6.1: The Standard Normal Distribution

The **standard normal distribution** is a normal distribution of **standardized values called z-scores**. **A z-score is measured in units of the standard deviation.**

The mean for the standard normal distribution is 0, and the standard deviation is 1. What this does is dramatically simplify the mathematical calculation of probabilities. Take a moment and substitute 0 and 1 in the appropriate places in the above formula and you can see that the equation collapses into one that can be much more easily solved using integral calculus. The transformation $z=\frac{x-\mu}{\sigma}$ produces the distribution $Z \sim N(0,1)$. The value x in the given equation comes from a known normal distribution with known mean μ and known standard deviation σ. The z-score tells how many standard deviations a particular x is away from the mean.

Z-Scores

If X is a normally distributed random variable and $X \sim N(\mu,\sigma)$, then the z-score for a particular value x is:

$$z=\frac{x-\mu}{\sigma}$$

The z-score tells you how many standard deviations the value x is above (to the right of) or below (to the left of) the mean, μ. Values of x that are larger than the mean have positive z-scores, and values of x that are smaller than the mean have negative z-scores. If x equals the mean, then x has a z-score of zero.

Example 6.1.1

Suppose $X \sim N(5,6)$. This says that X is a normally distributed random variable with mean $\mu=5$ and standard deviation $\sigma=6$. Suppose $x=17$. Then:

$$z=\frac{x-\mu}{\sigma}=\frac{17-5}{6}=2$$

This means that $x=17$ is **two standard deviations** (2σ) above or to the right of the mean $\mu=5$.

Now suppose $x=1$. Then: $z=\frac{x-\mu}{\sigma}=\frac{1-5}{6}=-0.67$ (rounded to two decimal places)

This means that $\mathbf{x=1}$ is 0.67 standard deviations (-0.67σ) below or to the left of the mean $\mu=5$.

The Empirical Rule

If X is a random variable and has a normal distribution with mean μ and standard deviation σ, then **the Empirical Rule** states the following:

- About 68% of the x values lie between -1σ and $+1\sigma$ of the mean μ (within one standard deviation of the mean).
- About 95% of the x values lie between -2σ and $+2\sigma$ of the mean μ (within two standard deviations of the mean).
- About 99.7% of the x values lie between -3σ and $+3\sigma$ of the mean μ (within three standard deviations of the mean). Notice that almost all the x values lie within three standard deviations of the mean.
- The z-scores for $+1\sigma$ and -1σ are $+1$ and -1, respectively.
- The z-scores for $+2\sigma$ and -2σ are $+2$ and -2, respectively.
- The z-scores for $+3\sigma$ and -3σ are $+3$ and -3 respectively.

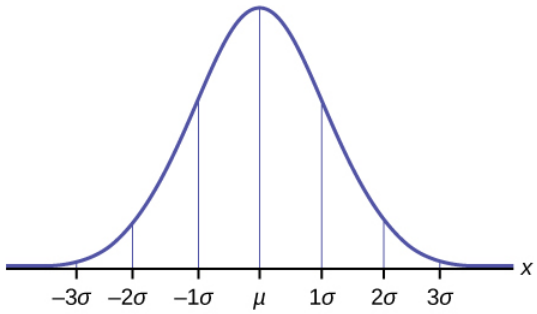

Figure 6.1.1

> **Example 6.1.2**
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> Suppose X has a normal distribution with mean 50 and standard deviation 6.
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> - About 68% of the X values lie within one standard deviation of the mean. Therefore, about 68% of the X values lie between $-1\sigma = (-1)(6) =- 6$ and $1\sigma = (1)(6) = 6$ of the mean 50. The values $50 - 6 = 44$ and $50 + 6 = 56$ are within one standard deviation from the mean 50. The z-scores are –1 and +1 for 44 and 56, respectively.
> - About 95% of the X values lie within two standard deviations of the mean. Therefore, about 95% of the X values lie between $-2\sigma = (-2)(6) =- 12$ and $2\sigma = (2)(6) = 12$. The values $50 - 12 = 38$ and $50 + 12 = 62$ are within two standard deviations from the mean 50. The z-scores are –2 and +2 for 38 and 62, respectively.
> - About 99.7% of the X values lie within three standard deviations of the mean. Therefore, about 99.7% of the X values lie between $-3\sigma = (-3)(6) =- 18$ and $3\sigma = (3)(6) = 18$ of the mean 50. The values $50 - 18 = 32$ and $50 + 18 = 68$ are within three standard deviations from the mean 50. The z-scores are –3 and +3 for 32 and 68, respectively.