6.2: Annuities and Sinking Funds

Learning Objectives

In this section, you will learn to:

1. Find the future value of an annuity.
2. Find the amount of payments to a sinking fund.

Ordinary Annuity

In the first two sections of this chapter, we examined problems where money was deposited in a lump sum in an account and left there for the entire period. Now, we will solve problems where timely payments are made in an account. When a sequence of payments of some fixed amount is made in an account at equal intervals, we call that an **annuity**. And this is the subject of this section.

To develop a formula to find the value of an annuity, we will need to recall the formula for the sum of a geometric series. A geometric series is of the form:

$$a+ax+ax^2+ax^3+\ldots+ax^n. \tag{6.2.1}$$

In a geometric series, each subsequent term is obtained by multiplying the preceding term by a number, called the **common ratio**. A geometric series is completely determined by knowing its first term, the common ratio, and the number of terms.

The first term of the series in Equation 6.2.1 is a, the common ratio is x, and the number of terms is $n+1$. The following are some examples of geometric series.

$$3+6+12+24+48$$

This above series has first term $a=3$ and common ratio $x=2$

$$2+6+18+54+162$$

This above series has first term $a=2$ and common ratio $x=3$

$$37+3.7+0.37+0.037+0.0037$$

This above series has first term $a=35$ and common ratio $x=0.1$

In your algebra class, you developed a formula for finding the sum of a geometric series. You probably used r as the symbol for the ratio, but we are using x because r is the symbol we have been using for the interest rate. The formula for the sum of the terms of a geometric series with first term a, common ratio x and last term ax^n is:

$$\frac{a\left(x^{n+1}-1\right)}{x-1}$$

We will use this formula to find the value of an annuity. Consider the following example.

Example 6.2.1

If at the end of each month a deposit of $500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

Solution

There are 60 deposits made in this account. The first payment stays in the account for 59 months, the second payment for 58 months, the third for 57 months, and so on.

- The first payment of $500 will accumulate to an amount of $\$500(1 + 0.08/12)^{59}$.
- The second payment of $500 will accumulate to an amount of $\$500(1 + 0.08/12)^{58}$.
- The third payment will accumulate to $\$500(1 + 0.08/12)^{57}$.
- The fourth payment will accumulate to $\$500(1 + 0.08/12)^{56}$.

And so on . . .

Finally the next to last (59^{th}) payment will accumulate to $\$500(1+0.08/12)^1$.

The last payment is taken out at the same time it is made and will not earn any interest.

To find the total amount in five years, we need to add the accumulated value of these sixty payments.

In other words, we must find the sum of the following series.

$$\$500(1+0.08/12)^{59}+\$500(1+0.08/12)^{58}+\$500(1+0.08/12)^{57}+\ldots+\$500$$

Written backward, we have

$$\$500+\$500(1+0.08/12)+\$500(1+0.08/12)^2+\ldots+\$500(1+0.08/12)^{59}$$

This is a geometric series with $a=\$500$, $r=(1+0.08/12$, and $n=59$. The sum is

$$\begin{aligned} \text{sum} &= \frac{\$500\left[(1+0.08/12)^{60}-1\right]}{0.08/12} \\ &= \$500(73.47686) \\ &= \$36,738.43 \end{aligned}$$

When the payments are made at the end of each period rather than the beginning, we call it an **ordinary annuity.**

Future Value of an Ordinary Annuity

If a payment of PMT dollars is made in account m times a year at an interest r, then the final amount FV after t years is

$$\mathbf{FV}=\frac{\mathbf{PMT}\left[(\mathbf{1}+\mathbf{r}/\mathbf{m})^{\mathbf{mt}}-\mathbf{1}\right]}{\mathbf{r}/\mathbf{m}}$$

The future value is also called the accumulated value

Example 6.2.2

Tanya deposits \$300 at the end of each quarter in her savings account. If the account earns 5.75% compounded quarterly, how much money will she have in 4 years?

Solution

The future value of this annuity can be found using the above formula.

$$\begin{aligned} FV &= \frac{\$300\left[(1+.0575/4)^{16}-1\right]}{0.0575/4} \\ &= \$300(17.8463) \\ &= \$5353.89 \end{aligned}$$

If Tanya deposits \$300 into a savings account earning 5.75% compounded quarterly for four years, then at the end of 4 years, she will have \$5,353.89

Example 6.2.3

Robert needs \$5,000 in three years. How much should he deposit monthly in an account that pays 8% compounded monthly to achieve his goal?

Solution

If Robert saves PMT dollars per month, after three years, he will have

$$\frac{PMT\left[(1+.08/12)^{36}-1\right]}{0.08/12}$$

But we'd like this amount to be $5,000. Therefore,

$$\begin{aligned}\frac{\text{PMT}\left[(1+.08/12)^{36}-1\right]}{.08/12} &= \$5000\\ \text{PMT}(40.5356) &= \$5000\\ \text{PMT} &= \frac{5000}{40.5356}\\ &= \$123.35\end{aligned}$$

Robert needs to deposit $123.35 at the end of each month for three years into an account paying 8% compounded monthly to have $5,000 at the end of 5 years.

Sinking Fund

When a business deposits money at regular intervals into an account to save for a future purchase of equipment, the savings fund is referred to as a "**sinking fund**." Calculating the sinking fund deposit uses the same method as the previous problem.

Example 6.2.4

A business needs $450,000 in five years. How much should be deposited each quarter in a sinking fund that earns 9% compounded quarterly to have this amount in five years?

Solution

Again, suppose that PMT dollars are deposited each quarter in the sinking fund. After five years, the future value of the fund should be $450,000. This suggests the following relationship:

$$\begin{aligned}\frac{\text{PMT}\left[(1+0.09/4)^{20}-1\right]}{0.09/4} &= \$450,000\\ \text{PMT}(24.9115) &= 450,000\\ \text{PMT} &= \frac{450000}{24.9115}\\ &= \$18,063.93\end{aligned}$$

The business needs to deposit $18,063.93 at the end of each quarter for five years into a sinking fund earning interest of 9% compounded quarterly to have $450,000 at the end of 5 years.

Annuity Due

If the payment is made at the beginning of each period rather than at the end, we call it an **annuity due**. The formula for the annuity due can be derived similarly. Reconsider Example 1, with the change that the deposits are made at the beginning of each month.

Example 6.2.5

If at the beginning of each month, a deposit of $500 is made in an account that pays 8% compounded monthly, what will the final amount be after five years?

Solution

There are 60 deposits made in this account. The first payment stays in the account for 60 months, the second payment for 59 months, the third for 58 months, and so on.

- The first payment of $500 will accumulate to an amount of $\$500(1+0.08/12)^{60}$.
- The second payment of $500 will accumulate to an amount of $\$500(1+.08/12)^{59}$.

- The third payment will accumulate to $\$500(1+0.08/12)^{58}$.

And so on . . .

The last payment is in the account for a month and accumulates to $\$500(1+0.08/12)$

To find the total amount in five years, we need to find the sum of the series:

$$\$500(1+0.08/12)^{60}+\$500(1+0.08/12)^{59}+\$500(1+0.08/12)^{58}+\ldots+\$500(1+0.08/12)$$

Written backward, we have

$$\$500(1+0.08/12)+\$500(1+0.08/12)^{2}+\ldots+\$500(1+0.08/12)^{60}$$

If we add \$500 to this series and later subtract that \$500, the value will not change. We get

$$\mathbf{\$500}+\$500(1+0.08/12)+\$500(1+0.08/12)^{2}+\ldots+\$500(1+0.08/12)^{60}-\mathbf{\$500}$$

Except for the last term, we have a geometric series with a = \$500, r = (1 + .08/12), and n = 60. Therefore, the sum is

$$\begin{aligned}\mathrm{A} &= \frac{\$500\left[(1+0.08/12)^{61}-1\right]}{0.08/12}-\$500\\ &= \$500(74.9667)-\$500\\ &= \$37483.35-\$500\\ &= \$36983.35\end{aligned}$$

So, in the case of an annuity due, to find the future value, we increase the number of periods n by one and subtract one payment.

Future Value of an "Annuity Due"

$$\mathrm{FV}=\frac{\mathrm{PMT}\left[(1+\mathrm{r}/\mathrm{m})^{\mathrm{mt}+1}-1\right]}{\mathrm{r}/\mathrm{m}}-\mathrm{PMT}$$

Summary

Finally, the author wishes that the students learn the concepts so they will not have to memorize every formula. It is for this reason formulas are kept at a minimum. But before we conclude this section, we will once again mention one single equation that will help us find the future value and the sinking fund payment.

If a payment of PMT dollars is made in an account m times a year at an interest r, then the future value FV after t years is

$$\mathbf{FV}=\frac{\mathbf{PMT}\left[(\mathbf{1}+\mathbf{r}/\mathbf{m})^{\mathbf{mt}}-\mathbf{1}\right]}{\mathbf{r}/\mathbf{m}}$$

Note that the formula assumes that the payment period is the same as the compounding period. If these are not the same, then this formula does not apply.