11.5: Independent Events

Learning Objectives

In this section, you will:

1. Define independent events
2. Identify whether two events are independent or dependent

In the last section, we considered conditional probabilities. In some examples, the probability of an event changed when additional information was provided. This is not always the case. The additional information may or may not alter the probability of the event.

In Example $11.5.1$ we revisit the discussion at the beginning of the previous section and then contrast that with Example $11.5.2$

Example $11.5.1$

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that the card is a face card.

Solution

a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | The card is a face card), we reason as follows:

There are 12 face cards in a deck of cards. There are 4 kings in a deck of cards.

$$P(\text{The card is a king} \mid \text{The card is a face card}) = 4/12 = 1/3.$$

The reader should observe that in the above example,

$$P(\text{The card is a king} \mid \text{The card is a face card}) \neq P(\text{The card is a king})$$

In other words, the additional information, knowing that the card selected is a face card changed the probability of obtaining a king.

Example $11.5.2$

A card is drawn from a deck. Find the following probabilities.

a. The card is a king.
b. The card is a king given that a red card has shown.

Solution

a. Clearly, P(The card is a king) = 4/52 = 1/13.

b. To find P(The card is a king | A red card has shown), we reason as follows:

Since a red card has shown, there are only twenty six possibilities. Of the 26 red cards, there are two kings. Therefore,

$$P(\text{The card is a king} \mid \text{A red card has shown}) = 2/26 = 1/13.$$

The reader should observe that in the above example,

$$P(\text{The card is a king} \mid \text{A red card has shown}) = P(\text{The card is a king})$$

In other words, the additional information, a red card has shown, did not affect the probability of obtaining a king.

Whenever the probability of an event E is not affected by the occurrence of another event F, and vice versa, we say that the two events E and F are **independent.** This leads to the following definition.

Definition: Independent

Two Events E and F are **independent** if and only if at least one of the following two conditions is true.

1. $\mathbf{P(E|F) = P(E)}$ or
2. $\mathbf{P(F|E) = P(F)}$

If the events are not independent, then they are dependent.

If one of these conditions is true, then both are true.

We can use the definition of independence to determine if two events are independent.

We can use that definition to develop another way to test whether two events are independent.

Recall the conditional probability formula:

$$\mathrm{P(E|F) = \frac{P(E \cap F)}{P(F)}}$$

Multiplying both sides by $\mathrm{P(F)}$, we get

$$\mathrm{P(E \cap F) = P(E|F)P(F)}$$

Now if the two events are independent, then by definition

$$\mathrm{P(E|F) = P(E)}$$

Substituting, $P(E \cap F) = P(E)P(F)$

We state it formally as follows.

Test For Independence

Two events E and F are independent if and only if

$$\mathbf{P(E \cap F) = P(E)P(F)}$$

In the Examples $11.5.3$ and $11.5.4$, we'll examine how to check for independence using both methods:

- Examine the probability of intersection of events to check whether $P(E \cap F) = P(E)P(F)$
- Examine conditional probabilities to check whether $P(E|F) = P(E)$ or $P(F|E) = P(F)$

We need to use only **one** of these methods. Both methods, if used properly, will always give results that are consistent with each other.

Use the method that seems easier based on the information given in the problem.

Example $11.5.3$

The table below shows the distribution of color-blind people by gender.

	Male(M)	Female(F)	Total
Color-Blind(C)	6	1	7
Not Color-Blind(N)	46	47	93
Total	52	48	100

where M represents male, F represents female, C represents color-blind, and N represents not color-blind. Are the events color-blind and male independent?

Solution 1: According to the test for independence, C and M are independent if and only if $\mathrm{P(C \cap M) = P(C)P(M)}$.

From the table: $P(C)$ = 7/100, $P(M)$ = 52/100 and $P(C \cap M)$ = 6/100

So $P(C)P(M)$ = (7/100)(52/100) = .0364

which is **not** equal to $P(C \cap M)$ = 6/100 = .06

Therefore, the two events are not independent. We may say they are dependent.

Solution 2: C and M are independent if and only if $P(C|M) = P(C)$.

From the total column $P(C)$ = 7/100 = 0.07

From the male column $P(C|M)$ = 6/52= 0.1154

Therefore $P(C|M) \neq P(C)$, indicating that the two events are not independent.

Example 11.5.4

In a city with two airports, 100 flights were surveyed. 20 of those flights departed late.

- 45 flights in the survey departed from airport A; 9 of those flights departed late.
- 55 flights in the survey departed from airport B; 11 flights departed late.

Are the events "depart from airport A" and "departed late" independent?

Solution 1

Let A be the event that a flight departs from airport A, and L the event that a flight departs late. We have

$P(A \cap L)$ = 9/100, $P(A)$ = 45/100 and $P(L)$ = 20/100

In order for two events to be independent, we must have $P(A \cap L) = P(A)P(L)$

Since $P(A \cap L)$ = 9/100 = 0.09

and $P(A)P(L)$ = (45/100)(20/100) = 900/10000 = 0.09

the two events "departing from airport A" and "departing late" are independent.

Solution 2

The definition of independent events states that two events are independent if $P(E|F) = P(E)$.

In this problem we are given that

$P(L|A)$ = 9/45= 0.2 and $P(L)$ = 20/100 = 0.2

$P(L|A) = P(L)$, so events "departing from airport A" and "departing late" are independent.

Example 11.5.5

A coin is tossed three times, and the events E, F and G are defined as follows:

E: The coin shows a head on the first toss.

F: At least two heads appear.

G: Heads appear in two successive tosses.

Determine whether the following events are independent.

a. E and F
b. F and G
c. E and G

Solution

We list the sample space, the events, their intersections and the probabilities.

$$\begin{array}{ll}
\mathrm{S}=\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}, \mathrm{TTT} & \\
\mathrm{E}=\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}\}, & \mathrm{P}(\mathrm{E})=4/8 \text{ or } 1/2 \\
\mathrm{F}=\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{THH}\}, & \mathrm{P}(\mathrm{F})=4/8 \text{ or } 1/2 \\
\mathrm{G}=\{\mathrm{HHT}, \mathrm{THH}\}, & \mathrm{P}(\mathrm{G})=2/8 \text{ or } 1/4 \\
\mathrm{E}\cap\mathrm{F}=\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}\}, & \mathrm{P}(\mathrm{E}\cap\mathrm{F})=3/8 \\
\mathrm{F}\cap\mathrm{G}=\{\mathrm{HHT}, \mathrm{THH}\}, & \mathrm{P}(\mathrm{F}\cap\mathrm{G})=2/8 \text{ or } 1/4 \\
\mathrm{E}\cap\mathrm{G}=\{\mathrm{HHT}\} & \mathrm{P}(\mathrm{E}\cap\mathrm{G})=1/8
\end{array}$$

a. E and F will be independent if and only if $P(E\cap F)=P(E)P(F)$

$$P(E\cap F)=3/8 \text{ and } P(E)P(F)=1/2\cdot 1/2=1/4.$$

Since 3/8 ≠ 1/4, we have $P(E\cap F)\neq P(E)P(F)$.

Events E and F are not independent.

b. F and G will be independent if and only if $P(F\cap G)=P(F)P(G)$.

$$P(F\cap G)=1/4 \text{ and } P(F)P(G)=1/2\cdot 1/4=1/8.$$

Since 3/8 ≠ 1/4, we have $P(F\cap G)\neq P(F)P(G)$.

Events F and G are not independent.

c. E and G will be independent if $P(E\cap G)=P(E)P(G)$

$$P(E\cap G)=1/8 \text{ and } P(E)P(G)=1/2\cdot 1/4=1/8$$

Events E and G are independent events because $P(E\cap G)=P(E)P(G)$

Example 11.5.6

The probability that Jaime will visit his aunt in Baltimore this year is .30, and the probability that he will go river rafting on the Colorado river is .50. If the two events are independent, what is the probability that Jaime will do both?

Solution

Let A be the event that Jaime will visit his aunt this year, and R be the event that he will go river rafting.

We are given $P(A)$ = .30 and $P(R)$ = .50, and we want to find $P(A\cap R)$.

Since we are told that the events A and R are independent,

$$P(A\cap R)=P(A)P(R)=(.30)(.50)=.15$$

Example 11.5.7

Given $P(B|A)=.4$. If A and B are independent, find $P(B)$.

Solution

If A and B are independent, then by definition $P(B|A)=P(B)$

Therefore, $P(B)=.4$

Example 11.5.8

Given $P(A)=.7$, $P(B|A)=.5$. Find $P(A\cap B)$.

Solution 1

By definition $P(B|A)=\frac{P(A\cap B)}{P(A)}$

Substituting, we have

$$.5=\frac{\mathrm{P}(\mathrm{A} \cap \mathrm{B})}{.7}$$

Therefore, $P(A \cap B)=.35$

Solution 2

Again, start with $P(B|A)=\frac{P(A \cap B)}{P(A)}$

Multiplying both sides by $P(A)$ gives

$$P(A \cap B)=P(B|A) P(A)=(.5)(.7)=.35$$

Both solutions to Example 11.5.8 are actually the same, except that in Solution 2 we delayed substituting the values into the equation until after we solved the equation for $P(A \cap B)$. That gives the following result:

Multiplication Rule for events that are NOT independent

If events E and F are not independent

$$\mathbf{P(E \cap F)=P(E|F) P(F)} \quad \text{and} \quad \mathbf{P(E \cap F)=P(F|E) P(E)}$$

Example 11.5.9

Given $P(A)=.5$, $P(A \cup B)=.7$, if A and B are independent, find $P(B)$.

Solution

The addition rule states that

$$\mathrm{P}(\mathrm{A} \cup \mathrm{B})=\mathrm{P}(\mathrm{A})+\mathrm{P}(\mathrm{B})-\mathrm{P}(\mathrm{A} \cap \mathrm{B})$$

Since A and B are independent, $P(A \cap B)=P(A) P(B)$

We substitute for $P(A \cap B)$ in the addition formula and get

$$\mathrm{P}(\mathrm{A} \cup \mathrm{B})=\mathrm{P}(\mathrm{A})+\mathrm{P}(\mathrm{B})-\mathrm{P}(\mathrm{A}) \mathrm{P}(\mathrm{B})$$

By letting $P(B)=x$, and substituting values, we get

$$\begin{aligned} .7&=.5+x-.5x \\ .7&=.5+.5x \\ .2&=.5x \\ .4&=x \end{aligned}$$

Therefore, $P(B)=.4$
