11.1: z-Test for Comparing Proportions

In Section 9.4 we covered a one-sample test for proportions using the z approximation to the binomial distribution. Here we want to compare a proportion p_1 in one population with p_2 in another population, a two-sample test for proportions, also using the z approximation to the binomial distribution. Define

$$\hat{p}_1 = \frac{x_1}{n_1}\backslash\ \backslash\ \backslash\ \backslash\ \text{and}\backslash\ \backslash\ \backslash\ \backslash\ \hat{p}_2 = \frac{x_2}{n_2} \tag{11.1.1}$$

where x_1 and x_2 are the number of items of interest in the samples from the two populations and n_1 and n_2 are their sample sizes. Also define the corresponding $q_1 = 1 - p_1$, $q_2 = 1 - p_2$, $\hat{q}_1 = 1 - \hat{p}_1$ and $\hat{q}_2 = 1 - \hat{}p_2$. The hypotheses we want to test is

H_{0}: p_{1} = p_{2} \hspace*{3em} H_{1}: p_{1} \neq p_{2}

which is equivalent to

H_{0}: p_{1} - p_{2} = 0 \hspace*{3em} H_{1}: p_{1} - p_{2} \neq 0

If $n_1 p_1, n_1 q_1, n_2 p_2,$ and $n_2 q_2$ are all > 5 then the appropriate normal distribution will provide a good approximation to the relevant binomial distribution and we can use the following test statistic to test the hypotheses

$$z_{\text{test}} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \tag{11.1.2}$$

where

\bar{p} = \frac{x_{1}+x_{2}}{n_{1}+n_{2}} \hspace*{3em} \bar{q} = 1 -\bar{p}

are the proportions of items of interest and not of interest in the two samples combined.

Example 11.1 : In a nursing home study we are interested in the proportions of nursing homes that have vaccination rates of less than 80%. The two populations we want to compare are small nursing homes and large nursing homes. In a sample of 34 small nursing homes, 12 were found to have a vaccination rate of less than 80%. In a sample of 24 large nursing homes, 17 were found to have a vaccination rate of less than 80%. At $\alpha = 0.05$ is there a difference in the proportions of small and large nursing homes with vaccination rates of less than 80%?

Solution :

0. Data reduction.

First define: population 1 = small nursing homes and population 2 = large nursing homes. Then compute the proportions:

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{12}{34} = 0.35 \qquad\qquad \hat{p}_2 = \frac{x_2}{n_2} = \frac{17}{24} = 0.71 \tag{11.1.3}$$

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{12 + 17}{34 + 24} = \frac{29}{58} = 0.5 \qquad\qquad \bar{q} = 1 - \bar{p} = 1 - 0.5 = 0.5 \tag{11.1.4}$$

1. Hypotheses.

H_{0}: p_{1} = p_{2} \hspace*{3em} H_{1}: p_{1} \neq p_{2}

2. Critical statistic.

Use Table F, the last (z) line in the column for a two-tailed test at $\alpha = 0.05$: $z_{\text{crit}} = \pm 1.96$

3. Test statistic.

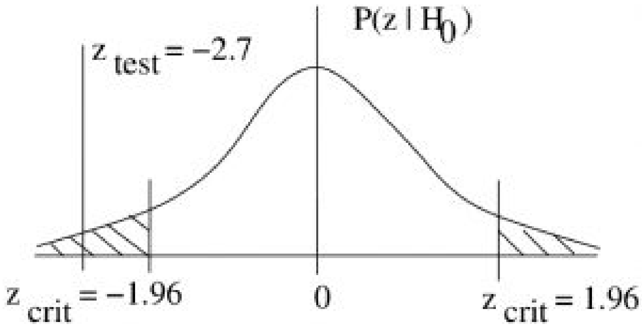
$$z_{\text{test}} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$z_{\text{test}} = \frac{0.35 - 0.71}{\sqrt{(0.5)(0.5)\left(\frac{1}{34} + \frac{1}{24}\right)}}$$

$$z_{\text{test}} = \frac{-0.36}{0.1333}$$

$$z_{\text{test}} = -2.7$$

4. Decision.



Reject H_0.

5. Interpretation.

There is enough evidence, from a z proportions test at $\alpha = 0.05$ to support the observation that large nursing homes have worse vaccination rates than small nursing homes. Make sure your parents end up in a small nursing home. (Note that rejection of H_0 in a one-tail test allows us to believe the direction of difference given by the sample data.)

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