11.2: Mutually Exclusive Events and the Addition Rule

Learning Objectives

In this section, you will learn to:

1. Define compound events using union, intersection, and complement.
2. Identify mutually exclusive events
3. Use the Addition Rule to calculate probability for unions of events.

In the last chapter, we learned to find the union, intersection, and complement of a set. We will now use these set operations to describe events.

- The **union** of two events E and F, E ∪ F, is the set of outcomes that are in E or in F or in both.
- The **intersection** of two events E and F, E ∩ F, is the set of outcomes that are in both E and F.
- The **complement** of an event E, denoted by E^c, is the set of outcomes in the sample space S that are not in E.

It is worth noting that $P(E^c) = 1 - P(E)$. This follows from the fact that if the sample space has n elements and E has k elements, then E^c has $n-k$ elements. Therefore,

$$P\left(E^{c}\right)=\frac{n-k}{n}=1-\frac{k}{n}=1-P(E)$$

Of particular interest to us are the events whose outcomes do not overlap. We call these events mutually exclusive.

Two events E and F are said to be **mutually exclusive** if they do not intersect: E ∩ F = ∅.

Next we'll determine whether a given pair of events are mutually exclusive.

Example 11.2.1

A card is drawn from a standard deck. Determine whether the pair of events given below is mutually exclusive.

E = {The card drawn is an Ace}

F = {The card drawn is a heart}

Solution

Clearly the ace of hearts belongs to both sets. That is

$$\mathrm{E} \cap \mathrm{F}=\{\text{ Ace of hearts }\} \neq \varnothing$$

Therefore, the events E and F are not mutually exclusive.

Example 11.2.2

Two dice are rolled. Determine whether the pair of events given below is mutually exclusive.

G = {The sum of the faces is six}

H = {One die shows a four}

Solution

For clarity, we list the elements of both sets.

G = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)} and H = {(2, 4), (4, 2)}

Clearly, G ∩ H = {(2, 4), (4, 2)} $\neq \varnothing$.

Therefore, the two sets are not mutually exclusive.

Example 11.2.3

A family has three children. Determine whether the following pair of events are mutually exclusive.

M = {The family has at least one boy}

N = {The family has all girls}

Solution

Although the answer may be clear, we list both the sets.

M = { BBB, BBG, BGB, BGG, GBB, GBG, GGB } and N = { GGG }

Clearly, M ∩ N = ∅

Therefore, events M and N are mutually exclusive.

We will now consider problems that involve the union of two events.

Given two events, E, F, then finding the probability of E ∪ F, is the same as finding the probability that E will happen, or F will happen, or both will happen.

Example 11.2.4

If a die is rolled, what is the probability of obtaining an even number or a number greater than four?

Solution

Let E be the event that the number shown on the die is an even number, and let F be the event that the number shown is greater than four.

The sample space S = { 1, 2, 3, 4, 5, 6 }. The event E = { 2, 4, 6 }, and event F = { 5, 6 }

We need to find P(E ∪ F).

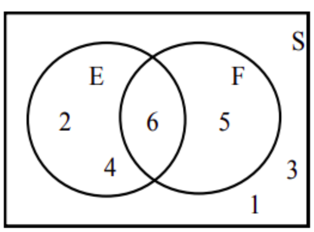
Since P(E) = 3/6, and P(F) = 2/6, a student may say P(E ∪ F) = 3/6 + 2/6. This will be incorrect because the element 6, which is in both E and F has been counted twice, once as an element of E and once as an element of F. In other words, the set E ∪ F has only four elements and not five: set E ∪ F = { 2,4,5,6 }

Therefore, P(E ∪ F) = 4/6 and not 5/6.

This can be illustrated by a Venn diagram. We'll use the Venn Diagram to re-examine Example $11.2.4$ and derive a probability rule that we can use to calculate probabilities for unions of events.

The sample space S, the events E and F, and E ∩ F are listed below.

$$\mathrm{S}=\{1,2,3,4,5,6\}, \mathrm{E}=\{2,4,6\}, \mathrm{F}=\{5,6\}, \text{ and } \mathrm{E} \cap \mathrm{F}=\{6\}.$$



The above figure shows S, E, F, and E ∩ F.

Finding the probability of E ∪ F, is the same as finding the probability that E will happen, or F will happen, or both will happen.

If we count the number of elements n(E) in E, and add to it the number of elements n(F) in F, the points in both E and F are counted twice, once as elements of E and once as elements of F. Now if we subtract from the sum, n(E) + n(F), the number n(E ∩ F), we remove the duplicity and get the correct answer. So as a rule,

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

By dividing the entire equation by n(S), we get

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

Since the probability of an event is the number of elements in that event divided by the number of all possible outcomes, we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Applying the above for Example $11.2.4$, we get

$$P(E \cup F) = 3/6 + 2/6 - 1/6 = 4/6$$

This is because, when we add P(E) and P(F), we have added P(E ∩ F) twice. Therefore, we must subtract P(E ∩ F), once.

This gives us the general formula, called **the Addition Rule**, for finding the probability of the union of two events. Because event E ∪ F is the event that E will happen, OR F will happen, OR both will happen, we sometimes call this the **Addition Rule for OR Events.** It states

Addition Rule

$$\mathbf{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

If, and only if, two events E and F are mutually exclusive, then $E \cap F = \varnothing$ and $P(E \cap F) = 0$, and we get $P(E \cup F) = P(E) + P(F)$

Example $11.2.5$

If a card is drawn from a deck, use the addition rule to find the probability of obtaining an ace or a heart.

Solution

Let A be the event that the card is an ace, and H the event that it is a heart.

Since there are four aces, and thirteen hearts in the deck,

P(A) = 4/52 and P(H) = 13/52.

Furthermore, since the intersection of two events consists of only one card, the ace of hearts, we now have:

P(A ∩ H) = 1/52

We need to find P(A ∪ H):

$$\begin{aligned} P(A \cup H) &= P(A) + P(H) - P(A \cap H) \\ &= 4/52 + 13/52 - 1/52 = 16/52 \end{aligned}$$

Example $11.2.6$

Two dice are rolled, and the events F and T are as follows:

F = {The sum of the dice is four} and T = {At least one die shows a three}

Find P(F ∪ T).

Solution

We list F and T, and F ∩ T as follows:

F = {(1, 3), (2, 2), (3, 1)}

T = {(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)}

F ∩ T = {(1, 3), (3, 1)}

Since P(F ∪ T) = P(F) + P(T) - P(F ∩ T)

We have P(F ∪ T) = 3/36 + 11/36 - 2/36 = 12/36.

Example 11.2.7

Mr. Washington is seeking a mathematics instructor's position at his favorite community college in Cupertino. His employment depends on two conditions: whether the board approves the position, and whether the hiring committee selects him. There is a 80% chance that the board will approve the position, and there is a 70% chance that the hiring committee will select him. If there is a 90% chance that at least one of the two conditions, the board approval or his selection, will be met, what is the probability that Mr. Washington will be hired?

Solution

Let A be the event that the board approves the position, and S be the event that Mr. Washington gets selected. We have,

P(A) = .80, P(S) = .70, and P(A ∪ S) = .90.

We need to find, P(A ∩ S).

The addition formula states that,

$$P(A \cup S) = P(A) + P(S) - P(A \cap S)$$

Substituting the known values, we get

$$.90 = .80 + .70 - P(\mathrm{A} \cap \mathrm{S})$$

Therefore, P(A ∩ S) = .60.

Example 11.2.8

The probability that this weekend will be cold is .6, the probability that it will be rainy is .7, and probability that it will be both cold and rainy is .5. What is the probability that it will be neither cold nor rainy?

Solution

Let C be the event that the weekend will be cold, and R be event that it will be rainy. We are given that

$$\mathrm{P(C)} = .6, \quad \mathrm{P(R)} = .7, \quad \mathrm{P(C \cap R)} = .5$$

First we find P(C ∪ R) using the Addition Rule.

$$P(C \cup R) = P(C) + P(R) - P(C \cap R) = .6 + .7 - .5 = .8$$

Then we find $P((C \cup R)^c)$ using the Complement Rule.

$$P\left((C \cup R)^c\right) = 1 - P(C \cup R) = 1 - .8 = .2$$

We summarize this section by listing the important rules.

Summary

The Addition Rule

For Two Events E and F, $\mathrm{P(E \cup F)} = \mathrm{P(E)} + \mathrm{P(F)} - \mathrm{P(E \cap F)}$

The Addition Rule for Mutually Exclusive Events

If Two Events E and F are Mutually Exclusive, then $\mathrm{P(E \cup F)} = \mathrm{P(E)} + \mathrm{P(F)}$

The Complement Rule

If E^c is the Complement of Event E, then $\mathrm{P}(\mathrm{E}^c) = 1 - \mathrm{P(E)}$