4.4: Logarithms and Logarithmic Functions

Learning Objectives

In this section you will learn

1. the definition of logarithmic function as the inverse of the exponential function
2. to write equivalent logarithmic and exponential expressions
3. the definition of common log and natural log
4. properties of logs
5. to evaluate logs using the change of base formula

The Logarithm

Suppose that a population of 50 flies is expected to double every week, leading to a function of the form $f(x) = 50(2)^x$, where x represents the number of weeks that have passed. When will this population reach 500?

Trying to solve this problem leads to

$$500 = 50(2)^x$$

Dividing both sides by 50 to isolate the exponential leads to

$$10 = 2^x.$$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$ above. We know that $2^3 = 8$ and $2^4 = 16$, so it is clear that x must be some value between 3 and 4 since $g(x) = 2^x$ is increasing. We could use technology to create a table of values or graph to better estimate the solution, but we would like to find an algebraic way to solve the equation.

We need an inverse operation to exponentiation in order to solve for the variable if the variable is in the exponent. As we learned in algebra class (prerequisite to this finite math course), the inverse function for an exponential function is a logarithmic function.

We also learned that an exponential function has an inverse function, because each output (y) value corresponds to only one input (x) value. The name given this property was "one-to-one".

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Logarithm

The logarithm (base b) function, written $\log_b(x)$, is the inverse of the exponential function (base b), b^x.

$$\mathbf{y = \log_b(x) \quad \text{is equivalent to} \quad b^y = x}$$

In general, the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Note: The base b must be positive: $b > 0$

Inverse Property of Logarithms

Since the logarithm and exponential are inverses, it follows that:

$$\log_b(b^x) \quad \text{and } b^{\log_b(x)} = x$$

Since log is a function, it is most correctly written as $\log_b(c)$, using parentheses to denote function evaluation, just as we would with $f(c)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_b c$.

Example 4.4.1

Write these exponential equations as logarithmic equations:

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-3} = \frac{1}{1000}$

Solution

a. $2^3 = 8$ can be written as a logarithmic equation as $\log_2(8) = 3$

b. $5^2 = 25$ can be written as a logarithmic equation as $\log_5(25) = 2$

c. $10^{-3} = \frac{1}{1000}$ can be written as a logarithmic equation as $\log_{10}\left(\frac{1}{1000}\right) = -3$

Example 4.4.2

Write these logarithmic equations as exponential equations:

a. $\log_6(\sqrt{6}) = \frac{1}{2}$
b. $\log_3(9) = 2$

Solution

a. $\log_6(\sqrt{6}) = \frac{1}{2}$ can be written as an exponential equation as $6^{\frac{1}{2}} = \sqrt{6}$
b. $\log_3(9) = 2$ can be written as an exponential equation as $3^2 = 9$

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

Example 4.4.3

Solve $\log_4(x) = 2$ for x.

Solution

By rewriting this expression as an exponential, $4^2 = x$, so $x = 16$

Example 4.4.4

Solve $2^x = 10$ for x.

Solution

By rewriting this expression as a logarithm, we get $x = \log_2(10)$

While this does define a solution, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful—often we really need a decimal approximation to the solution. Luckily, this is a task that calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases: base 10 and base e. Happily, this ends up not being a problem, as we'll see soon that we can use a "change of base" formula to evaluate logarithms for other bases.

Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written $\log(x)$ and sometimes like $\log_{10}(x)$. If the base is not indicated in the log function, then the base b used is $b = 10$.

The **natural log** is the logarithm with base e, and is typically written $\ln(x)$.

Note that for any other base b, other than 10, the base must be indicated in the notation $\log_b(x)$.

Example 4.4.5

Evaluate $\log(1000)$using the definition of the common log.

Solution

The table shows values of the common log

number	number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

To evaluate log(1000), we can say

$$x=\log(1000)$$

Then rewrite the equation in exponential form using the common log base of 10

$$10^x=1000$$

From this, we might recognize that 1000 is the cube of 10, so

$$x=3$$

Alternatively, we can use the inverse property of logs to write

$$\log_{10}(10^3)=3$$

Example 4.4.6

Evaluate $\log\left(\frac{1}{1,000,000}\right)$

Solution

To evaluate log(1/1,000,000), we can say

$$x=\log(1/1,000,000)=\log(1/10^6)=\log(10^{-6})$$

Then rewrite the equation in exponential form: $10^x=10^{-6}$

Therefore $x=-6$

Alternatively, we can use the inverse property of logs to find the answer:

$$\log_{10}(10^{-6})=-6$$

Example 4.4.7

Evaluate

a. $\ln e^5$
b. $\ln\sqrt{e}$

Solution

a. To evaluate $\ln e^5$, we can say

$$x=\ln e^5$$

Then rewrite into exponential form using the natural log base of e

$$e^x=e^5$$

Therefore $x=5$.

Alternatively, we can use the inverse property of logs to write $\ln\left(e^5\right)=5$.

b. To evaluate $\ln\sqrt{e}$, we recall that roots are represented by fractional exponents

$$\mathrm{x}=\ln\sqrt{e}=\ln(\sqrt{e})=\ln\left(e^{1/2}\right)$$

Then rewrite into exponential form using the natural log base of e

$$\mathrm{e}^x=\mathrm{e}^{1/2}$$

Therefore $x=1/2$

Alternatively, we can use the inverse property of logs to write

$$(\ln\left(\mathrm{e}^{1/2}\right)=1/2$$

Example 4.4.8

Evaluate the following using your calculator or computer:

a. $\log 500$
b. $\ln 500$

Solution

a. Using the LOG key on the calculator to evaluate logarithms in base 10, we evaluate LOG(500)

Answer: $\log 500\approx 2.69897$

b. Using the LN key on the calculator to evaluate natural logarithms, we evaluate LN(500)

Answer: $\ln 500\approx 6.214608$

Exercise 4.4.1

Convert $8^x = 272$ to an logarithmic function.

Answer

$\log_8(272) = x$

Exercise 4.4.2

Convert from $\log_e(456) = x$ to an exponential function.

Answer

$e^x = 456$

Some Properties of Logarithms

We often need to evaluate logarithms using a base other than 10 or e. To find a way to utilize the common or natural logarithm functions to evaluate expressions like $\log_2(10)$, we need some additional properties.

Properties of logs: Exponential Property

$$\log_b(A^q) = q\log_b(A)$$

The exponent property allows us to find a method for changing the base of a logarithmic expression.

Properties of Logs: Change of Base

$$\log_b(A) = \frac{\log_c(A)}{\log_c(b)} \text{ for any bases } b, c > 0$$

To show why these properties are true, we offer proofs.

Proof of Exponent Property: $\log_b(A^q) = q\log_b(A)$

Since the logarithmic and exponential functions are inverses,

$$\log_b(A^q) = \mathrm{A}$$

So

$$A^q = \left(b^{\log_b A}\right)^q$$

Utilizing the exponential rule that states $(x^p)^q = x^{pq}$, we get

$$A^q = \left(b^{\log_b A}\right)^q = b^{q\log_b A}$$

Then

$$\log_b A^q = \log_b b^{q\log_b A}$$

Again utilizing the inverse property on the right side yields the result

$$\log_b A^q = q\log_b A$$

Proof of Change of Base Property: $\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$ for any bases b, $c > 0$

Let $\log_b(A) = x$.

Rewriting as an exponential gives $b^x = A$.

Taking the log base c of both sides of this equation gives $\log_c b^x = \log_c A$.

Now utilizing the exponent property for logs on the left side,

$$x\log_c b = \log_c A$$

Dividing, we obtain $x = \frac{\log_c(A)}{\log_c(b)}$ which is the change of base formula.

Evaluating Logarithms

With the change of base formula, $\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$ for any bases b, $c > 0$, we can finally find a decimal approximation to our question from the beginning of the section.

Example 4.4.9

Solve $2^x = 10$ for x.

Solution

Rewrite exponential equation $2^x = 10$ as a logarithmic equation

$$x = \log_2(10)$$

Using the change of base formula, we can rewrite log base 2 as a logarithm of any other base. Since our calculators can evaluate natural log, we can choose to use the natural logarithm, which is the log base *e*:

Using our calculators to evaluate this, $\frac{\ln(10)}{\ln(2)} = \mathrm{LN}(10)/\mathrm{LN}(2) \approx 3.3219$

This finally allows us to answer our original question from the beginning of this section:
For the population of 50 flies that doubles every week, it will take approximately 3.32 weeks to grow to 500 flies.

Example 4.4.10

Evaluate $\log_5(100)$ using the change of base formula.

Solution

We can rewrite this expression using any other base.

Method 1: We can use natural logarithm base *e* with the change of base formula

$$\log_5(100) = \frac{\ln(100)}{\ln(5)} = \mathrm{LN}(100)/\mathrm{LN}(5) \approx 2.861$$

Method 2: We can use common logarithm base 10 with the change of base formula,

$$\log_5(100) = \frac{\log(100)}{\log(5)} = \mathrm{LOG}(100)/\mathrm{LOG}(5) \approx 2.861$$

We summarize the relationship between exponential and logarithmic functions

Logarithms

The logarithm (base *b*) function, written $\log_b(x)$, is the inverse of the exponential function (base *b*), b^x.

$$\mathbf{y = \log_b(x) \quad \text{is equivalent to} \quad b^y = x}$$

In general, the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Note: The base b must be positive: b>0

Inverse Property of Logarithms

Since the logarithm and exponential are inverses, it follows that:

$$\log_b(b^x) \quad \text{and } b^{\log_b(x)} = x$$

Properties of Logs: Exponential Property: $\log_b(A^q) = q\log_b(A)$

Properties of Logs: Change of Base: $\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$ for any base $b, c > 0$

The inverse, exponential and change of base properties above will allow us to solve the equations that arise in problems we encounter in this textbook. For completeness, we state a few more properties of logarithms

Sum of Logs Property: $\log_b(A) + \log_b(C) = \log_b(AC)$

Difference of Logs Property: $\log_b(A) - \log_b(C) = \log_b\left(\frac{A}{C}\right)$

Logs of Reciprocals: $\log_b\left(\frac{1}{C}\right) = -\log_b(C)$

Reciprocal Bases: $\log_{1/b} C = -\log_b(C)$

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