10.7: Binomial Theorem

We end this chapter with one more application of combinations. Combinations are used in determining the coefficients of a binomial expansion such as $(x+y)^n$. Expanding a binomial expression by multiplying it out is a very tedious task, and is not practiced. Instead, a formula known as the Binomial Theorem is utilized to determine such an expansion. Before we introduce the Binomial Theorem, however, consider the following expansions.

$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^2y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$
$$(x+y)^6 = x6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

We make the following observations.

1. There are $n+1$ terms in the expansion $(x+y)^n$
2. The sum of the powers of x and y is n.
3. The powers of x begin with n and decrease by one with each successive term.
   The powers of y begin with 0 and increase by one with each successive term.

Suppose we want to expand $(x+y)^3$. We first write the expansion without the coefficients. We temporarily substitute a blank in place of the coefficients.

$$(x+y)^3 = \square x^3+\square x^2y+\square xy^2+\square y^3 \tag{10.7.1}$$

Our next job is to replace each of the blanks in equation (10.7.1) with the corresponding coefficients that belong to this expansion. Clearly,

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

If we multiply the right side and do not collect terms, we get the following.

$$xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy$$

Each product in the above expansion is the result of multiplying three variables by picking one from each of the factors $(x+y)(x+y)(x+y)$. For example, the product xxy is gotten by choosing x from the first factor, x from the second factor, and y from the third factor. There are three such products that simplify to x^2y, namely xxy, xyx, and yxx. These products take place when we choose an x from two of the factors and choose a y from the other factor. Clearly this can be done in 3C2, or 3 ways. Therefore, the coefficient of the term x^2y is 3. The coefficients of the other terms are obtained in a similar manner.

We now replace the blanks with the coefficients in equation (10.7.1), and we get

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

Example 10.7.1

Find the coefficient of the term x^2y^5 in the expansion $(x+y)^7$.

Solution

The expansion $(x+y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$

In multiplying the right side, each product is gotten by picking an x or y from each of the seven factors $(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$.

The term x^2y^5 is obtained by choosing an x from two of the factors and a y from the other five factors. This can be done in 7C2, or 21 ways.

Therefore, the coefficient of the term x^2y^5 is 21.

✓ Example 10.7.2

Expand $(x+y)^7$

Solution

We first write the expansion without the coefficients.

$$(x+y)^7=\square x^7+\square x^6y+\square x^5y^2+\square x^4y^3+\square x^3y^4+\square x^2y^5+\square xy^6+\square y^7$$

Now we determine the coefficient of each term as we did in Example 10.7.1.

The coefficient of the term x^7 is 7C7 or 7CO which equals 1.

The coefficient of the term x^6y is 7C6 or 7C1 which equals 7.

The coefficient of the term x^5y^2 is 7C5 or 7C2 which equals 21.

The coefficient of the term x^4y^3 is 7C4 or 7C3 which equals 35,

and so on.

Substituting, we get: $(x+y)^7=x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$

We generalize the result.

Binomial Theorem

$$(x+y)^n={}_nC_0x^n+{}_nC_1x^{n-1}y+{}_nC_2x^{n-2}y^2+\cdots+{}_nC_{n-1}xy_n^{n-1+}C_ny^n$$

✓ Example 10.7.3

Expand $(3a-2b)^4$

Solution

If we let $x=3a$ and $y=-2b$, and apply the Binomial Theorem, we get

$$\begin{aligned}(3a-2b)^4 &= 4\,\mathrm{Co}(3a)^4+4Cl(3a)^3(-2b)+4C2(3a)^2(-2b)^2+4C3(3a)(-2b)^3+4C4(-2b)^4\\ &=1\left(81a^4\right)+4\left(27a^3\right)(-2b)+6\left(9a^2\right)\left(4b^2\right)+4(3a)\left(-8b^3\right)+1\left(16b^3\right)\\ &=81a^4-216a^3b+216a^2b^2-96ab^3+16b^4\end{aligned}$$

✓ Example 10.7.4

Find the fifth term of the expansion $(3a-2b)^7$.

Solution

The Binomial theorem tells us that in the r-th term of an expansion, the exponent of the y term is always one less than r, and, the coefficient of the term is ${}_n\mathrm{C}_{\mathrm{r}-1}$.

$n=7$ and $r-1=5-1=4$, so the coefficient is $7C4=35$

Thus, the fifth term is $(7\mathrm{C}4)(3\mathrm{a})^3(-2\mathrm{b})^4=35\left(27\mathrm{a}^3\right)\left(16\mathrm{b}^4\right)=15120\mathrm{a}^3\mathrm{b}^4$