10.4: Unpaired or Independent Sample t-Test

In comparing the variances of two populations we have one of two situations :

1. Homoscedasticity : $\sigma_1^2 = \sigma_2^2$
2. Heteroscedasticity : $\sigma_1^2 \neq \sigma_2^2$

These terms also apply when there are more than 2 populations. They either all have the same variance, or not. This affects how we do an independent sample t-test because we have two cases :

1. Variances of the two populations assumed unequal. $\boldsymbol{\sigma_1^2 \neq \sigma_2^2}$.

Then the test statistic is :

$$t_{\text{test}} = \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_1^2}{n_2}}} \tag{10.4.1}$$

This is the same formula as we used for the z-test. To find the critical statistic we will use, when solving problems by hand, degrees of freedom

$$\nu = \min(n_1 - 1, n_2 - 1).$$

This choice is a conservative approach (harder to reject H_0). SPSS uses a more accurate

$$\nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\left[\frac{\left(\frac{s_1^2}{n_1}\right)}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)}{n_2 - 1}\right]}$$

You will not need to use Equation (10.3), only Equation (10.2). Equation (10.3) gives fractional degrees of freedom. The t test statistic for this case and the degrees of freedom in Equation (10.3) is know as the Satterwaite approximation. The t-distributions are strictly only applicable if $\sigma_1 = \sigma_2$. The Satterwaite approximation is an adjustment to make the t-distributions fit this $\sigma_1 \neq \sigma_2$ case.

2. Variances of the two populations assumed equal. $\boldsymbol{\sigma_1 = \sigma_2 = \sigma}$.

In this case the test statistic is:

$$t_{\text{test}} = \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \tag{10.4.2}$$

This test statistic formula can be made more intuitive by defining

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

as the *pooled estimate of the variance*. s_p is the data estimate for the common population σ. s_p^2 is the weighted mean of the sample variances s_1^2 and s_2^2. Recall the generic weighted mean formula, Equation (3.2). The weights are $\nu_1 = n_1 - 1$ and $\nu_2 = n_1 - 1$; their sum is $\nu_1 + \nu_2 = n_1 - 1 + n_2 - 1 = n_1 + n_2 - 2$. In other words

$$s_p^2 = \frac{\nu_1 s_1^2 + \nu_2 s_2^2}{\nu_1 + \nu_2} \tag{10.4.3}$$

and we can write the test statistic as

$$t_{\text{test}} = \frac{(\bar{x}_1 - \bar{x}_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

See that $s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ is clearly a standard error of the mean.

10.4.1 General form of the *t* test statistic

All t statistics have the form :

$$t_{\text{test}} = \frac{\text{Difference of means}}{\text{Standard error of the mean}} = \frac{\text{Signal}}{\text{Noise}}. \tag{10.4.4}$$

Remember that! Memorizing complicated formulae is useless, but you should remember the basic form of a t test statistic.

10.4.2 Two step procedure for the independent samples *t* test

We will use the F test to decide whether to use case 1 or 2. SPSS uses a test called “Levine’s test” instead of the F test we developed to test $H_0 : \sigma_1^2 \neq \sigma_2^2$. Levine’s test also produces an F test statistic. It is a different F than our F but you interpret it in the same way. If the p-value of the F is high (larger than α) then assume $\sigma_1 = \sigma_2$, if the p-value is low (smaller than α) then assume $\sigma_1 \neq \sigma_2$. In real life, homoscedasticity is almost always assumed because the t-test is robust to violations of homoscedasticity until one sample set contains twice as many, or more, data points as the other.

Example 10.4: Case 1 example.

Given the following data summary :

$s_1 = 38$	$\bar{x}_1 = 191$	$n_1 = 8$
$s_2 = 12$	$\bar{x}_2 = 199$	$n_2 = 10$

(Note that $image$ (s_{2}=12)" title="Rendered by QuickLaTeX.com" height="19" width="164" style="vertical-align: -5px;">. If that wasn’t true, we could reverse the definitions of populations 1 and 2 so that $image$ 1" title="Rendered by QuickLaTeX.com" height="15" width="66" style="vertical-align: -3px;">.) Is $\overline{x}_1$ significantly different from $\overline{x}_2$? That is, is μ_1 different from μ_2? Test at $\alpha = 0.05$.

Solution :

So the question is to decide between

$$H_0 : \mu_1 = \mu_2 \qquad H_1 : \mu_1 \neq \mu_2 \tag{10.4.5}$$

a two-tailed test. But before we can test the question, we have to decide which t test statistic to use: case 1 or 2. So we need to do two hypotheses tests in a row. The first one to decide which t_{test} statistic to use, the second one to test the hypotheses of interest given above.

Test 1 : See if variances can be assumed equal or not.

1. Hypothesis.

$$H_0 : \sigma_1^2 = \sigma_2^2 \qquad H_1 : \sigma_1^2 \neq \sigma_2^2 \tag{10.4.6}$$

(Always use a two-tailed hypothesis when using the F test to decide between case 1 and 2 for the t test statistic.)

2. Critical statistic.

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(from the ***F* Distribution Table**)

(Here we used α given for the t-test question. But that is not necessary. You can use $\alpha = 0.05$ in general; the consequence of a type I error here is small because the t-test is robust to violations of the assumption of homoscedasticity.)

3. Test statistic.

$$F_{\text{test}} = F_{7,9} = \frac{s_1^2}{s_2^2} = \frac{38^2}{12^2} = 10.03 \tag{10.4.7}$$

4. Decision.

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5. Interpretation.

Assume the variances are unequal, $\sigma_1^2 \neq \sigma_2^2$, and use the t test statistic of case 1.

Test 2 : The question of interest.

1. Hypothesis.

$$H_0 : \mu_1 = \mu_2 \qquad H_1 : \mu_1 \neq \mu_2 \tag{10.4.8}$$

2. Critical statistic.

From the ***t* Distribution Table**, with $\nu = \min(n_1 - 1, n_2 - 1) = \min(8-1, 10-1)\min(7,9) = 7$, and a two-tailed test with $\alpha = 0.05$ we find

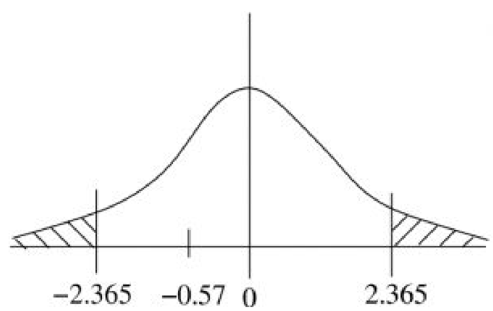
$$t_{\text{crit}} = \pm 2.365 \tag{10.4.9}$$

3. Test Statistic.

$$\begin{aligned} t &= \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \\ &= \frac{(191 - 199)}{\sqrt{\frac{38^2}{8} + \frac{12^2}{10}}} = -0.57 \end{aligned}$$

The p-value may be estimated from the ***t* Distribution Table** using the procedure given in Chapter 9: from the ***t* Distribution Table**, $\nu = 7$ line, find the values that bracket 0.57. There are none, the smallest value is 0.711 corresponding to $\alpha = 0.50$. So all we can say is $image$ 0.50" title="Rendered by QuickLaTeX.com" height="17" width="66" style="vertical-align: -4px;">.

4. Decision.



$t_{\text{test}} = -0.57$ is not in the rejection region so do not reject H_0. The estimate for the p-value confirms this decision.

5. Interpretation.

There is not enough evidence, at $\alpha = 0.05$ with the independent sample t-test, to conclude that the means of the populations are different.

□

Example 10.5 (Case 2 example) :

The following data seem to show that private nurses earn more than government nurses :

Private Nurses Salary	Government Nurses Salary
$\bar{x}_1 = 26,800$	$\bar{x}_2 = 25,400$
$s_1 = 600$	$s_2 = 450$
$n_1 = 10$	$n_2 = 8$

Testing at $\alpha = 0.01$, do private nurses earn more than government nurses?

Solution :

First confirm, or change, the population definitions so that $image$ s_{2}^{2}" title="Rendered by QuickLaTeX.com" height="20" width="55" style="vertical-align: -5px;">. This is already true so we are good to go.

Test 1 : See if variances can be assumed equal or not. This is a test of $H_0 : \sigma_1^2 = \sigma_2^2$ vs. $H_1 : \sigma_1^2 \neq \sigma_2^2$. After the test we find that we believe that $\sigma_1^2 = \sigma_2^2$ at $\alpha = 0.05$. So we will use the case 2, equal variances, t-test formula for test 2, the test of interest.

Test 2 : The question of interest.

1. Hypothesis.

$$H_0 : \mu_1 \leq \mu_2 \tag{10.4.10}$$

$H_1 : \mu_1 > \mu_2$ \mu_{2}\]" title="Rendered by QuickLaTeX.com">

(Note how H_1 reflects the face value of the data, that private nurses appear to earn more than government nurses in the population — it is true in the samples.)

2. Critical statistic.

Use the ***t* Distribution Table**, one-tailed test, $\alpha = 0.01$ (column) and $\nu = n_1 + n_2 - 2 = 10 + 8 - 2 = 16$ to find

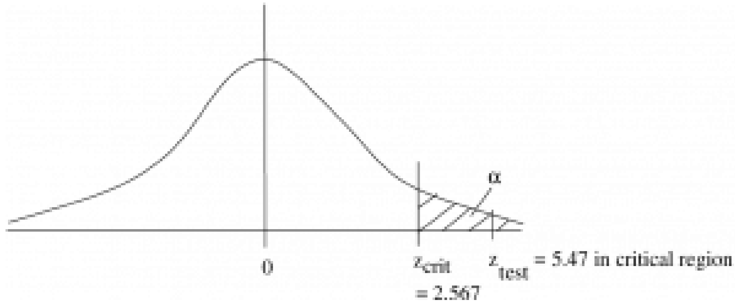
$$t_{\text{crit}} = 2.583 \tag{10.4.11}$$

3. Test statistic.

$$t_{\text{test}} = \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$
$$t_{\text{test}} = \frac{(26,800 - 25,400)}{\sqrt{\frac{(10-1)600^2 + (8-1)450^2}{10+8-2}}\sqrt{\frac{1}{10} + \frac{1}{8}}}$$
$$t_{\text{test}} = \frac{1400}{\sqrt{\frac{(9)(360000)+(7)(202500)}{16}}\sqrt{0.1 + 0.125}}$$
$$t_{\text{test}} = \frac{1400}{\sqrt{\frac{3240000+1417500}{16}}\sqrt{0.225}}$$
$$t_{\text{test}} = \frac{1400}{(\sqrt{291093.75})(\sqrt{0.225})} = 5.47$$

To estimate the p-value, look at the $\nu = 16$ line in the ***t* Distribution Table** to see if there are a pair of numbers that bracket $t_{\text{test}} = 5.47$. They are all smaller than 5.47 so p is less than the α associated with the largest number 2.921 whose α is 0.005 (one-tailed, remember). So $p < 0.005$.

4. Decision.



Reject H_0 since t_{test} is in the rejection region and $(p < 0.005) < (\alpha = 0.01)$.

$t_{test} > t_{crit}$ $(5.47 > 2.583)$ t_{crit} \hspace{.25in} (5.47 > 2.583)\]" title="Rendered by QuickLaTeX.com">

5. Interpretation.

From a t-test at $\alpha = 0.01$, there is enough evidence to conclude that private nurses earn more than government nurses.

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