6.1: Simple and Compound Interest

Learning Objectives

In this section, you will learn to:

1. Find simple interest.
2. Find the present value for simple interest.
3. Find discounts and proceeds for simple interest.
4. Find the future value of a lump-sum using compound interest.
5. Find the present value of a lump-sum using compound interest.
6. Find the effective interest rate using compound interest.

Simple Interest

It costs to borrow money. The rent one pays for the use of money is called the **interest**. The amount of money that is being borrowed or loaned is called the **principle** or **present value**. Simple interest is paid only on the original amount borrowed. When the money is loaned out, the person who borrows the money generally pays a fixed rate of interest on the principle for the time period he keeps the money. Although the interest rate is often specified for a year, it may be specified for a week, a month, or a quarter, etc. The credit card companies often list their charges as monthly rates, sometimes it is as high as 2.75% a month.

Definition: Simple Interest

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$I = P \cdot r \cdot t$$

Definition: Accumulated Value

The total amount FV, also called the **accumulated value** or the future value, is given by

$$\begin{aligned} FV &= P + I \\ &= P + Prt \end{aligned}$$

or

$$FV = P(1 + rt) \tag{6.1.1}$$

where interest rate r is expressed in decimals.

Example 6.1.1

Ursula borrows \$600 for 5 months at a simple interest rate of 15% per year. Find the interest, and the total amount she is obligated to pay?

Solution

The interest is computed by multiplying the principle with the interest rate and the time.

$$\begin{aligned} \mathrm{I} &= \mathrm{Prt} \\ &= \$600(0.15)\frac{5}{12} \\ &= \$37.50 \end{aligned}$$

The total amount is

$$\begin{aligned} \mathrm{FV} &= \mathrm{P}+\mathrm{I} \\ &= \$600+\$37.50 \\ &= \$637.50 \end{aligned}$$

Incidentally, the total amount can be computed directly via Equation 6.1.1 as

$$\begin{aligned} FV &= P(1+rt) \\ &= \$600[1+(0.15)(5/12)] \\ &= \$600(1+0.0625) \\ &= \$637.50 \end{aligned}$$

✓ Example 6.1.2

Jose deposited \$2500 in an account that pays 6% simple interest. How much money will he have at the end of 3 years?

Solution

The total amount or the future value is given by Equation 6.1.1.

$$\begin{aligned} FV &= P(1+rt) \\ &= \$2500[1+(.06)(3)] \\ \mathrm{FV} &= \$2950 \end{aligned}$$

✓ Example 6.1.3

Darnel owes a total of \$3060 which includes 12% interest for the three years he borrowed the money. How much did he originally borrow?

Solution

This time we are asked to compute the principle P via Equation 6.1.1.

$$\begin{aligned} \$3060 &= \mathrm{P}[1+(0.12)(3)] \\ \$3060 &= \mathrm{P}(1.36) \\ \frac{\$3060}{1.36} &= \mathrm{P} \\ \$2250 &= \mathrm{P} \quad \text{Darnel originally borrowed } \$2250. \end{aligned}$$

✓ Example 6.1.4

A Visa credit card company charges a 1.5% finance charge for each month on the unpaid balance. If Martha owed \$2350 and has not paid her bill for three months, how much does she owe now?

Solution

Before we attempt the problem, the reader should note that in this problem the rate of finance charge is given per month and not per year.

The total amount Martha owes is the previous unpaid balance plus the finance charge.

$$FV=\$2350+\$2350(.015)(3)=\$2350+\$105.75=\$2455.75$$

Alternatively, again, we can compute the amount directly by using formula $FV=P(1+rt)$

$$FV=\$2350[1+(.015)(3)]=\$2350(1.045)=\$2455.75$$

Discount and Proceeds

Banks often deduct the simple interest from the loan amount at the time that the loan is made. When this happens, we say the loan has been **discounted.** The interest that is deducted is called the **discount**, and the actual amount that is given to the borrower is called the **proceeds**. The amount the borrower is obligated to repay is called the **maturity value.**

Discount and Proceeds

If an amount M is borrowed for a time t at a discount rate of r per year, then the discount D is

$$D = M \cdot r \cdot t$$

The proceeds P, the actual amount the borrower gets, is given by

$$\begin{aligned} P &= M - D \\ &= M - Mrt \end{aligned}$$

or

$$P = M(1 - rt)$$

where interest rate r is expressed in decimals.

Example 6.1.5

Francisco borrows \$1200 for 10 months at a simple interest rate of 15% per year. Determine the discount and the proceeds.

Solution

The discount D is the interest on the loan that the bank deducts from the loan amount.

$$\begin{array}{l} \mathrm{D} = \mathrm{Mrt} \\ \mathrm{D} = \$1200(0.15)\left(\frac{10}{12}\right) = \$150 \end{array}$$

Therefore, the bank deducts \$150 from the maturity value of \$1200, and gives Francisco \$1050. Francisco is obligated to repay the bank \$1200.

In this case, the discount D = \$150, and the proceeds

$$P = \$1200 - \$150 = \$1050.$$

Example 6.1.6

If Francisco wants to receive \$1200 for 10 months at a simple interest rate of 15% per year, what amount of loan should he apply for?

Solution

In this problem, we are given the proceeds P and are being asked to find the maturity value M.

We have P = \$1200, $r = 0.15$, t = 10/12 . We need to find M.

We know $P = M - D$

but also $D = Mrt$

therefore

$$\begin{aligned} \mathrm{P} &= \mathrm{M} - \mathrm{Mrt} \\ &= \mathrm{M}(1 - \mathrm{rt}) \\ \$1200 &= \mathrm{M}\left[1 - (0.15)\left(\frac{10}{12}\right)\right] \end{aligned}$$

We need to solve for M.

$$\begin{aligned}\$1200 &= \mathrm{M}(1-0.125)\\ \$1200 &= \mathrm{M}(0.875)\\ \frac{\$1200}{0.875} &= \mathrm{M}\\ \$1371.43 &= \mathrm{M}\end{aligned}$$

Therefore, Francisco should ask for a loan for \$1371.43.

The bank will discount \$171.43 and Francisco will receive \$1200.

Simple Interest Summary

Below is a summary of the formulas we developed for calculations involving simple interest:

Simple interest

If an amount P is borrowed for a time t at an interest rate of r per time period, then the simple interest is given by

$$I = P \cdot r \cdot t$$

The total amount A, also called the accumulated value or the future value, is given by

$$FV = P + I = P + Prt$$

or

$$FV = P(1 + rt)$$

where interest rate r is expressed in decimals.

Discount and Proceeds

If an amount M is borrowed for a time t at a discount rate of r per year, then the discount D is

$$D = M \cdot r \cdot t$$

The proceeds P, the actual amount the borrower gets, is given by

$$\begin{gathered}P = M - D\\ P = M - Mrt\end{gathered}$$

or

$$P = M(1 - rt)$$

where interest rate r is expressed in decimals.

At the end of the loan's term, the borrower repays the entire maturity amount M.

Compound Interest

In the last section, we examined problems involving simple interest. Simple interest is generally charged when the lending period is short and often less than a year. When the money is loaned or borrowed for a longer time period, if the interest is paid (or charged) not only on the principle, but also on the past interest, then we say the interest is **compounded.**

Suppose we deposit \$200 in an account that pays 8% interest. At the end of one year, we will have \$200 + \$200(.08) = \$200(1 + .08) = \$216.

Now, suppose we put this amount, \$216, in the same account. After another year, we will have \$216 + \$216(.08) = \$216(1 + .08) = \$233.28.

So, an initial deposit of \$200 has accumulated to \$233.28 in two years. Further note that had it been simple interest, this amount would have accumulated to only \$232. The reason the amount is slightly higher is because the interest (\$16) we earned the first year, was put back into the account. And this \$16 amount itself earned for one year interest of \$16(.08) = \$1.28, thus resulting in the increase. So, we have earned interest on the principle as well as on the past interest, and that is why we call it compound interest.

Now suppose we leave this amount, \$233.28, in the bank for another year, the final amount will be \$233.28 + \$233.28(.08) = \$233.28(1 + .08) = \$251.94.

Now, let us look at the mathematical part of this problem to devise an easier way to solve these problems.

After one year, we had \$200(1 + .08) = \$216

After two years, we had \$216(1 + .08)

But \$216 = \$200(1 + .08), therefore, the above expression becomes

$$\$200(1+.08)(1+.08)=\$200(1+.08)^2=\$233.28$$

After three years, we get

$$\$233.28(1+.08)=\$200(1+.08)(1+.08)(1+.08)$$

which can be written as

$$\$200(1+.08)^3=\$251.94$$

Suppose we are asked to find the total amount at the end of 5 years, we will get

$$200(1+.08)^5=\$293.87$$

We summarize as follows:

The original amount	\$200	= \$200
The amount after one year	\$200(1 + .08)	= \$216
The amount after two years	$\$200(1+.08)^2$	= \$233.28
The amount after three years	$\$200(1+.08)^3$	= \$251.94
The amount after five years	$\$200(1+.08)^5$	= \$293.87
The amount after t years	$\$200(1+.08)^t$	

COMPOUNDING PERIODS

Banks often compound interest more than one time a year. Consider a bank that pays 8% interest but compounds it four times a year, or quarterly. This means that every quarter, the bank will pay an interest equal to one-fourth of 8%, or 2%.

Now if we deposit \$200 in the bank, after one quarter we will have $\$200\left(1+\frac{.08}{4}\right)$ or \$204.

After two quarters, we will have $\$200\left(1+\frac{.08}{4}\right)^2$ or \$208.08.

After one year, we will have $\$200\left(1+\frac{.08}{4}\right)^4$ or \$216.49.

After three years, we will have $\$200\left(1+\frac{.08}{4}\right)^{12}$ or \$253.65, etc.

The original amount	\$200	= \$200
The amount after one quarter	$\$200\left(1+\frac{.08}{4}\right)$	= \$204

The amount after two quarters	$\$200\left(1+\frac{.08}{4}\right)^{2}$	= \$208.08
The amount after one year	$\$200\left(1+\frac{.08}{4}\right)^{4}$	= \$216.49
The amount after two years	$\$200\left(1+\frac{.08}{4}\right)^{8}$	= \$234.31
The amount after three years	$\$200\left(1+\frac{.08}{4}\right)^{12}$	= \$253.65
The amount after five years	$\$200\left(1+\frac{.08}{4}\right)^{20}$	= \$297.19
The amount after t years	$\$200\left(1+\frac{.08}{4}\right)^{4t}$	

Therefore, if we invest a lump-sum amount of P dollars at an interest rate r, compounded n times a year, then after t years the final amount is given by

$$FV=P\left(1+\frac{r}{m}\right)^{mt}$$

The following examples use the compound interest formula $FV=P\left(1+\frac{r}{m}\right)^{mt}$

Example 6.1.7

If \$3500 is invested at 9% compounded monthly, what will the future value be in four years?

Solution

Clearly an interest of .09/12 is paid every month for four years. The interest is compounded $4\times 12=48$ times over the four-year period. We get

$$\mathrm{FV}=\$3500\left(1+\frac{.09}{12}\right)^{48}=\$3500(1.0075)^{48}=\$5009.92$$

\$3500 invested at 9% compounded monthly will accumulate to \$5009.92 in four years.

Example 6.1.8

How much should be invested in an account paying 9% compounded daily for it to accumulate to \$5,000 in five years?

Solution

We know the future value, but need to find the principle.

$$\begin{aligned}\$5000&=P\left(1+\tfrac{.09}{365}\right)^{365\times 5}\\ \$5000&=P(1.568225)\\ \$3188.32&=P\end{aligned}$$

\$3188,32 invested into an account paying 9% compounded daily will accumulate to \$5,000 in five years.

Example 6.1.9

If \$4,000 is invested at 4% compounded annually, how long will it take to accumulate to \$6,000?

Solution

$m=1$ because annual compounding means compounding only once per year. The formula simplifies to $FV=(1+r)^{t}$ when $m=1$.

$$
\begin{aligned}
\$6000 &= 4000(1+.04)^t \\
\frac{6000}{4000} &= 1.04^t \\
1.5 &= 1.04^t
\end{aligned}
$$

We use logarithms to solve for the value of t because the variable t is in the exponent.

$$t = \log_{1.04}(1.5)$$

Using the change of base formula we can solve for t:

$$t = \frac{\ln(1.5)}{\ln(1.04)} = 10.33 \text{ years}$$

It takes 10.33 years for $4000 to accumulate to $6000 if invested at 4% interest, compounded annually

Example 6.1.10

If $5,000 is invested now for 6 years what interest rate compounded quarterly is needed to obtain an accumulated value of $8000.

Solution

We have $m = 4$ for quarterly compounding.

$$
\begin{aligned}
\$8000 &= \$5000\left(1+\frac{r}{4}\right)^{4\times 6} \\
\frac{\$8000}{\$5000} &= \left(1+\frac{r}{4}\right)^{24} \\
1.6 &= \left(1+\frac{r}{4}\right)^{24}
\end{aligned}
$$

We use roots to solve for t because the variable r is in the base, whereas the exponent is a known number.

$$\sqrt[24]{1.6} = 1 + \frac{\mathrm{r}}{4}$$

Many calculators have a built in “nth root” key or function. In the TI-84 calculator, this is found in the Math menu. Roots can also be calculated as fractional exponents; if necessary, the previous step can be rewritten as

$$1.6^{1/24} = 1 + \frac{\mathrm{r}}{4}$$

Evaluating the left side of the equation gives

$$
\begin{aligned}
&1.0197765 = 1 + \tfrac{\mathrm{r}}{4} \\
&0.0197765 = \tfrac{\mathrm{r}}{4} \\
&\mathrm{r} = 4(0.0197765) = 0.0791
\end{aligned}
$$

An interest rate of (0.0791 * 100 = 7.91) 7.91% is needed in order for $5000 invested now to accumulate to $8000 at the end of 6 years, with interest compounded quarterly.

Effective Interest Rate

Banks are required to state their interest rate in terms of an **“effective yield”** ” or **“effective interest rate”**, for comparison purposes. The effective rate is also called the Annual Percentage Yield (APY) or Annual Percentage Rate (APR).

The effective rate is the interest rate compounded annually would be equivalent to the stated rate and compounding periods. The next example shows how to calculate the effective rate.

To examine several investments to see which has the best rate, we find and compare the effective rate for each investment.

Example $6.1.11$ illustrates how to calculate the effective rate.

Example 6.1.11

If Bank A pays 7.2% interest compounded monthly, what is the effective interest rate?
If Bank B pays 7.25% interest compounded semiannually, what is the effective interest rate? Which bank pays more interest?

Solution

Bank A: Suppose we deposit $1 in this bank and leave it for a year, we will get

$$1\left(1+\frac{0.072}{12}\right)^{12}=1.0744$$
$$\mathrm{r_{EFF}}=1.0744-1=0.0744$$

We earned interest of $1.0744 - $1.00 = $.0744 on an investment of $1.

The effective interest rate is 7.44%, often referred to as the APY or APR.

Bank B: The effective rate is calculated as

$$\mathbf{r}_{\mathrm{EFF}}=1\left(1+\frac{0.072}{2}\right)^{2}-1=.0738$$

The effective interest rate is 7.38%.

Bank A pays slightly higher interest, with an effective rate of 7.44%, compared to Bank B with effective rate 7.38%.

Continuous Compounding

Interest can be compounded yearly, semiannually, quarterly, monthly, and daily. Using the same calculation methods, we could compound every hour, every minute, and even every second. As the compounding period gets shorter and shorter, we move toward the concept of continuous compounding.

But what do we mean when we say the interest is compounded continuously, and how do we compute such amounts? When interest is compounded "infinitely many times", we say that the interest is **compounded continuously**. Our next objective is to derive a formula to model continuous compounding.

Suppose we put $1 in an account that pays 100% interest. If the interest is compounded once a year, the total amount after one year will be $\$1(1+1)=\$2$.

- If the interest is compounded semiannually, in one year we will have $\$1(1+1/2)^2=\$2.25$
- If the interest is compounded quarterly, in one year we will have $\$1(1+1/4)^4=\$2.44$
- If the interest is compounded monthly, in one year we will have $\$1(1+1/12)^{12}=\$2.61$
- If the interest is compounded daily, in one year we will have $\$1(1+1/365)^{365}=\$2.71$

We show the results as follows:

Frequency of compounding	Formula	Total amount
Annually	$\$1(1+1)$	$2
Semiannually	$\$1(1+1/2)^2$	$2.25
Quarterly	$\$1(1+1/4)^4=\$2.44$	$2.44140625
Monthly	$\$1(1+1/12)^{12}$	$2.61303529
Daily	$\$1(1+1/365)^{365}$	$2.71456748
Hourly	$\$1(1+1/8760)^{8760}$	$2.71812699
Every minute	$\$1(1+1/525600)^{525600}$	$2.71827922
Every Second	$\$1(1+1/31536000)^{31536000}$	$2.71828247
Continuously	$\$1(2.718281828\ldots)$	$2.718281828...

We have noticed that the \$1 we invested does not grow without bound. It starts to stabilize to an irrational number 2.718281828... given the name "*e*" after the great mathematician Euler.

In mathematics, we say that as m becomes infinitely large the expression equals $\left(1+\frac{1}{m}\right)^{m}$ *e*.

Therefore, it is natural that the number *e* play a part in continuous compounding.
It can be shown that as m becomes infinitely large the expression $\left(1+\frac{r}{m}\right)^{mt}=e^{rt}$

Therefore, it follows that if we invest $\$P$ at an interest rate r per year, compounded continuously, after t years the final amount will be given by

$$FV=P\cdot e^{rt}$$

Example \(\PageIndex{12}\)

\$3500 is invested at 9% compounded continuously. Find the future value in four years.

Solution

Using the formula for the continuous compounding, we get $FV=Pe^{rt}$.

$$\begin{aligned}FV&=\$3500e^{0.09\times 4}\\FV&=\$3500e^{0.36}\\FV&=\$5016.65\end{aligned}$$

Example 6.1.13

If an amount is invested at 7% compounded continuously, what is the effective interest rate?

Solution

If we deposit \$1 in the bank at 7% compounded continuously for one year, and subtract that \$1 from the final amount, we get the effective interest rate in decimals.

$$\begin{aligned}\mathrm{r_{EFF}}&=1\mathrm{e}^{0.07}-1\\\mathrm{r_{EFF}}&=1.0725-1\\\mathrm{r_{EFF}}&=.0725\text{ or }7.25\%\end{aligned}$$

Example 6.1.14

If an amount is invested at 7% compounded continuously, how long will it take to double?

We offer two solutions.

Solution 1 uses logarithms to calculate the exact answer, so it is preferred. We already used this method in Example 6.1.9 to solve for time needed for an investment to accumulate to a specified future value.

Solution 2 provides an estimated solution that is applicable only to doubling time, but not to other multiples. Students should find out from their instructor if there is a preference as to which solution method is to be used for doubling time problems.

Solution: Solution 1: **Calculating the answer exactly:** $Pe^{0.07t}=FV$.

We don't know the initial value of the priniciple but we do know that the accumulated value is double (twice) the principle.

$$\mathrm{P}_{e}^{0.07t}=2\mathrm{P}$$

We divide both sides by P

$$e^{.07t}=2$$

Using natural logarithm:

$$\begin{aligned}.07\mathrm{t}&=\ln(2)\\\mathrm{t}&=\ln(2)/.07=9.9\text{ years}\end{aligned}$$

It takes 9.9 years for money to double if invested at 7% continuous interest.

***Solution 2:* Estimating the answer using the Law of 72:**

The Law of 72 is a useful tool for estimating the time needed for an investment to double in value.

The Rule of 72

The **rule of 72** is a simple and often very useful mathematical shortcut that can help you estimate the impact **of** any interest or growth rate and can be used in situations ranging from financial calculations to projections **of** population growth. The formula for the **rule of 72** is expressed as the unknown (the required amount **of** time to double a value) calculated by taking the number **72** and dividing it by the known interest rate or growth rate. When using this formula, it is important to note that the rate should be expressed as a whole integer, not as a percentage. So, as a result, we have

Years for an Amount to Double=72/(Interest or Growth Rate)

This formula can be extremely practical when working with financial estimates or projections and for understanding how compound interest can have a dramatic effect on an original amount or monetary balance.

Following are just a few examples of how the rule **of 72** can help you solve problems very quickly and very easily, often enabling you to solve them “in your head,” without the need for a calculator or spreadsheet.

Let’s say you are interested in knowing how long it will take your savings account balance to double. If your account earns an interest rate of 9%, your money will take 72/9 or 8, years to double. However, if you are earning only 6% on this same investment, your money will take 72/6, or 12, years to double.

Now let’s say you have a specific future purchasing need and you know that you will need to double your money in five years. In this case, you would be required to invest it at an interest rate of 72/5, or 14.4%. Through these sample examples, it is easy to see how relatively small changes in a growth or interest rate can have significant impact on the time required for a balance to double in size.

The number of years required to double money ≈ 72 ÷ interest rate(%) = years.

With technology available to do calculations using logarithms, we would use the Law of 72 only for quick estimates of doubling times. Using the Law of 72 as an estimate works only for doubling times, but not other multiples, so it’s not a replacement for knowing how to find exact solutions.

However, the Law of 72 can be useful to help quickly estimate many “doubling time” problems mentally, which can be useful in compound interest applications as well as other applications involving exponential growth.

Example 6.1.15

a. At the peak growth rate in the 1960’s the world's population had a doubling time of 35 years. At that time, approximately what was the growth rate?
b. As of 2015, the world population’s annual growth rate was approximately 1.14%. Based on that rate, find the approximate doubling time.

Solution

a. According to the law of 72,

$$\text{doubling time} = 35 \approx 72 \div r$$

$$r \approx 2.057 \text{ expressed as a percent}$$

Therefore, the world population was growing at an approximate rate of 2.057% in the 1960’s.

b.. According to the law of 72,

$$\text{doubling time } t \approx 72 \div r = 72 \div 1.14 \approx 63.157 \text{ years}$$

If the world population were to continue to grow at the annual growth rate of 1.14% , it would take approximately 64 years for the population to double.

COMPOUND INTEREST SUMMARY

Below is a summary of the formulas we developed for calculations involving compound interest:

COMPOUND INTEREST $\mathbf{n}$ times per year

1. If an amount P is invested for t years at an interest rate r per year, compounded m times a year, then the future value is given by

$$FV = P\left(1+\frac{r}{m}\right)^{mt}$$

$\mathbf{P}$ is called the principle and is also called the present value.

2. If a bank pays an interest rate r per year, compounded m times a year, then the effective interest rate is given by

$$\mathbf{r}_{\mathrm{EFF}} = \left(1+\frac{r}{m}\right)^{m} - 1$$

CONTINUOUSLY COMPOUNDED INTEREST

3. If an amount P is invested for t years at an interest rate r per year, compounded continuously, then the future value is given by

$$\mathrm{FV} = \mathrm{P}e^{rt}$$

4. If a bank pays an interest rate r per year, compounded n times a year, then the effective interest rate is given by

$$\mathrm{r}_{\mathrm{EFF}} = e^{\mathbf{r}} - 1$$

5. The Law of 72 states that

The number of years to double money is approximately 72 ÷ interest rate