8.3: The t-Distributions

As a broad introduction, the t-distributions are family of distributions that give different approximations to the z-distribution as shown in Figure 8.5.

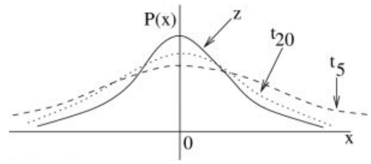

Figure 8.5 : The t-distributions are a family of distributions, labeled here by their degrees of freedom ν as in t_ν.

As the degrees of freedom, ν, increases, t_ν become closer to z, $\lim_{\nu\to\infty} t_\nu = z$. In practice, as reflected in the ***t* Distribution Table**, t_{30} is very very close to z.

The t-distributions arise as a corollary to the central limit theorem; they give the distribution of sample means when knowledge of the population σ is replaced by using the sample mean s. When we encounter the χ^2 distribution later, we will give a more exact mathematical specification of the t-distributions.

Similar, to the z-distribution case, the $\mathcal{C}$ confidence interval for the mean μ for small n samples is given by

$$\bar{x} - E < \mu < \bar{x} + E \tag{8.3.1}$$

where, now

$$E = t_{\nu,\mathcal{C}}\left(\frac{s}{\sqrt{n}}\right). \tag{8.3.2}$$

With this new formula for E we have replaced σ with s in comparison with the formula we used in Section 8.1: Confidence Intervals using the z-distribution and, of course, replaced $z_{\mathcal{C}}$ with $t_{\nu,\mathcal{C}}$. Some books use $t_{\nu,\mathcal{C}} = t_{\nu,\alpha/2}$ like the $z_{\mathcal{C}}$ of Section 8.1. We use $t_{\nu,\mathcal{C}}$ because we'll look up its value in the ***t* Distribution Table** in the column for $\mathcal{C}$ confidence intervals (just like we did with z) and with the degrees of freedom ν specifying the row. The formula for the degrees of freedom in this case is :

$$\nu = n - 1. \tag{8.3.3}$$

The $t_{\nu,\mathcal{C}}$ specify a probability $\mathcal{C}$ as shown in Figure 8.6. As before, the inverse z-transform, in the form $x = t_{\nu,\mathcal{C}} s + \bar{x}$ from the t-distribution on the left of Figure 8.6 to the distribution on the right of Figure 8.6 leads to our confidence interval formula for small means. And as before we should justify using that transform from a Bayesian perspective.

\(" class="alignnone size-medium wp-image-1620" height="145" width="300" src="/@api/deki/files/38426/fig57png-300x145.jpg" /> → inverse

Figure 8.6 : Derivation of confidence intervals for means of small samples.

Example 8.2 : Given the following data:

$$5460 \quad 5900 \quad 6090 \quad 6310 \quad 7160 \quad 8440 \quad 9930 \tag{8.3.4}$$

find the 99% confidence interval for the mean.

Solution : First count $n = 7$ and then, with your stats calculator compute

$$\bar{x} = 7041.4 \quad \text{and} \quad s = 1610.3. \tag{8.3.5}$$

Using the ***t* Distribution Table** with $\nu = n - 1 = 6$ in the 99% confidence interval column, find

$$t_{n-1,\mathcal{C}} = t_{6,99\%} = 3.707. \tag{8.3.6}$$

With these numbers, compute

$$E = t_{n-1,\mathcal{C}}\left(\frac{s}{\sqrt{n}}\right) = 3.707\left(\frac{1610.3}{\sqrt{7}}\right) = 2256.2 \tag{8.3.7}$$

so

$$\begin{aligned} \bar{x}-E & <\mu< & \bar{x}+E \\ 7041.4-2256.2 & <\mu< & 7041.4+2256.2 \\ 4785.2 & <\mu< & 9297.6 \end{aligned}$$

is the 99% confidence interval for μ.

□