4.1: Probability

The basic definition of probability is a ratio of things you can count (a ratio of their frequencies) :

$$P(E)=\frac{n(E)}{n(S)}$$

where

> $P(E)$ is the probability that event E happens,
> $n(E)$ is the number of ways E can happen and
> $n(S)$ is the total number of outcomes (all possibilities).

Example 4.1 : What is the probability of drawing a queen from a deck of cards :

$$P(E)=\frac{4}{52}=0.077\backslash\ \backslash\ \backslash\ (7.7\backslash\% \text{ if we were to express the result in percentages}) \tag{4.1.1}$$

□

To use $P(E)$ mathematically we set

$$0 \leq P(E) \leq 1 \tag{4.1.2}$$

Where, probability-wise:

> 0 means E definitely will not occur, and
> 1 means E definitely will occur.

This is a method we can use instead of using percent. To compute probabilities, we first need to know how to count.

Fundamental Counting Rule

Say you have n events in order, and for event i there are k_i ways for it to happen. Then the number of ways for the n events to play out is :

$$k_1 \cdot k_2 \cdot k_3 \backslash\text{hdots} k_n=\prod_{i=1}^{n} k_i \tag{4.1.3}$$

(The giant pi symbolizes a multiplication convention in the same way that a giant sigma symbolizes a summation convention as described in Section 1.3.)

Example 4.2 How many combinations are there on a lock with 3 numbers?

Lay out the events as : $k_1=10$, $k_2=10$, and $k_3=10$. Note that each number can be anything from 0 to 9 giving 10 possibilities ($k_i=10$) for each event. So the number of possible lock combinations is

$$k_1 k_2 k_3=10 \cdot 10 \cdot 10=10^3=1000 \tag{4.1.4}$$

Note that you could have guessed this because the combination range from 000 to 999 — counting in base 10.

□

Example 4.3 Suppose that a hardware store can produce paints with the following qualities :

> Colour : red, blue, white, black, green, brown, yellow (7 colours)
>
> Type : latex, oil (2 types)
>
> Texture : flat, semigloss, high-gloss (3 textures)
>
> Use : indoor, outdoor (2 uses)

How many ways are there to combine these qualities to produce a can of paint?

Answer : From the above list $k_1=7, k_2=2, k_3=3, k_4=2$ and the number of possible paint kinds is:

$$7 \cdot 2 \cdot 3 \cdot 2=84 \tag{4.1.5}$$

Applications of the Fundamental Counting Rule

We are interested in applying the fundamental counting rule to two special, important cases :

1. Permutations.
2. Combinations.

Let's define each one.

1. **Permutations**.

The number of ways, or permutations, of selecting r objects from a collection or n objects, *while keeping track of the order of selection* is [1]

$$ {}_nP_r = \frac{n!}{(n-r)!} \tag{4.1.6} $$

This formula follows from the fundamental counting rule. With n objects there are $k_1 = n$ ways to select the first object. After selecting the first object there are $n-1$ ways to choose the second object so $k_2 = n-1$, etc. up to $k_r = n-r+1$:

$$ {}_nP_r = (n)(n-1)(n-2)\hdots(n-r+1) \tag{4.1.7} $$

$$ = \frac{(n)(n-1)\hdots(2)(1)}{(n-r)(n-r-1)\hdots(2)(1)} \tag{4.1.8} $$

Example 4.4 : How many ways are there to choose 5 numbered balls from a bucket of 25 to make a lottery number?

Answer : $25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 = 6,375,600$ possibilities.

2. Combinations.

The number of ways of selecting x objects from a collection of n objects *without* caring about the order is :

\[{}_nC_x = \frac{n!}{(n-x)!x!} = \frac{3,628,800} = \underline{17.3 \times 10^{12}} \]

The symbol $\binom{n}{x}$ is also known as the *binomial coefficient* because it shows up in algebra when you expand expressions of the form $(x+y)^n$. For example[2]

$$ (x+y)^n = x^2 + 2xy + y^2 \tag{4.1.9} $$

$$ \begin{aligned} (x+y)^3 &= \binom{3}{0} x^3 + \binom{3}{1} x^2 y + \binom{3}{2} x y^2 + \binom{3}{3} y^3 \\ &= x^3 + 3x^2 y + 3xy^2 + y^3 \end{aligned} $$

The binomial coefficients can be quickly computed using Pascal's triangle :

$$ \begin{array}{ccccccccccccccc} & & & & & & & & & & & & & & n= \\ & & & & & & 1 & & & & & & & & 0 \\ & & & & & 1 & & 1 & & & & & & & 1 \\ & & & & 1 & & 2 & & 1 & & & & & & 2 \\ & & & 1 & & 3 & & 3 & & 1 & & & & & 3 \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & & & 4 \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & 5 \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & 6 \\ & & & & & & \text{etc.} & & & & & & & & \end{array} \tag{4.1.10} $$

Referring to Pascal's triangle we can quickly write

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 \tag{4.1.11}$$

for example.

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1. Recall that the definition of factorial follows $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ etc. ↵
2. You don't need this algebra for this statistics course. It's just interesting. ↵
