

8.3: Sampling Distributions and the Central Limit Theorem

The law of large numbers is a very powerful tool, but it's not going to be good enough to answer all our questions. Among other things, all it gives us is a "long run guarantee". In the long run, if we were somehow able to collect an infinite amount of data, then the law of large numbers guarantees that our sample statistics will be correct. But as John Maynard Keynes famously argued in economics, a long run guarantee is of little use in real life:

[The] long run is a misleading guide to current affairs. In the long run we are all dead. Economists set themselves too easy, too useless a task, if in tempestuous seasons they can only tell us, that when the storm is long past, the ocean is flat again. Keynes (1923)

As in economics, so too in psychology and statistics. It is not enough to know that we will *eventually* arrive at the right answer when calculating the sample mean. Knowing that an infinitely large data set will tell me the exact value of the population mean is cold comfort when my *actual* data set has a sample size of $N=100$. In real life, then, we must know something about the behaviour of the sample mean when it is calculated from a more modest data set!

8.3.1 Sampling distribution of the mean

With this in mind, let's abandon the idea that our studies will have sample sizes of 10000, and consider a very modest experiment indeed. This time around we'll sample $N=5$ people and measure their IQ scores. As before, I can simulate this experiment in R using the `rnorm()` function:

```
> IQ.1 <- round( rnorm(n=5, mean=100, sd=15 ))
> IQ.1
[1] 90 82 94 99 110
```

The mean IQ in this sample turns out to be exactly 95. Not surprisingly, this is much less accurate than the previous experiment. Now imagine that I decided to **replicate** the experiment. That is, I repeat the procedure as closely as possible: I randomly sample 5 new people and measure their IQ. Again, R allows me to simulate the results of this procedure:

```
> IQ.2 <- round( rnorm(n=5, mean=100, sd=15 ))
> IQ.2
[1] 78 88 111 111 117
```

This time around, the mean IQ in my sample is 101. If I repeat the experiment 10 times I obtain the results shown in Table ??, and as you can see the sample mean varies from one replication to the next.

NANA	Person.1	Person.2	Person.3	Person.4	Person.5	Sample.Mean	caption
Replication 1	90	82	94	99	110	95.0	Ten replications of the IQ experiment, each with a sample size of $N=5$.
Replication 2	78	88	111	111	117	101.0	Ten replications of the IQ experiment, each with a sample size of $N=5$.

NANA	Person.1	Person.2	Person.3	Person.4	Person.5	Sample.Mean	caption
Replication 3	111	122	91	98	86	101.6	Ten replications of the IQ experiment, each with a sample size of N=5.
Replication 4	98	96	119	99	107	103.8	Ten replications of the IQ experiment, each with a sample size of N=5.
Replication 5	105	113	103	103	98	104.4	Ten replications of the IQ experiment, each with a sample size of N=5.
Replication 6	81	89	93	85	114	92.4	Ten replications of the IQ experiment, each with a sample size of N=5.
Replication 7	100	93	108	98	133	106.4	Ten replications of the IQ experiment, each with a sample size of N=5.
Replication 8	107	100	105	117	85	102.8	Ten replications of the IQ experiment, each with a sample size of N=5.

NANA	Person.1	Person.2	Person.3	Person.4	Person.5	Sample.Mean	caption
Replication 9	86	119	108	73	116	100.4	Ten replications of the IQ experiment, each with a sample size of N=5.
Replication 10	95	126	112	120	76	105.8	Ten replications of the IQ experiment, each with a sample size of N=

Now suppose that I decided to keep going in this fashion, replicating this “five IQ scores” experiment over and over again. Every time I replicate the experiment I write down the sample mean. Over time, I’d be amassing a new data set, in which every experiment generates a single data point. The first 10 observations from my data set are the sample means listed in Table ??, so my data set starts out like this:

```
95.0 101.0 101.6 103.8 104.4 ...
```

What if I continued like this for 10,000 replications, and then drew a histogram? Using the magical powers of R that’s exactly what I did, and you can see the results in Figure 10.5. As this picture illustrates, the average of 5 IQ scores is usually between 90 and 110. But more importantly, what it highlights is that if we replicate an experiment over and over again, what we end up with is a *distribution* of sample means! This distribution has a special name in statistics: it’s called the **sampling distribution of the mean**.

Sampling distributions are another important theoretical idea in statistics, and they’re crucial for understanding the behaviour of small samples. For instance, when I ran the very first “five IQ scores” experiment, the sample mean turned out to be 95. What the sampling distribution in Figure 10.5 tells us, though, is that the “five IQ scores” experiment is not very accurate. If I repeat the experiment, the sampling distribution tells me that I can expect to see a sample mean anywhere between 80 and 120.

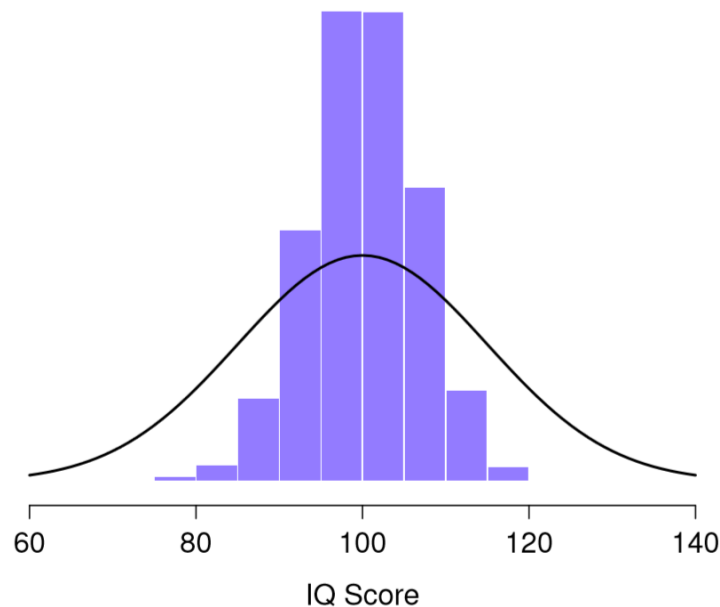


Figure 10.5: The sampling distribution of the mean for the “five IQ scores experiment”. If you sample 5 people at random and calculate their *average* IQ, you’ll almost certainly get a number between 80 and 120, even though there are quite a lot of individuals who have IQs above 120 or below 80. For comparison, the black line plots the population distribution of IQ scores.

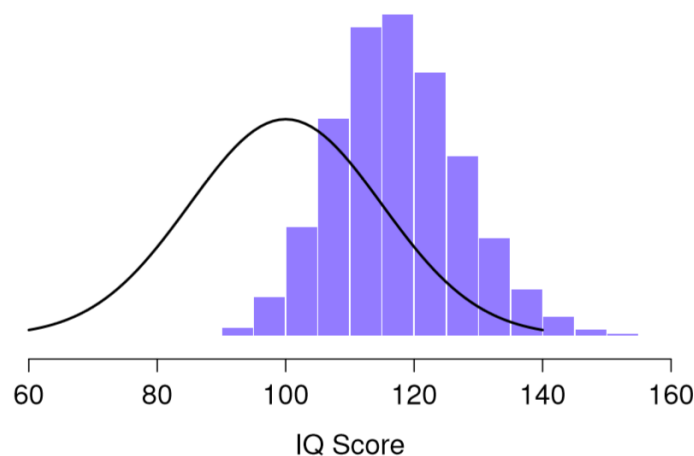


Figure 10.6: The sampling distribution of the *maximum* for the “five IQ scores experiment”. If you sample 5 people at random and select the one with the highest IQ score, you’ll probably see someone with an IQ between 100 and 140.

8.3.2 Sampling distributions exist for any sample statistic!

One thing to keep in mind when thinking about sampling distributions is that *any* sample statistic you might care to calculate has a sampling distribution. For example, suppose that each time I replicated the “five IQ scores” experiment I wrote down the largest IQ score in the experiment. This would give me a data set that started out like this:

110 117 122 119 113 ...

Doing this over and over again would give me a very different sampling distribution, namely the *sampling distribution of the maximum*. The sampling distribution of the maximum of 5 IQ scores is shown in Figure 10.6. Not surprisingly, if you pick 5 people at random and then find the person with the highest IQ score, they’re going to have an above average IQ. Most of the time you’ll end up with someone whose IQ is measured in the 100 to 140 range.

8.3.3 central limit theorem

An illustration of the how sampling distribution of the mean depends on sample size. In each panel, I generated 10,000 samples of IQ data, and calculated the mean IQ observed within each of these data sets. The histograms in these plots show the distribution of these means (i.e., the sampling distribution of the mean). Each individual IQ score was drawn from a normal distribution with mean 100 and standard deviation 15, which is shown as the solid black line).

Sample Size = 1

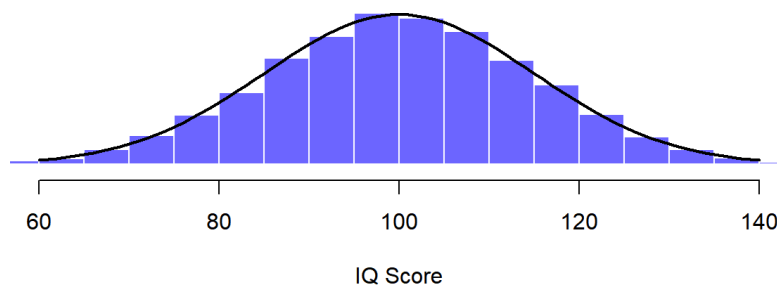


Figure 10.7: Each data set contained only a single observation, so the mean of each sample is just one person's IQ score. As a consequence, the sampling distribution of the mean is of course identical to the population distribution of IQ scores.

Sample Size = 2

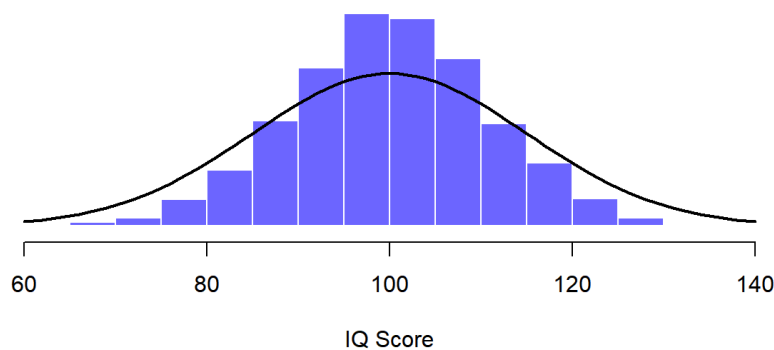


Figure 10.8: When we raise the sample size to 2, the mean of any one sample tends to be closer to the population mean than a one person's IQ score, and so the histogram (i.e., the sampling distribution) is a bit narrower than the population distribution.

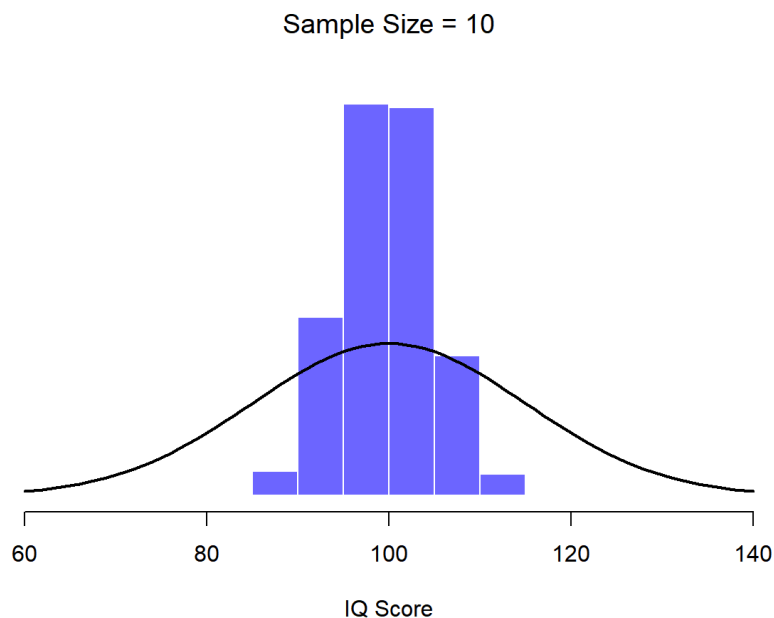


Figure 10.9: By the time we raise the sample size to 10, we can see that the distribution of sample means tend to be fairly tightly clustered around the true population mean.

At this point I hope you have a pretty good sense of what sampling distributions are, and in particular what the sampling distribution of the mean is. In this section I want to talk about how the sampling distribution of the mean changes as a function of sample size. Intuitively, you already know part of the answer: if you only have a few observations, the sample mean is likely to be quite inaccurate: if you replicate a small experiment and recalculate the mean you'll get a very different answer. In other words, the sampling distribution is quite wide. If you replicate a large experiment and recalculate the sample mean you'll probably get the same answer you got last time, so the sampling distribution will be very narrow. You can see this visually in Figures 10.7, 10.8 and 10.9: the bigger the sample size, the narrower the sampling distribution gets. We can quantify this effect by calculating the standard deviation of the sampling distribution, which is referred to as the **standard error**. The standard error of a statistic is often denoted SE, and since we're usually interested in the standard error of the sample *mean*, we often use the acronym SEM. As you can see just by looking at the picture, as the sample size N increases, the SEM decreases.

Okay, so that's one part of the story. However, there's something I've been glossing over so far. All my examples up to this point have been based on the "IQ scores" experiments, and because IQ scores are roughly normally distributed, I've assumed that the population distribution is normal. What if it isn't normal? What happens to the sampling distribution of the mean? The remarkable thing is this: no matter what shape your population distribution is, as N increases the sampling distribution of the mean starts to look more like a normal distribution. To give you a sense of this, I ran some simulations using R. To do this, I started with the "ramped" distribution shown in the histogram in Figure 10.10. As you can see by comparing the triangular shaped histogram to the bell curve plotted by the black line, the population distribution doesn't look very much like a normal distribution at all. Next, I used R to simulate the results of a large number of experiments. In each experiment I took $N=2$ samples from this distribution, and then calculated the sample mean. Figure ?? plots the histogram of these sample means (i.e., the sampling distribution of the mean for $N=2$). This time, the histogram produces a \cap -shaped distribution: it's still not normal, but it's a lot closer to the black line than the population distribution in Figure ???. When I increase the sample size to $N=4$, the sampling distribution of the mean is very close to normal (Figure ??), and by the time we reach a sample size of $N=8$ it's almost perfectly normal. In other words, as long as your sample size isn't tiny, the sampling distribution of the mean will be approximately normal no matter what your population distribution looks like!

```
# needed for printing
width <- 6
height <- 6

# parameters of the beta
a <- 2
b <- 1

# mean and standard deviation of the beta
s <- sqrt( a*b / (a+b)^2 / (a+b+1) )
m <- a / (a+b)

# define function to draw a plot
plotOne <- function(n,N=50000) {

  # generate N random sample means of size n
  X <- matrix(rbeta(n*N,a,b),n,N)
  X <- colMeans(X)

  # plot the data
  hist( X, breaks=seq(0,1,.025), border="white", freq=FALSE,
        col=ifelse(colour,emphCollLight,emphGrey),
        xlab="Sample Mean", ylab="", xlim=c(0,1.2),
        main=paste("Sample Size =",n), axes=FALSE,
        font.main=1, ylim=c(0,5)
  )
  box()
  axis(1)
  #axis(2)

  # plot the theoretical distribution
  lines( x <- seq(0,1.2,.01), dnorm(x,m,s/sqrt(n)),
        lwd=2, col="black", type="l"
  )
}

for( i in c(1,2,4,8)) {
  plotOne(i)}
```

Sample Size = 1

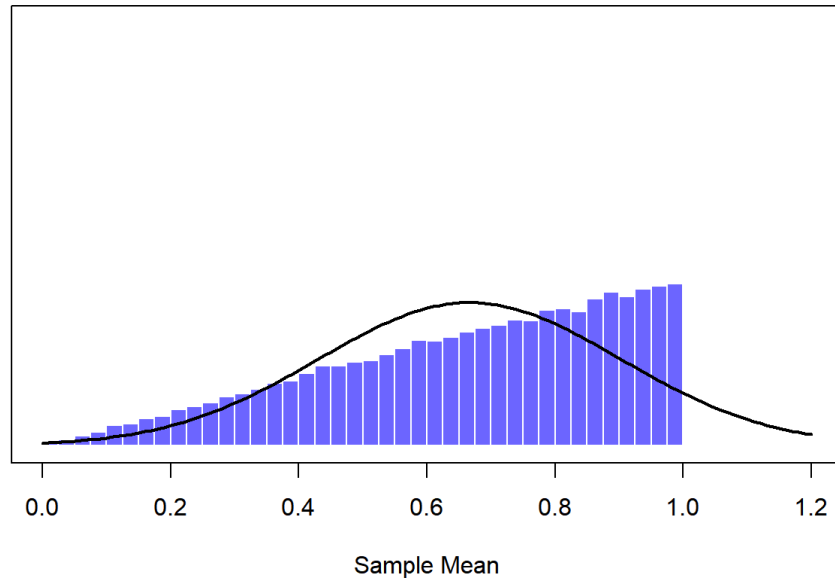


Figure 10.10: A demonstration of the central limit theorem. In panel a, we have a non-normal population distribution; and panels b-d show the sampling distribution of the mean for samples of size 2, 4 and 8, for data drawn from the distribution in panel a. As you can see, even though the original population distribution is non-normal, the sampling distribution of the mean becomes pretty close to normal by the time you have a sample of even 4 observations.

Sample Size = 2

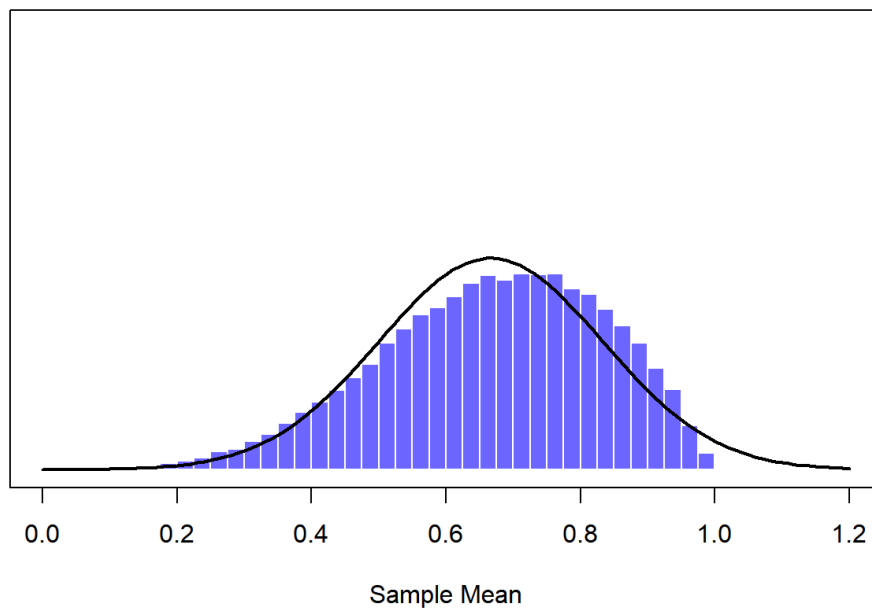


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Sample Size = 4

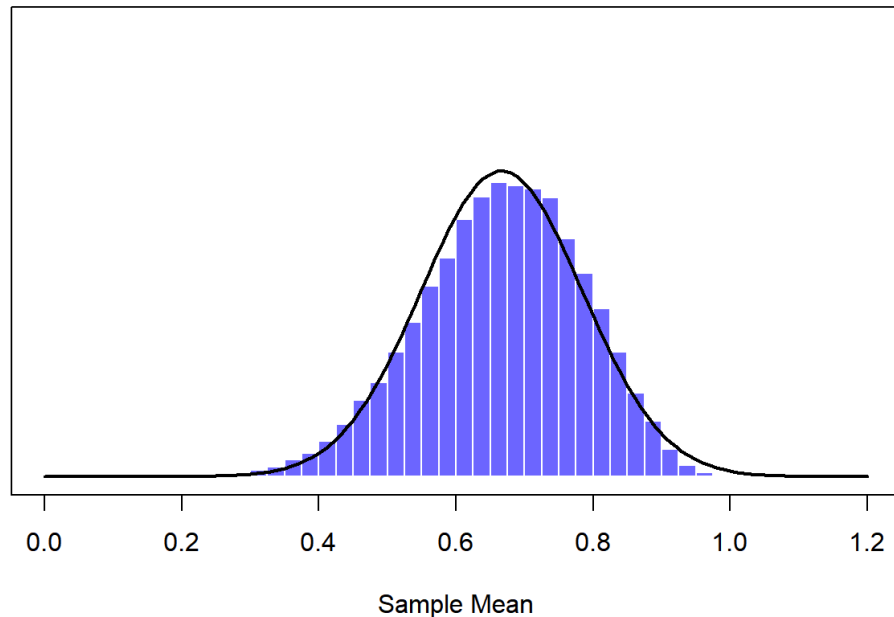


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Sample Size = 8

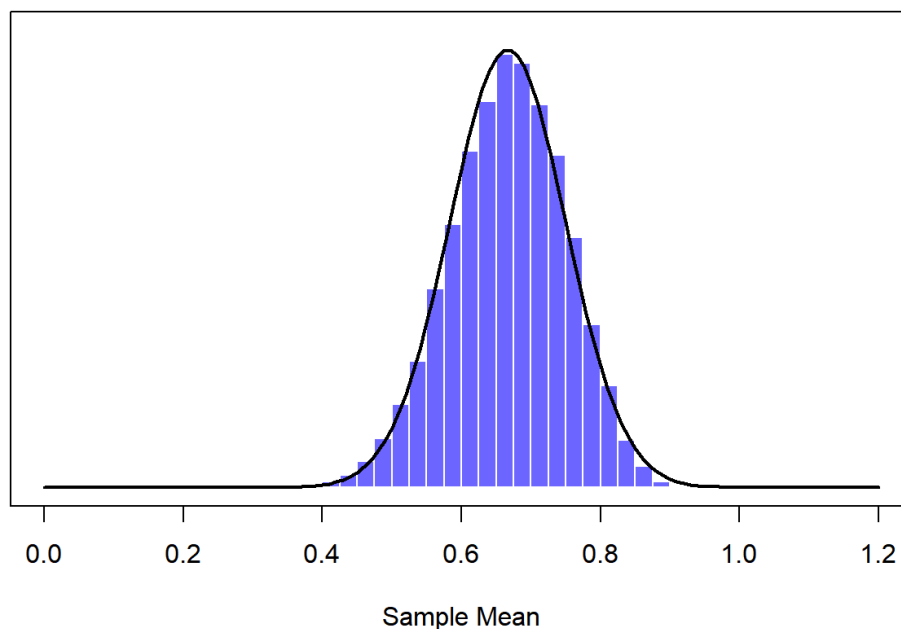


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On the basis of these figures, it seems like we have evidence for all of the following claims about the sampling distribution of the mean:

- The mean of the sampling distribution is the same as the mean of the population
- The standard deviation of the sampling distribution (i.e., the standard error) gets smaller as the sample size increases

- The shape of the sampling distribution becomes normal as the sample size increases

As it happens, not only are all of these statements true, there is a very famous theorem in statistics that proves all three of them, known as the **central limit theorem**. Among other things, the central limit theorem tells us that if the population distribution has mean μ and standard deviation σ , then the sampling distribution of the mean also has mean μ , and the standard error of the mean is

$$\text{SEM} = \frac{\sigma}{\sqrt{N}}$$

Because we divide the population standard deviation σ by the square root of the sample size N , the SEM gets smaller as the sample size increases. It also tells us that the shape of the sampling distribution becomes normal.¹⁵⁰

This result is useful for all sorts of things. It tells us why large experiments are more reliable than small ones, and because it gives us an explicit formula for the standard error it tells us *how much* more reliable a large experiment is. It tells us why the normal distribution is, well, *normal*. In real experiments, many of the things that we want to measure are actually averages of lots of different quantities (e.g., arguably, “general” intelligence as measured by IQ is an average of a large number of “specific” skills and abilities), and when that happens, the averaged quantity should follow a normal distribution. Because of this mathematical law, the normal distribution pops up over and over again in real data.

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