10.4: Circular Permutations and Permutations with Similar Elements

Learning Objectives

In this section you will learn to

1. Count the number of possible permutations of items arranged in a circle
2. Count the number of possible permutations when there are repeated items

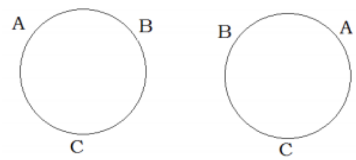
In this section we will address the following two problems.

1. In how many different ways can five people be seated in a circle?
2. In how many different ways can the letters of the word MISSISSIPPI be arranged?

The first problem comes under the category of Circular Permutations, and the second under Permutations with Similar Elements.

Circular Permutations

Suppose we have three people named A, B, and C. We have already determined that they can be seated in a straight line in 3! or 6 ways. Our next problem is to see how many ways these people can be seated in a circle. We draw a diagram.



It happens that there are only two ways we can seat three people in a circle, relative to each other's positions. This kind of permutation is called a circular permutation. In such cases, no matter where the first person sits, the permutation is not affected. Each person can shift as many places as they like, and the permutation will not be changed. We are interested in the position of each person in relation to the others. Imagine the people on a merry-go-round; the rotation of the permutation does not generate a new permutation. So in circular permutations, the first person is considered a place holder, and where he sits does not matter.

Definition: Circular Permutations

The number of permutations of n elements in a circle is $(n-1)!$

Example 10.4.1

In how many different ways can five people be seated at a circular table?

Solution

We have already determined that the first person is just a place holder. Therefore, there is only one choice for the first spot. We have

1	4	3	2	1

So the answer is 24.

Example 10.4.2

In how many ways can four couples be seated at a round table if the men and women want to sit alternately?

Solution

We again emphasize that the first person can sit anywhere without affecting the permutation.

So there is only one choice for the first spot. Suppose a man sat down first. The chair next to it must belong to a woman, and there are 4 choices. The next chair belongs to a man, so there are three choices and so on. We list the choices below.

1	4	3	3	2	2	1	1

So the answer is 144.

PERMUTATIONS WITH SIMILAR ELEMENTS

Let us determine the number of distinguishable permutations of the letters ELEMENT.

Suppose we make all the letters different by labeling the letters as follows.

$$E_1 L E_2 M E_3 N T$$

Since all the letters are now different, there are 7! different permutations.

Let us now look at one such permutation, say

$$L E_1 M E_2 N E_3 T$$

Suppose we form new permutations from this arrangement by only moving the E's. Clearly, there are 3! or 6 such arrangements. We list them below.

$$\begin{gathered}\mathrm{LE}_1\mathrm{ME}_2\mathrm{NE}_3\\ \mathrm{LE}_1\mathrm{ME}_3\mathrm{NE}_2\\ \mathrm{LE}_2\mathrm{ME}_1\mathrm{NE}_3\mathrm{T}\\ \mathrm{LE}_2\mathrm{ME}_3\mathrm{NE}_1\mathrm{T}\\ \mathrm{LE}_3\mathrm{ME}_2\mathrm{NE}_1\mathrm{T}\\ \mathrm{LE}_3\mathrm{ME}_I\mathrm{NE}_2\mathrm{T}\end{gathered}$$

Because the E's are not different, there is only one arrangement LEMENET and not six. This is true for every permutation.

Let us suppose there are n different permutations of the letters ELEMENT.

Then there are $n \cdot 3!$ permutations of the letters $E_1 L E_2 M E_3 N T$.

But we know there are 7! permutations of the letters $E_1 L E_2 M E_3 N T$.

Therefore, $n \cdot 3! = 7!$

Or $n = \frac{7!}{3!}$.

This gives us the method we are looking for.

Definition: Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, is

$$\frac{n!}{r_1! r_2! \ldots r_k!}$$

Example 10.4.3

Find the number of different permutations of the letters of the word MISSISSIPPI.

Solution

The word MISSISSIPPI has 11 letters. If the letters were all different there would have been 11! different permutations. But MISSISSIPPI has 4 S's, 4 I's, and 2 P's that are alike.

So the answer is $\frac{11!}{4!4!2!} = 34,650$.

Example 10.4.4

If a coin is tossed six times, how many different outcomes consisting of 4 heads and 2 tails are there?

Solution

Again, we have permutations with similar elements.

We are looking for permutations for the letters HHHHTT.

The answer is $\frac{6!}{4!2!} = 15$.

Example 10.4.5

In how many different ways can 4 nickels, 3 dimes, and 2 quarters be arranged in a row?

Solution

Assuming that all nickels are similar, all dimes are similar, and all quarters are similar, we have permutations with similar elements. Therefore, the answer is

$$\frac{9!}{4!3!2!} = 1260$$

Example 10.4.6

A stock broker wants to assign 20 new clients equally to 4 of its salespeople. In how many different ways can this be done?

Solution

This means that each sales person gets 5 clients. The problem can be thought of as an ordered partitions problem. In that case, using the formula we get

$$\frac{20!}{5!5!5!5!} = 11,732,745,024$$

Example 10.4.7

A shopping mall has a straight row of 5 flagpoles at its main entrance plaza. It has 3 identical green flags and 2 identical yellow flags. How many distinct arrangements of flags on the flagpoles are possible?

Solution

The problem can be thought of as distinct permutations of the letters GGGYY; that is arrangements of 5 letters, where 3 letters are similar, and the remaining 2 letters are similar:

$$\frac{5!}{3!2!} = 10$$

Just to provide a little more insight into the solution, we list all 10 distinct permutations:

GGGYY, GGYGY, GGYYG, GYGGY, GYGYG, GYYGG, YGGGY, YGGYG, YGYGY, YYGGG

We summarize.

Summary

1. Circular Permutations

The number of permutations of n elements in a circle is

$$(n-1)!$$

2. Permutations with Similar Elements

The number of permutations of n elements taken n at a time, with r_1 elements of one kind, r_2 elements of another kind, and so on, such that $\mathrm{n}=\mathrm{r}_1+\mathrm{r}_2+\ldots+\mathrm{r}_\mathrm{k}$ is

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

This is also referred to as **ordered partitions**.
