3.3: Counting and Compound Events

> **Learning Objectives**
>
> - Recognize distinct cases when counting
> - Define compound events
> - Understanding events using "or", "and", and "given"
> - Develop and use the Addition Rule for counting the number of possible outcomes
> - Develop and use a Multiplication Rule for counting the number of possible outcomes with compound events viewed sequentially
> - Count the number of possible outcomes for various compound events

Counting: Distinct Cases

Recall Text Exercise $3.2.1$b, where we were drawing two tiles in sequence without replacement from a bag consisting of 3 red, 2 white, and 1 blue tiles. We produced our sample space by constructing a tree diagram (reproduced below).

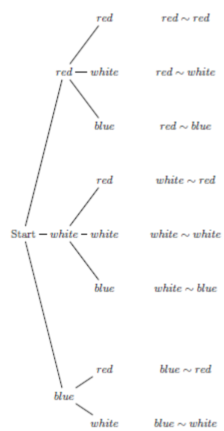

Figure $3.3.1$: Tree diagram of tile combinations

This tree diagram looks different from most of the tree diagrams from the previous section; two branches end with three options, but the bottom branch only has two options. The situation and the count differed based on which branch we initially started. We had to consider two cases: case 1 of blue initially drawn and case 2 of red/white initially drawn. Either case 1 or case 2 must happen because an initial tile must be drawn, but note that both cases are exclusive. If case 1 happens, blue is the only option for the initial draw, and then there are two options for the second tile, making our count for case 1 is $1 \cdot 2 = 2$. If case 2 happens, there are two options for the initial draw and three options for the second draw, making our count for case 2 $2 \cdot 3 = 6$. We have $1 \cdot 2 + 2 \cdot 3 = 8$ possible outcomes in the sample space between the two cases.

Here is a more challenging example. Suppose a committee of ten must be formed from a group of twenty hourly employees and five managers. How many different committees can be formed if at least three managers must be on the committee? We must think carefully about how the committee is formed to count correctly. The phrase "at least three managers" tells us our selected group can have three managers <u>or</u>, four managers <u>or</u>, five managers on the committee. The committee size remains at 10 regardless of the number of managers. We can construct a table to determine the different committee compositions at the manager and hourly employee levels.

Committee Total	10	10	10
Number of Managers	3	4	5
Number of Hourly Employees	$10-3=7$	$10-4=6$	$10-5=5$

Each column of the table represents a particular committee. We recognize these committee compositions as distinct cases, one of which must happen. To count the total number of committees, we count the number of committees in each case and add them together.

$$\begin{gathered}\text{total number of committees} = \\ \text{number of committees with three managers} + \text{number of committees with four managers} \\ + \text{number of committees with five managers}\end{gathered}$$

Each case is similar in the fact that both managers and hourly employees must be selected. Our multiplication rule applies nicely since we have each group of hourly workers for each group of managers. We proceed case by case.

If the number of managers is 3, we must choose 3 of the 5 managers, order of selection does not matter, yielding ${}_5C_3$ different possibilities for filling the manager positions. The remaining 7 committee members will be chosen from the 20 hourly employees, yielding ${}_{20}C_7$ different possibilities (again, we use combinations since order does not matter). Our multiplication rule informs us that there are ${}_5C_3 \cdot {}_{20}C_7 = 10 \cdot 77,520 = 775,200$ committees possible in which three of the members are managers.

Similarly for the other two cases, there are ${}_5C_4 \cdot {}_{20}C_6 = 5 \cdot 38,760 = 193,800$ possibilities with four managers and ${}_5C_5 \cdot {}_{20}C_5 = 1 \cdot 15,504 = 15,504$ possibilities with five managers.

$$\begin{aligned}\text{number of committees with three managers} + \text{ number of committees with four managers} + \text{number of committees with five managers} \\ = {}_5C_3 \cdot {}_{20}C_7 + {}_5C_4 \cdot {}_{20}C_6 + {}_5C_5 \cdot {}_{20}C_5 \\ = 775,200 + 193,800 + 15,504 \\ = 984,504\end{aligned}$$

There are $984,504$ possible committees that fit this description. From this work, we can also form some probability claims. If a committee is randomly formed given these restrictions, the probability that the committee will have five managers as members will be $\frac{15,504}{984,504} \approx 1.5748\%$ and the probability that the committee will have three managers as members will be $\frac{775,200}{984,504} \approx 78.7402\%$.

? Text Exercise 3.3.1

1. A food truck makes tacos with at least one but up to five fillings from eight ingredients: beef, fish, pork, beans, cheese, lettuce, tomatoes, and guacamole. How many ways can a taco be ordered from the food truck vendor?

Answer

We notice that we are given restrictions on the taco fillings. We must choose at least one filling for a taco while having at most five fillings. There are several distinct cases to consider, with only one taking place.

- One filling chosen
- Two fillings chosen
- Three fillings chosen
- Four fillings chosen
- Five fillings chosen

These are the only possible choices satisfying the requirements. We can determine the counts for each of these five cases and add those counts to find the total number of possibilities. Using our earlier reasoning that each of these five cases is a combination count (order of filling choices does not matter.)

$$\#\text{ of one filling} + \#\text{ of two fillings} + \#\text{ of three fillings} + \#\text{ of four fillings} + \#\text{ of five fillings}$$

$$_8C_1 + {_8C_2} + {_8C_3} + {_8C_4} + {_8C_5}$$

$$8 + 28 + 56 + 70 + 56$$

$$218$$

A customer has 218 possible tacos that could be ordered. No wonder it takes some customers so long to decide what to eat.

2. A ten-person committee to investigate possible corruption in U.S. military contracts is to be formed among the 59 members of the House of Representatives Armed Services Committee and the 25 members of the Senate Committee on Armed Services. The committee must have at least five House members and at least two senators. How many committees are possible?

Answer

We notice that we are given restrictions on the committee structure. We must have at least five house members while still having at least two senators on the committee of ten. We have several distinct cases to consider, with only one taking place.

- Five House members and five Senators
- Six House members and four Senators
- Seven House members and three Senators
- Eight House members and two Senators

These are the only four possible committee structures satisfying the requirements. We can determine the counts for each of these four cases and sum those counts to find the total number of possible committees. We use similar reasoning to compute:

$$_{59}C_5 \cdot {_{25}C_5} + {_{59}C_6} \cdot {_{25}C_4} + {_{59}C_7} \cdot {_{25}C_3} + {_{59}C_8} \cdot {_{25}C_2}$$

$$2.66 \times 10^{11} + 5.70 \times 10^{11} + 7.85 \times 10^{11} + 6.65 \times 10^{11}$$

$$2.29 \times 10^{12}$$

There are about $2,290,000,000,000$ committees that could be formed: an astounding number of possibilities. We are glad we only had to determine the number of possible committees and were not required to list the options by constructing a tree diagram.

Multiple counting strategies come into play as we explore possible outcomes. Counting can be pretty complicated. We are on the verge of a formal addition rule that will come in handy; we will formulate it within the context of compound events.

Connecting Events Using "or", "and", and "given"

When describing multiple events, we typically use one of the following: A or B, or A and B. Using multiple events to describe or form another event is called a **compound event**.

Consider rolling a single fair die. The sample space is the set: $\{1, 2, 3, 4, 5, 6\}$.

The compound event ODD involves three outcomes: 1, 3, and 5. We understand the event ODD to occur if when we roll the die, either a 1, 3, or 5 lands face up; there are three ways for our compound event to occur. Since the die is fair, we can compute the probability using the classical method: $P(\text{ODD}) = \frac{3}{6} = \frac{1}{2}$. The complement of ODD consists of the outcomes 2, 4, and 6 which we note is the compound event EVEN; so, $\overline{\text{ODD}} = \text{EVEN}$.

The compound event PERFECT SQUARE involves just two outcomes: 1 and 4. $P(\text{PERFECT SQUARE}) = \frac{2}{6} = \frac{1}{3}$.

Consider the compound event $A = \text{EVEN or PERFECT SQUARE}$. We refer to a single compound event A using two events connected with the word **"or."** We understand event A to occur if, when we roll the die, either EVEN occurs or PERFECT SQUARE occurs. Meaning, if 1, 2, 4, or 6 land face up, A occurs. $P(A) = P(\text{EVEN or PERFECT SQUARE}) = \frac{4}{6} = \frac{2}{3}$.

Side note: there are different ways to consider "or". Since we include the possibility that both A and B occur, our use of the word "or" is called **inclusive**. If we excluded the possibility that both events occur simultaneously, the word "or" would be called **exclusive**. Whenever we use the word "or" in this text, we are using it inclusively.

In our last example, notice that the outcome 4 was described by both EVEN and PERFECT SQUARE. When two events have outcomes in common, the events are not **mutually exclusive.**

Consider the compound event $B = \text{EVEN and PERFECT SQUARE}$. We are again referring to a single compound event B using two events; this time, the two events are connected with the word **"and"**. We understand event B to occur if, when we roll the die, both EVEN and PERFECT SQUARE occur; meaning B will occur only when 4 lands face up. So, $P(B) = P(\text{EVEN and PERFECT SQUARE}) = \frac{1}{6}$.

For a final example, consider the event $\text{EVEN given PERFECT SQUARE}$ which is often denoted $\text{EVEN}|\text{PERFECT SQUARE}$. For a third time, we refer to a single event, say event C, using two events connected with the word **"given."** When we use "given," we treat our situation as if we had additional knowledge about what is happening; that is, we assume that the die roll is guaranteed, having a probability of 1, to be a perfect square and are considering the likelihood, given this assumption, that it is even. We treat our situation **conditionally** by restricting our sample space to the event that follows the word "given." When examining the conditional event $\text{EVEN given PERFECT SQUARE}$, we do not consider the original sample space of $\{1, 2, 3, 4, 5, 6\}$. Instead, we only consider the perfect squares from that conditional sample space: 1 and 4. We have a "new" conditional sample space of size 2. We then ask, how many of these 2 possibilities satisfy the event EVEN? There is only one possibility of an even, 4. Now in computing the probability, we must remember that we only considered the outcomes in the event PERFECT SQUARE; so the total number of outcomes is not 6 but 2. $P(\text{EVEN}|\text{PERFECT SQUARE}) = \frac{1}{2}$.

Text Exercise $3.3.2$

Consider rolling two fair dice in sequence and the events below. For each event, identify the possible outcomes described and compute the probability.

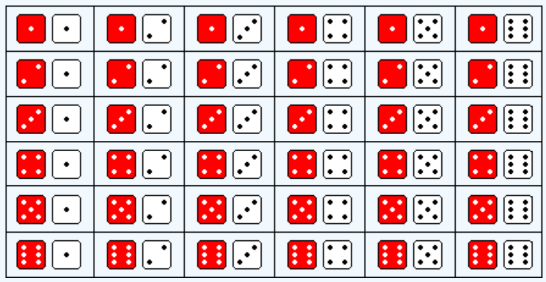

Figure $3.3.2$: Sample space of rolling two standard dice

1. Event: $\text{SUM OF THE TWO VALUES IS } 7$

Answer

The sum of the values will be 7 for the following pairs: 1 and 6, 2 and 5, 3 and 4, 6 and 1, 5 and 2, and 4 and 3. We have $P(\text{SUM OF THE TWO VALUES IS } 7) = \frac{6}{36} = \frac{1}{6}$. Notice all these outcomes fall on the diagonal from bottom left to top right.

2. Event: $\text{EITHER DIE IS } 2$

Answer

We can understand the compound event $\text{EITHER DIE IS } 2$ as $\text{FIRST DIE IS } 2$ or $\text{SECOND DIE IS } 2$. $\text{FIRST DIE IS } 2$ consists of the 6 outcomes of the second row, and $\text{SECOND DIE IS } 2$ consists of the 6 outcomes of second column. Note that these compound events are not mutually exclusive because the outcome where both dice are 2 is in both compound events. If we count the outcomes between both compound events, we arrive at 11 outcomes in the compound event $\text{EITHER DIE IS } 2$. Thus $P(\text{EITHER DIE IS } 2) = \frac{11}{36}$.

3. Event: $\text{SUM OF THE TWO VALUES IS } 7 \text{ and EITHER DIE IS } 2$

Answer

We need both $\text{SUM OF THE TWO VALUES IS } 7$ and $\text{EITHER DIE IS } 2$ to occur for our event to occur. We thus look for the overlap between the outcomes in each compound event. Only two of the outcomes from $\text{SUM OF THE TWO VALUES IS } 7$ have a 2 in them. Thus there are only two outcomes in the compound event $\text{SUM OF THE TWO VALUES IS } 7 \text{ and EITHER DIE IS } 2$. Thus $P(\text{SUM OF THE TWO VALUES IS } 7 \text{ and EITHER DIE IS } 2) = \frac{2}{36} = \frac{1}{18}$.

4. Event: $\text{SUM OF THE TWO VALUES IS } 7 | \text{EITHER DIE IS } 2$

Answer

We first recall that $|$ denotes "given." Thus our event of interest is $\text{SUM OF THE TWO VALUES IS } 7 \text{ given EITHER DIE IS } 2$. We know that there are only 11 outcomes in the event $\text{EITHER DIE IS } 2$ and that only 2 of those outcomes (2 and 5, and 5 and 2) add up to 7. Thus $P(\text{SUM OF THE TWO VALUES IS } 7 | \text{EITHER DIE IS } 2) = \frac{2}{11}$.

5. Event: $\text{EITHER DIE IS } 2 | \text{SUM OF THE TWO VALUES IS } 7$

Answer

Our event of interest is $\text{EITHER DIE IS } 2 \text{ given SUM OF THE TWO VALUES IS } 7$. From our previous exercise, we know that there are only 6 outcomes in the event $\text{SUM OF THE TWO VALUES IS } 7$ and that only 2 of those outcomes, 2 and 5, and 5 and 2, include 2. Thus $P(\text{EITHER DIE IS } 2 | \text{SUM OF THE TWO VALUES IS } 7) = \frac{2}{6} = \frac{1}{3}$.

Terminological Consideration: Simple Events

Some statisticians and educators introduce the term simple event to help students better handle complex event descriptions. The general idea is that we can better understand or more easily compute probabilities when complex event descriptions are understood using combinations of other most basic events called simple events. The difficulty in such a presentation lies in the subjective and experiential aspects of understanding descriptions as basic and straightforward. To avoid the technical difficulty of a sufficient definition and application, we avoid the term throughout the book but, hopefully, still manage to instill the underlying concept.

Compound Events and Counting

Again, we build our intuition by counting when rolling a single fair die. When we were considering the compound event $\text{EVEN or PERFECT SQUARE}$, computing the probability boiled down to two counting questions: how large is the sample space and how many outcomes comprise the compound event? The first counting question has an answer of 6. Having already worked the problem, we know the answer to the second question is 4, but how does this relate to the two separate events that form our compound event? In counting the outcomes in EVEN, we arrive at 3, and the number of outcomes in PERFECT SQUARE is 2. It would appear that the number of outcomes in $\text{EVEN or PERFECT SQUARE}$ is $3+2=5$. Our previous work shows that the answer is 4. We must have counted some outcome(s) twice, which can happen whenever the events are not mutually exclusive. Recall that the outcome 4 is both even and a perfect square but should only be counted once. The outcomes that are counted twice are in both $\text{EVEN and PERFECT SQUARE}$. The total count is the number of outcomes in EVEN plus the number of outcomes in PERFECT SQUARE minus the number of outcomes in $\text{EVEN and PERFECT SQUARE}$; namely, $3+2-1=4$.

Formally, this is known as the **Addition Rule for Counting**. Given events A and B,

$$\#\text{ of outcomes in }(A\text{ or }B)=\#\text{ of outcomes in }A+\#\text{ of outcomes in }B-\#\text{ of outcomes in }(A\text{ and }B)$$

Text Exercise 3.3.3

Explain how we used the Addition Rule for Counting at the beginning of this section when counting tiles and committees.

Answer

We were attempting to determine the many outcomes in a sample space. Each case can be identified as a compound event, and we joined these compound events using the word "or." We added each of the counts together to get the total sum. This looks similar to the addition rule, except we never had to subtract anything since the cases were mutually exclusive; no two cases shared an outcome. This would be like subtracting 0 outcomes. The addition rule was at play the whole time.

Greater care must be taken when counting outcomes involving compound events formed using "and." We cannot mindlessly look to some formula that will always work; we must assess the situation and check that certain conditions are met. To develop our intuition regarding counting A and B, we turn our attention to a more robust but familiar example, rolling two fair dice in sequence, and analyze: $\text{EVEN FIRST and PERFECT SQUARE SECOND}$.

Context matters tremendously; read carefully and understand the current situation. We have dealt with the events EVEN and PERFECT SQUARE in the context of a roll of a single fair die. The sample space consists of 36 pairs of dice values; we are counting the number of outcomes with an even value for the first die and a perfect square for the second die. The multiplication rule worked well for determining the size of sample space because we could understand our outcomes as pairs of outcomes in sequence. Suppose we can understand $\text{EVEN FIRST and PERFECT SQUARE SECOND}$ as two events in sequence, the same ideas apply. We apply the multiplication rule with the first activity, rolling the first die with an even face up (3 ways), and the second activity, rolling the second die with a perfect square face up (2 ways). We arrive at the conclusion there are $3\cdot 2=6$ outcomes in $\text{EVEN FIRST and PERFECT SQUARE SECOND}$. For this example, we can easily construct the outcomes to confirm our count: 2 and 1, 4 and 1, 6 and 1, 2 and 4, 4 and 4, and 6 and 4.

Text Exercise 3.3.4

Within the context of rolling two die in sequence, determine the number of outcomes in $\text{ONE DIE IS EVEN WHILE THE OTHER IS A PERFECT SQUARE}$.

Answer

To save space, let $\text{EVEN}=E$ and $\text{PERFECT SQUARE}=PS$. We can understand the event as follows: $(E\text{ FIRST and }PS\text{ SECOND})$ or $(PS\text{ FIRST and }E\text{ SECOND})$. We can count the number of outcomes using both addition and multiplication rules. Applying the same logic as in the previous example and recognizing that 4 satisfies both events, yields:

$$\begin{gathered}\#\text{ in }E\text{ and }PS\\ =\#\text{ in}\\ (E\text{ FIRST and }PS\text{ SECOND})+\#\text{ in }(PS\text{ FIRST and }E\text{ SECOND})-\#\text{ in }((E\text{ FIRST and }PS\text{ SECOND})\text{ and }(PS\text{ FIRST and }E\text{ SECOND}))\\ =3\cdot 2+2\cdot 3-1\\ =6+6-1=11\end{gathered}$$

Note that it might be tempting to describe $\text{ONE DIE IS EVEN WHILE THE OTHER IS A PERFECT SQUARE}$ as the compound event $\text{EVEN and PERFECT SQUARE}$. In the context of rolling two dice, we may interpret the event differently. For example, rolling a double 4 (satisfying both events), one die is even, and the other is a perfect square (one die satisfying one and the other die satisfying the other), or as long as both conditions are met between the two dice, the event occurs (a 4 with any other die value). Clarity in communication is key; put thought into the event description to protect it from ambiguity to the best of your ability.

To conclude this section and formalize one last component of counting compound events, we consider a familiar context in probability: drawing cards from a standard deck of playing cards. We draw two cards in sequence from a single deck of standard playing cards without replacing the cards drawn and examine the compound event $\text{SPADE FIRST and BLACK CARD SECOND}$.

Our counting procedure closely models the counting in the previous section. The phrasing of the compound event helps us readily understand our compound event as consisting of pairs of outcomes in sequence and uses the multiplication rule. There is, however, one stark contrast. In rolling two fair die and considering $\text{EVEN FIRST and PERFECT SQUARE SECOND}$, the value of the first die did not affect the possible values of the second die. Whatever card we draw first cannot be drawn again in our current context. There is a dependency between the events. This does not hinder our ability to count using the multiplication rule; we have run into this idea before in Text Exercise $3.2.2\text{b}$. There are 13 spades and 26 black cards in a standard deck of playing cards. There are 13 ways for SPADE FIRST to happen. Since we are counting the number of ways $\text{SPADE FIRST and BLACK CARD SECOND}$ can happen, we count the number of ways our second card can be black given our first card was a spade. Since spades are black, only 25 black cards are left. We have the total number of outcomes $13 \cdot 25 = 325$.

We provide a **Multiplication Rule for Counting Compound Events with "and."** Given events A and B,

$$\# \text{ of outcomes in } A \text{ and } B = (\# \text{ outcomes in A}) \cdot (\# \text{ outcomes in B}|\text{A})$$

Text Exercise 3.3.5

Within the context of drawing two cards in sequence from a single deck of standard playing cards without replacing the cards drawn, determine the number of outcomes in the event $\text{BLACK CARD FIRST and SPADE SECOND}$.

Answer

We are in a similar situation to the previous example; however, the order of the events has switched. We can easily understand the events as in sequence and apply the multiplication rule.

$$\# \text{ of outcomes in BLACK CARD FIRST and SPADE SECOND} = (\# \text{ outcomes in BLACK CARD FIRST}) \cdot (\# \text{ outcomes in SPADE SECOND}|\text{BLACK CARD FIRST})$$

The count in the second term depends on whether or not a spade was drawn the first time. We, therefore, need to reformulate our approach. The event BLACK CARD consists of all the spades and clubs; so we can think of the event BLACK CARD as the compound event SPADE or CLUB. Let us refer to these events as S and C, respectively. So we can understand our event as follows:

$$(\text{S or C FIRST}) \text{ and S SECOND}$$
$$(\text{S FIRST and S SECOND}) \text{ or } (\text{C FIRST and S SECOND})$$

We can thus apply both addition and multiplication rules to compute our total number of outcomes.

$$\# \text{ of outcomes in S FIRST and S SECOND} = 13 \cdot 12 = 156$$
$$\# \text{ of outcomes in C FIRST and S SECOND} = 13 \cdot 13 = 169$$
$$\# \text{ of outcomes in } (\text{S FIRST and S SECOND}) \text{ and } (\text{C FIRST and S SECOND}) = 0$$

Thus the total number of outcomes is $156 + 169 - 0 = 325$.

Text Exercise 3.3.6

A family of 5 is attending a convention on family life. The theme of this year's convention is nature and quality time. The opening banquet will have 4 door prizes related to the current theme. The door prizes, in order, are a camper, a smokeless fire pit and patio furniture, a trampoline, and a set of bicycles. Any person in attendance can win at most one prize. The family of 5 recently invested in their backyard patio with new furniture and have a trampoline but are very interested in the other two prizes. If there will be 400 people in attendance, how many ways can the door prizes be awarded so that this family gets the first and the fourth prizes?

Answer

While winning quality prizes even if you do not need them can be exciting, the question excludes the case where the family wins all four prizes. The event of interest is $\text{WINNING FIRST and LOSING SECOND and LOSING THIRD and WINNING FOURTH}$. There are 5 ways of winning the first prize (one of the five family members must be chosen the winner), 395 ways of not winning the second prize given a successful win of the first prize, and 394 ways of not winning the third prize given the desired outcomes of the first and second prizes, and just 4 ways to win the last prize given the outcomes of the first three prizes. There are $5 \cdot 395 \cdot 394 \cdot 4 = 3,112,600$ possibilities. This could happen in many ways, but what is the probability that it does happen? There are $400 \cdot 399 \cdot 398 \cdot 397 = 25,217,757,600$ ways the prizes could be awarded. The probability of this becoming a reality is very small. $\frac{3,112,600}{25,217,575,600} \approx 0.01234\%$. It would be highly unlikely for the family to win the first and fourth prizes.