8.1: Confidence Intervals Using the z-Distribution

With confidence intervals we will make our first statistical inference. Confidence intervals give us a direct inference about the population from a sample. The probability statement is one about hypotheses about the mean μ of the population based on the mean $\overline{x}$ and standard deviation s of the sample. This is a fine point. The frequentist definition of probability gives no way to assign a probability to a hypothesis. How do you count hypotheses? The central limit theorem makes a statement about the sample means $\overline{x}$ on the basis of a hypothesis about a population, about its mean μ and standard deviation σ. If the population is fixed then the central limit theorem gives the results of counting sample means, frequentist probabilities. If we let H represent a hypothesis about a population (i.e. that it is described by μ and σ) and let D represent data (with mean $\overline{x}$) then the central limit theorem gives the probability $P(D \mid H) = P(\overline{x} \mid \mu, \sigma)$. The confidence intervals that we'll look at first give $P(H \mid D) = P(\mu \mid \overline{x}, \sigma)$. We'll look at the recipe for computing confidence intervals for means first, then return to this discussion about probabilities for hypotheses.

Our goal is to define a symmetric interval about the population mean μ that will contain all potentially measured values of $\bar{x}$ with a probability[1] of $\mathcal{C}$.

Typically $\mathcal{C}$ will be

$$\mathcal{C} = 0.90 \qquad (90\backslash\% \text{ confidence}) \tag{8.1.1}$$

$$\mathcal{C} = 0.95 \qquad (95\backslash\% \text{ confidence}) \tag{8.1.2}$$

$$\mathcal{C} = 0.99 \qquad (99\backslash\% \text{ confidence}) \tag{8.1.3}$$

The assumptions that we need in order to use the z-distribution to compute confidence intervals for means are :

1. The population standard deviation, σ, is known (a somewhat artificial assumption since it is usually not known in an experimental situation) *or*
2. The sample size is greater than (or equal to) 30, $n \geq 30$ and we use $\sigma = s$, the sample standard deviation in our confidence interval formula.

Definition : Let $z_{\mathcal{C}} = z_{\alpha/2}$ where $\mathcal{C} = 1 - \alpha$ be the z-value, from the **Standard Normal Distribution Table** that corresponds to an area, between 0 and $z_{\mathcal{C}}$ of $\mathcal{C}/2$ as shown in Figure 8.1.

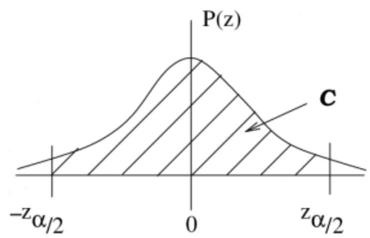

Figure 8.1 : The z-distribution areas of interest associated with $z_{\mathcal{C}} = z_{\alpha/2}$.

To get our confidence interval we simply inverse z-transform the picture of Figure 8.1, taking the mean of 0 to the sample mean $\bar{x}$ and the standard deviation of 1 to the standard error $\sigma/\sqrt{n}$ as shown in Figure 8.2.

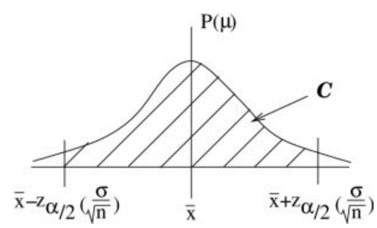

Figure 8.2 : The inverse z-transformation of Figure 8.1 gives the confidence interval for μ.

So here is our recipe from Figure 8.2. The $\mathcal{C}$-confidence interval for the mean, under one of the two assumptions given above, is :

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or using notation that we will use as a standard way of denoting symmetric confidence intervals

$$\bar{x} - E < \mu < \bar{x} + E$$

where

$$E=z_{\mathcal{C}}\left(\frac{\sigma}{\sqrt{n}}\right). \tag{8.1.4}$$

The notation $z_{\mathcal{C}}$ is more convenient for us than $z_{\alpha/2}$ because we will use the ***t* Distribution Table** in the Appendix to find $z_{\mathcal{C}}$ very quickly. We could equally well write

$$\mu=\bar{x}\pm E \tag{8.1.5}$$

but we will use Equation (8.1) because it explicitly gives the bounds for the confidence interval.

Notice how the confidence interval is *backwards* from the picture that the central limit theorem gives, the picture shown in Figure 8.3. We actually had no business using the inverse z-transformation $\mu=(z-\bar{x})/(\sigma/\sqrt{n})$ to arrive at Figure 8.2. It reverses the roles of μ and $\bar{x}$. We'll return to this point after we work through the mechanics of an example.

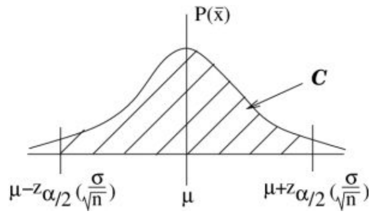

Figure 8.3 : The central limit theorem is about distributions of sample means.

Example 8.2 : What is the 95% confidence interval for student age if the population σ is 2 years, sample $n=50$, $\bar{x}=23.2$?

Solution : So $\mathcal{C}=0.95$. First write down the formula prescription so you can see with numbers you need:

$$\bar{x}-E<\mu<\bar{x}+E\backslash\text{hspace}\{2\text{em}\}\text{where}\backslash\text{hspace}\{2\text{em}\}E=z_{95\%}\frac{\sigma}{\sqrt{n}}. \tag{8.1.6}$$

First determine $z_{\mathcal{C}}=z_{\alpha/2}$. With the tables in the Appendices, there are two ways to do this. The first way is to use the **Standard Normal Distribution Table** noting that we need the z associated with a table area of $0.95/2=0.475$. Using the table backwards we find $z_{\mathcal{C}}=1.96$. The **second way**, *the recommended way* especially during exams, is to use the ***t* Distribution Table**. Simply find the column for the 95% confidence level and read the z from the last line of the table. We quickly find $z_{95\%}=1.960$.

Either way we now find

$$E=1.96\left(\frac{2}{\sqrt{50}}\right)=0.6 \tag{8.1.7}$$

so

$$\begin{aligned}\bar{x}-E &<\mu< \bar{x}+E\\ 23.2-0.6 &<\mu< 23.2+0.6\\ 22.6 &<\mu< 23.8\end{aligned}$$

with 95% confidence.

□

1. Because of this issue about probabilities of hypotheses, many prefer to say "confidence" and not probability. But we will learn enough about Bayesian probability to say "probability". ↩