substitute these points in the objective function to see which point gives us the minimum cost each week. The results are listed below.

Critical points	Income
(1, 3)	15(1) + 25(3) = $90
(4, 1)	15(4) + 25(1) = $85

The point (4, 1) gives the least cost, and that cost is $85. Therefore, we conclude that in order to minimize grading costs, Professor Symons should employ John 4 hours a week, and Mary 1 hour a week at a cost of $85 per week.

Example 8.3.2

Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals, Pasta or Tofu. The table below lists the amount of protein, carbohydrates, and vitamins each meal provides along with the amount of cholesterol he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of vitamins for lunch each month. Over this time period, how many days should he have the Pasta meal, and how many days the Tofu meal so that he gets the adequate amount of protein, carbohydrates, and vitamins and at the same time minimizes his cholesterol intake?

	PASTA	**TOFU**
PROTEIN	8g	16g
CARBOHYDRATES	60g	40g
VITAMIN C	2g	2g
CHOLESTEROL	60mg	50mg

Solution

We choose the variables as follows.

Let x = The number of days Mr. Hamer eats Pasta.

and y = The number of days Mr. Hamer eats Tofu.

Since he is trying to minimize his cholesterol intake, our objective function represents the total amount of cholesterol C provided by both meals.

$$C = 60x + 50y$$

The constraint associated with the total amount of protein provided by both meals is

$$8x + 16y \geq 200$$

Similarly, the two constraints associated with the total amount of carbohydrates and vitamins are obtained, and they are

$$\begin{aligned} &60x + 40y \geq 960 \\ &2x + 2y \geq 40 \end{aligned}$$

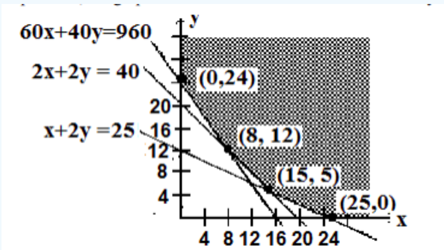
The constraints that state that x and y are non-negative are

$$x \geq 0, \text{ and } y \geq 0.$$

We summarize all information as follows:

$$\begin{array}{ll} \textbf{Minimize} & \mathrm{C = 60x + 50y} \\ \textbf{Subject to:} & \mathrm{8x + 16y \geq 200} \\ & \mathrm{60x + 40y \geq 960} \\ & \mathrm{2x + 2y \geq 40} \\ & \mathrm{x \geq 0; y \geq 0} \end{array}$$

To solve the problem, we graph the constraints and shade the feasibility region.



We have shaded the unbounded feasibility region, where all constraints are satisfied.

To minimize the objective function, we find the vertices of the feasibility region. These vertices are (0, 24), (8, 12), (15, 5) and (25, 0). To minimize cholesterol, we will substitute these points in the objective function to see which point gives us the smallest value. The results are listed below.

Critical points	Income
(0, 24)	60(0) + 50(24) = 1200
(8, 12)	60(8) + 50(12) = 1080
(15, 5)	60(15) + 50(5) = 1150
(25, 0)	60(25) + 50(0) = 1500

The point (8, 12) gives the least cholesterol, which is 1080 mg. This states that for every 20 meals, Professor Hamer should eat Pasta 8 days, and Tofu 12 days.

We must be aware that in some cases, a linear program may not have an optimal solution.

- A linear program can fail to have an optimal solution is if there is not a feasibility region. If the inequality constraints are not compatible, there may not be a region in the graph that satisfies **all** the constraints. If the linear program does not have a feasible solution satisfying all constraints, then it can not have an optimal solution.
- A linear program can fail to have an optimal solution if the feasibility region is unbounded.
  - The two minimization linear programs we examined had unbounded feasibility regions. The feasibility region was bounded by constraints on some sides but was not entirely enclosed by the constraints. Both of the minimization problems had optimal solutions.
  - However, if we were to consider a maximization problem with a similar unbounded feasibility region, the linear program would have no optimal solution. No matter what values of x and y were selected, we could always find other values of x and y that would produce a higher value for the objective function. In other words, if the value of the objective function can be increased without bound in a linear program with an unbounded feasible region, there is no optimal maximum solution.

Although the method of solving minimization problems is similar to that of the maximization problems, we still feel that we should summarize the steps involved.

Minimization Linear Programming Problems

1. Write the objective function.
2. Write the constraints.
   a. For standard minimization linear programming problems, constraints are of the form: $ax + by \geq c$
   b. Since the variables are non-negative, include the constraints: $x \geq 0$; $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the minimum value.
   a. This can be done by finding the value of the objective function at each corner point.
   b. This can also be done by moving the line associated with the objective function.
   c. There is the possibility that the problem has no solution

8.3.1: Minimization Applications (Exercises)

For each of the following minimization problems, choose your variables, write the objective function and the constraints, graph the constraints, shade the feasibility region, label all critical points, and determine the solution that optimizes the objective function.

1) A diet is to contain at least 2400 units of vitamins, 1800 units of minerals, and 1200 calories. Two foods, Food A and Food B are to be purchased. Each unit of Food A provides 50 units of vitamins, 30 units of minerals, and 10 calories. Each unit of Food B provides 20 units of vitamins, 20 units of minerals, and 40 calories. Food A costs \$2 per unit and Food B cost \$1 per unit. How many units of each food should be purchased to keep costs at a minimum?

2) A computer store sells two types of computers, laptops and desktops. The supplier demands that at least 150 computers be sold a month. Experience shows that most consumers prefer laptops, but some business customers require desktops. The result is that the number of laptops sold is at least twice of the number of desktops. The store pays its sales staff a \$60 commission for each laptop, and a \$40 commission for each desktop. Let x = the number of laptops and y = the number of desktop computers. How many of each type must be sold to minimize commission to its sales people?
What is the minimum commission?

3) An oil company has two refineries. Each day, Refinery A produces 200 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs \$12,000 to operate. Each day, Refinery B produces 100 barrels of high-grade oil, 100 barrels of medium-grade oil, and 200 barrels of low-grade oil and costs \$10,000 to operate. The company must produce at least 800 barrels of high-grade oil, 900 barrels of medium-grade oil, and 1,000 barrels of low-grade oil.
How many days should each refinery be operated to meet the goals at a minimum cost?

4) A print shop at a community college in Cupertino, California, employs two different contractors to maintain its copying machines. The print shop needs to have 12 IBM, 18 Xerox, and 20 Canon copying machines serviced. Contractor A can repair 2 IBM, 1 Xerox, and 2 Canon machines at a cost of \$800 per month, while Contractor B can repair 1 IBM, 3 Xerox, and 2 Canon machines at a cost of \$1000 per month. How many months should each of the two contractors be employed to minimize the cost?

8.4: Chapter Review

Solve the following linear programming problems by the graphical method.

1) Mr. Shoemacher has \$20,000 to invest in two types of mutual funds: a High-Yield Fund and an Equity Fund. The High-Yield fund has an annual yield of 12%, while the Equity fund earns 8%. He would like to invest at least \$3000 in the High-Yield fund and at least \$4000 in the Equity fund. How much should he invest in each to maximize his annual yield, and what is the maximum yield?

2) Dr. Lum teaches part-time at two community colleges, Hilltop College and Serra College. Dr. Lum can teach up to 5 classes per semester. For every class he teaches at Hilltop College, he needs to spend 3 hours per week preparing lessons and grading papers. For each class at Serra College, he must do 4 hours of work per week. He has determined that he cannot spend more than 18 hours per week preparing lessons and grading papers. If he earns \$6,000 per class at Hilltop College and \$7,500 per class at Serra College, how many classes should he teach at each college to maximize his income, and what will be his income?

3) Mr. Shamir employs two part-time typists, Inna and Jim, for his typing needs. Inna charges \$15 an hour and can type 6 pages an hour, while Jim charges \$18 an hour and can type 8 pages per hour. Each typist must be employed at least 8 hours per week to keep them on the payroll. If Mr. Shamir has at least 208 pages to be typed, how many hours per week should he employ each typist to minimize his typing costs, and what will be the total cost?

4) Mr. Boutros wants to invest up to \$20,000 in two stocks, Cal Computers and Texas Tools. The Cal Computers stock is expected to yield a 16% annual return, while the Texas Tools stock promises a 12% yield. Mr. Boutros would like to earn at least \$2,880 this year. According to Value Line Magazine's safety index (1 highest to 5 lowest), Cal Computers has a safety number of 3 and Texas Tools has a safety number of 2. How much money should he invest in each to minimize the safety number? Note: A lower safety number means less risk.

5) A store sells two types of copy machines: compact (low capacity) and standard (which takes more space). The store can sell up to 90 copiers a month. A maximum of 1080 cubic feet of storage space is available. A compact copier requires 6 cu. ft. of storage space, and a standard copier requires 18 cu. ft.. The compact and standard copy machines take, respectively, 1 and 1.5 sales hours of labor.
A maximum of 99 hours of labor is available. The profit from each of these copiers is \$60 and \$80, respectively, how many of each type should be sold to maximize profit, and what is the maximum profit?

6) A company manufactures two types of cell phones, a Basic model and a Pro model. The Basic model generates a profit of \$100 per phone and the Pro model has a profit of \$150 per phone. On the assembly line the Basic phone requires 7 hours, while the Pro model takes 11 hours. The Basic phone requires one hour and the Pro phone needs 3 hours for finishing, which includes loading software. Both phones require one hour for testing. On a particular production run the company has available 1,540 work hours on the assembly line, 360 work hours for finishing, and 200 work hours in the testing department. How many cell phones of each type should be produced to maximize profit, and what is that maximum profit?

7) John wishes to choose a combination of two types of cereals for breakfast - Cereal A and Cereal B. A small box (one serving) of Cereal A costs \$0.50 and contains 10 units of vitamins, 5 units of minerals, and 15 calories. A small box(one serving) of Cereal B costs \$0.40 and contains 5 units of vitamins, 10 units of minerals, and 15 calories. John wants to buy enough boxes to have at least 500 units of vitamins, 600 units of minerals, and 1200 calories. How many boxes of each food should he buy to minimize his cost, and what is the minimum cost?

8) Jessica needs at least 60 units of vitamin A, 40 units of vitamin B, and 140 units of vitamin C each week. She can choose between Costless brand or Savemore brand tablets. A Costless tablet costs 5 cents and contains 3 units of vitamin A, 1 unit of vitamin B, and 2 units of vitamin C. A Savemore tablet costs 7 cents and contains 1 unit of A, 1 of B, and 5 of C. How many tablets of each kind should she buy to minimize cost, and what is the minimum cost?

9) A small company manufactures two products: A and B. Each product requires three operations: Assembly, Finishing and Testing. Product A requires 1 hour of Assembly, 3 hours of Finishing, and 1 hour of Testing. Product B requires 3 hours of Assembly, 1 hour of Finishing, and 1 hour of Testing. The total work-hours available per week in the Assembly division is 60, in Finishing is 60, and in Testing is 24. Each item of product A has a profit of \$50, and each item of Product B has a profit of \$75. How many of each should be made to maximize profit? What is the maximum profit?

10) A factory manufactures two products, A and B. Each product requires the use of three machines, Machine I, Machine II, and Machine III. The time requirements and total hours available on each machine are listed below.

	Machine I	Machine II	Machine III
Product A	1	2	4
Product B	2	2	2
Total hours	70	90	160

If product A generates a profit of \$60 per unit and product B a profit of \$50 per unit, how many units of each product should be manufactured to maximize profit, and what is the maximum profit?

11) A company produces three types of shoes, formal, casual, and athletic, at its two factories, Factory I and Factory II. The company must produce at least 6000 pairs of formal shoes, 8000 pairs of casual shoes, and 9000 pairs of athletic shoes. Daily production of each factory for each type of shoe is:

	Factory I	Factory II
Formal	100	100
Casual	100	200
Athletic	300	100

Operating Factory I costs \$1500 per day and it costs \$2000 per day to operate Factory II. How many days should each factory operate to complete the order at a minimum cost, and what is the minimum cost?

12) A professor gives two types of quizzes, objective and recall. He plans to give at least 15 quizzes this quarter. The student preparation time for an objective quiz is 15 minutes and for a recall quiz 30 minutes. The professor would like a student to spend at least 5 hours (300 minutes) preparing for these quizzes above and beyond the normal study time. The average score on an objective quiz is 7, and on a recall type 5, and the professor would like the students to score at least 85 points on all quizzes. It takes the professor one minute to grade an objective quiz, and 1.5 minutes to grade a recall type quiz. How many of each type should he give in order to minimize his grading time?

13) A company makes two mixtures of nuts: Mixture A and Mixture B. Mixture A contains 30% peanuts, 30% almonds and 40% cashews and sells for \$5 per pound. Mixture B contains 30% peanuts, 60% almonds and 10% cashews and sells for \$3 a pound. The company has 540 pounds of peanuts, 900 pounds of almonds, 480 pounds of cashews. How many pounds of each of mixtures A and B should the company make to maximize profit, and what is the maximum profit?

CHAPTER OVERVIEW

9: Linear Programming - The Simplex Method

Learning Objectives

In this chapter, you will:

1. Investigate real world applications of linear programming and related methods.
2. Solve linear programming maximization problems using the simplex method.
3. Solve linear programming minimization problems using the simplex method.

9.1: Introduction to Linear Programming Applications in Business, Finance, Medicine, and Social Science

9.2: Maximization By The Simplex Method

9.2.1: Maximization By The Simplex Method (Exercises)

9.3: Minimization By The Simplex Method

9.3.1: Minimization By The Simplex Method (Exercises)

9.4: Chapter Review

9.1: Introduction to Linear Programming Applications in Business, Finance, Medicine, and Social Science

Learning Objectives

In this section, you will learn about real-world applications of linear programming and related methods.

The linear programs we solved in Chapter 7 contain only two variables, x and y, so that we could solve them graphically. In practice, linear programs can contain thousands of variables and constraints. Later in this chapter, we'll learn to solve linear programs with more than two variables using the simplex algorithm, which is a numerical solution method that uses matrices and row operations. However, in order to make the problems practical for learning purposes, our problems will still have only several variables. Now that we understand the main concepts behind linear programming, we can also consider how linear programming is currently used in large-scale real-world applications.

Linear programming is used in business and industry in production planning, transportation and routing, and various types of scheduling. Airlines use linear programs to schedule their flights, taking into account both scheduling aircraft and scheduling staff. Delivery services use linear programs to schedule and route shipments to minimize shipment time or minimize cost. Retailers use linear programs to determine how to order products from manufacturers and organize deliveries with their stores. Manufacturing companies use linear programming to plan and schedule production. Financial institutions use linear programming to determine the mix of financial products they offer, or to schedule payments transferring funds between institutions. Health-care institutions use linear programming to ensure the proper supplies are available when needed. And as we'll see below, linear programming has also been used to organize and coordinate life-saving health care procedures.

In some of the applications, the techniques used are related to linear programming but are more sophisticated than the methods we study in this class. One such technique is called integer programming. In these situations, answers must be integers to make sense, and can not be fractions. Problems, where solutions must be integers, are more difficult to solve than the linear programs we've worked with. In fact, many of our problems have been very carefully constructed for learning purposes so that the answers just happen to turn out to be integers, but in the real world unless we specify that as a restriction, there is no guarantee that a linear program will produce integer solutions. There are also related techniques that are called non-linear programs, where the functions defining the objective function and/or some or all of the constraints may be non-linear rather than straight lines.

Many large businesses that use linear programming and related methods have analysts on their staff who can perform the analyses needed, including linear programming and other mathematical techniques. Consulting firms specializing in use of such techniques also aid businesses who need to apply these methods to their planning and scheduling processes.

When used in business, many different terms may be used to describe the use of techniques such as linear programming as part of mathematical business models. Optimization, operations research, business analytics, data science, industrial engineering hand management science are among the terms used to describe mathematical modeling techniques that may include linear programming and related met

In the rest of this section, we'll explore six real-world applications, and investigate what they are trying to accomplish using optimization, as well as what their constraints might represent.

Airline Scheduling

Airlines use techniques that include and are related to linear programming to schedule their aircrafts to flights on various routes, and to schedule crews to the flights. In addition, airlines also use linear programming to determine ticket pricing for various types of seats and levels of service or amenities, as well as the timing at which ticket prices change.

The process of scheduling aircraft and departure times on flight routes can be expressed as a model that minimizes cost, of which the largest component is generally fuel costs.

Constraints involve considerations such as:

- Each aircraft needs to complete a daily or weekly tour to return back to its point of origin.
- Scheduling sufficient flights to meet demand on each route.
- Scheduling the right type and size of aircraft on each route to be appropriate for the route and for the demand for number of passengers.

- Aircraft must be compatible with the airports it departs from and arrives at - not all airports can handle all types of planes.

A model to accomplish this could contain thousands of variables and constraints. Highly trained analysts determine ways to translate all the constraints into mathematical inequalities or equations to put into the model.

After aircraft are scheduled, crews need to be assigned to flights. Each flight needs a pilot, a co-pilot, and flight attendants. Each crew member needs to complete a daily or weekly tour to return back to his or her home base. Additional constraints on flight crew assignments take into account factors such as:

- Pilot and co-pilot qualifications to fly the particular type of aircraft they are assigned to
- Flight crew have restrictions on the maximum amount of flying time per day and the length of mandatory rest periods between flights or per day that must meet certain minimum rest time regulations.
- Numbers of crew members required for a particular type or size of aircraft.

When scheduling crews to flights, the objective function would seek to minimize total flight crew costs, determined by the number of people on the crew and pay rates of the crew members. However, the cost for any particular route might not end up being the lowest possible for that route, depending on tradeoffs to the total cost of shifting different crews to different routes.

An airline can also use linear programming to revise schedules on short notice on an emergency basis when there is a schedule disruption, such as due to weather. In this case the considerations to be managed involve:

- Getting aircrafts and crews back on schedule as quickly as possible
- Moving aircraft from storm areas to areas with calm weather to keep the aircraft safe from damage and ready to come back into service as quickly and conveniently as possible
- Ensuring crews are available to operate the aircraft and that crews continue to meet mandatory rest period requirements and regulations.

Kidney Donation Chain

For patients who have kidney disease, a transplant of a healthy kidney from a living donor can often be a lifesaving procedure. Criteria for a kidney donation procedure include the availability of a donor who is healthy enough to donate a kidney, as well as a compatible match between the patient and donor for blood type and several other characteristics. Ideally, if a patient needs a kidney donation, a close relative may be a match and can be the kidney donor. However often there is not a relative who is a close enough match to be the donor. Considering donations from unrelated donor allows for a larger pool of potential donors. Kidney donations involving unrelated donors can sometimes be arranged through a chain of donations that pair patients with donors. For example a kidney donation chain with three donors might operate as follows:

- Donor A donates a kidney to Patient B.
- Donor B, who is related to Patient B, donates a kidney to Patient C.
- Donor C, who is related to Patient C, donates a kidney to Patient A, who is related to Donor A.

Linear programming is one of several mathematical tools that have been used to help efficiently identify a kidney donation chain. In this type of model, patient/donor pairs are assigned compatibility scores based on characteristics of patients and potential donors.

The objective is to maximize the total compatibility scores. Constraints ensure that donors and patients are paired only if compatibility scores are sufficiently high to indicate an acceptable match.

Advertisements in Online Marketing

Did you ever make a purchase online and then notice that as you browse websites, search, or use social media, you now see more ads related the item you purchased?

Marketing organizations use a variety of mathematical techniques, including linear programming, to determine individualized advertising placement purchases.

Instead of advertising randomly, online advertisers want to sell bundles of advertisements related to a particular product to batches of users who are more likely to purchase that product. Based on an individual’s previous browsing and purchase selections, he or she is assigned a “propensity score” for making a purchase if shown an ad for a certain product. The company placing the ad generally does not know individual personal information based on the history of items viewed and purchased, but instead has aggregated information for groups of individuals based on what they view or purchase. However, the company may know more about an individual's history if he or she logged into a website making that information identifiable, within the privacy provisions and terms of use of the site.

The company's goal is to buy ads to present to specified size batches of people who are browsing. The linear program would assign ads and batches of people to view the ads using an objective function that seeks to maximize advertising response modeled using the propensity scores. The constraints are to stay within the restrictions of the advertising budget.

Loans

A car manufacturer sells its cars through dealers. Dealers can offer loan financing to customers who need to take out loans to purchase a car. Here we will consider how car manufacturers can use linear programming to determine the specific characteristics of the loan they offer to a customer who purchases a car. In a past chapter, we will learn how to do the financial calculations related to loans.

A customer who applies for a car loan fills out an application. This provides the car dealer with information about that customer. In addition, the car dealer can access a credit bureau to obtain information about a customer's credit score.

Based on this information obtained about the customer, the car dealer offers a loan with certain characteristics, such as interest rate, loan amount, and length of loan repayment period.

Linear programming can be used as part of the process to determine the characteristics of the loan offer. The linear program seeks to maximize the profitability of its portfolio of loans. The constraints limit the risk that the customer will default and will not repay the loan. The constraints also seek to minimize the risk of losing the loan customer if the conditions of the loan are not favorable enough; otherwise, the customer may find another lender, such as a bank, which can offer a more favorable loan.

Production Planning and Scheduling in Manufacturing

Consider the example of a company that produces yogurt. There are different varieties of yogurt products in a variety of flavors. Yogurt products have a short shelf life; it must be produced on a timely basis to meet demand, rather than drawing upon a stockpile of inventory as can be done with a product that is not perishable. Most ingredients in yogurt also have a short shelf life, so can not be ordered and stored for long periods of time before use; ingredients must be obtained in a timely manner to be available when needed but still be fresh. Linear programming can be used in both production planning and scheduling.

To start the process, sales forecasts are developed to determine demand to know how much of each type of product to make.

There are often various manufacturing plants at which the products may be produced. The appropriate ingredients need to be at the production facility to produce the products assigned to that facility. Transportation costs must be considered, both for obtaining and delivering ingredients to the correct facilities, and for transport of finished product to the sellers.

The linear program that monitors production planning and scheduling must be updated frequently - daily or even twice each day - to take into account variations from a master plan.

Bike Share Programs

Over 600 cities worldwide have bikeshare programs. Although bikeshare programs have been around for a long time, they have proliferated in the past decade as technology has developed new methods for tracking the bicycles.

Bikeshare programs vary in the details of how they work, but most typically people pay a fee to join and then can borrow a bicycle from a bike share station and return the bike to the same or a different bike share station. Over time the bikes tend to migrate; there may be more people who want to pick up a bike at station A and return it at station B than there are people who want to do the opposite. In chapter 9, we'll investigate a technique that can be used to predict the distribution of bikes among the stations.

Once other methods are used to predict the actual and desired distributions of bikes among the stations, bikes may need to be transported between stations to even out the distribution. Bikeshare programs in large cities have used methods related to linear programming to help determine the best routes and methods for redistributing bicycles to the desired stations once the desire distributions have been determined. The optimization model would seek to minimize transport costs and/or time subject to constraints of having sufficient bicycles at the various stations to meet demand.

- **4.1: Introduction to Linear Programming Applications in Business, Finance, Medicine, and Social Science** by Rupinder Sekhon and Roberta Bloom is licensed CC BY 4.0. Original source: https://www.deanza.edu/faculty/bloomroberta/math11/afm3files.html.html.

9.2: Maximization By The Simplex Method

Learning Objectives

In this section, you will learn to solve linear programming maximization problems using the Simplex Method:

1. Identify and set up a linear program in standard maximization form
2. Convert inequality constraints to equations using slack variables
3. Set up the initial simplex tableau using the objective function and slack equations
4. Find the optimal simplex tableau by performing pivoting operations.
5. Identify the optimal solution from the optimal simplex tableau.

In the last chapter, we used the geometrical method to solve linear programming problems, but the geometrical approach will not work for problems that have more than two variables. In real-life situations, linear programming problems consist of literally thousands of variables and are solved by computers. We can solve these problems algebraically, but that will not be very efficient. Suppose we were given a problem with, say, 5 variables and 10 constraints. By choosing all combinations of five equations with five unknowns, we could find all the corner points, test them for feasibility, and come up with the solution, if it exists. But the trouble is that even for a problem with so few variables, we will get more than 250 corner points, and testing each point will be very tedious. So we need a method that has a systematic algorithm and can be programmed for a computer. The method has to be efficient enough so we wouldn't have to evaluate the objective function at each corner point. We have just such a method, and it is called the **simplex method**.

The simplex method was developed during the Second World War by Dr. George Dantzig. His linear programming models helped the Allied forces with transportation and scheduling problems. In 1979, a Soviet scientist named Leonid Khachian developed a method called the ellipsoid algorithm which was supposed to be revolutionary, but as it turned out it is not any better than the simplex method. In 1984, Narendra Karmarkar, a research scientist at AT&T Bell Laboratories developed Karmarkar's algorithm which has been proven to be four times faster than the simplex method for certain problems. But the simplex method still works the best for most problems.

The simplex method uses an approach that is very efficient. It does not compute the value of the objective function at every point; instead, it begins with a corner point of the feasibility region where all the main variables are zero and then systematically moves from corner point to corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found.

To learn the simplex method, we try a rather unconventional approach. We first list the algorithm, and then work a problem. We justify the reasoning behind each step during the process. A thorough justification is beyond the scope of this course.

We start out with an example we solved in the last chapter by the graphical method. This will provide us with some insight into the simplex method and at the same time give us the chance to compare a few of the feasible solutions we obtained previously by the graphical method. But first, we list the algorithm for the simplex method.

THE SIMPLEX METHOD

1. **Set up the problem.** That is, write the objective function and the inequality constraints.
2. **Convert the inequalities into equations.** This is done by adding one slack variable for each inequality.
3. **Construct the initial simplex tableau.** Write the objective function as the bottom row.
4. **The most negative entry in the bottom row identifies the pivot column.**
5. **Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.** The quotients are computed by dividing the far right column by the identified column in step 4. A quotient that is a zero, or a negative number, or that has a zero in the denominator, is ignored.
6. **Perform pivoting to make all other entries in this column zero.** This is done the same way as we did with the Gauss-Jordan method.
7. **When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.**
8. **Read off your answers.** Get the variables using the columns with 1 and 0s. All other variables are zero. The maximum value you are looking for appears in the bottom right-hand corner.

Now, we use the simplex method to solve Example 8.1.1 solved geometrically in section 8.1.

Example 9.2.1

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

In solving this problem, we will follow the algorithm listed above.

STEP 1. Set up the problem. Write the objective function and the constraints.

Since the simplex method is used for problems that consist of many variables, it is not practical to use the variables x, y, z etc. We use symbols x_1, x_2, x_3, and so on.

Let

- x_1 = The number of hours per week Niki will work at Job I and
- x_2 = The number of hours per week Niki will work at Job II.

It is customary to choose the variable that is to be maximized as Z.

The problem is formulated the same way as we did in the last chapter.

$$\begin{array}{ll} \textbf{Maximize} & \mathrm{Z}=40\mathrm{x}_1+30\mathrm{x}_2 \\ \textbf{Subject to:} & \mathrm{x}_1+\mathrm{x}_2 \leq 12 \\ & 2\mathrm{x}_1+\mathrm{x}_2 \leq 16 \\ & \mathrm{x}_1 \geq 0; \mathrm{x}_2 \geq 0 \end{array}$$

STEP 2. Convert the inequalities into equations. This is done by adding one slack variable for each inequality.

For example to convert the inequality $x_1+x_2 \leq 12$ into an equation, we add a non-negative variable s_1, and we get

$$x_1+x_2+s_1=12$$

Here the variable s_1 picks up the slack, and it represents the amount by which x_1+x_2 falls short of 12. In this problem, if Niki works fewer than 12 hours, say 10, then s_1 is 2. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

We rewrite the objective function $Z=40x_1+30x_2$ as $-40x_1-30x_2+Z=0$.

After adding the slack variables, our problem reads

$$\begin{array}{ll} \text{Objective function} & -40x_1-30x_2+Z=0 \\ \text{Subject to constraints:} & x_1+x_2+s_1=12 \\ & 2x_1+x_2+s_2=16 \\ & x1 \geq 0; x2 \geq 0 \end{array}$$

STEP 3. Construct the initial simplex tableau. Each inequality constraint appears in its own row. (The non-negativity constraints do *not* appear as rows in the simplex tableau.) Write the objective function as the bottom row.

Now that the inequalities are converted into equations, we can represent the problem into an augmented matrix called the initial simplex tableau as follows.

x1	x2	s1	s2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0

Here the vertical line separates the left-hand side of the equations from the right side. The horizontal line separates the constraints from the objective function. The right side of the equation is represented by the column C.

The reader needs to observe that the last four columns of this matrix look like the final matrix for the solution of a system of equations. If we arbitrarily choose $x_1 = 0$ and $x_2 = 0$, we get

$$\begin{bmatrix} s_1 & s_2 & Z & | & C \\ 1 & 0 & 0 & | & 12 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

which reads

$$s_1 = 12 \quad s_2 = 16 \quad Z = 0$$

The solution obtained by arbitrarily assigning values to some variables and then solving for the remaining variables is called the **basic solution** associated with the tableau. So the above solution is the basic solution associated with the initial simplex tableau. We can label the basic solution variable in the right of the last column as shown in the table below.

x1	x2	s1	s2	Z	C	
1	1	1	0	0	12	s1
2	1	0	1	0	16	s2
-40	-30	0	0	1	0	Z

STEP 4. The most negative entry in the bottom row identifies the pivot column.

The most negative entry in the bottom row is -40; therefore column 1 is identified.

x1	x2	s1	s2	Z	C	
1	1	1	0	0	12	s1
2	1	0	1	0	16	s2
-40	-30	0	0	1	0	Z
↑						

Question Why do we choose the most negative entry in the bottom row?

Answer The most negative entry in the bottom row represents the largest coefficient in the objective function - the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3 etc., are zero. It then moves from a corner point to the adjacent corner point always increasing the value of the objective function. In the case of the objective function $Z = 40x_1 + 30x2$, it will make more sense to increase the value of x_1 rather than x_2. The variable x_1 represents the number of hours per week Niki works at Job I. Since Job I pays $40 per hour as opposed to Job II which pays only $30, the variable x_1 will increase the objective function by $40 for a unit of increase in the variable x_1.

STEP 5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.

Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

x1	x2	s1	s2	Z	C		
1	1	1	0	0	12	s1	12/1 = 12
2	1	0	1	0	16	s2 ←	16/2 = 8
-40	-30	0	0	1	0	Z	
↑							

The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

Question Why do we find quotients, and why does the smallest quotient identify a row?

Answer When we choose the most negative entry in the bottom row, we are trying to increase the value of the objective function by bringing in the variable x_1. But we cannot choose any value for x_1. Can we let $x_1 = 100$? Definitely not! That is because Niki never wants to work for more than 12 hours at both jobs combined: $x_1 + x_2 \leq 12$. Can we let $x_1 = 12$? Again, the answer is no because the preparation time for Job I is two times the time spent on the job. Since Niki never wants to spend more than 16 hours for preparation, the maximum time she can work is 16 ÷ 2 = 8.

Now you see the purpose of computing the quotients; using the quotients to identify the pivot element guarantees that we do not violate the constraints.

Question Why do we identify the pivot element?

Answer As we have mentioned earlier, the simplex method begins with a corner point and then moves to the next corner point always improving the value of the objective function. The value of the objective function is improved by changing the number of units of the variables. We may add the number of units of one variable, while throwing away the units of another. Pivoting allows us to do just that.

The variable whose units are being added is called the **entering variable,** and the variable whose units are being replaced is called the **departing variable**. The entering variable in the above table is x_1, and it was identified by the most negative entry in the bottom row. The departing variable s_2 was identified by the lowest of all quotients.

STEP 6. Perform pivoting to make all other entries in this column zero.

In chapter 7, we used pivoting to obtain the row echelon form of an augmented matrix. Pivoting is a process of obtaining a 1 in the location of the pivot element, and then making all other entries zeros in that column. So now our job is to make our pivot element a 1 by dividing the entire second row by 2 (notation: (1/2)$R_2 \rightarrow R_2$). The result follows.

x1	x2	s1	s2	Z	C
1	1	1	0	0	12
1	1/2	0	1/2	0	8
-40	-30	0	0	1	0

To obtain a zero in the entry first above the pivot element, we multiply the second row by -1 and add it to row 1(notation: $-1R_2$ + R_1 → R_1). We get

x1	x2	s1	s2	Z	C
0	1/2	1	-1/2	0	4
1	1/2	0	1/2	0	8
-40	-30	0	0	1	0

To obtain a zero in the element below the pivot, we multiply the second row by 40 and add it to the last row(notation: $40R_2$ + R_3 → R_3).

x1	x2	s1	s2	Z	C	
0	1/2	1	-1/2	0	4	s1
1	1/2	0	1/2	0	8	x1
0	-10	0	20	1	320	Z

We now determine the basic solution associated with this tableau. By arbitrarily choosing $x_2 = 0$ and $s_2 = 0$, we obtain $x_1 = 8$, $s_1 = 4$, and $z = 320$. If we write the augmented matrix, whose left side is a matrix with columns that have one 1 and all other entries zeros, we get the following matrix stating the same thing.

$$\begin{bmatrix} \mathrm{x}_1 & \mathrm{s}_1 & \mathrm{Z} & | & \mathrm{C} \\ 0 & 1 & 0 & | & 4 \\ 1 & 0 & 0 & | & 8 \\ 0 & 0 & 1 & | & 320 \end{bmatrix}$$

We can restate the solution associated with this matrix as $x_1 = 8$, $x_2 = 0$, $s_1 = 4$, $s_2 = 0$ and $z = 320$. At this stage of the game, it reads that if Niki works 8 hours at Job I, and no hours at Job II, her profit Z will be $320. Recall from Example 8.1.1 in section 8.1 that (8, 0) was one of our corner points. Here $s_1 = 4$ and $s_2 = 0$ mean that she will be left with 4 hours of working time and no preparation time.

STEP 7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.

Since there is still a negative entry, -10, in the bottom row, we need to begin, again, from step 4. This time we will not repeat the details of every step, instead, we will identify the column and row that give us the pivot element, and highlight the pivot element. The result is as follows.

x1	x2	s1	s2	Z	C		
0	1/2	1	-1/2	0	4	x1 ←	4/(1/2) = 8
1	1/2	0	1/2	0	8	x1	8/(1/2) = 16
0	-10	0	20	1	320	Z	
	↑						

1

We make the pivot element 1 by multiplying row 1 by 2 (notation: $2R_1 \to R_1$), and we get

x1	x2	s1	s2	Z	C
0	1	2	-2	0	8
1	1/2	0	1/2	0	8
0	-10	0	20	1	320

Now to make all other entries as zeros in this column, we first multiply row 1 by -1/2 and add it to row 2(notation: (-1/2)R_1 + $R_2 \to R_2$), and then multiply row 1 by 10 (notation: $10R_1 + R_3 \to R_3$) and add it to the bottom row.

x1	x2	s1	s2	Z	C
0	1	2	-1	0	8
1	0	-1	1	0	4
0	0	20	10	1	400

We no longer have negative entries in the bottom row, therefore we are finished.

Question Why are we finished when there are no negative entries in the bottom row?

Answer The answer lies in the bottom row. The bottom row corresponds to the equation:

$$\begin{aligned} &0x_1 + 0x_2 + 20s_1 + 10s_2 + Z = 400 \quad \text{ or } \\ &z = 400 - 20s1 - 10s2 \end{aligned}$$

Since all variables are non-negative, the highest value Z can ever achieve is 400, and that will happen only when s_1 and s_2 are zero.

STEP 8. Read off your answers.

We now read off our answers, that is, we determine the basic solution associated with the final simplex tableau. Again, we look at the columns that have a 1 and all other entries zeros. Since the columns labeled s_1 and s_2 are not such columns, we arbitrarily choose $s_1 = 0$, and $s_2 = 0$, and we get

$$\left[\begin{array}{ccccc} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{Z} & | & \mathrm{C} \\ 0 & 1 & 0 & | & 8 \\ 1 & 0 & 0 & | & 4 \\ 0 & 0 & 1 & | & 400 \end{array}\right]$$

The matrix reads $x_1 = 4$, $x_2 = 8$ and $z = 400$.

The final solution says that if Niki works 4 hours at Job I and 8 hours at Job II, she will maximize her income to $400. Since both slack variables are zero, it means that she would have used up all the working time, as well as the preparation time, and none will be left.

9.2.1: Maximization By The Simplex Method (Exercises)

SECTION 9.2 PROBLEM SET: MAXIMIZATION BY THE SIMPLEX METHOD

Solve the following linear programming problems using the simplex method.

1)

$$\begin{array}{ll} \text{Maximize} & \mathrm{z = x_1 + 2x_2 + 3x_3} \\ \text{subject to} & \mathrm{x_1 + x_2 + x_3 \leq 12} \\ & \mathrm{2x_1 + x_2 + 3x_3 \leq 18} \\ & \mathrm{x_1, x_2, x_3 \geq 0} \end{array}$$

2)

$$\begin{array}{lll} \text{Maximize} & z = & x_1 + 2x_2 + x_3 \\ \text{subject to} & & x_1 + x_2 \leq 3 \\ & & x_2 + x_3 \leq 4 \\ & & x_1 + x_3 \leq 5 \\ & & x_1, x_2, x_3 \geq 0 \end{array}$$

3) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit?

SECTION 9.2 PROBLEM SET: MAXIMIZATION BY THE SIMPLEX METHOD

Solve the following linear programming problems using the simplex method.

4) A factory manufactures chairs, tables, and bookcases each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 600 hours; the second at most 500 hours; and the third at most 300 hours. A chair requires 1 hour of cutting, 1 hour of assembly, and 1 hour of finishing; a table needs 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; and a bookcase requires 3 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair, $30 for a table, and $25 for a bookcase, how many units of each should be manufactured to maximize profit?

5). The Acme Apple company sells its Pippin, Macintosh, and Fuji apples in mixes. Box I contains 4 apples of each kind; Box II contains 6 Pippin, 3 Macintosh, and 3 Fuji; and Box III contains no Pippin, 8 Macintosh and 4 Fuji apples. At the end of the season, the company has altogether 2800 Pippin, 2200 Macintosh, and 2300 Fuji apples left. Determine the maximum number of boxes that the company can make.

9.3: Minimization By The Simplex Method

Learning Objectives

In this section, you will learn to solve linear programming minimization problems using the simplex method.

1. Identify and set up a linear program in standard minimization form
2. Formulate a dual problem in standard maximization form
3. Use the simplex method to solve the dual maximization problem
4. Identify the optimal solution to the original minimization problem from the optimal simplex tableau.

In this section, we will solve the standard linear programming minimization problems using the simplex method. Once again, we remind the reader that in the standard minimization problems all constraints are of the form $ax + by \geq c$.

The procedure to solve these problems was developed by Dr. John Von Neuman. It involves solving an associated problem called the **dual problem.** To every minimization problem there corresponds a dual problem. The solution of the dual problem is used to find the solution of the original problem. The dual problem is a maximization problem, which we learned to solve in the last section. We first solve the dual problem by the simplex method.

From the final simplex tableau, we then extract the solution to the original minimization problem.

Before we go any further, however, we first learn to convert a minimization problem into its corresponding maximization problem called its **dual**.

Example 9.3.1

Convert the following minimization problem into its dual.

$$\begin{array}{ll}\textbf{Minimize} & \mathrm{Z}=12\mathrm{x}_1+16\mathrm{x}_2 \\ \textbf{Subject to:} & \mathrm{x}_1+2\mathrm{x}_2 \geq 40 \\ & \mathrm{x}_1+\mathrm{x}_2 \geq 30 \\ & \mathrm{x}_1 \geq 0; \mathrm{x}_2 \geq 0\end{array}$$

Solution

To achieve our goal, we first express our problem as the following matrix.

$$\begin{array}{cc|c}1 & 2 & 40 \\ 1 & 1 & 30 \\ \hline 12 & 16 & 0\end{array}$$

Observe that this table looks like an initial simplex tableau without the slack variables. Next, we write a matrix whose columns are the rows of this matrix, and the rows are the columns. Such a matrix is called a **transpose** of the original matrix (rows becomes columns). We get:

$$\begin{array}{cc|c}1 & 1 & 12 \\ 2 & 1 & 16 \\ \hline 40 & 30 & 0\end{array}$$

The following maximization problem associated with the above matrix is called its dual.

$$\begin{array}{ll}\textbf{Maximize} & \mathrm{Z}=40\mathrm{y}_1+30\mathrm{y}_2 \\ \textbf{Subject to:} & \mathrm{y}_1+\mathrm{y}_2 \leq 12 \\ & 2\mathrm{y}_1+\mathrm{y}_2 \leq 16 \\ & \mathrm{y}_1 \geq 0; \mathrm{y}_2 \geq 0\end{array}$$

Note that we have chosen the variables as y's, instead of x's, to distinguish the two problems.

✓ Example 9.3.2

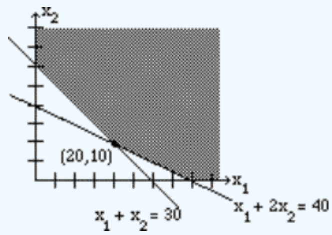
Solve graphically both the minimization problem and its dual maximization problem.

Solution

Our minimization problem is as follows.

$$\begin{array}{ll} \textbf{Minimize} & Z = 12x_1 + 16x_2 \\ \textbf{Subject to:} & x_1 + 2x_2 \geq 40 \\ & x_1 + x_2 \geq 30 \\ & x_1 \geq 0; x_2 \geq 0 \end{array}$$

We now graph the inequalities:

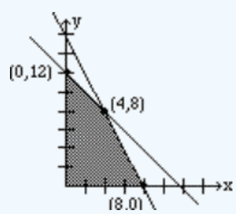


We have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (20, 10) gives the lowest value for the objective function and that value is 400.

Now its dual is:

$$\begin{array}{ll} \textbf{Maximize} & Z = 40y_1 + 30y_2 \\ \textbf{Subject to:} & y_1 + y_2 \leq 12 \\ & 2y_1 + y2 \leq 16 \\ & y_1 \geq 0; y_2 \geq 0 \end{array}$$

We graph the inequalities:



Again, we have plotted the graph, shaded the feasibility region, and labeled the corner points. The corner point (4, 8) gives the highest value for the objective function, with a value of 400.

The Duality Principle

✎ The Duality Principle

The objective function of the minimization problem reaches its minimum if and only if the objective function of its dual reaches its maximum. And when they do, they are equal.

Our next goal is to extract the solution for our minimization problem from the corresponding dual. To do this, we solve the dual by the simplex method.

Example 9.3.3

Find the solution to the minimization problem in Example 9.3.1 by solving its dual using the simplex method. We rewrite our problem.

$$\begin{array}{ll}\textbf{Minimize} & Z = 12x_1 + 16x_2\\ \textbf{Subject to:} & x_1 + 2x_2 \geq 40\\ & x_1 + x_2 \geq 30\\ & x_1 \geq 0; x_2 \geq 0\end{array}$$

Solution

$$\begin{array}{ll}\textbf{Maximize} & Z = 40y_1 + 30y_2\\ \textbf{Subject to:} & y_1 + y_2 \leq 12\\ & 2y_1 + y_2 \leq 16\\ & y_1 \geq 0; y_2 \geq 0\end{array}$$

Recall that we solved the above problem by the simplex method in Example 9.1.1, section 9.1. Therefore, we only show the initial and final simplex tableau.

The initial simplex tableau is

$$\begin{array}{ccccc|c} y_1 & y_2 & x_1 & x_2 & Z & C\\ 1 & 1 & 1 & 0 & 0 & 12\\ 2 & 1 & 0 & 1 & 0 & 16\\ \hline -40 & -30 & 0 & 0 & 1 & 0\end{array}$$

Observe an important change. Here our main variables are y_1 and y_2 and the slack variables are $x_1 and x_2$.

The final simplex tableau reads as follows:

$$\begin{array}{ccccc|c} y_1 & y_2 & x_1 & x_2 & Z & \\ 0 & 1 & 2 & -1 & 0 & 8\\ 1 & 0 & -1 & 1 & 0 & 4\\ \hline 0 & 0 & 20 & 10 & 1 & 400\end{array}$$

A closer look at this table reveals that the x_1 and x_2 values along with the minimum value for the minimization problem can be obtained from the last row of the final tableau. We have highlighted these values by the arrows.

$$\begin{array}{ccccc|c} y_1 & y_2 & x_1 & x_2 & Z & \\ 0 & 1 & 2 & -1 & 0 & 8\\ 1 & 0 & -1 & 1 & 0 & 4\\ \hline 0 & 0 & 20 & 10 & 1 & 400\\ & & \uparrow & \uparrow & & \uparrow\end{array}$$

We restate the solution as follows:

The minimization problem has a minimum value of 400 at the corner point (20, 10)

We now summarize our discussion.

Minimization by the Simplex Method

1. Set up the problem.
2. Write a matrix whose rows represent each constraint with the objective function as its bottom row.
3. Write the transpose of this matrix by interchanging the rows and columns.
4. Now write the dual problem associated with the transpose.
5. Solve the dual problem by the simplex method learned in section 9.1.

6. The optimal solution is found in the bottom row of the final matrix in the columns corresponding to the slack variables, and the minimum value of the objective function is the same as the maximum value of the dual.

9.3.1: Minimization By The Simplex Method (Exercises)

SECTION 9.3 PROBLEM SET: MINIMIZATION BY THE SIMPLEX METHOD

In problems 1-2, convert each minimization problem into a maximization problem, the dual, and then solve by the simplex method.

1)

$$\begin{array}{ll} \text{Minimize} & z = 6x_1 + 8x_2 \\ \text{subject to} & 2x_1 + 3x_2 \geq 7 \\ & 4x_1 + 5x_2 \geq 9 \\ & x_1, x_2 \geq 0 \end{array}$$

2)

$$\begin{array}{ll} \text{Minimize} & z = 5x_1 + 6x_2 + 7x_3 \\ \text{subject to} & 3x_1 + 2x_2 + 3x_3 \geq 10 \\ & 4x_1 + 3x_2 + 5x_3 \geq 12 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

SECTION 9.3 PROBLEM SET: MINIMIZATION BY THE SIMPLEX METHOD

In problems 3-4, convert each minimization problem into a maximization problem, the dual, and then solve by the simplex method.

3)

$$\begin{array}{ll} \text{Minimize} & z = 4x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \geq 10 \\ & 3x_1 + 2x_2 \geq 24 \\ & x_1, x_2 \geq 0 \end{array}$$

4) A diet is to contain at least 8 units of vitamins, 9 units of minerals, and 10 calories. Three foods, Food A, Food B, and Food C are to be purchased. Each unit of Food A provides 1 unit of vitamins, 1 unit of minerals, and 2 calories. Each unit of Food B provides 2 units of vitamins, 1 unit of minerals, and 1 calorie. Each unit of Food C provides 2 units of vitamins, 1 unit of minerals, and 2 calories. If Food A costs $3 per unit, Food B costs $2 per unit and Food C costs $3 per unit, how many units of each food should be purchased to keep costs at a minimum?

9.4: Chapter Review

SECTION 9.4 PROBLEM SET: CHAPTER REVIEW

Solve the following linear programming problems using the simplex method.

1)

$$\begin{array}{ll}\text{Maximize} & z=5x_1+3x_2\\ \text{subject to} & x_1+x_2\le 12\\ & 2x_1+x_2\le 16\\ & x_1\ge 0;\ x_2\ge 0\end{array}$$

2)

$$\begin{array}{ll}\text{Maximize} & z=5x_1+8x_2\\ \text{subject to} & x1+2x_2\le 30\\ & 3x_1+x_2\le 30\\ & x_1\ge 0;\ x_2\ge 0\end{array}$$

3)

$$\begin{array}{ll}\text{Maximize} & z=2x_1+3x_2+x_3\\ \text{subject to} & 4x_1+2x_2+5x_3\le 32\\ & 2x_1+4x_2+3x_3\le 28\\ & x_1,x_2,x_3\ge 0\end{array}$$

4)

$$\begin{array}{ll}\text{Maximize} & z=x_1+6x_2+8x_3\\ \text{subject to} & x_1+2x_2\le 1200\\ & 2x_2+x_3\le 1800\\ & 4x_1+x_3\le 3600\\ & x_1,x_2,x_3\ge 0\end{array}$$

5)

$$\begin{array}{ll}\text{Maximize} & z=6x_1+8x_2+5x_3\\ \text{subject to} & 4x_1+x_2+x_3\le 1800\\ & 2x_1+2x_2+x_3\le 2000\\ & 4x_1+2x_2+x_3\le 3200\\ & x_1,x_2,x_3\ge 0\end{array}$$

6)

$$\begin{array}{ll}\text{Minimize} & z=12x_1+10x_2\\ \text{subject to} & x_1+x_2\ge 6\\ & 2x_1+x_2\ge 8\\ & x_1\ge 0;\ x_2\ge 0\end{array}$$

7)

$$\begin{array}{ll}\text{Minimize} & z=4x_1+6x_2+7x_3\\ \text{subject to} & x_1+x_2+2x_3\\ & x_1+2x_2+x_3\ge 30\\ & x_1,x_2,x_3\ge 0\end{array}$$

8)

$$\begin{array}{ll}\text{Minimize} & z=40x_1+48x_2+30x_3\\ \text{subject to} & 2x_1+2x_2+x_3\ge 25\\ & x_1+3x_2+2x_3\ge 30\\ & x_1,x_2,x_3\ge 0\end{array}$$

9) An appliance store sells three different types of ovens: small, medium, and large. The small, medium, and large ovens require, respectively, 3, 5, and 6 cubic feet of storage space; a maximum of 1,000 cubic feet of storage space is available. Each oven takes 1hour of sales time; there is a maximum of 200 hours of sales labor time available for ovens. The small, medium, and large ovens require, respectively, 1, 1, and 2 hours of installation time; a maximum of 280 hours of installer labor for ovens is available monthly.
If the profit made from sales of small, medium and large ovens is \$50, \$100, and \$150, respectively, how many of each type of oven should be sold to maximize profit, and what is the maximum profit?

SECTION 9.4 PROBLEM SET: CHAPTER REVIEW

10) A factory manufactures three products, A, B, and C. Each product requires the use of two machines, Machine I and Machine II. The total hours available, respectively, on Machine I and Machine II per month are 180 and 300. The time requirements and profit per unit for each product are listed below.

	A	B	C
Machine I	1	2	2
Machine II	2	2	4
Profit	20	30	40

How many units of each product should be manufactured to maximize profit, and what is the maximum profit?

11) A company produces three products, A, B, and C, at its two factories, Factory I and Factory II. Daily production of each factory for each product is listed below.

	Factory I	Factory II
Product A	10	20
Product B	20	20
Product C	20	10

The company must produce at least 1000 units of product A, 1600 units of B, and 700 units of C. If the cost of operating Factory I is $4,000 per day and the cost of operating Factory II is $5000, how many days should each factory operate to complete the order at a minimum cost, and what is the minimum cost?

12) For his classes, Professor Wright gives three types of quizzes, objective, recall, and recall-plus.
To keep his students on their toes, he has decided to give at least 20 quizzes next quarter.

The three types, objective, recall, and recall-plus quizzes, require the students to spend, respectively, 10 minutes, 30 minutes, and 60 minutes for preparation, and Professor Wright would like them to spend at least 12 hours(720 minutes) preparing for these quizzes above and beyond the normal study time.

An average score on an objective quiz is 5, on a recall type 6, and on a recall-plus 7, and Dr. Wright would like the students to score at least 130 points on all quizzes.

It takes the professor one minute to grade an objective quiz, 2 minutes to grade a recall type quiz, and 3 minutes to grade a recall-plus quiz.

How many of each type should he give in order to minimize his grading time?

CHAPTER OVERVIEW

10: Sets and Counting

Learning Objectives

In this chapter, you will learn to:

- Use set theory and Venn diagrams to solve counting problems.
- Use the Multiplication Axiom to solve counting problems.
- Use Permutations to solve counting problems.
- Use Combinations to solve counting problems.
- Use the Binomial Theorem to expand $(x+y)^n$

10.1: Sets and Venn Diagrams
10.1.1: Sets and Venn Diagrams (Exercises)
10.2: Tree Diagrams and the Multiplication Axiom
10.2.1: Tree Diagrams and the Multiplication Axiom (Exercises)
10.3: Permutations
10.3.1: Permutations (Exercises)
10.4: Circular Permutations and Permutations with Similar Elements
10.4.1: Circular Permutations and Permutations with Similar Elements (Exercises)
10.5: Combinations
10.5.1: Combinations (Exercises)
10.6: Combinations- Involving Several Sets
10.6.1: Combinations- Involving Several Sets (Exercises)
10.7: Binomial Theorem
10.7.1: Binomial Theorem (Exercises)
10.8: Chapter Review

10.1: Sets and Venn Diagrams

Learning Objectives

In this section, you will learn to:

1. Use set notation to represent unions, intersections, and complements of sets
2. Use Venn diagrams to solve counting problems.

Introduction to Sets

In this section, we will familiarize ourselves with set operations and notations, so that we can apply these concepts to both counting and probability problems. We begin by defining some terms.

Definition: Set and Elements

A **set** is a collection of objects, and its members are called the **elements** of the set.

We name the set by using capital letters, and enclose its members in braces. Suppose we need to list the members of the chess club. We use the following set notation.

C = { Ken, Bob, Tran, Shanti, Eric }

Definition: Set Notation

Set Builder Notation: Is a way to describe the set in mathematics terms. D = {2, 3, 4, 5, 6, 7, 8}

The set builder notation is a written description of the set. D = {x| x is a integer between and including 2 and 8}.

Definition: Roster Notation

Let G be the set of integers from 6 to 9 inclusive. The roster notation is a list of elements in a set. G = {6, 7, 8, 9}

Definition: Number of elements in a set

Given the set G= {6, 7, 8, 9}, the number of elements in the set is denoted as n(G) = 4.

Definition: Empty Set

A set that has no members is called an **empty set**. The empty set is denoted by the symbol ∅.

Definition: Set Equality

Two sets are **equal** if they have the same elements.

Definition: Subset

Set A is a **subset** of a set B if every member of A is also a member of B.

Suppose C = { Al, Bob, Chris, David, Ed } and A = { Bob, David }. Then A is a subset of C, written as A $\subseteq$ C.

Every set is a subset of itself, and the empty set is a subset of every set.

Definition: Number of Subsets possible

Then number of subsets in a set is 2^k where k is the number of elements in the set.

F = {2, 3, 4} there are 3 elements in the set. Therefore, there are $2^3 = 8$. The subsets are listed below.

- $\varnothing$
- {2}
- {3}
- {4}
- {2, 3}
- {2, 4}
- {3, 4}
- {2, 3, 4}

Definition: Union of Two Sets

Let A and B be two sets, then the **union** of A and B, written as $A \cup B$, is the set of all elements that are either in A or in B, or in both A and B.

Definition: Intersection of Two Sets

Let A and \(B\0 be two sets, then the **intersection** of A and B, written as $A \cap B$, is the set of all elements that are common to both sets A and B.

Definition: Universal Set

A **universal set** U is the set consisting of all elements under consideration.

Definition: Complement of a Set and Disjoint Sets

Let A be any set, then the **complement** of set A, written as $\bar{A}$, is the set consisting of elements in the universal set U that are not in A.

Two sets A and B are called **disjoint sets** if their intersection is an empty set. Clearly, a set and its complement are disjoint; however two sets can be disjoint and not be complements.

Example 10.1.1

List all the subsets of the set of primary colors { red, yellow, blue }.

Solution

The subsets are $\varnothing$, {red}, {yellow}, {blue}, {red, yellow}, {red, blue}, {yellow, blue}, {red, yellow, blue}

Note that the empty set is a subset of every set, and a set is a subset of itself.

Example 10.1.2

Let F = { Aikman, Jackson, Rice, Sanders, Young }, and let B = { Griffey, Jackson, Sanders, Thomas }.
Find the intersection of the sets F and B.

Solution

The intersection of the two sets is the set whose elements belong to both sets. Therefore, F ∩ B = { Jackson, Sanders }

Example 10.1.3

Find the union of the sets F and B given as follows.

- F = { Aikman, Jackson, Rice, Sanders, Young }
- B = { Griffey, Jackson, Sanders, Thomas }

Solution

The union of two sets is the set whose elements are either in A or in B or in both A and B. Observe that when writing the union of two sets, the repetitions are avoided.

F ∪ B = { Aikman, Griffey, Jackson, Rice, Sanders, Thomas, Young }

Example 10.1.4

Let the universal set U = { red, orange, yellow, green, blue, indigo, violet }, and P = { red, yellow, blue }. Find the complement of P.

Solution

The complement of a set P is the set consisting of elements in the universal set U that are not in P. Therefore,

$\bar{P}$ = { orange, green, indigo, violet }

To achieve a better understanding, let us suppose that the universal set U represents the colors of the spectrum, and P the primary colors, then $\bar{P}$ represents those colors of the spectrum that are not primary colors.

Example 10.1.5

Let the universal set U = { red, orange, yellow, green, blue, indigo, violet }, and P = { red, yellow, blue } . Find a set R so that R is not the complement of P but R and P are disjoint.

Solution

R = { orange, green } and P = { red, yellow, blue } are disjoint because the intersection of the two sets is the empty set. The sets have no elements in common. However they are not complements because their union P ∪ R = { red, yellow, blue, orange, green } is not equal to the universal set U.

Example 10.1.6

Let U = { red, orange, yellow, green, blue, indigo, violet }, P = { red, yellow, blue }, Q = { red, green }, and R = { orange, green, indigo }. Find $\overline{P \cup Q} \cap \overline{R}$.

Solution

We do the problems in steps:

P ∪ Q = { red, yellow, blue, green }
$\overline{P \cup Q}$ = { orange, indigo, violet }
$\overline{R}$ = { red, yellow, blue, violet }
$\overline{P \cup Q} \cap \overline{R}$ = {violet}

Definition: Addition Rule

If we have two sets $A = \{1, 2, 3, 5, 7\}$ and $B = \{2, 3, 4, 5, 8, 9\}$, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

$n(A \cup B) = 8$.

$n(A \cap B) = n(A) + n(B) - n(A \cap B) = 5 + 6 - 3 = 11 - 3 = 8$

Venn Diagrams

We now use **Venn diagrams** to illustrate the relations between sets. In the late 1800s, an English logician named John Venn developed a method to represent relationship between sets. He represented these relationships using diagrams, which are now known as Venn diagrams.

A Venn diagram represents a set as the interior of a circle. Often two or more circles are enclosed in a rectangle where the rectangle represents the universal set. To visualize an intersection or union of a set is easy. In this section, we will mainly use Venn diagrams to sort various populations and count objects.

Example 10.1.7

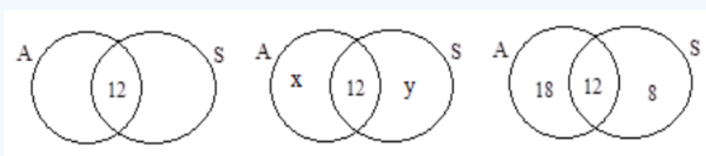
Suppose a survey of car enthusiasts showed that over a certain time period, 30 drove cars with automatic transmissions, 20 drove cars with standard transmissions, and 12 drove cars of both types. If everyone in the survey drove cars with one of these transmissions, how many people participated in the survey?

Solution

We will use Venn diagrams to solve this problem.

Let the set A represent those car enthusiasts who drove cars with automatic transmissions, and set S represent the car enthusiasts who drove the cars with standard transmissions. Now we use Venn diagrams to sort out the information given in this problem.

Since 12 people drove both cars, we place the number 12 in the region common to both sets.



Because 30 people drove cars with automatic transmissions, the circle A must contain 30 elements. This means that

$x + 12 = 30$, or $x = 18$.

Similarly, since 20 people drove cars with standard transmissions, the circle B must contain 20 elements.

Thus, $y + 12 = 20$ which in turn makes $y = 8$.

Now that all the information is sorted out, it is easy to read from the diagram that 18 people drove cars with automatic transmissions only, 12 people drove both types of cars, and 8 drove cars with standard transmissions only.

Therefore, $18 + 12 + 8 = 38$ people took part in the survey.

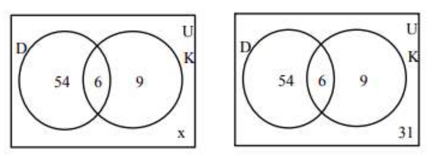
Example 10.1.8

A survey of 100 people in California indicates that 60 people have visited Disneyland, 15 have visited Knott's Berry Farm, and 6 have visited both. How many people have visited neither place?

Solution

The problem is similar to the one in the previous example.

Let the set D represent the people who have visited Disneyland, and K the set of people who have visited Knott's Berry Farm.



We fill the three regions associated with the sets D and K in the same manner as before. Since 100 people participated in the survey, the rectangle representing the universal set U must contain 100 objects. Let x represent those people in the universal set that are neither in the set D nor in K. This means 54 + 6 + 9 + x = 100, or x = 31.

Therefore, there are 31 people in the survey who have visited neither place.

✓ Example 10.1.9

A survey of 100 exercise conscious people resulted in the following information:

- 50 jog, 30 swim, and 35 cycle
- 14 jog and swim
- 7 swim and cycle
- 9 jog and cycle
- 3 people take part in all three activities

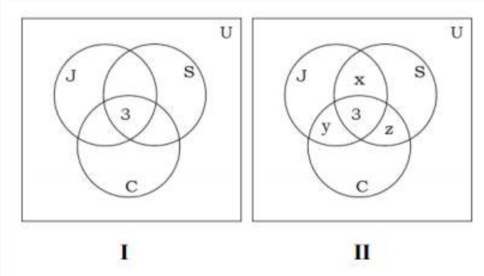
a. How many jog but do not swim or cycle?
b. How many take part in only one of the activities?
c. How many do not take part in any of these activities?

Solution

Let J represent the set of people who jog, S the set of people who swim, and C who cycle.

In using Venn diagrams, our ultimate aim is to assign a number to each region. We always begin by first assigning the number to the innermost region and then working our way out.

We'll show the solution step by step. As you practice working out such problems, you will find that with practice you will not need to draw multiple copies of the diagram.



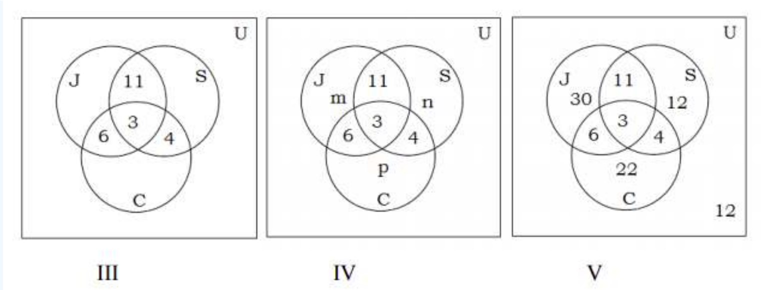
We place a 3 in the innermost region of figure I because it represents the number of people who participate in all three activities. Next we use figure II to compute x, y and z.

Since 14 people jog and swim, x + 3 = 14, or x = 11.

The fact that 9 people jog and cycle results in y + 3 = 9, or y = 6.

Since 7 people swim and cycle, z + 3 = 7, or z = 4.

This information is depicted in figure III.



Now we proceed to find the unknowns m, n and p, as shown in figure IV

Since 50 people jog, m + 11 + 6 + 3 = 50, or m = 30.

30 people swim, therefore, n + 11 + 4 + 3 = 30, or n = 12.

35 people cycle, therefore, p + 6 + 4 + 3 = 35, or p = 22.

By adding all the entries in all three sets, we get a sum of 88.
Since 100 people were surveyed, the number inside the universal set but outside of all three sets is 100 - 88, or 12.

In figure V, all the information is sorted out, and the questions can readily be answered.

a. The number of people who jog but do not swim or cycle is 30.
b. The number who take part in only one of these activities is 30 + 12 + 22 = 64.
c. The number of people who do not take part in any of these activities is 12.

10.1.1: Sets and Venn Diagrams (Exercises)

SECTION 10.1 PROBLEM SET: SETS AND Venn Diagrams

Find the indicated sets.

1. List all subsets of the following set. { Al, Bob }	2. List all subsets of the following set. { Al, Bob, Chris }
3. List the elements of the following set. { Al, Bob, Chris, Dave } ∩ { Bob, Chris, Dave, Ed }	4. List the elements of the following set. { Al, Bob, Chris, Dave } ∪ {Bob, Chris, Dave, Ed }

Problems 5 – 8: Let Universal set U = { a, b, c, d, e, f, g, h, i, j }, sets V = { a, e, i, f, h }, W = { a, c, e, g, i }. List the members of the following sets.

5. V ∪ W	6. V ∩ W
7. $\overline{V \cup W}$	8. $\bar{V} \cap \bar{W}$

Problems 9 – 12: Let Universal set U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 } and sets A = { 1, 2, 3, 4, 5 }, B = { 1, 3, 4, 6 }, and C = { 2, 4, 6 }.

List the members of the following sets.

9. A ∪ B	10. A ∩ C
11. $\overline{A \cup B} \cap C$	12. $\bar{A} \cup \overline{B \cap C}$

Use Venn Diagrams to find the number of elements in the following sets.

13. In Mrs. Yamamoto's class of 35 students, 12 students are taking history, 18 are taking English, and 4 are taking both. Draw a Venn diagram and use it determine how many students are taking neither history nor English.	14. In a survey of 1200 college students, 700 used Spotify to listen to music and 400 used iTunes to listen to music; of these, 100 used both. a. Draw a Venn Diagram and find the number of people in each region of the diagram. b. How many used either Spotify or iTunes?
15. A survey of athletes revealed that for their minor aches and pains, 30 used aspirin, 50 used ibuprofen, and 15 used both. How many athletes were surveyed?	16. In 2016, 80 college students were surveyed about what video services they subscribed to. Suppose the survey showed that 50 use Amazon Prime, 30 use Netflix, 20 use Hulu. Of those, 13 use Amazon Prime and Netflix, 9 use Amazon Prime and Hulu, 7 use Netflix and Hulu. 3 students use all three services. a. Draw a Venn Diagram and use it to determine the number of people in each region of the diagram. b. How many use at least one of these? c. How many use none of these?

17. A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.
    a. Find the number of students taking math but not the other two subjects.
    b. The number of students taking English or math but not history.
    c. The number of students taking none of these subjects.

18. In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. Draw a Venn diagram and find the numbers in each region. Use the diagram to answer the questions below.
    a. Find the number of investors that participated in the survey.
    b. How many invested in stocks or mutual funds but not in bonds?
    c. How many invested in exactly one type of investment?

19. A survey of 100 students at a college finds that 50 take math, 40 take English, and 30 take history. Of these 15 take English and math, 10 take English and history, 10 take math and history, and 5 take all three subjects. (This question relates back to question #17.) For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

a. Students who take at least one of these classes	b. Students who take exactly one of these classes
c. Students who take at least two of these classes	d. Students who take exactly two of these classes
e. Students who take at most two of these classes	f. Students who take English or Math but not both
g. Students who take Math or History but not English	h. Students who take all of these classes

20. In a survey of investors it was found that 100 invested in stocks, 60 in mutual funds, and 50 in bonds. Of these, 35 invested in stocks and mutual funds, 30 in mutual funds and bonds, 28 in stocks and bonds, and 20 in all three. (This question relates back to question #18.) For each of the following draw a Venn Diagram and shade the indicated sets and determine the number of students in the set.

a. Investors who invested in mutual funds only	b. Investors who invested in stocks and bonds but not mutual funds
c. Investors who invested in exactly one of these investments	d. Investors who invested in exactly two of these investments
e. Investors who invested in at least two of these investments	f. Investors who invested in at most two of these investments
g. Investors who did not invest in bonds	h. Investors who invested in all three investments

10.2: Tree Diagrams and the Multiplication Axiom

Learning Objectives

In this section you will learn to

1. Use trees to count possible outcomes in a multi-step process
2. Use the multiplication axiom to count possible outcomes in a multi-step process.

In this chapter, we are trying to develop counting techniques that will be used in the next chapter to study probability. One of the most fundamental of such techniques is called the Multiplication Axiom. Before we introduce the multiplication axiom, we first look at some examples.

Example 10.2.1

If a woman has two blouses and three skirts, how many different outfits consisting of a blouse and a skirt can she wear?

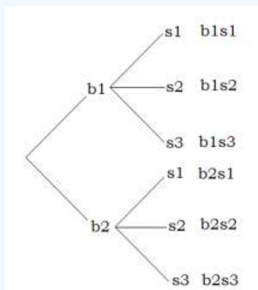
Solution

Suppose we call the blouses b_1 and b_2, and skirts s_1, s_2, and s_3.

We can have the following six outfits.

$$b_1s_1, b_1s_2, b_1s_3, b_2s_1, b_2s_2, b_2s_3$$

Alternatively, we can draw a tree diagram:



The tree diagram gives us all six possibilities. The method involves two steps. First the woman chooses a blouse. She has two choices: blouse one or blouse two. If she chooses blouse one, she has three skirts to match it with; skirt one, skirt two, or skirt three. Similarly if she chooses blouse two, she can match it with each of the three skirts, again. The tree diagram helps us visualize these possibilities.

The reader should note that the process involves two steps. For the first step of choosing a blouse, there are two choices, and for each choice of a blouse, there are three choices of choosing a skirt. So altogether there are $2 \cdot 3 = 6$ possibilities.

If, in the previous example, we add the shoes to the outfit, we have the following problem.

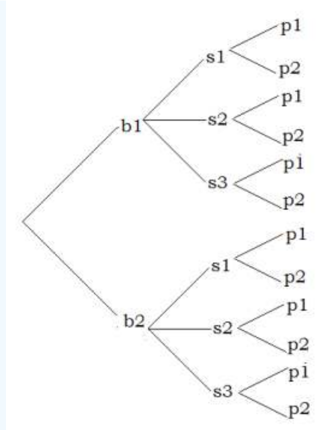
Example 10.2.2

If a woman has two blouses, three skirts, and two pumps, how many different outfits consisting of a blouse, a skirt, and a pair of pumps can she wear?

Solution

Suppose we call the blouses b_1 and b_2, the skirts s_1, s_2, and s_3, and the pumps p_1, and p_2.

The following tree diagram results.



We count the number of branches in the tree, and see that there are 12 different possibilities.

This time the method involves three steps. First, the woman chooses a blouse. She has two choices: blouse one or blouse two. Now suppose she chooses blouse one. This takes us to step two of the process which consists of choosing a skirt. She has three choices for a skirt, and let us suppose she chooses skirt two. Now that she has chosen a blouse and a skirt, we have moved to the third step of choosing a pair of pumps. Since she has two pairs of pumps, she has two choices for the last step. Let us suppose she chooses pumps two. She has chosen the outfit consisting of blouse one, skirt two, and pumps two, or $b_1 s_2 p_2$. By looking at the different branches on the tree, one can easily see the other possibilities.

The important thing to observe here, again, is that this is a three step process. There are two choices for the first step of choosing a blouse. For each choice of a blouse, there are three choices of choosing a skirt, and for each combination of a blouse and a skirt, there are two choices of selecting a pair of pumps.

All in all, we have $2 \cdot 3 \cdot 2 = 12$ different possibilities.

Tree diagrams help us visualize the different possibilities, but they are not practical when the possibilities are numerous. Besides, we are mostly interested in finding the number of elements in the set and not the actual list of all possibilities; once the problem is envisioned, we can solve it without a tree diagram. The two examples we just solved may have given us a clue to do just that.

Let us now try to solve Example $10.2.2$ without a tree diagram. The problem involves three steps: choosing a blouse, choosing a skirt, and choosing a pair of pumps. The number of ways of choosing each are listed below. By multiplying these three numbers we get 12, which is what we got when we did the problem using a tree diagram.

The number of ways of choosing a blouse	The number of ways of choosing a skirt	The number of ways of choosing pumps
2	3	2

The procedure we just employed is called the multiplication axiom.

The Multiplication Axiom

If a task can be done in m ways, and a second task can be done in n ways, then the operation involving the first task followed by the second can be performed in $m \cdot n$ ways.

The general multiplication axiom is not limited to just two tasks and can be used for any number of tasks.

Example $10.2.3$

A truck license plate consists of a letter followed by four digits. How many such license plates are possible?

Solution

Since there are 26 letters and 10 digits, we have the following choices for each.

Letter	Digit	Digit	Digit	Digit
26	10	10	10	10

Therefore, the number of possible license plates is $26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 260,000$.

Example 10.2.4

In how many different ways can a 3-question true-false test be answered?

Solution

Since there are two choices for each question, we have

Question 1	Question 2	Question 3
2	2	2

Applying the multiplication axiom, we get $2 \cdot 2 \cdot 2 = 8$ different ways.

We list all eight possibilities: TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF

The reader should note that the first letter in each possibility is the answer corresponding to the first question, the second letter corresponds to the answer to the second question, and so on. For example, TFF, says that the answer to the first question is given as true, and the answers to the second and third questions false.

Example 10.2.5

In how many different ways can four people be seated in a row?

Solution

Suppose we put four chairs in a row, and proceed to put four people in these seats.

There are four choices for the first chair we choose. Once a person sits down in that chair, there are only three choices for the second chair, and so on. We list as shown below.

4	3	2	1

So there are altogether $4 \cdot 3 \cdot 2 \cdot 1 = 24$ different ways.

Example 10.2.6

How many three-letter word sequences can be formed using the letters { A, B, C } if no letter is to be repeated?

Solution

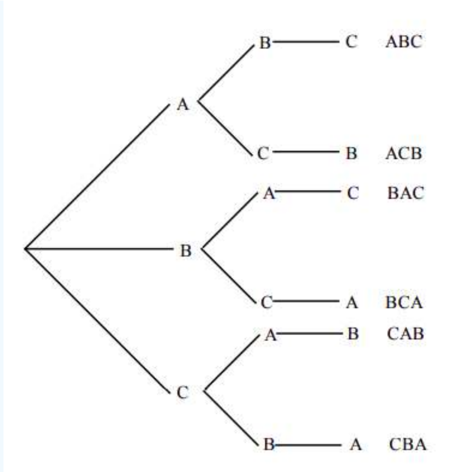
The problem is very similar to the previous example.

Imagine a child having three building blocks labeled A, B, and C. Suppose he puts these blocks on top of each other to make word sequences. For the first letter he has three choices, namely A, B, or C. Let us suppose he chooses the first letter to be a B, then for the second block which must go on top of the first, he has only two choices: A or C. And for the last letter he has only one choice. We list the choices below.

3	2	1

Therefore, 6 different word sequences can be formed.

Finally, we'd like to illustrate this with a tree diagram showing all six possibilities.


A
B
C
B
C
C
B
A
C
C
A
A
B
B
A
ABC
ACB
BAC
BCA
CAB
CBA

10.2.1: Tree Diagrams and the Multiplication Axiom (Exercises)

Do the following problems using a tree diagram or the multiplication axiom.

1. A man has 3 shirts, and 2 pairs of pants. Use a tree diagram to determine the number of possible outfits.
2. In a city election, there are 2 candidates for mayor, and 3 for supervisor. Use a tree diagram to find the number of ways to fill the two offices.
3. There are 4 roads from Town A to Town B, 2 roads from Town B to Town C. Use a tree diagram to find the number of ways one can travel from Town A to Town C.
4. Brown Home Construction offers a selection of 3 floor plans, 2 roof types, and 2 exterior wall types. Use a tree diagram to determine the number of possible homes available.
5. For lunch, a small restaurant offers 2 types of soups, three kinds of sandwiches, and two types of soft drinks. Use a tree diagram to determine the number of possible meals consisting of a soup, sandwich, and a soft drink.
6. A California license plate consists of a number from 1 to 5, then three letters followed by three digits. How many such plates are possible?

Do the following problems using the Multiplication Axiom

7. A license plate consists of three letters followed by three digits. How many license plates are possible if no letter may be repeated?
8. How many different 4-letter radio station call letters can be made if the first letter must be K or W and no letters can be repeated?
9. How many seven-digit telephone numbers are possible if the first two digits cannot be ones or zeros?
10. How many 3-letter word sequences can be formed using the letters {a, b, c, d} if no letter is to be repeated?

Use a tree diagram for questions 11 and 12:

11. A family has two children, use a tree diagram to determine all four possibilities of outcomes by gender.
12. A coin is tossed three times and the sequence of heads and tails is recorded. Use a tree diagram to list all the possible outcomes.

Do the following problems using the Multiplication Axiom

13. In how many ways can a 4-question true-false test be answered?
14. In how many ways can three people be arranged to stand in a straight line?
15. A combination lock is opened by first turning to the left, then to the right, and then to the left again. If there are 30 digits on the dial, how many possible combinations are there?
16. How many different answers are possible for a multiple-choice test with 10 questions and five possible answers for each question?
17. In the past, a college required students to use a 4 digit PIN (Personal Identification Number) as their password for its registration system. How many different PINs are possible if each must have 4 digits with no restrictions on selection or arrangement of the digits used?
18. The college decided that a more secure password system is needed. New passwords must have 3 numerical digits followed by 6 letters. There are no restrictions on the selection of the numerical digits. However, the letters I and O are not permitted. How many different passwords are possible?

10.3: Permutations

Learning Objectives

In this section you will learn to

1. Count the number of possible permutations (ordered arrangement) of n items taken r at a time
2. Count the number of possible permutations when there are conditions imposed on the arrangements
3. Perform calculations using factorials

In Example 10.2.6 of section 10.2, we were asked to find the word sequences formed by using the letters { A, B, C } if no letter is to be repeated. The tree diagram gave us the following six arrangements.

ABC, ACB, BAC, BCA, CAB, and CBA.

Arrangements like these, where order is important and no element is repeated, are called permutations.

Definition: Permutations

A permutation of a set of elements is an ordered arrangement where each element is used once.

Example 10.3.1

How many three-letter word sequences can be formed using the letters { A, B, C, D }?

Solution

There are four choices for the first letter of our word, three choices for the second letter, and two choices for the third.

4	3	2

Applying the multiplication axiom, we get $4 \cdot 3 \cdot 2 = 24$ different arrangements.

Example 10.3.2

How many permutations of the letters of the word ARTICLE have consonants in the first and last positions?

Solution

In the word ARTICLE, there are 4 consonants.

Since the first letter must be a consonant, we have four choices for the first position, and once we use up a consonant, there are only three consonants left for the last spot. We show as follows:

4						3

Since there are no more restrictions, we can go ahead and make the choices for the rest of the positions.

So far we have used up 2 letters, therefore, five remain. So for the next position there are five choices, for the position after that there are four choices, and so on. We get

4	5	4	3	2	1	3

So the total permutations are $4 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 = 1440$.

Example 10.3.3

Given five letters { A, B, C, D, E }. Find the following:

a. The number of four-letter word sequences.
b. The number of three-letter word sequences.
c. The number of two-letter word sequences.

Solution

The problem is easily solved by the multiplication axiom, and answers are as follows:

a. The number of four-letter word sequences is $5 \cdot 4 \cdot 3 \cdot 2 = 120$.
b. The number of three-letter word sequences is $5 \cdot 4 \cdot 3 = 60$.
c. The number of two-letter word sequences is $5 \cdot 4 = 20$.

We often encounter situations where we have a set of n objects and we are selecting r objects to form permutations. We refer to this as **permutations of n objects taken r at a time**, and we write it as **nPr**.

Therefore, the above example can also be answered as listed below.

a. The number of four-letter word sequences is ${}_5P_4$ = 120.
b. The number of three-letter word sequences is ${}_5P_3$ = 60.
c. The number of two-letter word sequences is ${}_5P_2$ = 20.

Before we give a formula for nPr, we'd like to introduce a symbol that we will use a great deal in this as well as in the next chapter.

Definition: Factorial

$$\mathrm{n}! = \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)\cdots 3 \cdot 2 \cdot 1$$

where n is a natural number.

$$0! = 1$$

Now we define nPr.

Definition: nPr

The Number of Permutations of n Objects Taken r at a Time

\[\mathrm{${}_n$P$_r$}=\mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3) \cdots(\mathrm{n}-\mathrm{r}+1) \nonumber \]

or

\[\mathrm{${}_n$P$_r$}=\frac{\mathrm{n} !}{(\mathrm{n}-\mathrm{r}) !} \nonumber \]

where n and r are natural numbers.

The reader should become familiar with both formulas and should feel comfortable in applying either.

Example 10.3.4

Compute the following using both formulas.

a. ${}_6P_3$
b. ${}_7P_2$

Solution

We will identify n and r in each case and solve using the formulas provided.

a. ${}_6P_3$ = $6 \cdot 5 \cdot 4 = 120$, alternately

$$6P3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 120$$

b. ${}_7P_2 = 7 \cdot 6 = 42$, or

$$7P2 = \frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 42$$

Next we consider some more permutation problems to get further insight into these concepts.

Example 10.3.5

In how many different ways can 4 people be seated in a straight line if two of them insist on sitting next to each other?

Solution

Let us suppose we have four people A, B, C, and D. Further suppose that A and B want to sit together. For the sake of argument, we tie A and B together and treat them as one person.

The four people are $\boxed{AB}$ CD. Since $\boxed{AB}$ is treated as one person, we have the following possible arrangements.

$$\boxed{AB}CD, \boxed{AB}DC, C\boxed{AB}D, D\boxed{AB}C, CD\boxed{AB}, DC\boxed{AB}$$

Note that there are six more such permutations because A and B could also be tied in the order BA. And they are

$$\boxed{BA}CD, \boxed{BA}DC, C\boxed{BA}D, D\boxed{BA}C, CD\boxed{BA}, DC\boxed{BA}$$

So altogether there are 12 different permutations.

Let us now do the problem using the multiplication axiom.

After we tie two of the people together and treat them as one person, we can say we have only three people. The multiplication axiom tells us that three people can be seated in 3! ways. Since two people can be tied together 2! ways, there are 3! 2! = 12 different arrangements

Example 10.3.6

You have 4 math books and 5 history books to put on a shelf that has 5 slots. In how many ways can the books be shelved if the first three slots are filled with math books and the next two slots are filled with history books?

Solution

We first do the problem using the multiplication axiom.

Since the math books go in the first three slots, there are 4 choices for the first slot,
3 choices for the second and 2 choices for the third.

The fourth slot requires a history book, and has five choices. Once that choice is made, there are 4 history books left, and therefore, 4 choices for the last slot. The choices are shown below.

4	3	2	5	4

Therefore, the number of permutations are $4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 = 480$.

Alternately, we can see that $4 \cdot 3 \cdot 2$ is really same as ${}_4P3$, and $5 \cdot 4$ is ${}_5P_2$.

So the answer can be written as $({}_4P_3)$ $({}_5P2)$ = 480.

Clearly, this makes sense. For every permutation of three math books placed in the first three slots, there are ${}_5P2$ permutations of history books that can be placed in the last two slots. Hence the multiplication axiom applies, and we have the answer $({}_4P_3)$ $({}_5P_2)$.

We summarize the concepts of this section:

Note

1. Permutations

A permutation of a set of elements is an ordered arrangement where each element is used once.

2. Factorial

$$\mathrm{n}! = \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)\cdots 3 \cdot 2 \cdot 1$$

Where n is a natural number.

$$0! = 1$$

3. Permutations of n Objects Taken r at a Time

$$\mathrm{nPr} = \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)\cdots(\mathrm{n}-\mathrm{r}+1)$$

or

$$\mathrm{nPr} = \frac{\mathrm{n}!}{(\mathrm{n}-\mathrm{r})!}$$

where n and r are natural numbers.

10.3.1: Permutations (Exercises)

Do the following problems using permutations.

1. How many three-letter words can be made using the letters { a, b, c, d, e } if no repetitions are allowed?	2. A grocery store has five checkout counters, and seven clerks. How many different ways can the 7 clerks be assigned to the 5 counters?
3. A group of fifteen people who are members of an investment club wish to choose a president, and a secretary. How many different ways can this be done?	4. Compute the following. a. ${}_9P_2$ b. ${}_6P_4$ c. ${}_8P_3$ d. ${}_7P_4$
5. In how many ways can the letters of the word CUPERTINO be arranged if each letter is used only once in each arrangement?	6. How many permutations of the letters of the word PROBLEM end in a vowel?
7. How many permutations of the letters of the word SECURITY end in a consonant?	8. How many permutations of the letters PRODUCT have consonants in the second and third positions?
9. How many three-digit numbers are there?	10. How many three-digit odd numbers are there?
11. In how many different ways can five people be seated in a row if two of them insist on sitting next to each other?	12. In how many different ways can five people be seated in a row if two of them insist on not sitting next to each other?
13. In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if the English books are set on the left, history books in the middle, and math books on the right?	14. In how many ways can 3 English, 3 history, and 2 math books be set on a shelf, if they are grouped by subject?
15. You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with math books and the next three with history books?	16. You have 5 math books and 6 history books to put on a shelf with five slots. In how many ways can you put the books on the shelf if the first two slots are to be filled with the books of one subject and the next three slots are to be filled with the books of the other subject?
17. A bakery has 9 different fancy cakes. In how many ways can 5 of the 9 fancy cakes be lined up in a row in the bakery display case?	18. A landscaper has 6 different flowering plants. She needs to plant 4 of them in a row in a garden. How many different ways can 4 of the 6 plants be arranged in a row?
19. At an auction of used construction vehicles, there are 7 different vehicles for sale. In how many orders could these 7 vehicles be listed in the auction program?	20. A landscaper has 6 different flowering plants and 4 different non-flowering bushes. She needs to plant a row of 6 plants in a garden. There must be a bush at each end, and four flowering plants in a row in between the bushes. How many different arrangements in a row are possible?
21. In how many ways can all 7 letters of the word QUIETLY be arranged if the letters Q and U must be next to each other in the order QU?	22. a. In how many ways can the letters ABCDEXY be arranged if the X and Y must be next to each other in either order XY or YX? b. In how many ways can the letters ABCDEXY be arranged if the X and Y can not be next to each other?

10.4: Circular Permutations and Permutations with Similar Elements

Learning Objectives

In this section you will learn to

1. Count the number of possible permutations of items arranged in a circle
2. Count the number of possible permutations when there are repeated items

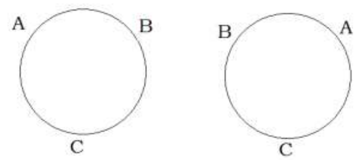
In this section we will address the following two problems.

1. In how many different ways can five people be seated in a circle?
2. In how many different ways can the letters of the word MISSISSIPPI be arranged?

The first problem comes under the category of Circular Permutations, and the second under Permutations with Similar Elements.

Circular Permutations

Suppose we have three people named A, B, and C. We have already determined that they can be seated in a straight line in 3! or 6 ways. Our next problem is to see how many ways these people can be seated in a circle. We draw a diagram.



It happens that there are only two ways we can seat three people in a circle, relative to each other's positions. This kind of permutation is called a circular permutation. In such cases, no matter where the first person sits, the permutation is not affected. Each person can shift as many places as they like, and the permutation will not be changed. We are interested in the position of each person in relation to the others. Imagine the people on a merry-go-round; the rotation of the permutation does not generate a new permutation. So in circular permutations, the first person is considered a place holder, and where he sits does not matter.

Definition: Circular Permutations

The number of permutations of n elements in a circle is $(n-1)!$

Example 10.4.1

In how many different ways can five people be seated at a circular table?

Solution

We have already determined that the first person is just a place holder. Therefore, there is only one choice for the first spot. We have

1	4	3	2	1

So the answer is 24.

Example 10.4.2

In how many ways can four couples be seated at a round table if the men and women want to sit alternately?

Solution

We again emphasize that the first person can sit anywhere without affecting the permutation.

So there is only one choice for the first spot. Suppose a man sat down first. The chair next to it must belong to a woman, and there are 4 choices. The next chair belongs to a man, so there are three choices and so on. We list the choices below.

1	4	3	3	2	2	1	1

So the answer is 144.

PERMUTATIONS WITH SIMILAR ELEMENTS

Let us determine the number of distinguishable permutations of the letters ELEMENT.

Suppose we make all the letters different by labeling the letters as follows.

$$E_1LE_2ME_3NT$$

Since all the letters are now different, there are 7! different permutations.

Let us now look at one such permutation, say

$$LE_1ME_2NE_3T$$

Suppose we form new permutations from this arrangement by only moving the E's. Clearly, there are 3! or 6 such arrangements. We list them below.

$$\begin{array}{l} \text{LE}_1\text{ME}_2\text{NE}_3 \\ \text{LE}_1\text{ME}_3\text{NE}_2 \\ \text{LE}_2\text{ME}_1\text{NE}_3\text{T} \\ \text{LE}_2\text{ME}_3\text{NE}_1\text{T} \\ \text{LE}_3\text{ME}_2\text{NE}_1\text{T} \\ \text{LE}_3\text{ME}_I\text{NE}_2\text{T} \end{array}$$

Because the E's are not different, there is only one arrangement LEMENET and not six. This is true for every permutation.

Let us suppose there are n different permutations of the letters ELEMENT.

Then there are $n \cdot 3!$ permutations of the letters $E_1LE_2ME_3NT$.

But we know there are 7! permutations of the letters $E_1LE_2ME_3NT$.

Therefore, $n \cdot 3! = 7!$

Or $n = \frac{7!}{3!}$.

This gives us the method we are looking for.

Definition: Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, is

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

Example 10.4.3

Find the number of different permutations of the letters of the word MISSISSIPPI.

Solution

The word MISSISSIPPI has 11 letters. If the letters were all different there would have been 11! different permutations. But MISSISSIPPI has 4 S's, 4 I's, and 2 P's that are alike.

So the answer is $\frac{11!}{4!4!2!} = 34,650$.

Example 10.4.4

If a coin is tossed six times, how many different outcomes consisting of 4 heads and 2 tails are there?

Solution

Again, we have permutations with similar elements.

We are looking for permutations for the letters HHHHTT.

The answer is $\frac{6!}{4!2!} = 15$.

Example 10.4.5

In how many different ways can 4 nickels, 3 dimes, and 2 quarters be arranged in a row?

Solution

Assuming that all nickels are similar, all dimes are similar, and all quarters are similar, we have permutations with similar elements. Therefore, the answer is

$$\frac{9!}{4!3!2!} = 1260$$

Example 10.4.6

A stock broker wants to assign 20 new clients equally to 4 of its salespeople. In how many different ways can this be done?

Solution

This means that each sales person gets 5 clients. The problem can be thought of as an ordered partitions problem. In that case, using the formula we get

$$\frac{20!}{5!5!5!5!} = 11,732,745,024$$

Example 10.4.7

A shopping mall has a straight row of 5 flagpoles at its main entrance plaza. It has 3 identical green flags and 2 identical yellow flags. How many distinct arrangements of flags on the flagpoles are possible?

Solution

The problem can be thought of as distinct permutations of the letters GGGYY; that is arrangements of 5 letters, where 3 letters are similar, and the remaining 2 letters are similar:

$$\frac{5!}{3!2!} = 10$$

Just to provide a little more insight into the solution, we list all 10 distinct permutations:

GGGYY, GGYGY, GGYYG, GYGGY, GYGYG, GYYGG, YGGGY, YGGYG, YGYGY, YYGGG

We summarize.

Summary

1. Circular Permutations

The number of permutations of n elements in a circle is

$$(n-1)!$$

2. Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, such that $\mathrm{n}=\mathrm{r}_1+\mathrm{r}_2+\ldots+\mathrm{r}_\mathrm{k}$ is

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

This is also referred to as **ordered partitions**.

10.4.1: Circular Permutations and Permutations with Similar Elements (Exercises)

Do the following problems using the techniques learned in this section.

1. In how many different ways can five children hold hands to play "Ring Around the Rosy"?	2. In how many ways can three people be made to sit at a round table?
3. In how many different ways can six children ride a "Merry Go Around" with six horses?	4. In how many ways can three couples be seated at a round table, so that men and women sit alternately?
5. In how many ways can six trinkets be arranged on a chain?	6. In how many ways can five keys be put on a key ring?
7. Find the number of different permutations of the letters of the word MASSACHUSETTS.	8. Find the number of different permutations of the letters of the word MATHEMATICS.
9. Seven flags are to be flown on seven poles: 3 flags are red, 2 are white, and 2 are blue,. How many different arrangements are possible?	10. How many different ways can 3 pennies, 2 nickels and 5 dimes be arranged in a row?
11. How many four-digit numbers can be made using two 2's and two 3's?	12. How many five-digit numbers can be made using two 6's and three 7's?
13. If a coin is tossed 5 times, how many different outcomes of 3 heads and 2 tails are possible?	14. If a coin is tossed 10 times, how many different outcomes of 7 heads and 3 tails are possible?
15. If a team plays ten games, how many different outcomes of 6 wins and 4 losses are possible?	16. If a team plays ten games, how many different ways can the team have a winning season?

10.5: Combinations

Learning Objectives

In this section you will learn to

1. Count the number of combinations of r out of n items (selections without regard to arrangement)
2. Use factorials to perform calculations involving combinations

Suppose we have a set of three letters { A, B, C }, and we are asked to make two-letter word sequences. We have the following six permutations.

AB BA BC CB AC CA

Now suppose we have a group of three people { A, B, C } as Al, Bob, and Chris, respectively, and we are asked to form committees of two people each. This time we have only three committees, namely,

AB BC AC

When forming committees, the order is not important, because the committee that has Al and Bob is no different than the committee that has Bob and Al. As a result, we have only three committees and not six.

Forming word sequences is an example of permutations, while forming committees is an example of **combinations** - the topic of this section.

Permutations are those arrangements where order is important, while combinations are those arrangements where order is not significant. From now on, this is how we will tell permutations and combinations apart.

In the above example, there were six permutations, but only three combinations.

Just as the symbol ${}_nP_r$ represents the number of permutations of n objects taken r at a time, ${}_nC_r$ represents the number of combinations of n objects taken r at a time.

So in the above example, ${}_3P_2 = 6$, and ${}_3C_2 = 3$.

Our next goal is to determine the relationship between the number of combinations and the number of permutations in a given situation.

In the above example, if we knew that there were three combinations, we could have found the number of permutations by multiplying this number by 2!. That is because each combination consists of two letters, and that makes 2! permutations.

Example 10.5.1

Given the set of letters { A, B, C, D }. Write the number of combinations of three letters, and then from these combinations determine the number of permutations.

Solution

We have the following four combinations.

ABC BCD CDA BDA

Since every combination has three letters, there are 3! permutations for every combination. We list them below.

$$\begin{array}{llll} \mathrm{ABC} & \mathrm{BCD} & \mathrm{CDA} & \mathrm{BDA} \\ \mathrm{ACB} & \mathrm{BDC} & \mathrm{CAD} & \mathrm{BAD} \\ \mathrm{BAC} & \mathrm{CDB} & \mathrm{DAC} & \mathrm{DAB} \\ \mathrm{BCA} & \mathrm{CBD} & \mathrm{DCA} & \mathrm{DBA} \\ \mathrm{CAB} & \mathrm{DCB} & \mathrm{ACD} & \mathrm{ADB} \\ \mathrm{CBA} & \mathrm{DBC} & \mathrm{ADC} & \mathrm{ABD} \end{array}$$

The number of permutations are 3! times the number of combinations; that is

${}_4P_3 = 3! \cdot {}_4C_3$

or

$${}_4C_3 = \frac{4P3}{3!}$$

In general,

$$nCr = \frac{nPr}{r!}$$

Since

$$nPr = \frac{n!}{(n-r)!}$$

We have,

$$nCr = \frac{n!}{(n-r)!r!}$$

Summarizing,

Note

1. Combinations

A combination of a set of elements is an arrangement where each element is used once, and order is not important.

2. The Number of Combinations of n Objects Taken r at a Time

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

where n and r are natural numbers.

Example \(\PageIndex{3}\)Example 10.5.2

Compute:

a. ${}_5C_3$
b. ${}_7C_3$

Solution

We use the above formula.

$${}_5C_3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = 10$$

$${}_7C_3 = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = 35$$

Example 10.5.3

In how many different ways can a student select to answer five questions from a test that has seven questions, if the order of the selection is not important?

Solution

Since the order is not important, it is a combination problem, and the answer is

$$ {}_{7}C_{5} = 21 $$

Example 10.5.4

How many line segments can be drawn by connecting any two of the six points that lie on the circumference of a circle?

Solution

Since the line that goes from point A to point B is same as the one that goes from B to A, this is a combination problem.

It is a combination of 6 objects taken 2 at a time. Therefore, the answer is

$$ {}_{6}C_{2} = \frac{6!}{(6-2)!2!} = \frac{6!}{4!2!} = 15 $$

Example 10.5.5

There are ten people at a party. If they all shake hands, how many hand-shakes are possible?

Solution

Note that between any two people there is only one hand shake. Therefore, we have

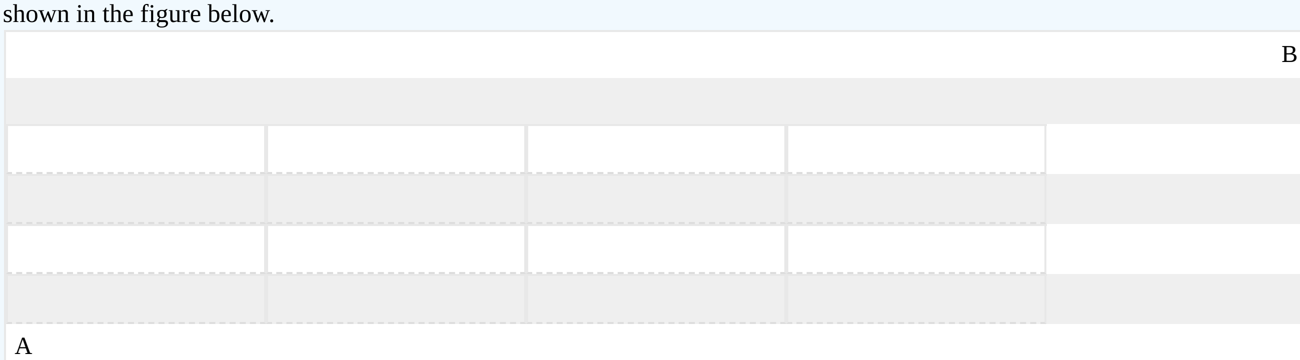
$$ {}_{10}C_{2} = 45 \text{ hand-shakes.} $$

Example 10.5.6

The shopping area of a town is in the shape of square that is 5 blocks by 5 blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?

Solution

Let us suppose the taxi driver drives from the point A, the lower left hand corner, to the point B, the upper right hand corner as shown in the figure below.



To reach his destination, he has to travel ten blocks; five horizontal, and five vertical. So if out of the ten blocks he chooses any five horizontal, the other five will have to be the vertical blocks, and vice versa.

Therefore, all he has to do is to choose 5 out of ten to be the horizontal blocks

The answer is ${}_{10}C_{5}$, or 252.

Alternately, the problem can be solved by permutations with similar elements.

The taxi driver's route consists of five horizontal and five vertical blocks. If we call a horizontal block H, and a vertical block a V, then one possible route may be as follows.

HHHHHVVVVV

Clearly there are $\frac{10!}{5!5!} = 252$ permutations.

Further note that by definition ${}_{10}C_{5} = \frac{10!}{5!5!}$.

Example 10.5.7

If a coin is tossed six times, in how many ways can it fall four heads and two tails?

Solution

First we solve this problem using section 10.2 technique-permutations with similar elements.

We need 4 heads and 2 tails, that is

$$\text{HHHHTT}$$

There are $\frac{6!}{4!2!} = 15$ permutations.

Now we solve this problem using combinations.

Suppose we have six spots to put the coins on. If we choose any four spots for heads, the other two will automatically be tails. So the problem is simply

$$_6C_4 = 15.$$

Incidentally, we could have easily chosen the two tails, instead. In that case, we would have gotten

$$_6C_2 = 15.$$

Further observe that by definition

$$_6C_4 = \frac{6!}{2!4!}$$

and

$$_6C_2 = \frac{6!}{4!2!}$$

Which implies $_6C_4 = {}_6C_2$.

10.5.1: Combinations (Exercises)

Do the following problems using combinations.

1. How many different 3-people committees can be chosen from ten people?	2. How many different 5-player teams can be chosen from eight players?
3. In how many ways can a person chose to vote for three out of five candidates on a ballot for a school board election?	4. Compute the following: a. ${}_9C_2$ b. ${}_6C_4$ c. ${}_8C_3$ d. ${}_7C_4$
5. How many 5-card hands can be chosen from a deck of cards?	6. How many 13-card bridge hands can be chosen from a deck of cards?
7. There are twelve people at a party. If they all shake hands, how many different hand-shakes are there?	8. In how many ways can a student choose to do four questions out of five on a test?
9. Five points lie on a circle. How many chords can be drawn through them?	10. How many diagonals does a hexagon have?
11. There are five team in a league. How many games are played if every team plays each other twice?	12. A team plays 15 games a season. In how many ways can it have 8 wins and 7 losses?
13. In how many different ways can a 4-child family have 2 boys and 2 girls?	14. A coin is tossed five times. In how many ways can it fall three heads and two tails?
15. The shopping area of a town is a square that is six blocks by six blocks. How many different routes can a taxi driver take to go from one corner of the shopping area to the opposite cater-corner?	16. If the shopping area in the previous problem has a rectangular form of 5 blocks by 3 blocks, then how many different routes can a taxi driver take to drive from one end of the shopping area to the opposite kitty corner end?
17. A team of 7 workers is assigned to a project. In how many ways can 3 of the 7 workers be selected to make a presentation to the management about their progress on the project?	18. A real estate company has 12 houses listed for sale by their clients. In how many ways can 5 of the 12 houses be selected to be featured in an advertising brochures?
19. A frozen yogurt store has 9 toppings to choose from. In how many ways can 3 of the 9 toppings be selected ?	20. A kindergarten teacher has 14 books about a holiday. In how many ways can she select 4 of the books to read to her class in the week before the holiday?

10.6: Combinations- Involving Several Sets

Learning Objectives

In this section you will learn to

1. count the number of items selected from more than one set
2. count the number of items selected when there are restrictions on the selections

So far we have solved the basic combination problem of r objects chosen from n different objects. Now we will consider certain variations of this problem.

Example 10.6.1

How many five-people committees consisting of 2 men and 3 women can be chosen from a total of 4 men and 4 women?

Solution

We list 4 men and 4 women as follows:

$$M_1 M_2 M_3 M_4 W_1 W_2 W_3 W_4$$

Since we want 5-people committees consisting of 2 men and 3 women, we'll first form all possible two-man committees and all possible three-woman committees. Clearly there are 4C2 = 6 two-man committees, and 4C3 = 4 three-woman committees, we list them as follows:

2-Man Committees	3-Woman Committees
M_1M_2 M_1M_3 M_1M_4 M_2M_3 M_2M_4 M_3M_4	$W_1W_2W_3$ $W_1W_2W_4$ $W_1W_3W_4$ $W_2W_3W_4$

For every 2-man committee there are four 3-woman committees that can be chosen to make a 5-person committee. If we choose $M_1 M_2$ as our 2-man committee, then we can choose any of $W_1 W_2 W_3$, $W_1 W_2 W_4$, $W_1 W_3 W_4$, or $W_2 W_3 W_4$ as our 3-woman committees. As a result, we get

$$\boxed{M_1 M_2} W_1 W_2 W_3, \boxed{M_1 M_2} W_1 W_2 W_4, \boxed{M_1 M_2} W_1 W_3 W_4, \boxed{M_1 M_2} W_2 W_3 W_4$$

Similarly, if we choose $M_1 M_3$ as our 2-man committee, then, again, we can choose any of $W_1 W_2 W_3$, $W_1 W_2 W_4$, $W_1 W_3 W_4$, or $W_2 W_3 W_4$ as our 3-woman committees.

$$\boxed{M_1 M_3} W_1 W_2 W_3, \boxed{M_1 M_3} W_1 W_2 W_4, \boxed{M_1 M_3} W_1 W_3 W_4, \boxed{M_1 M_3} W_2 W_3 W_4$$

And so on.

Since there are six 2-man committees, and for every 2-man committee there are four 3-woman committees, there are altogether $6 \cdot 4 = 24$ five-people committees.

In essence, we are applying the multiplication axiom to the different combinations.

Example 10.6.2

A high school club consists of 4 freshmen, 5 sophomores, 5 juniors, and 6 seniors. How many ways can a committee of 4 people be chosen that includes

a. One student from each class?
b. All juniors?
c. Two freshmen and 2 seniors?
d. No freshmen?
e. At least three seniors?

Solution

a. Applying the multiplication axiom to the combinations involved, we get

$$(4C1)(5C1)(5C1)(6C1) = 600$$

b. We are choosing all 4 members from the 5 juniors, and none from the others.

$$5C4 = 5$$

c. $4C2 \cdot 6C2 = 90$

d. Since we don't want any freshmen on the committee, we need to choose all members from the remaining 16. That is

$$16C4 = 1820$$

e. Of the 4 people on the committee, we want at least three seniors. This can be done in two ways. We could have three seniors, and one non-senior, or all four seniors.

$$(6C3)(14C1) + 6C4 = 295$$

✓ Example 10.6.3

How many five-letter word sequences consisting of 2 vowels and 3 consonants can be formed from the letters of the word INTRODUCE?

Solution

First we select a group of five letters consisting of 2 vowels and 3 consonants.
Since there are 4 vowels and 5 consonants, we have

$$(4C2)(5C3)$$

Since our next task is to make word sequences out of these letters, we multiply these by 5!.

$$(4C2)(5C3)(5!) = 7200.$$

✓ Example 10.6.4

A standard deck of playing cards has 52 cards consisting of 4 suits each with 13 cards. In how many different ways can a 5-card hand consisting of four cards of one suit and one of another be drawn?

Solution

We will do the problem using the following steps.
Step 1. Select a suit.
Step 2. Select four cards from this suit.
Step 3. Select another suit.
Step 4. Select a card from that suit.

Applying the multiplication axiom, we have

Ways of selecting the first suit	Ways of selecting 4 cards from this suit	Ways of selecting the next suit	Ways of selecting a card from that suit
4C1	13C4	3C1	13C1

$(4C1)(13C4)(3C1)(13C1) = 111,540.$

A STANDARD DECK OF 52 PLAYING CARDS

As in the previous example, many examples and homework problems in this book refer to a standard deck of 52 playing cards. Before we end this section, we take a minute to describe a standard deck of playing cards, as some readers may not be familiar with this.

A standard deck of 52 playing cards has 4 suits with 13 cards in each suit.

♦ diamonds ♥ hearts ♠ spades ♣ clubs

Each suit is associated with a color, either black (spades, clubs) or red (diamonds, hearts)

Each suit contains 13 denominations (or values) for cards:

nine numbers 2, 3, 4, …., 10 and Jack(J), Queen (Q), King (K), Ace (A).

The Jack, Queen and King are called “face cards” because they have pictures on them. Therefore a standard deck has 12 face cards: (3 values JQK) x (4 suits ♦♥♠ ♣)

We can visualize the 52 cards by the following display

Suit	Color	Values (Denominations)
♦ Diamonds	Red	2 3 4 5 6 7 8 9 10 J Q K A
♥ Hearts	Red	2 3 4 5 6 7 8 9 10 J Q K A
♠ Spades	Black	2 3 4 5 6 7 8 9 10 J Q K A
♣ Clubs	Black	2 3 4 5 6 7 8 9 10 J Q K A

10.6.1: Combinations- Involving Several Sets (Exercises)

Following problems involve combinations from several different sets.

1. How many 5-people committees consisting of three boys and two girls can be chosen from a group of four boys and four girls?	2. A club has 4 men, 5 women, 8 boys and 10 girls as members. In how many ways can a group of 2 men, 3 women, 4 boys and 4 girls be chosen?
3. How many 4-people committees chosen from 4 men and 6 women will have at least 3 men?	4. A batch contains 10 transistors of which three are defective. If three are chosen, in how many ways can they be selected with two defective?
5. In how many ways can five counters labeled A, B, C, D and E at a store be staffed by two men and three women chosen from a group of four men and six women?	6. How many 4-letter word sequences consisting of two vowels and two consonants can be made from the letters of the word PHOENIX if no letter is repeated?

Three marbles are chosen from an urn that contains 5 red, 4 white, and 3 blue marbles. How many samples of the following type are possible?

7. All three white.	8. Two blue and one white
9. One of each color.	10. All three of the same color.
11. At least two red.	12. None red.

The following problems involve combinations from several different sets.

Five coins are chosen from a bag that contains 4 dimes, 5 nickels, and 6 pennies. How many samples of five coins of the following types are possible?

13. At least four nickels.	14. No pennies.
15. Five of a kind.	16. Four of a kind.
17. Two of one kind and two of another kind.	18. Three of one kind and two of another kind.

Find the number of different ways draw a 5-card hand from a deck to have the following combinations.

19. Three face cards.	20. A heart flush (all hearts).
21. Two hearts and three diamonds	22. Two cards of one suit, and three of another suit.
23. Two kings and three queesns.	24. 2 cards of one value and 3 of another value

The party affiliation of the 100 United States Senators in the 114^{th} Congress, January 2015, was:

44 Democrats, 54 Republicans, and 2 Independents.

25. In how many ways could a 10 person committee be selected if it is to contain 4 Democrats, 5 Republicans, and 1 Independent?	26. In how many different ways could a 10 person committee be selected with 6 or 7 Republicans and the Democrats (with no Independents)?

The 100 United States Senators in the 114^{th} Congress, January 2015, included 80 men and 20 women. Suppose a committee senators is working on legislation about wage discrimination by gender.

27. In how many ways could a 12 person committee be selected to contain equal numbers of men and women.	28. In how many ways could a 6 person committee be selected to contain fewer women than men?

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises. He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises.

How many different playlists of 6 songs can be selected that satisfy each of the following: (We care which songs are selected to be on the playlist, but not what order they are selected or listed in.)

29. Playlist has 2 songs of each type	30. Playlist has no country songs
31. Playlist has 3 rocks, 2 raps, and 1 country song	32. Playlist has 3 or 4 rock songs and all the rest are rap songs

10.7: Binomial Theorem

We end this chapter with one more application of combinations. Combinations are used in determining the coefficients of a binomial expansion such as $(x+y)^n$. Expanding a binomial expression by multiplying it out is a very tedious task, and is not practiced. Instead, a formula known as the Binomial Theorem is utilized to determine such an expansion. Before we introduce the Binomial Theorem, however, consider the following expansions.

$$
\begin{aligned}
(x+y)^2 &= x^2+2xy+y^2 \\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3 \\
(x+y)^4 &= x^4+4x^2y+6x^2y^2+4xy^3+y^4 \\
(x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5 \\
(x+y)^6 &= x6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6
\end{aligned}
$$

We make the following observations.

1. There are $n+1$ terms in the expansion $(x+y)^n$
2. The sum of the powers of x and y is n.
3. The powers of x begin with n and decrease by one with each successive term.
   The powers of y begin with 0 and increase by one with each successive term.

Suppose we want to expand $(x+y)^3$. We first write the expansion without the coefficients. We temporarily substitute a blank in place of the coefficients.

$$(x+y)^3 = \square x^3 + \square x^2 y + \square x y^2 + \square y^3 \tag{10.7.1}$$

Our next job is to replace each of the blanks in equation (10.7.1) with the corresponding coefficients that belong to this expansion. Clearly,

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

If we multiply the right side and do not collect terms, we get the following.

$$xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy$$

Each product in the above expansion is the result of multiplying three variables by picking one from each of the factors $(x+y)(x+y)(x+y)$. For example, the product xxy is gotten by choosing x from the first factor, x from the second factor, and y from the third factor. There are three such products that simplify to x^2y, namely xxy, xyx, and yxx. These products take place when we choose an x from two of the factors and choose a y from the other factor. Clearly this can be done in 3C2, or 3 ways. Therefore, the coefficient of the term x^2y is 3. The coefficients of the other terms are obtained in a similar manner.

We now replace the blanks with the coefficients in equation (10.7.1), and we get

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

Example 10.7.1

Find the coefficient of the term x^2y^5 in the expansion $(x+y)^7$.

Solution

The expansion $(x+y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$

In multiplying the right side, each product is gotten by picking an x or y from each of the seven factors $(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$.

The term x^2y^5 is obtained by choosing an x from two of the factors and a y from the other five factors. This can be done in 7C2, or 21 ways.

Therefore, the coefficient of the term x^2y^5 is 21.

Example 10.7.2

Expand $(x+y)^7$

Solution

We first write the expansion without the coefficients.

$$(x+y)^7 = \square x^7 + \square x^6 y + \square x^5 y^2 + \square x^4 y^3 + \square x^3 y^4 + \square x^2 y^5 + \square x y^6 + \square y^7$$

Now we determine the coefficient of each term as we did in Example 10.7.1.

The coefficient of the term x^7 is 7C7 or 7CO which equals 1.

The coefficient of the term $x^6 y$ is 7C6 or 7C1 which equals 7.

The coefficient of the term $x^5 y^2$ is 7C5 or 7C2 which equals 21.

The coefficient of the term $x^4 y^3$ is 7C4 or 7C3 which equals 35,

and so on.

Substituting, we get: $(x+y)^7 = x^7 + 7x^6 y + 21x^5 y^2 + 35x^4 y^3 + 35x^3 y^4 + 21x^2 y^5 + 7xy^6 + y^7$

We generalize the result.

Binomial Theorem

$$(x+y)^n =_n C_0 x^n +_n C_1 x^{n-1} y +_n C_2 x^{n-2} y^2 + \cdots +_n C_{n-1} x y_n^{n-1+} C_n y^n$$

Example 10.7.3

Expand $(3a-2b)^4$

Solution

If we let $x = 3a$ and $y = -2b$, and apply the Binomial Theorem, we get

$$\begin{aligned}(3a-2b)^4 &= 4\,\mathrm{Co}(3a)^4 + 4Cl(3a)^3(-2b) + 4C2(3a)^2(-2b)^2 + 4C3(3a)(-2b)^3 + 4C4(-2b)^4 \\ &= 1\left(81a^4\right) + 4\left(27a^3\right)(-2b) + 6\left(9a^2\right)\left(4b^2\right) + 4(3a)\left(-8b^3\right) + 1\left(16b^3\right) \\ &= 81a^4 - 216a^3 b + 216a^2 b^2 - 96ab^3 + 16b^4\end{aligned}$$

Example 10.7.4

Find the fifth term of the expansion $(3a-2b)^7$.

Solution

The Binomial theorem tells us that in the r-th term of an expansion, the exponent of the y term is always one less than r, and, the coefficient of the term is ${}_{\mathrm{n}}\mathrm{C}_{\mathrm{r}-1}$.

$n = 7$ and $r - 1 = 5 - 1 = 4$, so the coefficient is $7\mathrm{C}4 = 35$

Thus, the fifth term is $(7\mathrm{C}4)(3\mathrm{a})^3(-2\mathrm{b})^4 = 35\left(27\mathrm{a}^3\right)\left(16\mathrm{b}^4\right) = 15120\mathrm{a}^3\mathrm{b}^4$

10.7.1: Binomial Theorem (Exercises)

Use the Binomial Theorem to do the following problems.

1. Expand $(a+b)^5$.	2. Expand $(a-b)^6$.
3. Expand $(x-2y)^5$.	4. Expand $(2x-3y)^4$.
5. Find the third term of $(2x-3y)^6$.	6. Find the sixth term of $(5x+y)^8$.
7. Find the coefficient of the x^3y^4 term in the expansion of $(2x+y)^7$.	8. Find the coefficient of the a^4b^6 term in the expansion of $(3a-b)^{10}$.
9. A coin is tossed 5 times, in how many ways is it possible to get three heads and two tails?	10. A coin is tossed 10 times, in how many ways is it possible to get seven heads and three tails?
11. How many subsets are there of a set that has 6 elements?	12. How many subsets are there of a set that has n elements?

10.8: Chapter Review

1. Suppose of the 4,000 freshmen at a college everyone is enrolled in a mathematics or an English class during a given quarter. If 2,000 are enrolled in a mathematics class, and 3,000 in an English class, how many are enrolled in both a mathematics class and an English class?
2. In a survey of 250 people, it was found that 125 had read Time magazine, 175 had read Newsweek, 100 had read U. S. News, 75 had read Time and Newsweek, 60 had read Newsweek and U. S. News, 55 had read Time and U. S. News, and 25 had read all three.
   a. How many had read Time but not the other two?
   b. How many had read Time or Newsweek but not the U. S. News And World Report?
   c. How many had read none of these three magazines?
3. At a manufacturing plant, a product goes through assembly, testing, and packing. If a plant has three assembly stations, two testing stations, and two packing stations, in how many different ways can a product achieve its completion?
4. Six people are to line up for a photograph. How many different lineups are possible if three of them insist on standing next to each other?
5. How many four-letter word sequences can be made from the letters of the word CUPERTINO?
6. In how many different ways can a 20-question multiple choice test be designed so that its answers contain 2 A's, 4 B's, 9 C's, 3 D's, and 2 E's?
7. The U. S. Supreme Court has nine judges. In how many different ways can the judges cast a six-tothree decision in favor of a ruling?
8. In how many different ways can a coach choose a linebacker, a guard, and a tackle from five players on the bench, if all five can play any of the three positions?
9. How many three digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if no repetitions are allowed?
10. Compute:
    a. $9C4$
    b. $8P3$
    c. $\frac{10!}{4!(10-4)!}$
11. In how many ways can 3 English, 3 Math, and 4 Spanish books be set on a shelf if the books are grouped by subject?
12. In how many ways can a 10-question multiple choice test with four possible answers for each question be answered?
13. On a soccer team three fullbacks can play any of the three fullback positions, left, center, and right. The three halfbacks can play any of the three halfback positions, the four forwards can play any of the four positions, and the goalkeeper plays only his position. How many different arrangements of the 11 players are possible?
14. From a group of 6 people, 3 are assigned to cleaning, 2 to hauling and one to garbage collecting. How many different ways can this be done?
15. How many three-letter word sequences can be made from the letters of the word OXYGEN?
16. In how many ways can 3 books be selected from 4 English and 2 History books if at least one English book must be chosen?
17. Five points lie on the rim of a circle. Choosing the points as vertices, how many different triangles can be drawn?
18. A club consists of six men and nine women. In how many ways can a president, a vice president and a treasurer be chosen if the two of the officers must be women?
19. Of its 12 sales people, a company wants to assign 4 to its Western territory, 5 to its Northern territory, and 3 to its Southern territory. How many ways can this be done?
20. How many permutations of the letters of the word OUTSIDE have consonants in the first and last place?
21. How many distinguishable permutations are there in the word COMMUNICATION?
22. How many five-card poker hands consisting of the following distribution are there?
    a. A flush(all five cards of a single suit)
    b. Three of a kind(e.g. three aces and two other cards)
    c. Two pairs(e.g. two aces, two kings and one other card)
    d. A straight(all five cards in a sequence)
23. Company stocks on an exchange are given symbols consisting of three letters. How many different three-letter symbols are possible?
24. How many four-digit odd numbers are there?

25. In how many ways can 7 people be made to stand in a straight line? In a circle?
26. A United Nations delegation consists of 6 Americans, 5 Russians, and 4 Chinese. Answer the following questions.
    a. How many committees of five people are there?
    b. How many committees of three people consisting of at least one American are there?
    c. How many committees of four people having no Russians are there?
    d. How many committees of three people have more Americans than Russians?
    e. How many committees of three people do not have all three Americans?
27. If a coin is flipped five times, in how many different ways can it show up three heads?
28. To reach his destination, a man is to walk three blocks north and four blocks west. How many different routes are possible?
29. All three players of the women's beach volleyball team, and all three players of the men's beach volleyball team are to line up for a picture with all members of the women's team lined together and all members of men's team lined up together. How many ways can this be done?
30. From a group of 6 Americans, 5 Japanese and 4 German delegates, two Americans, two Japanese and a German are chosen to line up for a photograph. In how many different ways can this be done?
31. Find the fourth term of the expansion $(2x-3y)^8$.
32. Find the coefficient of the a^5b^4 term in the expansion of $(a – 2b)^9$.

CHAPTER OVERVIEW

11: Probability

Learning Objectives

In this chapter, you will learn to:

1. Write sample spaces.
2. Determine whether two events are mutually exclusive.
3. Use the Addition Rule.
4. Calculate probabilities using both tree diagrams and combinations.
5. Do problems involving conditional probability.
6. Determine whether two events are independent.

11.1: Sample Spaces and Probability
- 11.1.1: Sample Spaces and Probability (Exercises)

11.2: Mutually Exclusive Events and the Addition Rule
- 11.2.1: Mutually Exclusive Events and the Addition Rule (Exercises)

11.3: Probability Using Tree Diagrams and Combinations
- 11.3.1: Probability Using Tree Diagrams and Combinations (Exercises)

11.4: Conditional Probability
- 11.4.1: Conditional Probability (Exercises)

11.5: Independent Events
- 11.5.1: Independent Events (Exercises)

11.6: Chapter Review

11.1: Sample Spaces and Probability

Learning Objectives

In this section, you will learn to:

1. Write sample spaces.
2. Calculate probabilities by examining simple events in sample spaces.

If two coins are tossed, what is the probability that both coins will fall heads? The problem seems simple enough, but it is not uncommon to hear the incorrect answer 1/3. A student may incorrectly reason that if two coins are tossed there are three possibilities, one head, two heads, or no heads. Therefore, the probability of two heads is one out of three. The answer is wrong because if we toss two coins there are four possibilities and not three. For clarity, assume that one coin is a penny and the other a nickel. Then we have the following four possibilities.

HH HT TH TT

The possibility HT, for example, indicates a head on the penny and a tail on the nickel, while TH represents a tail on the penny and a head on the nickel. It is for this reason, we emphasize the need for understanding sample spaces.

Sample Spaces

An act of flipping coins, rolling dice, drawing cards, or surveying people are referred to as a probability **experiment**. A **sample space** of an experiment is the set of all possible outcomes.

Example 11.1.1

If a die is rolled, write a sample space.

Solution

A die has six faces each having an equally likely chance of appearing. Therefore, the set of all possible outcomes S is

{ 1, 2, 3, 4, 5, 6 }.

Example 11.1.2

A family has three children. Write a sample space.

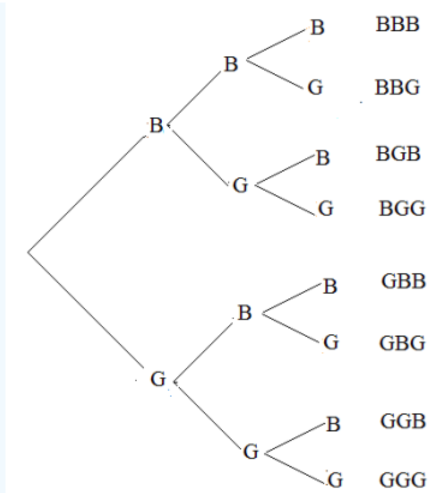
Solution

The sample space consists of eight possibilities.

{ BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG }

The possibility BGB, for example, indicates that the first born is a boy, the second born a girl, and the third a boy.

We illustrate these possibilities with a tree diagram.



✔ Example 11.1.3

Two dice are rolled. Write the sample space.

Solution

We assume one of the dice is red, and the other green. We have the following 36 possibilities.

Green						
Red	**1**	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The entry (2, 5), for example, indicates that the red die shows a 2, and the green a 5.

Probability

Now that we understand the concept of a sample space, we will define probability.

Definition: Probability

For a sample space S, and an outcome A of S, the following two properties are satisfied.

1. If A is an outcome of a sample space, then the probability of A, denoted by $P(A)$, is between 0 and 1, inclusive.

$$0 \leq P(A) \leq 1$$

2. The sum of the probabilities of all the outcomes in S equals 1.

The probability $P(A)$ of an event A describes the chance or likelihood of that event occurring.

- If $P(A)=0$, event A is certain not to occur. If $P(A)=1$, event A is certain to occur.
- If $P(A)=0.5$, then event A is equally likely to occur or not occur.
- If we toss a fair coin that is equally likely to land on heads or tails, then P(Head) = 0.50.

- If the weather forecast says there is a 70% chance of rain today, then P(Rain) = 0.70, indicating is it more likely to rain than to not rain.

Example 11.1.4

If two dice, one red and one green, are rolled, find the probability that the red die shows a 3 and the green shows a six.

Solution

Since two dice are rolled, there are 36 possibilities. The probability of each outcome, listed in Example $11.1.3$, is equally likely.

Since (3, 6) is one such outcome, the probability of obtaining (3, 6) is 1/36.

The example we just considered consisted of only one outcome of the sample space. We are often interested in finding probabilities of several outcomes represented by an event.

An **event** is a subset of a sample space. If an event consists of only one outcome, it is called a **simple event.**

Example 11.1.5

If two dice are rolled, find the probability that the sum of the faces of the dice is 7.

Solution

Let E represent the event that the sum of the faces of two dice is 7.

The possible cases for the sum to be equal to 7 are: (1, 6), (2,5), (3, 4), (4, 3), (5, 2), and (6, 1), so event E is

E = {(1, 6), (2,5), (3, 4), (4, 3), (5, 2), (6, 1)}

The probability of the event E is

P(E) = 6/36 or 1/6.

Example 11.1.6

A jar contains 3 red, 4 white, and 3 blue marbles. If a marble is chosen at random, what is the probability that the marble is a red marble or a blue marble?

Solution

We assume the marbles are r_1, r_2, r_3, w_1, w_2, w_3, w_4, b_1, b_2, b_3. Let the event C represent that the marble is red or blue.

The sample space $\mathrm{S} = \{\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3, \mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3, \mathrm{w}_4, \mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3\}$.

And the event $\mathrm{C} = \{\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3, \mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3\}$

Therefore, the probability of C,

$$\mathrm{P(C)} = 6/10 \text{ or } 3/5$$

Example 11.1.7

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **without replacement**. That means that after a marble is drawn it is not replaced in the jar, and therefore is no longer available to select on the second draw.

Solution

Since two marbles are drawn without replacement, the sample space consists of the following six possibilities.

$$\mathrm{S} = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$$

Note that (1,1), (2,2) and (3,3) are not listed in the sample space. These outcomes are not possible when drawing without replacement, because once the first marble is drawn but not replaced into the jar, that marble is not available in the jar to be selected again on the second draw.

Let the event E represent that the sum of the numbers is five. Then

$$\mathrm{E} = \{(2,3),(3,2)\}$$

Therefore, the probability of E is

$$\mathrm{P(E)} = 2/6 \text{ or } 1/3.$$

✓ Example 11.1.8

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **without replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution

The sample space, as in Example 11.1.7, consists of the following six possibilities.

$$\mathrm{S} = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$$

Let the event F represent that the sum of the numbers is at least four. Then

$$\mathrm{F} = \{(1,3),(3,1),(2,3),(3,2)\}$$

Therefore, the probability of F is

$$\mathrm{P(F)} = 4/6 \text{ or } 2/3$$

✓ Example 11.1.9

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is 5?

Note: The two marbles in this example are drawn consecutively **with replacement**. That means that after a marble is drawn it IS replaced in the jar, and therefore is available to select again on the second draw.

Solution

When two marbles are drawn with replacement, the sample space consists of the following nine possibilities.

$$\mathrm{S} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

Note that (1,1), (2,2) and (3,3) are listed in the sample space. These outcomes are possible when drawing with replacement, because once the first marble is drawn and replaced, that marble is not available in the jar to be drawn again.

Let the event E represent that the sum of the numbers is four. Then

$$\mathrm{E} = (2,3),(3,2)$$

Therefore, the probability of F is $\mathrm{P(E)} = 2/9$

Note that in Example 11.1.9 when we selected marbles with replacement, the probability has changed from Example 11.1.7 where we selected marbles without replacement.

✓ Example 11.1.10

A jar contains three marbles numbered 1, 2, and 3. If two marbles are drawn **with replacement**, what is the probability that the sum of the numbers is *at least* 4?

Solution

The sample space when drawing with replacement consists of the following nine possibilities.

$$\mathrm{S} = (1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$$

Let the event F represent that the sum of the numbers is at least four. Then

$$\mathrm{F} = (1,3),(3,1),(2,3),(3,2),(2,2),(3,3)$$

Therefore, the probability of F is

$$\mathrm{P}(\mathrm{F}) = 6/9 \text{ or } 2/3.$$

Note that in Example $11.1.10$ when we selected marbles with replacement, the probability is the same as in Example $11.1.8$ where we selected marbles without replacement.

Thus sampling with or without replacement MAY change the probabilities, but may not, depending on the situation in the particular problem under consideration. We'll re-examine the concepts of sampling with and without replacement in Section 8.3.

Example 11.1.11

One 6 sided die is rolled once. Find the probability that the result is greater than 4.

Solution

The sample space consists of the following six possibilities in set S: $\mathrm{S} = 1,2,3,4,5,6$

Let E be the event that the number rolled is greater than four: $\mathrm{E} = 5,6$

Therefore, the probability of E is: $\mathrm{P}(\mathrm{E}) = 2/6 \text{ or } 1/3$.

11.1.1: Sample Spaces and Probability (Exercises)

SECTION 11.1 PROBLEM SET: SAMPLE SPACES AND PROBABILITY

In problems 1 - 6, write a sample space for the given experiment.

1) A die is rolled.	2) A penny and a nickel are tossed.
3) A die is rolled, and a coin is tossed.	4) Three coins are tossed.
5) Two dice are rolled.	6) A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn.

In problems 7 - 12, one card is randomly selected from a deck. Find the following probabilities.

7) P(an ace)	8) P(a red card)
9) P(a club)	10) P(a face card)
11) P(a jack or a spade)	12) P(a jack and a spade)

For problems 13 - 16: A jar contains 6 red, 7 white, and 7 blue marbles. If one marble is chosen at random, find the following probabilities.

13) P(red)	14) P(white)
15) P(red or blue)	16) P(red and blue)

For problems 17 - 22: Consider a family of three children. Find the following probabilities.

17) P(two boys and a girl)	18) P(at least one boy)
19) P(children of both sexes)	20) P(at most one girl)
21) P(first and third children are male)	22) P(all children are of the same gender)

For problems 23 - 27: Two dice are rolled. Find the following probabilities.

23) P(the sum of the dice is 5)	24) P(the sum of the dice is 8)
25) P(the sum is 3 or 6)	26) P(the sum is more than 10)
27) P(the result is a double) (Hint: a double means that both dice show the same value)	

For problems 28-31: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITHOUT REPLACEMENT. That means that after a marble is drawn it is NOT replaced in the jar before the second marble is selected. Find the following probabilities.

28) P(the sum of the numbers is 5)	29) P(the sum of the numbers is odd)
30) P(the sum of the numbers is 9)	31) P(one of the numbers is 3)

For problems, 32-33: A jar contains four marbles numbered 1, 2, 3, and 4. Two marbles are drawn randomly WITH REPLACEMENT. That means that after a marble is drawn it is replaced in the jar before the second marble is selected. Find the following probabilities.

32) P(the sum of the numbers is 5)	33) P(the sum of the numbers is 2)

This page titled 11.1.1: Sample Spaces and Probability (Exercises) is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Rupinder Sekhon and Roberta Bloom via source content that was edited to the style and standards of the LibreTexts platform.

- **8.1.1: Sample Spaces and Probability (Exercises)** by Rupinder Sekhon and Roberta Bloom is licensed CC BY 4.0. Original source: https://www.deanza.edu/faculty/bloomroberta/math11/afm3files.html.html.

11.2: Mutually Exclusive Events and the Addition Rule

Learning Objectives

In this section, you will learn to:

1. Define compound events using union, intersection, and complement.
2. Identify mutually exclusive events
3. Use the Addition Rule to calculate probability for unions of events.

In the last chapter, we learned to find the union, intersection, and complement of a set. We will now use these set operations to describe events.

- The **union** of two events E and F, E ∪ F, is the set of outcomes that are in E or in F or in both.
- The **intersection** of two events E and F, E ∩ F, is the set of outcomes that are in both E and F.
- The **complement** of an event E, denoted by E^c, is the set of outcomes in the sample space S that are not in E.

It is worth noting that $P(E^c)$ = 1 - P(E). This follows from the fact that if the sample space has n elements and E has k elements, then E^c has $n-k$ elements. Therefore,

$$P\left(E^{c}\right)=\frac{n-k}{n}=1-\frac{k}{n}=1-P(E)$$

Of particular interest to us are the events whose outcomes do not overlap. We call these events mutually exclusive.

Two events E and F are said to be **mutually exclusive** if they do not intersect: E ∩ F = ∅.

Next we'll determine whether a given pair of events are mutually exclusive.

Example 11.2.1

A card is drawn from a standard deck. Determine whether the pair of events given below is mutually exclusive.

E = {The card drawn is an Ace}

F = {The card drawn is a heart}

Solution

Clearly the ace of hearts belongs to both sets. That is

$$\mathrm{E} \cap \mathrm{F}=\{\text { Ace of hearts }\} \neq \varnothing$$

Therefore, the events E and F are not mutually exclusive.

Example 11.2.2

Two dice are rolled. Determine whether the pair of events given below is mutually exclusive.

G = {The sum of the faces is six}

H = {One die shows a four}

Solution

For clarity, we list the elements of both sets.

G = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)} and H = {(2, 4), (4, 2)}

Clearly, G ∩ H = {(2, 4), (4, 2)} $\neq \varnothing$.

Therefore, the two sets are not mutually exclusive.

Example 11.2.3

A family has three children. Determine whether the following pair of events are mutually exclusive.

M = {The family has at least one boy}

N = {The family has all girls}

Solution

Although the answer may be clear, we list both the sets.

M = { BBB, BBG, BGB, BGG, GBB, GBG, GGB } and N = { GGG }

Clearly, M ∩ N = ∅

Therefore, events M and N are mutually exclusive.

We will now consider problems that involve the union of two events.

Given two events, E, F, then finding the probability of E ∪ F, is the same as finding the probability that E will happen, or F will happen, or both will happen.

Example 11.2.4

If a die is rolled, what is the probability of obtaining an even number or a number greater than four?

Solution

Let E be the event that the number shown on the die is an even number, and let F be the event that the number shown is greater than four.

The sample space S = { 1, 2, 3, 4, 5, 6 }. The event E = { 2, 4, 6 }, and event F = { 5, 6 }

We need to find P(E ∪ F).

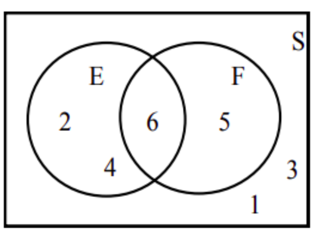
Since P(E) = 3/6, and P(F) = 2/6, a student may say P(E ∪ F) = 3/6 + 2/6. This will be incorrect because the element 6, which is in both E and F has been counted twice, once as an element of E and once as an element of F. In other words, the set E ∪ F has only four elements and not five: set E ∪ F = { 2,4,5,6 }

Therefore, P(E ∪ F) = 4/6 and not 5/6.

This can be illustrated by a Venn diagram. We'll use the Venn Diagram to re-examine Example 11.2.4 and derive a probability rule that we can use to calculate probabilities for unions of events.

The sample space S, the events E and F, and E ∩ F are listed below.

$$\mathrm{S}=\{1,2,3,4,5,6\}, \mathrm{E}=\{2,4,6\}, \mathrm{F}=\{5,6\}, \text{ and } \mathrm{E} \cap \mathrm{F}=\{6\}.$$



The above figure shows S, E, F, and E ∩ F.

Finding the probability of E ∪ F, is the same as finding the probability that E will happen, or F will happen, or both will happen.

If we count the number of elements n(E) in E, and add to it the number of elements n(F) in F, the points in both E and F are counted twice, once as elements of E and once as elements of F. Now if we subtract from the sum, n(E) + n(F), the number n(E ∩ F), we remove the duplicity and get the correct answer. So as a rule,

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

By dividing the entire equation by n(S), we get

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

Since the probability of an event is the number of elements in that event divided by the number of all possible outcomes, we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Applying the above for Example $11.2.4$, we get

$$P(E \cup F) = 3/6 + 2/6 - 1/6 = 4/6$$

This is because, when we add P(E) and P(F), we have added P(E ∩ F) twice. Therefore, we must subtract P(E ∩ F), once.

This gives us the general formula, called **the Addition Rule**, for finding the probability of the union of two events. Because event E ∪ F is the event that E will happen, OR F will happen, OR both will happen, we sometimes call this the **Addition Rule for OR Events.** It states

Addition Rule

$$\mathbf{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

If, and only if, two events E and F are mutually exclusive, then $E \cap F = \varnothing$ and $P(E \cap F) = 0$, and we get $P(E \cup F) = P(E) + P(F)$

Example $11.2.5$

If a card is drawn from a deck, use the addition rule to find the probability of obtaining an ace or a heart.

Solution

Let A be the event that the card is an ace, and H the event that it is a heart.

Since there are four aces, and thirteen hearts in the deck,

P(A) = 4/52 and P(H) = 13/52.

Furthermore, since the intersection of two events consists of only one card, the ace of hearts, we now have:

P(A ∩ H) = 1/52

We need to find P(A ∪ H):

$$\begin{aligned} P(A \cup H) &= P(A) + P(H) - P(A \cap H) \\ &= 4/52 + 13/52 - 1/52 = 16/52 \end{aligned}$$

Example $11.2.6$

Two dice are rolled, and the events F and T are as follows:

F = {The sum of the dice is four} and T = {At least one die shows a three}

Find P(F ∪ T).

Solution

We list F and T, and F ∩ T as follows:

F = {(1, 3), (2, 2), (3, 1)}

T = {(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)}

F ∩ T = {(1, 3), (3, 1)}

Since P(F ∪ T) = P(F) + P(T) - P(F ∩ T)

We have P(F ∪ T) = 3/36 + 11/36 - 2/36 = 12/36.

Example 11.2.7

Mr. Washington is seeking a mathematics instructor's position at his favorite community college in Cupertino. His employment depends on two conditions: whether the board approves the position, and whether the hiring committee selects him. There is a 80% chance that the board will approve the position, and there is a 70% chance that the hiring committee will select him. If there is a 90% chance that at least one of the two conditions, the board approval or his selection, will be met, what is the probability that Mr. Washington will be hired?

Solution

Let A be the event that the board approves the position, and S be the event that Mr. Washington gets selected. We have,

P(A) = .80, P(S) = .70, and P(A ∪ S) = .90.

We need to find, P(A ∩ S).

The addition formula states that,

$$P(A \cup S) = P(A) + P(S) - P(A \cap S)$$

Substituting the known values, we get

$$.90 = .80 + .70 - P(\mathrm{A} \cap \mathrm{S})$$

Therefore, P(A ∩ S) = .60.

Example 11.2.8

The probability that this weekend will be cold is .6, the probability that it will be rainy is .7, and probability that it will be both cold and rainy is .5. What is the probability that it will be neither cold nor rainy?

Solution

Let C be the event that the weekend will be cold, and R be event that it will be rainy. We are given that

$$\mathrm{P(C)} = .6, \quad \mathrm{P(R)} = .7, \quad \mathrm{P(C \cap R)} = .5$$

First we find P(C ∪ R) using the Addition Rule.

$$P(C \cup R) = P(C) + P(R) - P(C \cap R) = .6 + .7 - .5 = .8$$

Then we find P((C ∪ R)^c) using the Complement Rule.

$$P\left((C \cup R)^c\right) = 1 - P(C \cup R) = 1 - .8 = .2$$

We summarize this section by listing the important rules.

Summary

The Addition Rule

For Two Events E and F, $\mathrm{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$

The Addition Rule for Mutually Exclusive Events

If Two Events E and F are Mutually Exclusive, then $\mathrm{P(E \cup F) = P(E) + P(F)}$

The Complement Rule

If E^c is the Complement of Event E, then $\mathrm{P(E^c) = 1 - P(E)}$

11.2.1: Mutually Exclusive Events and the Addition Rule (Exercises)

SECTION 11.2 PROBLEM SET: MUTUALLY EXCLUSIVE EVENTS AND THE ADDITION RULE

Determine whether the following pair of events are mutually exclusive.

1) A = {A person earns more than $25,000} B = {A person earns less than $20,000}	2) A card is drawn from a deck. C = {It is a King} D = {It is a heart}.
3) A die is rolled. E = {An even number shows} F = {A number greater than 3 shows}	4) Two dice are rolled. G = {The sum of dice is 8} H = {One die shows a 6}
5) Three coins are tossed. I = {Two heads come up} J = {At least one tail comes up}	6) A family has three children. K = {First born is a boy} L = {The family has children of both sexes}

Use the Addition Rule to find the following probabilities.

7) A card is drawn from a deck. Events C and D are: C = {It is a king} D = {It is a heart} Find P(C or D).	8) A die is rolled. The events E and F are: E = {An even number shows} F = {A number greater than 3 shows} Find P(E or F).
9) Two dice are rolled. Events G and H are: G = {The sum of dice is 8} H ={Exactly one die shows a 6} Find P(G or H).	10) Three coins are tossed. Events I and J are: I = {Two heads come up} J = {At least one tail comes up} Find P(I or J).
11) At a college, 20% of the students take Finite Mathematics, 30% take Statistics and 10% take both. What percent of students take Finite Mathematics or Statistics?	12) This quarter, there is a 50% chance that Jason will pass Accounting, a 60% chance that he will pass English, and 80% chance that he will pass at least one of these two courses. What is the probability that he will pass both Accounting and English?

Questions 13 - 20 refer to the following: The table shows the distribution of Democratic and Republican U.S by gender in the 114th Congress as of January 2015.

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities.

13) P(M and D)	14) P(F and R)
15) P(M or D)	16) P(F or R)
17) P(Mc or R)	18) P(M or F)
19) Are the events F, R mutually exclusive? Use probabilities to support your conclusions.	20) Are the events F, T mutually exclusive? Use probabilities to support your conclusion.

SECTION 11.2 PROBLEM SET: MUTUALLY EXCLUSIVE EVENTS AND THE ADDITION RULE

Use the Addition Rule to find the following probabilities.

21) If P(E) = .5 , P(F) = .4 , E and F are mutually exclusive, find P(E and F).	22) If P(E) = .4 , P(F) = .2 , E and F are mutually exclusive, find P(E or F).
23) If P(E) = .3, P(E or F) = .6 , P(E and F) = .2, find P(F).	24) If P(E) = .4, P(F) = .5 , P(E or F) = .7, find P(E and F).

25) In a box of assorted cookies, 36% of cookies contain chocolate and 12% of cookies contain nuts. 8% of cookies have both chocolats and nuts. Sean is allergic to chocolate and nuts. Find the probability that a cookie has chocolate chips **or** nuts (he can't eat it).	26) At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper. Find the probability that a course requires a final exam **or** a research paper.

Questions 25 and 26 are adapted from Introductory Statistics from OpenStax under a creative Commons Attribution 3.0 Unported License, available for download free atcnx.org/content/col11562/latest u

---

11.3: Probability Using Tree Diagrams and Combinations

Learning Objectives

In this section, you will learn to:

1. Use probability tree diagrams to calculate probabilities
2. Use combinations to calculate probabilities

In this section, we will apply previously learnt counting techniques in calculating probabilities, and use tree diagrams to help us gain a better understanding of what is involved.

USING TREE DIAGRAMS TO CALCULATE PROBABILITIES

We already used tree diagrams to list events in a sample space. Tree diagrams can be helpful in organizing information in probability problems; they help provide a structure for understanding probability. In this section we expand our previous use of tree diagrams to situations in which the events in the sample space are not all equally likely.

We assign the appropriate probabilities to the events shown on the branches of the tree.
By multiplying probabilities along a path through the tree, we can find probabilities for "and" events, which are intersections of events.

We begin with an example.

Example 11.3.1

Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn with replacement, what is the probability that both marbles are red?

Solution

Let E be the event that the first marble drawn is red, and let F be the event that the second marble drawn is red.

We need to find $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$.

By the statement, "two marbles are drawn with replacement," we mean that the first marble is replaced before the second marble is drawn.

There are 7 choices for the first draw. And since the first marble is replaced before the second is drawn, there are, again, seven choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 49 ordered pairs. Of the 49 ordered pairs, there are $3 \times 3=9$ ordered pairs that show red on the first draw and, also, red on the second draw. Therefore,

$$P(E \cap F)=\frac{9}{49}$$

Further note that in this particular case

$$P(E \cap F)=\frac{9}{49}=\frac{3}{7} \cdot \frac{3}{7}$$

giving us the result that in this example: $\mathbf{P}(\mathbf{E} \cap \mathbf{F})=\mathbf{P}(\mathbf{E}) \cdot \mathbf{P}(\mathbf{F})$

Example 11.3.2

If in Example $11.3.1$, the two marbles are drawn without replacement, then what is the probability that both marbles are red?

Solution

By the statement, "two marbles are drawn without replacement," we mean that the first marble is not replaced before the second marble is drawn.

Again, we need to find $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$.

There are, again, 7 choices for the first draw. And since the first marble is not replaced before the second is drawn, there are only six choices for the second draw. Using the multiplication axiom, we conclude that the sample space S consists of 42 ordered pairs. Of the 42 ordered pairs, there are $3 \times 2 = 6$ ordered pairs that show red on the first draw and red on the second draw. Therefore,

$$P(E \cap F) = \frac{6}{42}$$

Note that we can break this calculation down as

$$P(E \cap F) = \frac{6}{42} = \frac{3}{7} \cdot \frac{2}{6}$$

.

Here 3/7 represents $\mathrm{P}(\mathrm{E})$, and 2/6 represents the probability of drawing a red on the second draw, given that the first draw resulted in a red.

We write the latter as P(red on the second | red on first) or $\mathrm{P}(\mathrm{F} \mid \mathrm{E})$. The "|" represents the word "given" or “if”. This leads to the result that:

$$\mathbf{P}(\mathbf{E} \cap \mathbf{F}) = \mathbf{P}(\mathbf{E}) \cdot \mathbf{P}(\mathbf{F} \mid \mathbf{E})$$

The is an important result, called the **Multiplication Rule,** which will appear again in later sections.

We now demonstrate the above results with a tree diagram.

Example 11.3.3

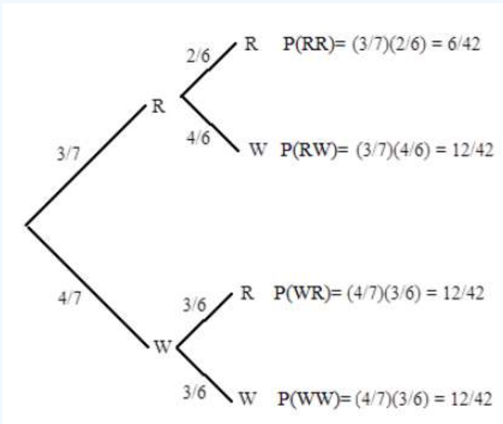
Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn without replacement, find the following probabilities using a tree diagram.

a. The probability that both marbles are red.
b. The probability that the first marble is red and the second white.
c. The probability that one marble is red and the other white.

Solution

Let R be the event that the marble drawn is red, and let W be the event that the marble drawn is white.

We draw the following tree diagram.



a. The probability that both marbles are red is $\mathrm{P}(\mathrm{RR}) = 6/42$
b. The probability that the first marble is red and the second is white is $\mathrm{P}(\mathrm{RW}) = 12/42$
c. For the probability that one marble is red and the other is white, we observe that this can be satisfied if the first is red and the second is white, **or** if the first is white and the second is red. The “or” tells us we’ll be using the Addition Rule from Section 7.2.

Furthermore events RW and WR are mutually exclusive events, so we use the form of the Addition Rule that applies to mutually exclusive events.

Therefore

P(one marble is red and the other marble is white)

$$\begin{aligned} &= \mathrm{P}(\mathrm{RW} \text{ or } \mathrm{WR}) \\ &= \mathrm{P}(\mathrm{RW}) + \mathrm{P}(\mathrm{WR}) \\ &= 12/42 + 12/42 = 24/42 \end{aligned}$$

USING COMBINATIONS TO FIND PROBABILITIES

Although the tree diagrams give us better insight into a problem, they are not practical for problems where more than two or three things are chosen. In such cases, we use the concept of combinations that we learned in the last chapter. This method is best suited for problems where the order in which the objects are chosen is not important, and the objects are chosen without replacement.

Example 11.3.4

Suppose a jar contains 3 red, 2 white, and 3 blue marbles. If three marbles are drawn without replacement, find the following probabilities.

a. P(Two red and one white)
b. P(One of each color)
c. P(None blue)
d. P(At least one blue)

Solution

Let us suppose the marbles are labeled as $R_1, R_2, R_3, W_1, W_2, B_1, B_2, B_3$.

a. P(Two red and one white)

Since we are choosing 3 marbles from a total of 8, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are $3\mathrm{C}2 \times 2\mathrm{C}1 = 6$ combinations consisting of 2 red and one white. Therefore,

$$P(\text{ Two red and one white }) = \frac{3\mathrm{C}2 \times 2\mathrm{C}1}{8\mathrm{C}3} = \frac{6}{56}.$$

b. P(One of each color)

Again, there are 8C3 = 56 possible combinations. Of these 56 combinations, there are $3\mathrm{Cl} \times 2\mathrm{Cl} \times 3\mathrm{Cl} = 18$ combinations consisting of one red, one white, and one blue. Therefore,

$$P(\text{ One of each color }) = \frac{3\mathrm{C}1 \times 2\mathrm{C}1 \times 3\mathrm{C}1}{8\mathrm{C}3} = \frac{18}{56}$$

c. P(None blue)

There are 5 non-blue marbles, therefore

$$\mathrm{P}(\text{ None blue }) = \frac{5\mathrm{C}3}{8\mathrm{C}3} = \frac{10}{56} = \frac{5}{28}$$

d. P(At least one blue)

By "at least one blue marble," we mean the following: one blue marble and two non-blue marbles, *OR* two blue marbles and one non-blue marble, *OR* all three blue marbles. So we have to find the sum of the probabilities of all three cases.

$$\mathrm{P}(\text{At least one blue }) = \mathrm{P}(1 \text{ blue, } 2 \text{ non-blue}) + \mathrm{P}(2 \text{ blue, } 1 \text{ non-blue}) + \mathrm{P}(3 \text{ blue})$$

$$P(\text{ At least one blue }) = \frac{3\mathrm{C}1 \times 5\mathrm{C}2}{8\mathrm{C}3} + \frac{3\mathrm{C}2 \times 5\mathrm{C}1}{8\mathrm{C}3} + \frac{3\mathrm{C}3}{8\mathrm{C}3}$$

$$P(\text{ At least one blue }) = 30/56 + 15/56 + 1/56 = 46/56 = 23/28$$

Alternately, we can use the fact that $\mathrm{P(E)} = 1 - \mathrm{P(E^c)}$. If the event E = At least one blue, then E^c = None blue.

But from part c of this example, we have $(\mathrm{E}^c) = 5/28$, so $\mathrm{P(E)} = 1 - 5/28 = 23/28$.

Example 11.3.5

Five cards are drawn from a deck. Find the probability of obtaining two pairs, that is, two cards of one value, two of another value, and one other card.

Solution

Let us first do an easier problem-the probability of obtaining a pair of kings and queens.

Since there are four kings, and four queens in the deck, the probability of obtaining two kings, two queens and one other card is

$$\mathrm{P(A\ pair\ of\ kings\ and\ queens\ )} = \frac{4\mathrm{C}2 \times 4\mathrm{C}2 \times 44\mathrm{C}1}{52\mathrm{C}5}$$

To find the probability of obtaining two pairs, we have to consider all possible pairs.

Since there are altogether 13 values, that is, aces, deuces, and so on, there are 13C2 different combinations of pairs.

$$P(\text{ Two pairs }) = 13\mathrm{C}2 \cdot \frac{4\mathrm{C}2 \times 4\mathrm{C}2 \times 44\mathrm{C}1}{52\mathrm{C}5} = .04754$$

Example 11.3.6

A cell phone store receives a shipment of 15 cell phones that contains 8 iPhones and 7 Android phones. Suppose that 6 cell phones are randomly selected from this shipment. Find the probability that a randomly selected set of 6 cell phones consists of 2 iPhones and 4 Android phones.

Solution

There are 8C2 ways of selecting 2 out of the 8 iPhones.

and 7C4 ways of selecting 4 out of the 7 Android phones

But altogether there are 15C6 ways of selecting 6 out of 15 cell phones.

Therefore we have

$$P(2\text{ iPhones and }4\text{ Android phones }) = \frac{8\mathrm{C}2 \times 7\mathrm{C}4}{15\mathrm{C}6} = \frac{(28)(35)}{5005} = \frac{980}{5005} = 0.1958$$

Example 11.3.7

One afternoon, a bagel store still has 53 bagels remaining: 20 plain, 15 poppyseed, and 18 sesame seed bagels. Suppose that the store owner packages up a bag of 9 bagels to bring home for tomorrow's breakfast, and selects the bagels randomly. Find the probability that the bag contains 4 plain, 3 poppyseed, and 2 sesame seed.

Solution

There are 20C4 ways of selecting 4 out of the 20 plain bagels,

and 15C3 ways of selecting 3 out of the 15 poppyseed bagels,

and 18C2 ways of selecting 2 out of the 18 sesame seed bagels.

But altogether there are 53C9 ways of selecting 9 out of the 53 bagels.

$$\begin{aligned} \mathrm{P(4\ plain,\ 3\ poppyseed,\ and\ 2\ sesame\ seed)} &= \tfrac{20\mathrm{C}4 \times 15\mathrm{C}3 \times 18\mathrm{C}2}{53\mathrm{C}9} \\ &= \tfrac{(4845)(455)(153)}{4431613550} \\ &= 0.761 \end{aligned} \tag{11.3.1}$$

We end the section by solving a famous problem called the **Birthday Problem**.

Example 11.3.8: Birthday Problem

If there are 25 people in a room, what is the probability that at least two people have the same birthday?

Solution

Let event E represent that at least two people have the same birthday.

We first find the probability that no two people have the same birthday.

We analyze as follows.

Suppose there are 365 days to every year. According to the multiplication axiom, there are 365^{25} possible birthdays for 25 people. Therefore, the sample space has 365^{25} elements. We are interested in the probability that no two people have the same birthday. There are 365 possible choices for the first person and since the second person must have a different birthday, there are 364 choices for the second, 363 for the third, and so on. Therefore,

$$\mathrm{P}(\text{No two have the same birthday }) = \frac{365 \cdot 364 \cdot 363 \cdots 341}{365^{25}} = \frac{365\mathrm{P}25}{365^{25}}$$

Since P(at least two people have the same birthday) = 1 - P(No two have the same birthday),

$$\mathrm{P} \text{ at least two people have the same birthday }) = 1 - \frac{365\mathrm{P}25}{365^{25}} = .5687$$

11.3.1: Probability Using Tree Diagrams and Combinations (Exercises)

SECTION 11.3 PROBLEM SET: PROBABILITIES USING TREE DIAGRAMS AND COMBINATIONS

Two apples are chosen from a basket containing five red and three yellow apples.
Draw a tree diagram below, and find the following probabilities.

1) P(both red)	2) P(one red, one yellow)
3) P(both yellow)	4) P(First red and second yellow)

A basket contains six red and four blue marbles. Three marbles are drawn at random.
Find the following probabilities using the method shown in Example 8.3.2. Do not use combinations.

5) P(All three red)	6) P(two red, one blue)
7) P(one red, two blue)	8) P(first red, second blue, third red)

Three marbles are drawn from a jar containing five red, four white, and three blue marbles.
Find the following probabilities using combinations.

9) P(all three red)	10) P(two white and 1 blue)
11) P(none white)	12) P(at least one red)

A committee of four is selected from a total of 4 freshmen, 5 sophomores, and 6 juniors. Find the probabilities for the following events.

13) At least three freshmen.	14) No sophomores.
15) All four of the same class.	16) Not all four from the same class.
17) Exactly three of the same class.	18) More juniors than freshmen and sophomores combined.

Five cards are drawn from a deck. Find the probabilities for the following events.

19) Two hearts, two spades, and one club.	20) A flush of any suit (*all cards of a single suit*).
21) A full house of nines and tens (*3 nines and 2 tens*).	22) Any full house.
23) A pair of nines and a pair of tens (*and the fifth card is not a nine or ten*).	24) Any two pairs (*two cards of one value, two more cards of another value, and the fifth card does not have the same value as either pair*).

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises.
He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises.

Find the following probabilities:

25) P(playlist has 2 songs of each type)	26) P(playlist has no country songs)
27) P(playlist has 3 rock, 2 rap, and 1 country song)	28) P(playlist has 3 or 4 rock songs and the rest are rap songs)

A project is staffed 12 people: 5 engineers, 4 salespeople, and 3 customer service representatives.
A committee of 5 people is selected to make a presentation to senior management.

Find the probabilities of the following events.

29) The committee has 2 engineers, 2 salespeople, and 1 customer service representative.	30) The committee contains 3 engineer and 2 salespeople.
31) The committee has no engineers.	32) The committee has all salespeople.

Do the following birthday problems.

33) If there are 5 people in a room, what is the probability that no two have the same birthday?	34) If there are 5 people in a room, find the probability that at least 2 have the same birthday.

11.4: Conditional Probability

Learning Objectives

In this section, you will learn to:

1. recognize situations involving conditional probability
2. calculate conditional probabilities

Suppose a friend asks you the probability that it will snow today.

If you are in Boston, Massachusetts in the winter, the probability of snow today might be quite substantial. If you are in Cupertino, California in summer, the probability of snow today is very tiny, this probability is pretty much 0.

Let:

- A = the event that it will snow today
- B = the event that today you are in Boston in wintertime
- C = the event that today you are in Cupertino in summertime

Because the probability of snow is affected by the location and time of year, we can't just write $\mathrm{P}(\mathrm{A})$ for the probability of snow. We need to indicate the other information we know -location and time of year. We need to use **conditional probability**.

The event we are interested in is event A for snow. The other event is called the condition, representing location and time of year in this case.

We represent conditional probability using a vertical line | that means "if", or "given that", or "if we know that". The event of interest appears on the left of the |. The condition appears on the right side of the |.

The probability it will snow given that (if) you are in Boston in the winter is represented by $\mathbf{P}\left(\mathbf{A} | \mathbf{B}\right)$. In this case, the condition is B.

The probability that it will snow given that (if) you are in Cupertino in the summer is represented by $\mathbf{P}(\mathbf{A} | \mathbf{C})$. In this case, the condition is C.

Now, let's examine a situation where we can calculate some probabilities.

Suppose you and a friend play a game that involves choosing a single card from a well-shuffled deck. Your friend deals you one card, face down, from the deck and offers you the following deal: If the card is a king, he will pay you \$5, otherwise, you pay him \$1. Should you play the game?

You reason in the following manner. Since there are four kings in the deck, the probability of obtaining a king is 4/52 or 1/13. So, probability of not obtaining a king is 12/13. This implies that the ratio of your winning to losing is 1 to 12, while the payoff ratio is only \$1 to \$5. Therefore, you determine that you should not play.

But consider the following scenario. While your friend was dealing the card, you happened to get a glance of it and noticed that the card was a face card. Should you, now, play the game?

Since there are 12 face cards in the deck, the total elements in the sample space are no longer 52, but just 12. This means the chance of obtaining a king is 4/12 or 1/3. So your chance of winning is 1/3 and of losing 2/3. This makes your winning to losing ratio 1 to 2 which fares much better with the payoff ratio of \$1 to \$5. This time, you determine that you should play.

In the second part of the above example, we were finding the probability of obtaining a king knowing that a face card had shown. This is an example of **conditional probability**. Whenever we are finding the probability of an event E under the condition that another event F has happened, we are finding conditional probability.

The symbol $\mathrm{P}(\mathrm{E} | \mathrm{F})$ denotes the problem of finding the probability of E given that F has occurred. We read $\mathrm{P}(\mathrm{E} | \mathrm{F})$ as "the probability of E, given F."

Example 11.4.1

A family has three children. Find the conditional probability of having two boys and a girl given that the first born is a boy.

Solution

Let event E be that the family has two boys and a girl, and F that the first born is a boy.

First, we the sample space for a family of three children as follows.

$$S=\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$

Since we know the first born is a boy, our possibilities narrow down to four outcomes: BBB, BBG, BGB, and BGG.

Among the four, BBG and BGB represent two boys and a girl.

Therefore, $\mathrm{P}(\mathrm{E} \mid \mathrm{F})$ = 2/4 or 1/2.

Example 11.4.2

One six sided die is rolled once.

a. Find the probability that the result is even.
b. Find the probability that the result is even given that the result is greater than three.

Solution

The sample space is $\mathrm{S}=1,2,3,4,5,6$

Let event E be that the result is even and T be that the result is greater than 3.

a. $\mathrm{P}(\mathrm{E})$ = 3/6 because $\mathrm{E}=2,4,6$

b. Because $T=4,5,6$, we know that 1, 2, 3 cannot occur; only outcomes 4, 5, 6 are possible. Therefore of the values in E, only 4, 6 are possible.

Therefore, $\mathrm{P}(\mathrm{E} \mid \mathrm{T})$ = 2/3

Example 11.4.3

A fair coin is tossed twice.

a. Find the probability that the result is is two heads.
b. Find the probability that the result is two heads given that at least one head is obtained.

Solution

The sample space is$S=HH, HT, TH, TT$

Let event E be that the two heads are obtained and F be at least one head is obtained

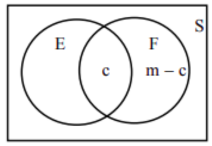
a. $\mathrm{P}(\mathrm{E})$ = 1/4 because $\mathrm{E}=HH$ and the sample space S has 4 outcomes.

b. $\mathrm{F}=HH, HT, TH$. Since at least one head was obtained, TT did not occur.
We are interested in the probability event $\mathrm{E}=HH$ out of the 3 outcomes in the reduced sample space F.

Therefore, $\mathrm{P}(\mathrm{E} \mid \mathrm{F})$ = 1/3

Let us now develop a formula for the conditional probability $\mathrm{P}(\mathrm{E} \mid \mathrm{F})$.

Suppose an experiment consists of n equally likely events. Further suppose that there are m elements in F, and c elements in $\mathrm{E} \cap \mathrm{F}$, as shown in the following Venn diagram.



If the event F has occurred, the set of all possible outcomes is no longer the entire sample space, but instead, the subset F. Therefore, we only look at the set F and at nothing outside of F. Since F has m elements, the denominator in the calculation of $\mathrm{P}(\mathrm{E}|\mathrm{F})$ is m. We may think that the numerator for our conditional probability is the number of elements in E. But clearly we cannot consider the elements of E that are not in F. We can only count the elements of E that are in F, that is, the elements in $\mathrm{E} \cap \mathrm{F}$. Therefore,

$$\mathrm{P}(\mathrm{E}|\mathrm{F})=\frac{\mathrm{c}}{\mathrm{m}}$$

Dividing both the numerator and the denominator by n, we get

$$\mathrm{P}(\mathrm{E}|\mathrm{F})=\frac{c/n}{m/n}$$

But $c/n=\mathrm{P}(\mathrm{E} \cap \mathrm{F})$, and $m/n=\mathrm{P}(\mathrm{F})$.

Substituting, we derive the following formula for $\mathrm{P}(\mathrm{E}|\mathrm{F})$.

Conditional Probability Rule

For two events E and F, the probability of "E Given F" is

$$\mathbf{P}(\mathbf{E}|\mathbf{F})=\frac{\mathbf{P}(\mathbf{E} \cap \mathbf{F})}{\mathbf{P}(\mathbf{F})}$$

Example 11.4.4

A single die is rolled. Use the above formula to find the conditional probability of obtaining an even number given that a number greater than three has shown.

Solution

Let E be the event that an even number shows, and F be the event that a number greater than three shows. We want $\mathrm{P}(\mathrm{E}|\mathrm{F})$.

$\mathrm{E}=2,4,6$ and $\mathrm{F}=4,5,6$. Which implies, $\mathrm{E} \cap \mathrm{F}=4,6$

Therefore, $\mathrm{P}(\mathrm{F})$ = 3/6, and $\mathrm{P}(\mathrm{E} \cap \mathrm{F})$ = 2/6

$$P(E|F)=\frac{P(E \cap F)}{P(F)}=\frac{2/6}{3/6}=\frac{2}{3}.$$

Example 11.4.5

The following table shows the distribution by gender of students at a community college who take public transportation and the ones who drive to school.

	Male(M)	Female(F)	Total
Public Transportation(T)	8	13	21
Drive(D)	39	40	79
Total	47	53	100

The events M, F, T, and D are self explanatory. Find the following probabilities.

a. $\mathrm{P(D|M)}$
b. $\mathrm{P(F|D)}$
c. $\mathrm{P(M|T)}$

Solution 1

Conditional probabilities can often be found directly from a contingency table. If the condition corresponds to only one row or only one column in the table, then you can ignore the rest of the table and read the conditional probability directly from the row or column indicated by the condition.

a. The condition is event M; we can look at only the “Male” column of the table and ignore the rest of the table: $\mathrm{P(D|M)}=\frac{39}{47}$.
b. The condition is event D; we can look at only the “Drive” row of the table and ignore the rest of the table: $\mathrm{P(F|D)}=\frac{40}{79}$.
c. The condition is event T; we can look at only the “Public Transportation” row of the table and ignore the rest of the table: $\mathrm{P(M|T)}=\frac{8}{21}$.

Solution 2

We use the conditional probability formula $\mathrm{P(E|F)}=\frac{\mathrm{P(E\cap F)}}{\mathrm{P(F)}}$.

a.
$$P(D|M)=\frac{P(D\cap M)}{P(M)}=\frac{39/100}{47/100}=\frac{39}{47}.$$

b.
$$P(F|D)=\frac{P(F\cap D)}{P(D)}=\frac{40/100}{79/100}=\frac{40}{79}.$$

c.
$$\mathrm{P(M|T)}=\frac{P(M\cap T)}{P(T)}=\frac{8/100}{21/100}=\frac{8}{21}$$

✓ Example 11.4.6

Given $\mathrm{P(E)}$ = .5, $\mathrm{P(F)}$ = .7, and $\mathrm{P(E\cap F)}$ = .3. Find the following:

a. $\mathrm{P(E|F)}$
b. $\mathrm{P(F|E)}$

Solution

We use the conditional probability formula.

a. $P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{3}{7}=\frac{3}{7}$
b. $P(F|E)=\frac{P(E\cap F)}{P(E)}=.3/.5=3/5$

✓ Example 11.4.7

E and F are mutually exclusive events such that $\mathrm{P(E)}$ = .4, $\mathrm{P(F)}$ = .9. Find $\mathrm{P(E|F)}$.

Solution

E and F are mutually exclusive, so $\mathrm{P(E\cap F)}$ = 0.
Therefore $\mathrm{P(E|F)}=\frac{\mathrm{P(E\cap F)}}{\mathrm{P(F)}}=\frac{0}{9}=0$.

✓ Example 11.4.8

Given $\mathrm{P(F|E)}$ = .5, and $\mathrm{P(E\cap F)}$ = .3. Find $\mathrm{P(E)}$.

Solution

Using the conditional probability formula $\mathrm{P(E|F)}=\frac{\mathrm{P(E\cap F)}}{\mathrm{P(F)}}$, we get

$$P(F|E)=\frac{P(E\cap F)}{P(E)}$$

Substituting and solving:

$$.5=\frac{.3}{P(E)} \quad \text{or} \quad P(E)=3/5$$

Example 11.4.9

In a family of three children, find the conditional probability of having two boys and a girl, given that the family has at least two boys.

Solution

Let event E be that the family has two boys and a girl, and let F be the probability that the family has at least two boys. We want $P(E|F)$.

We list the sample space along with the events E and F.

$$\begin{array}{l} S=\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\} \\ E=\{BBG, BGB, GBB\} \text{ and } F=\{BBB, BBG, BGB, GBB\} \\ E\cap F=\{BBG, BGB, GBB\} \end{array}$$

Therefore, $P(F)$ = 4/8, and $P(E\cap F)$ = 3/8, and

$$P(E|F)=\frac{P(E\cap F)}{P(E)}=\frac{3/8}{4/8}=\frac{3}{4}.$$

Example 11.4.10

At a community college 65% of the students subscribe to Amazon Prime, 50% subscribe to Netflix, and 20% subscribe to both. If a student is chosen at random, find the following probabilities:

a. the student subscribes to Amazon Prime given that he subscribes to Netflix
b. the student subscribes to Netflix given that he subscribes to Amazon Prime

Solution

Let A be the event that the student subscribes to Amazon Prime, and N be the event that the student subscribes to Netflix.

First identify the probabilities and events given in the problem.

P(student subscribes to Amazon Prime) = $P(A)$ = 0.65

P(student subscribes to Netflix) = $P(N)$ = 0.50

P(student subscribes to both Amazon Prime and Netflix) = $P(A\cap N)$ = 0.20

Then use the conditional probability rule:

a. $P(A|N)=\frac{P(A\cap N)}{P(N)}=\frac{.20}{.50}=\frac{2}{5}$
b. $P(N|A)=\frac{P(A\cap N)}{P(A)}=\frac{.20}{.65}=\frac{4}{13}$

11.4.1: Conditional Probability (Exercises)

SECTION 11.4 PROBLEM SET: CONDITIONAL PROBABILITY

Questions 1 - 4: Do these problems using the conditional probability formula: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

1. A card is drawn from a deck. Find the conditional probability of P(a queen | a face card).	2. A card is drawn from a deck. Find the conditional probability of P(a queen | a club).
3. A die is rolled. Find the conditional probability that it shows a three if it is known that an odd number has shown.	4. If $P(A)$ = .3 , $P(B)$ = .4, $P(A$ and $B)$ = .12, find: a. $P(A\|B)$ b. $P(B\|A)$

Questions 5 - 8 refer to the following: The table shows the distribution of Democratic and Republican U.S. Senators by gender in the 114th Congress as of January 2015.

	MALE(M)	FEMALE(F)	TOTAL
DEMOCRATS (D)	30	14	44
REPUBLICANS(R)	48	6	54
OTHER (T)	2	0	2
TOTALS	80	20	100

Use this table to determine the following probabilities:

5. $P(M\|D)$	6. $P(D\|M)$
7. $P(F\|R)$	8. $P(R\|F)$

Do the following conditional probability problems.

9. At a college, 20% of the students take Finite Math, 30% take History, and 5% take both Finite Math and History. If a student is chosen at random, find the following conditional probabilities. a. He is taking Finite Math given that he is taking History. b. He is taking History assuming that he is taking Finite Math.	10. At a college, 60% of the students pass Accounting, 70% pass English, and 30% pass both of these courses. If a student is selected at random, find the following conditional probabilities. a. He passes Accounting given that he passed English. b. He passes English assuming that he passed Accounting.
11. If $P(F) = .4$, $P(E\|F) = .3$, find $P(E$ and $F)$.	12. $P(E) = .3$, $P(F) = .3$; E and F are mutually exclusive. Find $P(E\|F)$.
13. If $P(E) = .6$, $P(E$ and $F)$ = .24, find $P(F\|E)$.	14. If $P(E$ and $F)$ = .04, $P(E\|F) = .1$, find $P(F)$.

At a college, 72% of courses have final exams and 46% of courses require research papers. 32% of courses have both a research paper and a final exam. Let F be the event that a course has a final exam and R be the event that a course requires a research paper.

15. Find the probability that a course has a final exam given that it has a research paper.	16. Find the probability that a course has a research paper if it has a final exam.

SECTION 11.4 PROBLEM SET: CONDITIONAL PROBABILITY

Consider a family of three children. Find the following probabilities.

17. P(two boys | first born is a boy)	18. P(all girls | at least one girl is born)

19. P(children of both sexes | first born is a boy)	20. P(all boys | there are children of both sexes)

Questions 21 - 26 refer to the following:
The table shows highest attained educational status for a sample of US residents age 25 or over:

	(D) Did not Complete High School	(H) High School Graduate	(C) Some College	(A) Associate Degree	(B) Bachelor Degree	(G) Graduate Degree	TOTAL
25-44 (R)	95	228	143	81	188	61	796
45-64 (S)	83	256	136	80	150	67	772
65+ (T)	96	191	84	36	80	41	528
Total	274	675	363	197	418	169	2096

Use this table to determine the following probabilities:

21. $P(C \mid T)$	22. $P(S \mid A)$	23. $P(CandT)$
24. $P(R \mid B)$	25. $P(B \mid R)$	26. $P(G \mid S)$

11.5: Independent Events

Learning Objectives

In this section, you will:

1. Define independent events
2. Identify whether two events are independent or dependent

In the last section, we considered conditional probabilities. In some examples, the probability of an event changed when additional information was provided. This is not always the case. The additional information may or may not alter the probability of the event.

In Example $11.5.1$ we revisit the discussion at the beginning of the previous section and then contrast that with Example $11.5.2$

Example $11.5.1$

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that the card is a face card.

Solution

a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | The card is a face card), we reason as follows:

There are 12 face cards in a deck of cards. There are 4 kings in a deck of cards.

$$P(\text{The card is a king} \mid \text{The card is a face card}) = 4/12 = 1/3.$$

The reader should observe that in the above example,

$$P(\text{The card is a king} \mid \text{The card is a face card}) \neq P(\text{The card is a king})$$

In other words, the additional information, knowing that the card selected is a face card changed the probability of obtaining a king.

Example $11.5.2$

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that a red card has shown.

Solution

a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | A red card has shown), we reason as follows:

Since a red card has shown, there are only twenty six possibilities. Of the 26 red cards, there are two kings. Therefore,

$$P(\text{The card is a king} \mid \text{A red card has shown}) = 2/26 = 1/13.$$

The reader should observe that in the above example,

$$P(\text{The card is a king} \mid \text{A red card has shown}) = P(\text{The card is a king})$$

In other words, the additional information, a red card has shown, did not affect the probability of obtaining a king.

Whenever the probability of an event E is not affected by the occurrence of another event F, and vice versa, we say that the two events E and F are **independent.** This leads to the following definition.

Definition: Independent

Two Events E and F are **independent** if and only if at least one of the following two conditions is true.

1. $\mathbf{P(E|F) = P(E)}$ or
2. $\mathbf{P(F|E) = P(F)}$

If the events are not independent, then they are dependent.

If one of these conditions is true, then both are true.

We can use the definition of independence to determine if two events are independent.

We can use that definition to develop another way to test whether two events are independent.

Recall the conditional probability formula:

$$\mathrm{P(E|F) = \frac{P(E \cap F)}{P(F)}}$$

Multiplying both sides by $\mathrm{P(F)}$, we get

$$\mathrm{P(E \cap F) = P(E|F)P(F)}$$

Now if the two events are independent, then by definition

$$\mathrm{P(E|F) = P(E)}$$

Substituting, $P(E \cap F) = P(E)P(F)$

We state it formally as follows.

Test For Independence

Two events E and F are independent if and only if

$$\mathbf{P(E \cap F) = P(E)P(F)}$$

In the Examples $11.5.3$ and $11.5.4$, we'll examine how to check for independence using both methods:

- Examine the probability of intersection of events to check whether $P(E \cap F) = P(E)P(F)$
- Examine conditional probabilities to check whether $P(E|F) = P(E)$ or $P(F|E) = P(F)$

We need to use only **one** of these methods. Both methods, if used properly, will always give results that are consistent with each other.

Use the method that seems easier based on the information given in the problem.

Example $11.5.3$

The table below shows the distribution of color-blind people by gender.

	Male(M)	Female(F)	Total
Color-Blind(C)	6	1	7
Not Color-Blind(N)	46	47	93
Total	52	48	100

where M represents male, F represents female, C represents color-blind, and N represents not color-blind. Are the events color-blind and male independent?

Solution 1: According to the test for independence, C and M are independent if and only if $\mathrm{P(C \cap M) = P(C)P(M)}$.

From the table: $P(C)$ = 7/100, $P(M)$ = 52/100 and $P(C \cap M)$ = 6/100

So $P(C)P(M)$ = (7/100)(52/100) = .0364

which is **not** equal to $P(C \cap M)$ = 6/100 = .06

Therefore, the two events are not independent. We may say they are dependent.

Solution 2: C and M are independent if and only if $P(C|M) = P(C)$.

From the total column $P(C)$ = 7/100 = 0.07

From the male column $P(C|M)$ = 6/52= 0.1154

Therefore $P(C|M) \neq P(C)$, indicating that the two events are not independent.

Example 11.5.4

In a city with two airports, 100 flights were surveyed. 20 of those flights departed late.

- 45 flights in the survey departed from airport A; 9 of those flights departed late.
- 55 flights in the survey departed from airport B; 11 flights departed late.

Are the events "depart from airport A" and "departed late" independent?

Solution 1

Let A be the event that a flight departs from airport A, and L the event that a flight departs late. We have

$P(A \cap L)$ = 9/100, $P(A)$ = 45/100 and $P(L)$ = 20/100

In order for two events to be independent, we must have $P(A \cap L) = P(A)P(L)$

Since $P(A \cap L)$ = 9/100 = 0.09

and $P(A)P(L)$ = (45/100)(20/100) = 900/10000 = 0.09

the two events "departing from airport A" and "departing late" are independent.

Solution 2

The definition of independent events states that two events are independent if $P(E|F) = P(E)$.

In this problem we are given that

$P(L|A)$ = 9/45= 0.2 and $P(L)$ = 20/100 = 0.2

$P(L|A) = P(L)$, so events "departing from airport A" and "departing late" are independent.

Example 11.5.5

A coin is tossed three times, and the events E, F and G are defined as follows:

E: The coin shows a head on the first toss.

F: At least two heads appear.

G: Heads appear in two successive tosses.

Determine whether the following events are independent.

a. E and F
b. F and G
c. E and G

Solution

We list the sample space, the events, their intersections and the probabilities.

$$\begin{aligned}
&S=\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\\
&E=\{HHH, HHT, HTH, HTT\}, \quad P(E)=4/8 \text{ or } 1/2\\
&F=\{HHH, HHT, HTH, THH\}, \quad P(F)=4/8 \text{ or } 1/2\\
&G=\{HHT, THH\}, \quad P(G)=2/8 \text{ or } 1/4\\
&E\cap F=\{HHH, HHT, HTH\}, \quad P(E\cap F)=3/8\\
&F\cap G=\{HHT, THH\}, \quad P(F\cap G)=2/8 \text{ or } 1/4\\
&E\cap G=\{HHT\} \quad P(E\cap G)=1/8
\end{aligned}$$

a. E and F will be independent if and only if $P(E\cap F)=P(E)P(F)$

$$P(E\cap F)=3/8 \text{ and } P(E)P(F)=1/2\cdot 1/2=1/4.$$

Since 3/8 ≠ 1/4, we have $P(E\cap F)\neq P(E)P(F)$.

Events E and F are not independent.

b. F and G will be independent if and only if $P(F\cap G)=P(F)P(G)$.

$$P(F\cap G)=1/4 \text{ and } P(F)P(G)=1/2\cdot 1/4=1/8.$$

Since 3/8 ≠ 1/4, we have $P(F\cap G)\neq P(F)P(G)$.

Events F and G are not independent.

c. E and G will be independent if $P(E\cap G)=P(E)P(G)$

$$P(E\cap G)=1/8 \text{ and } P(E)P(G)=1/2\cdot 1/4=1/8$$

Events E and G are independent events because $P(E\cap G)=P(E)P(G)$

Example 11.5.6

The probability that Jaime will visit his aunt in Baltimore this year is .30, and the probability that he will go river rafting on the Colorado river is .50. If the two events are independent, what is the probability that Jaime will do both?

Solution

Let A be the event that Jaime will visit his aunt this year, and R be the event that he will go river rafting.

We are given $P(A)$ = .30 and $P(R)$ = .50, and we want to find $P(A\cap R)$.

Since we are told that the events A and R are independent,

$$P(A\cap R)=P(A)P(R)=(.30)(.50)=.15$$

Example 11.5.7

Given $P(B|A)=.4$. If A and B are independent, find $P(B)$.

Solution

If A and B are independent, then by definition $P(B|A)=P(B)$

Therefore, $P(B)=.4$

Example 11.5.8

Given $P(A)=.7$, $P(B|A)=.5$. Find $P(A\cap B)$.

Solution 1

By definition $P(B|A)=\frac{P(A\cap B)}{P(A)}$

Substituting, we have

$$.5 = \frac{\mathrm{P(A \cap B)}}{.7}$$

Therefore, $P(A \cap B) = .35$

Solution 2

Again, start with $P(B|A) = \frac{P(A \cap B)}{P(A)}$

Multiplying both sides by $P(A)$ gives

$$P(A \cap B) = P(B|A)P(A) = (.5)(.7) = .35$$

Both solutions to Example 11.5.8 are actually the same, except that in Solution 2 we delayed substituting the values into the equation until after we solved the equation for $P(A \cap B)$. That gives the following result:

Multiplication Rule for events that are NOT independent

If events E and F are not independent

$$\mathbf{P(E \cap F) = P(E|F)P(F)} \quad \text{and} \quad \mathbf{P(E \cap F) = P(F|E)P(E)}$$

Example 11.5.9

Given $P(A) = .5$, $P(A \cup B) = .7$, if A and B are independent, find $P(B)$.

Solution

The addition rule states that

$$\mathrm{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Since A and B are independent, $P(A \cap B) = P(A)P(B)$

We substitute for $P(A \cap B)$ in the addition formula and get

$$\mathrm{P(A \cup B) = P(A) + P(B) - P(A)P(B)}$$

By letting $P(B) = x$, and substituting values, we get

$$\begin{aligned} .7 &= .5 + x - .5x \\ .7 &= .5 + .5x \\ .2 &= .5x \\ .4 &= x \end{aligned}$$

Therefore, $P(B) = .4$

11.5.1: Independent Events (Exercises)

SECTION 11.5 PROBLEM SET: INDEPENDENT EVENTS

The distribution of the number of fiction and non-fiction books checked out at a city's main library and at a smaller branch on a given day is as follows.

	MAIN (M)	BRANCH (B)	TOTAL
FICTION (F)	300	100	400
NON-FICTION (N)	150	50	200
TOTALS	450	150	600

Use this table to determine the following probabilities:

1. $P(F)$	2. $P(M \mid F)$
3. $P(N \mid B)$	4. Is the fact that a person checks out a fiction book independent of the main library? Use probabilities to justify your conclusion.

For a two-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

5. Find the following. a. $P(E)$ b. $P(F)$ c. $P(E \cap F)$ d. Are E and F independent? Use probabilities to justify your conclusion.	6. Find the following. a. $P(F)$ b. $P(G)$ c. $P(F \cap G)$ d. Are F and G independent? Use probabilities to justify your conclusion.

Do the following problems involving independence.

7. If $P(E) = .6$, $P(F) = .2$, and E and F are independent, find $P(E$ and $F)$.	8. If $P(E) = .6$, $P(F) = .2$, and E and F are independent, find $P(E$ or $F)$.
9. If $P(E) = .9$, $P(F \mid E) = .36$, and E and F are independent, find $P(F)$.	10. If $P(E) = .6$, $P(E$ or $F)$ = .8, and E and F are independent, find $P(F)$.
11. In a survey of 100 people, 40 were casual drinkers, and 60 did not drink. Of the ones who drank, 6 had minor headaches. Of the non-drinkers, 9 had minor headaches. Are the events "drinkers" and "had headaches" independent?	12. It is known that 80% of the people wear seat belts, and 5% of the people quit smoking last year. If 4% of the people who wear seat belts quit smoking, are the events, wearing a seat belt and quitting smoking, independent?

13. John's probability of passing statistics is 40%, and Linda's probability of passing the same course is 70%. If the two events are independent, find the following probabilities. a. $P($ both of them will pass statistics) b. $P($at least one of them will pass statistics)	14. Jane is flying home for the Christmas holidays. She has to change planes twice. There is an 80% chance that she will make the first connection, and a 90% chance that she will make the second connection. If the two events are independent, find the probabilities: a. $P($ Jane will make both connections) b. $P($Jane will make at least one connection)

For a three-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

15. Find the following.
    a. $P(E)$
    b. $P(F)$
    c. $P(E \cap F)$
    d. Are E and F independent?

16. Find the following.
    a. $P(F)$
    b. $P(G)$
    c. $P(F \cap G)$
    d. Are F and G independent?

SECTION 11.5 PROBLEM SET: INDEPENDENT EVENTS

17. $P(K|D) = 0.7$, $P(D) = 0.25$ and $P(K) = 0.7$
    a. Are events K and D independent? Use probabilities to justify your conclusion.
    b. Find $P(K \cap D)$

18. $P(R|S) = 0.4$, $P(S) = 0.2$ and $P(R) = 0.3$
    a. Are events R and S independent? Use probabilities to justify your conclusion.
    b. Find $P(R \cap S)$

19. At a college:
    54% of students are female
    25% of students are majoring in engineering.
    15% of female students are majoring in engineering.
    Event E = student is majoring in engineering
    Event F = student is female
    a. Are events E and F independent? Use probabilities to justify your conclusion.
    b. Find $P(E \cap F)$

20. At a college:
    54% of all students are female
    60% of all students receive financial aid.
    60% of female students receive financial aid.
    Event A = student receives financial aid
    Event F = student is female
    a. Are events A and F independent? Use probabilities to justify your conclusion.
    b. Find $P(A \cap F)$

11.6: Chapter Review

SECTION 11.6 PROBLEM SET: CHAPTER REVIEW

1. Two dice are rolled. Find the probability that the sum of the dice is
   a. four
   b. five
2. A jar contains 3 red, 4 white, and 5 blue marbles. If a marble is chosen at random, find the following probabilities:
   a. P(red or blue)
   b. P(not blue)
3. A card is drawn from a standard deck. Find the following probabilities:
   a. P(a jack or a king)
   b. P(a jack or a spade)
4. A basket contains 3 red and 2 yellow apples. Two apples are chosen at random. Find the following probabilities:
   a. P(one red, one yellow)
   b. P(at least one red)
5. A basket contains 4 red, 3 white, and 3 blue marbles. Three marbles are chosen at random. Find the following probabilities:
   a. P(two red, one white)
   b. P(first red, second white, third blue)
   c. P(at least one red)
   d. P(none red)
6. Given a family of four children. Find the following probabilities:
   a. P(All boys)
   b. P(1 boy and 3 girls)
7. Consider a family of three children. Find the following:
   a. P(children of both sexes | first born is a boy)
   b. P(all girls | children of both sexes)
8. Mrs. Rossetti is flying from San Francisco to New York. On her way to the San Francisco Airport she encounters heavy traffic and determines that there is a 20% chance that she will be late to the airport and will miss her flight. Even if she makes her flight, there is a 10% chance that she will miss her connecting flight at Chicago. What is the probability that she will make it to New York as scheduled?
9. At a college, twenty percent of the students take history, thirty percent take math, and ten percent take both. What percent of the students take at least one of these two courses?
10. In a T-maze, a mouse may run to the right (R) or may run to the left (L). A mouse goes up the maze three times, and events E and F are described as follows:

    $$\text{E: Runs to the right on the first trial} \quad \text{F: Runs to the left two consecutive times}$$

    Determine whether the events E and F are independent.
11. A college has found that 20% of its students take advanced math courses, 40% take advanced English courses and 15% take both advanced math and advanced English courses. If a student is selected at random, what is the probability that
    a. he is taking English given that he is taking math?
    b. he is taking math or English?
12. If there are 35 students in a class, what is the probability that at least two have the same birthday?
13. A student feels that her probability of passing accounting is .62, of passing mathematics is .45, and her passing accounting or mathematics is .85. Find the probability that the student passes both accounting and math.
14. There are nine judges on the U. S. Supreme Court. Suppose that five are conservative and four are liberal. This year the court will act on six major cases. What is the probability that out of six cases the court will favor the conservatives in at least four?
15. Five cards are drawn from a deck. Find the probability of obtaining
    a. four cards of a single suit

b. two cards of one suit, two of another suit, and one from the remaining
c. a pair(e.g. two aces and three other cards)
d. a straight flush(five in a row of a single suit but not a royal flush)

16. The following table shows a distribution of drink preferences by gender.

	Coke(C)	Pepsi(P)	Seven Up(S)	TOTALS
Male(M)	60	50	22	132
Female(F)	50	40	18	108
TOTALS	110	90	40	240

The events M, F, C, P and S are defined as Male, Female, Coca Cola, Pepsi, and Seven Up, respectively. Find the following:

a. P(F | S)
b. P(P | F)
c. P(C | M)
d. P(M | P ∪ C)
e. Are the events F and S mutually exclusive?
f. Are the events F and S independent?

17. At a clothing outlet 20% of the clothes are irregular, 10% have at least a button missing and 4% are both irregular and have a button missing. If Martha found a dress that has a button missing, what is the probability that it is irregular?
18. A trade delegation consists of four Americans, three Japanese and two Germans. Three people are chosen at random. Find the following probabilities:

a. P(two Americans and one Japanese)
b. P(at least one American)
c. P(One of each nationality)
d. P(no German)

19. A coin is tossed three times, and the events E and F are as follows.

$$\text{E: It shows a head on the first toss} \qquad \text{F: Never turns up a tail}$$

Are the events E and F independent?
20. If $P(E)=.6$ and $P(F)=.4$ and E and F are mutually exclusive, find P(E and F).
21. If $P(E)=.5$ and $P(F)=.3$ and E and F are independent, find $P(E\cup F)$.
22. If $P(F)=.9$ and $P(E|F)=.36$ and E and F are independent, find $P(E)$.
23. If $P(E)=.4$ and P(E or F) =.9 and E and F are independent, find $P(F)$.
24. If $P(E)=.4$ and $P(F|E)=.5$, find P(E and F).
25. If $P(E)=.6$ and P(E and F) = .3, find $P(F|E)$.
26. If $P(E)=.3$ and $P(F)=.4$ and E and F are independent, find $P(E|F)$.

CHAPTER OVERVIEW

12: More Probability

> **Learning Objectives**
>
> In this chapter, you will learn to:
>
> 1. Find the probability of a binomial experiment.
> 2. Find probabilities using Bayes' Formula.
> 3. Find the expected value or payoff in a game of chance.
> 4. Find probabilities using tree diagrams.

- 12.1: Binomial Probability
  - 12.1.1: Binomial Probability (Exercises)
- 12.2: Bayes' Formula
  - 12.2.1: Bayes' Formula (Exercises)
- 12.3: Expected Value
  - 12.3.1: Expected Value (Exercises)
- 12.4: Probability Using Tree Diagrams
  - 12.4.1: Probability Using Tree Diagrams (Exercises)
- 12.5: Chapter Review

12.1: Binomial Probability

Learning Objectives

In this section, you will learn to:

1. Recognize when to use the binomial probability distribution
2. Derive the formula for the binomial probability distribution
3. Calculate probabilities for a binomial probability experiment

In this section, we consider problems that involve a sequence of trials, where each trial has only two outcomes, a *success* or a *failure*. These trials are independent, that is, the outcome of one does not affect the outcome of any other trial. The probability of success, p, and the probability of failure, $(1-p)$, remains the same throughout the experiment. These problems are called **binomial probability** problems. Since these problems were researched by Swiss mathematician Jacques Bernoulli around 1700, they are also called **Bernoulli trials.**

We give the following definition:

Binomial Experiment

A binomial experiment satisfies the following four conditions:

1. There are only two outcomes, a success or a failure, for each trial.
2. The same experiment is repeated several times.
3. The trials are independent; that is, the outcome of a particular trial does not affect the outcome of any other trial.
4. The probability of success remains the same for every trial.

This probability model that will give us the tools to solve many real-life problems , such as:

1. If a coin is flipped 10 times, what is the probability that it will fall heads 3 times?
2. If a basketball player makes 3 out of every 4 free throws, what is the probability that he will make 7 out of 10 free throws in a game?
3. If a medicine cures 80% of the people who take it, what is the probability that among the ten people who take the medicine, 6 will be cured?
4. If a microchip manufacturer claims that only 4% of his chips are defective, what is the probability that among the 60 chips chosen, exactly three are defective?
5. If a telemarketing executive has determined that 15% of the people contacted will purchase the product, what is the probability that among the 12 people who are contacted, 2 will buy the product?

We now consider the following example to develop a formula for finding the probability of k successes in n Bernoulli trials.

Example 12.1.1

A baseball player has a batting average of .300. If he bats four times in a game, find the probability that he will have

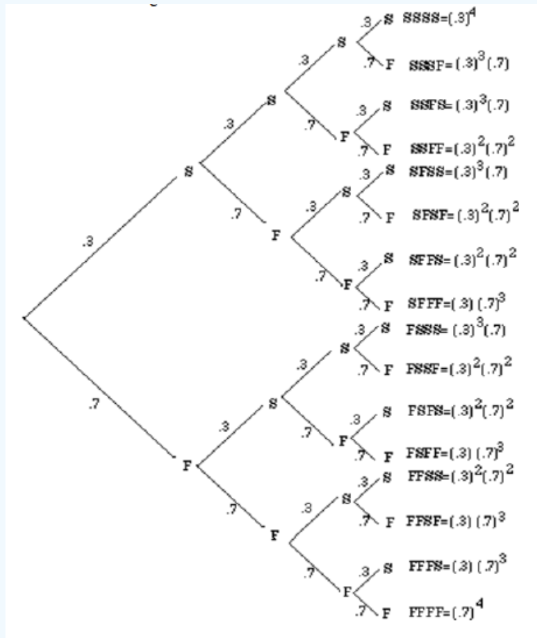
a. 4 hits
b. 3 hits
c. 2 hits
d. 1 hit
e. no hits.

Solution

Let S denote that the player gets a hit, and F denote that he does not get a hit.

This is a binomial experiment because it meets all four conditions. First, there are only two outcomes, S or F. Clearly the experiment is repeated four times. Lastly, if we assume that the player's skillfulness to get a hit does not change each time he comes to bat, the trials are independent with a probability of .3 of getting a hit during each trial.

We draw a tree diagram to show all situations.



Let us first find the probability of getting, for example, two hits. We will have to consider the six possibilities, SSFF, SFSF, SFFS, FSSF, FSFS, FFSS, as shown in the above tree diagram. We list the probabilities of each below.

$$\begin{aligned}
\mathrm{P(SSFF)} &= (.3)(.3)(.7)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(SFSF)} &= (.3)(.7)(.3)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(SFFS)} &= (.3)(.7)(.7)(.3) = (.3)^2(.7)^2 \\
\mathrm{P(FSSF)} &= (.7)(.3)(.3)(.7) = (.3)^2(.7)^2 \\
\mathrm{P(FSFS)} &= (.7)(.3)(.7)(.3) = (.3)^2(.7)^2 \\
\mathrm{P(FFSS)} &= (.7)(.7)(.3)(.3) = (.3)^2(.7)^2
\end{aligned}$$

Since the probability of each of these six outcomes is $(.3)^2(.7)^2$, the probability of obtaining two successes is $6(.3)^2(.7)^2$.

The probability of getting one hit can be obtained in the same way. Since each permutation has one S and three F's, there are four such outcomes: SFFF, FSFF, FFSF, and FFFS.

And since the probability of each of the four outcomes is $(.3)(.7)^3$, the probability of getting one hit is $4(.3)(.7)^3$.

The table below lists the probabilities for all cases, and shows a comparison with the binomial expansion of fourth degree. Again, p denotes the probability of success, and $q = (1-p)$ the probability of failure.

Outcome	Four Hits	Three Hits	Two Hits	One Hit	No Hits
Probability	$(.3)^4$	$4(.3)^3(.7)$	$6(.3)^2(.7)^2$	$4(.3)(.7)^3$	$(.7)^4$

$$\begin{aligned}
(.3+.7)^4 &= (.3)^4 + 4(.3)^3(.7) + 6(.3)^2(.7)^2 + 4(.3)(.7)^3 + (.7)^4 \\
(p+q)^4 &= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4
\end{aligned}$$

This gives us the following theorem:

Binomial Probability Theorem

The probability of obtaining k successes in n independent Bernoulli trials is given by

$$\mathbf{P(n, k; p) = nCk\ p^k q^{n-k}}$$

where p denotes the probability of success and $q = (1 - p)$ the probability of failure.

We use the binomial probability formula to solve the following examples.

Example 12.1.2

If a coin is flipped 10 times, what is the probability that it will fall heads 3 times?

Solution

Let S denote the probability of obtaining a head, and F the probability of getting a tail.

Clearly, $n = 10$, $k = 3$, $p = 1/2$, and $q = 1/2$.

Therefore, $\mathrm{b}(10, 3; 1/2) = 10\mathrm{C}3\ (1/2)^3(1/2)^7 = .1172$

Example 12.1.3

If a basketball player makes 3 out of every 4 free throws, what is the probability that he will make 6 out of 10 free throws in a game?

Solution

The probability of making a free throw is 3/4.
Therefore, $p = 3/4$, $q = 1/4$, $n = 10$, and $k = 6$.

Therefore, $b(10, 6; 3/4) = 10\mathrm{C}6\ (3/4)^6(1/4)^4 = .1460$

Example 12.1.4

If a medicine cures 80% of the people who take it, what is the probability that of the eight people who take the medicine, 5 will be cured?

Solution

Here $p = .80$, $q = .20$, $n = 8$, and $k = 5$.

$$b(8, 5; .80) = 8\mathrm{C}5\ (.80)^5(.20)^3 = .1468$$

Example 12.1.5

If a microchip manufacturer claims that only 4% of his chips are defective, what is the probability that among the 60 chips chosen, exactly three are defective?

Solution

If S denotes the probability that the chip is defective, and F the probability that the chip is not defective, then $p = .04$, $q = .96$, $n = 60$, and $k = 3$.

$$b(60, 3; .04) = 60\mathrm{C}3\ (.04)^3(.96)^{57} = .2138$$

Example 12.1.6

A telemarketing executive has determined that 15% of people contacted will purchase the product. 12 people are contacted about this product.

a. Find the probability that among 12 people contacted, 2 will buy the product.
b. Find the probability that among 12 people contacted, at most 2 will buy the product?

Solution

a. If S denoted the probability that a person will buy the product, and F the probability that the person will not buy the product, then $p = .15$, $q = .85$, $n = 12$, and $k = 2$.

$$b(12, 2; .15) = 12\mathrm{C}2\ (.15)^2(.85)^{10} = .2924.$$

The probability that 2 people buy the product is 0.2924.

b. Again $p = .15$, $q = .85$, $n = 12$. But to find the probability that **at most 2** buy the product, we need to find the probabilities for $k = 0$, $k = 1$, $k = 2$ and add them together.

$$\begin{aligned} \mathrm{b}(12, 0; .15) = 12\mathrm{C}0\ (.15)^0(.85)^{12} = .1422 \\ \mathrm{b}(12, 1; .15) = 12\mathrm{C}1\ (.15)^1(.85)^{11} = .3012 \end{aligned}$$

Adding all three probabilities gives: .1422 +0.3012 +.2924 = .7358

The probability that at most 2 people buy the product is 0.7358.

12.1.1: Binomial Probability (Exercises)

SECTION 12.1 PROBLEM SET: BINOMIAL PROBABILITY

Do the following problems using the binomial probability formula.

1. A coin is tossed ten times. Find the probability of getting six heads and four tails.	2. A family has three children. Find the probability of having one boy and two girls.
3. What is the probability of getting three aces(ones) if a die is rolled five times?	4. A baseball player has a .250 batting average. What is the probability that he will have three hits in five times at bat?
5. A basketball player has an 80% chance of sinking a basket on a free throw. What is the probability that he will sink at least three baskets in five free throws?	6. With a new flu vaccination, 85% of the people in the high risk group can go through the entire winter without contracting the flu. In a group of six people who were vaccinated with this drug, what is the probabi
7. A transistor manufacturer has known that 5% of the transistors produced are defective. What is the probability that a batch of twenty five transistors will have two defective?	8. It has been determined that only 80% of the people wear seat belts. If a police officer stops a car with four people, what is the probability that at least one person will not be wearing a seat belt?
9. What is the probability that a family of five children will have at least three boys?	10. What is the probability that a toss of four coins will yield at most two heads?
11. A telemarketing executive has determined that for a particular product, 20% of the people contacted will purchase the product. If 10 people are contacted, what is the probability that at most 2 will buy the product?	12. To the problem: "Five cards are dealt from a deck of cards, find the probability that three of them are kings," the following incorrect answer was offered by a student. $$5\mathrm{C}3\ (1/13)^3(12/13)^2$$ What change would you make in the wording of the problem for the given answer to be correct?
13. 63% of all registered voters in a large city voted in the last election. 20 registered voters from this city are randomly selected. Find the probability that a. exactly half of them voted in the last election b. all of them voted	14. 30% of customers at BigMart pay cash for their purchases. Suppose that 15 customers are randomly selected. Find the probability that a. 5 or 6 of them pay cash b. at most 1 pays cash
15. 12% of all cars on Brighton Expressway exceed the speed limit. If 10 vehicles on this road are randomly selected and their speed is recorded by radar, find the probability that a. none of them are exceeding the speed limit b. 1 or 2 are exceeding the speed limit	16. Suppose that 73% of all people taking a professional certification exam pass the exam. If 12 people who take this exam are randomly selected, find the probability that a. exactly half of them pass the exam b. all of them pass the exam c. 8 or 9 of them pass the exam

12.2: Bayes' Formula

Learning Objectives

In this section, you will learn to:

1. Find probabilities using Bayes' formula
2. Use a probability tree to find and represent values needed when using Bayes' formula.

In this section, we will develop and use Bayes' Formula to solve an important type of probability problem. Bayes' formula is a method of calculating the conditional probability $P(F|E)$ from $P(E|F)$. The ideas involved here are not new, and most of these problems can be solved using a tree diagram. However, Bayes' formula does provide us with a tool with which we can solve these problems without a tree diagram.

We begin with an example.

Example 12.2.1

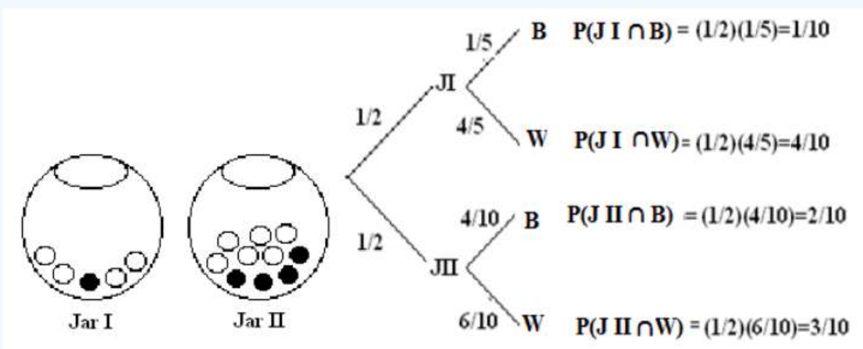
Suppose you are given two jars. Jar I contains one black and 4 white marbles, and Jar II contains 4 black and 6 white marbles. If a jar is selected at random and a marble is chosen,

a. What is the probability that the marble chosen is a black marble?
b. If the chosen marble is black, what is the probability that it came from Jar I?
c. If the chosen marble is black, what is the probability that it came from Jar II?

Solution

Let JI be the event that Jar I is chosen, JII be the event that Jar II is chosen, B be the event that a black marble is chosen and W the event that a white marble is chosen.

We illustrate using a tree diagram.



a. The probability that a black marble is chosen is $P(B)$ = 1/10 + 2/10 = 3/10.
b. To find $P(JI|B)$, we use the definition of conditional probability, and we get

$$P(JI|B) = \frac{P(JI \cap B)}{P(B)} = \frac{1/10}{3/10} = \frac{1}{3}$$

c. Similarly, $\mathrm{P(JII|B)} = \frac{\mathrm{P(JII \cap B)}}{\mathrm{P(B)}} = \frac{2/10}{3/10} = \frac{2}{3}$

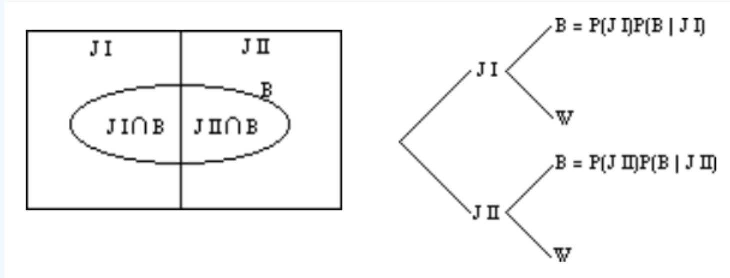
In parts b and c, the reader should note that the denominator is the sum of all probabilities of all branches of the tree that produce a black marble, while the numerator is the branch that is associated with the particular jar in question.

We will soon discover that this is a statement of Bayes' formula .

Let us first visualize the problem.

We are given a sample space S and two mutually exclusive events JI and JII. That is, the two events, JI and JII, divide the sample space into two parts such that $\mathrm{JI} \cup \mathrm{JII} = \mathrm{S}$. Furthermore, we are given an event B that has elements in both JI and

JII, as shown in the Venn diagram below.



From the Venn diagram, we can see that $\mathrm{B} = (\mathrm{B} \cap \mathrm{JI}) \cup (\mathrm{B} \cap \mathrm{JII})$ Therefore:

$$P(B) = P(B \cap JI) + P(B \cap JII) \tag{12.2.1}$$

But the product rule in chapter 7 gives us

$$P(B \cap JI) = P(JI) \cdot P(B|JI) \quad \text{and} \quad P(B \cap JII) = P(JII) \cdot P(B|JII)$$

Substituting in 12.2.1, we get

$$P(B) = P(JI) \cdot P(B|JI) + P(JII) \cdot P(B|JII)$$

The conditional probability formula gives us

$$P(JI|B) = \frac{P(JI \cap B)}{P(B)}$$

Therefore, $P(JI|B) = \frac{P(JI) \cdot P(B|JI)}{P(B)}$

or

$$P(JI|B) = \frac{P(JI) \cdot P(B|JI)}{P(JI) \cdot P(B|JI) + P(JII) \cdot P(B|JII)}$$

The last statement is Bayes' Formula for the case where the sample space is divided into two partitions.

The following is the generalization of Bayes’ formula for n partitions.

Bayes' Formula for n partitions

Let S be a sample space that is divided into n partitions, A_1, A_2, . . . A_n. If E is any event in S, then

$$\mathbf{P}\left(\mathbf{A_i}|\mathbf{E}\right) = \frac{\mathbf{P}\left(\mathbf{A_i}\right)\mathbf{P}\left(\mathbf{E}|\mathbf{A_i}\right)}{\mathbf{P}\left(\mathbf{A_1}\right)\mathbf{P}\left(\mathbf{E}|\mathbf{A_1}\right) + \mathbf{P}\left(\mathbf{A}_2\right)\mathbf{P}\left(\mathbf{E}|\mathbf{A}_2\right) + \cdots + \mathbf{P}\left(\mathbf{A_n}\right)\mathbf{P}\left(\mathbf{E}|\mathbf{A_n}\right)}$$

We begin with the following example.

Example 12.2.2

A department store buys 50% of its appliances from Manufacturer A, 30% from Manufacturer B, and 20% from Manufacturer C. It is estimated that 6% of Manufacturer A's appliances, 5% of Manufacturer B's appliances, and 4% of Manufacturer C's appliances need repair before the warranty expires. An appliance is chosen at random. If the appliance chosen needed repair before the warranty expired, what is the probability that the appliance was manufactured by Manufacturer A? Manufacturer B? Manufacturer C?

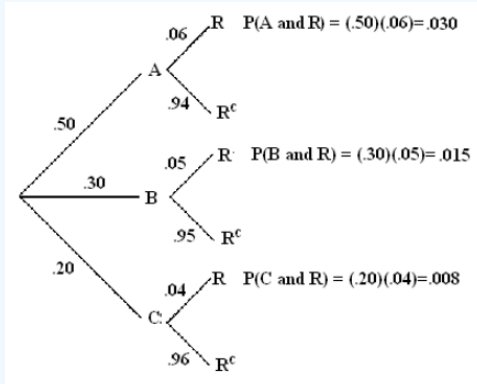
Solution

Let A, B and C be the events that the appliance is manufactured by Manufacturer A, Manufacturer B, and Manufacturer C, respectively. Further, suppose that the event R denotes that the appliance needs repair before the warranty expires.

We need to find P(A | R), P(B | R) and P(C | R).

We will do this problem both by using a tree diagram and by using Bayes' formula.

We draw a tree diagram.



The probability P(A | R), for example, is a fraction whose denominator is the sum of all probabilities of all branches of the tree that result in an appliance that needs repair before the warranty expires, and the numerator is the branch that is associated with Manufacturer A. P(B | R) and P(C | R) are found in the same way.

$$P(A|R) = \frac{.030}{(.030)+(.015)+(.008)} = \frac{.030}{.053} = .566$$

$$P(B|R) = \frac{.015}{.053} = .283 \text{ and } P(C|R) = \frac{.008}{.053} = .151$$

Alternatively, using Bayes' formula,

$$\begin{aligned} \mathrm{P(A|R)} &= \frac{\mathrm{P(A)P(R|A)}}{\mathrm{P(A)P(R|A)+P(B)P(R|B)+P(C)P(R|C)}} \\ &= \frac{.030}{(.030)+(.015)+(.008)} = \frac{.030}{.053} = .566 \end{aligned}$$

P(B | R) and P(C | R) can be determined in the same manner.

Example 12.2.3

There are five Jacy's department stores in San Jose. The distribution of number of employees by gender is given in the table below.

Store Number	Number of Employees	Percent of Women Employees
1	300	.40
2	150	.65
3	200	.60
4	250	.50
5	100	.70
	Total = 1000	

If an employee chosen at random is a woman, what is the probability that the employee works at store III?

Solution

Let k = 1, 2, . . . , 5 be the event that the employee worked at store k, and W be the event that the employee is a woman. Since there are a total of 1000 employees at the five stores,

$$P(1)=.30 \quad P(2)=.15 \quad P(3)=.20 \quad P(4)=.25 \quad P(5)=.10$$

Using Bayes' formula,

$$\begin{aligned} \mathrm{P}(3 \mid \mathrm{W}) &=\frac{\mathrm{P}(3) \mathrm{P}(\mathrm{W} \mid 3)}{\mathrm{P}(1) \mathrm{P}(\mathrm{W} \mid 1)+\mathrm{P}(2) \mathrm{P}(\mathrm{W} \mid 2)+\mathrm{P}(3) \mathrm{P}(\mathrm{W} \mid 3)+\mathrm{P}(4) \mathrm{P}(\mathrm{W} \mid 4)+\mathrm{P}(5) \mathrm{P}(\mathrm{W} \mid 5)} \\ &=\frac{(.20)(.60)}{(.30)(.40)+(.15)(.65)+(.20)(.60)+(.25)(.50)+(.10)(.70)} \\ &=.2254 \end{aligned}$$

12.2.1: Bayes' Formula (Exercises)

SECTION 12.2 PROBLEM SET: BAYES' FORMULA

1. Jar I contains five red and three white marbles, and Jar II contains four red and two white marbles. A jar is picked at random and a marble is drawn. Draw a tree diagram below, and find the following probabilities.
   a. P(marble is red)
   b. P(It came from Jar II | marble is white)
   c. P(Red | Jar I)

2. In Mr. Symons' class, if a student does homework most days, the chance of passing the course is 90%. On the other hand, if a student does not do homework most days, the chance of passing the course is only 20%.
   H = event that the student did homework
   C = event that the student passed the course
   Mr. Symons claims that 80% of his students do homework on a regular basis. If a student is chosen at random from Mr. Symons' class, find the following probabilities.
   a. P(C)
   b. P(H|C)
   c. P(C|H)

3. A city has 60% Democrats, and 40% Republicans. In the last mayoral election, 60% of the Democrats voted for their Democratic candidate while 95% of the Republicans voted for their candidate. Which party's mayor runs city hall?

4. In a certain population of 48% men and 52% women, 56% of the men and 8% of the women are color-blind.
   a. What percent of the people are color-blind?
   b. If a person is found to be color-blind, what is the probability that the person is a male?

5. A test for a certain disease gives a positive result 95% of the time if the person actually carries the disease. However, the test also gives a positive result 3% of the time when the individual is not carrying the disease. It is known that 10% of the population carries the disease. If a person tests positive, what is the probability that he or she has the disease?

6. A person has two coins: a fair coin and a two-headed coin. A coin is selected at random, and tossed. If the coin shows a head, what is the probability that the coin is fair?

7. A computer company buys its chips from three different manufacturers. Manufacturer I provides 60% of the chips and is known to produce 5% defective; Manufacturer II supplies 30% of the chips and makes 4% defective; while the rest are supplied by Manufacturer III with 3% defective chips. If a chip is chosen at random, find the following probabilities:
   a. P(the chip is defective)
   b. P(chip is from Manufacturer II | defective)
   c. P(defective |chip is from manufacturer III)

8. Lincoln Union High School District is made up of three high schools: Monterey, Fremont, and Kennedy, with an enrollment of 500, 300, and 200, respectively. On a given day, the percentage of students absent at Monterey High School is 6%, at Fremont 4%, and at Kennedy 5%. If a student is chosen at random, find the probabilities below: *Hint: Convert the enrollments into percentages.*
   a. P(the student is absent)
   b. P(student is from Kennedy | student is absent)
   c. P(student is absent | student is from Fremont)

9. At a retail store, 20% of customers use the store’s online app to assist them when shopping in the store ; 80% of store shoppers don’t use the app. Of those customers that use the online app while in the store, 50% are very satisfied with their purchases, 40% are moderately satisfied, and 10% are dissatisfied. Of those customers that do not use the online app while in the store, 30% are very satisfied with their purchases, 50% are moderately satisfied and 20% are dissatisfied. Indicate the events by the following: A = shopper uses the app in the store N = shopper does not use the app in the store V = very satisfied with purchase M = moderately satisfied D = dissatisfied a. Find P(A and D), the probability that a store customer uses the app and is dissatisfied b. Find P(A|D), the probability that a store customer uses the app if the customer is dissatisfied.	10. A medical clinic uses a pregnancy test to confirm pregnancy in patients who suspect they are pregnant. Historically data has shown that overall, 70% of the women at this clinic who are given the pregnancy test are pregnant, but 30% are not. The test's manufacturer indicates that if a woman is pregnant, the test will be positive 92% of the time. But if a woman is not pregnant, the test will be positive only 2% of the time and will be negative 98% of the time. a. Find the probability that a woman at this clinic is pregnant and tests positive. b. Find the probability that a woman at this clinic is actually pregnant given that she tests positive.

12.3: Expected Value

Learning Objectives

In this section, you will learn to:

1. Find the expected value of a discrete probability distribution
2. Interpret expected value as a long-run average

An expected gain or loss in a game of chance is called **Expected Value**. The concept of expected value is closely related to a *weighted average*. Consider the following situations.

1. Suppose you and your friend play a game that consists of rolling a die. Your friend offers you the following deal: If the die shows any number from 1 to 5, he will pay you the face value of the die in dollars, that is, if the die shows a 4, he will pay you $4. But if the die shows a 6, you will have to pay him $18.

Before you play the game you decide to find the expected value. You analyze as follows.

Since a die will show a number from 1 to 6, with an equal probability of 1/6, your chance of winning $1 is 1/6, winning $2 is 1/6, and so on up to the face value of 5. But if the die shows a 6, you will lose $18. You write the expected value.

$$\mathrm{E}=\$1(1/6)+\$2(1/6)+\$3(1/6)+\$4(1/6)+\$5(1/6)-\$18(1/6)=-\$.50$$

This means that every time you play this game, you can expect to lose 50 cents. In other words, if you play this game 100 times, theoretically you will lose $50. Obviously, it is not to your interest to play.

2. Suppose of the ten quizzes you took in a course, on eight quizzes you scored 80, and on two you scored 90. You wish to find the average of the ten quizzes.
The average is

$$\mathrm{A}=\frac{(80)(8)+(90)(2)}{10}=(80)\frac{8}{10}+(90)\frac{2}{10}=82$$

It should be observed that it would be incorrect to take the average of 80 and 90 because you scored 80 on eight quizzes, and 90 on only two of them. Therefore, you take a "weighted average" of 80 and 90. That is, the average of 8 parts of 80 and 2 parts of 90, which is 82.

In the first situation, to find the expected value, we multiplied each payoff by the probability of its occurrence, and then added up the amounts calculated for all possible cases. In the second example, if we consider our test score a payoff, we did the same. This leads us to the following definition.

Expected Value

If an experiment has the following probability distribution,

Payoff	x_1	x_2	x_3	$\ldots$	x_n
Probability	$p(x_1)$	$p(x_2)$	$p(x_3)$	$\ldots$	$p(x_n)$

then the expected value of the experiment is

$$\text{Expected Value }=x_1p\left(x_1\right)+x_2p\left(x_2\right)+x_3p\left(x_3\right)+\cdots+x_np\left(x_n\right)$$

Example 12.3.1

In a town, 10% of the families have three children, 60% of the families have two children, 20% of the families have one child, and 10% of the families have no children. What is the expected number of children to a family?

Solution

We list the information in the following table.

Number of children	3	2	1	0
Probability	0.10	0.60	0.20	0.10

Expected Value = $x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3) + x_4 p(x_4)$

$E = 3(.10) + 2(.60) + 1(.20) + 0(.10) = 1.7$

So on average, there are 1.7 children to a family.

Example 12.3.2

To sell an average house, a real estate broker spends $1200 for advertisement expenses. If the house sells in three months, the broker makes $8,000. Otherwise, the broker loses the listing. If there is a 40% chance that the house will sell in three months, what is the expected payoff for the real estate broker?

Solution

The broker makes $8,000 with a probability of .40, but he loses $1200 whether the house sells or not.

$$E = (\$8000)(.40) - (\$1200) = \$2,000.$$

Alternatively, the broker makes $(8000 - 1200) with a probability of .40, but loses $1200 with a probability of .60. Therefore,

$$E = (\$6800)(.40) - (\$1200)(.60) = \$2,000.$$

Example 12.3.3

In a town, the attendance at a football game depends on the weather. On a sunny day the attendance is 60,000, on a cold day the attendance is 40,000, and on a stormy day the attendance is 30,000. If for the next football season, the weatherman has predicted that 30% of the days will be sunny, 50% of the days will be cold, and 20% days will be stormy, what is the expected attendance for a single game?

Solution

Using the expected value formula, we get

$$E = (60,000)(.30) + (40,000)(.50) + (30,000)(.20) = 44,000.$$

Example 12.3.4

A lottery consists of choosing 6 numbers from a total of 51 numbers. The person who matches all six numbers wins $2 million. If the lottery ticket costs $1, what is the expected payoff?

Solution

Since there are $51 \mathrm{C} 6 = 18,009,460$combinations of six numbers from a total of 51 numbers, the chance of choosing the winning number is 1 out of 18,009,460.

So the expected payoff is: $\mathrm{E} = (\$2 \text{ million }) \left(\frac{1}{18009460}\right) - \$1 = -\$0.89$

This means that every time a person spends $1 to buy a ticket, he or she can expect to lose 89 cents.

12.3.1: Expected Value (Exercises)

SECTION 12.3 PROBLEM SET: EXPECTED VALUE

Do the following problems using the expected value concepts learned in this section,

1. You are about to make an investment which gives you a 30% chance of making $60,000 and 70% chance of losing $ 30,000. Should you invest? Explain.
2. In a town, 40% of the men and 30% of the women are overweight. If the town has 46% men and 54% women, what percent of the people are overweight?
3. A game involves rolling a Korean die (4 faces). If a one, two, or three shows, the player receives the face value of the die in dollars, but if a four shows, the player is obligated to pay $4. What is the expected value of the game?
4. A game involves rolling a single die. One receives the face value of the die in dollars. How much should one be willing to pay to roll the die to make the game fair?
5. In a European country, 20% of the families have three children, 40% have two children, 30% have one child, and 10% have no children. On average, how many children are there to a family?
6. A game involves drawing a single card from a standard deck. One receives 60 cents for an ace, 30 cents for a king, and 5 cents for a red card that is neither an ace nor a king. If the cost of each draw is 10 cents, should one play? Explain.
7. Hillview Church plans to raise money by raffling a television worth $500. A total of 3000 tickets are sold at $1 each. Find the expected value of the winnings for a person who buys a ticket in the raffle.
8. During her four years at college, Niki received A's in 30% of her courses, B's in 60% of her courses, and C's in the remaining 10%. If A = 4, B = 3, and C = 2, find her grade point average.
9. Attendance at a Stanford football game depends upon which team Stanford is playing against. If the game is against U. C. Berkeley,attendance will be 70,000; if it is against another California team, it will be 40,000; and if it is against an out of state team, it will be 30,000. If the probability of playing against U. C. Berkeley is 10%, against a California team 50% and against an out of state team 40%, how many fans are expected to attend a game?
10. A Texas oil drilling company has determined that it costs $25,000 to sink a test well. If oil is hit, the revenue for the company will be $500,000. If natural gas is found, the revenue will be $150,000. If the probability of hitting oil is 3% and of hitting gas is 6%, find the expected value of sinking a test well.
11. A $1 lottery ticket offers a grand prize of $10,000; 10 runner-up prizes each pay $1000; 100 third-place prizes each pay $100; and 1,000 fourth-place prizes each pay $10. Find the expected value of entering this contest if 1 million tickets are sold.
12. Assume that for the next heavyweight fight the odds of current champion winning are 15 to 2. A gambler bets $10 that the current champion will lose. If current champion loses, how much can the gambler hope to receive?
13. In a housing development, 35% of households have no school age children, 20% of households have 1 school age child, 25% of households have 2 school age children, 15% have 3, and 5% have 4 school age children.
    a. Find the average number of children per household
    b. If there are 300 homes in this housing development, what is the total number of children expected to attend school?
14. At a large community college, 30% of students take one course, 15% take two courses, 25% take three courses and 20% take four courses. The rest of the students take five courses.
    a. What percent of students take 5 courses?
    b. Find the average number of courses that students take.

12.4: Probability Using Tree Diagrams

Learning Objectives

In this section, you will learn to:

1. Use probability trees to organize information in probability problems
2. Use probability trees to calculate probabilities

As we have already seen, tree diagrams play an important role in solving probability problems. A tree diagram helps us not only visualize, but also list all possible outcomes in a systematic fashion. Furthermore, when we list various outcomes of an experiment and their corresponding probabilities on a tree diagram, we gain a better understanding of when probabilities are multiplied and when they are added.

The meanings of the words *and* and *or* become clear when we learn to multiply probabilities horizontally across branches, and add probabilities vertically down the tree.

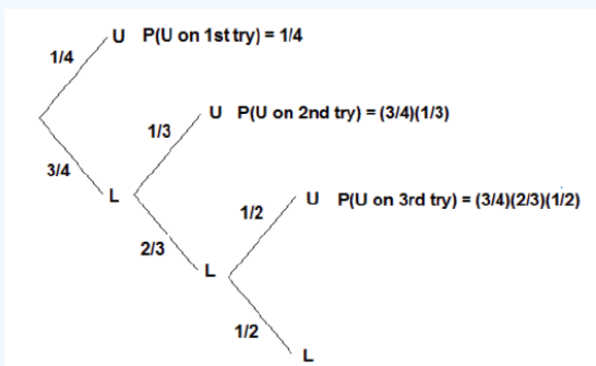
Although tree diagrams are not practical in situations where the possible outcomes become large, they are a significant tool in breaking the problem down in a schematic way. We consider some examples that may seem difficult at first, but with the help of a tree diagram, they can easily be solved.

Example 12.4.1

A person has four keys and only one key fits to the lock of a door. What is the probability that the locked door can be unlocked in at most three tries?

Solution

Let U be the event that the door has been unlocked and L be the event that the door has not been unlocked. We illustrate with a tree diagram.



The probability of unlocking the door in the first try = 1/4

The probability of unlocking the door in the second try = (3/4)(1/3) = 1/4

The probability of unlocking the door in the third try = (3/4)(2/3)(1/2) = 1/4

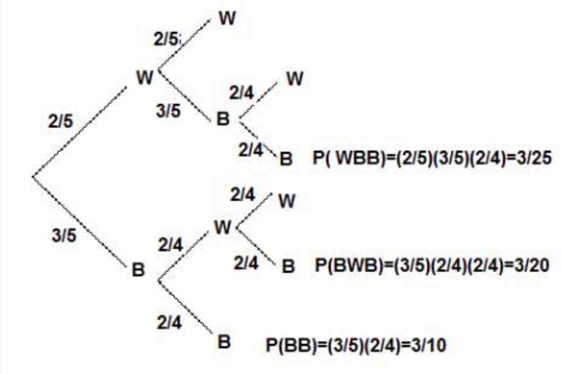
Therefore, the probability of unlocking the door in at most three tries = 1/4 + 1/4 + 1/4 = 3/4.

Example 12.4.2

A jar contains 3 black and 2 white marbles. We continue to draw marbles one at a time until two black marbles are drawn. If a white marble is drawn, the outcome is recorded and the marble is put back in the jar before drawing the next marble. What is the probability that we will get exactly two black marbles in at most three tries?

Solution

We illustrate using a tree diagram.



The probability that we will get two black marbles in the first two tries is listed adjacent to the lowest branch, and it = 3/10.

The probability of getting first black, second white, and third black = 3/20.

Similarly, the probability of getting first white, second black, and third black = 3/25.

Therefore, the probability of getting exactly two black marbles in at most three tries = 3/10 + 3/20 + 3/25 = 57/100.

Example 12.4.3

A circuit consists of three resistors: resistor R_1, resistor R_2, and resistor R_3, joined in a series. If one of the resistors fails, the circuit stops working. The probabilities that resistors R_1, R_2, or R_3 will fail are .07, .10, and .08, respectively. Find the probability that at least one of the resistors will fail?

Solution

The probability that at least one of the resistors fails = 1 - none of the resistors fails.

It is quite easy to find the probability of the event that none of the resistors fails.
We don't even need to draw a tree because we can visualize the only branch of the tree that assures this outcome.

The probabilities that R_1, R_2, R_3 will not fail are .93, .90, and .92 respectively. Therefore, the probability that none of the resistors fails = (.93)(.90)(.92) = .77.

Thus, the probability that at least one of them will fail = 1 - .77 = .23.

12.4.1: Probability Using Tree Diagrams (Exercises)

SECTION 12.4 PROBLEM SET: PROBABILITY USING TREE DIAGRAM

Use a tree diagram to solve the following problems.

1. Suppose you have five keys and only one key fits to the lock of a door. What is the probability that you can open the door in at most three tries?
2. A coin is tossed until a head appears. What is the probability that a head will appear in at most three tries?
3. A basketball player has an 80% chance of making a basket on a free throw. If he makes the basket on the first throw, he has a 90% chance of making it on the second. However, if he misses on the first try, there is only a 70% chance he will make it on the second. If he gets two free throws, what is the probability that he will make at least one of them?
4. You are to play three games. In the first game, you draw a card, and you win if the card is a heart. In the second game, you toss two coins, and you win if one head and one tail are shown. In the third game, two dice are rolled and you win if the sum of the dice is 7 or 11. What is the probability that you win all three games? What is the probability that you win exactly two games?
5. John's car is in the garage, and he has to take a bus to get to school. He needs to make all three connections on time to get to his class. If the chance of making the first connection on time is 80%, the second 80%, and the third 70%, what is the chance that John will make it to his class on time?
6. For a real estate exam the probability of a person passing the test on the first try is .70. The probability that a person who fails on the first try will pass on each of the successive attempts is .80. What is the probability that a person passes the test in at most three attempts?
7. On a Christmas tree with lights, if one bulb goes out, the entire string goes out. If there are twelve bulbs on a string, and the probability of any one going out is .04, what is the probability that the string will not go out?
8. The Long Life Light Bulbs claims that the probability that a light bulb will go out when first used is 15%, but if it does not go out on the first use the probability that it will last the first year is 95%, and if it lasts the first year, there is a 90% probability that it will last two years. Find the probability that a new bulb will last 2 years.
9. A die is rolled until an ace (1) shows. What is the probability that an ace will show on the fourth try?
10. If there are four people in a room, what is the probability that no two have the same birthday?
11. Dan forgets to set his alarm 60% of the time. If he hears the alarm, he turns it off and goes back to sleep 20% of the time, and even if he does wake up on time, he is late getting ready 30% of the time. What is the probability that Dan will be late to school?
12. It has been estimated that 20% of the athletes take some type of drugs. A drug test is 90% accurate, that is, the probability of a falsenegative is 10%. Furthermore, for this test the probability of a false-positive is 20%. If an athlete tests positive, what is the probability that he is a drug user?

12.5: Chapter Review

SECTION 12.5 PROBLEM SET: CHAPTER REVIEW

1. A coin is tossed five times. Find the following
   a. P(2 heads and 3 tails)
   b. P(at least 4 tails)
2. A dandruff shampoo helps 80% of the people who use it. If 10 people apply this shampoo to their hair, what is the probability that 6 will be dandruff free?
3. A baseball player has a .250 batting average. What is the probability that he will have 2 hits in 4 times at bat?
4. Suppose that 60% of the voters in California intend to vote Democratic in the next election. If we choose five people at random, what is the probability that at least four will vote Democratic?
5. A basketball player has a .70 chance of sinking a basket on a free throw. What is the probability that he will sink at least 4 baskets in six shots?
6. During an archery competition, Stan has a 0.8 chance of hitting a target. If he shoots three times, what is the probability that he will hit the target all three times?
7. A company finds that one out of four new applicants overstate their work experience. If ten people apply for a job at this company, what is the probability that at most two will overstate their work experience?
8. A missile has a 70% chance of hitting a target. How many missiles should be fired to make sure that the target is destroyed with a probability of .99 or more?
9. Jar I contains 4 red and 5 white marbles, and Jar II contains 2 red and 4 white marbles. A jar is picked at random and a marble is drawn. Draw a tree diagram and find,
   a. P(Marble is red)
   b. P(It is white given that it came from Jar II)
   c. P(It came from Jar II knowing that the marble drawn is white)
10. Suppose a test is given to determine if a person is infected with HIV. If a person is infected with HIV, the test will detect it in 90% of the cases; and if the person is not infected with HIV, the test will show a positive result 3% of the time. If we assume that 2% of the population is actually infected with HIV, what is the probability that a person obtaining a positive result is actually infected with HIV?
11. A car dealer's inventory consists of 70% cars and 30% trucks. 20% of the cars and 10% of the trucks are used vehicles. If a vehicle chosen at random is used, find the probability that it is a car.
12. Two machines make all the products in a factory, with the first machine making 30% of the products and the second 70%. The first machine makes defective products 3% of the time and the second machine 5% of the time.
    a. Overall what percent of the products made are defective?
    b. If a defective product is found, what is the probability that it was made on the second machine?
    c. If it was made on the second machine, what is the probability that it is defective?
13. An instructor in a finite math course estimates that a student who does his homework has a 90% of chance of passing the course, while a student who does not do the homework has only a 20% chance of passing the course. It has been determined that 60% of the students in a large class do their homework.
    a. What percent of all the students will pass?
    b. If a student passes, what is the probability that he did the homework?
14. Cars are produced at three factories. Factory I produces 10% of the cars and it is known that 2% are defective. Factory II produces 20% of the cars and 3% are defective. Factory III produces 70% of the cars and 4% of those are defective. A car is chosen at random. Find the following probabilities:
    a. P(The car is defective)
    b. P(The car came from Factory III | the car is defective)
15. A stock has a 50% chance of a 10% gain, a 30% chance of no gain, and otherwise it will lose 8%. Find the expected return.
16. A game involves rolling a pair of dice. One receives the sum of the face value of both dice in dollars. How much should one be willing to pay to roll the dice to make the game fair?
17. A roulette wheel consists of numbers 1 through 36, 0, and 00. If the wheel shows an odd number you win a dollar, otherwise you lose a dollar. If you play the game ten times, what is your expectation?

18. A student takes a 100-question multiple-choice exam in which there are four choices to each question. If the student is just guessing the answers, what score can he expect?
19. Mr. Shaw invests 50% of his money in stocks, 30% in mutual funds, and the remaining 20% in bonds. If the annual yield from stocks is 10%, from mutual funds 12%, and from bonds 7%, what percent return can Mr. Shaw expect on his money?
20. An insurance company is planning to insure a group of surgeons against medical malpractice. Its research shows that two surgeons in every fifteen are involved in a medical malpractice suit each year where the average award to the victim is $450,000. How much minimum annual premium should the insurance company charge each doctor?
21. In an evening finite math class of 30 students, it was discovered that 5 students were of age 20, 8 students were about 25 years old, 10 students were close to 30, 4 students were 35, 2 students were 40 and one student 55. What is the average age of a student in this class?
22. Jar I contains 4 marbles of which one is red, and Jar II contains 6 marbles of which 3 are red. Katy selects a jar and then chooses a marble. If the marble is red, she gets paid 3 dollars, otherwise she loses a dollar. If she plays this game ten times, what is her expected payoff?
23. Jar I contains 1 red and 3 white, and Jar II contains 2 red and 3 white marbles. A marble is drawn from Jar I and put in Jar II. Now if one marble is drawn from Jar II, what is the probability that it is a red marble?
24. Let us suppose there are three traffic lights between your house and the school. The chance of finding the first light green is 60%, the second 50%, and the third 30%. What is the probability that on your way to school, you will find at least two lights green?
25. Sonya has just earned her law degree and is planning to take the bar exam. If her chance of passing the bar exam is 65% on each try, what is the probability that she will pass the exam in at least three tries?
26. Every time a particular baseball player is at bat, his probability of getting a hit is .3, his probability of walking is .1, and his probability of being struck out is .4. If he is at bat three times, what is the probability that he will get two hits and one walk?
27. Jar I contains 4 marbles of which none are red, and Jar II contains 6 marbles of which 4 are red. Juan first chooses a jar and then from it he chooses a marble. After the chosen marble is replaced, Mary repeats the same experiment. What is the probability that at least one of them chooses a red marble?
28. Andre and Pete are two tennis players with equal ability. Andre makes the following offer to Pete: We will not play more than four games, and anytime I win more games than you, I am declared a winner and we stop. Draw a tree diagram and determine Andre's probability of winning.

Index

A

absolute value
3.4: Absolute Value

absolute value equations
3.5: Absolute Value Equations

Absolute Value Function
3.1: Functions and Function Notation

accumulated value (annuity)
6.2: Annuities and Sinking Funds

Addition Property
2.1: Properties of Inequalities

annuity
6.2: Annuities and Sinking Funds

augmented matrix
7.2: Systems of Linear Equations and the Gauss-Jordan Method

average rate of change
5.5: Average rate of Change and Derivatives

C

Circular Permutations
10.4: Circular Permutations and Permutations with Similar Elements

close model (economy)
7.6: Applications – Leontief Models

coefficient
3.10: Power Functions and Polynomial Functions

coefficient matrix
7.1: Introduction to Matrices

column vector
7.1: Introduction to Matrices

Combinations
10.5: Combinations

common logarithms
4.4: Logarithms and Logarithmic Functions

common ratio
6.2: Annuities and Sinking Funds

complement
11.2: Mutually Exclusive Events and the Addition Rule

compound interest
6.1: Simple and Compound Interest

conditional probability
11.4: Conditional Probability

constant function
3.1: Functions and Function Notation
3.3: Piecewise-Defined Functions

constraints
8.2: Maximization Applications

continuity
5.4: Continuity

continuous compounding
6.1: Simple and Compound Interest

continuous function
3.10: Power Functions and Polynomial Functions

Cube root function
3.1: Functions and Function Notation

cubic function
3.1: Functions and Function Notation

D

definition of a logarithm
4.6: Exponential and Logarithmic Equations

degree
3.10: Power Functions and Polynomial Functions

dependent system
7.3: Systems of Linear Equations – Special Cases

dependent variable
3.1: Functions and Function Notation

Derivatives
5.5: Average rate of Change and Derivatives

Differentiability (two variables)
5.5: Average rate of Change and Derivatives

dimension
7.1: Introduction to Matrices

discontinuous function
5.4: Continuity

Division Property
2.1: Properties of Inequalities

domain
3.1: Functions and Function Notation
3.2: Domain and Range
3.12: Graphing Rational Functions

dual
9.3: Minimization By The Simplex Method

dual problem
9.3: Minimization By The Simplex Method

E

elements
7.1: Introduction to Matrices
10.1: Sets and Venn Diagrams

elimination method
1.5: More Applications
3.8: More Applications

empty set
10.1: Sets and Venn Diagrams

end behavior
3.10: Power Functions and Polynomial Functions
3.11: Graphs of Polynomial Functions

entries
7.1: Introduction to Matrices

event
11.1: Sample Spaces and Probability

F

factorial
10.3: Permutations

fixed cost
1.4: Applications
3.7: Applications

function
3.1: Functions and Function Notation

G

general form of a polynomial function
3.10: Power Functions and Polynomial Functions

geometric series
6.2: Annuities and Sinking Funds

global maximum
3.11: Graphs of Polynomial Functions

global minimum
3.11: Graphs of Polynomial Functions

H

horizontal line test
3.1: Functions and Function Notation

I

identity function
3.1: Functions and Function Notation

identity matrix
7.1: Introduction to Matrices

inconsistent system
7.3: Systems of Linear Equations – Special Cases

independent events
11.5: Independent Events

independent system
7.3: Systems of Linear Equations – Special Cases

independent variable
3.1: Functions and Function Notation

input
3.1: Functions and Function Notation

instantaneous velocity
5.5: Average rate of Change and Derivatives

intercepts
1.1: Graphing a Linear Equation

Intermediate Value Theorem
3.11: Graphs of Polynomial Functions

intersection
11.2: Mutually Exclusive Events and the Addition Rule

Interval Notation
3.2: Domain and Range

inverse of a matrix
7.4: Inverse Matrices

J

jump discontinuity
5.4: Continuity

L

leading coefficient
3.10: Power Functions and Polynomial Functions

leading term
3.10: Power Functions and Polynomial Functions

Leontief models
7.6: Applications – Leontief Models

limit of a polynomial
5.3: Finding Limits - Properties of Limits

limit of a power
5.3: Finding Limits - Properties of Limits

limit of a quotient
5.3: Finding Limits - Properties of Limits

limit of a sum
5.3: Finding Limits - Properties of Limits

limits
5.2: Finding Limits - Numerical and Graphical Approaches

linear equation
1.1: Graphing a Linear Equation

linear programming
8.2: Maximization Applications

logarithmic function
4.4: Logarithms and Logarithmic Functions

M

mixed constraints
8.2: Maximization Applications

Multiplication Property
2.1: Properties of Inequalities

multiplication rule
11.3: Probability Using Tree Diagrams and Combinations
multiplicity
3.11: Graphs of Polynomial Functions
mutually exclusive
11.2: Mutually Exclusive Events and the Addition Rule

N

natural logarithm
4.4: Logarithms and Logarithmic Functions
number e
4.2: Definition of Exponential Function, Exponential Growth and Decay Models

O

objective function
8.2: Maximization Applications
open model (economy)
7.6: Applications – Leontief Models
ordinary annuity
6.2: Annuities and Sinking Funds
output
3.1: Functions and Function Notation
outstanding balance of a loan
6.4: Miscellaneous Application Problems

P

parameter
1.1: Graphing a Linear Equation
permutation
10.3: Permutations
piecewise function
3.3: Piecewise-Defined Functions
pivot element
7.2: Systems of Linear Equations and the Gauss-Jordan Method
pivot row
7.2: Systems of Linear Equations and the Gauss-Jordan Method
pivoting
7.2: Systems of Linear Equations and the Gauss-Jordan Method
polynomial function
3.10: Power Functions and Polynomial Functions
Power Functions
3.10: Power Functions and Polynomial Functions
probability
11.1: Sample Spaces and Probability
profit
1.5: More Applications
3.8: More Applications
properties of limits
5.3: Finding Limits - Properties of Limits

Q

quadratic function
3.1: Functions and Function Notation
3.9: Quadratic Functions

R

Range
3.1: Functions and Function Notation
3.2: Domain and Range
Rational Functions
3.12: Graphing Rational Functions
Reciprocal function
3.1: Functions and Function Notation
Reciprocal squared function
3.1: Functions and Function Notation
reduced row echelon form
7.2: Systems of Linear Equations and the Gauss-Jordan Method
relation
3.1: Functions and Function Notation
removable discontinuity
5.4: Continuity
revenue
1.5: More Applications
3.8: More Applications
row operation
7.2: Systems of Linear Equations and the Gauss-Jordan Method
row vector
7.1: Introduction to Matrices

S

sample space
11.1: Sample Spaces and Probability
sampled with replacement
11.1: Sample Spaces and Probability
sampled without replacement
11.1: Sample Spaces and Probability
secant
5.5: Average rate of Change and Derivatives
set
10.1: Sets and Venn Diagrams
simple event
11.1: Sample Spaces and Probability
simplex algorithm
9.2: Maximization By The Simplex Method
Simplex method
9.3: Minimization By The Simplex Method
sinking fund
6.2: Annuities and Sinking Funds
slope
1.2: Slope of a Line
smooth curve
3.10: Power Functions and Polynomial Functions
square matrix
7.1: Introduction to Matrices
square root function
3.1: Functions and Function Notation
standard form
1.3: Determining the Equation of a Line
standard minimization linear program
8.3: Minimization Applications
subset
10.1: Sets and Venn Diagrams
Subtraction Property
2.1: Properties of Inequalities

T

tangent line
5.5: Average rate of Change and Derivatives
target row
7.2: Systems of Linear Equations and the Gauss-Jordan Method
term of a polynomial function
3.10: Power Functions and Polynomial Functions
Transitive Property
2.1: Properties of Inequalities
tree diagrams
11.3: Probability Using Tree Diagrams and Combinations
turning point
3.10: Power Functions and Polynomial Functions

U

union
11.2: Mutually Exclusive Events and the Addition Rule

V

variable cost
1.4: Applications
3.7: Applications
Venn diagram
10.1: Sets and Venn Diagrams
vertical line test
3.1: Functions and Function Notation

Z

zero matrix
7.1: Introduction to Matrices

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: FCC DS 21 Finite Mathematics - Fall 2024

Webpages: 144

Applicable Restrictions: Noncommercial

All licenses found:

- CC BY 4.0: 83.3% (120 pages)
- Undeclared: 9% (13 pages)
- CC BY-SA 4.0: 4.2% (6 pages)
- CC BY-NC-SA 2.5: 2.8% (4 pages)
- CC BY-NC-SA 4.0: 0.7% (1 page)

By Page

- FCC DS 21 Finite Mathematics - Fall 2024 - *Undeclared*
  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Linear Equations - *CC BY 4.0*
    - 1.1: Graphing a Linear Equation - *CC BY 4.0*
      - 1.1.1: Graphing a Linear Equation (Exercises) - *CC BY 4.0*
    - 1.2: Slope of a Line - *CC BY 4.0*
      - 1.2.1: Slope of a Line (Exercises) - *CC BY 4.0*
    - 1.3: Determining the Equation of a Line - *CC BY 4.0*
      - 1.3.1: Determining the Equation of a Line (Exercises) - *CC BY 4.0*
    - 1.4: Applications - *CC BY 4.0*
      - 1.4.1: Applications (Exercises) - *CC BY 4.0*
    - 1.5: More Applications - *CC BY 4.0*
      - 1.5.1: More Applications (Exercises) - *CC BY 4.0*
    - 1.6: Chapter Review - *CC BY 4.0*
  - 2: Inequalities - *CC BY-SA 4.0*
    - 2.1: Properties of Inequalities - *CC BY-SA 4.0*
    - 2.2: Solving and Graphing Inequalities, and Writing Answers in Interval Notation - *CC BY-SA 4.0*
    - 2.3: Set Builder Notation - *Undeclared*
    - 2.4: Rational Inequalities - *CC BY-SA 4.0*
  - 3: Functions - *Undeclared*
    - 3.1: Functions and Function Notation - *CC BY 4.0*
    - 3.2: Domain and Range - *CC BY 4.0*
    - 3.3: Piecewise-Defined Functions - *CC BY-NC-SA 2.5*
    - 3.4: Absolute Value - *CC BY-NC-SA 2.5*
    - 3.5: Absolute Value Equations - *CC BY-NC-SA 2.5*
    - 3.6: Break-Even Analysis (Sink or Swim) - *CC BY-NC-SA 4.0*
    - 3.7: Applications - *CC BY 4.0*
      - 3.7.1: Applications (Exercises) - *CC BY 4.0*
    - 3.8: More Applications - *CC BY 4.0*
      - 3.8.1: More Applications (Exercises) - *CC BY 4.0*
    - 3.9: Quadratic Functions - *CC BY-SA 4.0*
      - 3.9E: Quadratic Functions (Exercises) - *CC BY-SA 4.0*
    - 3.10: Power Functions and Polynomial Functions - *CC BY 4.0*
    - 3.11: Graphs of Polynomial Functions - *CC BY 4.0*
    - 3.12: Graphing Rational Functions - *CC BY-NC-SA 2.5*
  - 4: Exponential and Logarithmic Functions - *CC BY 4.0*
    - 4.1: Prelude to Exponential and Logarithmic Functions including Exponential properties - *CC BY 4.0*
    - 4.2: Definition of Exponential Function, Exponential Growth and Decay Models - *CC BY 4.0*
      - 4.2.1: Exponential Growth and Decay Models (Exercises) - *CC BY 4.0*
    - 4.3: Graphs and Properties of Exponential Growth and Decay Functions/Use TI-84 calculator - *CC BY 4.0*
      - 4.3.1: Graphs and Properties of Exponential Growth and Decay Functions (Exercises) - *CC BY 4.0*
    - 4.4: Logarithms and Logarithmic Functions - *CC BY 4.0*
      - 4.4.1: Logarithms and Logarithmic Functions (Exercises) - *CC BY 4.0*

- 4.5: Graphs and Properties of Logarithmic Functions - *CC BY 4.0*
  - 4.5.1: Graphs and Properties of Logarithmic Functions (Exercises) - *CC BY 4.0*
- 4.6: Exponential and Logarithmic Equations - *CC BY 4.0*
- 4.7: Chapter Review - *CC BY 4.0*
- 4.E: Exponential and Logarithmic Functions (Exercises) - *CC BY 4.0*

- 5: Introduction to Calculus - *CC BY 4.0*
  - 5.1: Prelude to Calculus - *CC BY 4.0*
  - 5.2: Finding Limits - Numerical and Graphical Approaches - *CC BY 4.0*
  - 5.3: Finding Limits - Properties of Limits - *CC BY 4.0*
  - 5.4: Continuity - *CC BY 4.0*
  - 5.5: Average rate of Change and Derivatives - *CC BY 4.0*
  - 5.E: Introduction to Calculus (Exercises) - *CC BY 4.0*
    - 1.1: 5.6 Derivatives of Polynomials - *Undeclared*
  - 5.R: Introduction to Calculus (Review) - *CC BY 4.0*
- 6: Mathematics of Finance - *CC BY 4.0*
  - 6.1: Simple and Compound Interest - *CC BY 4.0*
    - 6.1.1: Simple and Compound Interest (Exercises) - *CC BY 4.0*
  - 6.2: Annuities and Sinking Funds - *CC BY 4.0*
    - 6.2.1: Annuities and Sinking Funds (Exercises) - *CC BY 4.0*
  - 6.3: Present Value of an Annuity and Installment Payment - *CC BY 4.0*
    - 6.3.1: Present Value of an Annuity and Installment Payment (Exercises) - *CC BY 4.0*
  - 6.4: Miscellaneous Application Problems - *CC BY 4.0*
    - 6.4.1: Miscellaneous Application Problems (Exercises) - *CC BY 4.0*
  - 6.5: Classification of Finance Problems - *CC BY 4.0*
    - 6.5.1: Classification of Finance Problems (Exercises) - *CC BY 4.0*
  - 6.6: Chapter Review - *CC BY 4.0*
- 7: Matrices - *CC BY 4.0*
  - 7.1: Introduction to Matrices - *CC BY 4.0*
    - 7.1.1: Introduction to Matrices (Exercises) - *CC BY 4.0*
  - 7.2: Systems of Linear Equations and the Gauss-Jordan Method - *CC BY 4.0*
    - 7.2.1: Systems of Linear Equations and the Gauss-Jordan Method (Exercises) - *CC BY 4.0*
  - 7.3: Systems of Linear Equations – Special Cases - *CC BY 4.0*
    - 7.3.1: Systems of Linear Equations – Special Cases (Exercises) - *CC BY 4.0*
  - 7.4: Inverse Matrices - *CC BY 4.0*
    - 7.4.1: Inverse Matrices (Exercises) - *CC BY 4.0*
  - 7.5: Application of Matrices in Cryptography - *CC BY 4.0*
    - 7.5.1: Application of Matrices in Cryptography (Exercises) - *CC BY 4.0*
  - 7.6: Applications – Leontief Models - *CC BY 4.0*
    - 7.6.1: Applications – Leontief Models (Exercises) - *CC BY 4.0*
  - 7.7: Chapter Review - *CC BY 4.0*
- 8: Linear Programming - A Geometric Approach - *CC BY 4.0*
  - 8.1: Graph Linear Inequalities in Two Variables - *CC BY 4.0*
    - 8.1E: Exercises - *CC BY 4.0*
  - 8.2: Maximization Applications - *CC BY 4.0*
    - 8.2.1: Maximization Applications (Exercises) - *CC BY 4.0*
  - 8.3: Minimization Applications - *CC BY 4.0*
    - 8.3.1: Minimization Applications (Exercises) - *CC BY 4.0*
  - 8.4: Chapter Review - *CC BY 4.0*
- 9: Linear Programming - The Simplex Method - *CC BY 4.0*
  - 9.1: Introduction to Linear Programming Applications in Business, Finance, Medicine, and Social Science - *CC BY 4.0*
  - 9.2: Maximization By The Simplex Method - *CC BY 4.0*
    - 9.2.1: Maximization By The Simplex Method (Exercises) - *CC BY 4.0*
  - 9.3: Minimization By The Simplex Method - *CC BY 4.0*
    - 9.3.1: Minimization By The Simplex Method (Exercises) - *CC BY 4.0*
  - 9.4: Chapter Review - *CC BY 4.0*
- 10: Sets and Counting - *CC BY 4.0*
  - 10.1: Sets and Venn Diagrams - *CC BY 4.0*
    - 10.1.1: Sets and Venn Diagrams (Exercises) - *CC BY 4.0*
  - 10.2: Tree Diagrams and the Multiplication Axiom - *CC BY 4.0*
    - 10.2.1: Tree Diagrams and the Multiplication Axiom (Exercises) - *CC BY 4.0*
  - 10.3: Permutations - *CC BY 4.0*

- 10.3.1: Permutations (Exercises) - *CC BY 4.0*
- 10.4: Circular Permutations and Permutations with Similar Elements - *CC BY 4.0*
  - 10.4.1: Circular Permutations and Permutations with Similar Elements (Exercises) - *CC BY 4.0*
- 10.5: Combinations - *CC BY 4.0*
  - 10.5.1: Combinations (Exercises) - *CC BY 4.0*
- 10.6: Combinations- Involving Several Sets - *CC BY 4.0*
  - 10.6.1: Combinations- Involving Several Sets (Exercises) - *CC BY 4.0*
- 10.7: Binomial Theorem - *CC BY 4.0*
  - 10.7.1: Binomial Theorem (Exercises) - *CC BY 4.0*
- 10.8: Chapter Review - *CC BY 4.0*
- 11: Probability - *CC BY 4.0*
  - 11.1: Sample Spaces and Probability - *CC BY 4.0*
    - 11.1.1: Sample Spaces and Probability (Exercises) - *CC BY 4.0*
  - 11.2: Mutually Exclusive Events and the Addition Rule - *CC BY 4.0*
    - 11.2.1: Mutually Exclusive Events and the Addition Rule (Exercises) - *CC BY 4.0*
  - 11.3: Probability Using Tree Diagrams and Combinations - *CC BY 4.0*
    - 11.3.1: Probability Using Tree Diagrams and Combinations (Exercises) - *CC BY 4.0*
  - 11.4: Conditional Probability - *CC BY 4.0*
    - 11.4.1: Conditional Probability (Exercises) - *CC BY 4.0*
  - 11.5: Independent Events - *CC BY 4.0*
    - 11.5.1: Independent Events (Exercises) - *CC BY 4.0*
  - 11.6: Chapter Review - *CC BY 4.0*
- 12: More Probability - *CC BY 4.0*
  - 12.1: Binomial Probability - *CC BY 4.0*
    - 12.1.1: Binomial Probability (Exercises) - *CC BY 4.0*
  - 12.2: Bayes' Formula - *CC BY 4.0*
    - 12.2.1: Bayes' Formula (Exercises) - *CC BY 4.0*
  - 12.3: Expected Value - *CC BY 4.0*
    - 12.3.1: Expected Value (Exercises) - *CC BY 4.0*
  - 12.4: Probability Using Tree Diagrams - *CC BY 4.0*
    - 12.4.1: Probability Using Tree Diagrams (Exercises) - *CC BY 4.0*
  - 12.5: Chapter Review - *CC BY 4.0*
- Back Matter - *Undeclared*
  - Index - *Undeclared*
  - Glossary - *Undeclared*
  - Detailed Licensing - *Undeclared*