12.4: The F Distribution and the F-Ratio

The distribution used for the hypothesis test is a new one. It is called the **F distribution**, invented by George Snedecor but named in honor of Sir Ronald Fisher, an English statistician. The F statistic is a ratio (a fraction). There are two sets of degrees of freedom; one for the numerator and one for the denominator.

For example, if F follows an F distribution and the number of degrees of freedom for the numerator is four, and the number of degrees of freedom for the denominator is ten, then $F \sim F_{4,10}$.

To calculate the **F ratio**, two estimates of the variance are made.

1. Variance between samples: An estimate of σ^2 that is the variance of the sample means multiplied by n (when the sample sizes are the same.). If the samples are different sizes, the variance between samples is weighted to account for the different sample sizes. The variance is also called variation due to treatment or explained variation.
2. Variance within samples: An estimate of σ^2 that is the average of the sample variances (also known as a pooled variance). When the sample sizes are different, the variance within samples is weighted. The variance is also called the variation due to error or unexplained variation.

- $SS_{\text{between}} =$ the sum of squares that represents the variation among the different samples
- $SS_{\text{within}} =$ the sum of squares that represents the variation within samples that is due to chance.

To find a "sum of squares" means to add together squared quantities that, in some cases, may be weighted. We used sum of squares to calculate the sample variance and the sample standard deviation in Descriptive Statistics.

MS means "**mean square**." MS_{between} is the variance between groups, and MS_{within} is the variance within groups.

Calculation of Sum of Squares and Mean Square

- $k =$ the number of different groups
- $n_j =$ the size of the j^{th} group
- $s_j =$ the sum of the values in the j^{th} group
- $n =$ total number of all the values combined (total sample size: $\sum n_j$)
- $x =$ one value: $\sum x = \sum s_j$
- Sum of squares of all values from every group combined: $\sum x^2$
- Total sum of squares: $SS_{\text{total}} = \sum x^2 \frac{\left(\sum x\right)^2}{n}$
- Explained variation: sum of squares representing variation among the different samples:

$$SS_{\text{between}} = \sum \left[\frac{(s_j)^2}{n_j}\right] - \frac{\left(\sum s_j\right)^2}{n} \tag{12.4.1}$$

- Unexplained variation: sum of squares representing variation within samples due to chance:

$$SS_{\text{within}} = SS_{\text{total}} - SS_{\text{between}} \tag{12.4.2}$$

- df' s for different groups (df' s for the numerator): $df_{\text{between}} = k-1$
- Equation for errors within samples (df 's for the denominator): $df_{\text{within}} = n-k$
- Mean square (variance estimate) explained by the different groups: $MS_{\text{between}} = \frac{SS_{\text{between}}}{df_{\text{between}}}$
- Mean square (variance estimate) that is due to chance (unexplained): $MS_{\text{within}} = \frac{SS_{\text{within}}}{df_{\text{within}}}$

MS_{between} and MS_{within} can be written as follows:

- $MS_{\text{between}} = \frac{SS_{\text{between}}}{df_{\text{between}}} = \frac{SS_{\text{between}}}{k-1}$
- $MS_{\text{within}} = \frac{SS_{\text{within}}}{df_{\text{within}}} = \frac{SS_{\text{within}}}{n-k}$

The one-way ANOVA test depends on the fact that MS_{between} can be influenced by population differences among means of the several groups. Since MS_{within} compares values of each group to its own group mean, the fact that group means might be different does not affect MS_{within}.

The null hypothesis says that all groups are samples from populations having the same normal distribution. The alternate hypothesis says that at least two of the sample groups come from populations with different normal distributions. If the null hypothesis is true, MS_{between} and MS_{within} should both estimate the same value.

> **Note**
>
> The null hypothesis says that all the group population means are equal. The hypothesis of equal means implies that the populations have the same normal distribution, because it is assumed that the populations are normal and that they have equal variances.

F-Ratio or F-Statistic

$$F = \frac{MS_{\text{between}}}{MS_{\text{within}}} \tag{12.4.3}$$

If MS_{between} and MS_{within} estimate the same value (following the belief that H_0 is true), then the F-ratio should be approximately equal to one. Mostly, just sampling errors would contribute to variations away from one. As it turns out, MS_{between} consists of the population variance plus a variance produced from the differences between the samples. MS_{within} is an estimate of the population variance. Since variances are always positive, if the null hypothesis is false, MS_{between} will generally be larger than $MS_{\text{within.}}$. Then the F-ratio will be larger than one. However, if the population effect is small, it is not unlikely that MS_{within} will be larger in a given sample.

The foregoing calculations were done with groups of different sizes. If the groups are the same size, the calculations simplify somewhat and the F-ratio can be written as:

F-Ratio Formula when the groups are the same size

$$F = \frac{n \cdot s_{\bar{x}}^2}{s_{\text{pooled}}^2} \tag{12.4.4}$$

where ...

- $n =$ the sample size
- $df_{\text{numerator}} = k-1$
- $df_{\text{denominator}} = n-k$
- s^2 pooled $=$ the mean of the sample variances (pooled variance)
- $s_{\bar{x}}^2 =$ the variance of the sample means

Data are typically put into a table for easy viewing. One-Way ANOVA results are often displayed in this manner by computer software.

12.4.1

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (df)	Mean Square (MS)	F
Factor (Between)	SS (Factor)	$k - 1$	MS (Factor) = SS (Factor)/(k – 1)	F = MS(Factor)/MS(Error)
Error (Within)	SS (Error)	$n - k$	MS (Error) = SS (Error)/(n – k)	
Total	SS (Total)	$n - 1$		

Exercise 12.4.1

Three different diet plans are to be tested for mean weight loss. The entries in the table are the weight losses for the different plans. The one-way ANOVA results are shown in Table 12.4.2

Table 12.4.2

Plan 1: $n_1 = 4$	Plan 2: $n_2 = 3$	Plan 3: $n_3 = 3$
5	3.5	8
4.5	7	4
4		3.5
3	4.5	

Answer

$$s_1 = 16.5, s_2 = 15, s_3 = 15.5 \tag{12.4.5}$$

Following are the calculations needed to fill in the one-way ANOVA table. The table is used to conduct a hypothesis test.

$$\begin{aligned} SS(\text{ between }) &= \sum\left[\frac{(s_j)^2}{n_j}\right] - \frac{\left(\sum s_j\right)^2}{n} \\ &= \frac{s_1^2}{4} + \frac{s_2^2}{3} + \frac{s_3^2}{3} - \frac{(s_1+s_2+s_3)^2}{10} \end{aligned} \tag{12.4.6}$$

where $n_1 = 4, n_2 = 3, n_3 = 3$ and $n = n_1 + n_2 + n_3 = 10$

$$\begin{gathered} = \frac{(16.5)^2}{4} + \frac{(15)^2}{3} + \frac{(15.5)^2}{3} - \frac{(16.5+15+15.5)^2}{10} \\ SS(\text{ between }) = 2.2458 \\ S(\text{ total }) = \sum x^2 - \frac{\left(\sum x\right)^2}{n} \\ = \left(5^2 + 4.5^2 + 4^2 + 3^2 + 3.5^2 + 7^2 + 4.5^2 + 8^2 + 4^2 + 3.5^2\right) \\ -\frac{(5+4.5+4+3+3.5+7+4.5+8+4+3.5)^2}{10} \\ = 244 - \frac{47^2}{10} = 244 - 220.9 \\ SS(\text{ total }) = 23.1 \\ SS(\text{ within }) = SS(\text{ total }) - SS(\text{ between }) \\ = 23.1 - 2.2458 \\ SS(\text{ within }) = 20.8542 \end{gathered} \tag{12.4.7}$$

Table 12.4.3

Source of variation	Sum of squares (SS)	Degrees of freedom (df)	Mean square (MS)	F
Factor (Between)	SS (Factor) = SS (Between) = 2.2458	k – 1 = 3 groups – 1 = 2	MS (Factor) = SS (Factor)/(k – 1) = 2.2458/2 = 1.1229	F = MS (Factor)/MS (Error) = 1.1229/2.9792 = 0.3769
Error (Within)	SS (Error) = SS (Within) = 20.8542	n – k = 10 total data – 3 groups = 7	MS (Error) = SS (Error)/(n – k) = 20.8542/7 = 2.9792	
Total	SS (Total) = 2.2458 + 20.8542 = 23.1	n – 1 = 10 total data – 1 = 9		

Try It 12.4.1

As part of an experiment to see how different types of soil cover would affect slicing tomato production, Marist College students grew tomato plants under different soil cover conditions. Groups of three plants each had one of the following treatments

- bare soil
- a commercial ground cover
- black plastic
- straw
- compost

All plants grew under the same conditions and were the same variety. Students recorded the weight (in grams) of tomatoes produced by each of the n = 15 plants:

Table 12.4.4

Bare: $n_1 = 3$	Ground Cover: $n_2 = 3$	Plastic: $n_3 = 3$	Straw: $n_4 = 3$	Compost: $n_5 = 3$
2,625	5,348	6,583	7,285	6,277
2,997	5,682	8,560	6,897	7,818
4,915	5,482	3,830	9,230	8,677

Create the one-way ANOVA table.

The one-way ANOVA hypothesis test is always right-tailed because larger *F*-values are way out in the right tail of the *F*-distribution curve and tend to make us reject H_0.

Exercise 12.4.2

Let's return to the slicing tomato exercise in Try It 12.4.1. The means of the tomato yields under the five mulching conditions are represented by $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$. We will conduct a hypothesis test to determine if all means are the same or at least one is different. Using a significance level of 5%, test the null hypothesis that there is no difference in mean yields among the five groups against the alternative hypothesis that at least one mean is different from the rest.

Answer

The null and alternative hypotheses are:

$$\begin{aligned} &H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 \\ &H_a: \mu_i \neq \mu_j \text{ some } i \neq j \end{aligned} \tag{12.4.8}$$

The one-way ANOVA results are shown in Table 12.7

Table 12.4.5

Source of variation	Sum of squares (*SS*)	Degrees of freedom (*df*)	Mean square (*MS*)	*F*
Factor (Between)	36,648,561	5 – 1 = 4	$\frac{36,648,561}{4} = 9,162,140$	$\frac{9,162,140}{2,044,672.6} = 4.48$
Error (Within)	20,446,726	15 – 5 = 10	$\frac{20,446,726}{10} = 2,044,672.6$	
Total	57,095,287	15 – 1 = 14		

Distribution for the test: $F_{4,10}$

$$\begin{aligned} &df(\text{ num }) = 5 - 1 = 4 \\ &df(\text{ denom }) = 15 - 5 = 10 \end{aligned} \tag{12.4.9}$$

Test statistic: $F = 4.4810$

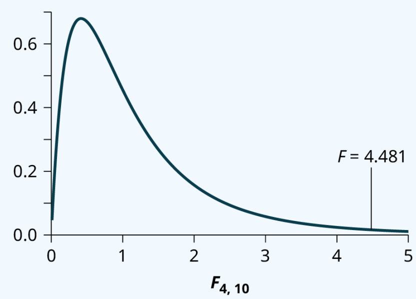

Figure 12.4.1

Probability Statement: *p*-value $= P(F > 4.481) = 0.0248$.

Compare α and the *p*-value: $\alpha = 0.05$, *p*-value $= 0.0248$

Make a decision: Since $\alpha > p$-value, we cannot accept H_0.

Conclusion: At the 5\% significance level, we have reasonably strong evidence that differences in mean yields for slicing tomato plants grown under different mulching conditions are unlikely to be due to chance alone. We may conclude that at least some of mulches led to different mean yields.

Try It 12.4.2

There are multiple variants of the virus that causes COVID-19. The length of hospital stays for patients afflicted with various strains of COVID-19 is shown in Table 12.4.6.

Table 12.4.6

Delta Strain	Omicron Strain	Alpha Strain	Gamma Strain	Beta Strain
13.9	11.7	18.2	16.9	9.3
14.9	15.1	14.6	12.8	15.8
16.8	9.9	10.1	11.2	16.4

Test whether the mean length of hospital stay is the same or different for the various strains of COVID-19. Construct the ANOVA table, find the *p*-value, and state your conclusion. Use a 5% significance level.

Exercise 12.4.3

Four sororities took a random sample of sisters regarding their grade means for the past term. The results are shown in Table 12.4.7.

Table 12.4.7: Mean grades for four sororities

Sorority 1	Sorority 2	Sorority 3	Sorority 4
2.17	2.63	2.63	3.79
1.85	1.77	3.78	3.45
2.83	3.25	4.00	3.08
1.69	1.86	2.55	2.26
3.33	2.21	2.45	3.18

Problem

Using a significance level of 1%, is there a difference in mean grades among the sororities?

Answer

Add texts here. Do not delete this text first.

Try It 12.4.3

Four sports teams took a random sample of players regarding their GPAs for the last year. The results are shown in Table 12.4.8.

Table 12.4.8: GPAs for four sports teams

Basketball	Baseball	Hockey	Lacrosse
3.6	2.1	4.0	2.0
2.9	2.6	2.0	3.6
2.5	3.9	2.6	3.9
3.3	3.1	3.2	2.7
3.8	3.4	3.2	2.5

GPAs for four sports teams

Use a significance level of 5%, and determine if there is a difference in GPA among the teams.

Exercise 12.4.4

A fourth grade class is studying the environment. One of the assignments is to grow bean plants in different soils. Tommy chose to grow his bean plants in soil found outside his classroom mixed with dryer lint. Tara chose to grow her bean plants in potting soil bought at the local nursery. Nick chose to grow his bean plants in soil from his mother's garden. No chemicals were used on the plants, only water. They were grown inside the classroom next to a large window. Each child grew five plants. At the end of the growing period, each plant was measured, producing the data (in inches) in Table 12.4.9.

Table 12.4.9

Tommy's plants	Tara's plants	Nick's plants
24	25	23
21	31	27
23	23	22
30	20	30
23	28	20

Problem

Does it appear that the three media in which the bean plants were grown produce the same mean height? Test at a 3% level of significance.

Answer

Add texts here. Do not delete this text first.

Try It 12.4.4

Another fourth grader also grew bean plants, but this time in a jelly-like mass. The heights were (in inches) 24, 28, 25, 30, and 32. Do a one-way ANOVA test on the four groups. Are the heights of the bean plants different? Use the same method as shown in Example 12.5.

Notation

The notation for the F distribution is $F \sim F_{df(\text{ num }), df(\text{ denom })}$ where $df(\text{ num }) = df_{\text{between}}$ and $df(\text{ denom }) = df_{\text{within}}$ The mean for the F distribution is $\mu = \frac{df(\text{ num })}{df(\text{ denom }) - 2}$

Example 13.1

Three different diet plans are to be tested for mean weight loss. The entries in the table are the weight losses for the different plans. The one-way ANOVA results are shown in Table 13.2.

Plan 1: $n_1 = 4$	Plan 2: $n_2 = 3$	Plan 3: $n_3 = 3$
5	3.5	8
4.5	7	4
4		3.5
3	4.5	

Table 13.2

$s_1 = 16.5$, $s_2 = 15$, $s_3 = 15.5$

Following are the calculations needed to fill in the one-way ANOVA table. The table is used to conduct a hypothesis test.

SS(between)=∑[(sj)2nj]−(∑sj)2n SS(between)=∑[(sj)2nj]−(∑sj)2n " role="presentation">

$$SS(between) = \sum\left[\frac{(s_j)^2}{n_j}\right] - \frac{\left(\sum s_j\right)^2}{n}$$

= s124+s223+s323−(s1+s2+s3)210= s124+s223+s323−(s1+s2+s3)210" role="presentation">

$$= \frac{s_1^2}{4} + \frac{s_2^2}{3} + \frac{s_3^2}{3} - \frac{(s_1+s_2+s_3)^2}{10}$$

where $n_1 = 4$, $n_2 = 3$, $n_3 = 3$ and $n = n_1 + n_2 + n_3 = 10$

  =(16.5)24+(15)23+(15.5)23−(16.5+15+15.5)210 =(16.5)24+(15)23+(15.5)23−(16.5+15+15.5)210" role="presentation">

$$= \frac{(16.5)^2}{4} + \frac{(15)^2}{3} + \frac{(15.5)^2}{3} - \frac{(16.5+15+15.5)^2}{10}$$

SS(between)=2.2458SS(between)=2.2458" role="presentation">

$$SS(between) = 2.2458$$

S(total)=∑x2−(∑x)2nS(total)=∑x2−(∑x)2n" role="presentation">

$$S(total) = \sum x^2 - \frac{\left(\sum x\right)^2}{n}$$

 =(52+4.52+42+32+3.52+72+4.52+82+42+3.52) =(52+4.52+42+32+3.52+72+4.52+82+42+3.52)" role="presentation">

$$= \left(5^2 + 4.5^2 + 4^2 + 3^2 + 3.5^2 + 7^2 + 4.5^2 + 8^2 + 4^2 + 3.5^2\right)$$

−(5+4.5+4+3+3.5+7+4.5+8+4+3.5)210−(5+4.5+4+3+3.5+7+4.5+8+4+3.5)210" role="presentation">

$$-\frac{(5+4.5+4+3+3.5+7+4.5+8+4+3.5)^2}{10}$$

=244−47210=244−220.9=244−47210=244−220.9" role="presentation">

$$= 244 - \frac{47^2}{10} = 244 - 220.9$$

SS(total)=23.1SS(total)=23.1" role="presentation">

$$SS(total) = 23.1$$

SS(within)=SS(total)−SS(between)SS(within)=SS(total)−SS(between)" role="presentation">

$$SS(within) = SS(total) - SS(between)$$

= 23.1−2.2458= 23.1−2.2458" role="presentation">

$$= 23.1 - 2.2458$$

SS(within)=20.8542SS(within)=20.8542" role="presentation">

$$SS(within) = 20.8542$$

Using the TI-83, 83+, 84, 84+ Calculator

One-Way ANOVA Table: The formulas for `SS(Total)` , `SS(Factor) = SS(Between) and SS(Error) = SS(Within)` as shown previously. The same information is provided by the TI calculator hypothesis test function `ANOVA` in STAT TESTS (syntax is `ANOVA(L1, L2, L3)` where L1, L2, L3 have the data from Plan 1, Plan 2, Plan 3 respectively).

Source of Variation	Sum of Squares (*SS*)	Degrees of Freedom (*df*)	Mean Square (*MS*)	*F*
Factor (Between)	*SS*(Factor) = *SS*(Between) = 2.2458	*k* – 1 = 3 groups – 1 = 2	*MS*(Factor) = *SS*(Factor)/(*k* – 1) = 2.2458/2 = 1.1229	*F* = *MS*(Factor)/*MS*(Error) = 1.1229/2.9792 = 0.3769
Error (Within)	*SS*(Error) = *SS*(Within) = 20.8542	*n* – *k* = 10 total data – 3 groups = 7	*MS*(Error) = *SS*(Error)/(*n* – *k*) = 20.8542/7 = 2.9792	
Total	*SS*(Total) = 2.2458 + 20.8542 = 23.1	*n* – 1 = 10 total data – 1 = 9		

Table 13.3

Try It 13.1

As part of an experiment to see how different types of soil cover would affect slicing tomato production, Marist College students grew tomato plants under different soil cover conditions. Groups of three plants each had one of the following treatments

- bare soil
- a commercial ground cover
- black plastic
- straw
- compost

All plants grew under the same conditions and were the same variety. Students recorded the weight (in grams) of tomatoes produced by each of the $n = 15$ plants:

Bare: $n_1 = 3$	Ground Cover: $n_2 = 3$	Plastic: $n_3 = 3$	Straw: $n_4 = 3$	Compost: $n_5 = 3$

Bare: $n_1 = 3$	Ground Cover: $n_2 = 3$	Plastic: $n_3 = 3$	Straw: $n_4 = 3$	Compost: $n_5 = 3$
2,625	5,348	6,583	7,285	6,277
2,997	5,682	8,560	6,897	7,818
4,915	5,482	3,830	9,230	8,677

Table 13.4

Create the one-way ANOVA table.

The one-way ANOVA hypothesis test is always right-tailed because larger *F*-values are way out in the right tail of the *F*-distribution curve and tend to make us reject H_0.

Notation

The notation for the *F* distribution is $F \sim F_{df(num),df(denom)}$

where $df(num) = df_{\text{between}}$ and $df(denom) = df_{\text{within}}$

The mean for the *F* distribution is μ=df(denom)df(denom)–2μ=df(denom)df(denom)–2" role="presentation" style="position: relative;"> $\mu = \frac{df(denom)}{df(denom)-2}$