APPLIED STATISTICS FOR SOCIAL SCIENCE (19-20)

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CHAPTER OVERVIEW

1: Sampling and Data

Included in this chapter are the basic ideas and words of probability and statistics. You will soon understand that statistics and probability work together. You will also learn how data are gathered and what "good" data can be distinguished from "bad."

Contributors

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1.1: Introduction

Learning Objectives

By the end of this chapter, the student should be able to:

- Recognize and differentiate between key terms.
- Apply various types of sampling methods to data collection.
- Create and interpret frequency tables.

You are probably asking yourself the question, "When and where will I use statistics?" If you read any newspaper, watch television, or use the Internet, you will see statistical information. There are statistics about crime, sports, education, politics, and real estate. Typically, when you read a newspaper article or watch a television news program, you are given sample information. With this information, you may make a decision about the correctness of a statement, claim, or "fact." Statistical methods can help you make the "best educated guess."

Figure 1.1.1: We encounter statistics in our daily lives more often than we probably realize and from many different sources, like the news. (credit: David Sim)

Since you will undoubtedly be given statistical information at some point in your life, you need to know some techniques for analyzing the information thoughtfully. Think about buying a house or managing a budget. Think about your chosen profession. The fields of economics, business, psychology, education, biology, law, computer science, police science, and early childhood development require at least one course in statistics.

Included in this chapter are the basic ideas and words of probability and statistics. You will soon understand that statistics and probability work together. You will also learn how data are gathered and what "good" data can be distinguished from "bad."

1.2: Definitions of Statistics, Probability, and Key Terms

The science of statistics deals with the collection, analysis, interpretation, and presentation of data. We see and use data in our everyday lives.

Collaborative Exercise

In your classroom, try this exercise. Have class members write down the average time (in hours, to the nearest half-hour) they sleep per night. Your instructor will record the data. Then create a simple graph (called a **dot plot**) of the data. A dot plot consists of a number line and dots (or points) positioned above the number line. For example, consider the following data:

5; 5.5; 6; 6; 6; 6.5; 6.5; 6.5; 6.5; 7; 7; 8; 8; 9

The dot plot for this data would be as follows:

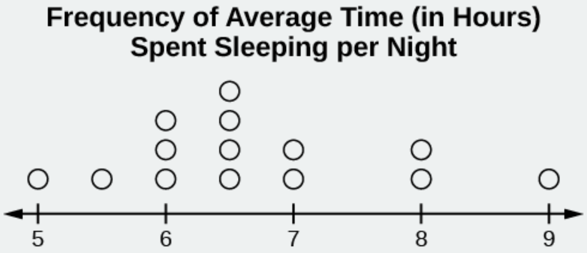


Figure 1.2.1

- Does your dot plot look the same as or different from the example? Why?
- If you did the same example in an English class with the same number of students, do you think the results would be the same? Why or why not?
- Where do your data appear to cluster? How might you interpret the clustering?

The questions above ask you to analyze and interpret your data. With this example, you have begun your study of statistics.

In this course, you will learn how to organize and summarize data. Organizing and summarizing data is called **descriptive statistics**. Two ways to summarize data are by graphing and by using numbers (for example, finding an average). After you have studied probability and probability distributions, you will use formal methods for drawing conclusions from "good" data. The formal methods are called **inferential statistics**. Statistical inference uses probability to determine how confident we can be that our conclusions are correct.

Effective interpretation of data (inference) is based on good procedures for producing data and thoughtful examination of the data. You will encounter what will seem to be too many mathematical formulas for interpreting data. The goal of statistics is not to perform numerous calculations using the formulas, but to gain an understanding of your data. The calculations can be done using a calculator or a computer. The understanding must come from you. If you can thoroughly grasp the basics of statistics, you can be more confident in the decisions you make in life.

Probability

Probability is a mathematical tool used to study randomness. It deals with the chance (the likelihood) of an event occurring. For example, if you toss a fair coin four times, the outcomes may not be two heads and two tails. However, if you toss the same coin 4,000 times, the outcomes will be close to half heads and half tails. The expected theoretical probability of heads in any one toss is $\frac{1}{2}$ or 0.5. Even though the outcomes of a few repetitions are uncertain, there is a regular pattern of outcomes when there are many repetitions. After reading about the English statistician Karl Pearson who tossed a coin 24,000 times with a result of 12,012 heads, one of the authors tossed a coin 2,000 times. The results were 996 heads. The fraction $\frac{996}{2000}$ is equal to 0.498 which is very close to 0.5, the expected probability.

The theory of probability began with the study of games of chance such as poker. Predictions take the form of probabilities. To predict the likelihood of an earthquake, of rain, or whether you will get an A in this course, we use probabilities. Doctors use probability to determine the chance of a vaccination causing the disease the vaccination is supposed to prevent. A stockbroker uses probability to determine the rate of return on a client's investments. You might use probability to decide to buy a lottery ticket or not. In your study of statistics, you will use the power of mathematics through probability calculations to analyze and interpret your data.

Key Terms

In statistics, we generally want to study a **population**. You can think of a population as a collection of persons, things, or objects under study. To study the population, we select a **sample**. The idea of **sampling** is to select a portion (or subset) of the larger population and study that portion (the sample) to gain information about the population. Data are the result of sampling from a population.

Because it takes a lot of time and money to examine an entire population, sampling is a very practical technique. If you wished to compute the overall grade point average at your school, it would make sense to select a sample of students who attend the school. The data collected from the sample would be the students' grade point averages. In presidential elections, opinion poll samples of 1,000–2,000 people are taken. The opinion poll is supposed to represent the views of the people in the entire country. Manufacturers of canned carbonated drinks take samples to determine if a 16 ounce can contains 16 ounces of carbonated drink.

From the sample data, we can calculate a statistic. A **statistic** is a number that represents a property of the sample. For example, if we consider one math class to be a sample of the population of all math classes, then the average number of points earned by students in that one math class at the end of the term is an example of a statistic. The statistic is an estimate of a population parameter. A **parameter** is a number that is a property of the population. Since we considered all math classes to be the population, then the average number of points earned per student over all the math classes is an example of a parameter.

One of the main concerns in the field of statistics is how accurately a statistic estimates a parameter. The accuracy really depends on how well the sample represents the population. The sample must contain the characteristics of the population in order to be a **representative sample**. We are interested in both the sample statistic and the population parameter in inferential statistics. In a later chapter, we will use the sample statistic to test the validity of the established population parameter.

A **variable**, notated by capital letters such as X and Y, is a characteristic of interest for each person or thing in a population. Variables may be **numerical** or **categorical**. **Numerical variables** take on values with equal units such as weight in pounds and time in hours. **Categorical variables** place the person or thing into a category. If we let X equal the number of points earned by one math student at the end of a term, then X is a numerical variable. If we let Y be a person's party affiliation, then some examples of Y include Republican, Democrat, and Independent. Y is a categorical variable. We could do some math with values of X (calculate the average number of points earned, for example), but it makes no sense to do math with values of Y (calculating an average party affiliation makes no sense).

Data are the actual values of the variable. They may be numbers or they may be words. **Datum** is a single value.

Two words that come up often in statistics are **mean** and **proportion**. If you were to take three exams in your math classes and obtain scores of 86, 75, and 92, you would calculate your mean score by adding the three exam scores and dividing by three (your mean score would be 84.3 to one decimal place). If, in your math class, there are 40 students and 22 are men and 18 are women, then the proportion of men students is $\frac{22}{40}$ and the proportion of women students is $\frac{18}{40}$. Mean and proportion are discussed in more detail in later chapters.

> The words "**mean**" and "**average**" are often used interchangeably. The substitution of one word for the other is common practice. The technical term is "arithmetic mean," and "average" is technically a center location. However, in practice among non-statisticians, "average" is commonly accepted for "arithmetic mean."

✓ Example 1.2.1

Determine what the key terms refer to in the following study. We want to know the average (mean) amount of money first year college students spend at ABC College on school supplies that do not include books. We randomly survey 100 first year students at the college. Three of those students spent $150, $200, and $225, respectively.

Answer

- The **population** is all first year students attending ABC College this term.
- The **sample** could be all students enrolled in one section of a beginning statistics course at ABC College (although this sample may not represent the entire population).

- The **parameter** is the average (mean) amount of money spent (excluding books) by first year college students at ABC College this term.
- The **statistic** is the average (mean) amount of money spent (excluding books) by first year college students in the sample.
- The **variable** could be the amount of money spent (excluding books) by one first year student. Let X = the amount of money spent (excluding books) by one first year student attending ABC College.
- The **data** are the dollar amounts spent by the first year students. Examples of the data are \$150, \$200, and \$225.

? Exercise 1.2.1

Determine what the key terms refer to in the following study. We want to know the average (mean) amount of money spent on school uniforms each year by families with children at Knoll Academy. We randomly survey 100 families with children in the school. Three of the families spent \$65, \$75, and \$95, respectively.

Answer

- The **population** is all families with children attending Knoll Academy.
- The **sample** is a random selection of 100 families with children attending Knoll Academy.
- The **parameter** is the average (mean) amount of money spent on school uniforms by families with children at Knoll Academy.
- The **statistic** is the average (mean) amount of money spent on school uniforms by families in the sample.
- The **variable** is the amount of money spent by one family. Let X = the amount of money spent on school uniforms by one family with children attending Knoll Academy.
- The **data** are the dollar amounts spent by the families. Examples of the data are \$65, \$75, and \$95.

✓ Example 1.2.2

Determine what the key terms refer to in the following study.

A study was conducted at a local college to analyze the average cumulative GPA’s of students who graduated last year. Fill in the letter of the phrase that best describes each of the items below.

1._____ Population 2._____ Statistic 3._____ Parameter 4._____ Sample 5._____ Variable 6._____ Data

a. all students who attended the college last year
b. the cumulative GPA of one student who graduated from the college last year
c. 3.65, 2.80, 1.50, 3.90
d. a group of students who graduated from the college last year, randomly selected
e. the average cumulative GPA of students who graduated from the college last year
f. all students who graduated from the college last year
g. the average cumulative GPA of students in the study who graduated from the college last year

Answer

1. f; 2. g; 3. e; 4. d; 5. b; 6. c

✓ Example 1.2.3

Determine what the key terms refer to in the following study.

As part of a study designed to test the safety of automobiles, the National Transportation Safety Board collected and reviewed data about the effects of an automobile crash on test dummies. Here is the criterion they used:

Speed at which Cars Crashed	Location of “drive” (i.e. dummies)
35 miles/hour	Front Seat

Cars with dummies in the front seats were crashed into a wall at a speed of 35 miles per hour. We want to know the proportion of dummies in the driver’s seat that would have had head injuries, if they had been actual drivers. We start with a simple random sample of 75 cars.

Answer

- The **population** is all cars containing dummies in the front seat.
- The **sample** is the 75 cars, selected by a simple random sample.
- The **parameter** is the proportion of driver dummies (if they had been real people) who would have suffered head injuries in the population.
- The **statistic** is proportion of driver dummies (if they had been real people) who would have suffered head injuries in the sample.
- The **variable** X = the number of driver dummies (if they had been real people) who would have suffered head injuries.
- The **data** are either: yes, had head injury, or no, did not.

✓ Example 1.2.4

Determine what the key terms refer to in the following study.

An insurance company would like to determine the proportion of all medical doctors who have been involved in one or more malpractice lawsuits. The company selects 500 doctors at random from a professional directory and determines the number in the sample who have been involved in a malpractice lawsuit.

Answer

- The **population** is all medical doctors listed in the professional directory.
- The **parameter** is the proportion of medical doctors who have been involved in one or more malpractice suits in the population.
- The **sample** is the 500 doctors selected at random from the professional directory.
- The **statistic** is the proportion of medical doctors who have been involved in one or more malpractice suits in the sample.
- The **variable** X = the number of medical doctors who have been involved in one or more malpractice suits.
- The **data** are either: yes, was involved in one or more malpractice lawsuits, or no, was not.

Collaborative Exercise

Do the following exercise collaboratively with up to four people per group. Find a population, a sample, the parameter, the statistic, a variable, and data for the following study: You want to determine the average (mean) number of glasses of milk college students drink per day. Suppose yesterday, in your English class, you asked five students how many glasses of milk they drank the day before. The answers were 1, 0, 1, 3, and 4 glasses of milk.

References

1. The Data and Story Library, https://dasl.datadescription.com/ (accessed May 1, 2013).

Practice

Use the following information to answer the next five exercises. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average (mean) length of time in months patients live once they start the treatment. Two researchers each follow a different set of 40 patients with AIDS from the start of treatment until their deaths. The following data (in months) are collected.

Researcher A:

3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34

Researcher B:

3; 14; 11; 5; 16; 17; 28; 41; 31; 18; 14; 14; 26; 25; 21; 22; 31; 2; 35; 44; 23; 21; 21; 16; 12; 18; 41; 22; 16; 25; 33; 34; 29; 13; 18; 24; 23; 42; 33; 29

Determine what the key terms refer to in the example for Researcher A.

? Exercise 1.2.2

population

Answer

AIDS patients.

? Exercise 1.2.3

sample

? Exercise 1.2.4

parameter

Answer

The average length of time (in months) AIDS patients live after treatment.

? Exercise 1.2.5

statistic

? Exercise 1.2.6

variable

Answer

$X =$ the length of time (in months) AIDS patients live after treatment

Glossary

The mathematical theory of statistics is easier to learn when you know the language. This module presents important terms that will be used throughout the text.

Average

also called mean; a number that describes the central tendency of the data

Categorical Variable

variables that take on values that are names or labels

Data

a set of observations (a set of possible outcomes); most data can be put into two groups: **qualitative**(an attribute whose value is indicated by a label) or **quantitative** (an attribute whose value is indicated by a number). Quantitative data can be separated into two subgroups: **discrete** and **continuous**. Data is discrete if it is the result of counting (such as the number of students of a given ethnic group in a class or the number of books on a shelf). Data is continuous if it is the result of measuring (such as distance traveled or weight of luggage)

Numerical Variable

variables that take on values that are indicated by numbers

Parameter

a number that is used to represent a population characteristic and that generally cannot be determined easily

Population

all individuals, objects, or measurements whose properties are being studied

Probability

a number between zero and one, inclusive, that gives the likelihood that a specific event will occur

Proportion

the number of successes divided by the total number in the sample

Representative Sample

a subset of the population that has the same characteristics as the population

Sample

a subset of the population studied

Statistic

a numerical characteristic of the sample; a statistic estimates the corresponding population parameter.

Variable

a characteristic of interest for each person or object in a population

1.3: Data, Sampling, and Variation in Data and Sampling

Data may come from a population or from a sample. Small letters like x or y generally are used to represent data values. Most data can be put into the following categories:

- Qualitative
- Quantitative

Qualitative data are the result of categorizing or describing attributes of a population. Hair color, blood type, ethnic group, the car a person drives, and the street a person lives on are examples of qualitative data. Qualitative data are generally described by words or letters. For instance, hair color might be black, dark brown, light brown, blonde, gray, or red. Blood type might be AB+, O-, or B+. Researchers often prefer to use quantitative data over qualitative data because it lends itself more easily to mathematical analysis. For example, it does not make sense to find an average hair color or blood type.

Quantitative data are always numbers. Quantitative data are the result of *counting* or *measuring* attributes of a population. Amount of money, pulse rate, weight, number of people living in your town, and number of students who take statistics are examples of quantitative data. Quantitative data may be either *discrete* or *continuous*.

All data that are the result of counting are called *quantitative discrete data*. These data take on only certain numerical values. If you count the number of phone calls you receive for each day of the week, you might get values such as zero, one, two, or three.

All data that are the result of measuring are *quantitative continuous data* assuming that we can measure accurately. Measuring angles in radians might result in such numbers as $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π, $\frac{3\pi}{4}$, and so on. If you and your friends carry backpacks with books in them to school, the numbers of books in the backpacks are discrete data and the weights of the backpacks are continuous data.

Sample of Quantitative Discrete Data

The data are the number of books students carry in their backpacks. You sample five students. Two students carry three books, one student carries four books, one student carries two books, and one student carries one book. The numbers of books (three, four, two, and one) are the **quantitative discrete data**.

Exercise 1.3.1

The data are the number of machines in a gym. You sample five gyms. One gym has 12 machines, one gym has 15 machines, one gym has ten machines, one gym has 22 machines, and the other gym has 20 machines. What type of data is this?

Answer

quantitative discrete data

Sample of Quantitative Continuous Data

The data are the weights of backpacks with books in them. You sample the same five students. The weights (in pounds) of their backpacks are 6.2, 7, 6.8, 9.1, 4.3. Notice that backpacks carrying three books can have different weights. Weights are **quantitative continuous data** because weights are measured.

Exercise 1.3.2

The data are the areas of lawns in square feet. You sample five houses. The areas of the lawns are 144 sq. feet, 160 sq. feet, 190 sq. feet, 180 sq. feet, and 210 sq. feet. What type of data is this?

Answer

quantitative continuous data

Exercise 1.3.3

You go to the supermarket and purchase three cans of soup (19 ounces) tomato bisque, 14.1 ounces lentil, and 19 ounces Italian wedding), two packages of nuts (walnuts and peanuts), four different kinds of vegetable (broccoli, cauliflower, spinach, and carrots), and two desserts (16 ounces Cherry Garcia ice cream and two pounds (32 ounces chocolate chip cookies).

Name data sets that are quantitative discrete, quantitative continuous, and qualitative.

Solution

One Possible Solution:

- The three cans of soup, two packages of nuts, four kinds of vegetables and two desserts are quantitative discrete data because you count them.
- The weights of the soups (19 ounces, 14.1 ounces, 19 ounces) are quantitative continuous data because you measure weights as precisely as possible.
- Types of soups, nuts, vegetables and desserts are qualitative data because they are categorical.

Try to identify additional data sets in this example.

Sample of qualitative data

The data are the colors of backpacks. Again, you sample the same five students. One student has a red backpack, two students have black backpacks, one student has a green backpack, and one student has a gray backpack. The colors red, black, black, green, and gray are **qualitative data**.

Exercise 1.3.4

The data are the colors of houses. You sample five houses. The colors of the houses are white, yellow, white, red, and white. What type of data is this?

Answer

qualitative data

Collaborative Exercise 1.3.1

Work collaboratively to determine the correct data type (quantitative or qualitative). Indicate whether quantitative data are continuous or discrete. Hint: Data that are discrete often start with the words "the number of."

a. the number of pairs of shoes you own
b. the type of car you drive
c. where you go on vacation
d. the distance it is from your home to the nearest grocery store
e. the number of classes you take per school year.
f. the tuition for your classes
g. the type of calculator you use
h. movie ratings
i. political party preferences
j. weights of sumo wrestlers
k. amount of money (in dollars) won playing poker
l. number of correct answers on a quiz
m. peoples’ attitudes toward the government
n. IQ scores (This may cause some discussion.)

Answer

Items a, e, f, k, and l are quantitative discrete; items d, j, and n are quantitative continuous; items b, c, g, h, i, and m are qualitative.

? Exercise 1.3.5

Determine the correct data type (quantitative or qualitative) for the number of cars in a parking lot. Indicate whether quantitative data are continuous or discrete.

Answer

quantitative discrete

? Exercise 1.3.6

A statistics professor collects information about the classification of her students as freshmen, sophomores, juniors, or seniors. The data she collects are summarized in the pie chart Figure 1.3.1. What type of data does this graph show?

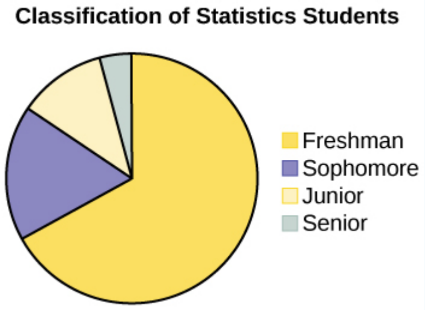


Figure 1.3.1

Answer

This pie chart shows the students in each year, which is **qualitative data**.

? Exercise 1.3.7

The registrar at State University keeps records of the number of credit hours students complete each semester. The data he collects are summarized in the histogram. The class boundaries are 10 to less than 13, 13 to less than 16, 16 to less than 19, 19 to less than 22, and 22 to less than 25.

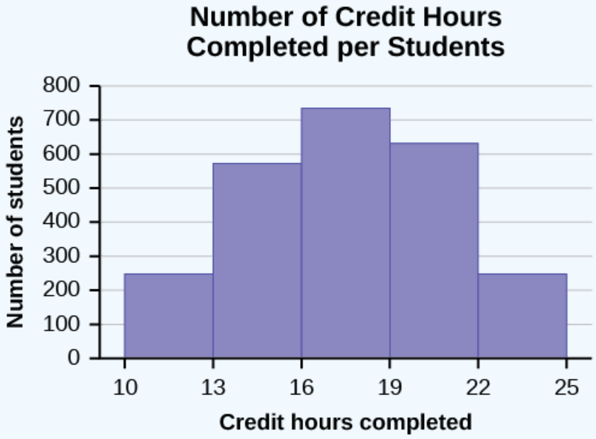


Figure 1.3.2

What type of data does this graph show?

Answer

A histogram is used to display quantitative data: the numbers of credit hours completed. Because students can complete only a whole number of hours (no fractions of hours allowed), this data is quantitative discrete.

Qualitative Data Discussion

Below are tables comparing the number of part-time and full-time students at De Anza College and Foothill College enrolled for the spring 2010 quarter. The tables display counts (frequencies) and percentages or proportions (relative frequencies). The percent columns make comparing the same categories in the colleges easier. Displaying percentages along with the numbers is often helpful, but it is particularly important when comparing sets of data that do not have the same totals, such as the total enrollments for both colleges in this example. Notice how much larger the percentage for part-time students at Foothill College is compared to De Anza College.

Table 1.3.1: Fall Term 2007 (Census day)

De Anza College				Foothill College		
	Number	Percent			Number	Percent
Full-time	9,200	40.9%		Full-time	4,059	28.6%
Part-time	13,296	59.1%		Part-time	10,124	71.4%
Total	22,496	100%		Total	14,183	100%

Tables are a good way of organizing and displaying data. But graphs can be even more helpful in understanding the data. There are no strict rules concerning which graphs to use. Two graphs that are used to display qualitative data are pie charts and bar graphs.

- In a **pie chart**, categories of data are represented by wedges in a circle and are proportional in size to the percent of individuals in each category.
- In a **bar graph**, the length of the bar for each category is proportional to the number or percent of individuals in each category. Bars may be vertical or horizontal.
- A **Pareto chart** consists of bars that are sorted into order by category size (largest to smallest).

Look at Figures 1.3.3 and 1.3.4 and determine which graph (pie or bar) you think displays the comparisons better.

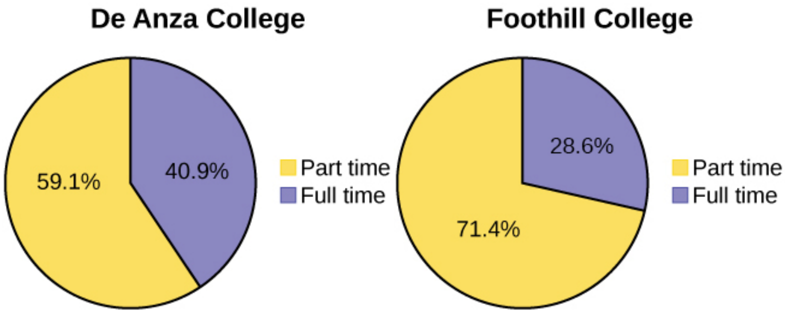


Figure 1.3.3: Pie Charts

It is a good idea to look at a variety of graphs to see which is the most helpful in displaying the data. We might make different choices of what we think is the “best” graph depending on the data and the context. Our choice also depends on what we are using the data for.

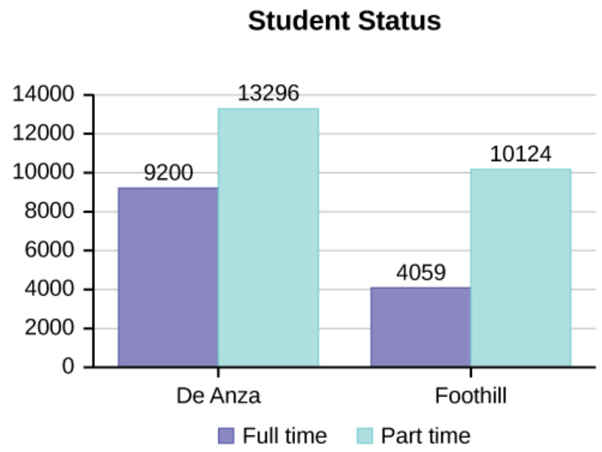


Figure 1.3.4: Bar chart

Percentages That Add to More (or Less) Than 100%

Sometimes percentages add up to be more than 100% (or less than 100%). In the graph, the percentages add to more than 100% because students can be in more than one category. A bar graph is appropriate to compare the relative size of the categories. A pie chart cannot be used. It also could not be used if the percentages added to less than 100%.

Table 1.3.2: De Anza College Spring 2010

Characteristic/Category	Percent
Full-Time Students	40.9%
Students who intend to transfer to a 4-year educational institution	48.6%
Students under age 25	61.0%
TOTAL	150.5%

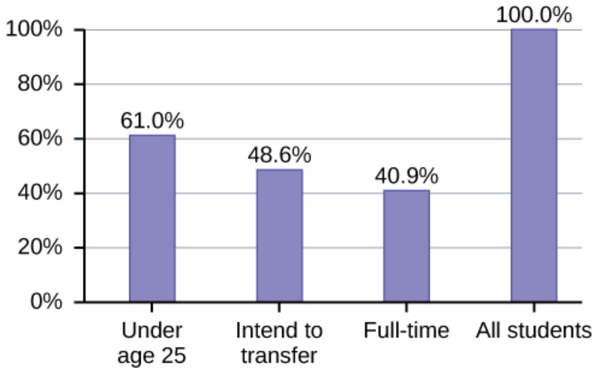


Figure 1.3.2: Bar chart of data in Table 1.3.2.

Omitting Categories/Missing Data

The table displays Ethnicity of Students but is missing the "Other/Unknown" category. This category contains people who did not feel they fit into any of the ethnicity categories or declined to respond. Notice that the frequencies do not add up to the total number of students. In this situation, create a bar graph and not a pie chart.

Table 1.3.2: Ethnicity of Students at De Anza College Fall Term 2007 (Census Day)

	Frequency	Percent
Asian	8,794	36.1%
Black	1,412	5.8%

	Frequency	Percent
Filipino	1,298	5.3%
Hispanic	4,180	17.1%
Native American	146	0.6%
Pacific Islander	236	1.0%
White	5,978	24.5%
TOTAL	22,044 out of 24,382	90.4% out of 100%

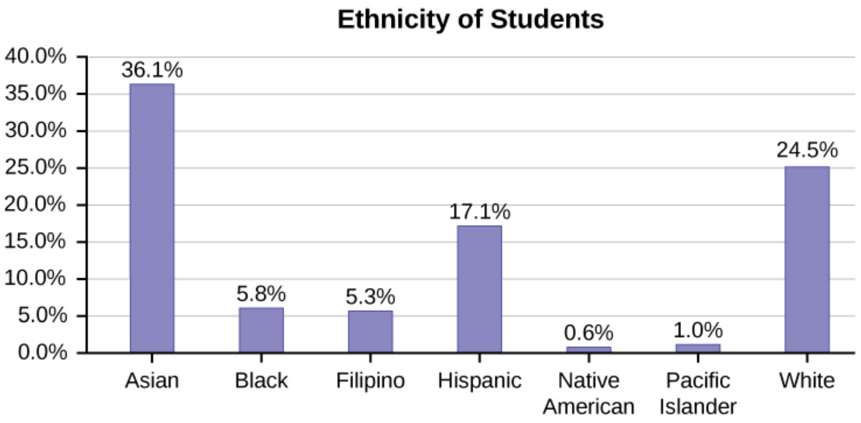


Figure 1.3.3: Enrollment of De Anza College (Spring 2010)

The following graph is the same as the previous graph but the “Other/Unknown” percent (9.6%) has been included. The “Other/Unknown” category is large compared to some of the other categories (Native American, 0.6%, Pacific Islander 1.0%). This is important to know when we think about what the data are telling us.

This particular bar graph in Figure 1.3.4 can be difficult to understand visually. The graph in Figure 1.3.5 is a *Pareto chart*. The Pareto chart has the bars sorted from largest to smallest and is easier to read and interpret.

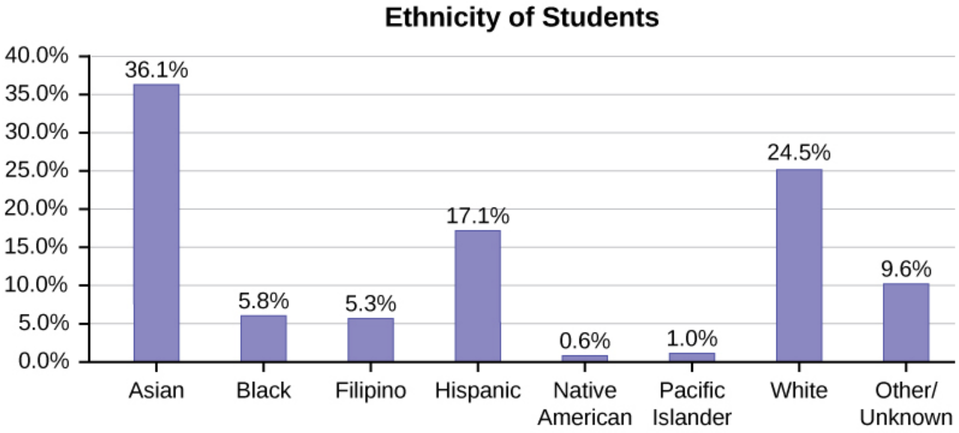


Figure 1.3.4: Bar Graph with Other/Unknown Category

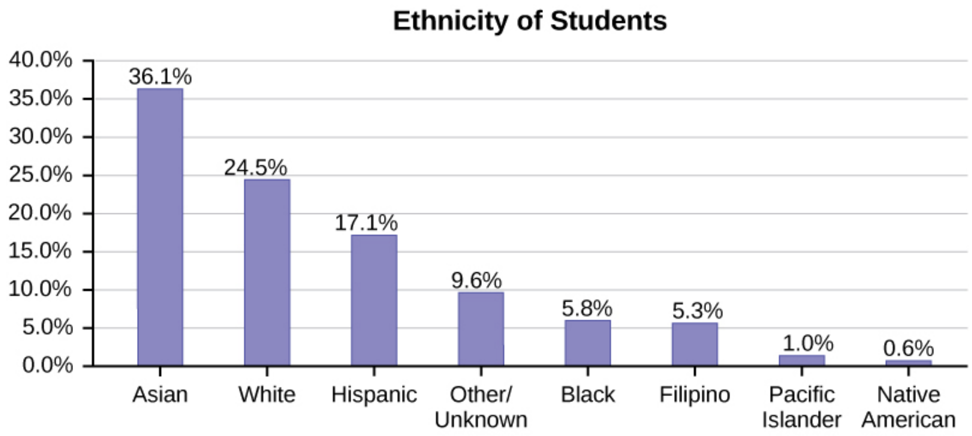


Figure 1.3.5: Pareto Chart With Bars Sorted by Size

Pie Charts: No Missing Data

The following pie charts have the "Other/Unknown" category included (since the percentages must add to 100%). The chart in Figure 1.3.6 is organized by the size of each wedge, which makes it a more visually informative graph than the unsorted, alphabetical graph in Figure 1.3.6.

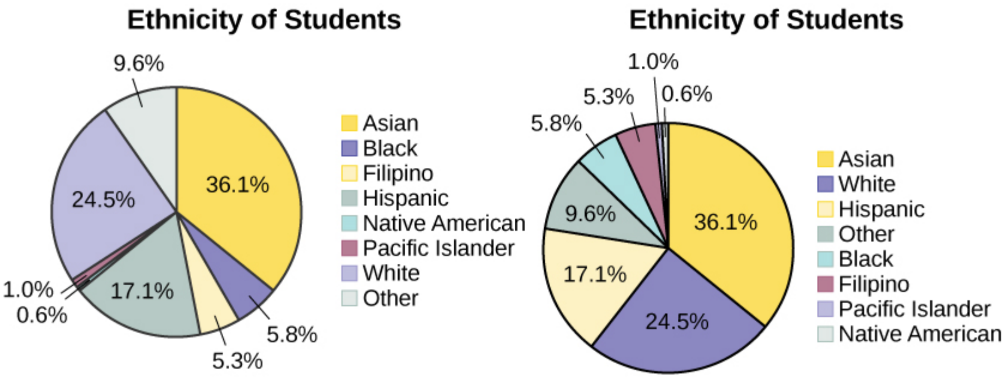


Figure 1.3.6.

Sampling

Gathering information about an entire population often costs too much or is virtually impossible. Instead, we use a sample of the population. **A sample should have the same characteristics as the population it is representing.** Most statisticians use various methods of random sampling in an attempt to achieve this goal. This section will describe a few of the most common methods. There are several different methods of **random sampling**. In each form of random sampling, each member of a population initially has an equal chance of being selected for the sample. Each method has pros and cons. The easiest method to describe is called a **simple random sample**. Any group of *n* individuals is equally likely to be chosen by any other group of *n* individuals if the simple random sampling technique is used. In other words, each sample of the same size has an equal chance of being selected. For example, suppose Lisa wants to form a four-person study group (herself and three other people) from her pre-calculus class, which has 31 members not including Lisa. To choose a simple random sample of size three from the other members of her class, Lisa could put all 31 names in a hat, shake the hat, close her eyes, and pick out three names. A more technological way is for Lisa to first list the last names of the members of her class together with a two-digit number, as in Table 1.3.2:

Table 1.3.3: Class Roster

ID	Name	ID	Name	ID	Name
00	Anselmo	11	King	21	Roquero
01	Bautista	12	Legeny	22	Roth
02	Bayani	13	Lundquist	23	Rowell

ID	Name	ID	Name	ID	Name
03	Cheng	14	Macierz	24	Salangsang
04	Cuarismo	15	Motogawa	25	Slade
05	Cuningham	16	Okimoto	26	Stratcher
06	Fontecha	17	Patel	27	Tallai
07	Hong	18	Price	28	Tran
08	Hoobler	19	Quizon	29	Wai
09	Jiao	20	Reyes	30	Wood
10	Khan				

Lisa can use a table of random numbers (found in many statistics books and mathematical handbooks), a calculator, or a computer to generate random numbers. For this example, suppose Lisa chooses to generate random numbers from a calculator. The numbers generated are as follows:

0.94360; 0.99832; 0.14669; 0.51470; 0.40581; 0.73381; 0.04399

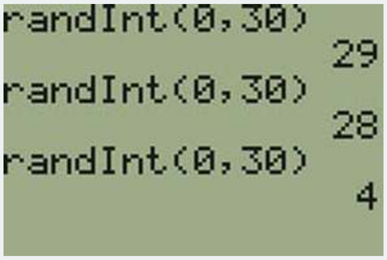
Lisa reads two-digit groups until she has chosen three class members (that is, she reads 0.94360 as the groups 94, 43, 36, 60). Each random number may only contribute one class member. If she needed to, Lisa could have generated more random numbers.

The random numbers 0.94360 and 0.99832 do not contain appropriate two digit numbers. However the third random number, 0.14669, contains 14 (the fourth random number also contains 14), the fifth random number contains 05, and the seventh random number contains 04. The two-digit number 14 corresponds to Macierz, 05 corresponds to Cuningham, and 04 corresponds to Cuarismo. Besides herself, Lisa's group will consist of Marcierz, Cuningham, and Cuarismo.

To generate random numbers:

- Press MATH.
- Arrow over to PRB.
- Press 5:randInt(. Enter 0, 30).
- Press ENTER for the first random number.
- Press ENTER two more times for the other 2 random numbers. If there is a repeat press ENTER again.

Note: randInt(0, 30, 3) will generate 3 random numbers.



```
randInt(0,30)
           29
randInt(0,30)
           28
randInt(0,30)
            4
```

Figure 1.3.7

Besides simple random sampling, there are other forms of sampling that involve a chance process for getting the sample. **Other well-known random sampling methods are the stratified sample, the cluster sample, and the systematic sample.**

To choose a **stratified sample**, divide the population into groups called strata and then take a **proportionate** number from each stratum. For example, you could stratify (group) your college population by department and then choose a proportionate simple random sample from each stratum (each department) to get a stratified random sample. To choose a simple random sample from each department, number each member of the first department, number each member of the second department, and do the same for the remaining departments. Then use simple random sampling to choose proportionate numbers from the first department and do the same for each of the remaining departments. Those numbers picked from the first department, picked from the second department, and so on represent the members who make up the stratified sample.

To choose a **cluster sample**, divide the population into clusters (groups) and then randomly select some of the clusters. All the members from these clusters are in the cluster sample. For example, if you randomly sample four departments from your college population, the four departments make up the cluster sample. Divide your college faculty by department. The departments are the clusters. Number each department, and then choose four different numbers using simple random sampling. All members of the four departments with those numbers are the cluster sample.

To choose a **systematic sample**, randomly select a starting point and take every n^{th} piece of data from a listing of the population. For example, suppose you have to do a phone survey. Your phone book contains 20,000 residence listings. You must choose 400 names for the sample. Number the population 1–20,000 and then use a simple random sample to pick a number that represents the first name in the sample. Then choose every fiftieth name thereafter until you have a total of 400 names (you might have to go back to the beginning of your phone list). Systematic sampling is frequently chosen because it is a simple method.

A type of sampling that is non-random is convenience sampling. **Convenience sampling** involves using results that are readily available. For example, a computer software store conducts a marketing study by interviewing potential customers who happen to be in the store browsing through the available software. The results of convenience sampling may be very good in some cases and highly biased (favor certain outcomes) in others.

Sampling data should be done very carefully. Collecting data carelessly can have devastating results. Surveys mailed to households and then returned may be very biased (they may favor a certain group). It is better for the person conducting the survey to select the sample respondents.

True random sampling is done **with replacement**. That is, once a member is picked, that member goes back into the population and thus may be chosen more than once. However for practical reasons, in most populations, simple random sampling is done **without replacement**. Surveys are typically done without replacement. That is, a member of the population may be chosen only once. Most samples are taken from large populations and the sample tends to be small in comparison to the population. Since this is the case, sampling without replacement is approximately the same as sampling with replacement because the chance of picking the same individual more than once with replacement is very low.

In a college population of 10,000 people, suppose you want to pick a sample of 1,000 randomly for a survey. **For any particular sample of 1,000**, if you are sampling **with replacement**,

- the chance of picking the first person is 1,000 out of 10,000 (0.1000);
- the chance of picking a different second person for this sample is 999 out of 10,000 (0.0999);
- the chance of picking the same person again is 1 out of 10,000 (very low).

If you are sampling **without replacement**,

- the chance of picking the first person for any particular sample is 1000 out of 10,000 (0.1000);
- the chance of picking a different second person is 999 out of 9,999 (0.0999);
- you do not replace the first person before picking the next person.

Compare the fractions 999/10,000 and 999/9,999. For accuracy, carry the decimal answers to four decimal places. To four decimal places, these numbers are equivalent (0.0999).

Sampling without replacement instead of sampling with replacement becomes a mathematical issue only when the population is small. For example, if the population is 25 people, the sample is ten, and you are sampling **with replacement for any particular sample**, then the chance of picking the first person is ten out of 25, and the chance of picking a different second person is nine out of 25 (you replace the first person).

If you sample **without replacement**, then the chance of picking the first person is ten out of 25, and then the chance of picking the second person (who is different) is nine out of 24 (you do not replace the first person).

Compare the fractions 9/25 and 9/24. To four decimal places, 9/25 = 0.3600 and 9/24 = 0.3750. To four decimal places, these numbers are not equivalent.

When you analyze data, it is important to be aware of **sampling errors** and nonsampling errors. The actual process of sampling causes sampling errors. For example, the sample may not be large enough. Factors not related to the sampling process cause **nonsampling errors**. A defective counting device can cause a nonsampling error.

In reality, a sample will never be exactly representative of the population so there will always be some sampling error. As a rule, the larger the sample, the smaller the sampling error.

In statistics, **a sampling bias** is created when a sample is collected from a population and some members of the population are not as likely to be chosen as others (remember, each member of the population should have an equally likely chance of being chosen). When a sampling bias happens, there can be incorrect conclusions drawn about the population that is being studied.

? Exercise 1.3.8

A study is done to determine the average tuition that San Jose State undergraduate students pay per semester. Each student in the following samples is asked how much tuition he or she paid for the Fall semester. What is the type of sampling in each case?

a. A sample of 100 undergraduate San Jose State students is taken by organizing the students' names by classification (freshman, sophomore, junior, or senior), and then selecting 25 students from each.
b. A random number generator is used to select a student from the alphabetical listing of all undergraduate students in the Fall semester. Starting with that student, every 50th student is chosen until 75 students are included in the sample.
c. A completely random method is used to select 75 students. Each undergraduate student in the fall semester has the same probability of being chosen at any stage of the sampling process.
d. The freshman, sophomore, junior, and senior years are numbered one, two, three, and four, respectively. A random number generator is used to pick two of those years. All students in those two years are in the sample.
e. An administrative assistant is asked to stand in front of the library one Wednesday and to ask the first 100 undergraduate students he encounters what they paid for tuition the Fall semester. Those 100 students are the sample.

Answer

a. stratified; b. systematic; c. simple random; d. cluster; e. convenience

✓ Example 1.3.9: Calculator

You are going to use the random number generator to generate different types of samples from the data. This table displays six sets of quiz scores (each quiz counts 10 points) for an elementary statistics class.

#1	#2	#3	#4	#5	#6
5	7	10	9	8	3
10	5	9	8	7	6
9	10	8	6	7	9
9	10	10	9	8	9
7	8	9	5	7	4
9	9	9	10	8	7
7	7	10	9	8	8
8	8	9	10	8	8
9	7	8	7	7	8
8	8	10	9	8	7

Instructions: Use the Random Number Generator to pick samples.

a. Create a stratified sample by column. Pick three quiz scores randomly from each column.
   - Number each row one through ten.
   - On your calculator, press Math and arrow over to PRB.

- For column 1, Press 5:randInt(and enter 1,10). Press ENTER. Record the number. Press ENTER 2 more times (even the repeats). Record these numbers. Record the three quiz scores in column one that correspond to these three numbers.
- Repeat for columns two through six.
- These 18 quiz scores are a stratified sample.

b. Create a cluster sample by picking two of the columns. Use the column numbers: one through six.

- Press MATH and arrow over to PRB.
- Press 5:randInt(and enter 1,6). Press ENTER. Record the number. Press ENTER and record that number.
- The two numbers are for two of the columns.
- The quiz scores (20 of them) in these 2 columns are the cluster sample.

c. Create a simple random sample of 15 quiz scores.

- Use the numbering one through 60.
- Press MATH. Arrow over to PRB. Press 5:randInt(and enter 1, 60).
- Press ENTER 15 times and record the numbers.
- Record the quiz scores that correspond to these numbers.
- These 15 quiz scores are the systematic sample.

d. Create a systematic sample of 12 quiz scores.

- Use the numbering one through 60.
- Press MATH. Arrow over to PRB. Press 5:randInt(and enter 1, 60).
- Press ENTER. Record the number and the first quiz score. From that number, count ten quiz scores and record that quiz score. Keep counting ten quiz scores and recording the quiz score until you have a sample of 12 quiz scores. You may wrap around (go back to the beginning).

✓ Example 1.3.10

Determine the type of sampling used (simple random, stratified, systematic, cluster, or convenience).

a. A soccer coach selects six players from a group of boys aged eight to ten, seven players from a group of boys aged 11 to 12, and three players from a group of boys aged 13 to 14 to form a recreational soccer team.
b. A pollster interviews all human resource personnel in five different high tech companies.
c. A high school educational researcher interviews 50 high school female teachers and 50 high school male teachers.
d. A medical researcher interviews every third cancer patient from a list of cancer patients at a local hospital.
e. A high school counselor uses a computer to generate 50 random numbers and then picks students whose names correspond to the numbers.
f. A student interviews classmates in his algebra class to determine how many pairs of jeans a student owns, on the average.

Answer

a. stratified; b. cluster; c. stratified; d. systematic; e. simple random; f.convenience

? Exercise 1.3.11

Determine the type of sampling used (simple random, stratified, systematic, cluster, or convenience).

A high school principal polls 50 freshmen, 50 sophomores, 50 juniors, and 50 seniors regarding policy changes for after school activities.

Answer

stratified

If we were to examine two samples representing the same population, even if we used random sampling methods for the samples, they would not be exactly the same. Just as there is variation in data, there is variation in samples. As you become accustomed to sampling, the variability will begin to seem natural.

✓ Example 1.3.12: Sampling

Suppose ABC College has 10,000 part-time students (the population). We are interested in the average amount of money a part-time student spends on books in the fall term. Asking all 10,000 students is an almost impossible task. Suppose we take two different samples.

First, we use convenience sampling and survey ten students from a first term organic chemistry class. Many of these students are taking first term calculus in addition to the organic chemistry class. The amount of money they spend on books is as follows:

$128; $87; $173; $116; $130; $204; $147; $189; $93; $153

The second sample is taken using a list of senior citizens who take P.E. classes and taking every fifth senior citizen on the list, for a total of ten senior citizens. They spend:

$50; $40; $36; $15; $50; $100; $40; $53; $22; $22

a. Do you think that either of these samples is representative of (or is characteristic of) the entire 10,000 part-time student population?

Answer

a. No. The first sample probably consists of science-oriented students. Besides the chemistry course, some of them are also taking first-term calculus. Books for these classes tend to be expensive. Most of these students are, more than likely, paying more than the average part-time student for their books. The second sample is a group of senior citizens who are, more than likely, taking courses for health and interest. The amount of money they spend on books is probably much less than the average parttime student. Both samples are biased. Also, in both cases, not all students have a chance to be in either sample.

b. Since these samples are not representative of the entire population, is it wise to use the results to describe the entire population?

Answer

b. No. For these samples, each member of the population did not have an equally likely chance of being chosen.

Now, suppose we take a third sample. We choose ten different part-time students from the disciplines of chemistry, math, English, psychology, sociology, history, nursing, physical education, art, and early childhood development. (We assume that these are the only disciplines in which part-time students at ABC College are enrolled and that an equal number of part-time students are enrolled in each of the disciplines.) Each student is chosen using simple random sampling. Using a calculator, random numbers are generated and a student from a particular discipline is selected if he or she has a corresponding number. The students spend the following amounts:

$180; $50; $150; $85; $260; $75; $180; $200; $200; $150

c. Is the sample biased?

Answer

Students often ask if it is "good enough" to take a sample, instead of surveying the entire population. If the survey is done well, the answer is yes.

? Exercise 1.3.12

A local radio station has a fan base of 20,000 listeners. The station wants to know if its audience would prefer more music or more talk shows. Asking all 20,000 listeners is an almost impossible task.

The station uses convenience sampling and surveys the first 200 people they meet at one of the station’s music concert events. 24 people said they’d prefer more talk shows, and 176 people said they’d prefer more music.

Do you think that this sample is representative of (or is characteristic of) the entire 20,000 listener population?

Answer

The sample probably consists more of people who prefer music because it is a concert event. Also, the sample represents only those who showed up to the event earlier than the majority. The sample probably doesn't represent the entire fan base and is probably biased towards people who would prefer music.

Collaborative Exercise 1.3.8

As a class, determine whether or not the following samples are representative. If they are not, discuss the reasons.

a. To find the average GPA of all students in a university, use all honor students at the university as the sample.
b. To find out the most popular cereal among young people under the age of ten, stand outside a large supermarket for three hours and speak to every twentieth child under age ten who enters the supermarket.
c. To find the average annual income of all adults in the United States, sample U.S. congressmen. Create a cluster sample by considering each state as a stratum (group). By using simple random sampling, select states to be part of the cluster. Then survey every U.S. congressman in the cluster.
d. To determine the proportion of people taking public transportation to work, survey 20 people in New York City. Conduct the survey by sitting in Central Park on a bench and interviewing every person who sits next to you.
e. To determine the average cost of a two-day stay in a hospital in Massachusetts, survey 100 hospitals across the state using simple random sampling.

Variation in Data

Variation is present in any set of data. For example, 16-ounce cans of beverage may contain more or less than 16 ounces of liquid. In one study, eight 16 ounce cans were measured and produced the following amount (in ounces) of beverage:

15.8; 16.1; 15.2; 14.8; 15.8; 15.9; 16.0; 15.5

Measurements of the amount of beverage in a 16-ounce can may vary because different people make the measurements or because the exact amount, 16 ounces of liquid, was not put into the cans. Manufacturers regularly run tests to determine if the amount of beverage in a 16-ounce can falls within the desired range.

Be aware that as you take data, your data may vary somewhat from the data someone else is taking for the same purpose. This is completely natural. However, if two or more of you are taking the same data and get very different results, it is time for you and the others to reevaluate your data-taking methods and your accuracy.

Variation in Samples

It was mentioned previously that two or more samples from the same population, taken randomly, and having close to the same characteristics of the population will likely be different from each other. Suppose Doreen and Jung both decide to study the average amount of time students at their college sleep each night. Doreen and Jung each take samples of 500 students. Doreen uses systematic sampling and Jung uses cluster sampling. Doreen's sample will be different from Jung's sample. Even if Doreen and Jung used the same sampling method, in all likelihood their samples would be different. Neither would be wrong, however.

Think about what contributes to making Doreen's and Jung's samples different.

If Doreen and Jung took larger samples (i.e. the number of data values is increased), their sample results (the average amount of time a student sleeps) might be closer to the actual population average. But still, their samples would be, in all likelihood, different from each other. This *variability in samples* cannot be stressed enough.

Size of a Sample

The size of a sample (often called the number of observations) is important. The examples you have seen in this book so far have been small. Samples of only a few hundred observations, or even smaller, are sufficient for many purposes. In polling, samples that are from 1,200 to 1,500 observations are considered large enough and good enough if the survey is random and is well done. You will learn why when you study confidence intervals.

Be aware that many large samples are biased. For example, call-in surveys are invariably biased, because people choose to respond or not.

Collaborative Exercise 1.3.8

Divide into groups of two, three, or four. Your instructor will give each group one six-sided die. Try this experiment twice. Roll one fair die (six-sided) 20 times. Record the number of ones, twos, threes, fours, fives, and sixes you get in the following table ("frequency" is the number of times a particular face of the die occurs):

First Experiment (20 rolls)			Second Experiment (20 rolls)	
Face on Die	**Frequency**		**Face on Die**	**Frequency**
1				
2				
3				
4				
5				
6				

Did the two experiments have the same results? Probably not. If you did the experiment a third time, do you expect the results to be identical to the first or second experiment? Why or why not?

Which experiment had the correct results? They both did. The job of the statistician is to see through the variability and draw appropriate conclusions.

Critical Evaluation

We need to evaluate the statistical studies we read about critically and analyze them before accepting the results of the studies. Common problems to be aware of include

- Problems with samples: A sample must be representative of the population. A sample that is not representative of the population is biased. Biased samples that are not representative of the population give results that are inaccurate and not valid.
- Self-selected samples: Responses only by people who choose to respond, such as call-in surveys, are often unreliable.
- Sample size issues: Samples that are too small may be unreliable. Larger samples are better, if possible. In some situations, having small samples is unavoidable and can still be used to draw conclusions. Examples: crash testing cars or medical testing for rare conditions
- Undue influence: collecting data or asking questions in a way that influences the response
- Non-response or refusal of subject to participate: The collected responses may no longer be representative of the population. Often, people with strong positive or negative opinions may answer surveys, which can affect the results.
- Causality: A relationship between two variables does not mean that one causes the other to occur. They may be related (correlated) because of their relationship through a different variable.
- Self-funded or self-interest studies: A study performed by a person or organization in order to support their claim. Is the study impartial? Read the study carefully to evaluate the work. Do not automatically assume that the study is good, but do not automatically assume the study is bad either. Evaluate it on its merits and the work done.
- Misleading use of data: improperly displayed graphs, incomplete data, or lack of context
- Confounding: When the effects of multiple factors on a response cannot be separated. Confounding makes it difficult or impossible to draw valid conclusions about the effect of each factor.

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Review

Data are individual items of information that come from a population or sample. Data may be classified as qualitative, quantitative continuous, or quantitative discrete.

Because it is not practical to measure the entire population in a study, researchers use samples to represent the population. A random sample is a representative group from the population chosen by using a method that gives each individual in the population an equal chance of being included in the sample. Random sampling methods include simple random sampling, stratified sampling, cluster sampling, and systematic sampling. Convenience sampling is a nonrandom method of choosing a sample that often produces biased data.

Samples that contain different individuals result in different data. This is true even when the samples are well-chosen and representative of the population. When properly selected, larger samples model the population more closely than smaller samples. There are many different potential problems that can affect the reliability of a sample. Statistical data needs to be critically analyzed, not simply accepted.

Footnotes

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Glossary

Cluster Sampling

a method for selecting a random sample and dividing the population into groups (clusters); use simple random sampling to select a set of clusters. Every individual in the chosen clusters is included in the sample.

Continuous Random Variable

a random variable (RV) whose outcomes are measured; the height of trees in the forest is a continuous RV.

Convenience Sampling

a nonrandom method of selecting a sample; this method selects individuals that are easily accessible and may result in biased data.

Discrete Random Variable

a random variable (RV) whose outcomes are counted

Nonsampling Error

an issue that affects the reliability of sampling data other than natural variation; it includes a variety of human errors including poor study design, biased sampling methods, inaccurate information provided by study participants, data entry errors, and poor analysis.

Qualitative Data

See Data.

Quantitative Data

See Data.

Random Sampling

a method of selecting a sample that gives every member of the population an equal chance of being selected.

Sampling Bias

not all members of the population are equally likely to be selected

Sampling Error

the natural variation that results from selecting a sample to represent a larger population; this variation decreases as the sample size increases, so selecting larger samples reduces sampling error.

Sampling with Replacement

Once a member of the population is selected for inclusion in a sample, that member is returned to the population for the selection of the next individual.

Sampling without Replacement

A member of the population may be chosen for inclusion in a sample only once. If chosen, the member is not returned to the population before the next selection.

Simple Random Sampling

a straightforward method for selecting a random sample; give each member of the population a number. Use a random number generator to select a set of labels. These randomly selected labels identify the members of your sample.

Stratified Sampling

a method for selecting a random sample used to ensure that subgroups of the population are represented adequately; divide the population into groups (strata). Use simple random sampling to identify a proportionate number of individuals from each stratum.

Systematic Sampling

a method for selecting a random sample; list the members of the population. Use simple random sampling to select a starting point in the population. Let k = (number of individuals in the population)/(number of individuals needed in the sample). Choose every kth individual in the list starting with the one that was randomly selected. If necessary, return to the beginning of the population list to complete your sample.

1.4: Frequency, Frequency Tables, and Levels of Measurement

Once you have a set of data, you will need to organize it so that you can analyze how frequently each datum occurs in the set. However, when calculating the frequency, you may need to round your answers so that they are as precise as possible.

Answers and Rounding Off

A simple way to round off answers is to carry your final answer one more decimal place than was present in the original data. Round off only the final answer. Do not round off any intermediate results, if possible. If it becomes necessary to round off intermediate results, carry them to at least twice as many decimal places as the final answer. For example, the average of the three quiz scores four, six, and nine is 6.3, rounded off to the nearest tenth, because the data are whole numbers. Most answers will be rounded off in this manner.

It is not necessary to reduce most fractions in this course. Especially in Probability Topics, the chapter on probability, it is more helpful to leave an answer as an unreduced fraction.

Levels of Measurement

The way a set of data is measured is called its **level of measurement**. Correct statistical procedures depend on a researcher being familiar with levels of measurement. Not every statistical operation can be used with every set of data. Data can be classified into four levels of measurement. They are (from lowest to highest level):

- **Nominal scale level**
- **Ordinal scale level**
- **Interval scale level**
- **Ratio scale level**

Data that is measured using a **nominal scale** is **qualitative**. Categories, colors, names, labels and favorite foods along with yes or no responses are examples of nominal level data. Nominal scale data are not ordered. For example, trying to classify people according to their favorite food does not make any sense. Putting pizza first and sushi second is not meaningful.

Smartphone companies are another example of nominal scale data. Some examples are Sony, Motorola, Nokia, Samsung and Apple. This is just a list and there is no agreed upon order. Some people may favor Apple but that is a matter of opinion. Nominal scale data cannot be used in calculations.

Data that is measured using an **ordinal scale** is similar to nominal scale data but there is a big difference. The ordinal scale data can be ordered. An example of ordinal scale data is a list of the top five national parks in the United States. The top five national parks in the United States can be ranked from one to five but we cannot measure differences between the data.

Another example of using the ordinal scale is a cruise survey where the responses to questions about the cruise are “excellent,” “good,” “satisfactory,” and “unsatisfactory.” These responses are ordered from the most desired response to the least desired. But the differences between two pieces of data cannot be measured. Like the nominal scale data, ordinal scale data cannot be used in calculations.

Data that is measured using the **interval scale** is similar to ordinal level data because it has a definite ordering but there is a difference between data. The differences between interval scale data can be measured though the data does not have a starting point.

Temperature scales like Celsius (C) and Fahrenheit (F) are measured by using the interval scale. In both temperature measurements, 40° is equal to 100° minus 60°. Differences make sense. But 0 degrees does not because, in both scales, 0 is not the absolute lowest temperature. Temperatures like -10° F and -15° C exist and are colder than 0.

Interval level data can be used in calculations, but one type of comparison cannot be done. 80° C is not four times as hot as 20° C (nor is 80° F four times as hot as 20° F). There is no meaning to the ratio of 80 to 20 (or four to one).

Data that is measured using the **ratio scale** takes care of the ratio problem and gives you the most information. Ratio scale data is like interval scale data, but it has a 0 point and ratios can be calculated. For example, four multiple choice statistics final exam scores are 80, 68, 20 and 92 (out of a possible 100 points). The exams are machine-graded.

The data can be put in order from lowest to highest: 20, 68, 80, 92.

The differences between the data have meaning. The score 92 is more than the score 68 by 24 points. Ratios can be calculated. The smallest score is 0. So 80 is four times 20. The score of 80 is four times better than the score of 20.

Frequency

Twenty students were asked how many hours they worked per day. Their responses, in hours, are as follows:

5; 6; 3; 3; 2; 4; 7; 5; 2; 3; 5; 6; 5; 4; 4; 3; 5; 2; 5; 3.

Table lists the different data values in ascending order and their frequencies.

Table 1.4.1: Frequency Table of Student Work Hours

DATA VALUE	FREQUENCY
2	3
3	5
4	3
5	6
6	2
7	1

Definition: Relative Frequency

A frequency is the number of times a value of the data occurs. According to Table Table 1.4.1, there are three students who work two hours, five students who work three hours, and so on. The sum of the values in the frequency column, 20, represents the total number of students included in the sample.

Definition: Relative frequencies

A *relative frequency* is the ratio (fraction or proportion) of the number of times a value of the data occurs in the set of all outcomes to the total number of outcomes. To find the relative frequencies, divide each frequency by the total number of students in the sample–in this case, 20. Relative frequencies can be written as fractions, percents, or decimals.

Table 1.4.2: Frequency Table of Student Work Hours with Relative Frequencies

DATA VALUE	FREQUENCY	RELATIVE FREQUENCY
2	3	$\frac{3}{20}$ or 0.15
3	5	$\frac{5}{20}$ or 0.25
4	3	$\frac{3}{20}$ or 0.15
5	6	$\frac{6}{20}$ or 0.30
6	2	$\frac{2}{20}$ or 0.10
7	1	$\frac{1}{20}$ or 0.05

The sum of the values in the relative frequency column of Table 1.4.2 is $\frac{20}{20}$, or 1.

Definition: Cumulative Relative Frequency

Cumulative relative frequency is the accumulation of the previous relative frequencies. To find the cumulative relative frequencies, add all the previous relative frequencies to the relative frequency for the current row, as shown in Table 1.4.3.

Table 1.4.3: Frequency Table of Student Work Hours with Relative and Cumulative Relative Frequencies

DATA VALUE	FREQUENCY	RELATIVE FREQUENCY	CUMULATIVE RELATIVE FREQUENCY
2	3	$\frac{3}{20}$ or 0.15	0.15
3	5	$\frac{5}{20}$ or 0.25	0.15 + 0.25 = 0.40
4	3	$\frac{3}{20}$ or 0.15	0.40 + 0.15 = 0.55
5	6	$\frac{6}{20}$ or 0.30	0.55 + 0.30 = 0.85
6	2	$\frac{2}{20}$ or 0.10	0.85 + 0.10 = 0.95
7	1	$\frac{1}{20}$ or 0.05	0.95 + 0.05 = 1.00

The last entry of the cumulative relative frequency column is one, indicating that one hundred percent of the data has been accumulated.

> Because of rounding, the relative frequency column may not always sum to one, and the last entry in the cumulative relative frequency column may not be one. However, they each should be close to one.

Table 1.4.4 represents the heights, in inches, of a sample of 100 male semiprofessional soccer players.

Table 1.4.4: Frequency Table of Soccer Player Height

HEIGHTS (INCHES)	FREQUENCY	RELATIVE FREQUENCY	CUMULATIVE RELATIVE FREQUENCY
59.95–61.95	5	$\frac{5}{100} = 0.05$	0.05
61.95–63.95	3	$\frac{3}{100} = 0.03$	$0.05 + 0.03 = 0.08$
63.95–65.95	15	$\frac{15}{100} = 0.15$	$0.08 + 0.15 = 0.23$
65.95–67.95	40	$\frac{40}{100} = 0.40$	$0.23 + 0.40 = 0.63$
67.95–69.95	17	$\frac{17}{100} = 0.17$	$0.63 + 0.17 = 0.80$
69.95–71.95	12	$\frac{12}{100} = 0.12$	$0.80 + 0.12 = 0.92$
71.95–73.95	7	$\frac{7}{100} = 0.07$	$0.92 + 0.07 = 0.99$
73.95–75.95	1	$\frac{1}{100} = 0.01$	$0.99 + 0.01 = 1.00$
	Total = 100	**Total = 1.00**	

The data in this table have been **grouped** into the following intervals:

- 61.95 to 63.95 inches
- 63.95 to 65.95 inches
- 65.95 to 67.95 inches
- 67.95 to 69.95 inches
- 69.95 to 71.95 inches
- 71.95 to 73.95 inches
- 73.95 to 75.95 inches

> This example is used again in Descriptive Statistics, where the method used to compute the intervals will be explained.

In this sample, there are **five** players whose heights fall within the interval 59.95–61.95 inches, **three** players whose heights fall within the interval 61.95–63.95 inches, **15** players whose heights fall within the interval 63.95–65.95 inches, **40** players whose heights fall within the interval 65.95–67.95 inches, **17** players whose heights fall within the interval 67.95–69.95 inches, **12** players whose heights fall within the interval 69.95–71.95, **seven** players whose heights fall within the interval 71.95–73.95, and **one** player whose heights fall within the interval 73.95–75.95. All heights fall between the endpoints of an interval and not at the endpoints.

Exercise 1.4.1

a. From the Table 1.4.4, find the percentage of heights that are less than 65.95 inches.
b. Find the percentage of heights that fall between 61.95 and 65.95 inches.

Answer

a. If you look at the first, second, and third rows, the heights are all less than 65.95 inches. There are $5+3+15=23$ players whose heights are less than 65.95 inches. The percentage of heights less than 65.95 inches is then $\frac{23}{100}$ or 23%. This percentage is the cumulative relative frequency entry in the third row.
b. Add the relative frequencies in the second and third rows: $0.03+0.15=0.18$ or 18%.

Exercise 1.4.2

Table $1.4.5$ shows the amount, in inches, of annual rainfall in a sample of towns.

a. Find the percentage of rainfall that is less than 9.01 inches.
b. Find the percentage of rainfall that is between 6.99 and 13.05 inches.

Table 1.4.5

Rainfall (Inches)	Frequency	Relative Frequency	Cumulative Relative Frequency
2.95–4.97	6	$\frac{6}{50}=0.12$	0.12
4.97–6.99	7	$\frac{7}{50}=0.14$	$0.12+0.14=0.26$
6.99–9.01	15	$\frac{15}{50}=0.30$	$0.26+0.30=0.56$
9.01–11.03	8	$\frac{8}{50}=0.16$	$0.56+0.16=0.72$
11.03–13.05	9	$\frac{9}{50}=0.18$	$0.72+0.18=0.90$
13.05–15.07	5	$\frac{5}{50}=0.10$	$0.90+0.10=1.00$
	Total = 50	Total = 1.00	

Answer

a. 0.56or 56
b. $0.30+0.16+0.18=0.64$or 64

Exercise 1.4.3

Use the heights of the 100 male semiprofessional soccer players in Table $1.4.4$. Fill in the blanks and check your answers.

a. The percentage of heights that are from 67.95 to 71.95 inches is: ____.
b. The percentage of heights that are from 67.95 to 73.95 inches is: ____.
c. The percentage of heights that are more than 65.95 inches is: ____.
d. The number of players in the sample who are between 61.95 and 71.95 inches tall is: ____.
e. What kind of data are the heights?
f. Describe how you could gather this data (the heights) so that the data are characteristic of all male semiprofessional soccer players.

Remember, you **count frequencies**. To find the relative frequency, divide the frequency by the total number of data values. To find the cumulative relative frequency, add all of the previous relative frequencies to the relative frequency for the current row.

Answer

a. 29%
b. 36%
c. 77%
d. 87
e. quantitative continuous
f. get rosters from each team and choose a simple random sample from each

? Exercise 1.4.4

From Table 1.4.5, find the number of towns that have rainfall between 2.95 and 9.01 inches.

Answer

$$6+7+15=28 \text{ towns}$$

Collaborative Exercise 1.4.7

In your class, have someone conduct a survey of the number of siblings (brothers and sisters) each student has. Create a frequency table. Add to it a relative frequency column and a cumulative relative frequency column. Answer the following questions:

a. What percentage of the students in your class have no siblings?
b. What percentage of the students have from one to three siblings?
c. What percentage of the students have fewer than three siblings?

✓ Example 1.4.7

Nineteen people were asked how many miles, to the nearest mile, they commute to work each day. The data are as follows: 2; 5; 7; 3; 2; 10; 18; 15; 20; 7; 10; 18; 5; 12; 13; 12; 4; 5; 10. Table 1.4.6 was produced:

Table 1.4.6: Frequency of Commuting Distances

DATA	FREQUENCY	RELATIVE FREQUENCY	CUMULATIVE RELATIVE FREQUENCY
3	3	$\frac{3}{19}$	0.1579
4	1	$\frac{1}{19}$	0.2105
5	3	$\frac{3}{19}$	0.1579
7	2	$\frac{2}{19}$	0.2632
10	3	$\frac{3}{19}$	0.4737
12	2	$\frac{2}{19}$	0.7895
13	1	$\frac{1}{19}$	0.8421
15	1	$\frac{1}{19}$	0.8948
18	1	$\frac{1}{19}$	0.9474
20	1	$\frac{1}{19}$	1.0000

a. Is the table correct? If it is not correct, what is wrong?
b. True or False: Three percent of the people surveyed commute three miles. If the statement is not correct, what should it be? If the table is incorrect, make the corrections.

c. What fraction of the people surveyed commute five or seven miles?
d. What fraction of the people surveyed commute 12 miles or more? Less than 12 miles? Between five and 13 miles (not including five and 13 miles)?

Answer

a. No. The frequency column sums to 18, not 19. Not all cumulative relative frequencies are correct.
b. False. The frequency for three miles should be one; for two miles (left out), two. The cumulative relative frequency column should read: 0.1052, 0.1579, 0.2105, 0.3684, 0.4737, 0.6316, 0.7368, 0.7895, 0.8421, 0.9474, 1.0000.
c. $\frac{5}{19}$
d. $\frac{7}{19}$, $\frac{12}{19}$, $\frac{7}{19}$

? Exercise 1.4.8

Table $1.4.5$ represents the amount, in inches, of annual rainfall in a sample of towns. What fraction of towns surveyed get between 11.03 and 13.05 inches of rainfall each year?

Answer

$\frac{9}{50}$

✓ Example 1.4.9

Table $1.4.7$ contains the total number of deaths worldwide as a result of earthquakes for the period from 2000 to 2012.

Table $1.4.7$: Total Number of Deaths Worldwide as a Result of Earthquakes

Year	Total Number of Deaths
2000	231
2001	21,357
2002	11,685
2003	33,819
2004	228,802
2005	88,003
2006	6,605
2007	712
2008	88,011
2009	1,790
2010	320,120
2011	21,953
2012	768
Total	823,356

Answer the following questions.

a. What is the frequency of deaths measured from 2006 through 2009?
b. What percentage of deaths occurred after 2009?
c. What is the relative frequency of deaths that occurred in 2003 or earlier?
d. What is the percentage of deaths that occurred in 2004?

e. What kind of data are the numbers of deaths?
f. The Richter scale is used to quantify the energy produced by an earthquake. Examples of Richter scale numbers are 2.3, 4.0, 6.1, and 7.0. What kind of data are these numbers?

Answer

a. 97,118 (11.8%)
b. 41.6%
c. 67,092/823,356 or 0.081 or 8.1 %
d. 27.8%
e. Quantitative discrete
f. Quantitative continuous

? Exercise 1.4.10

Table 1.4.8 contains the total number of fatal motor vehicle traffic crashes in the United States for the period from 1994 to 2011.

Table 1.4.8:

Year	Total Number of Crashes	Year	Total Number of Crashes
1994	36,254	2004	38,444
1995	37,241	2005	39,252
1996	37,494	2006	38,648
1997	37,324	2007	37,435
1998	37,107	2008	34,172
1999	37,140	2009	30,862
2000	37,526	2010	30,296
2001	37,862	2011	29,757
2002	38,491	Total	653,782
2003	38,477		

Answer the following questions.

a. What is the frequency of deaths measured from 2000 through 2004?
b. What percentage of deaths occurred after 2006?
c. What is the relative frequency of deaths that occurred in 2000 or before?
d. What is the percentage of deaths that occurred in 2011?
e. What is the cumulative relative frequency for 2006? Explain what this number tells you about the data.

Answer

a. 190,800 (29.2%)
b. 24.9%
c. 260,086/653,782 or 39.8%
d. 4.6%
e. 75.1% of all fatal traffic crashes for the period from 1994 to 2011 happened from 1994 to 2006.

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Review

Some calculations generate numbers that are artificially precise. It is not necessary to report a value to eight decimal places when the measures that generated that value were only accurate to the nearest tenth. Round off your final answer to one more decimal place than was present in the original data. This means that if you have data measured to the nearest tenth of a unit, report the final statistic to the nearest hundredth.

In addition to rounding your answers, you can measure your data using the following four levels of measurement.

- **Nominal scale level:** data that cannot be ordered nor can it be used in calculations
- **Ordinal scale level:** data that can be ordered; the differences cannot be measured
- **Interval scale level:** data with a definite ordering but no starting point; the differences can be measured, but there is no such thing as a ratio.
- **Ratio scale level:** data with a starting point that can be ordered; the differences have meaning and ratios can be calculated.

When organizing data, it is important to know how many times a value appears. How many statistics students study five hours or more for an exam? What percent of families on our block own two pets? Frequency, relative frequency, and cumulative relative frequency are measures that answer questions like these.

Exercise 1.4.11

What type of measure scale is being used? Nominal, ordinal, interval or ratio.

a. High school soccer players classified by their athletic ability: Superior, Average, Above average
b. Baking temperatures for various main dishes: 350, 400, 325, 250, 300
c. The colors of crayons in a 24-crayon box
d. Social security numbers
e. Incomes measured in dollars
f. A satisfaction survey of a social website by number: 1 = very satisfied, 2 = somewhat satisfied, 3 = not satisfied
g. Political outlook: extreme left, left-of-center, right-of-center, extreme right
h. Time of day on an analog watch
i. The distance in miles to the closest grocery store
j. The dates 1066, 1492, 1644, 1947, and 1944
k. The heights of 21–65 year-old women
l. Common letter grades: A, B, C, D, and F

Answer

a. ordinal
b. interval
c. nominal
d. nominal
e. ratio
f. ordinal
g. nominal
h. interval
i. ratio
j. interval
k. ratio
l. ordinal

Glossary

Cumulative Relative Frequency

The term applies to an ordered set of observations from smallest to largest. The cumulative relative frequency is the sum of the relative frequencies for all values that are less than or equal to the given value.

Frequency

the number of times a value of the data occurs

Relative Frequency

the ratio of the number of times a value of the data occurs in the set of all outcomes to the number of all outcomes to the total number of outcomes

1.5: Experimental Design and Ethics

Does aspirin reduce the risk of heart attacks? Is one brand of fertilizer more effective at growing roses than another? Is fatigue as dangerous to a driver as the influence of alcohol? Questions like these are answered using randomized experiments. In this module, you will learn important aspects of experimental design. Proper study design ensures the production of reliable, accurate data.

The purpose of an experiment is to investigate the relationship between two variables. When one variable causes change in another, we call the first variable the **explanatory variable**. The affected variable is called the response variable. In a randomized experiment, the researcher manipulates values of the explanatory variable and measures the resulting changes in the response variable. The different values of the explanatory variable are called **treatments**. An **experimental unit** is a single object or individual to be measured.

You want to investigate the effectiveness of vitamin E in preventing disease. You recruit a group of subjects and ask them if they regularly take vitamin E. You notice that the subjects who take vitamin E exhibit better health on average than those who do not. Does this prove that vitamin E is effective in disease prevention? It does not. There are many differences between the two groups compared in addition to vitamin E consumption. People who take vitamin E regularly often take other steps to improve their health: exercise, diet, other vitamin supplements, choosing not to smoke. Any one of these factors could be influencing health. As described, this study does not prove that vitamin E is the key to disease prevention.

Additional variables that can cloud a study are called **lurking variables**. In order to prove that the explanatory variable is causing a change in the response variable, it is necessary to isolate the explanatory variable. The researcher must design her experiment in such a way that there is only one difference between groups being compared: the planned treatments. This is accomplished by the **random assignment** of experimental units to treatment groups. When subjects are assigned treatments randomly, all of the potential lurking variables are spread equally among the groups. At this point the only difference between groups is the one imposed by the researcher. Different outcomes measured in the response variable, therefore, must be a direct result of the different treatments. In this way, an experiment can prove a cause-and-effect connection between the explanatory and response variables.

The power of suggestion can have an important influence on the outcome of an experiment. Studies have shown that the expectation of the study participant can be as important as the actual medication. In one study of performance-enhancing drugs, researchers noted:

*Results showed that believing one had taken the substance resulted in [*performance*] times almost as fast as those associated with consuming the drug itself. In contrast, taking the drug without knowledge yielded no significant performance increment.*[1]

When participation in a study prompts a physical response from a participant, it is difficult to isolate the effects of the explanatory variable. To counter the power of suggestion, researchers set aside one treatment group as a **control group**. This group is given a **placebo** treatment–a treatment that cannot influence the response variable. The control group helps researchers balance the effects of being in an experiment with the effects of the active treatments. Of course, if you are participating in a study and you know that you are receiving a pill which contains no actual medication, then the power of suggestion is no longer a factor. **Blinding** in a randomized experiment preserves the power of suggestion. When a person involved in a research study is blinded, he does not know who is receiving the active treatment(s) and who is receiving the placebo treatment. A **double-blind experiment** is one in which both the subjects and the researchers involved with the subjects are blinded.

✓ Example 1.5.1

Researchers want to investigate whether taking aspirin regularly reduces the risk of heart attack. Four hundred men between the ages of 50 and 84 are recruited as participants. The men are divided randomly into two groups: one group will take aspirin, and the other group will take a placebo. Each man takes one pill each day for three years, but he does not know whether he is taking aspirin or the placebo. At the end of the study, researchers count the number of men in each group who have had heart attacks.

Identify the following values for this study: population, sample, experimental units, explanatory variable, response variable, treatments.

Answer

- The *population* is men aged 50 to 84.
- The *sample* is the 400 men who participated.

- The *experimental units* are the individual men in the study.
- The *explanatory variable* is oral medication.
- The *treatments* are aspirin and a placebo.
- The *response variable* is whether a subject had a heart attack.

✓ Example 1.5.2

The Smell & Taste Treatment and Research Foundation conducted a study to investigate whether smell can affect learning. Subjects completed mazes multiple times while wearing masks. They completed the pencil and paper mazes three times wearing floral-scented masks, and three times with unscented masks. Participants were assigned at random to wear the floral mask during the first three trials or during the last three trials. For each trial, researchers recorded the time it took to complete the maze and the subject's impression of the mask's scent: positive, negative, or neutral.

a. Describe the explanatory and response variables in this study.
b. What are the treatments?
c. Identify any lurking variables that could interfere with this study.
d. Is it possible to use blinding in this study?

Answer

a. The explanatory variable is scent, and the response variable is the time it takes to complete the maze.
b. There are two treatments: a floral-scented mask and an unscented mask.
c. All subjects experienced both treatments. The order of treatments was randomly assigned so there were no differences between the treatment groups. Random assignment eliminates the problem of lurking variables.
d. Subjects will clearly know whether they can smell flowers or not, so subjects cannot be blinded in this study. Researchers timing the mazes can be blinded, though. The researcher who is observing a subject will not know which mask is being worn.

✓ Example 1.5.3

A researcher wants to study the effects of birth order on personality. Explain why this study could not be conducted as a randomized experiment. What is the main problem in a study that cannot be designed as a randomized experiment?

Answer

The explanatory variable is birth order. You cannot randomly assign a person's birth order. Random assignment eliminates the impact of lurking variables. When you cannot assign subjects to treatment groups at random, there will be differences between the groups other than the explanatory variable.

? Exercise 1.5.4

You are concerned about the effects of texting on driving performance. Design a study to test the response time of drivers while texting and while driving only. How many seconds does it take for a driver to respond when a leading car hits the brakes?

a. Describe the explanatory and response variables in the study.
b. What are the treatments?
c. What should you consider when selecting participants?
d. Your research partner wants to divide participants randomly into two groups: one to drive without distraction and one to text and drive simultaneously. Is this a good idea? Why or why not?
e. Identify any lurking variables that could interfere with this study.
f. How can blinding be used in this study?

Answer

a. Explanatory: presence of distraction from texting; response: response time measured in seconds
b. Driving without distraction and driving while texting
c. Answers will vary. Possible responses: Do participants regularly send and receive text messages? How long has the subject been driving? What is the age of the participants? Do participants have similar texting and driving experience?

d. This is not a good plan because it compares drivers with different abilities. It would be better to assign both treatments to each participant in random order.
e. Possible responses include: texting ability, driving experience, type of phone.
f. The researchers observing the trials and recording response time could be blinded to the treatment being applied.

Ethics

The widespread misuse and misrepresentation of statistical information often gives the field a bad name. Some say that "numbers don't lie," but the people who use numbers to support their claims often do.

A recent investigation of famous social psychologist, Diederik Stapel, has led to the retraction of his articles from some of the world's top journals including *Journal of Experimental Social Psychology, Social Psychology, Basic and Applied Social Psychology, British Journal of Social Psychology,* and the magazine *Science*. Diederik Stapel is a former professor at Tilburg University in the Netherlands. Over the past two years, an extensive investigation involving three universities where Stapel has worked concluded that the psychologist is guilty of fraud on a colossal scale. Falsified data taints over 55 papers he authored and 10 Ph.D. dissertations that he supervised.

Stapel did not deny that his deceit was driven by ambition. But it was more complicated than that, he told me. He insisted that he loved social psychology but had been frustrated by the messiness of experimental data, which rarely led to clear conclusions. His lifelong obsession with elegance and order, he said, led him to concoct sexy results that journals found attractive. "It was a quest for aesthetics, for beauty—instead of the truth," he said. He described his behavior as an addiction that drove him to carry out acts of increasingly daring fraud, like a junkie seeking a bigger and better high.[2]

The committee investigating Stapel concluded that he is guilty of several practices including:

- creating datasets, which largely confirmed the prior expectations,
- altering data in existing datasets,
- changing measuring instruments without reporting the change, and
- misrepresenting the number of experimental subjects.

Clearly, it is never acceptable to falsify data the way this researcher did. Sometimes, however, violations of ethics are not as easy to spot.

Researchers have a responsibility to verify that proper methods are being followed. The report describing the investigation of Stapel's fraud states that, "statistical flaws frequently revealed a lack of familiarity with elementary statistics."[3] Many of Stapel's co-authors should have spotted irregularities in his data. Unfortunately, they did not know very much about statistical analysis, and they simply trusted that he was collecting and reporting data properly.

Many types of statistical fraud are difficult to spot. Some researchers simply stop collecting data once they have just enough to prove what they had hoped to prove. They don't want to take the chance that a more extensive study would complicate their lives by producing data contradicting their hypothesis.

Professional organizations, like the American Statistical Association, clearly define expectations for researchers. There are even laws in the federal code about the use of research data.

When a statistical study uses human participants, as in medical studies, both ethics and the law dictate that researchers should be mindful of the safety of their research subjects. The U.S. Department of Health and Human Services oversees federal regulations of research studies with the aim of protecting participants. When a university or other research institution engages in research, it must ensure the safety of all human subjects. For this reason, research institutions establish oversight committees known as **Institutional Review Boards (IRB)**. All planned studies must be approved in advance by the IRB. Key protections that are mandated by law include the following:

- Risks to participants must be minimized and reasonable with respect to projected benefits.
- Participants must give **informed consent**. This means that the risks of participation must be clearly explained to the subjects of the study. Subjects must consent in writing, and researchers are required to keep documentation of their consent.
- Data collected from individuals must be guarded carefully to protect their privacy.

These ideas may seem fundamental, but they can be very difficult to verify in practice. Is removing a participant's name from the data record sufficient to protect privacy? Perhaps the person's identity could be discovered from the data that remains. What happens if the study does not proceed as planned and risks arise that were not anticipated? When is informed consent really necessary? Suppose your doctor wants a blood sample to check your cholesterol level. Once the sample has been tested, you expect the lab to dispose of the remaining blood. At that point the blood becomes biological waste. Does a researcher have the right to take it for use in a study?

It is important that students of statistics take time to consider the ethical questions that arise in statistical studies. How prevalent is fraud in statistical studies? You might be surprised—and disappointed. There is a website (www.retractionwatch.com) dedicated to cataloging retractions of study articles that have been proven fraudulent. A quick glance will show that the misuse of statistics is a bigger problem than most people realize.

Vigilance against fraud requires knowledge. Learning the basic theory of statistics will empower you to analyze statistical studies critically.

✓ Example 1.5.5

Describe the unethical behavior in each example and describe how it could impact the reliability of the resulting data. Explain how the problem should be corrected.

A researcher is collecting data in a community.

a. She selects a block where she is comfortable walking because she knows many of the people living on the street.
b. No one seems to be home at four houses on her route. She does not record the addresses and does not return at a later time to try to find residents at home.
c. She skips four houses on her route because she is running late for an appointment. When she gets home, she fills in the forms by selecting random answers from other residents in the neighborhood.

Answer

a. By selecting a convenient sample, the researcher is intentionally selecting a sample that could be biased. Claiming that this sample represents the community is misleading. The researcher needs to select areas in the community at random.
b. Intentionally omitting relevant data will create bias in the sample. Suppose the researcher is gathering information about jobs and child care. By ignoring people who are not home, she may be missing data from working families that are relevant to her study. She needs to make every effort to interview all members of the target sample.
c. It is never acceptable to fake data. Even though the responses she uses are “real” responses provided by other participants, the duplication is fraudulent and can create bias in the data. She needs to work diligently to interview everyone on her route.

? Exercise 1.5.6

Describe the unethical behavior, if any, in each example and describe how it could impact the reliability of the resulting data. Explain how the problem should be corrected.

A study is commissioned to determine the favorite brand of fruit juice among teens in California.

a. The survey is commissioned by the seller of a popular brand of apple juice.
b. There are only two types of juice included in the study: apple juice and cranberry juice.
c. Researchers allow participants to see the brand of juice as samples are poured for a taste test.
d. Twenty-five percent of participants prefer Brand X, 33% prefer Brand Y and 42% have no preference between the two brands. Brand X references the study in a commercial saying “Most teens like Brand X as much as or more than Brand Y.”

Answer

a. This is not necessarily a problem. The study should be monitored carefully, however, to ensure that the company is not pressuring researchers to return biased results.
b. If the researchers truly want to determine the favorite brand of juice, then researchers should ask teens to compare different brands of the same type of juice. Choosing a sweet juice to compare against a sharp-flavored juice will not lead to an accurate comparison of brand quality.
c. Participants could be biased by the knowledge. The results may be different from those obtained in a blind taste test.
d. The commercial tells the truth, but not the whole truth. It leads consumers to believe that Brand X was preferred by more participants than Brand Y while the opposite is true.

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Review

A poorly designed study will not produce reliable data. There are certain key components that must be included in every experiment. To eliminate lurking variables, subjects must be assigned randomly to different treatment groups. One of the groups must act as a control group, demonstrating what happens when the active treatment is not applied. Participants in the control group receive a placebo treatment that looks exactly like the active treatments but cannot influence the response variable. To preserve the integrity of the placebo, both researchers and subjects may be blinded. When a study is designed properly, the only difference between treatment groups is the one imposed by the researcher. Therefore, when groups respond differently to different treatments, the difference must be due to the influence of the explanatory variable.

“An ethics problem arises when you are considering an action that benefits you or some cause you support, hurts or reduces benefits to others, and violates some rule.”[4] Ethical violations in statistics are not always easy to spot. Professional associations and federal agencies post guidelines for proper conduct. It is important that you learn basic statistical procedures so that you can recognize proper data analysis.

? Exercise 1.5.7

Design an experiment. Identify the explanatory and response variables. Describe the population being studied and the experimental units. Explain the treatments that will be used and how they will be assigned to the experimental units. Describe how blinding and placebos may be used to counter the power of suggestion.

? Exercise 1.5.7

Discuss potential violations of the rule requiring informed consent.

a. Inmates in a correctional facility are offered good behavior credit in return for participation in a study.
b. A research study is designed to investigate a new children’s allergy medication.

c. Participants in a study are told that the new medication being tested is highly promising, but they are not told that only a small portion of participants will receive the new medication. Others will receive placebo treatments and traditional treatments.

Answer

a. Inmates may not feel comfortable refusing participation, or may feel obligated to take advantage of the promised benefits. They may not feel truly free to refuse participation.
b. Parents can provide consent on behalf of their children, but children are not competent to provide consent for themselves.
c. All risks and benefits must be clearly outlined. Study participants must be informed of relevant aspects of the study in order to give appropriate consent.

Footnotes

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Glossary

Explanatory Variable
the independent variable in an experiment; the value controlled by researchers

Treatments
different values or components of the explanatory variable applied in an experiment

Response Variable
the dependent variable in an experiment; the value that is measured for change at the end of an experiment

Experimental Unit
any individual or object to be measured

Lurking Variable
a variable that has an effect on a study even though it is neither an explanatory variable nor a response variable

Random Assignment
the act of organizing experimental units into treatment groups using random methods

Control Group
a group in a randomized experiment that receives an inactive treatment but is otherwise managed exactly as the other groups

Informed Consent
Any human subject in a research study must be cognizant of any risks or costs associated with the study. The subject has the right to know the nature of the treatments included in the study, their potential risks, and their potential benefits. Consent must be given freely by an informed, fit participant.

Institutional Review Board
a committee tasked with oversight of research programs that involve human subjects

Placebo

an inactive treatment that has no real effect on the explanatory variable

Blinding

not telling participants which treatment a subject is receiving

Double-blinding

the act of blinding both the subjects of an experiment and the researchers who work with the subjects

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CHAPTER OVERVIEW

2: Descriptive Statistics

In this chapter, you will study numerical and graphical ways to describe and display your data. This area of statistics is called "Descriptive Statistics." You will learn how to calculate, and even more importantly, how to interpret these measurements and graphs.

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2.1: Prelude to Descriptive Statistics

Learning Objectives

By the end of this chapter, the student should be able to:

- Display data graphically and interpret graphs: stemplots, histograms, and box plots.
- Recognize, describe, and calculate the measures of location of data: quartiles and percentiles.
- Recognize, describe, and calculate the measures of the center of data: mean, median, and mode.
- Recognize, describe, and calculate the measures of the spread of data: variance, standard deviation, and range.

Once you have collected data, what will you do with it? Data can be described and presented in many different formats. For example, suppose you are interested in buying a house in a particular area. You may have no clue about the house prices, so you might ask your real estate agent to give you a sample data set of prices. Looking at all the prices in the sample often is overwhelming. A better way might be to look at the median price and the variation of prices. The median and variation are just two ways that you will learn to describe data. Your agent might also provide you with a graph of the data.

Figure 2.1.1: When you have large amounts of data, you will need to organize it in a way that makes sense. These ballots from an election are rolled together with similar ballots to keep them organized. (credit: William Greeson)

In this chapter, you will study numerical and graphical ways to describe and display your data. This area of statistics is called **"Descriptive Statistics."** You will learn how to calculate, and even more importantly, how to interpret these measurements and graphs.

A statistical graph is a tool that helps you learn about the shape or distribution of a sample or a population. A graph can be a more effective way of presenting data than a mass of numbers because we can see where data clusters and where there are only a few data values. Newspapers and the Internet use graphs to show trends and to enable readers to compare facts and figures quickly. Statisticians often graph data first to get a picture of the data. Then, more formal tools may be applied.

Some of the types of graphs that are used to summarize and organize data are the dot plot, the bar graph, the histogram, the stem-and-leaf plot, the frequency polygon (a type of broken line graph), the pie chart, and the box plot. In this chapter, we will briefly look at stem-and-leaf plots, line graphs, and bar graphs, as well as frequency polygons, and time series graphs. Our emphasis will be on histograms and box plots.

> This book contains instructions for constructing a histogram and a box plot for the TI-83+ and TI-84 calculators. The Texas Instruments (TI) website provides additional instructions for using these calculators.

2.2: Histograms, Frequency Polygons, and Time Series Graphs

For most of the work you do in this book, you will use a histogram to display the data. One advantage of a histogram is that it can readily display large data sets. A rule of thumb is to use a histogram when the data set consists of 100 values or more.

A histogram consists of contiguous (adjoining) boxes. It has both a horizontal axis and a vertical axis. The horizontal axis is labeled with what the data represents (for instance, distance from your home to school). The vertical axis is labeled either frequency or relative frequency (or percent frequency or probability). The graph will have the same shape with either label. The histogram (like the stemplot) can give you the shape of the data, the center, and the spread of the data.

The relative frequency is equal to the frequency for an observed value of the data divided by the total number of data values in the sample.(Remember, frequency is defined as the number of times an answer occurs.) If:

- f is frequency
- n is total number of data values (or the sum of the individual frequencies), and
- RF is relative frequency,

then:

$$RF = \frac{f}{n} \tag{2.2.1}$$

For example, if three students in Mr. Ahab's English class of 40 students received from 90% to 100%, then, f = 3, n = 40, and RF = fn = 340 = 0.075. 7.5% of the students received 90–100%. 90–100% are quantitative measures.

To construct a histogram, first decide how many bars or intervals, also called classes, represent the data. Many histograms consist of five to 15 bars or classes for clarity. The number of bars needs to be chosen. Choose a starting point for the first interval to be less than the smallest data value. A convenient starting point is a lower value carried out to one more decimal place than the value with the most decimal places. For example, if the value with the most decimal places is 6.1 and this is the smallest value, a convenient starting point is $6.05(6.1–0.05 = 6.05)$ We say that 6.05 has more precision. If the value with the most decimal places is 2.23 and the lowest value is 1.5, a convenient starting point is $1.495(1.5–0.005 = 1.495)$ If the value with the most decimal places is 3.234 and the lowest value is 1.0, a convenient starting point is $0.9995(1.0–0.0005 = 0.9995)$If all the data happen to be integers and the smallest value is two, then a convenient starting point is $1.5(2-0.5 = 1.5)$. Also, when the starting point and other boundaries are carried to one additional decimal place, no data value will fall on a boundary. The next two examples go into detail about how to construct a histogram using continuous data and how to create a histogram using discrete data.

✓ Example 2.2.1

The following data are the heights (in inches to the nearest half inch) of 100 male semiprofessional soccer players. The heights are **continuous** data, since height is measured.

60; 60.5; 61; 61; 61.5

63.5; 63.5; 63.5

64; 64; 64; 64; 64; 64; 64; 64.5; 64.5; 64.5; 64.5; 64.5; 64.5; 64.5; 64.5

66; 66; 66; 66; 66; 66; 66; 66; 66; 66; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 66.5; 67; 67; 67; 67; 67; 67; 67; 67; 67; 67; 67; 67; 67.5; 67.5; 67.5; 67.5; 67.5; 67.5; 67.5

68; 68; 69; 69; 69; 69; 69; 69; 69; 69; 69; 69; 69.5; 69.5; 69.5; 69.5; 69.5

70; 70; 70; 70; 70; 70; 70.5; 70.5; 70.5; 71; 71; 71

72; 72; 72; 72.5; 72.5; 73; 73.5

74

The smallest data value is 60. Since the data with the most decimal places has one decimal (for instance, 61.5), we want our starting point to have two decimal places. Since the numbers 0.5, 0.05, 0.005, etc. are convenient numbers, use 0.05 and subtract it from 60, the smallest value, for the convenient starting point.

60 – 0.05 = 59.95 which is more precise than, say, 61.5 by one decimal place. The starting point is, then, 59.95.

The largest value is 74, so 74 + 0.05 = 74.05 is the ending value.

Next, calculate the width of each bar or class interval. To calculate this width, subtract the starting point from the ending value and divide by the number of bars (you must choose the number of bars you desire). Suppose you choose eight bars.

$$\frac{74.05 - 59.95}{8} = 1.76 \tag{2.2.2}$$

We will round up to two and make each bar or class interval two units wide. Rounding up to two is one way to prevent a value from falling on a boundary. Rounding to the next number is often necessary even if it goes against the standard rules of rounding. For this example, using 1.76 as the width would also work. A guideline that is followed by some for the width of a bar or class interval is to take the square root of the number of data values and then round to the nearest whole number, if necessary. For example, if there are 150 values of data, take the square root of 150 and round to 12 bars or intervals.

The boundaries are:

- 59.95
- 59.95 + 2 = 61.95
- 61.95 + 2 = 63.95
- 63.95 + 2 = 65.95
- 65.95 + 2 = 67.95
- 67.95 + 2 = 69.95
- 69.95 + 2 = 71.95
- 71.95 + 2 = 73.95
- 73.95 + 2 = 75.95

The heights 60 through 61.5 inches are in the interval 59.95–61.95. The heights that are 63.5 are in the interval 61.95–63.95. The heights that are 64 through 64.5 are in the interval 63.95–65.95. The heights 66 through 67.5 are in the interval 65.95–67.95. The heights 68 through 69.5 are in the interval 67.95–69.95. The heights 70 through 71 are in the interval 69.95–71.95. The heights 72 through 73.5 are in the interval 71.95–73.95. The height 74 is in the interval 73.95–75.95.

The following histogram displays the heights on the *x*-axis and relative frequency on the *y*-axis.

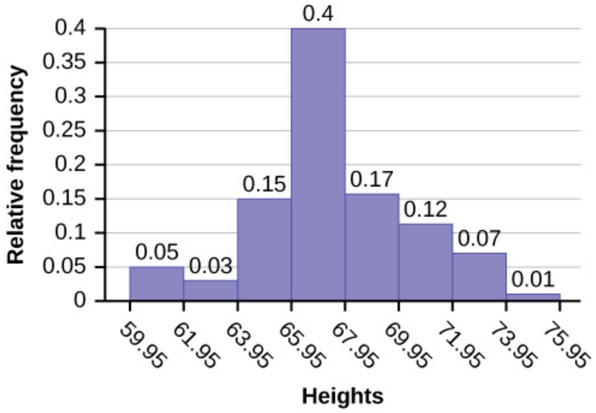

Figure 2.2.1: Histogram of something

? Exercise 2.2.1

The following data are the shoe sizes of 50 male students. The sizes are discrete data since shoe size is measured in whole and half units only. Construct a histogram and calculate the width of each bar or class interval. Suppose you choose six bars.

9; 9; 9.5; 9.5; 10; 10; 10; 10; 10; 10; 10.5; 10.5; 10.5; 10.5; 10.5; 10.5; 10.5; 10.5
11; 11; 11; 11; 11; 11; 11; 11; 11; 11; 11; 11; 11; 11.5; 11.5; 11.5; 11.5; 11.5; 11.5; 11.5
12; 12; 12; 12; 12; 12; 12; 12.5; 12.5; 12.5; 12.5; 14

Answer

Smallest value: 9

Largest value: 14

Convenient starting value: 9 – 0.05 = 8.95

Convenient ending value: 14 + 0.05 = 14.05

$$\frac{14.05-8.95}{6} = 0.85$$

The calculations suggests using 0.85 as the width of each bar or class interval. You can also use an interval with a width equal to one.

✓ Example 2.2.2

The following data are the number of books bought by 50 part-time college students at ABC College. The number of books is **discrete data**, since books are counted.

1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1
2; 2; 2; 2; 2; 2; 2; 2; 2; 2
3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3; 3
4; 4; 4; 4; 4; 4
5; 5; 5; 5; 5
6; 6

Eleven students buy one book. Ten students buy two books. Sixteen students buy three books. Six students buy four books. Five students buy five books. Two students buy six books.

Because the data are integers, subtract 0.5 from 1, the smallest data value and add 0.5 to 6, the largest data value. Then the starting point is 0.5 and the ending value is 6.5.

Next, calculate the width of each bar or class interval. If the data are discrete and there are not too many different values, a width that places the data values in the middle of the bar or class interval is the most convenient. Since the data consist of the numbers 1, 2, 3, 4, 5, 6, and the starting point is 0.5, a width of one places the 1 in the middle of the interval from 0.5 to 1.5, the 2 in the middle of the interval from 1.5 to 2.5, the 3 in the middle of the interval from 2.5 to 3.5, the 4 in the middle of the interval from _______ to _______, the 5 in the middle of the interval from _______ to _______, and the _______ in the middle of the interval from _______ to _______ .

Answer

Calculate the number of bars as follows:

$$\frac{6.5-0.5}{\text{number of bars}} = 1$$

where 1 is the width of a bar. Therefore, bars = 6.

The following histogram displays the number of books on the *x*-axis and the frequency on the *y*-axis.

Figure 2.2.2: Histogram consists of 6 bars with the y-axis in increments of 2 from 0-16 and the x-axis in intervals of 1 from 0.5-6.5.

Note

Go to [link]. There are calculator instructions for entering data and for creating a customized histogram. Create the histogram for Example.

- Press Y=. Press CLEAR to delete any equations.
- Press STAT 1:EDIT. If L1 has data in it, arrow up into the name L1, press CLEAR and then arrow down. If necessary, do the same for L2.
- Into L1, enter 1, 2, 3, 4, 5, 6.
- Into L2, enter 11, 10, 16, 6, 5, 2.
- Press WINDOW. Set Xmin = .5, Xscl = (6.5 – .5)/6, Ymin = –1, Ymax = 20, Yscl = 1, Xres = 1.
- Press 2nd Y=. Start by pressing 4:Plotsoff ENTER.

- Press 2nd Y=. Press 1:Plot1. Press ENTER. Arrow down to TYPE. Arrow to the 3rdpicture (histogram). Press ENTER.
- Arrow down to Xlist: Enter L1 (2nd 1). Arrow down to Freq. Enter L2 (2nd 2).
- Press GRAPH.
- Use the TRACE key and the arrow keys to examine the histogram.

? Exercise 2.2.2

The following data are the number of sports played by 50 student athletes. The number of sports is discrete data since sports are counted.

1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1

2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2; 2
3; 3; 3; 3; 3; 3; 3; 3

20 student athletes play one sport. 22 student athletes play two sports. Eight student athletes play three sports.

Fill in the blanks for the following sentence. Since the data consist of the numbers 1, 2, 3, and the starting point is 0.5, a width of one places the 1 in the middle of the interval 0.5 to _____, the 2 in the middle of the interval from _____ to _____, and the 3 in the middle of the interval from _____ to _____.

Answer

1.5

1.5 to 2.5

2.5 to 3.5

✓ Example 2.2.3

Using this data set, construct a histogram.

Number of Hours My Classmates Spent Playing Video Games on Weekends

9.95	10	2.25	16.75	0
19.5	22.5	7.5	15	12.75
5.5	11	10	20.75	17.5
23	21.9	24	23.75	18
20	15	22.9	18.8	20.5

Answer

Figure 2.2.3: This is a histogram that matches the supplied data. The x-axis consists of 5 bars in intervals of 5 from 0 to 25. The y-axis is marked in increments of 1 from 0 to 10. The x-axis shows the number of hours spent playing video games on the weekends, and the y-axis shows the number of students.

Some values in this data set fall on boundaries for the class intervals. A value is counted in a class interval if it falls on the left boundary, but not if it falls on the right boundary. Different researchers may set up histograms for the same data in different ways. There is more than one correct way to set up a histogram.

? Exercise 2.2.3

The following data represent the number of employees at various restaurants in New York City. Using this data, create a histogram.

22; 35; 15; 26; 40; 28; 18; 20; 25; 34; 39; 42; 24; 22; 19; 27; 22; 34; 40; 20; 38 and 28

Use 10–19 as the first interval.

Count the money (bills and change) in your pocket or purse. Your instructor will record the amounts. As a class, construct a histogram displaying the data. Discuss how many intervals you think is appropriate. You may want to experiment with the number of intervals.

Frequency Polygons

Frequency polygons are analogous to line graphs, and just as line graphs make continuous data visually easy to interpret, so too do frequency polygons. To construct a frequency polygon, first examine the data and decide on the number of intervals, or class intervals, to use on the *x*-axis and *y*-axis. After choosing the appropriate ranges, begin plotting the data points. After all the points are plotted, draw line segments to connect them.

✓ Example 2.2.4

A frequency polygon was constructed from the frequency table below.

Frequency Distribution for Calculus Final Test Scores

Lower Bound	Upper Bound	Frequency	Cumulative Frequency
49.5	59.5	5	5
59.5	69.5	10	15
69.5	79.5	30	45
79.5	89.5	40	85
89.5	99.5	15	100

Figure 2.2.4: A frequency polygon was constructed from the frequency table above.

The first label on the *x*-axis is 44.5. This represents an interval extending from 39.5 to 49.5. Since the lowest test score is 54.5, this interval is used only to allow the graph to touch the *x*-axis. The point labeled 54.5 represents the next interval, or the first “real” interval from the table, and contains five scores. This reasoning is followed for each of the remaining intervals with the point 104.5 representing the interval from 99.5 to 109.5. Again, this interval contains no data and is only used so that the graph will touch the *x*-axis. Looking at the graph, we say that this distribution is skewed because one side of the graph does not mirror the other side.

? Exercise 2.2.4

Construct a frequency polygon of U.S. Presidents’ ages at inauguration shown in the Table.

Age at Inauguration	Frequency
41.5–46.5	4
46.5–51.5	11
51.5–56.5	14
56.5–61.5	9
61.5–66.5	4
66.5–71.5	2

Answer

The first label on the *x*-axis is 39. This represents an interval extending from 36.5 to 41.5. Since there are no ages less than 41.5, this interval is used only to allow the graph to touch the *x*-axis. The point labeled 44 represents the next interval, or the first “real” interval from the table, and contains four scores. This reasoning is followed for each of the remaining intervals with the point 74 representing the interval from 71.5 to 76.5. Again, this interval contains no data and is only used so that the graph will touch the *x*-axis. Looking at the graph, we say that this distribution is skewed because one side of the graph does not mirror the other side.

Figure 2.2.5: This figure shows a graph entitled, 'President's Age at Inauguration.' The x-axis is labeled 'Ages' and is marked off at 39, 44, 49, 54, 59, 64, 69 and 74. The y-axis is labeled, 'Frequency,' and is marked off in intervals of 1 from 0 to 15. The following points are plotted and a line connects one to the other to create the frequency polygon: (39, 0), (44, 4), (49, 11), (54, 14), (59, 9), (64, 4), (69, 2), (74, 0).

Frequency polygons are useful for comparing distributions. This is achieved by overlaying the frequency polygons drawn for different data sets.

✓ Example 2.2.5

We will construct an overlay frequency polygon comparing the scores from Example with the students' final numeric grade.

Frequency Distribution for Calculus Final Test Scores

Lower Bound	Upper Bound	Frequency	Cumulative Frequency
49.5	59.5	5	5
59.5	69.5	10	15
69.5	79.5	30	45
79.5	89.5	40	85
89.5	99.5	15	100

Frequency Distribution for Calculus Final Grades

Lower Bound	Upper Bound	Frequency	Cumulative Frequency
49.5	59.5	10	10
59.5	69.5	10	20
69.5	79.5	30	50
79.5	89.5	45	95
89.5	99.5	5	100

Figure 2.2.6: This is an overlay frequency polygon that matches the supplied data. The x-axis shows the grades, and the y-axis shows the frequency.

Suppose that we want to study the temperature range of a region for an entire month. Every day at noon we note the temperature and write this down in a log. A variety of statistical studies could be done with this data. We could find the mean or the median temperature for the month. We could construct a histogram displaying the number of days that temperatures reach a certain range of values. However, all of these methods ignore a portion of the data that we have collected.

One feature of the data that we may want to consider is that of time. Since each date is paired with the temperature reading for the day, we don't have to think of the data as being random. We can instead use the times given to impose a chronological order on the data. A graph that recognizes this ordering and displays the changing temperature as the month progresses is called a time series graph.

Constructing a Time Series Graph

To construct a time series graph, we must look at both pieces of our **paired data set**. We start with a standard Cartesian coordinate system. The horizontal axis is used to plot the date or time increments, and the vertical axis is used to plot the values of the variable that we are measuring. By doing this, we make each point on the graph correspond to a date and a measured quantity. The points on the graph are typically connected by straight lines in the order in which they occur.

✓ Example 2.2.6

The following data shows the Annual Consumer Price Index, each month, for ten years. Construct a time series graph for the Annual Consumer Price Index data only.

Year	Jan	Feb	Mar	Apr	May	Jun	Jul
2003	181.7	183.1	184.2	183.8	183.5	183.7	183.9
2004	185.2	186.2	187.4	188.0	189.1	189.7	189.4
2005	190.7	191.8	193.3	194.6	194.4	194.5	195.4
2006	198.3	198.7	199.8	201.5	202.5	202.9	203.5
2007	202.416	203.499	205.352	206.686	207.949	208.352	208.299
2008	211.080	211.693	213.528	214.823	216.632	218.815	219.964
2009	211.143	212.193	212.709	213.240	213.856	215.693	215.351
2010	216.687	216.741	217.631	218.009	218.178	217.965	218.011
2011	220.223	221.309	223.467	224.906	225.964	225.722	225.922
2012	226.665	227.663	229.392	230.085	229.815	229.478	229.104

Year	Aug	Sep	Oct	Nov	Dec	Annual
2003	184.6	185.2	185.0	184.5	184.3	184.0
2004	189.5	189.9	190.9	191.0	190.3	188.9
2005	196.4	198.8	199.2	197.6	196.8	195.3
2006	203.9	202.9	201.8	201.5	201.8	201.6
2007	207.917	208.490	208.936	210.177	210.036	207.342
2008	219.086	218.783	216.573	212.425	210.228	215.303
2009	215.834	215.969	216.177	216.330	215.949	214.537
2010	218.312	218.439	218.711	218.803	219.179	218.056
2011	226.545	226.889	226.421	226.230	225.672	224.939
2012	230.379	231.407	231.317	230.221	229.601	229.594

Answer

Figure 2.2.7: This is a times series graph that matches the supplied data. The x-axis shows years from 2003 to 2012, and the y-axis shows the annual CPI.

? Exercise 2.2.5

The following table is a portion of a data set from www.worldbank.org. Use the table to construct a time series graph for CO_2 emissions for the United States.

CO_2 Emissions

	Ukraine	United Kingdom	United States
2003	352,259	540,640	5,681,664
2004	343,121	540,409	5,790,761

	Ukraine	United Kingdom	United States
2005	339,029	541,990	5,826,394
2006	327,797	542,045	5,737,615
2007	328,357	528,631	5,828,697
2008	323,657	522,247	5,656,839
2009	272,176	474,579	5,299,563

Figure 2.2.8: This is a times series graph that matches the supplied data. The x-axis shows years from 2003 to 2012, and the y-axis shows the annual CPI.

Uses of a Time Series Graph

Time series graphs are important tools in various applications of statistics. When recording values of the same variable over an extended period of time, sometimes it is difficult to discern any trend or pattern. However, once the same data points are displayed graphically, some features jump out. Time series graphs make trends easy to spot.

Review

A **histogram** is a graphic version of a frequency distribution. The graph consists of bars of equal width drawn adjacent to each other. The horizontal scale represents classes of quantitative data values and the vertical scale represents frequencies. The heights of the bars correspond to frequency values. Histograms are typically used for large, continuous, quantitative data sets. A frequency polygon can also be used when graphing large data sets with data points that repeat. The data usually goes on *y*-axis with the frequency being graphed on the *x*-axis. Time series graphs can be helpful when looking at large amounts of data for one variable over a period of time.Glossary

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Frequency

the number of times a value of the data occurs

Histogram

a graphical representation in $x - y$ form of the distribution of data in a data set; x represents the data and y represents the frequency, or relative frequency. The graph consists of contiguous rectangles.

Relative Frequency

the ratio of the number of times a value of the data occurs in the set of all outcomes to the number of all outcomes

2.3: Measures of the Location of the Data

The common measures of location are **quartiles** and **percentiles.** Quartiles are special percentiles. The first quartile, Q_1, is the same as the 25th percentile, and the third quartile, Q_3, is the same as the 75th percentile. The median, M, is called both the second quartile and the 50th percentile.

To calculate quartiles and percentiles, the data must be ordered from smallest to largest. Quartiles divide ordered data into quarters. Percentiles divide ordered data into hundredths. To score in the 90th percentile of an exam does not mean, necessarily, that you received 90% on a test. It means that 90% of test scores are the same or less than your score and 10% of the test scores are the same or greater than your test score.

Percentiles are useful for comparing values. For this reason, universities and colleges use percentiles extensively. One instance in which colleges and universities use percentiles is when SAT results are used to determine a minimum testing score that will be used as an acceptance factor. For example, suppose Duke accepts SAT scores at or above the 75th percentile. That translates into a score of at least 1220.

Percentiles are mostly used with very large populations. Therefore, if you were to say that 90% of the test scores are less (and not the same or less) than your score, it would be acceptable because removing one particular data value is not significant.

The **median** is a number that measures the "center" of the data. You can think of the median as the "middle value," but it does not actually have to be one of the observed values. It is a number that separates ordered data into halves. Half the values are the same number or smaller than the median, and half the values are the same number or larger. For example, consider the following data.

1; 11.5; 6; 7.2; 4; 8; 9; 10; 6.8; 8.3; 2; 2; 10; 1

Ordered from smallest to largest:

1; 1; 2; 2; 4; 6; 6.8; 7.2; 8; 8.3; 9; 10; 10; 11.5

Since there are 14 observations, the median is between the seventh value, 6.8, and the eighth value, 7.2. To find the median, add the two values together and divide by two.

$$\frac{6.8+7.2}{2}=7 \tag{2.3.1}$$

The median is seven. Half of the values are smaller than seven and half of the values are larger than seven.

Quartiles are numbers that separate the data into quarters. Quartiles may or may not be part of the data. To find the quartiles, first find the median or second quartile. The first quartile, Q_1, is the middle value of the lower half of the data, and the third quartile, Q_3, is the middle value, or median, of the upper half of the data. To get the idea, consider the same data set:

1; 1; 2; 2; 4; 6; 6.8; 7.2; 8; 8.3; 9; 10; 10; 11.5

The median or **second quartile** is seven. The lower half of the data are 1, 1, 2, 2, 4, 6, 6.8. The middle value of the lower half is two.

1; 1; 2; 2; 4; 6; 6.8

The number two, which is part of the data, is the **first quartile**. One-fourth of the entire sets of values are the same as or less than two and three-fourths of the values are more than two.

The upper half of the data is 7.2, 8, 8.3, 9, 10, 10, 11.5. The middle value of the upper half is nine.

The **third quartile**, $Q3$, is nine. Three-fourths (75%) of the ordered data set are less than nine. One-fourth (25%) of the ordered data set are greater than nine. The third quartile is part of the data set in this example.

The **interquartile range** is a number that indicates the spread of the middle half or the middle 50% of the data. It is the difference between the third quartile (Q_3) and the first quartile (Q_1).

$$IQR=Q_3-Q_1 \tag{2.4.1}$$

The IQR can help to determine potential **outliers**. **A value is suspected to be a potential outlier if it is less than (1.5)(IQR) below the first quartile or more than (1.5)(IQR) above the third quartile**. Potential outliers always require further investigation.

Definition: Outliers

A potential outlier is a data point that is significantly different from the other data points. These special data points may be errors or some kind of abnormality or they may be a key to understanding the data.

Example 2.4.1

For the following 13 real estate prices, calculate the *IQR* and determine if any prices are potential outliers. Prices are in dollars.

389,950; 230,500; 158,000; 479,000; 639,000; 114,950; 5,500,000; 387,000; 659,000; 529,000; 575,000; 488,800; 1,095,000

Answer

Order the data from smallest to largest.

114,950; 158,000; 230,500; 387,000; 389,950; 479,000; 488,800; 529,000; 575,000; 639,000; 659,000; 1,095,000; 5,500,000

$$M = 488,800$$

$$Q_1 = \frac{230,500 + 387,000}{2} = 308,750$$

$$Q_3 = \frac{639,000 + 659,000}{2} = 649,000$$

$$IQR = 649,000 - 308,750 = 340,250$$

$$(1.5)(IQR) = (1.5)(340,250) = 510,375$$

$$Q_1 - (1.5)(IQR) = 308,750 - 510,375 = -201,625$$

$$Q_3 + (1.5)(IQR) = 649,000 + 510,375 = 1,159,375$$

No house price is less than –201,625. However, 5,500,000 is more than 1,159,375. Therefore, 5,500,000 is a potential **outlier**.

Exercise 2.3.1

For the following 11 salaries, calculate the *IQR* and determine if any salaries are outliers. The salaries are in dollars.

$33,000; $64,500; $28,000; $54,000; $72,000; $68,500; $69,000; $42,000; $54,000; $120,000; $40,500

Answer

Order the data from smallest to largest.

$28,000; $33,000; $40,500; $42,000; $54,000; $54,000; $64,500; $68,500; $69,000; $72,000; $120,000

Median = $54,000

$$Q_1 = \$40,500$$

$$Q_3 = \$69,000$$

$$IQR = \$69,000 - \$40,500 = \$28,500$$

$$(1.5)(IQR) = (1.5)(\$28,500) = \$42,750$$

$$Q_1 - (1.5)(IQR) = \$40,500 - \$42,750 = -\$2,250$$

$$Q_3 + (1.5)(IQR) = \$69,000 + \$42,750 = \$111,750$$

No salary is less than –$2,250. However, $120,000 is more than $11,750, so $120,000 is a potential outlier.

✓ Example 2.4.2

For the two data sets in the test scores example, find the following:

a. The interquartile range. Compare the two interquartile ranges.
b. Any outliers in either set.

Answer

The five number summary for the day and night classes is

	Minimum	Q_1	Median	Q_3	Maximum
Day	32	56	74.5	82.5	99
Night	25.5	78	81	89	98

a. The *IQR* for the day group is $Q_3 - Q_1 = 82.5 - 56 = 26.5$

The *IQR* for the night group is $Q_3 - Q_1 = 89 - 78 = 11$

The interquartile range (the spread or variability) for the day class is larger than the night class *IQR*. This suggests more variation will be found in the day class's class test scores.

b. Day class outliers are found using the IQR times 1.5 rule. So,

- $Q_1 - IQR(1.5) = 56 – 26.5(1.5) = 16.25$
- $Q_3 + IQR(1.5) = 82.5 + 26.5(1.5) = 122.25$

Since the minimum and maximum values for the day class are greater than 16.25 and less than 122.25, there are no outliers.

Night class outliers are calculated as:

- $Q_1 - IQR(1.5) = 78 – 11(1.5) = 61.5$
- $Q_3 + IQR(1.5) = 89 + 11(1.5) = 105.5$

For this class, any test score less than 61.5 is an outlier. Therefore, the scores of 45 and 25.5 are outliers. Since no test score is greater than 105.5, there is no upper end outlier.

? Exercise 2.3.2

Find the interquartile range for the following two data sets and compare them.

Test Scores for Class *A*

69; 96; 81; 79; 65; 76; 83; 99; 89; 67; 90; 77; 85; 98; 66; 91; 77; 69; 80; 94

Test Scores for Class *B*

90; 72; 80; 92; 90; 97; 92; 75; 79; 68; 70; 80; 99; 95; 78; 73; 71; 68; 95; 100

Answer

Class *A*

Order the data from smallest to largest.

65; 66; 67; 69; 69; 76; 77; 77; 79; 80; 81; 83; 85; 89; 90; 91; 94; 96; 98; 99

$$Median = \frac{80+81}{2} = 80.5$$

$$Q_1 = \frac{69+76}{2} = 72.5$$

$$Q_3 = \frac{90+91}{2} = 90.5$$

$$IQR = 90.5 - 72.5 = 18$$

Class *B*

Order the data from smallest to largest.

68; 68; 70; 71; 72; 73; 75; 78; 79; 80; 80; 90; 90; 92; 92; 95; 95; 97; 99; 100

$$Median = \frac{80+80}{2} = 80$$

$$Q_1 = \frac{72+73}{2} = 72.5$$

$$Q_3 = \frac{92+95}{2} = 93.5$$

$$IQR = 93.5 - 72.5 = 21$$

The data for Class *B* has a larger *IQR*, so the scores between Q_3 and Q_1 (middle 50%) for the data for Class *B* are more spread out and not clustered about the median.

✓ Example 2.3.3

Fifty statistics students were asked how much sleep they get per school night (rounded to the nearest hour). The results were:

AMOUNT OF SLEEP PER SCHOOL NIGHT (HOURS)	FREQUENCY	RELATIVE FREQUENCY	CUMULATIVE RELATIVE FREQUENCY
4	2	0.04	0.04
5	5	0.10	0.14
6	7	0.14	0.28
7	12	0.24	0.52
8	14	0.28	0.80
9	7	0.14	0.94
10	3	0.06	1.00

Find the 28th percentile. Notice the 0.28 in the "cumulative relative frequency" column. Twenty-eight percent of 50 data values is 14 values. There are 14 values less than the 28th percentile. They include the two 4s, the five 5s, and the seven 6s. The 28th percentile is between the last six and the first seven. **The 28th percentile is 6.5.**

Find the median. Look again at the "cumulative relative frequency" column and find 0.52. The median is the 50th percentile or the second quartile. 50% of 50 is 25. There are 25 values less than the median. They include the two 4s, the five 5s, the seven 6s, and eleven of the 7s. The median or 50th percentile is between the 25th, or seven, and 26th, or seven, values. **The median is seven.**

Find the third quartile. The third quartile is the same as the 75th percentile. You can "eyeball" this answer. If you look at the "cumulative relative frequency" column, you find 0.52 and 0.80. When you have all the fours, fives, sixes and sevens, you have 52% of the data. When you include all the 8s, you have 80% of the data. **The 75th percentile, then, must be an eight**. Another way to look at the problem is to find 75% of 50, which is 37.5, and round up to 38. The third quartile, Q_3, is the 38th value, which is an eight. You can check this answer by counting the values. (There are 37 values below the third quartile and 12 values above.)

? Exercise 2.3.3

Forty bus drivers were asked how many hours they spend each day running their routes (rounded to the nearest hour). Find the 65th percentile.

Amount of time spent on route (hours)	Frequency	Relative Frequency	Cumulative Relative Frequency
2	12	0.30	0.30
3	14	0.35	0.65
4	10	0.25	0.90
5	4	0.10	1.00

Answer

The 65th percentile is between the last three and the first four.

The 65th percentile is 3.5.

✓ Example 2.4.4

Using the table above in Example $2.3.3$

a. Find the 80th percentile.
b. Find the 90th percentile.
c. Find the first quartile. What is another name for the first quartile?

Solution

Using the data from the frequency table, we have:

a. The 80th percentile is between the last eight and the first nine in the table (between the 40th and 41st values). Therefore, we need to take the mean of the 40th an 41st values. The 80th percentile $= \dfrac{8+9}{2} = 8.5$
b. The 90th percentile will be the 45th data value (location is $0.90(50) = 45$) and the 45th data value is nine.
c. Q_1 is also the 25th percentile. The 25th percentile location calculation: $P_{25} = 0.25(50) = 12.5 \approx 13$ the 13th data value. Thus, the 25th percentile is six.

? Exercise $2.3.4$

Refer to the table above in Exercise $2.3.3$. Find the third quartile. What is another name for the third quartile?

Answer

The third quartile is the 75th percentile, which is four. The 65th percentile is between three and four, and the 90th percentile is between four and 5.75. The third quartile is between 65 and 90, so it must be four.

COLLABORATIVE STATISTICS

Your instructor or a member of the class will ask everyone in class how many sweaters they own. Answer the following questions:

a. How many students were surveyed?
b. What kind of sampling did you do?
c. Construct two different histograms. For each, starting value = _____ ending value = ____.
d. Find the median, first quartile, and third quartile.
e. Construct a table of the data to find the following:
    i. the 10th percentile
    ii. the 70th percentile
    iii. the percent of students who own less than four sweaters

A Formula for Finding the *k*th Percentile

If you were to do a little research, you would find several formulas for calculating the kth percentile. Here is one of them.

- $k =$ the kth percentile. It may or may not be part of the data.
- $i =$ the index (ranking or position of a data value)
- $n =$ the total number of data

Order the data from smallest to largest.

Calculate $i = \frac{k}{100}(n+1)$

If i is an integer, then the k^{th} percentile is the data value in the i^{th} position in the ordered set of data.

If i is not an integer, then round i up and round i down to the nearest integers. Average the two data values in these two positions in the ordered data set. This is easier to understand in an example.

✓ Example 2.4.5

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

a. Find the 70th percentile.
b. Find the 83rd percentile.

Solution

a.
- $k = 70$
- i = the index
- $n = 29$

$i = \frac{k}{100}(n+1) = \frac{70}{100}(29+1) = 21$. Twenty-one is an integer, and the data value in the 21st position in the ordered data set is 64. The 70th percentile is 64 years.

b.
- k = 83rd percentile
- i= the index
- $n = 29$

$i = \frac{k}{100}(n+1) = (\frac{83}{100})(29+1) = 24.9$, which is NOT an integer. Round it down to 24 and up to 25. The age in the 24th position is 71 and the age in the 25th position is 72. Average 71 and 72. The 83rd percentile is 71.5 years.

? Exercise 2.3.5

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

Calculate the 20th percentile and the 55th percentile.

Answer

$k = 20$. Index $= i = \frac{k}{100}(n+1) = \frac{20}{100}(29+1) = 6$. The age in the sixth position is 27. The 20th percentile is 27 years.

$k = 55$. Index $= i = \frac{k}{100}(n+1) = \frac{55}{100}(29+1) = 16.5$. Round down to 16 and up to 17. The age in the 16th position is 52 and the age in the 17th position is 55. The average of 52 and 55 is 53.5. The 55th percentile is 53.5 years.

Note 2.4.2

You can calculate percentiles using calculators and computers. There are a variety of online calculators.

A Formula for Finding the Percentile of a Value in a Data Set

- Order the data from smallest to largest.
- $x =$ the number of data values counting from the bottom of the data list up to but not including the data value for which you want to find the percentile.
- $y =$ the number of data values equal to the data value for which you want to find the percentile.
- $n =$ the total number of data.
- Calculate $\frac{x+0.5y}{n}(100)$. Then round to the nearest integer.

✓ Example 2.4.6

Listed are 29 ages for Academy Award winning best actors *in order from smallest to largest.*

18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

a. Find the percentile for 58.
b. Find the percentile for 25.

Solution

a. Counting from the bottom of the list, there are 18 data values less than 58. There is one value of 58.

$x = 18$ and $y = 1$. $\frac{x+0.5y}{n}(100) = \frac{18+0.5(1)}{29}(100) = 63.80$. 58 is the 64th percentile.

b. Counting from the bottom of the list, there are three data values less than 25. There is one value of 25.

$x = 3$ and $y = 1$. $\frac{x+0.5y}{n}(100) = \frac{3+0.5(1)}{29}(100) = 12.07$. Twenty-five is the 12thpercentile.

? Exercise 2.3.6

Listed are 30 ages for Academy Award winning best actors in order from smallest to largest.

18; 21; 22; 25; 26; 27; 29; 30; 31, 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77

Find the percentiles for 47 and 31.

Answer

Percentile for 47: Counting from the bottom of the list, there are 15 data values less than 47. There is one value of 47.

$x = 15$ and $y = 1$. $\frac{x+0.5y}{n}(100) = \frac{15+0.5(1)}{30}(100) = 51.67$. 47 is the 52nd percentile.

Percentile for 31: Counting from the bottom of the list, there are eight data values less than 31. There are two values of 31.

$x = 8$ and $y = 2$. $\frac{x+0.5y}{n}(100) = \frac{8+0.5(2)}{30}(100) = 30$. 31 is the 30th percentile.

Interpreting Percentiles, Quartiles, and Median

A percentile indicates the relative standing of a data value when data are sorted into numerical order from smallest to largest. Percentages of data values are less than or equal to the pth percentile. For example, 15% of data values are less than or equal to the 15th percentile.

- Low percentiles always correspond to lower data values.
- High percentiles always correspond to higher data values.

A percentile may or may not correspond to a value judgment about whether it is "good" or "bad." The interpretation of whether a certain percentile is "good" or "bad" depends on the context of the situation to which the data applies. In some situations, a low percentile would be considered "good;" in other contexts a high percentile might be considered "good". In many situations, there is no value judgment that applies.

Understanding how to interpret percentiles properly is important not only when describing data, but also when calculating probabilities in later chapters of this text.

GUIDELINE

When writing the interpretation of a percentile in the context of the given data, the sentence should contain the following information.

- information about the context of the situation being considered
- the data value (value of the variable) that represents the percentile
- the percent of individuals or items with data values below the percentile
- the percent of individuals or items with data values above the percentile.

On a timed math test, the first quartile for time it took to finish the exam was 35 minutes. Interpret the first quartile in the context of this situation.

Answer

- Twenty-five percent of students finished the exam in 35 minutes or less.
- Seventy-five percent of students finished the exam in 35 minutes or more.
- A low percentile could be considered good, as finishing more quickly on a timed exam is desirable. (If you take too long, you might not be able to finish.)

? Exercise 2.3.7

For the 100-meter dash, the third quartile for times for finishing the race was 11.5 seconds. Interpret the third quartile in the context of the situation.

Answer

Twenty-five percent of runners finished the race in 11.5 seconds or more. Seventy-five percent of runners finished the race in 11.5 seconds or less. A lower percentile is good because finishing a race more quickly is desirable.

On a 20 question math test, the 70th percentile for number of correct answers was 16. Interpret the 70th percentile in the context of this situation.

Answer

- Seventy percent of students answered 16 or fewer questions correctly.
- Thirty percent of students answered 16 or more questions correctly.
- A higher percentile could be considered good, as answering more questions correctly is desirable.

? Exercise 2.3.8

On a 60 point written assignment, the 80th percentile for the number of points earned was 49. Interpret the 80th percentile in the context of this situation.

Answer

Eighty percent of students earned 49 points or fewer. Twenty percent of students earned 49 or more points. A higher percentile is good because getting more points on an assignment is desirable.

✓ Example 2.4.9

At a community college, it was found that the 30th percentile of credit units that students are enrolled for is seven units. Interpret the 30th percentile in the context of this situation.

Answer

- Thirty percent of students are enrolled in seven or fewer credit units.
- Seventy percent of students are enrolled in seven or more credit units.
- In this example, there is no "good" or "bad" value judgment associated with a higher or lower percentile. Students attend community college for varied reasons and needs, and their course load varies according to their needs.

? Exercise 2.3.9

During a season, the 40th percentile for points scored per player in a game is eight. Interpret the 40th percentile in the context of this situation.

Answer

Forty percent of players scored eight points or fewer. Sixty percent of players scored eight points or more. A higher percentile is good because getting more points in a basketball game is desirable.

✓ Example 2.4.10

Sharpe Middle School is applying for a grant that will be used to add fitness equipment to the gym. The principal surveyed 15 anonymous students to determine how many minutes a day the students spend exercising. The results from the 15 anonymous students are shown.

0 minutes; 40 minutes; 60 minutes; 30 minutes; 60 minutes

10 minutes; 45 minutes; 30 minutes; 300 minutes; 90 minutes;

30 minutes; 120 minutes; 60 minutes; 0 minutes; 20 minutes

Determine the following five values.

- Min = 0
- Q_1 = 20
- Med = 40
- Q_3 = 60
- Max = 300

If you were the principal, would you be justified in purchasing new fitness equipment? Since 75% of the students exercise for 60 minutes or less daily, and since the *IQR* is 40 minutes (60 – 20 = 40), we know that half of the students surveyed exercise between 20 minutes and 60 minutes daily. This seems a reasonable amount of time spent exercising, so the principal would be justified in purchasing the new equipment.

However, the principal needs to be careful. The value 300 appears to be a potential outlier.

$$Q_3 + 1.5(IQR) = 60 + (1.5)(40) = 120 \tag{2.3.2}$$

.

The value 300 is greater than 120 so it is a potential outlier. If we delete it and calculate the five values, we get the following values:

- Min = 0
- Q_1 = 20
- Q_3 = 60
- Max = 120

We still have 75% of the students exercising for 60 minutes or less daily and half of the students exercising between 20 and 60 minutes a day. However, 15 students is a small sample and the principal should survey more students to be sure of his survey results.

References

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2. Data from the United States Department of Commerce: United States Census Bureau. Available online at http://www.census.gov/ (accessed April 3, 2013).
3. “1990 Census.” United States Department of Commerce: United States Census Bureau. Available online at http://www.census.gov/main/www/cen1990.html (accessed April 3, 2013).
4. Data from *San Jose Mercury News*.
5. Data from *Time Magazine*; survey by Yankelovich Partners, Inc.

Review

The values that divide a rank-ordered set of data into 100 equal parts are called percentiles. Percentiles are used to compare and interpret data. For example, an observation at the 50th percentile would be greater than 50 percent of the other obeservations in the set. Quartiles divide data into quarters. The first quartile (Q_1) is the 25th percentile,the second quartile (Q_2 or median) is 50th percentile, and the third quartile (Q_3) is the the 75th percentile. The interquartile range, or *IQR*, is the range of the middle 50 percent of the data values. The *IQR* is found by subtracting Q_1 from Q_3, and can help determine outliers by using the following two expressions.

- $Q_3 + IQR(1.5)$
- $Q_1 - IQR(1.5)$

Formula Review

$$i = \frac{k}{100}(n+1)$$

where i = the ranking or position of a data value,

- k = the kth percentile,
- n = total number of data.

Expression for finding the percentile of a data value: $\left(\frac{x+0.5y}{n}\right)(100)$

where $x =$ the number of values counting from the bottom of the data list up to but not including the data value for which you want to find the percentile,

$y =$ the number of data values equal to the data value for which you want to find the percentile,

$n =$ total number of data

Glossary

Interquartile Range

or *IQR*, is the range of the middle 50 percent of the data values; the *IQR* is found by subtracting the first quartile from the third quartile.

Outlier

an observation that does not fit the rest of the data

Percentile

a number that divides ordered data into hundredths; percentiles may or may not be part of the data. The median of the data is the second quartile and the 50th percentile. The first and third quartiles are the 25th and the 75th percentiles, respectively.

Quartiles

the numbers that separate the data into quarters; quartiles may or may not be part of the data. The second quartile is the median of the data.

- **2.4: Measures of the Location of the Data** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/introductory-statistics.

2.4: Measures of the Center of the Data

The "center" of a data set is also a way of describing location. The two most widely used measures of the "center" of the data are the **mean** (average) and the median. To calculate the **mean weight** of 50 people, add the 50 weights together and divide by 50. To find the **median weight** of the 50 people, order the data and find the number that splits the data into two equal parts. The median is generally a better measure of the center when there are extreme values or outliers because it is not affected by the precise numerical values of the outliers. The mean is the most common measure of the center.

Note

The words “mean” and “average” are often used interchangeably. The substitution of one word for the other is common practice. The technical term is “arithmetic mean” and “average” is technically a center location. However, in practice among non-statisticians, “average" is commonly accepted for “arithmetic mean.”

When each value in the data set is not unique, the mean can be calculated by multiplying each distinct value by its frequency and then dividing the sum by the total number of data values. The letter used to represent the sample mean is an x with a bar over it (pronounced “x bar”): $\bar{x}$.

The Greek letter μ (pronounced "mew") represents the **population mean**. One of the requirements for the **sample mean** to be a good estimate of the **population mean** is for the sample taken to be truly random.

To see that both ways of calculating the mean are the same, consider the sample:

1; 1; 1; 2; 2; 3; 4; 4; 4; 4; 4

$$\bar{x}=\frac{1+1+1+2+2+3+4+4+4+4+4}{11}=2.7 \tag{2.4.1}$$

$$\bar{x}=\frac{3(1)+2(2)+1(3)+5(4)}{11}=2.7 \tag{2.4.2}$$

In the second calculation, the frequencies are 3, 2, 1, and 5.

You can quickly find the location of the median by using the expression

$$\frac{n+1}{2} \tag{2.4.3}$$

The letter n is the total number of data values in the sample. If n is an odd number, the median is the middle value of the ordered data (ordered smallest to largest). If n is an even number, the median is equal to the two middle values added together and divided by two after the data has been ordered. For example, if the total number of data values is 97, then

$$\frac{n+1}{2}=\frac{97+1}{2}=49. \tag{2.4.4}$$

The median is the 49th value in the ordered data. If the total number of data values is 100, then

$$\frac{n+1}{2}=\frac{100+1}{2}=50.5. \tag{2.4.5}$$

The median occurs midway between the 50th and 51st values. The location of the median and the value of the median are **not** the same. The upper case letter M is often used to represent the median. The next example illustrates the location of the median and the value of the median.

Example 2.4.1

AIDS data indicating the number of months a patient with AIDS lives after taking a new antibody drug are as follows (smallest to largest):

3; 4; 8; 8; 10; 11; 12; 13; 14; 15; 15; 16; 16; 17; 17; 18; 21; 22; 22; 24; 24; 25; 26; 26; 27; 27; 29; 29; 31; 32; 33; 33; 34; 34; 35; 37; 40; 44; 44; 47

Calculate the mean and the median.

Answer

The calculation for the mean is:

$$\bar{x} = \frac{[3+4+(8)(2)+10+11+12+13+14+(15)(2)+(16)(2)+\ldots+35+37+40+(44)(2)+47]}{40} = 23.6 \tag{2.4.6}$$

To find the median, M, first use the formula for the location. The location is:

$$\frac{n+1}{2} = \frac{40+1}{2} = 20.5 \tag{2.4.7}$$

Starting at the smallest value, the median is located between the 20th and 21st values (the two 24s):

3; 4; 8; 8; 10; 11; 12; 13; 14; 15; 15; 16; 16; 17; 17; 18; 21; 22; 22; 24; 24; 25; 26; 26; 27; 27; 29; 29; 31; 32; 33; 33; 34; 34; 35; 37; 40; 44; 44; 47

$$M = \frac{24+24}{2} = 24 \tag{2.4.8}$$

Calculator

To find the mean and the median:

Clear list L1. Pres STAT 4:ClrList. Enter 2nd 1 for list L1. Press ENTER.

Enter data into the list editor. Press STAT 1:EDIT.

Put the data values into list L1.

Press STAT and arrow to CALC. Press 1:1-VarStats. Press 2nd 1 for L1 and then ENTER.

Press the down and up arrow keys to scroll.

$\bar{x}$ = 23.6, M = 24

Exercise 2.4.1

The following data show the number of months patients typically wait on a transplant list before getting surgery. The data are ordered from smallest to largest. Calculate the mean and median.

3; 4; 5; 7; 7; 7; 7; 8; 8; 9; 9; 10; 10; 10; 10; 10; 11; 12; 12; 13; 14; 14; 15; 15; 17; 17; 18; 19; 19; 19; 21; 21; 22; 22; 23; 24; 24; 24; 24

Answer

Mean: $3+4+5+7+7+7+7+8+8+9+9+10+10+10+10+10+11+12+12+13+14+14+15+15$
$+17+17+18+19+19+19+21+21+22+22+23+24+24+24=544$

$$\frac{544}{39} = 13.95 \tag{2.4.9}$$

Median: Starting at the smallest value, the median is the 20th term, which is 13.

Example 2.4.2

Suppose that in a small town of 50 people, one person earns $5,000,000 per year and the other 49 each earn $30,000. Which is the better measure of the "center": the mean or the median?

Solution

$$\bar{x} = \frac{5,000,000+49(30,000)}{50} = 129,400 \tag{2.4.10}$$

$M = 30,000$

(There are 49 people who earn $30,000 and one person who earns $5,000,000.)

The median is a better measure of the "center" than the mean because 49 of the values are 30,000 and one is 5,000,000. The 5,000,000 is an outlier. The 30,000 gives us a better sense of the middle of the data.

? Exercise 2.4.2

In a sample of 60 households, one house is worth $2,500,000. Half of the rest are worth $280,000, and all the others are worth $315,000. Which is the better measure of the “center”: the mean or the median?

Answer

The median is the better measure of the “center” than the mean because 59 of the values are $280,000 and one is $2,500,000. The $2,500,000 is an outlier. Either $280,000 or $315,000 gives us a better sense of the middle of the data.

Another measure of the center is the mode. The **mode** is the most frequent value. There can be more than one mode in a data set as long as those values have the same frequency and that frequency is the highest. A data set with two modes is called bimodal.

✓ Example 2.4.3

Statistics exam scores for 20 students are as follows:

50; 53; 59; 59; 63; 63; 72; 72; 72; 72; 72; 76; 78; 81; 83; 84; 84; 84; 90; 93

Find the mode.

Answer

The most frequent score is 72, which occurs five times. Mode = 72.

? Exercise 2.4.3

The number of books checked out from the library from 25 students are as follows:

0; 0; 0; 1; 2; 3; 3; 4; 4; 5; 5; 7; 7; 7; 7; 8; 8; 8; 9; 10; 10; 11; 11; 12; 12

Find the mode.

Answer

The most frequent number of books is 7, which occurs four times. Mode = 7.

✓ Example 2.4.4

Five real estate exam scores are 430, 430, 480, 480, 495. The data set is bimodal because the scores 430 and 480 each occur twice.

When is the mode the best measure of the "center"? Consider a weight loss program that advertises a mean weight loss of six pounds the first week of the program. The mode might indicate that most people lose two pounds the first week, making the program less appealing.

> The mode can be calculated for qualitative data as well as for quantitative data. For example, if the data set is: red, red, red, green, green, yellow, purple, black, blue, the mode is red.

Statistical software will easily calculate the mean, the median, and the mode. Some graphing calculators can also make these calculations. In the real world, people make these calculations using software.

? Exercise 2.4.4

Five credit scores are 680, 680, 700, 720, 720. The data set is bimodal because the scores 680 and 720 each occur twice. Consider the annual earnings of workers at a factory. The mode is $25,000 and occurs 150 times out of 301. The median is $50,000 and the mean is $47,500. What would be the best measure of the “center”?

Answer

Because $25,000 occurs nearly half the time, the mode would be the best measure of the center because the median and mean don’t represent what most people make at the factory.

The Law of Large Numbers and the Mean

The Law of Large Numbers says that if you take samples of larger and larger size from any population, then the mean $\bar{x}$ of the sample is very likely to get closer and closer to μ. This is discussed in more detail later in the text.

Sampling Distributions and Statistic of a Sampling Distribution

You can think of a sampling distribution as a **relative frequency distribution** with a great many samples. (See **Sampling and Data** for a review of relative frequency). Suppose thirty randomly selected students were asked the number of movies they watched the previous week. The results are in the **relative frequency table** shown below.

# of movies	Relative Frequency
0	$\frac{5}{30}$
1	$\frac{15}{30}$
2	$\frac{6}{30}$
3	$\frac{3}{30}$
4	$\frac{1}{30}$

If you let the number of samples get very large (say, 300 million or more), the relative frequency table becomes a relative frequency distribution.

A statistic is a number calculated from a sample. Statistic examples include the mean, the median and the mode as well as others. The sample mean $\bar{x}$ is an example of a statistic which estimates the population mean μ.

Calculating the Mean of Grouped Frequency Tables

When only grouped data is available, you do not know the individual data values (we only know intervals and interval frequencies); therefore, you cannot compute an exact mean for the data set. What we must do is estimate the actual mean by calculating the mean of a frequency table. A frequency table is a data representation in which grouped data is displayed along with the corresponding frequencies. To calculate the mean from a grouped frequency table we can apply the basic definition of mean:

$$mean = \frac{\text{data sum}}{\text{number of data values}}. \tag{2.4.11}$$

We simply need to modify the definition to fit within the restrictions of a frequency table.

Since we do not know the individual data values we can instead find the midpoint of each interval. The midpoint is

$$\frac{\text{lower boundary+upper boundary}}{2}. \tag{2.4.12}$$

We can now modify the mean definition to be

$$\text{Mean of Frequency Table} = \frac{\sum fm}{\sum f} \tag{2.4.13}$$

where f is the frequency of the interval and m is the midpoint of the interval.

Example 2.4.5

A frequency table displaying professor Blount’s last statistic test is shown. Find the best estimate of the class mean.

Grade Interval	Number of Students
50–56.5	1
56.5–62.5	0
62.5–68.5	4

Grade Interval	Number of Students
68.5–74.5	4
74.5–80.5	2
80.5–86.5	3
86.5–92.5	4
92.5–98.5	1

Solution

- Find the midpoints for all intervals

Grade Interval	Midpoint
50–56.5	53.25
56.5–62.5	59.5
62.5–68.5	65.5
68.5–74.5	71.5
74.5–80.5	77.5
80.5–86.5	83.5
86.5–92.5	89.5
92.5–98.5	95.5

- Calculate the sum of the product of each interval frequency and midpoint. $53.25(1)+59.5(0)+65.5(4)+71.5(4)+77.5(2)+83.5(3)+89.5(4)+95.5(1)=1460.25$
- $\mu=\frac{\sum fm}{\sum f}=\frac{1460.25}{19}=76.86$

Exercise 2.4.5

Maris conducted a study on the effect that playing video games has on memory recall. As part of her study, she compiled the following data:

Hours Teenagers Spend on Video Games	Number of Teenagers
0–3.5	3
3.5–7.5	7
7.5–11.5	12
11.5–15.5	7
15.5–19.5	9

What is the best estimate for the mean number of hours spent playing video games?

Answer

Find the midpoint of each interval, multiply by the corresponding number of teenagers, add the results and then divide by the total number of teenagers

The midpoints are 1.75, 5.5, 9.5, 13.5,17.5.

$$Mean=(1.75)(3)+(5.5)(7)+(9.5)(12)+(13.5)(7)+(17.5)(9)=409.75 \tag{2.4.14}$$

References

1. Data from The World Bank, available online at http://www.worldbank.org (accessed April 3, 2013).
2. “Demographics: Obesity – adult prevalence rate.” Indexmundi. Available online at http://www.indexmundi.com/g/r.aspx?t=50&v=2228&l=en (accessed April 3, 2013).

Review

The mean and the median can be calculated to help you find the "center" of a data set. The mean is the best estimate for the actual data set, but the median is the best measurement when a data set contains several outliers or extreme values. The mode will tell you the most frequently occuring datum (or data) in your data set. The mean, median, and mode are extremely helpful when you need to analyze your data, but if your data set consists of ranges which lack specific values, the mean may seem impossible to calculate. However, the mean can be approximated if you add the lower boundary with the upper boundary and divide by two to find the midpoint of each interval. Multiply each midpoint by the number of values found in the corresponding range. Divide the sum of these values by the total number of data values in the set.

Formula Review

$$\mu = \frac{\sum fm}{\sum f} \tag{2.4.15}$$

where f = interval frequencies and m = interval midpoints.

Exercise 2.6.6

Find the mean for the following frequency tables.

a.

Grade	Frequency
49.5–59.5	2
59.5–69.5	3
69.5–79.5	8
79.5–89.5	12
89.5–99.5	5

b.

Daily Low Temperature	Frequency
49.5–59.5	53
59.5–69.5	32
69.5–79.5	15
79.5–89.5	1
89.5–99.5	0

c.

Points per Game	Frequency
49.5–59.5	14
59.5–69.5	32
69.5–79.5	15
79.5–89.5	23
89.5–99.5	2

Use the following information to answer the next three exercises: The following data show the lengths of boats moored in a marina. The data are ordered from smallest to largest: 16; 17; 19; 20; 20; 21; 23; 24; 25; 25; 25; 26; 26; 27; 27; 27; 28; 29; 30; 32; 33; 33; 34; 35; 37; 39; 40

Exercise 2.6.7

Calculate the mean.

Answer

Mean: $16+17+19+20+20+21+23+24+25+25+25+26+26+27+27+27+28+29+30+32+33+33$; $+34+35+37+39+40=738$

$\frac{738}{27}=27.33$

Exercise 2.6.8

Identify the median.

Exercise 2.6.9

Identify the mode.

Answer

The most frequent lengths are 25 and 27, which occur three times. Mode = 25, 27

Use the following information to answer the next three exercises: Sixty-five randomly selected car salespersons were asked the number of cars they generally sell in one week. Fourteen people answered that they generally sell three cars; nineteen generally sell four cars; twelve generally sell five cars; nine generally sell six cars; eleven generally sell seven cars. Calculate the following:

Exercise 2.6.10

sample mean = $\bar{x}$ = _______

Exercise 2.6.11

median = _______

Answer

4

Bringing It Together

Exercise 2.6.12

Javier and Ercilia are supervisors at a shopping mall. Each was given the task of estimating the mean distance that shoppers live from the mall. They each randomly surveyed 100 shoppers. The samples yielded the following information.

	Javier	Ercilia
$\bar{x}$	6.0 miles	6.0 miles
s	4.0 miles	7.0 miles

a. How can you determine which survey was correct ?
b. Explain what the difference in the results of the surveys implies about the data.
c. If the two histograms depict the distribution of values for each supervisor, which one depicts Ercilia's sample? How do you know?

This shows two histograms. The first histogram shows a fairly symmetrical distribution with a mode of 6. The second histogram shows a uniform distribution.

Figure 2.4.1: This shows two histograms. The first histogram shows a fairly symmetrical distribution with a mode of 6. The second histogram shows a uniform distribution.

1. If the two box plots depict the distribution of values for each supervisor, which one depicts Ercilia’s sample? How do you know?

This shows two horizontal boxplots. The first boxplot is graphed over a number line from 0 to 21. The first whisker extends from 0 to 1. The box begins at the first quartile, 1, and ends at the third quartile, 14. A vertical, dashed line marks the median at 6. The second whisker extends from the third quartile to the largest value, 21. The second boxplot is graphed over a number line from 0 to 12. The first whisker extends from 0 to 4. The box begins at the first quartile, 4, and ends at the third quartile, 9. A vertical, dashed line marks the median at 6. The second whisker extends from the third quartile to the largest value, 12.

Figure 2.4.2: This shows two horizontal boxplots. The first boxplot is graphed over a number line from 0 to 21. The first whisker extends from 0 to 1. The box begins at the first quartile, 1, and ends at the third quartile, 14. A vertical, dashed line marks the median at 6. The second whisker extends from the third quartile to the largest value, 21. The second boxplot is graphed over a number line from 0 to 12. The first whisker extends from 0 to 4. The box begins at the first quartile, 4, and ends at the third quartile, 9. A vertical, dashed line marks the median at 6. The second whisker extends from the third quartile to the largest value, 12.

Use the following information to answer the next three exercises: We are interested in the number of years students in a particular elementary statistics class have lived in California. The information in the following table is from the entire section.

Number of years	Frequency	Number of years	Frequency
7	1	22	1
14	3	23	1
15	1	26	1
18	1	40	2
19	4	42	2
20	3		
			Total = 20

? Exercise 2.6.13

What is the *IQR*?

a. 8
b. 11
c. 15
d. 35

Answer

a

? Exercise 2.6.14

What is the mode?

a. 19
b. 19.5
c. 14 and 20
d. 22.65

? Exercise 2.6.15

Is this a sample or the entire population?

a. sample
b. entire population
c. neither

Answer

b

Glossary

Frequency Table

a data representation in which grouped data is displayed along with the corresponding frequencies

Mean

a number that measures the central tendency of the data; a common name for mean is 'average.' The term 'mean' is a shortened form of 'arithmetic mean.' By definition, the mean for a sample (denoted by $\bar{x}$) is $\bar{x} = \frac{\text{Sum of all values in the sample}}{\text{Number of values in the sample}}$, and the mean for a population (denoted by μ) is $\mu = \frac{\text{Sum of all values in the population}}{\text{Number of values in the population}}$.

Median

a number that separates ordered data into halves; half the values are the same number or smaller than the median and half the values are the same number or larger than the median. The median may or may not be part of the data.

Midpoint

the mean of an interval in a frequency table

Mode

the value that appears most frequently in a set of data

2.5: Skewness and the Mean, Median, and Mode

Consider the following data set.

4; 5; 6; 6; 6; 7; 7; 7; 7; 7; 7; 8; 8; 8; 9; 10

This data set can be represented by following histogram. Each interval has width one, and each value is located in the middle of an interval.

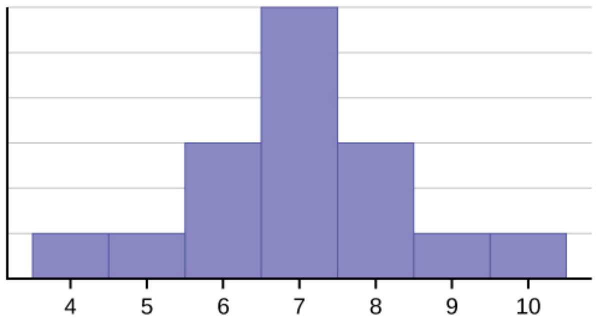


Figure 2.5.1

The histogram displays a **symmetrical** distribution of data. A distribution is symmetrical if a vertical line can be drawn at some point in the histogram such that the shape to the left and the right of the vertical line are mirror images of each other. The mean, the median, and the mode are each seven for these data. **In a perfectly symmetrical distribution, the mean and the median are the same.** This example has one mode (unimodal), and the mode is the same as the mean and median. In a symmetrical distribution that has two modes (bimodal), the two modes would be different from the mean and median.

The histogram for the data: 4; 5; 6; 6; 6; 7; 7; 7; 7; 8 is not symmetrical. The right-hand side seems "chopped off" compared to the left side. A distribution of this type is called **skewed to the left** because it is pulled out to the left.

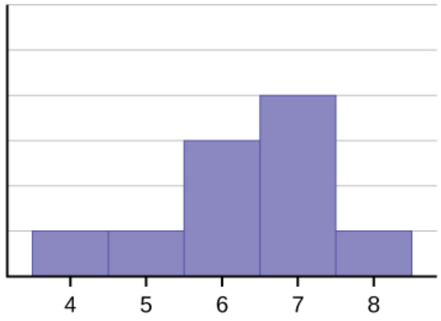


Figure 2.5.2

The mean is 6.3, the median is 6.5, and the mode is seven. **Notice that the mean is less than the median, and they are both less than the mode.** The mean and the median both reflect the skewing, but the mean reflects it more so.

The histogram for the data: 6; 7; 7; 7; 7; 8; 8; 8; 9; 10, is also not symmetrical. It is **skewed to the right**.

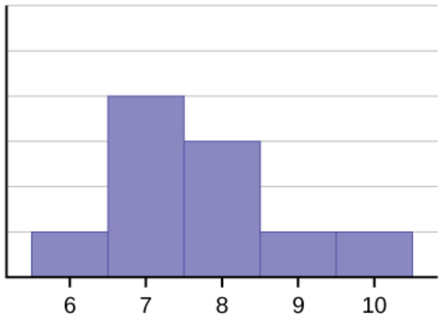


Figure 2.5.3

The mean is 7.7, the median is 7.5, and the mode is seven. Of the three statistics, **the mean is the largest, while the mode is the smallest**. Again, the mean reflects the skewing the most.

> *Generally, if the distribution of data is skewed to the left, the mean is less than the median, which is often less than the mode. If the distribution of data is skewed to the right, the mode is often less than the median, which is less than the mean.*

Skewness and symmetry become important when we discuss probability distributions in later chapters.

✓ Example 2.5.1

Statistics are used to compare and sometimes identify authors. The following lists shows a simple random sample that compares the letter counts for three authors.

- Terry: 7; 9; 3; 3; 3; 4; 1; 3; 2; 2
- Davis: 3; 3; 3; 4; 1; 4; 3; 2; 3; 1
- Maris: 2; 3; 4; 4; 4; 6; 6; 6; 8; 3

a. Make a dot plot for the three authors and compare the shapes.
b. Calculate the mean for each.
c. Calculate the median for each.
d. Describe any pattern you notice between the shape and the measures of center.

Solution

a.

Figure 2.5.4: This dot plot matches the supplied data for Terry. The plot uses a number line from 1 to 10. It shows one x over 1, two x's over 2, four x's over 3, one x over 4, one x over 7, and one x over 9. There are no x's over the numbers 5, 6, 8, and 10.

Figure 2.5.5: Copy and Paste Caption here. (Copyright; author via source)

Figure 2.5.6: Copy and Paste Caption here. (Copyright; author via source)

- Terry's mean is 3.7, Davis' mean is 2.7, Maris' mean is 4.6.
- Terry's median is three, Davis' median is three. Maris' median is four.
- It appears that the median is always closest to the high point (the mode), while the mean tends to be farther out on the tail. In a symmetrical distribution, the mean and the median are both centrally located close to the high point of the distribution.

? Exercise 2.5.1

Discuss the mean, median, and mode for each of the following problems. Is there a pattern between the shape and measure of the center?

a.

Figure 2.5.7: This dot plot matches the supplied data. The plot uses a number line from 0 to 14. It shows two x's over 0, four x's over 1, three x's over 2, one x over 3, two x's over the number 4, 5, 6, and 9, and 1 x each over 10 and 14. There are no x's over the numbers 7, 8, 11, 12, and 13.

b.

The Ages Former U.S Presidents Died	
4	6 9
5	3 6 7 7 7 8
6	0 0 3 3 4 4 5 6 7 7 7 8
7	0 1 1 2 3 4 7 8 8 9
8	0 1 3 5 8
9	0 0 3 3

The Ages Former U.S Presidents Died
Key: 8|0 means 80.

c.

Figure 2.5.8: This is a histogram titled Hours Spent Playing Video Games on Weekends. The x-axis shows the number of hours spent playing video games with bars showing values at intervals of 5. The y-axis shows the number of students. The first bar for 0 - 4.99 hours has a height of 2. The second bar from 5 - 9.99 has a height of 3. The third bar from 10 - 14.99 has a height of 4. The fourth bar from 15 - 19.99 has a height of 7. The fifth bar from 20 - 24.99 has a height of 9.

Review

Looking at the distribution of data can reveal a lot about the relationship between the mean, the median, and the mode. There are three types of distributions. A **left (or negative) skewed** distribution has a shape like Figure 2.5.2. A **right (or positive) skewed** distribution has a shape like Figure 2.5.3. A **symmetrical** distribution looks like Figure 2.5.1.

Use the following information to answer the next three exercises: State whether the data are symmetrical, skewed to the left, or skewed to the right.

? Exercise 2.7.2

1; 1; 1; 2; 2; 2; 2; 3; 3; 3; 3; 3; 3; 3; 3; 4; 4; 4; 5; 5

Answer

The data are symmetrical. The median is 3 and the mean is 2.85. They are close, and the mode lies close to the middle of the data, so the data are symmetrical.

? Exercise 2.7.3

16; 17; 19; 22; 22; 22; 22; 22; 23

? Exercise 2.7.4

87; 87; 87; 87; 87; 88; 89; 89; 90; 91

Answer

The data are skewed right. The median is 87.5 and the mean is 88.2. Even though they are close, the mode lies to the left of the middle of the data, and there are many more instances of 87 than any other number, so the data are skewed right.

? Exercise 2.7.5

When the data are skewed left, what is the typical relationship between the mean and median?

? Exercise 2.7.6

When the data are symmetrical, what is the typical relationship between the mean and median?

Answer

When the data are symmetrical, the mean and median are close or the same.

? Exercise 2.7.7

What word describes a distribution that has two modes?

? Exercise 2.7.8

Describe the shape of this distribution.

Figure 2.5.9: This is a historgram which consists of 5 adjacent bars with the x-axis split into intervals of 1 from 3 to 7. The bar heights peak at the first bar and taper lower to the right.

Answer

The distribution is skewed right because it looks pulled out to the right.

? Exercise 2.7.9

Describe the relationship between the mode and the median of this distribution.

Figure 2.5.10: This is a histogram which consists of 5 adjacent bars with the x-axis split into intervals of 1 from 3 to 7. The bar heights peak at the first bar and taper lower to the right. The bar heighs from left to right are: 8, 4, 2, 2, 1.

? Exercise 2.7.10

Describe the relationship between the mean and the median of this distribution.

Figure 2.5.11: This is a histogram which consists of 5 adjacent bars with the x-axis split into intervals of 1 from 3 to 7. The bar heights peak at the first bar and taper lower to the right. The bar heights from left to right are: 8, 4, 2, 2, 1.

Answer

The mean is 4.1 and is slightly greater than the median, which is four.

? Exercise 2.7.11

Describe the shape of this distribution.

Figure 2.5.12

? Exercise 2.7.12

Describe the relationship between the mode and the median of this distribution.

Figure 2.5.13

Answer

The mode and the median are the same. In this case, they are both five.

? Exercise 2.7.13

Are the mean and the median the exact same in this distribution? Why or why not?

Figure 2.5.14

? Exercise 2.7.14

Describe the shape of this distribution.

Figure 2.5.15

Answer

The distribution is skewed left because it looks pulled out to the left.

? Exercise 2.7.15

Describe the relationship between the mode and the median of this distribution.

Figure 2.5.16: Copy and Paste Caption here. (Copyright; author via source)

? Exercise 2.7.16

Describe the relationship between the mean and the median of this distribution.

Figure 2.5.17

Answer

The mean and the median are both six.

? Exercise 2.7.17

The mean and median for the data are the same.

3; 4; 5; 5; 6; 6; 6; 6; 7; 7; 7; 7; 7; 7; 7

Is the data perfectly symmetrical? Why or why not?

? Exercise 2.7.18

Which is the greatest, the mean, the mode, or the median of the data set?

11; 11; 12; 12; 12; 12; 13; 15; 17; 22; 22; 22

Answer

The mode is 12, the median is 12.5, and the mean is 15.1. The mean is the largest.

? Exercise 2.7.19

Which is the least, the mean, the mode, and the median of the data set?

56; 56; 56; 58; 59; 60; 62; 64; 64; 65; 67

? Exercise 2.7.20

Of the three measures, which tends to reflect skewing the most, the mean, the mode, or the median? Why?

Answer

The mean tends to reflect skewing the most because it is affected the most by outliers.

? Exercise 2.7.21

In a perfectly symmetrical distribution, when would the mode be different from the mean and median?

2.6: Measures of the Spread of the Data

An important characteristic of any set of data is the variation in the data. In some data sets, the data values are concentrated closely near the mean; in other data sets, the data values are more widely spread out from the mean. The most common measure of variation, or spread, is the standard deviation. The standard deviation is a number that measures how far data values are from their mean.

> **The standard deviation**
>
> - provides a numerical measure of the overall amount of variation in a data set, and
> - can be used to determine whether a particular data value is close to or far from the mean.

The standard deviation provides a measure of the overall variation in a data set

The standard deviation is always positive or zero. The standard deviation is small when the data are all concentrated close to the mean, exhibiting little variation or spread. The standard deviation is larger when the data values are more spread out from the mean, exhibiting more variation.

Suppose that we are studying the amount of time customers wait in line at the checkout at supermarket *A* and supermarket *B*. the average wait time at both supermarkets is five minutes. At supermarket *A*, the standard deviation for the wait time is two minutes; at supermarket *B* the standard deviation for the wait time is four minutes.

Because supermarket *B* has a higher standard deviation, we know that there is more variation in the wait times at supermarket *B*. Overall, wait times at supermarket *B* are more spread out from the average; wait times at supermarket *A* are more concentrated near the average.

The standard deviation can be used to determine whether a data value is close to or far from the mean.

Suppose that Rosa and Binh both shop at supermarket *A*. Rosa waits at the checkout counter for seven minutes and Binh waits for one minute. At supermarket *A*, the mean waiting time is five minutes and the standard deviation is two minutes. The standard deviation can be used to determine whether a data value is close to or far from the mean.

Rosa waits for seven minutes:

- Seven is two minutes longer than the average of five; two minutes is equal to one standard deviation.
- Rosa's wait time of seven minutes is **two minutes longer than the average** of five minutes.
- Rosa's wait time of seven minutes is **one standard deviation above the average** of five minutes.

Binh waits for one minute.

- One is four minutes less than the average of five; four minutes is equal to two standard deviations.
- Binh's wait time of one minute is **four minutes less than the average** of five minutes.
- Binh's wait time of one minute is **two standard deviations below the average** of five minutes.
- A data value that is two standard deviations from the average is just on the borderline for what many statisticians would consider to be far from the average. Considering data to be far from the mean if it is more than two standard deviations away is more of an approximate "rule of thumb" than a rigid rule. In general, the shape of the distribution of the data affects how much of the data is further away than two standard deviations. (You will learn more about this in later chapters.)

The number line may help you understand standard deviation. If we were to put five and seven on a number line, seven is to the right of five. We say, then, that seven is **one** standard deviation to the **right** of five because $5+(1)(2)=7$.

If one were also part of the data set, then one is **two** standard deviations to the **left** of five because $5+(-2)(2)=1$.

Figure 2.6.1

- In general, a **value = mean + (#ofSTDEV)(standard deviation)**
- where #ofSTDEVs = the number of standard deviations
- #ofSTDEV does not need to be an integer

- One is **two standard deviations less than the mean** of five because: $1 = 5 + (-2)(2)$.

The equation value = mean + (#ofSTDEVs)(standard deviation) can be expressed for a sample and for a population.

- sample:

$$x = \bar{x} + (\#\text{ofSTDEV})(\text{s}) \tag{2.6.1}$$

- Population:

$$x = \mu + (\#\text{ofSTDEV})(\text{s}) \tag{2.6.2}$$

The lower case letter s represents the sample standard deviation and the Greek letter σ (sigma, lower case) represents the population standard deviation.

The symbol $\bar{x}$ is the sample mean and the Greek symbol μ is the population mean.

Calculating the Standard Deviation

If x is a number, then the difference "x – mean" is called its **deviation**. In a data set, there are as many deviations as there are items in the data set. The deviations are used to calculate the standard deviation. If the numbers belong to a population, in symbols a deviation is $x - \mu$. For sample data, in symbols a deviation is $x - \bar{x}$.

The procedure to calculate the standard deviation depends on whether the numbers are the entire population or are data from a sample. The calculations are similar, but not identical. Therefore the symbol used to represent the standard deviation depends on whether it is calculated from a population or a sample. The lower case letter s represents the sample standard deviation and the Greek letter σ (sigma, lower case) represents the population standard deviation. If the sample has the same characteristics as the population, then s should be a good estimate of σ.

To calculate the standard deviation, we need to calculate the variance first. The variance is the **average of the squares of the deviations** (the $x - \bar{x}$ values for a sample, or the $x - \mu$ values for a population). The symbol σ^2 represents the population variance; the population standard deviation σ is the square root of the population variance. The symbol s^2 represents the sample variance; the sample standard deviation s is the square root of the sample variance. You can think of the standard deviation as a special average of the deviations.

If the numbers come from a census of the entire **population** and not a sample, when we calculate the average of the squared deviations to find the variance, we divide by N, the number of items in the population. If the data are from a **sample** rather than a population, when we calculate the average of the squared deviations, we divide by ***n* – 1**, one less than the number of items in the sample.

Formulas for the Sample Standard Deviation

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n-1}} \tag{2.6.3}$$

or

$$s = \sqrt{\frac{\sum f(x - \bar{x})^2}{n-1}} \tag{2.6.4}$$

For the sample standard deviation, the denominator is $n - 1$, that is the sample size MINUS 1.

Formulas for the Population Standard Deviation

$$\sigma = \sqrt{\frac{\sum (x - \mu)^2}{N}} \tag{2.6.5}$$

or

$$\sigma = \sqrt{\frac{\sum f(x - \mu)^2}{N}} \tag{2.6.6}$$

> For the population standard deviation, the denominator is N, the number of items in the population.

In Equations 2.6.4 and 2.6.6, f represents the frequency with which a value appears. For example, if a value appears once, f is one. If a value appears three times in the data set or population, f is three.

Sampling Variability of a Statistic

The statistic of a sampling distribution was discussed in Section 2.6. How much the statistic varies from one sample to another is known as the sampling variability of a statistic. You typically measure the **sampling variability of a statistic** by its standard error.

The **standard error of the mean** is an example of a standard error. It is a special standard deviation and is known as the standard deviation of the sampling distribution of the mean. You will cover the standard error of the mean in Chapter 7. The notation for the standard error of the mean is $\frac{\sigma}{\sqrt{n}}$ where σ is the standard deviation of the population and n is the size of the sample.

> In practice, USE A CALCULATOR OR COMPUTER SOFTWARE TO CALCULATE THE STANDARD DEVIATION. If you are using a TI-83, 83+, 84+ calculator, you need to select the appropriate standard deviation σ_x or s_x from the summary statistics. We will concentrate on using and interpreting the information that the standard deviation gives us. However you should study the following step-by-step example to help you understand how the standard deviation measures variation from the mean. (The calculator instructions appear at the end of this example.)

✓ Example 2.6.1

In a fifth grade class, the teacher was interested in the average age and the sample standard deviation of the ages of her students. The following data are the ages for a SAMPLE of n = 20 fifth grade students. The ages are rounded to the nearest half year:

9; 9.5; 9.5; 10; 10; 10; 10; 10.5; 10.5; 10.5; 10.5; 11; 11; 11; 11; 11; 11; 11.5; 11.5; 11.5;

$$\bar{x} = \frac{9 + 9.5(2) + 10(4) + 10.5(4) + 11(6) + 11.5(3)}{20} = 10.525$$

The average age is 10.53 years, rounded to two places.

The variance may be calculated by using a table. Then the standard deviation is calculated by taking the square root of the variance. We will explain the parts of the table after calculating *s*.

Data	Freq.	Deviations	*Deviations*^{2}	(Freq.)(*Deviations*^{2})
x	*f*	(x – $\bar{x}$)	(x – $\bar{x}$)2	(f)(x – $\bar{x}$)2
9	1	9 – 10.525 = –1.525	(–1.525)2 = 2.325625	1 × 2.325625 = 2.325625
9.5	2	9.5 – 10.525 = –1.025	(–1.025)^2 = 1.050625	2 × 1.050625 = 2.101250
10	4	10 – 10.525 = –0.525	(–0.525)^2 = 0.275625	4 × 0.275625 = 1.1025
10.5	4	10.5 – 10.525 = –0.025	(–0.025)^2 = 0.000625	4 × 0.000625 = 0.0025
11	6	11 – 10.525 = 0.475	(0.475)^2 = 0.225625	6 × 0.225625 = 1.35375
11.5	3	11.5 – 10.525 = 0.975	(0.975)^2 = 0.950625	3 × 0.950625 = 2.851875
				The total is 9.7375

The sample variance, s^2, is equal to the sum of the last column (9.7375) divided by the total number of data values minus one (20 – 1):

$$s^2 = \frac{9.7375}{20-1} = 0.5125$$

The **sample standard deviation** s is equal to the square root of the sample variance:

$$s = \sqrt{0.5125} = 0.715891$$

and this is rounded to two decimal places, $s = 0.72$.

Typically, you do the calculation for the standard deviation on your calculator or computer. The intermediate results are not rounded. This is done for accuracy.

- For the following problems, recall that **value = mean + (#ofSTDEVs)(standard deviation)**. Verify the mean and standard deviation or a calculator or computer.
- For a sample: x = $\bar{x}$ + (#ofSTDEVs)(s)
- For a population: x = μ + (#ofSTDEVs)σ
- For this example, use x = $\bar{x}$ + (#ofSTDEVs)(s) because the data is from a sample

a. Verify the mean and standard deviation on your calculator or computer.
b. Find the value that is one standard deviation above the mean. Find ($\bar{x}$ + 1s).
c. Find the value that is two standard deviations below the mean. Find ($\bar{x}$ – 2s).
d. Find the values that are 1.5 standard deviations **from** (below and above) the mean.

Solution

a.
- Clear lists L1 and L2. Press STAT 4:ClrList. Enter 2nd 1 for L1, the comma (,), and 2nd 2 for L2.
- Enter data into the list editor. Press STAT 1:EDIT. If necessary, clear the lists by arrowing up into the name. Press CLEAR and arrow down.
- Put the data values (9, 9.5, 10, 10.5, 11, 11.5) into list L1 and the frequencies (1, 2, 4, 4, 6, 3) into list L2. Use the arrow keys to move around.
- Press STAT and arrow to CALC. Press 1:1-VarStats and enter L1 (2nd 1), L2 (2nd 2). Do not forget the comma. Press ENTER.
- $\bar{x}$ = 10.525
- Use Sx because this is sample data (not a population): Sx=0.715891

b. $(\bar{x} + 1s) = 10.53 + (1)(0.72) = 11.25$
c. $(\bar{x} - 2s) = 10.53 – (2)(0.72) = 9.09$
d.
- $(\bar{x} - 1.5s) = 10.53 – (1.5)(0.72) = 9.45$
- $(\bar{x} + 1.5s) = 10.53 + (1.5)(0.72) = 11.61$

? Exercise 2.8.1

On a baseball team, the ages of each of the players are as follows:

21; 21; 22; 23; 24; 24; 25; 25; 28; 29; 29; 31; 32; 33; 33; 34; 35; 36; 36; 36; 36; 38; 38; 38; 40

Use your calculator or computer to find the mean and standard deviation. Then find the value that is two standard deviations above the mean.

Answer

μ = 30.68

$s = 6.09$

$(\bar{x} + 2s = 30.68 + (2)(6.09) = 42.86$.

Explanation of the standard deviation calculation shown in the table

The deviations show how spread out the data are about the mean. The data value 11.5 is farther from the mean than is the data value 11 which is indicated by the deviations 0.97 and 0.47. A positive deviation occurs when the data value is greater than the mean, whereas a negative deviation occurs when the data value is less than the mean. The deviation is –1.525 for the data value nine. **If you add the deviations, the sum is always zero**. (For Example $2.6.1$, there are $n = 20$ deviations.) So you cannot simply add the deviations to get the spread of the data. By squaring the deviations, you make them positive numbers, and the sum will also be positive. The variance, then, is the average squared deviation.

The variance is a squared measure and does not have the same units as the data. Taking the square root solves the problem. The standard deviation measures the spread in the same units as the data.

Notice that instead of dividing by $n = 20$, the calculation divided by $n - 1 = 20 - 1 = 19$ because the data is a sample. For the **sample** variance, we divide by the sample size minus one ($n - 1$). Why not divide by n? The answer has to do with the population variance. **The sample variance is an estimate of the population variance.** Based on the theoretical mathematics that lies behind these calculations, dividing by ($n - 1$) gives a better estimate of the population variance.

> Your concentration should be on what the standard deviation tells us about the data. The standard deviation is a number which measures how far the data are spread from the mean. Let a calculator or computer do the arithmetic.

The standard deviation, s or σ, is either zero or larger than zero. When the standard deviation is zero, there is no spread; that is, all the data values are equal to each other. The standard deviation is small when the data are all concentrated close to the mean, and is larger when the data values show more variation from the mean. When the standard deviation is a lot larger than zero, the data values are very spread out about the mean; outliers can make s or σ very large.

The standard deviation, when first presented, can seem unclear. By graphing your data, you can get a better "feel" for the deviations and the standard deviation. You will find that in symmetrical distributions, the standard deviation can be very helpful but in skewed distributions, the standard deviation may not be much help. The reason is that the two sides of a skewed distribution have different spreads. In a skewed distribution, it is better to look at the first quartile, the median, the third quartile, the smallest value, and the largest value. Because numbers can be confusing, **always graph your data**. Display your data in a histogram or a box plot.

✓ Example $2.6.2$

Use the following data (first exam scores) from Susan Dean's spring pre-calculus class:

33; 42; 49; 49; 53; 55; 55; 61; 63; 67; 68; 68; 69; 69; 72; 73; 74; 78; 80; 83; 88; 88; 88; 90; 92; 94; 94; 94; 94; 96; 100

a. Create a chart containing the data, frequencies, relative frequencies, and cumulative relative frequencies to three decimal places.

b. Calculate the following to one decimal place using a TI-83+ or TI-84 calculator:

   i. The sample mean
   ii. The sample standard deviation
   iii. The median
   iv. The first quartile
   v. The third quartile
   vi. *IQR*

c. Construct a box plot and a histogram on the same set of axes. Make comments about the box plot, the histogram, and the chart.

Answer

a. See Table

b. i. The sample mean = 73.5
   ii. The sample standard deviation = 17.9
   iii. The median = 73
   iv. The first quartile = 61
   v. The third quartile = 90
   vi. *IQR* = 90 – 61 = 29

c. The x-axis goes from 32.5 to 100.5; y-axis goes from -2.4 to 15 for the histogram. The number of intervals is five, so the width of an interval is $(100.5 - 32.5)$ divided by five, is equal to 13.6. Endpoints of the intervals are as follows: the starting point is 32.5, $32.5 + 13.6 = 46.1$, $46.1 + 13.6 = 59.7$, $59.7 + 13.6 = 73.3$, $73.3 + 13.6 = 86.9$, $86.9 + 13.6 = 100.5 =$ the ending value; No data values fall on an interval boundary.

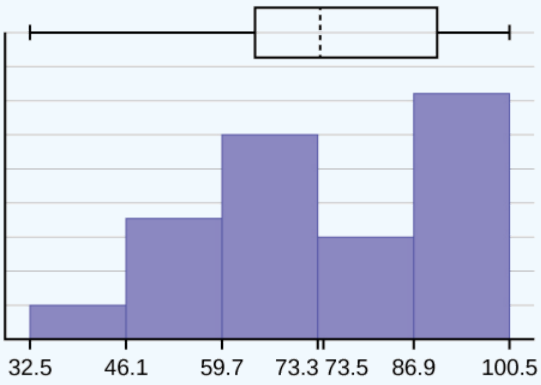


Figure 2.6.2.

The long left whisker in the box plot is reflected in the left side of the histogram. The spread of the exam scores in the lower 50% is greater $(73-33=40)$ than the spread in the upper 50% $(100-73=27)$. The histogram, box plot, and chart all reflect this. There are a substantial number of A and B grades (80s, 90s, and 100). The histogram clearly shows this. The box plot shows us that the middle 50% of the exam scores (*IQR* = 29) are Ds, Cs, and Bs. The box plot also shows us that the lower 25% of the exam scores are Ds and Fs.

Data	Frequency	Relative Frequency	Cumulative Relative Frequency
33	1	0.032	0.032
42	1	0.032	0.064
49	2	0.065	0.129
53	1	0.032	0.161
55	2	0.065	0.226
61	1	0.032	0.258
63	1	0.032	0.29
67	1	0.032	0.322
68	2	0.065	0.387
69	2	0.065	0.452
72	1	0.032	0.484
73	1	0.032	0.516
74	1	0.032	0.548
78	1	0.032	0.580
80	1	0.032	0.612
83	1	0.032	0.644
88	3	0.097	0.741
90	1	0.032	0.773
92	1	0.032	0.805
94	4	0.129	0.934
96	1	0.032	0.966

Data	Frequency	Relative Frequency	Cumulative Relative Frequency
100	1	0.032	0.998 (Why isn't this value 1?)

? Exercise 2.6.2

The following data show the different types of pet food stores in the area carry.

6; 6; 6; 6; 7; 7; 7; 7; 7; 8; 9; 9; 9; 9; 10; 10; 10; 10; 10; 11; 11; 11; 11; 12; 12; 12; 12; 12; 12;

Calculate the sample mean and the sample standard deviation to one decimal place using a TI-83+ or TI-84 calculator.

Answer

$\mu = 9.3$ and $s = 2.2$

Standard deviation of Grouped Frequency Tables

Recall that for grouped data we do not know individual data values, so we cannot describe the typical value of the data with precision. In other words, we cannot find the exact mean, median, or mode. We can, however, determine the best estimate of the measures of center by finding the mean of the grouped data with the formula:

$$\text{Mean of Frequency Table} = \frac{\sum fm}{\sum f} \tag{2.6.7}$$

where f interval frequencies and $m =$ interval midpoints.

Just as we could not find the exact mean, neither can we find the exact standard deviation. Remember that standard deviation describes numerically the expected deviation a data value has from the mean. In simple English, the standard deviation allows us to compare how “unusual” individual data is compared to the mean.

✓ Example 2.6.3

Find the standard deviation for the data in Table $2.6.3$.

Table 2.6.3

Class	Frequency, *f*	Midpoint, *m*	m^2	$\bar{x}$	fm^2	Standard Deviation
0–2	1	1	1	7.58	1	3.5
3–5	6	4	16	7.58	96	3.5
6–8	10	7	49	7.58	490	3.5
9–11	7	10	100	7.58	700	3.5
12–14	0	13	169	7.58	0	3.5
15–17	2	16	256	7.58	512	3.5

For this data set, we have the mean, $\bar{x}$ = 7.58 and the standard deviation, s_x = 3.5. This means that a randomly selected data value would be expected to be 3.5 units from the mean. If we look at the first class, we see that the class midpoint is equal to one. This is almost two full standard deviations from the mean since 7.58 – 3.5 – 3.5 = 0.58. While the formula for calculating the standard deviation is not complicated, $s_x = \sqrt{\frac{f(m-\bar{x})^2}{n-1}}$ where s_x = sample standard deviation, $\bar{x}$ = sample mean, the calculations are tedious. It is usually best to use technology when performing the calculations.

Find the standard deviation for the data from the previous example

Class	0-2	3-5	6-8	9–11	12–14	15–17
Frequency, *f*	1	6	10	7	0	2

First, press the **STAT** key and select **1:Edit**

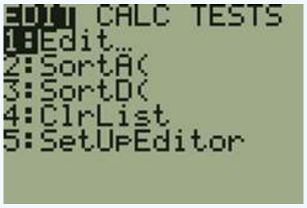
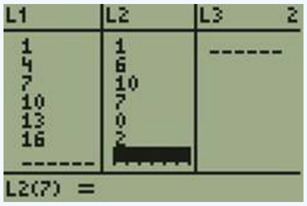


Figure 2.6.3

Input the midpoint values into **L1** and the frequencies into **L2**



```
L1      L2      L3     2
1       1       ------
4       6
7       10
10      7
13      0
16      2
------
L2(7) =
```

Figure 2.6.4

Select **STAT**, **CALC**, and **1: 1-Var Stats**

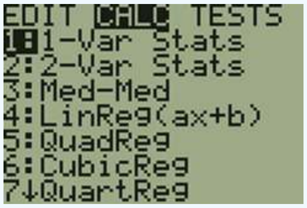
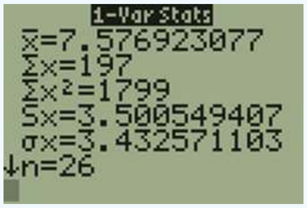


Figure 2.6.5

Select **2$^{\text{nd}}$** then **1** then , **2$^{\text{nd}}$** then **2 Enter**



```
1-Var Stats
x̄=7.576923077
Σx=197
Σx²=1799
Sx=3.500549407
σx=3.432571103
↓n=26
```

Figure 2.6.6

You will see displayed both a population standard deviation, σ_x, and the sample standard deviation, s_x.

Comparing Values from Different Data Sets

The standard deviation is useful when comparing data values that come from different data sets. If the data sets have different means and standard deviations, then comparing the data values directly can be misleading.

- For each data value, calculate how many standard deviations away from its mean the value is.
- Use the formula: value = mean + (#ofSTDEVs)(standard deviation); solve for #ofSTDEVs.
- $\#\text{ofSTDEVs} = \dfrac{\text{value-mean}}{\text{standard deviation}}$
- Compare the results of this calculation.

#ofSTDEVs is often called a "z-score"; we can use the symbol z. In symbols, the formulas become:

Sample	$x = \bar{x} + zs$	$z = \dfrac{x - \bar{x}}{s}$

Population	$x=\mu+z\sigma$	$z=\frac{x-\mu}{\sigma}$

Example 2.6.4

Two students, John and Ali, from different high schools, wanted to find out who had the highest GPA when compared to his school. Which student had the highest GPA when compared to his school?

Student	GPA	School Mean GPA	School Standard Deviation
John	2.85	3.0	0.7
Ali	77	80	10

Answer

For each student, determine how many standard deviations (#ofSTDEVs) his GPA is away from the average, for his school. Pay careful attention to signs when comparing and interpreting the answer.

$$z=\#\text{ofSTDEVs}=\left(\frac{\text{value-mean}}{\text{standard deviation}}\right)=\left(\frac{x+\mu}{\sigma}\right)$$

For John,

$$z=\#\text{ofSTDEVs}=\left(\frac{2.85-3.0}{0.7}\right)=-0.21$$

For Ali,

$$z=\#\text{ofSTDEVs}=(\frac{77-80}{10})=-0.3$$

John has the better GPA when compared to his school because his GPA is 0.21 standard deviations **below** his school's mean while Ali's GPA is 0.3 standard deviations **below** his school's mean.

John's *z*-score of –0.21 is higher than Ali's *z*-score of –0.3. For GPA, higher values are better, so we conclude that John has the better GPA when compared to his school.

Exercise 2.6.4

Two swimmers, Angie and Beth, from different teams, wanted to find out who had the fastest time for the 50 meter freestyle when compared to her team. Which swimmer had the fastest time when compared to her team?

Swimmer	Time (seconds)	Team Mean Time	Team Standard Deviation
Angie	26.2	27.2	0.8
Beth	27.3	30.1	1.4

Answer

For Angie:

$$z=\left(\frac{26.2-27.2}{0.8}\right)=-1.25$$

For Beth:

$$z=\left(\frac{27.3-30.1}{1.4}\right)=-2$$

The following lists give a few facts that provide a little more insight into what the standard deviation tells us about the distribution of the data.

For ANY data set, no matter what the distribution of the data is:

- At least 75% of the data is within two standard deviations of the mean.
- At least 89% of the data is within three standard deviations of the mean.
- At least 95% of the data is within 4.5 standard deviations of the mean.
- This is known as Chebyshev's Rule.

For data having a distribution that is BELL-SHAPED and SYMMETRIC:

- Approximately 68% of the data is within one standard deviation of the mean.
- Approximately 95% of the data is within two standard deviations of the mean.
- More than 99% of the data is within three standard deviations of the mean.
- This is known as the Empirical Rule.
- It is important to note that this rule only applies when the shape of the distribution of the data is bell-shaped and symmetric. We will learn more about this when studying the "Normal" or "Gaussian" probability distribution in later chapters.

References

1. Data from Microsoft Bookshelf.
2. King, Bill.“Graphically Speaking.” Institutional Research, Lake Tahoe Community College. Available online at www.ltcc.edu/web/about/institutional-research (accessed April 3, 2013).

Review

The standard deviation can help you calculate the spread of data. There are different equations to use if are calculating the standard deviation of a sample or of a population.

- The Standard Deviation allows us to compare individual data or classes to the data set mean numerically.
- $s=\sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$ or $s=\sqrt{\frac{\sum f(x-\bar{x})^2}{n-1}}$ is the formula for calculating the standard deviation of a sample. To calculate the standard deviation of a population, we would use the population mean, μ, and the formula $\sigma=\sqrt{\frac{\sum(x-\mu)^2}{N}}$ or $\sigma=\sqrt{\frac{\sum f(x-\mu)^2}{N}}$. .

Formula Review

$$s_x=\sqrt{\frac{\sum fm^2}{n}-\bar{x}^2} \tag{2.6.8}$$

where $s_x=\text{sample standard deviation}$ and $\bar{x}=\text{sample mean}$

Use the following information to answer the next two exercises: The following data are the distances between 20 retail stores and a large distribution center. The distances are in miles.

29; 37; 38; 40; 58; 67; 68; 69; 76; 86; 87; 95; 96; 96; 99; 106; 112; 127; 145; 150

? Exercise 2.8.4

Use a graphing calculator or computer to find the standard deviation and round to the nearest tenth.

Answer

s = 34.5

Exercise 2.8.5

Find the value that is one standard deviation below the mean.

Exercise 2.8.6

Two baseball players, Fredo and Karl, on different teams wanted to find out who had the higher batting average when compared to his team. Which baseball player had the higher batting average when compared to his team?

Baseball Player	Batting Average	Team Batting Average	Team Standard Deviation
Fredo	0.158	0.166	0.012
Karl	0.177	0.189	0.015

Answer

For Fredo:

$$z = \frac{0.158 - 0.166}{0.012} = –0.67$$

For Karl:

$$z = \frac{0.177 - 0.189}{0.015} = –0.8$$

Fredo's *z*-score of –0.67 is higher than Karl's *z*-score of –0.8. For batting average, higher values are better, so Fredo has a better batting average compared to his team.

Exercise 2.8.7

Use Table to find the value that is three standard deviations:

- above the mean
- below the mean

Find the standard deviation for the following frequency tables using the formula. Check the calculations with the TI 83/84.

Exercise 2.8.5

Find the standard deviation for the following frequency tables using the formula. Check the calculations with the TI 83/84.

a.

Grade	Frequency
49.5–59.5	2
59.5–69.5	3
69.5–79.5	8
79.5–89.5	12
89.5–99.5	5

b.

Daily Low Temperature	Frequency
49.5–59.5	53
59.5–69.5	32
69.5–79.5	15

Daily Low Temperature	Frequency
79.5–89.5	1
89.5–99.5	0

c.

Points per Game	Frequency
49.5–59.5	14
59.5–69.5	32
69.5–79.5	15
79.5–89.5	23
89.5–99.5	2

Answer

a. $s_x = \sqrt{\frac{\sum fm^2}{n} - \bar{x}^2} = \sqrt{\frac{193157.45}{30} - 79.5^2} = 10.88$

b. $s_x = \sqrt{\frac{\sum fm^2}{n} - \bar{x}^2} = \sqrt{\frac{380945.3}{101} - 60.94^2} = 7.62$

c. $s_x = \sqrt{\frac{\sum fm^2}{n} - \bar{x}^2} = \sqrt{\frac{440051.5}{86} - 70.66^2} = 11.14$

Bringing It Together

Exercise 2.8.7

Twenty-five randomly selected students were asked the number of movies they watched the previous week. The results are as follows:

# of movies	Frequency
0	5
1	9
2	6
3	4
4	1

a. Find the sample mean $\bar{x}$.
b. Find the approximate sample standard deviation, s.

Answer

a. 1.48
b. 1.12

Exercise 2.8.8

Forty randomly selected students were asked the number of pairs of sneakers they owned. Let $X =$ the number of pairs of sneakers owned. The results are as follows:

X	Frequency

X	Frequency
1	2
2	5
3	8
4	12
5	12
6	0
7	1

a. Find the sample mean $\bar{x}$
b. Find the sample standard deviation, s
c. Construct a histogram of the data.
d. Complete the columns of the chart.
e. Find the first quartile.
f. Find the median.
g. Find the third quartile.
h. Construct a box plot of the data.
i. What percent of the students owned at least five pairs?
j. Find the 40th percentile.
k. Find the 90th percentile.
l. Construct a line graph of the data
m. Construct a stemplot of the data

? Exercise 2.8.9

Following are the published weights (in pounds) of all of the team members of the San Francisco 49ers from a previous year.

177; 205; 210; 210; 232; 205; 185; 185; 178; 210; 206; 212; 184; 174; 185; 242; 188; 212; 215; 247; 241; 223; 220; 260; 245; 259; 278; 270; 280; 295; 275; 285; 290; 272; 273; 280; 285; 286; 200; 215; 185; 230; 250; 241; 190; 260; 250; 302; 265; 290; 276; 228; 265

a. Organize the data from smallest to largest value.
b. Find the median.
c. Find the first quartile.
d. Find the third quartile.
e. Construct a box plot of the data.
f. The middle 50% of the weights are from _______ to _______.
g. If our population were all professional football players, would the above data be a sample of weights or the population of weights? Why?
h. If our population included every team member who ever played for the San Francisco 49ers, would the above data be a sample of weights or the population of weights? Why?
i. Assume the population was the San Francisco 49ers. Find:
   i. the population mean, μ.
   ii. the population standard deviation, σ.
   iii. the weight that is two standard deviations below the mean.
   iv. When Steve Young, quarterback, played football, he weighed 205 pounds. How many standard deviations above or below the mean was he?

j. That same year, the mean weight for the Dallas Cowboys was 240.08 pounds with a standard deviation of 44.38 pounds. Emmit Smith weighed in at 209 pounds. With respect to his team, who was lighter, Smith or Young? How did you determine your answer?

Answer

a. 174; 177; 178; 184; 185; 185; 185; 185; 188; 190; 200; 205; 205; 206; 210; 210; 210; 212; 212; 215; 215; 220; 223; 228; 230; 232; 241; 241; 242; 245; 247; 250; 250; 259; 260; 260; 265; 265; 270; 272; 273; 275; 276; 278; 280; 280; 285; 285; 286; 290; 290; 295; 302
b. 241
c. 205.5
d. 272.5
e. A box plot with a whisker between 174 and 205.5, a solid line at 205.5, a dashed line at 241, a solid line at 272.5, and a whisker between 272.5 and 302.
f. 205.5, 272.5
g. sample
h. population
i. i. 236.34
 ii. 37.50
 iii. 161.34
 iv. 0.84 std. dev. below the mean
j. Young

? Exercise 2.8.10

One hundred teachers attended a seminar on mathematical problem solving. The attitudes of a representative sample of 12 of the teachers were measured before and after the seminar. A positive number for change in attitude indicates that a teacher's attitude toward math became more positive. The 12 change scores are as follows:

3; 8; –1; 2; 0; 5; –3; 1; –1; 6; 5; –2

a. What is the mean change score?
b. What is the standard deviation for this population?
c. What is the median change score?
d. Find the change score that is 2.2 standard deviations below the mean.

? Exercise 2.8.11

Refer to Figure determine which of the following are true and which are false. Explain your solution to each part in complete sentences.

<figure >

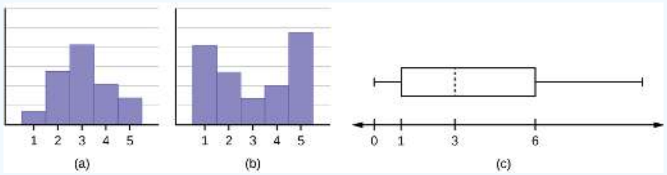


Figure 2.6.6.

</figure>

a. The medians for all three graphs are the same.
b. We cannot determine if any of the means for the three graphs is different.
c. The standard deviation for graph b is larger than the standard deviation for graph a.
d. We cannot determine if any of the third quartiles for the three graphs is different.

Answer

a. True
b. True
c. True
d. False

Exercise 2.8.12

In a recent issue of the *IEEE Spectrum*, 84 engineering conferences were announced. Four conferences lasted two days. Thirty-six lasted three days. Eighteen lasted four days. Nineteen lasted five days. Four lasted six days. One lasted seven days. One lasted eight days. One lasted nine days. Let X = the length (in days) of an engineering conference.

a. Organize the data in a chart.
b. Find the median, the first quartile, and the third quartile.
c. Find the 65th percentile.
d. Find the 10th percentile.
e. Construct a box plot of the data.
f. The middle 50% of the conferences last from _______ days to _______ days.
g. Calculate the sample mean of days of engineering conferences.
h. Calculate the sample standard deviation of days of engineering conferences.
i. Find the mode.
j. If you were planning an engineering conference, which would you choose as the length of the conference: mean; median; or mode? Explain why you made that choice.
k. Give two reasons why you think that three to five days seem to be popular lengths of engineering conferences.

Exercise 2.8.13

A survey of enrollment at 35 community colleges across the United States yielded the following figures:

6414; 1550; 2109; 9350; 21828; 4300; 5944; 5722; 2825; 2044; 5481; 5200; 5853; 2750; 10012; 6357; 27000; 9414; 7681; 3200; 17500; 9200; 7380; 18314; 6557; 13713; 17768; 7493; 2771; 2861; 1263; 7285; 28165; 5080; 11622

a. Organize the data into a chart with five intervals of equal width. Label the two columns "Enrollment" and "Frequency."
b. Construct a histogram of the data.
c. If you were to build a new community college, which piece of information would be more valuable: the mode or the mean?
d. Calculate the sample mean.
e. Calculate the sample standard deviation.
f. A school with an enrollment of 8000 would be how many standard deviations away from the mean?

Answer

a.

Enrollment	Frequency
1000-5000	10
5000-10000	16
10000-15000	3
15000-20000	3
20000-25000	1
25000-30000	2

b. Check student’s solution.
c. mode
d. 8628.74
e. 6943.88

f. –0.09

Use the following information to answer the next two exercises. $X =$ the number of days per week that 100 clients use a particular exercise facility.

x	**Frequency**
0	3
1	12
2	33
3	28
4	11
5	9
6	4

? Exercise 2.8.14

The 80th percentile is _____

a. 5
b. 80
c. 3
d. 4

? Exercise 2.8.15

The number that is 1.5 standard deviations BELOW the mean is approximately _____

a. 0.7
b. 4.8
c. –2.8
d. Cannot be determined

Answer

a

? Exercise 2.8.16

Suppose that a publisher conducted a survey asking adult consumers the number of fiction paperback books they had purchased in the previous month. The results are summarized in the Table.

# of books	Freq.	Rel. Freq.
0	18	
1	24	
2	24	
3	22	
4	15	
5	10	
7	5	

# of books	Freq.	Rel. Freq.
9	1	

a. Are there any outliers in the data? Use an appropriate numerical test involving the *IQR*to identify outliers, if any, and clearly state your conclusion.
b. If a data value is identified as an outlier, what should be done about it?
c. Are any data values further than two standard deviations away from the mean? In some situations, statisticians may use this criteria to identify data values that are unusual, compared to the other data values. (Note that this criteria is most appropriate to use for data that is mound-shaped and symmetric, rather than for skewed data.)
d. Do parts a and c of this problem give the same answer?
e. Examine the shape of the data. Which part, a or c, of this question gives a more appropriate result for this data?
f. Based on the shape of the data which is the most appropriate measure of center for this data: mean, median or mode?

Glossary

Standard Deviation

a number that is equal to the square root of the variance and measures how far data values are from their mean; notation: *s* for sample standard deviation and σ for population standard deviation.

Contributors and Attributions

Variance

mean of the squared deviations from the mean, or the square of the standard deviation; for a set of data, a deviation can be represented as $x - \bar{x}$ where x is a value of the data and $\bar{x}$ is the sample mean. The sample variance is equal to the sum of the squares of the deviations divided by the difference of the sample size and one.

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CHAPTER OVERVIEW

3: The Normal Distribution

In this chapter, you will study the normal distribution, the standard normal distribution, and applications associated with them. The normal distribution has two parameters (two numerical descriptive measures), the mean (μ) and the standard deviation (σ).

Contributors

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3.1: Prelude to The Normal Distribution

Learning Objectives

By the end of this chapter, the student should be able to:

- Recognize the normal probability distribution and apply it appropriately.
- Recognize the standard normal probability distribution and apply it appropriately.
- Compare normal probabilities by converting to the standard normal distribution.

The normal, a continuous distribution, is the most important of all the distributions. It is widely used and even more widely abused. Its graph is bell-shaped. You see the bell curve in almost all disciplines. Some of these include psychology, business, economics, the sciences, nursing, and, of course, mathematics. Some of your instructors may use the normal distribution to help determine your grade. Most IQ scores are normally distributed. Often real-estate prices fit a normal distribution. The normal distribution is extremely important, but it cannot be applied to everything in the real world.

Figure $3.1.1$: If you ask enough people about their shoe size, you will find that your graphed data is shaped like a bell curve and can be described as normally distributed. (credit: Ömer Ünlü)

In this chapter, you will study the normal distribution, the standard normal distribution, and applications associated with them. The normal distribution has two parameters (two numerical descriptive measures), the mean (μ) and the standard deviation (σ). If X is a quantity to be measured that has a normal distribution with mean (μ) and standard deviation (σ), we designate this by writing

$$f(x)=\frac{1}{\sigma\cdot\sqrt{2\cdot\pi}}\cdot e^{\left(-\frac{1}{2}\right)\cdot\left(\frac{x-\mu}{\sigma}\right)^{2}} \tag{3.1.1}$$

The probability density function is a rather complicated function. **Do not memorize it**. It is not necessary.

The cumulative distribution function is $P(X<x)$. It is calculated either by a calculator or a computer, or it is looked up in a table. Technology has made the tables virtually obsolete. For that reason, as well as the fact that there are various table formats, we are not including table instructions.

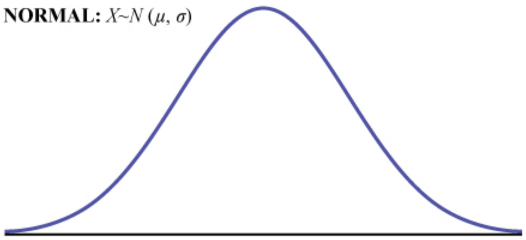

Figure $3.1.2$: The standard normal distribution

The curve is symmetrical about a vertical line drawn through the mean, μ. In theory, the mean is the same as the median, because the graph is symmetric about μ. As the notation indicates, the normal distribution depends only on the mean and the standard deviation. Since the area under the curve must equal one, a change in the standard deviation, σ, causes a change in the shape of the curve; the curve becomes fatter or skinnier depending on σ. A change in μ causes the graph to shift to the left or right. This means there are an infinite number of normal probability distributions. One of special interest is called the **standard normal distribution**.

COLLABORATIVE CLASSROOM ACTIVITY

Your instructor will record the heights of both men and women in your class, separately. Draw histograms of your data. Then draw a smooth curve through each histogram. Is each curve somewhat bell-shaped? Do you think that if you had recorded 200 data values for men and 200 for women that the curves would look bell-shaped? Calculate the mean for each data set. Write the means on the *x*-axis of the appropriate graph below the peak. Shade the approximate area that represents the probability that one randomly chosen male is taller than 72 inches. Shade the approximate area that represents the probability that one randomly chosen female is shorter than 60 inches. If the total area under each curve is one, does either probability appear to be more than 0.5?

Formula Review

- $X \sim N(\mu, \sigma)$
- $\mu =$ the mean $\sigma =$ the standard deviation

Glossary

Normal Distribution

a continuous random variable (RV) with pdf

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{(x\cdot\mu)^{2}}{2\sigma^{2}}} \tag{3.1.2}$$

, where μ is the mean of the distribution and σ is the standard deviation; notation: $X \sim N(\mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, the RV is called the **standard normal distribution**.

3.2: The Standard Normal Distribution

Z-Scores

The standard normal distribution is a normal distribution of standardized values called *z-scores*. A z-score is measured in units of the standard deviation.

Definition: Z-Score

If X is a normally distributed random variable and $X \sim N(\mu, \sigma)$, then the z-score is:

$$z = \frac{x-\mu}{\sigma} \tag{3.2.1}$$

The *z*-score tells you how many standard deviations the value x is above (to the right of) or below (to the left of) the mean, μ. Values of x that are larger than the mean have positive z-scores, and values of x that are smaller than the mean have negative z-scores. If x equals the mean, then x has a z-score of zero. For example, if the mean of a normal distribution is five and the standard deviation is two, the value 11 is three standard deviations above (or to the right of) the mean. The calculation is as follows:

$$\begin{aligned} x &= \mu + (z)(\sigma) \\ &= 5 + (3)(2) = 11 \end{aligned}$$

The *z*-score is three.

Since the mean for the standard normal distribution is zero and the standard deviation is one, then the transformation in Equation 3.2.1 produces the distribution $Z \sim N(0,1)$. The value x comes from a normal distribution with mean μ and standard deviation σ.

> *A z-score is measured in units of the standard deviation.*

Example 3.2.1

Suppose $X \sim N(5,6)$. This says that x is a normally distributed random variable with mean $\mu = 5$ and standard deviation $\sigma = 6$. Suppose $x = 17$. Then (via Equation 3.2.1):

$$z = \frac{x-\mu}{\sigma} = \frac{17-5}{6} = 2$$

This means that $x = 17$ is **two** standard deviations (2σ) above or to the right of the mean $\mu = 5$. The standard deviation is $\sigma = 6$.

Notice that: $5 + (2)(6) = 17$ (The pattern is $\mu + z\sigma = x$)

Now suppose $x = 1$. Then:

$$z = \frac{x-\mu}{\sigma} = \frac{1-5}{6} = -0.67$$

(rounded to two decimal places)

This means that $x = 1$ is 0.67 standard deviations (-0.67σ) below or to the left of the mean $\mu = 5$. Notice that: $5 + (-0.67)(6)$ is approximately equal to one (This has the pattern $\mu + (-0.67)\sigma = 1$)

Summarizing, when z is positive, x is above or to the right of μ and when z is negative, x is to the left of or below μ. Or, when z is positive, x is greater than μ, and when z is negative x is less than μ.

Exercise 3.2.1

What is the z-score of x, when $x = 1$ and $X \sim N(12,3)$?

Answer

$z = \frac{1-12}{3} \approx -3.67$

✓ Example 3.2.2

Some doctors believe that a person can lose five pounds, on the average, in a month by reducing his or her fat intake and by exercising consistently. Suppose weight loss has a normal distribution. Let $X =$ the amount of weight lost(in pounds) by a person in a month. Use a standard deviation of two pounds. $X \sim N(5,2)$. Fill in the blanks.

a. Suppose a person **lost** ten pounds in a month. The z-score when $x = 10$ pounds is $x = 2.5$ (verify). This z-score tells you that $x = 10$ is ________ standard deviations to the ________ (right or left) of the mean _____ (What is the mean?).
b. Suppose a person **gained** three pounds (a negative weight loss). Then $z =$ __________. This z-score tells you that $x = -3$ is ________ standard deviations to the __________ (right or left) of the mean.

Answers

a. This z-score tells you that $x = 10$ is 2.5 standard deviations to the right of the mean five.

b. Suppose the random variables X and Y have the following normal distributions: $X \sim N(5,6)$ and $Y \sim N(2,1)$. If $x = 17$, then $z = 2$. (This was previously shown.) If $y = 4$, what is z?

$$z = \frac{y-\mu}{\sigma} = \frac{4-2}{1} = 2$$

where $\mu = 2$ and $\sigma = 1$.

The z-score for $y = 4$ is $z = 2$. This means that four is $z = 2$ standard deviations to the right of the mean. Therefore, $x = 17$ and $y = 4$ are both two (of their own) standard deviations to the right of their respective means.

The z-score allows us to compare data that are scaled differently. To understand the concept, suppose $X \sim N(5,6)$ represents weight gains for one group of people who are trying to gain weight in a six week period and $Y \sim N(2,1)$ measures the same weight gain for a second group of people. A negative weight gain would be a weight loss. Since $x = 17$ and $y = 4$ are each two standard deviations to the right of their means, they represent the same, standardized weight gain **relative to their means**.

? Exercise 3.2.2

Fill in the blanks.

Jerome averages 16 points a game with a standard deviation of four points. $X \sim N(16,4)$. Suppose Jerome scores ten points in a game. The z–score when $x = 10$ is -1.5. This score tells you that $x = 10$ is _____ standard deviations to the ______(right or left) of the mean______(What is the mean?).

Answer

1.5, left, 16

The Empirical Rule

If X is a random variable and has a normal distribution with mean μ and standard deviation σ, then the *Empirical Rule* says the following:

- About 68% of the x values lie between -1σ and $+1\sigma$ of the mean μ (within one standard deviation of the mean).
- About 95% of the x values lie between -2σ and $+2\sigma$ of the mean μ (within two standard deviations of the mean).
- About 99.7% of the x values lie between -3σ and $+3\sigma$ of the mean μ (within three standard deviations of the mean). Notice that almost all the x values lie within three standard deviations of the mean.
- The z-scores for $+1\sigma$ and -1σ are +1 and –1, respectively.
- The z-scores for $+2\sigma$ and -2σ are +2 and –2, respectively.
- The z-scores for $+3\sigma$ and -3σ are +3 and –3 respectively.

The empirical rule is also known as the 68-95-99.7 rule.

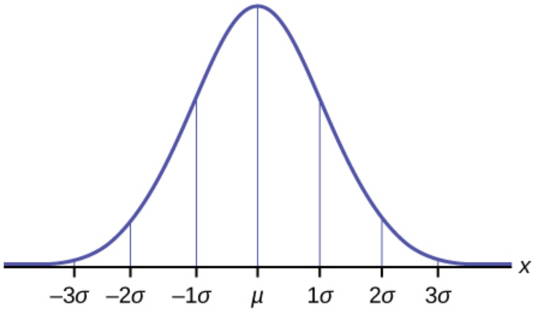

Figure 3.2.1

✓ Example 3.2.3

The mean height of 15 to 18-year-old males from Chile from 2009 to 2010 was 170 cm with a standard deviation of 6.28 cm. Male heights are known to follow a normal distribution. Let $X =$ the height of a 15 to 18-year-old male from Chile in 2009 to 2010. Then $X \sim N(170, 6.28)$.

a. Suppose a 15 to 18-year-old male from Chile was 168 cm tall from 2009 to 2010. The z-score when $x = 168$ cm is $z =$ _______. This z-score tells you that $x = 168$ is ________ standard deviations to the ________ (right or left) of the mean _____ (What is the mean?).
b. Suppose that the height of a 15 to 18-year-old male from Chile from 2009 to 2010 has a z-score of $z = 1.27$. What is the male's height? The z-score ($z = 1.27$) tells you that the male's height is ________ standard deviations to the __________ (right or left) of the mean.

Answers

a. –0.32, 0.32, left, 170
b. 177.98, 1.27, right

? Exercise 3.2.3

Use the information in Example $3.2.3$ to answer the following questions.

a. Suppose a 15 to 18-year-old male from Chile was 176 cm tall from 2009 to 2010. The z-score when $x = 176$ cm is $z =$ _______. This z-score tells you that $x = 176$ cm is ________ standard deviations to the ________ (right or left) of the mean _____ (What is the mean?).
b. Suppose that the height of a 15 to 18-year-old male from Chile from 2009 to 2010 has a z-score of $z =-2$. What is the male's height? The z-score ($z =-2$) tells you that the male's height is ________ standard deviations to the __________ (right or left) of the mean.

Answer

Solve the equation $z = \dfrac{x-\mu}{\sigma}$ for z. $x = \mu + (z)(\sigma)$

$z = \dfrac{176-170}{6.28}$, This z-score tells you that $x = 176$ cm is 0.96 standard deviations to the right of the mean 170 cm.

Answer

Solve the equation $z = \dfrac{x-\mu}{\sigma}$ for z. $x = \mu + (z)(\sigma)$

$X = 157.44$cm, The z-score($z =-2$) tells you that the male's height is two standard deviations to the left of the mean.

✓ Example 3.2.4

From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was 172.36 cm, and the standard deviation was 6.34 cm. Let $Y =$ the height of 15 to 18-year-old males from 1984 to 1985. Then $Y \sim N(172.36, 6.34)$.

The mean height of 15 to 18-year-old males from Chile from 2009 to 2010 was 170 cm with a standard deviation of 6.28 cm. Male heights are known to follow a normal distribution. Let $X =$ the height of a 15 to 18-year-old male from Chile in 2009 to 2010. Then $X \sim N(170, 6.28)$.

Find the z-scores for $x = 160.58$ cm and $y = 162.85$ cm. Interpret each z-score. What can you say about $x = 160.58$ cm and $y = 162.85$ cm?

Answer

- The z-score (Equation 3.2.1) for $x = 160.58$ is $z = -1.5$.
- The z-score for $y = 162.85$ is $z = -1.5$.

Both $x = 160.58$ and $y = 162.85$ deviate the same number of standard deviations from their respective means and in the same direction.

? Exercise 3.2.4

In 2012, 1,664,479 students took the SAT exam. The distribution of scores in the verbal section of the SAT had a mean $\mu = 496$ and a standard deviation $\sigma = 114$. Let $X =$ a SAT exam verbal section score in 2012. Then $X \sim N(496, 114)$.

Find the z-scores for $x_1 = 325$ and $x_2 = 366.21$. Interpret each z-score. What can you say about $x_1 = 325$ and $x_2 = 366.21$?

Answer

The z-score (Equation 3.2.1) for $x_1 = 325$ is $z_1 = -1.15$.

The z-score (Equation 3.2.1) for $x_2 = 366.21$ is $z_2 = -1.14$.

Student 2 scored closer to the mean than Student 1 and, since they both had negative z-scores, Student 2 had the better score.

✓ Example 3.2.5

Suppose *x* has a normal distribution with mean 50 and standard deviation 6.

- About 68% of the *x* values lie within one standard deviation of the mean. Therefore, about 68% of the *x* values lie between –1σ = (–1)(6) = –6 and 1σ = (1)(6) = 6 of the mean 50. The values 50 – 6 = 44 and 50 + 6 = 56 are within one standard deviation from the mean 50. The z-scores are –1 and +1 for 44 and 56, respectively.
- About 95% of the x values lie within two standard deviations of the mean. Therefore, about 95% of the x values lie between –2σ = (–2)(6) = –12 and 2σ = (2)(6) = 12. The values 50 – 12 = 38 and 50 + 12 = 62 are within two standard deviations from the mean 50. The z-scores are –2 and +2 for 38 and 62, respectively.
- About 99.7% of the x values lie within three standard deviations of the mean. Therefore, about 99.7% of the x values lie between –3σ = (–3)(6) = –18 and 3σ = (3)(6) = 18 from the mean 50. The values 50 – 18 = 32 and 50 + 18 = 68 are within three standard deviations of the mean 50. The z-scores are –3 and +3 for 32 and 68, respectively.

? Exercise 3.2.5

Suppose X has a normal distribution with mean 25 and standard deviation five. Between what values of x do 68% of the values lie?

Answer

between 20 and 30.

✓ Example 3.2.6

From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was 172.36 cm, and the standard deviation was 6.34 cm. Let $Y =$ the height of 15 to 18-year-old males in 1984 to 1985. Then $Y \sim N(172.36, 6.34)$.

a. About 68% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.
b. About 95% of the y values lie between what two values? These values are ________________. The z-scores are ________________ respectively.
c. About 99.7% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.

Answer

a. About 68% of the values lie between 166.02 and 178.7. The z-scores are –1 and 1.
b. About 95% of the values lie between 159.68 and 185.04. The z-scores are –2 and 2.
c. About 99.7% of the values lie between 153.34 and 191.38. The z-scores are –3 and 3.

? Exercise 3.2.6

The scores on a college entrance exam have an approximate normal distribution with mean, $\mu = 52$ points and a standard deviation, $\sigma = 11$ points.

a. About 68% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.
b. About 95% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.
c. About 99.7% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.

Answer a

About 68% of the values lie between the values 41 and 63. The z-scores are –1 and 1, respectively.

Answer b

About 95% of the values lie between the values 30 and 74. The z-scores are –2 and 2, respectively.

Answer c

About 99.7% of the values lie between the values 19 and 85. The z-scores are –3 and 3, respectively.

Summary

A z-score is a standardized value. Its distribution is the standard normal, $Z \sim N(0, 1)$. The mean of the z-scores is zero and the standard deviation is one. If y is the z-score for a value x from the normal distribution $N(\mu, \sigma)$ then z tells you how many standard deviations x is above (greater than) or below (less than) μ.

Formula Review

$Z \sim N(0, 1)$

$z = a$ standardized value (z-score)

mean = 0; standard deviation = 1

To find the K^{th} percentile of X when the z-scores is known:

$k = \mu + (z)\sigma$

z-score: $z = \dfrac{x - \mu}{\sigma}$

$Z =$ the random variable for *z*-scores

$Z \sim N(0, 1)$

Glossary

Standard Normal Distribution

a continuous random variable (RV) $X \sim N(0, 1)$; when X follows the standard normal distribution, it is often noted as \(Z \sim N(0, 1)\.

***z*-score**

the linear transformation of the form $z = \frac{x - \mu}{\sigma}$; if this transformation is applied to any normal distribution $X \sim N(\mu, \sigma$ the result is the standard normal distribution $Z \sim N(0, 1)$. If this transformation is applied to any specific value x of the RV with mean μ and standard deviation σ, the result is called the z-score of x. The z-score allows us to compare data that are normally distributed but scaled differently.

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3.3: Using the Normal Distribution

The shaded area in the following graph indicates the area to the left of x. This area is represented by the probability $P(X < x)$. Normal tables, computers, and calculators provide or calculate the probability $P(X < x)$.

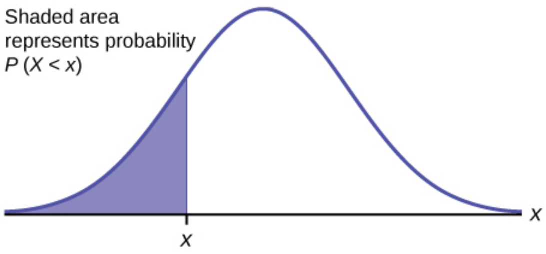

Figure $3.3.1$.

The area to the right is then $P(X > x) = 1 – P(X < x)$. Remember, $P(X < x) =$ **Area to the left** of the vertical line through x. $P(X > x) = 1 – P(X < x) =$ **Area to the right** of the vertical line through x. $P(X < x)$ is the same as $P(X \leq x)$ and $P(X > x)$ is the same as $P(X \geq x)$ for continuous distributions.

Calculations of Probabilities

Probabilities are calculated using technology. There are instructions given as necessary for the TI-83+ and TI-84 calculators.To calculate the probability, use the probability tables provided in [link] without the use of technology. The tables include instructions for how to use them.

✓ Example $3.3.1$

If the area to the left is 0.0228, then the area to the right is $1 - 0.0228 = 0.9772$.

? Exercise $3.3.1$

If the area to the left of x is 0.012, then what is the area to the right?

Answer

$1 - 0.012 = 0.988$

✓ Example $3.3.2$

The final exam scores in a statistics class were normally distributed with a mean of 63 and a standard deviation of five.

a. Find the probability that a randomly selected student scored more than 65 on the exam.
b. Find the probability that a randomly selected student scored less than 85.
c. Find the 90th percentile (that is, find the score k that has 90% of the scores below k and 10% of the scores above k).
d. Find the 70th percentile (that is, find the score k such that 70% of scores are below k and 30% of the scores are above k).

Answer

a. Let X = a score on the final exam. $X \sim N(63, 5)$, where $\mu = 63$ and $\sigma = 5$

Draw a graph.

Then, find $P(x > 65)$.

$$P(x > 65) = 0.3446$$

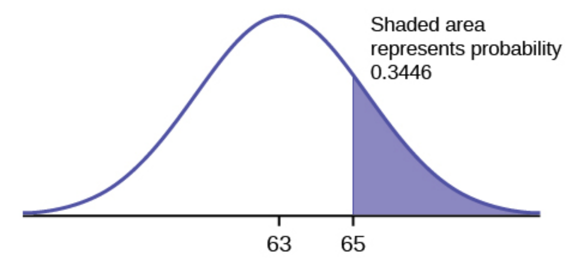

Figure 3.3.2.

The probability that any student selected at random scores more than 65 is 0.3446.

USING THE TI-83, 83+, 84, 84+ CALCULATOR

Go into `2nd DISTR` .

After pressing `2nd DISTR` , press `2:normalcdf` .

The syntax for the instructions are as follows:

normalcdf(lower value, upper value, mean, standard deviation) For this problem: normalcdf(65,1E99,63,5) = 0.3446. You get 1E99 (= 10^{99}) by pressing `1` , the `EE` key (a 2nd key) and then `99` . Or, you can enter `10^ 99` instead. The number 10^{99} is way out in the right tail of the normal curve. We are calculating the area between 65 and 10^{99}. In some instances, the lower number of the area might be –1E99 (= -10^{99}). The number -10^{99} is way out in the left tail of the normal curve.

Historical Note

The TI probability program calculates a z-score and then the probability from the z-score. Before technology, the z-score was looked up in a standard normal probability table (because the math involved is too cumbersome) to find the probability. In this example, a standard normal table with area to the left of the z-score was used. You calculate the z-score and look up the area to the left. The probability is the area to the right.

$$z = 65-63565-635 = 0.4$$

Area to the left is 0.6554.

$$P(x > 65) = P(z > 0.4) = 1 - 0.6554 = 0.3446$$

USING THE TI-83, 83+, 84, 84+ CALCULATOR

Find the percentile for a student scoring 65:

*Press `2nd Distr`
*Press `2:normalcdf` (
*Enter lower bound, upper bound, mean, standard deviation followed by)
*Press `ENTER` .
For this Example, the steps are
`2nd Distr`
`2:normalcdf` (65,1,2nd EE,99,63,5) `ENTER`

The probability that a selected student scored more than 65 is 0.3446.
To find the probability that a selected student scored *more than* 65, subtract the percentile from 1.

Answer

b. Draw a graph.

Then find $P(x < 85)$, and shade the graph.

Using a computer or calculator, find $P(x < 85) = 1$.

$\text{normalcdf}(0, 85, 63, 5) = 1 \text{(rounds to one)}$

The probability that one student scores less than 85 is approximately one (or 100%).

Answer

c. Find the 90th percentile. For each problem or part of a problem, draw a new graph. Draw the x-axis. Shade the area that corresponds to the 90th percentile.

Let $k =$ the 90th percentile. The variable k is located on the x-axis. $P(x < k)$ is the area to the left of k. The 90th percentile k separates the exam scores into those that are the same or lower than k and those that are the same or higher. Ninety percent of the test scores are the same or lower than k, and ten percent are the same or higher. The variable k is often called a critical value.

$k = 69.4$

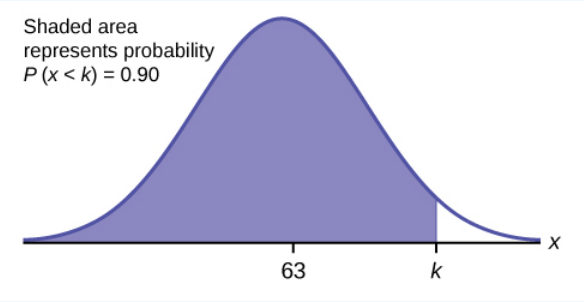

Figure $3.3.3$.

The 90th percentile is 69.4. This means that 90% of the test scores fall at or below 69.4 and 10% fall at or above. To get this answer on the calculator, follow this step:

`invNorm` in `2nd DISTR`. invNorm(area to the left, mean, standard deviation)

For this problem, $\text{invNorm}(0.90, 63, 5) = 69.4$

Answer

d. Find the 70th percentile.

Draw a new graph and label it appropriately. $k = 65.6$

The 70th percentile is 65.6. This means that 70% of the test scores fall at or below 65.6 and 30% fall at or above.

$$\text{invNorm}(0.70, 63, 5) = 65.6$$

? Exercise $3.3.2$

The golf scores for a school team were normally distributed with a mean of 68 and a standard deviation of three. Find the probability that a randomly selected golfer scored less than 65.

Answer

$\text{normalcdf}(10^{99}, 65, 68, 3) = 0.1587$

✓ Example $3.3.3$

A personal computer is used for office work at home, research, communication, personal finances, education, entertainment, social networking, and a myriad of other things. Suppose that the average number of hours a household personal computer is used for entertainment is two hours per day. Assume the times for entertainment are normally distributed and the standard deviation for the times is half an hour.

a. Find the probability that a household personal computer is used for entertainment between 1.8 and 2.75 hours per day.

b. Find the maximum number of hours per day that the bottom quartile of households uses a personal computer for entertainment.

Answer

a. Let $X =$ the amount of time (in hours) a household personal computer is used for entertainment. $X \sim N(2, 0.5)$ where $\mu = 2$ and $\sigma = 0.5$.

Find $P(1.8 < x < 2.75)$.

The probability for which you are looking is the area **between** $x = 1.8$ and $x = 2.75$. $P(1.8 < x < 2.75) = 0.5886$

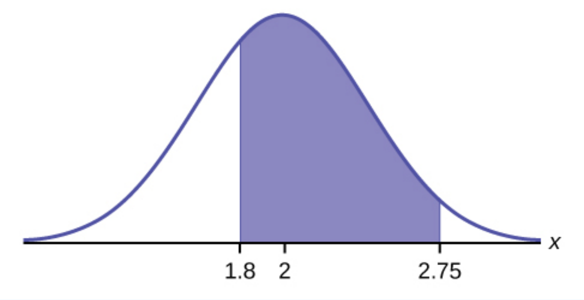

Figure 3.3.4.

$$\text{normalcdf}(1.8, 2.75, 2, 0.5) = 0.5886$$

The probability that a household personal computer is used between 1.8 and 2.75 hours per day for entertainment is 0.5886.

b.

To find the maximum number of hours per day that the bottom quartile of households uses a personal computer for entertainment, **find the 25th percentile,** k, where $P(x < k) = 0.25$.

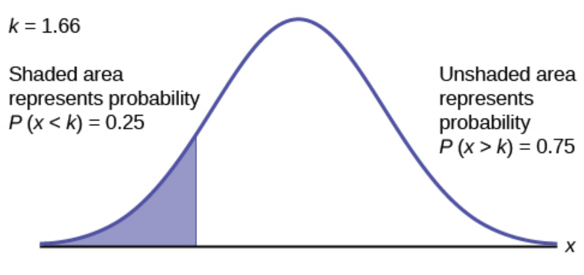

Figure 3.3.5.

$$\text{invNorm}(0.25, 2, 0.5) = 1.66$$

The maximum number of hours per day that the bottom quartile of households uses a personal computer for entertainment is 1.66 hours.

? Exercise 3.3.3

The golf scores for a school team were normally distributed with a mean of 68 and a standard deviation of three. Find the probability that a golfer scored between 66 and 70.

Answer

$$\text{normalcdf}(66, 70, 68, 3) = 0.4950$$

✓ Example 3.3.4

There are approximately one billion smartphone users in the world today. In the United States the ages 13 to 55+ of smartphone users approximately follow a normal distribution with approximate mean and standard deviation of 36.9 years and 13.9 years, respectively.

a. Determine the probability that a random smartphone user in the age range 13 to 55+ is between 23 and 64.7 years old.
b. Determine the probability that a randomly selected smartphone user in the age range 13 to 55+ is at most 50.8 years old.
c. Find the 80th percentile of this distribution, and interpret it in a complete sentence.

Answer

a. $\text{normalcdf}(23, 64.7, 36.9, 13.9) = 0.8186$
b. $\text{normalcdf}(-10^{99}, 50.8, 36.9, 13.9) = 0.8413$
c. $\text{invNorm}(0.80, 36.9, 13.9) = 48.6$

The 80th percentile is 48.6 years.

80% of the smartphone users in the age range 13 – 55+ are 48.6 years old or less.

Use the information in Example to answer the following questions.

? Exercise 3.3.4

a. Find the 30th percentile, and interpret it in a complete sentence.
b. What is the probability that the age of a randomly selected smartphone user in the range 13 to 55+ is less than 27 years old and at least 0 years old?

70.

Answer

Let $X =$ a smart phone user whose age is 13 to 55+. $X \sim N(36.9, 13.9)$

To find the 30th percentile, find k such that $P(x < k) = 0.30$.
$\text{invNorm}(0.30, 36.9, 13.9) = 29.6$years
Thirty percent of smartphone users 13 to 55+ are at most 29.6 years and 70% are at least 29.6 years. Find $P(x < 27)$
(Note that $\text{normalcdf}(-10^{99}, 27, 36.9, 13.9) = 0.2382$The two answers differ only by 0.0040.)

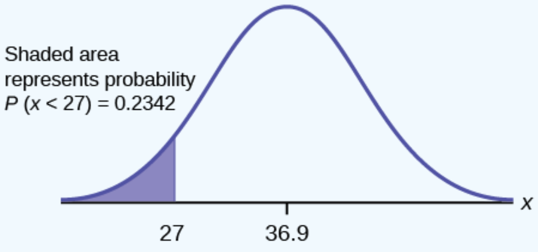

Figure 3.3.6.

$$\text{normalcdf}(0, 27, 36.9, 13.9) = 0.2342$$

✓ Example 3.3.5

In the United States the ages 13 to 55+ of smartphone users approximately follow a normal distribution with approximate mean and standard deviation of 36.9 years and 13.9 years respectively. Using this information, answer the following questions (round answers to one decimal place).

a. Calculate the interquartile range (IQR).
b. Forty percent of the ages that range from 13 to 55+ are at least what age?

Answer

a.

$$IQR = Q_3 - Q_1$$

Calculate $Q_3 = 75^{\text{th}}$ percentile and $Q_1 = 25^{\text{th}}$ percentile.

$$\text{invNorm}(0.75, 36.9, 13.9) = Q_3 = 46.2754$$
$$\text{invNorm}(0.25, 36.9, 13.9) = Q_1 = 27.5246$$
$$IQR = Q_3 - Q_1 = 18.7508$$

b.

Find k where $P(x > k) = 0.40$ ("At least" translates to "greater than or equal to.")

$0.40 =$ the area to the right.

Area to the left $= 1 - 0.40 = 0.60$.

The area to the left of $k = 0.60$.

$\text{invNorm}(0.60, 36.9, 13.9) = 40.4215$

$k = 40.42$.

Forty percent of the smartphone users from 13 to 55+ are at least 40.4 years.

Exercise 3.3.5

Two thousand students took an exam. The scores on the exam have an approximate normal distribution with a mean $\mu = 81$ points and standard deviation $\sigma = 15$ points.

a. Calculate the first- and third-quartile scores for this exam.
b. The middle 50% of the exam scores are between what two values?

Answer

a. $Q_1 = 25^{\text{th}}$ percentile $= \text{invNorm}(0.25, 81, 15) = 70.9$
   $Q_3 = 75^{\text{th}}$ percentile $= \text{invNorm}(0.75, 81, 15) = 91.1$
b. The middle 50% of the scores are between 70.9 and 91.1.

Example 3.3.6

A citrus farmer who grows mandarin oranges finds that the diameters of mandarin oranges harvested on his farm follow a normal distribution with a mean diameter of 5.85 cm and a standard deviation of 0.24 cm.

a. Find the probability that a randomly selected mandarin orange from this farm has a diameter larger than 6.0 cm. Sketch the graph.
b. The middle 20% of mandarin oranges from this farm have diameters between ______ and ______.
c. Find the 90^{th} percentile for the diameters of mandarin oranges, and interpret it in a complete sentence.

Answer

a. $\text{normalcdf}(6, 10^{99}, 5.85, 0.24) = 0.2660$

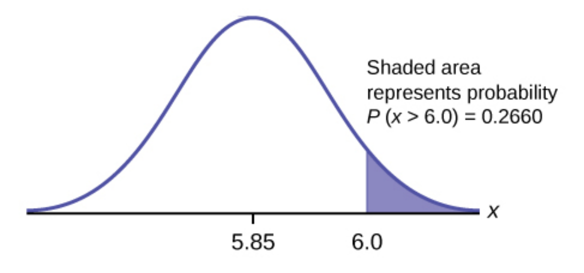

Figure 3.3.7.

Answer

b.

$1 – 0.20 = 0.80$

The tails of the graph of the normal distribution each have an area of 0.40.

Find $k1$, the 40th percentile, and $k2$, the 60th percentile $(0.40 + 0.20 = 0.60)$.

$k1 = \text{invNorm}(0.40, 5.85, 0.24) = 5.79\text{cm}$

$k2 = \text{invNorm}(0.60, 5.85, 0.24) = 5.91\text{cm}$

Answer

c. 6.16: Ninety percent of the diameter of the mandarin oranges is at most 6.15 cm.

? Exercise 3.3.6

Using the information from Example, answer the following:

a. The middle 45% of mandarin oranges from this farm are between ______ and ______.
b. Find the 16th percentile and interpret it in a complete sentence.

Answer a

The middle area $= 0.40$, so each tail has an area of 0.30.

$– 0.40 = 0.60$

The tails of the graph of the normal distribution each have an area of 0.30.

Find $k1$, the 30th percentile and $k2$, the 70th percentile $(0.40 + 0.30 = 0.70)$.

$k1 = \text{invNorm}(0.30, 5.85, 0.24) = 5.72\text{cm}$

$k2 = \text{invNorm}(0.70, 5.85, 0.24) = 5.98\text{cm}$

Answer b

$\text{normalcdf}(5, 10^{99}, 5.85, 0.24) = 0.9998$

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Review

The normal distribution, which is continuous, is the most important of all the probability distributions. Its graph is bell-shaped. This bell-shaped curve is used in almost all disciplines. Since it is a continuous distribution, the total area under the curve is one. The parameters of the normal are the mean μ and the standard deviation σ. A special normal distribution, called the standard normal distribution is the distribution of *z*-scores. Its mean is zero, and its standard deviation is one.

Formula Review

- Normal Distribution: $X \sim N(\mu, \sigma)$ where μ is the mean and σ is the standard deviation.
- Standard Normal Distribution: $Z \sim N(0, 1)$.
- Calculator function for probability: normalcdf (lower x value of the area, upper x value of the area, mean, standard deviation)
- Calculator function for the k^{th} percentile: $k = \text{invNorm}$ (area to the left of k, mean, standard deviation)

Exercise 3.3.7

How would you represent the area to the left of one in a probability statement?

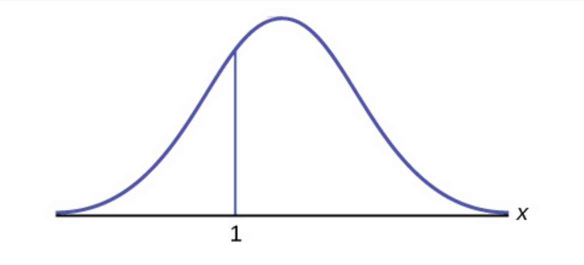

Figure 3.3.8.

Answer

$P(x < 1)$

Exercise 3.3.8

Is $P(x < 1)$ equal to $P(x \leq 1)$? Why?

Answer

Yes, because they are the same in a continuous distribution: $P(x = 1) = 0$

Exercise 3.3.9

How would you represent the area to the left of three in a probability statement?

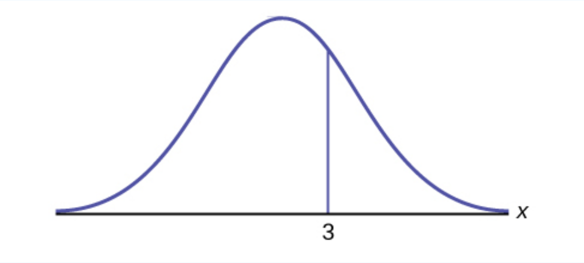

Figure 3.3.10.

Exercise 3.3.10

What is the area to the right of three?

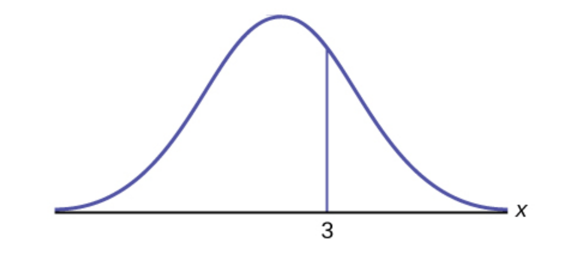

Figure 3.3.11.

Answer

$1 - P(x < 3)$ or $P(x > 3)$

Exercise 3.3.11

If the area to the left of x in a normal distribution is 0.123, what is the area to the right of x?

Exercise 3.3.12

If the area to the right of x in a normal distribution is 0.543, what is the area to the left of x?

Answer

$1 - 0.543 = 0.457$

Use the following information to answer the next four exercises:

$X \sim N(54, 8)$

Exercise 3.3.13

Find the probability that $x > 56$.

Exercise 3.3.14

Find the probability that $x < 30$.

Answer

0.0013

Exercise 3.3.15

Find the 80th percentile.

Exercise 3.3.16

Find the 60th percentile.

Answer

56.03

Exercise 3.3.17

$X \sim N(6, 2)$

Find the probability that x is between three and nine.

Exercise 3.3.18

$X \sim N(-3, 4)$

Find the probability that x is between one and four.

Answer

0.1186

Exercise 3.3.19

$X \sim N(4, 5)$

Find the maximum of x in the bottom quartile.

Exercise 3.3.20

Use the following information to answer the next three exercise: The life of Sunshine CD players is normally distributed with a mean of 4.1 years and a standard deviation of 1.3 years. A CD player is guaranteed for three years. We are interested in the length of time a CD player lasts. Find the probability that a CD player will break down during the guarantee period.

a. Sketch the situation. Label and scale the axes. Shade the region corresponding to the probability.

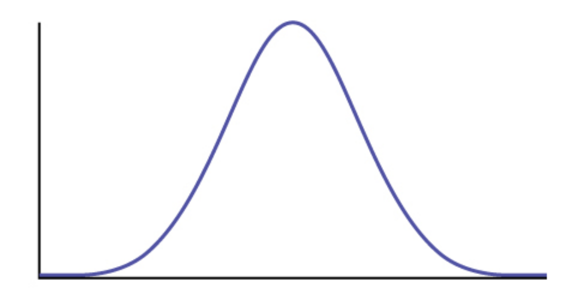

Figure **3.3.12.**

$P(0 < x <$ ____________$)=$ ___________ (Use zero for the minimum value of x.)

Answer

a. Check student’s solution.
b. 3, 0.1979

Exercise 3.3.21

Find the probability that a CD player will last between 2.8 and six years.

a. Sketch the situation. Label and scale the axes. Shade the region corresponding to the probability.

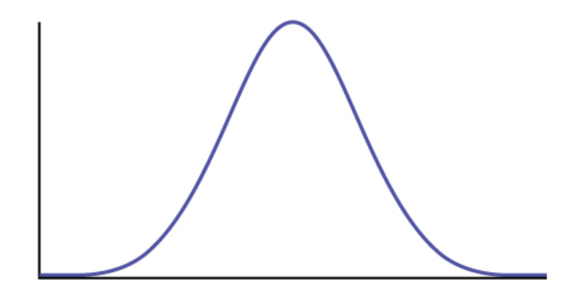

Figure **3.3.13.**

$P(\_\_\_\_\_\_\_\_\_\_ < x < \_\_\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_\_\_$

? Exercise 3.3.22

Find the 70th percentile of the distribution for the time a CD player lasts.

a. Sketch the situation. Label and scale the axes. Shade the region corresponding to the lower 70%.

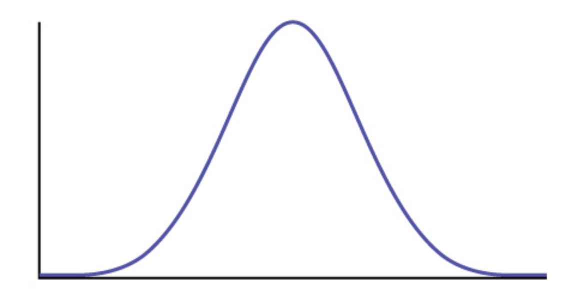

Figure **3.3.14.**

$P(x < k) =$ __________ Therefore, $k =$ __________

Answer

a. Check student’s solution.
b. 0.70, 4.78 years

CHAPTER OVERVIEW

4: The Central Limit Theorem

In this chapter, you will study means and the **central limit theorem**, which is one of the most powerful and useful ideas in all of statistics. There are two alternative forms of the theorem, and both alternatives are concerned with drawing finite samples size n from a population with a known mean, μ, and a known standard deviation, σ. The first alternative says that if we collect samples of size n with a "large enough n," calculate each sample's mean, and create a histogram of those means, then the resulting histogram will tend to have an approximate normal bell shape. The second alternative says that if we again collect samples of size n that are "large enough," calculate the sum of each sample and create a histogram, then the resulting histogram will again tend to have a normal bell-shape.

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4.1: Prelude to the Central Limit Theorem

Learning Objectives

By the end of this chapter, the student should be able to:

- Recognize central limit theorem problems.
- Classify continuous word problems by their distributions.
- Apply and interpret the central limit theorem for means.
- Apply and interpret the central limit theorem for sums.

Why are we so concerned with means? Two reasons are: they give us a middle ground for comparison, and they are easy to calculate. In this chapter, you will study means and the **central limit theorem**. The **central limit theorem** (clt for short) is one of the most powerful and useful ideas in all of statistics. There are two alternative forms of the theorem, and both alternatives are concerned with drawing finite samples size n from a population with a known mean, μ, and a known standard deviation, σ. The first alternative says that if we collect samples of size n with a "large enough n," calculate each sample's mean, and create a histogram of those means, then the resulting histogram will tend to have an approximate normal bell shape. The second alternative says that if we again collect samples of size n that are "large enough," calculate the sum of each sample and create a histogram, then the resulting histogram will again tend to have a normal bell-shape.

In either case, it does not matter what the distribution of the original population is, or whether you even need to know it. The important fact is that the distribution of sample means and the sums tend to follow the normal distribution.

The size of the sample, n, that is required in order to be "large enough" depends on the original population from which the samples are drawn (the sample size should be at least 30 or the data should come from a normal distribution). If the original population is far from normal, then more observations are needed for the sample means or sums to be normal. **Sampling is done with replacement.**

Figure 4.1.1. If you want to figure out the distribution of the change people carry in their pockets, using the central limit theorem and assuming your sample is large enough, you will find that the distribution is normal and bell-shaped. (credit: John Lodder)

COLLABORATIVE CLASSROOM ACTIVITY

Suppose eight of you roll one fair die ten times, seven of you roll two fair dice ten times, nine of you roll five fair dice ten times, and 11 of you roll ten fair dice ten times.

Each time a person rolls more than one die, he or she calculates the sample **mean** of the faces showing. For example, one person might roll five fair dice and get 2, 2, 3, 4, 6 on one roll.

The mean is $\frac{2+2+3+4+6}{5} = 3.4$. The 3.4 is one mean when five fair dice are rolled. This same person would roll the five dice nine more times and calculate nine more means for a total of ten means.

Your instructor will pass out the dice to several people. Roll your dice ten times. For each roll, record the faces, and find the mean. Round to the nearest 0.5.

Your instructor (and possibly you) will produce one graph (it might be a histogram) for one die, one graph for two dice, one graph for five dice, and one graph for ten dice. Since the "mean" when you roll one die is just the face on the die, what distribution do these **means** appear to be representing?

- **Draw the graph for the means using two dice.** Do the sample means show any kind of pattern?
- **Draw the graph for the means using five dice.** Do you see any pattern emerging?
- **Finally, draw the graph for the means using ten dice.** Do you see any pattern to the graph? What can you conclude as you increase the number of dice?

As the number of dice rolled increases from one to two to five to ten, the following is happening:

1. The mean of the sample means remains approximately the same.
2. The spread of the sample means (the standard deviation of the sample means) gets smaller.
3. The graph appears steeper and thinner.

You have just demonstrated the central limit theorem (clt). The central limit theorem tells you that as you increase the number of dice, **the sample means tend toward a normal distribution (the sampling distribution).**

Glossary

Sampling Distribution

Given simple random samples of size n from a given population with a measured characteristic such as mean, proportion, or standard deviation for each sample, the probability distribution of all the measured characteristics is called a sampling distribution.

4.2: The Central Limit Theorem for Sample Means (Averages)

Suppose X is a random variable with a distribution that may be known or unknown (it can be any distribution). Using a subscript that matches the random variable, suppose:

a. $\mu_x =$ the mean of X
b. $\sigma_x =$ the standard deviation of X

If you draw random samples of size n, then as n increases, the random variable $\bar{X}$ which consists of sample means, tends to be normally distributed and

$$\bar{X} \sim N\left(\mu_x, \frac{\sigma_x}{\sqrt{n}}\right). \tag{4.2.1}$$

The central limit theorem for sample means says that if you keep drawing larger and larger samples (such as rolling one, two, five, and finally, ten dice) and calculating their means, the sample means form their own *normal distribution* (the sampling distribution). The normal distribution has the same mean as the original distribution and a variance that equals the original variance divided by, the sample size. The variable n is the number of values that are averaged together, not the number of times the experiment is done.

To put it more formally, if you draw random samples of size n, the distribution of the random variable $\bar{X}$, which consists of sample means, is called the *sampling distribution of the mean*. The sampling distribution of the mean approaches a normal distribution as n, the sample size, increases.

The random variable $\bar{X}$ has a different z-score associated with it from that of the random variable X. The mean $\bar{x}$ is the value of $\bar{X}$ in one sample.

$$z = \frac{\bar{x} - \mu_x}{\left(\frac{\sigma_x}{\sqrt{n}}\right)} \tag{4.2.2}$$

- μ_x is the average of both X and $\bar{X}$.
- $\sigma\bar{x} = \frac{\sigma_x}{\sqrt{n}} =$ standard deviation of $\bar{X}$ and is called the standard error of the mean.

Howto: Find probabilities for means on the calculator

2nd DISTR

2:normalcdf

$$\text{normalcdf}\left(\text{lower value of the area, upper value of the area, mean, } \frac{\text{standard deviation}}{\sqrt{\text{sample size}}}\right)$$

where:

- *mean* is the mean of the original distribution
- *standard deviation* is the standard deviation of the original distribution
- *sample size* $= n$

Example $4.2.1$

An unknown distribution has a mean of 90 and a standard deviation of 15. Samples of size $n = 25$ are drawn randomly from the population.

a. Find the probability that the sample mean is between 85 and 92.
b. Find the value that is two standard deviations above the expected value, 90, of the sample mean.

Answer

a.

Let $X =$ one value from the original unknown population. The probability question asks you to find a probability for the **sample mean**.

Let $\bar{X} =$ the mean of a sample of size 25. Since $\mu_x = 90, \sigma_x = 15$, and $n = 25$,

$$\bar{X} \sim N(90, \frac{15}{\sqrt{25}}).$$

Find $P(85 < x < 92)$. Draw a graph.

$$P(85 < x < 92) = 0.6997$$

The probability that the sample mean is between 85 and 92 is 0.6997.

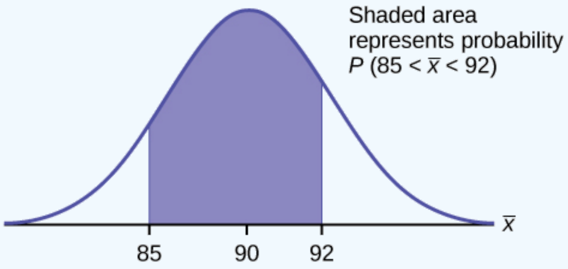

Figure 4.2.1.

`normalcdf` (lower value, upper value, mean, standard error of the mean)

The parameter list is abbreviated (lower value, upper value, μ, $\frac{\sigma}{\sqrt{n}}$)

`normalcdf` $(85, 92, 90, \frac{15}{\sqrt{25}}) = 0.6997$

b.

To find the value that is two standard deviations above the expected value 90, use the formula:

$$\text{value } = \mu_x + (\#\text{ofTSDEVs}) \left(\frac{\sigma_x}{\sqrt{n}}\right)$$

$$= 90 + 2\left(\frac{15}{\sqrt{25}}\right) = 96$$

The value that is two standard deviations above the expected value is 96.

The standard error of the mean is

$$\frac{\sigma_x}{\sqrt{n}} = \frac{15}{\sqrt{25}} = 3.$$

Recall that the standard error of the mean is a description of how far (on average) that the sample mean will be from the population mean in repeated simple random samples of size n.

? Exercise 4.2.1

An unknown distribution has a mean of 45 and a standard deviation of eight. Samples of size n = 30 are drawn randomly from the population. Find the probability that the sample mean is between 42 and 50.

Answer

$$P(42 < \bar{x} < 50) = \left(42, 50, 45, \frac{8}{\sqrt{30}}\right) = 0.9797$$

Example 4.2.2

The length of time, in hours, it takes an "over 40" group of people to play one soccer match is normally distributed with a **mean of two hours** and a **standard deviation of 0.5 hours**. A **sample of size $n = 50$** is drawn randomly from the population. Find the probability that the **sample mean** is between 1.8 hours and 2.3 hours.

Answer

Let $X =$ the time, in hours, it takes to play one soccer match.

The probability question asks you to find a probability for the **sample mean time, in hours**, it takes to play one soccer match.

Let $\bar{X} =$ the mean time, in hours, it takes to play one soccer match.

If $\mu_x =$ _________, $\sigma_x =$ __________, and $n =$ ___________, then $X \sim N($______, ______) by the central limit theorem for means.

$\mu_x = 2, \sigma_x = 0.5, n = 50$, and $X \sim N\left(2, \frac{0.5}{\sqrt{50}}\right)$

Find $P(1.8 < \bar{x} < 2.3)$. Draw a graph.

$P(1.8 < \bar{x} < 2.3) = 0.9977$

$\texttt{normalcdf}\left(1.8, 2.3, 2, \frac{.5}{\sqrt{50}}\right) = 0.9977$

The probability that the mean time is between 1.8 hours and 2.3 hours is 0.9977.

Exercise 4.2.2

The length of time taken on the SAT for a group of students is normally distributed with a mean of 2.5 hours and a standard deviation of 0.25 hours. A sample size of $n = 60$ is drawn randomly from the population. Find the probability that the sample mean is between two hours and three hours.

Answer

$$P(2 < \bar{x} < 3) = \text{normalcdf}\left(2, 3, 2.5, \frac{0.25}{\sqrt{60}}\right) = 1$$

Calculator SKills

To find percentiles for means on the calculator, follow these steps.

- 2^{nd} DIStR
- 3:invNorm

$$k = \text{invNorm}\left(\text{area to the left of} k, \text{mean}, \frac{\text{standard deviation}}{\sqrt{samplesize}}\right)$$

where:

- k = the k^{th} percentile
- *mean* is the mean of the original distribution
- *standard deviation* is the standard deviation of the original distribution
- *sample size* = n

Example 4.2.3

In a recent study reported Oct. 29, 2012 on the Flurry Blog, the mean age of tablet users is 34 years. Suppose the standard deviation is 15 years. Take a sample of size $n = 100$.

a. What are the mean and standard deviation for the sample mean ages of tablet users?
b. What does the distribution look like?
c. Find the probability that the sample mean age is more than 30 years (the reported mean age of tablet users in this particular study).
d. Find the 95th percentile for the sample mean age (to one decimal place).

Answer

a. Since the sample mean tends to target the population mean, we have $\mu_x = \mu = 34$. The sample standard deviation is given by:

$$\sigma_x = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{100}} = \frac{15}{10} = 1.5$$

b. The central limit theorem states that for large sample sizes (n), the sampling distribution will be approximately normal.
c. The probability that the sample mean age is more than 30 is given by:

$$P(\mathrm{X} > 30) = \mathrm{normalcdf}(30, E99, 34, 1.5) = 0.9962$$

d. Let k = the 95th percentile.

$$k = \mathrm{invNorm}\left(0.95, 34, \frac{15}{\sqrt{100}}\right) = 36.5$$

Exercise 4.2.3

In an article on Flurry Blog, a gaming marketing gap for men between the ages of 30 and 40 is identified. You are researching a startup game targeted at the 35-year-old demographic. Your idea is to develop a strategy game that can be played by men from their late 20s through their late 30s. Based on the article's data, industry research shows that the average strategy player is 28 years old with a standard deviation of 4.8 years. You take a sample of 100 randomly selected gamers. If your target market is 29- to 35-year-olds, should you continue with your development strategy?

Answer

You need to determine the probability for men whose mean age is between 29 and 35 years of age wanting to play a strategy game.

$$P(29 < \bar{x} < 35) = \mathrm{normalcdf}\left(29, 35, 28, \frac{4.8}{\sqrt{100}}\right) = 0.0186 \tag{4.2.3}$$

You can conclude there is approximately a 1.9% chance that your game will be played by men whose mean age is between 29 and 35.

Example 4.2.4

The mean number of minutes for app engagement by a tablet user is 8.2 minutes. Suppose the standard deviation is one minute. Take a sample of 60.

a. What are the mean and standard deviation for the sample mean number of app engagement by a tablet user?
b. What is the standard error of the mean?
c. Find the 90th percentile for the sample mean time for app engagement for a tablet user. Interpret this value in a complete sentence.
d. Find the probability that the sample mean is between eight minutes and 8.5 minutes.

Answer

a. $\mu = \mu = 8.2\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{60}} = 0.13$
b. This allows us to calculate the probability of sample means of a particular distance from the mean, in repeated samples of size 60.

c. Let k = the 90th percentile

$k=\text{invNorm}\left(0.90, 8.2, \frac{1}{\sqrt{60}}\right)=8.37$. This values indicates that 90 percent of the average app engagement time for table users is less than 8.37 minutes.

d. $P(8<\bar{x}<8.5)=\text{normalcdf}\left(8, 8.5, 8.2, \frac{1}{\sqrt{60}}\right)=0.9293$

? Exercise 4.2.4

Cans of a cola beverage claim to contain 16 ounces. The amounts in a sample are measured and the statistics are $n=34$, $\bar{x}=16.01$ ounces. If the cans are filled so that $\mu=16.00$ ounces (as labeled) and $\sigma=0.143$ ounces, find the probability that a sample of 34 cans will have an average amount greater than 16.01 ounces. Do the results suggest that cans are filled with an amount greater than 16 ounces?

Answer

We have $P(\bar{x}>16.01)=\text{normalcdf}\left(16.01, E99, 16, \frac{0.143}{\sqrt{34}}\right)=0.3417$Since there is a 34.17% probability that the average sample weight is greater than 16.01 ounces, we should be skeptical of the company's claimed volume. If I am a consumer, I should be glad that I am probably receiving free cola. If I am the manufacturer, I need to determine if my bottling processes are outside of acceptable limits.

Summary

In a population whose distribution may be known or unknown, if the size (n) of samples is sufficiently large, the distribution of the sample means will be approximately normal. The mean of the sample means will equal the population mean. The standard deviation of the distribution of the sample means, called the standard error of the mean, is equal to the population standard deviation divided by the square root of the sample size (n).

Formula Review

- The Central Limit Theorem for Sample Means:

$$\bar{X}\sim N\left(\mu_x, \frac{\sigma_x}{\sqrt{n}}\right)$$

- The Mean $\bar{X}$: σ_x
- Central Limit Theorem for Sample Means z-score and standard error of the mean:

$$z=\frac{\bar{x}-\mu_x}{\left(\frac{\sigma_x}{\sqrt{n}}\right)}$$

- Standard Error of the Mean (Standard Deviation ($\bar{X}$)):

$$\frac{\sigma_x}{\sqrt{n}}$$

Glossary

Average

a number that describes the central tendency of the data; there are a number of specialized averages, including the arithmetic mean, weighted mean, median, mode, and geometric mean.

Central Limit Theorem

Given a random variable (RV) with known mean μ and known standard deviation, σ, we are sampling with size n, and we are interested in two new RVs: the sample mean, $\bar{X}$, and the sample sum, $\sum X$. If the size (n) of the sample is sufficiently large, then $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ and $\sum X \sim N(n\mu, (\sqrt{n})(\sigma))$. If the size (n) of the sample is sufficiently large, then the distribution of the sample means and the distribution of the sample sums will approximate a normal distributions regardless of the shape of the population. The mean of the sample means will equal the population mean, and the mean of the sample sums will equal n times the population mean. The standard deviation of the distribution of the sample means, $\frac{\sigma}{\sqrt{n}}$, is called the standard error of the mean.

Normal Distribution

a continuous random variable (RV) with pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, where μ is the mean of the distribution and σ is the standard deviation; notation: $X \sim N(\mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, the RV is called a **standard normal distribution**.

Standard Error of the Mean

the standard deviation of the distribution of the sample means, or $\frac{\sigma}{\sqrt{n}}$.

References

1. Baran, Daya. “20 Percent of Americans Have Never Used Email.”WebGuild, 2010. Available online at www.webguild.org/20080519/20-...ver-used-email (accessed May 17, 2013).
2. Data from The Flurry Blog, 2013. Available online at blog.flurry.com (accessed May 17, 2013).
3. Data from the United States Department of Agriculture.

4.3: Using the Central Limit Theorem

It is important for you to understand when to use the central limit theorem (clt). If you are being asked to find the probability of the mean, use the clt for the mean. If you are being asked to find the probability of a sum or total, use the clt for sums. This also applies to percentiles for means and sums.

> If you are being asked to find the probability of an individual value, do not use the clt. Use the distribution of its random variable.

Law of Large Numbers

The law of large numbers says that if you take samples of larger and larger size from any population, then the mean $\bar{x}$ of the sample tends to get closer and closer to μ. From the central limit theorem, we know that as n gets larger and larger, the sample means follow a normal distribution. The larger n gets, the smaller the standard deviation gets. (Remember that the standard deviation for $\bar{X}$ is $\frac{\sigma}{\sqrt{n}}$.) This means that the sample mean $\bar{x}$ must be close to the population mean μ. We can say that μ is the value that the sample means approach as n gets larger. The central limit theorem illustrates the law of large numbers.

✓ Example 4.3.1

A study involving stress is conducted among the students on a college campus. The stress scores follow a uniform distribution with the lowest stress score equal to one and the highest equal to five. Using a sample of 75 students, find:

a. The probability that the **mean stress score** for the 75 students is less than two.
b. The 90th percentile for the **mean stress score** for the 75 students.
c. The probability that the **total of the 75 stress scores** is less than 200.
d. The 90th percentile for the **total stress score** for the 75 students.

Solutions

Let $X =$ one stress score.

Problems a and b ask you to find a probability or a percentile for a mean. Problems c and d ask you to find a probability or a percentile for a **total or sum**. The sample size, n, is equal to 75.

Since the individual stress scores follow a uniform distribution, $X \sim U(1,5)$ where $a=1$ and $b=5$.

$$\mu_x = \frac{a+b}{2} = \frac{1+5}{2} = 3 \tag{4.3.1}$$

$$\sigma_x = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(5-1)^2}{12}} = 1.15 \tag{4.3.2}$$

For problems 1. and 2., let $\bar{X} =$ the mean stress score for the 75 students. Then,

$$\bar{X} \sim N\left(3, \frac{1,15}{\sqrt{75}}\right) \tag{4.3.3}$$

where $n=75$.

a. Find $P(\bar{x}<2)$. Draw the graph.
b. Find the 90th percentile for the mean of 75 stress scores. Draw a graph.
c. Find $P(\sum x < 2000)$. Draw the graph.
d. Find the 90th percentile for the total of 75 stress scores. Draw a graph.

Answers

a. $\boldsymbol{P(\bar{x}<2)=0}$

The probability that the mean stress score is less than two is about zero.

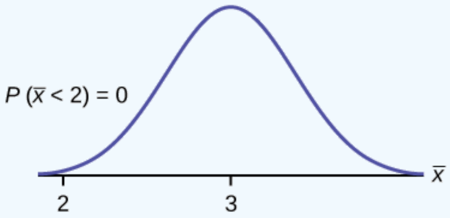

Figure 4.3.1.

$$\texttt{normalcdf}\left(1, 2, 3, \frac{1.15}{\sqrt{75}}\right) = 0$$

REMINDER

The smallest stress score is one

b. Let $k =$ the 90th percentile.

Find k, where $P(\bar{x} < k) = 0.90$.

$k = 3.2$

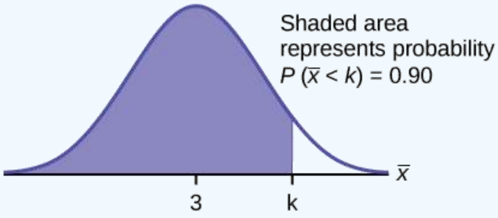

Figure 4.3.2.

The 90th percentile for the mean of 75 scores is about 3.2. This tells us that 90% of all the means of 75 stress scores are at most 3.2, and that 10% are at least 3.2.

$$\texttt{invNorm}\left(0.90, 3, 1.\frac{1.15}{\sqrt{75}}\right) = 3.2$$

For problems c and d, let $\sum X =$ the sum of the 75 stress scores. Then,

$$\sum X \sim N((75)(3), (\sqrt{75})(1.15)) \tag{4.3.4}$$

c. The mean of the sum of 75 stress scores is $(75)(3) = 225$

The standard deviation of the sum of 75 stress scores is $(\sqrt{75})(1.15) = 9.96$

$P(\sum x < 200)$

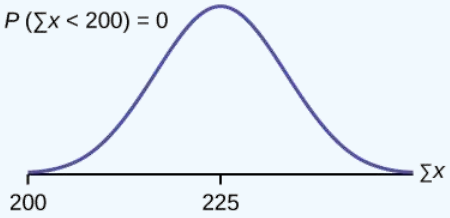

Figure 4.3.3.

The probability that the total of 75 scores is less than 200 is about zero.

$$\texttt{normalcdf}\ 75, 200, (75)(3), (\sqrt{75})(1.15)$$

REMINDER

The smallest total of 75 stress scores is 75, because the smallest single score is one.

d. Let $k =$ the 90th percentile.

Find k where $P(\sum x < k) = 0.90$.

$k = 237.8$

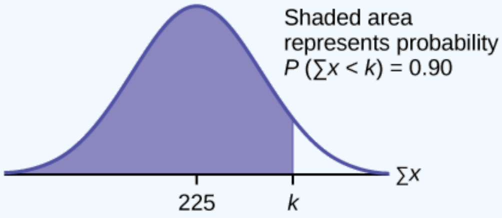

Figure 4.3.4.

The 90th percentile for the sum of 75 scores is about 237.8. This tells us that 90% of all the sums of 75 scores are no more than 237.8 and 10% are no less than 237.8.

$\texttt{invNorm}\ \left(0.90, (75)(3), (\sqrt{75})(1.15)\right) = 237.8$

? Exercise 4.3.1

Use the information in Example 4.3.1, but use a sample size of 55 to answer the following questions.

a. Find $P(\bar{x} < 7)$.
b. Find $P(\sum x < 7)$.
c. Find the 80th percentile for the mean of 55 scores.
d. Find the 85th percentile for the sum of 55 scores.

Answer

a. 0.0265
b. 0.2789
c. 3.13
d. 173.84

✓ Example 4.3.2

Suppose that a market research analyst for a cell phone company conducts a study of their customers who exceed the time allowance included on their basic cell phone contract; the analyst finds that for those people who exceed the time included in their basic contract, the *excess time used* follows an exponential distribution with a mean of 22 minutes.

Consider a random sample of 80 customers who exceed the time allowance included in their basic cell phone contract.

Let $X =$ the excess time used by one INDIVIDUAL cell phone customer who exceeds his contracted time allowance.

$X \sim Exp\left(\frac{1}{22}\right)$. From previous chapters, we know that $\mu = 22$ and $\sigma = 22$.

Let $\bar{X}$ = the mean excess time used by a sample of $n = 80$ customers who exceed their contracted time allowance.

$$\bar{X} \sim N\left(22, \frac{22}{\sqrt{80}}\right) \tag{4.3.5}$$

by the central limit theorem for sample means

a. Find the probability that the mean excess time used by the 80 customers in the sample is longer than 20 minutes. This is asking us to find $P(\bar{x} > 20)$. Draw the graph.
b. Suppose that one customer who exceeds the time limit for his cell phone contract is randomly selected. Find the probability that this individual customer's excess time is longer than 20 minutes. This is asking us to find $P(x > 20)$.
c. Explain why the probabilities in parts a and b are different.
d. Find the 95th percentile for the **sample mean excess time** for samples of 80 customers who exceed their basic contract time allowances. Draw a graph.

Answer

a. Find: $P(\bar{x} > 20)$

$P(\bar{x} > 20) = 0.79199$ using `normalcdf` $\left(20, 1E99, 22, \frac{22}{\sqrt{80}}\right)$

The probability is 0.7919 that the mean excess time used is more than 20 minutes, for a sample of 80 customers who exceed their contracted time allowance.

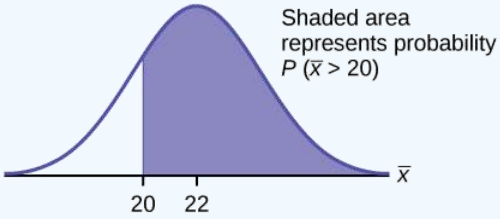

***Figure* 4.3.5.**

REMINDER

1E99 = 10^{99} and –1E99 = –10^{99}. Press the `EE` key for E. Or just use 10^{99} instead of 1E99.

b. Find $P(x > 20)$. Remember to use the exponential distribution for an **individual:** $X \sim Exp\left(\frac{1}{22}\right)$.

$P(x > 20) = e^{\left(-\left(\frac{1}{22}\right)(20)\right)}$ or $e^{(-0.04545(20))} = 0.4029$

c. i. $P(x > 20) = 0.4029$ but $P(\bar{x} > 20) = 0.7919$

ii. The probabilities are not equal because we use different distributions to calculate the probability for individuals and for means.

iii. When asked to find the probability of an individual value, use the stated distribution of its random variable; do not use the clt. Use the clt with the normal distribution when you are being asked to find the probability for a mean.

d. Let k = the 95th percentile. Find k where $P(\bar{x} < k) = 0.95$

$k = 26.0$ using `invNorm` $\left(0.95, 22, \frac{22}{\sqrt{80}}\right) = 26.0$

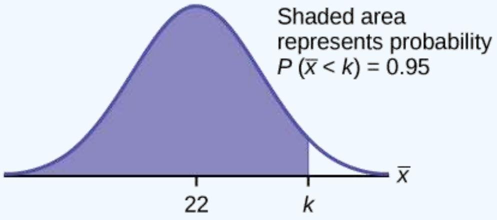

***Figure* 4.3.6.**

The 95th percentile for the **sample mean excess time used** is about 26.0 minutes for random samples of 80 customers who exceed their contractual allowed time.

Ninety five percent of such samples would have means under 26 minutes; only five percent of such samples would have means above 26 minutes.

? Exercise 4.3.2

Use the information in Example 4.3.2, but change the sample size to 144.

a. Find $P(20 < \bar{x} < 30)$.
b. Find $P(\sum x$ is at least $3,000)$.
c. Find the 75th percentile for the sample mean excess time of 144 customers.

d. Find the 85th percentile for the sum of 144 excess times used by customers.

Answer

a. 0.8623
b. 0.7377
c. 23.2
d. 3,441.6

✓ Example 4.3.3

In the United States, someone is sexually assaulted every two minutes, on average, according to a number of studies. Suppose the standard deviation is 0.5 minutes and the sample size is 100.

a. Find the median, the first quartile, and the third quartile for the sample mean time of sexual assaults in the United States.
b. Find the median, the first quartile, and the third quartile for the sum of sample times of sexual assaults in the United States.
c. Find the probability that a sexual assault occurs on the average between 1.75 and 1.85 minutes.
d. Find the value that is two standard deviations above the sample mean.
e. Find the *IQR* for the sum of the sample times.

Answer

a. We have, $\mu_x = \mu = 2$ and $\sigma_x = \frac{\sigma}{\sqrt{n}} = \frac{0.5}{10} = 0.05$. Therefore:

   a. 50th percentile $= \mu_x = \mu = 2$
   b. 25th percentile $= \text{invNorm}(0.25, 2, 0.05) = 1.97$
   c. 75th percentile $= \text{invNorm}(0.75, 2, 0.05) = 2.03$

b. We have $\mu_{\sum X} = n(\mu_x) = 100(2)$ and $\sigma_{\mu X} = \sqrt{n}(\sigma_x) = 10(0.5) = 5$. Therefore

   a. 50th percentile = $\mu_{\sum X} = n(\mu_X) = 100(2) = 200$
   b. 25th percentile $= \text{invNorm}(0.25, 200, 5) = 196.63$
   c. 75th percentile $= \text{invNorm}(0.75, 200, 5) = 203.37$

c. $P(1.75 < barx < 1.85) =$ `normalcdf` $(1.75, 1.85, 2, 0.05) = 0.0013$

d. Using the z-score equation, $z = \frac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}}$, and solving for x, we have $x = 2(0.05) + 2 = 2.1$

e. The IQR is 75th percentile – 25th percentile $= 203.37 - 196.63 = 6.74$

? Exercise 4.3.3

Based on data from the National Health Survey, women between the ages of 18 and 24 have an average systolic blood pressures (in mm Hg) of 114.8 with a standard deviation of 13.1. Systolic blood pressure for women between the ages of 18 to 24 follow a normal distribution.

a. If one woman from this population is randomly selected, find the probability that her systolic blood pressure is greater than 120.
b. If 40 women from this population are randomly selected, find the probability that their mean systolic blood pressure is greater than 120.
c. If the sample were four women between the ages of 18 to 24 and we did not know the original distribution, could the central limit theorem be used?

Answer

a. $P(x > 120) =$ `normalcdf` $(120, 99, 114.8, 13.1) = 0.0272$There is about a 3%, that the randomly selected woman will have systolics blood pressure greater than 120.

b. $P(\bar{x} > 120) =$ `normalcdf` $\left(120, 114.8, \frac{13.1}{\sqrt{40}}\right) = 0.006$ There is only a 0.6% chance that the average systolic blood pressure for the randomly selected group is greater than 120.

c. The central limit theorem could not be used if the sample size were four and we did not know the original distribution was normal. The sample size would be too small.

✓ Example 4.3.4

A study was done about violence against prostitutes and the symptoms of the posttraumatic stress that they developed. The age range of the prostitutes was 14 to 61. The mean age was 30.9 years with a standard deviation of nine years.

a. In a sample of 25 prostitutes, what is the probability that the mean age of the prostitutes is less than 35?
b. Is it likely that the mean age of the sample group could be more than 50 years? Interpret the results.
c. In a sample of 49 prostitutes, what is the probability that the sum of the ages is no less than 1,600?
d. Is it likely that the sum of the ages of the 49 prostitutes is at most 1,595? Interpret the results.
e. Find the 95th percentile for the sample mean age of 65 prostitutes. Interpret the results.
f. Find the 90th percentile for the sum of the ages of 65 prostitutes. Interpret the results.

Answer

1. $P(\bar{x} < 35) =$ `normalcdf` $(-E99, 35, 30.9, 1.8) = 0.9886$
2. $P(\bar{x} > 50) =$ `normalcdf` $(50, E99, 30.9, 1.8) \approx 0$ For this sample group, it is almost impossible for the group's average age to be more than 50. However, it is still possible for an individual in this group to have an age greater than 50.
3. $P(\sum x \geq 1,600) =$ `normalcdf` $(1600, E99, 1514.10, 63) = 0.0864$
4. $P(\sum x \leq 1,595) =$ `normalcdf` $(-E99, 1595, 1514.10, 63) = 0.9005$This means that there is a 90% chance that the sum of the ages for the sample group $n = 49$ is at most 1595.
5. The 95th percentile = `invNorm` $(0.95, 30.9, 1.1) = 32.7$This indicates that 95% of the prostitutes in the sample of 65 are younger than 32.7 years, on average.
6. The 90th percentile = `invNorm` $(0.90, 2008.5, 72.56) = 2101.5$This indicates that 90% of the prostitutes in the sample of 65 have a sum of ages less than 2,101.5 years.

? Exercise 4.3.4

According to Boeing data, the 757 airliner carries 200 passengers and has doors with a mean height of 72 inches. Assume for a certain population of men we have a mean of 69.0 inches and a standard deviation of 2.8 inches.

a. What mean doorway height would allow 95% of men to enter the aircraft without bending?
b. Assume that half of the 200 passengers are men. What mean doorway height satisfies the condition that there is a 0.95 probability that this height is greater than the mean height of 100 men?
c. For engineers designing the 757, which result is more relevant: the height from part a or part b? Why?

Answer

a. We know that $\mu_x = \mu = 69$ and we have $\sigma_x = 2.8$. The height of the doorway is found to be `invNorm` $(0.95, 69, 2.8) = 73.61$
b. We know that $\mu_x = \mu = 69$ and we have $\sigma_x = 2.8$. So, `invNorm` $(0.95, 69, 0.28) = 69.49$
c. When designing the doorway heights, we need to incorporate as much variability as possible in order to accommodate as many passengers as possible. Therefore, we need to use the result based on part a.

Historical Note: Normal Approximation to the Binomial

Historically, being able to compute binomial probabilities was one of the most important applications of the central limit theorem. Binomial probabilities with a small value for n(say, 20) were displayed in a table in a book. To calculate the probabilities with large values of n, you had to use the binomial formula, which could be very complicated. Using the normal approximation to the binomial distribution simplified the process. To compute the normal approximation to the binomial distribution, take a simple random sample from a population. You must meet the conditions for a binomial distribution:

- there are a certain number n of independent trials
- the outcomes of any trial are success or failure
- each trial has the same probability of a success p

Recall that if X is the binomial random variable, then $X \sim B(n, p)$. The shape of the binomial distribution needs to be similar to the shape of the normal distribution. To ensure this, the quantities np and nq must both be greater than five ($np > 5$ and $nq > 5$); the approximation is better if they are both greater than or equal to 10). Then the binomial can be approximated by the normal distribution with mean $\mu = np$ and standard deviation $\sigma = \sqrt{npq}$. Remember that $q = 1 - p$. In order to get the best approximation, add 0.5 to x or subtract 0.5 from x (use $x + 0.5$ or $x - 0.5$). The number 0.5 is called the continuity correction factor and is used in the following example.

✓ Example $4.3.5$

Suppose in a local Kindergarten through 12th grade (K - 12) school district, 53 percent of the population favor a charter school for grades K through 5. A simple random sample of 300 is surveyed.

a. Find the probability that **at least 150** favor a charter school.
b. Find the probability that **at most 160** favor a charter school.
c. Find the probability that **more than 155** favor a charter school.
d. Find the probability that **fewer than 147** favor a charter school.
e. Find the probability that **exactly 175** favor a charter school.

Let $X =$ the number that favor a charter school for grades K trough 5. $X \sim B(n, p)$ where $n = 300$ and $p = 0.53$. Since $np > 5$ and $nq > 5$, use the normal approximation to the binomial. The formulas for the mean and standard deviation are $\mu = np$ and $\sigma = \sqrt{npq}$. The mean is 159 and the standard deviation is 8.6447. The random variable for the normal distribution is X. $Y \sim N(159, 8.6447)$. See The Normal Distribution for help with calculator instructions.

For part a, you **include 150** so $P(X \geq 150)$ has normal approximation $P(Y \geq 149.5) = 0.8641$.

`normalcdf` $(149.5, 10^{99}, 159, 8.6447) = 0.8641$

For part b, you **include 160** so $P(X \leq 160)$ has normal approximation $P(Y \leq 160.5) = 0.5689$.

`normalcdf` $(0, 160.5, 159, 8.6447) = 0.5689$

For part c, you **exclude 155** so $P(X > 155)$ has normal approximation $P(y > 155.5) = 0.6572$.

`normalcdf` $(155.5, 10^{99}, 159, 8.6447) = 0.6572$

For part d, you **exclude 147** so $P(X < 147)$ has normal approximation $P(Y < 146.5) = 0.0741$.

`normalcdf` $(0, 146.5, 159, 8.6447) = 0.0741$

For part e, $P(X = 175)$ has normal approximation $P(174.5 < Y < 175.5) = 0.0083$.

`normalcdf` $(174.5, 175.5, 159, 8.6447) = 0.0083$

Because of calculators and computer software that let you calculate binomial probabilities for large values of n easily, it is not necessary to use the the normal approximation to the binomial distribution, provided that you have access to these technology tools. Most school labs have Microsoft Excel, an example of computer software that calculates binomial probabilities. Many students have access to the TI-83 or 84 series calculators, and they easily calculate probabilities for the binomial distribution. If you type in "binomial probability distribution calculation" in an Internet browser, you can find at least one online calculator for the binomial.

For Example, the probabilities are calculated using the following binomial distribution: ($n = 300 and p = 0.53$). Compare the binomial and normal distribution answers. See Discrete Random Variables for help with calculator instructions for the binomial.

$P(X \geq 150)$: `1 - binomialcdf` $(300, 0.53, 149) = 0.8641$

$P(X \leq 160)$: `binomialcdf` $(300, 0.53, 160) = 0.5684$

$P(X > 155)$: `1 - binomialcdf` $(300, 0.53, 155) = 0.6576$

$P(X < 147)$: `binomialcdf` $(300, 0.53, 146) = 0.0742$

$P(X = 175)$:(You use the binomial pdf.) `binomialpdf` $(300, 0.53, 175) = 0.0083$

? Exercise 4.3.5

In a city, 46 percent of the population favor the incumbent, Dawn Morgan, for mayor. A simple random sample of 500 is taken. Using the continuity correction factor, find the probability that at least 250 favor Dawn Morgan for mayor.

Answer

0.0401

References

- Data from the Wall Street Journal.
- “National Health and Nutrition Examination Survey.” Center for Disease Control and Prevention. Available online at http://www.cdc.gov/nchs/nhanes.htm (accessed May 17, 2013).

Glossary

Exponential Distribution

a continuous random variable (RV) that appears when we are interested in the intervals of time between some random events, for example, the length of time between emergency arrivals at a hospital, notation: $X \sim Exp(m)$. The mean is $\mu = \frac{1}{m}$ and the standard deviation is $\sigma = \frac{1}{m}$. The probability density function is $f(x) = me^{-mx}$, $x \geq 0$ and the cumulative distribution function is $P(X \leq x) = 1 - e^{-mx}$.

Mean

a number that measures the central tendency; a common name for mean is "average." The term "mean" is a shortened form of "arithmetic mean." By definition, the mean for a sample (denoted by $\bar{x}$) is $\bar{x} = \frac{\text{Sum of all values in the sample}}{\text{Number of values in the sample}}$, and the mean for a population (denoted by μ) is $\mu = \frac{\text{Sum of all values in the population}}{\text{Number of values in the population}}$.

Normal Distribution

a continuous random variable (RV) with pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e \frac{(x-\mu)^2}{2\sigma^2}$, where μ is the mean of the distribution and σ is the standard deviation.; notation: $X \sim N(\mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, the RV is called the **standard normal distribution**.

Uniform Distribution

a continuous random variable (RV) that has equally likely outcomes over the domain, \(a < x < b\); often referred as the **Rectangular Distribution** because the graph of the pdf has the form of a rectangle. Notation: $X \sim U(a, b)$. The mean is $\mu = \frac{a+b}{2}$ and the standard deviation is $\sigma = \sqrt{\frac{(b-a)^2}{12}}$. The probability density function is $f(x) = \frac{a+b}{2}$ for $a < x < b$ or $a \leq x \leq b$. The cumulative distribution is $P(X \leq x) = \frac{x-a}{b-a}$.

CHAPTER OVERVIEW

5: Confidence Intervals

In this chapter, you will learn to construct and interpret confidence intervals. You will also learn a new distribution, the Student's-t, and how it is used with these intervals. Throughout the chapter, it is important to keep in mind that the confidence interval is a random variable. It is the population parameter that is fixed.

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5.1: Prelude to Confidence Intervals

Learning Objectives

By the end of this chapter, the student should be able to:

- Calculate and interpret confidence intervals for estimating a population mean and a population proportion.
- Interpret the Student's t probability distribution as the sample size changes.
- Discriminate between problems applying the normal and the Student's *t* distributions.
- Calculate the sample size required to estimate a population mean and a population proportion given a desired confidence level and margin of error.

Suppose you were trying to determine the mean rent of a two-bedroom apartment in your town. You might look in the classified section of the newspaper, write down several rents listed, and average them together. You would have obtained a point estimate of the true mean. If you are trying to determine the percentage of times you make a basket when shooting a basketball, you might count the number of shots you make and divide that by the number of shots you attempted. In this case, you would have obtained a point estimate for the true proportion.

We use sample data to make generalizations about an unknown population. This part of statistics is called **inferential statistics**. The sample data help us to make an estimate of a population parameter. We realize that the point estimate is most likely not the exact value of the population parameter, but close to it. After calculating point estimates, we construct interval estimates, called confidence intervals.

Figure 5.1.1. Have you ever wondered what the average number of M&Ms in a bag at the grocery store is? You can use confidence intervals to answer this question. (credit: comedy_nose/flickr)

In this chapter, you will learn to construct and interpret confidence intervals. You will also learn a new distribution, the Student's-t, and how it is used with these intervals. Throughout the chapter, it is important to keep in mind that the confidence interval is a random variable. It is the population parameter that is fixed.

If you worked in the marketing department of an entertainment company, you might be interested in the mean number of songs a consumer downloads a month from iTunes. If so, you could conduct a survey and calculate the sample mean, $\bar{x}$, and the sample standard deviation, s. You would use $\bar{x}$ to estimate the population mean and s to estimate the population standard deviation. The sample mean, $\bar{x}$, is the point estimate for the population mean, μ. The sample standard deviation, s, is the point estimate for the population standard deviation, σ.

Each of $\bar{x}$ and s is called a statistic.

A confidence interval is another type of estimate but, instead of being just one number, it is an interval of numbers. The interval of numbers is a range of values calculated from a given set of sample data. The confidence interval is likely to include an unknown population parameter.

Suppose, for the iTunes example, we do not know the population mean μ, but we do know that the population standard deviation is $\sigma = 1$ and our sample size is 100. Then, by the central limit theorem, the standard deviation for the sample mean is

$$\frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{100}} = 0.1. \tag{5.1.1}$$

The empirical rule, which applies to bell-shaped distributions, says that in approximately 95% of the samples, the sample mean, $\bar{x}$, will be within two standard deviations of the population mean μ. For our iTunes example, two standard deviations is (2)(0.1) = 0.2. The sample mean $\bar{x}$ is likely to be within 0.2 units of μ.

Because $\bar{x}$ is within 0.2 units of μ, which is unknown, then μ is likely to be within 0.2 units of $\bar{x}$ in 95% of the samples. The population mean μ is contained in an interval whose lower number is calculated by taking the sample mean and subtracting two standard deviations (2)(0.1) and whose upper number is calculated by taking the sample mean and adding two standard deviations. In other words, μ is between $\bar{x}-0.2$ and $\bar{x}+0.2$ in 95% of all the samples.

For the iTunes example, suppose that a sample produced a sample mean $\bar{x}=2$. Then the unknown population mean μ is between

$$\bar{x}-0.2=2-0.2=1.8 \tag{5.1.2}$$

and

$$\bar{x}+0.2=2+0.2=2.2 \tag{5.1.3}$$

We say that we are 95% confident that the unknown population mean number of songs downloaded from iTunes per month is between 1.8 and 2.2. The 95% confidence interval is (1.8, 2.2). This 95% confidence interval implies two possibilities. Either the interval (1.8, 2.2) contains the true mean μ or our sample produced an $\bar{x}$ that is not within 0.2 units of the true mean μ. The second possibility happens for only 5% of all the samples (95–100%).

Remember that a confidence interval is created for an unknown population parameter like the population mean, $\bar{x}$. Confidence intervals for some parameters have the form:

(point estimate – margin of error, point estimate + margin of error)

The margin of error depends on the confidence level or percentage of confidence and the standard error of the mean.

When you read newspapers and journals, some reports will use the phrase "margin of error." Other reports will not use that phrase, but include a confidence interval as the point estimate plus or minus the margin of error. These are two ways of expressing the same concept.

> Although the text only covers symmetrical confidence intervals, there are non-symmetrical confidence intervals (for example, a confidence interval for the standard deviation).

Collaborative Exercise

Have your instructor record the number of meals each student in your class eats out in a week. Assume that the standard deviation is known to be three meals. Construct an approximate 95% confidence interval for the true mean number of meals students eat out each week.

1. Calculate the sample mean.
2. Let $\sigma=3$ and n = the number of students surveyed.
3. Construct the interval $\left(\bar{x}-2\cdot\frac{\sigma}{\sqrt{n}}, \bar{x}+2\cdot\frac{\sigma}{\sqrt{n}}\right)$.

We say we are approximately 95% confident that the true mean number of meals that students eat out in a week is between __________ and ___________.

Glossary

Confidence Interval (CI)

an interval estimate for an unknown population parameter. This depends on:

- the desired confidence level,
- information that is known about the distribution (for example, known standard deviation),
- the sample and its size.

Inferential Statistics

also called statistical inference or inductive statistics; this facet of statistics deals with estimating a population parameter based on a sample statistic. For example, if four out of the 100 calculators sampled are defective we might infer that four percent of the production is defective.

Parameter

a numerical characteristic of a population

Point Estimate

a single number computed from a sample and used to estimate a population parameter

5.2: A Single Population Mean using the Normal Distribution

A confidence interval for a population mean with a known standard deviation is based on the fact that the sample means follow an approximately normal distribution. Suppose that our sample has a mean of $\bar{x} = 10$ and we have constructed the 90% confidence interval (5, 15) where $EBM = 5$.

Calculating the Confidence Interval

To construct a confidence interval for a single unknown population mean μ, where the population standard deviation is known, we need $\bar{x}$ as an estimate for μ and we need the margin of error. Here, the margin of error (EBM) is called the error bound for a population mean (abbreviated *EBM*). The sample mean $\bar{x}$ is the point estimate of the unknown population mean μ.

The confidence interval estimate will have the form:

$$(\text{point estimate} - \text{error bound}, \text{point estimate} + \text{error bound})$$

or, in symbols,

$$(\bar{x} - EBM, \bar{x} + EBM)$$

The **margin of error** (EBM) depends on the confidence level (abbreviated CL). The confidence level is often considered the probability that the calculated confidence interval estimate will contain the true population parameter. However, it is more accurate to state that the confidence level is the percent of confidence intervals that contain the true population parameter when repeated samples are taken. Most often, it is the choice of the person constructing the confidence interval to choose a confidence level of 90% or higher because that person wants to be reasonably certain of his or her conclusions.

There is another probability called alpha (α). α is related to the confidence level, CL. α is the probability that the interval does not contain the unknown population parameter. Mathematically,

$$\alpha + CL = 1.$$

✓ Example 5.2.1

Suppose we have collected data from a sample. We know the sample mean but we do not know the mean for the entire population. The sample mean is seven, and the error bound for the mean is 2.5: $\bar{x} = 7$ and $EBM = 2.5$

The confidence interval is (7 – 2.5, 7 + 2.5) and calculating the values gives (4.5, 9.5). If the confidence level (CL) is 95%, then we say that, "We estimate with 95% confidence that the true value of the population mean is between 4.5 and 9.5."

? Exercise 5.2.1

Suppose we have data from a sample. The sample mean is 15, and the error bound for the mean is 3.2. What is the confidence interval estimate for the population mean?

Answer

(11.8, 18.2)

A confidence interval for a population mean with a known standard deviation is based on the fact that the sample means follow an approximately normal distribution. Suppose that our sample has a mean of $\bar{x} = 10$, and we have constructed the 90% confidence interval (5, 15) where $EBM = 5$. To get a 90% confidence interval, we must include the central 90% of the probability of the normal distribution. If we include the central 90%, we leave out a total of $\alpha = 10$ in both tails, or 5% in each tail, of the normal distribution.

This is a normal distribution curve. The peak of the curve coincides with the point 10 on the horizontal axis. The points 5 and 15 are labeled on the axis. Vertical lines are drawn from these points to the curve, and the region between the lines is shaded. The shaded region has area equal to 0.90.

Figure 5.2.1

To capture the central 90%, we must go out 1.645 "standard deviations" on either side of the calculated sample mean. The value 1.645 is the *z*-score from a standard normal probability distribution that puts an area of 0.90 in the center, an area of 0.05 in the far left tail, and an area of 0.05 in the far right tail.

It is important that the "standard deviation" used must be appropriate for the parameter we are estimating, so in this section we need to use the standard deviation that applies to sample means, which is

$$\frac{\sigma}{\sqrt{n}}$$

This fraction is commonly called the "standard error of the mean" to distinguish clearly the standard deviation for a mean from the population standard deviation σ.

In summary, as a result of the central limit theorem:

- $\bar{X}$ is normally distributed, that is, $\bar{X} \sim N(\mu_x, \frac{\sigma}{\sqrt{n}})$.
- When the population standard deviation σ is known, we use a normal distribution to calculate the error bound.

Calculating the Confidence Interval

To construct a confidence interval estimate for an unknown population mean, we need data from a random sample. The steps to construct and interpret the confidence interval are:

- Calculate the sample mean $\bar{x}$ from the sample data. Remember, in this section we already know the population standard deviation σ.
- Find the z-score that corresponds to the confidence level.
- Calculate the error bound EBM.
- Construct the confidence interval.
- Write a sentence that interprets the estimate in the context of the situation in the problem. (Explain what the confidence interval means, in the words of the problem.)

We will first examine each step in more detail, and then illustrate the process with some examples.

Finding the z-score for the Stated Confidence Level

When we know the population standard deviation σ, we use a standard normal distribution to calculate the error bound EBM and construct the confidence interval. We need to find the value of z that puts an area equal to the confidence level (in decimal form) in the middle of the standard normal distribution $Z \sim N(0,1)$.

The confidence level, CL, is the area in the middle of the standard normal distribution. $CL = 1 - \alpha$, so α is the area that is split equally between the two tails. Each of the tails contains an area equal to $\frac{\alpha}{2}$.

The z-score that has an area to the right of $\frac{\alpha}{2}$ is denoted by $z_{\frac{\alpha}{2}}$.

For example, when $CL = 0.95, \alpha = 0.05$ and $\frac{\alpha}{2} = 0.025$; we write $z_{\frac{\alpha}{2}} = z_{0.025}$.

The area to the right of $z_{0.025}$ is 0.025 and the area to the left of $z_{0.025}$ is $1 - 0.025 = 0.975$.

$$z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$$

using a calculator, computer or a standard normal probability table.

`invNorm` $(0.975, 0, 1) = 1.96$

> Remember to use the area to the LEFT of $z_{\frac{\alpha}{2}}$; in this chapter the last two inputs in the invNorm command are 0, 1, because you are using a standard normal distribution $Z \sim N(0,1)$.

Calculating the Error Bound

The error bound formula for an unknown population mean μ when the population standard deviation σ is known is

$$EBM = z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

Constructing the Confidence Interval

The confidence interval estimate has the format $(\bar{x} - EBM, \bar{x} + EBM)$.

The graph gives a picture of the entire situation.

$$CL + \frac{\alpha}{2} + \frac{\alpha}{2} = CL + \alpha = 1.$$

This is a normal distribution curve. The peak of the curve coincides with the point x-bar on the horizontal axis. The points x-bar - EBM and x-bar + EBM are labeled on the axis. Vertical lines are drawn from these points to the curve, and the region between the lines is shaded. The shaded region has area equal to 1 - a and represents the confidence level. Each unshaded tail has area a/2.

Figure 8.2.2.

Writing the Interpretation

The interpretation should clearly state the confidence level (CL), explain what population parameter is being estimated (here, a **population mean**), and state the confidence interval (both endpoints). "We estimate with ___% confidence that the true population mean (include the context of the problem) is between ___ and ___ (include appropriate units)."

✓ Example $5.2.2$

Suppose scores on exams in statistics are normally distributed with an unknown population mean and a population standard deviation of three points. A random sample of 36 scores is taken and gives a sample mean (sample mean score) of 68. Find a confidence interval estimate for the population mean exam score (the mean score on all exams).

Find a 90% confidence interval for the true (population) mean of statistics exam scores.

Answer

- You can use technology to calculate the confidence interval directly.
- The first solution is shown step-by-step (Solution A).
- The second solution uses the TI-83, 83+, and 84+ calculators (Solution B).

Solution A

To find the confidence interval, you need the sample mean, $\bar{x}$, and the EBM.

$$\bar{x} = 68$$

$$EBM = \left(z_{\frac{\alpha}{2}}\right)\left(\frac{\sigma}{\sqrt{n}}\right)$$

$$\sigma = 3; n = 36$$

The confidence level is 90% (CL = 0.90)

$$CL = 0.90$$

so

$$\alpha = 1 – CL = 1 – 0.90 = 0.10$$

$$\frac{\alpha}{2} = 0.05 z_{\frac{\alpha}{2}} = z_{0.05}$$

The area to the right of $z_{0.05}$ is 0.05 and the area to the left of $z_{0.05}$ is $1 - 0.05 = 0.95$.

$$z_{\frac{\alpha}{2}} = z_{0.05} = 1.645$$

using $\text{invNorm}(0.95, 0, 1)$ on the TI-83,83+, and 84+ calculators. This can also be found using appropriate commands on other calculators, using a computer, or using a probability table for the standard normal distribution.

$$EBM = (1.645)\left(\frac{3}{\sqrt{36}}\right) = 0.8225$$

$$\bar{x} - EBM = 68 - 0.8225 = 67.1775$$

$$\bar{x} + EBM = 68 + 0.8225 = 68.8225$$

The 90% confidence interval is **(67.1775, 68.8225).**

Solution B

Press `STAT` and arrow over to `TESTS` .

Arrow down to `7:ZInterval` .
Press `ENTER` .
Arrow to `Stats` and press `ENTER` .
Arrow down and enter three for σ, 68 for $\bar{x}$, 36 for n, and .90 for `C-level` .
Arrow down to `Calculate` and press `ENTER` .
The confidence interval is (to three decimal places)(67.178, 68.822).

Interpretation

We estimate with 90% confidence that the true population mean exam score for all statistics students is between 67.18 and 68.82.

Explanation of 90% Confidence Level

Ninety percent of all confidence intervals constructed in this way contain the true mean statistics exam score. For example, if we constructed 100 of these confidence intervals, we would expect 90 of them to contain the true population mean exam score.

? Exercise 5.2.2

Suppose average pizza delivery times are normally distributed with an unknown population mean and a population standard deviation of six minutes. A random sample of 28 pizza delivery restaurants is taken and has a sample mean delivery time of 36 minutes. Find a 90% confidence interval estimate for the population mean delivery time.

Answer

(34.1347, 37.8653)

✓ Example 5.2.3: Specific Absorption Rate

The Specific Absorption Rate (SAR) for a cell phone measures the amount of radio frequency (RF) energy absorbed by the user’s body when using the handset. Every cell phone emits RF energy. Different phone models have different SAR measures. To receive certification from the Federal Communications Commission (FCC) for sale in the United States, the SAR level for a cell phone must be no more than 1.6 watts per kilogram. Table shows the highest SAR level for a random selection of cell phone models as measured by the FCC.

Phone Model	SAR	Phone Model	SAR	Phone Model	SAR
Apple iPhone 4S	1.11	LG Ally	1.36	Pantech Laser	0.74
BlackBerry Pearl 8120	1.48	LG AX275	1.34	Samsung Character	0.5
BlackBerry Tour 9630	1.43	LG Cosmos	1.18	Samsung Epic 4G Touch	0.4
Cricket TXTM8	1.3	LG CU515	1.3	Samsung M240	0.867

Phone Model	SAR	Phone Model	SAR	Phone Model	SAR
HP/Palm Centro	1.09	LG Trax CU575	1.26	Samsung Messager III SCH-R750	0.68
HTC One V	0.455	Motorola Q9h	1.29	Samsung Nexus S	0.51
HTC Touch Pro 2	1.41	Motorola Razr2 V8	0.36	Samsung SGH-A227	1.13
Huawei M835 Ideos	0.82	Motorola Razr2 V9	0.52	SGH-a107 GoPhone	0.3
Kyocera DuraPlus	0.78	Motorola V195s	1.6	Sony W350a	1.48
Kyocera K127 Marbl	1.25	Nokia 1680	1.39	T-Mobile Concord	1.38

Find a 98% confidence interval for the true (population) mean of the Specific Absorption Rates (SARs) for cell phones. Assume that the population standard deviation is $\sigma = 0.337$.

Solution A

To find the confidence interval, start by finding the point estimate: the sample mean.

$$\bar{x} = 1.024$$

Next, find the EBM. Because you are creating a 98% confidence interval, $CL = 0.98$.

This is a normal distribution curve. The point z0.01 is labeled at the right edge of the curve and the region to the right of this point is shaded. The area of this shaded region equals 0.01. The unshaded area equals 0.99.

Figure 8.2.3.

You need to find $z_{0.01}$ having the property that the area under the normal density curve to the right of $z_{0.01}$ is 0.01 and the area to the left is 0.99. Use your calculator, a computer, or a probability table for the standard normal distribution to find $z_{0.01} = 2.326$.

$$EBM = (z_{0.01})\frac{\sigma}{\sqrt{n}} = (2.326)\frac{0.337}{\sqrt{30}} = 0.1431$$

To find the 98% confidence interval, find $\bar{x} \pm EBM$.

$\bar{x} - EBM = 1.024 – 0.1431 = 0.8809$

$\bar{x} - EBM = 1.024 – 0.1431 = 1.1671$

We estimate with 98% confidence that the true SAR mean for the population of cell phones in the United States is between 0.8809 and 1.1671 watts per kilogram.

Solution B

- Press STAT and arrow over to TESTS.
- Arrow down to 7:Z Interval.
- Press ENTER.
- Arrow to Stats and press ENTER.
- Arrow down and enter the following values:
  - $\sigma : 0.337$
  - $\bar{x} : 1024$
  - $n : 30$
  - C-level: 0.98
- Arrow down to Calculate and press ENTER.
- The confidence interval is (to three decimal places) (0.881, 1.167).

? Exercise 5.2.3

Table shows a different random sampling of 20 cell phone models. Use this data to calculate a 93% confidence interval for the true mean SAR for cell phones certified for use in the United States. As previously, assume that the population standard deviation is $\sigma = 0.337$.

Phone Model	SAR	Phone Model	SAR
Blackberry Pearl 8120	1.48	Nokia E71x	1.53
HTC Evo Design 4G	0.8	Nokia N75	0.68
HTC Freestyle	1.15	Nokia N79	1.4
LG Ally	1.36	Sagem Puma	1.24
LG Fathom	0.77	Samsung Fascinate	0.57
LG Optimus Vu	0.462	Samsung Infuse 4G	0.2
Motorola Cliq XT	1.36	Samsung Nexus S	0.51
Motorola Droid Pro	1.39	Samsung Replenish	0.3
Motorola Droid Razr M	1.3	Sony W518a Walkman	0.73
Nokia 7705 Twist	0.7	ZTE C79	0.869

Answer

$$\bar{x} = 0.940$$

$$\frac{\alpha}{2} = \frac{1 - CL}{2} = \frac{1 - 0.93}{2} = 0.035$$

$$z_{0.035} = 1.812$$

$$EBM = (z_{0.035})\left(\frac{\sigma}{\sqrt{n}}\right) = (1.812)\left(\frac{0.337}{\sqrt{20}}\right) = 0.1365$$

$$\bar{x} - EBM = 0.940 - 0.1365 = 0.8035$$

$$\bar{x} + EBM = 0.940 + 0.1365 = 1.0765$$

We estimate with 93% confidence that the true SAR mean for the population of cell phones in the United States is between 0.8035 and 1.0765 watts per kilogram.

Notice the difference in the confidence intervals calculated in Example and the following Try It exercise. These intervals are different for several reasons: they were calculated from different samples, the samples were different sizes, and the intervals were calculated for different levels of confidence. Even though the intervals are different, they do not yield conflicting information. The effects of these kinds of changes are the subject of the next section in this chapter.

Changing the Confidence Level or Sample Size

✓ Example 5.2.4

Suppose we change the original problem in Example by using a 95% confidence level. Find a 95% confidence interval for the true (population) mean statistics exam score.

Answer

To find the confidence interval, you need the sample mean, $\bar{x}$, and the EBM.

$$\bar{x} = 68$$

$$EBM = \left(z_{\frac{\alpha}{2}}\right)\left(\frac{\sigma}{\sqrt{n}}\right)$$

$\sigma = 3; n = 36$; The confidence level is 95% (*CL* = 0.95).

$$CL = 0.95 \text{ so } \alpha = 1 - CL = 1 - 0.95 = 0.05$$

$$\frac{\alpha}{2} = 0.025 z_{\frac{\alpha}{2}} = z_{0.025}$$

The area to the right of $z_{0.025}$ is 0.025 and the area to the left of $z_{0.025}$ is $1 - 0.025 = 0.975$

$$z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$$

when using invnorm(0.975,0,1) on the TI-83, 83+, or 84+ calculators. (This can also be found using appropriate commands on other calculators, using a computer, or using a probability table for the standard normal distribution.)

$$EBM = (1.96)\left(\frac{3}{\sqrt{36}}\right) = 0.98$$

$$\bar{x} - EBM = 68 - 0.98 = 67.02$$

$$\bar{x} + EBM = 68 + 0.98 = 68.98$$

Notice that the EBM is larger for a 95% confidence level in the original problem.

Interpretation

We estimate with 95% confidence that the true population mean for all statistics exam scores is between 67.02 and 68.98.

Explanation of 95% Confidence Level

Ninety-five percent of all confidence intervals constructed in this way contain the true value of the population mean statistics exam score.

Comparing the results

The 90% confidence interval is (67.18, 68.82). The 95% confidence interval is (67.02, 68.98). The 95% confidence interval is wider. If you look at the graphs, because the area 0.95 is larger than the area 0.90, it makes sense that the 95% confidence interval is wider. To be more confident that the confidence interval actually does contain the true value of the population mean for all statistics exam scores, the confidence interval necessarily needs to be wider.

Part (a) shows a normal distribution curve. A central region with area equal to 0.90 is shaded. Each unshaded tail of the curve has area equal to 0.05. Part (b) shows a normal distribution curve. A central region with area equal to 0.95 is shaded. Each unshaded tail of the curve has area equal to 0.025.

Figure 8.2.4.

Summary: Effect of Changing the Confidence Level

- Increasing the confidence level increases the error bound, making the confidence interval wider.
- Decreasing the confidence level decreases the error bound, making the confidence interval narrower.

? Exercise 5.2.4

Refer back to the pizza-delivery Try It exercise. The population standard deviation is six minutes and the sample mean deliver time is 36 minutes. Use a sample size of 20. Find a 95% confidence interval estimate for the true mean pizza delivery time.

Answer

(33.37, 38.63)

Example 5.2.5

Suppose we change the original problem in Example to see what happens to the error bound if the sample size is changed.

Leave everything the same except the sample size. Use the original 90% confidence level. What happens to the error bound and the confidence interval if we increase the sample size and use $n = 100$ instead of $n = 36$? What happens if we decrease the sample size to $n = 25$ instead of $n = 36$?

- $\bar{x} = 68$
- $EBM = \left(z_{\frac{\alpha}{2}}\right)\left(\frac{\sigma}{\sqrt{n}}\right)$
- $\sigma = 3$; The confidence level is 90% (*CL*=0.90); $z_{\frac{\alpha}{2}} = z_{0.05} = 1.645$.

Answer

Solution A

If we **increase** the sample size n to 100, we **decrease** the error bound.

When $n = 100 : EBM = \left(z_{\frac{\alpha}{2}}\right)\left(\frac{\sigma}{\sqrt{n}}\right) = (1.645)\left(\frac{3}{\sqrt{100}}\right) = 0.4935.$

Solution B

If we **decrease** the sample size n to 25, we **increase** the error bound.

When $n = 25 : EBM = \left(z_{\frac{\alpha}{2}}\right)\left(\frac{\sigma}{\sqrt{n}}\right) = (1.645)\left(\frac{3}{\sqrt{25}}\right) = 0.987.$

Summary: Effect of Changing the Sample Size

- Increasing the sample size causes the error bound to decrease, making the confidence interval narrower.
- Decreasing the sample size causes the error bound to increase, making the confidence interval wider.

Exercise 5.2.5

Refer back to the pizza-delivery Try It exercise. The mean delivery time is 36 minutes and the population standard deviation is six minutes. Assume the sample size is changed to 50 restaurants with the same sample mean. Find a 90% confidence interval estimate for the population mean delivery time.

Answer

(34.6041, 37.3958)

Working Backwards to Find the Error Bound or Sample Mean

When we calculate a confidence interval, we find the sample mean, calculate the error bound, and use them to calculate the confidence interval. However, sometimes when we read statistical studies, the study may state the confidence interval only. If we know the confidence interval, we can work backwards to find both the error bound and the sample mean.

Finding the Error Bound

- From the upper value for the interval, subtract the sample mean,
- OR, from the upper value for the interval, subtract the lower value. Then divide the difference by two.

Finding the Sample Mean

- Subtract the error bound from the upper value of the confidence interval,
- OR, average the upper and lower endpoints of the confidence interval.

Notice that there are two methods to perform each calculation. You can choose the method that is easier to use with the information you know.

Example 5.2.6

Suppose we know that a confidence interval is **(67.18, 68.82)** and we want to find the error bound. We may know that the sample mean is 68, or perhaps our source only gave the confidence interval and did not tell us the value of the sample mean.

Calculate the Error Bound:

- If we know that the sample mean is 68 : $EBM = 68.82 – 68 = 0.82$.
- If we don't know the sample mean: $EBM = \frac{(68.82 - 67.18)}{2} = 0.82$.

Calculate the Sample Mean:

- If we know the error bound: $\bar{x} = 68.82 – 0.82 = 68$
- If we don't know the error bound: $\bar{x} = \frac{(67.18 + 68.82)}{2} = 68$.

Exercise 5.2.6

Suppose we know that a confidence interval is (42.12, 47.88). Find the error bound and the sample mean.

Answer

Sample mean is 45, error bound is 2.88

Calculating the Sample Size n

If researchers desire a specific margin of error, then they can use the error bound formula to calculate the required sample size. The error bound formula for a population mean when the population standard deviation is known is

$$EBM = \left(z_{\frac{a}{2}}\right)\left(\frac{\sigma}{\sqrt{n}}\right)$$

The formula for sample size is $n = \frac{z^2\sigma^2}{EBM^2}$, found by solving the error bound formula for n. In Equation ???, z is $z_{\frac{a}{2}}$, corresponding to the desired confidence level. A researcher planning a study who wants a specified confidence level and error bound can use this formula to calculate the size of the sample needed for the study.

Example 5.2.7

The population standard deviation for the age of Foothill College students is 15 years. If we want to be 95% confident that the sample mean age is within two years of the true population mean age of Foothill College students, how many randomly selected Foothill College students must be surveyed?

Solution

- From the problem, we know that $\sigma = 15$ and $EBM = 2$.
- $z = z_{0.025} = 1.96$, because the confidence level is 95%.
- $n = \frac{z^2\sigma^2}{EBM^2} = \frac{(1.96)^2(15)^2}{2^2}$ using the sample size equation.
- Use $n = 217$: Always round the answer UP to the next higher integer to ensure that the sample size is large enough.

Therefore, 217 Foothill College students should be surveyed in order to be 95% confident that we are within two years of the true population mean age of Foothill College students.

? Exercise 5.2.7

The population standard deviation for the height of high school basketball players is three inches. If we want to be 95% confident that the sample mean height is within one inch of the true population mean height, how many randomly selected students must be surveyed?

Answer

35 students

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Glossary

Confidence Level (CL)

the percent expression for the probability that the confidence interval contains the true population parameter; for example, if the $CL = 90$, then in 90 out of 100 samples the interval estimate will enclose the true population parameter.

Error Bound for a Population Mean (EBM)

the margin of error; depends on the confidence level, sample size, and known or estimated population standard deviation.

5.3: A Single Population Mean using the Student t-Distribution

In practice, we rarely know the population standard deviation. In the past, when the sample size was large, this did not present a problem to statisticians. They used the sample standard deviation s as an estimate for σ and proceeded as before to calculate a confidence interval with close enough results. However, statisticians ran into problems when the sample size was small. A small sample size caused inaccuracies in the confidence interval.

William S. Goset (1876–1937) of the Guinness brewery in Dublin, Ireland ran into this problem. His experiments with hops and barley produced very few samples. Just replacing σ with s did not produce accurate results when he tried to calculate a confidence interval. He realized that he could not use a normal distribution for the calculation; he found that the actual distribution depends on the sample size. This problem led him to "discover" what is called the Student's t-distribution. The name comes from the fact that Gosset wrote under the pen name "Student."

Up until the mid-1970s, some statisticians used the normal distribution approximation for large sample sizes and only used the Student's t-distribution only for sample sizes of at most 30. With graphing calculators and computers, the practice now is to use the Student's t-distribution whenever s is used as an estimate for σ. If you draw a simple random sample of size n from a population that has an approximately a normal distribution with mean μ and unknown population standard deviation σ and calculate the t-score

$$t=\frac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n}}\right)}, \tag{5.3.1}$$

then the t-scores follow a Student's t-distribution with $n-1$ degrees of freedom. The t-score has the same interpretation as the z-score. It measures how far $\bar{x}$ is from its mean μ. For each sample size n, there is a different Student's t-distribution.

The degrees of freedom, $n-1$, come from the calculation of the sample standard deviation s. Previously, we used n deviations ($x-\bar{x}$ values) to calculate s. Because the sum of the deviations is zero, we can find the last deviation once we know the other $n-1$ deviations. The other $n-1$ deviations can change or vary freely. We call the number $n-1$ the degrees of freedom (df).

> *For each sample size n, there is a different Student's t-distribution.*

Properties of the Student's t-Distribution

- The graph for the Student's t-distribution is similar to the standard normal curve.
- The mean for the Student's t-distribution is zero and the distribution is symmetric about zero.
- The Student's t-distribution has more probability in its tails than the standard normal distribution because the spread of the t-distribution is greater than the spread of the standard normal. So the graph of the Student's t-distribution will be thicker in the tails and shorter in the center than the graph of the standard normal distribution.
- The exact shape of the Student's t-distribution depends on the degrees of freedom. As the degrees of freedom increases, the graph of Student's t-distribution becomes more like the graph of the standard normal distribution.
- The underlying population of individual observations is assumed to be normally distributed with unknown population mean μ and unknown population standard deviation σ. The size of the underlying population is generally not relevant unless it is very small. If it is bell shaped (normal) then the assumption is met and doesn't need discussion. Random sampling is assumed, but that is a completely separate assumption from normality.

Calculators and computers can easily calculate any Student's t-probabilities. The TI-83,83+, and 84+ have a tcdf function to find the probability for given values of t. The grammar for the tcdf command is tcdf(lower bound, upper bound, degrees of freedom). However for confidence intervals, we need to use **inverse** probability to find the value of t when we know the probability.

For the TI-84+ you can use the invT command on the DISTRibution menu. The invT command works similarly to the invnorm. The invT command requires two inputs: **invT(area to the left, degrees of freedom)** The output is the t-score that corresponds to the area we specified.

The TI-83 and 83+ do not have the invT command. (The TI-89 has an inverse T command.)

A probability table for the Student's t-distribution can also be used. The table gives t-scores that correspond to the confidence level (column) and degrees of freedom (row). (The TI-86 does not have an invT program or command, so if you are using that calculator, you need to use a probability table for the Student's t-Distribution.) When using a t-table, note that some tables are formatted to show the confidence level in the column headings, while the column headings in some tables may show only corresponding area in one or both tails.

A Student's t-table gives t-scores given the degrees of freedom and the right-tailed probability. The table is very limited. **Calculators and computers can easily calculate any Student's t-probabilities.**

The notation for the Student's t-distribution (using T as the random variable) is:

- $T \sim t_{df}$ where $df = n – 1$.
- For example, if we have a sample of size $n = 20$ items, then we calculate the degrees of freedom as $df = n - 1 = 20 - 1 = 19$ and we write the distribution as $T \sim t_{19}$.

If the population standard deviation is not known, the error bound for a population mean is:

- $EBM = \left(t_{\frac{\alpha}{2}}\right)\left(\frac{s}{\sqrt{n}}\right)$,
- $t_{\frac{\alpha}{2}}$ is the t-score with area to the right equal to $\frac{\alpha}{2}$,
- use $df = n – 1$ degrees of freedom, and
- $s =$ sample standard deviation.

The format for the confidence interval is:

$$(\bar{x} - EBM, \bar{x} + EBM). \tag{5.3.2}$$

To calculate the confidence interval directly:

Press STAT.
Arrow over to TESTS.
Arrow down to 8:TInterval and press ENTER (or just press 8).

✔ Example 5.3.1: Acupuncture

Suppose you do a study of acupuncture to determine how effective it is in relieving pain. You measure sensory rates for 15 subjects with the results given. Use the sample data to construct a 95% confidence interval for the mean sensory rate for the population (assumed normal) from which you took the data.

The solution is shown step-by-step and by using the TI-83, 83+, or 84+ calculators.

8.6; 9.4; 7.9; 6.8; 8.3; 7.3; 9.2; 9.6; 8.7; 11.4; 10.3; 5.4; 8.1; 5.5; 6.9

Answer

- The first solution is step-by-step (Solution A).
- The second solution uses the TI-83+ and TI-84 calculators (Solution B).

Solution A

To find the confidence interval, you need the sample mean, $\bar{x}$, and the EBM.

$$\bar{x} = 8.2267$$

$$s = 1.6722 \; n = 15$$

$$df = 15 – 1 = 14 CL so \alpha = 1 – CL = 1 – 0.95 = 0.05$$

$$\frac{\alpha}{2} = 0.025 t_{\frac{\alpha}{2}} = t_{0.025}$$

The area to the right of $t_{0.025}$ is 0.025, and the area to the left of $t_{0.025}$ is 1 – 0.025 = 0.975

$$t_{\frac{\alpha}{2}} = t_{0.025} = 2.14 \text{ using invT(.975,14) on the TI-84+ calculator.}$$

$$EBM = \left(t_{\frac{\alpha}{2}}\right)\left(\frac{s}{\sqrt{n}}\right)$$

$$= (2.14)\left(\frac{1.6722}{\sqrt{15}}\right) = 0.924$$

Now it is just a direct application of Equation 5.3.2:

$$\bar{x} - EBM = 8.2267 - 0.9240 = 7.3$$
$$\bar{x} + EBM = 8.2267 + 0.9240 = 9.15$$

The 95% confidence interval is (7.30, 9.15).

We estimate with 95% confidence that the true population mean sensory rate is between 7.30 and 9.15.

Solution B

Press `STAT` and arrow over to `TESTS` .

Arrow down to `8:TInterval` and press `ENTER` (or you can just press `8`).
Arrow to `Data` and press `ENTER` .
Arrow down to `List` and enter the list name where you put the data.
There should be a 1 after `Freq` .
Arrow down to `C-level` and enter 0.95
Arrow down to `Calculate` and press `ENTER` .
The 95% confidence interval is (7.3006, 9.1527)

When calculating the error bound, a probability table for the Student's t-distribution can also be used to find the value of t. The table gives t-scores that correspond to the confidence level (column) and degrees of freedom (row); the t-score is found where the row and column intersect in the table.

? Exercise 5.3.1

You do a study of hypnotherapy to determine how effective it is in increasing the number of hourse of sleep subjects get each night. You measure hours of sleep for 12 subjects with the following results. Construct a 95% confidence interval for the mean number of hours slept for the population (assumed normal) from which you took the data.

8.2; 9.1; 7.7; 8.6; 6.9; 11.2; 10.1; 9.9; 8.9; 9.2; 7.5; 10.5

Answer

(8.1634, 9.8032)

✓ Example 5.3.2: The Human Toxome Project

The Human Toxome Project (HTP) is working to understand the scope of industrial pollution in the human body. Industrial chemicals may enter the body through pollution or as ingredients in consumer products. In October 2008, the scientists at HTP tested cord blood samples for 20 newborn infants in the United States. The cord blood of the "In utero/newborn" group was tested for 430 industrial compounds, pollutants, and other chemicals, including chemicals linked to brain and nervous system toxicity, immune system toxicity, and reproductive toxicity, and fertility problems. There are health concerns about the effects of some chemicals on the brain and nervous system. Table 5.3.1 shows how many of the targeted chemicals were found in each infant's cord blood.

Table 5.3.1

79	145	147	160	116	100	159	151	156	126
137	83	156	94	121	144	123	114	139	99

Use this sample data to construct a 90% confidence interval for the mean number of targeted industrial chemicals to be found in an in infant's blood.

Solution A

From the sample, you can calculate $\bar{x} = 127.45$ and $s = 25.965$. There are 20 infants in the sample, so $n = 20$, and $df = 20 - 1 = 19$.

You are asked to calculate a 90% confidence interval: $CL = 0.90$, so

$$\alpha = 1 - CL = 1 - 0.90 = 0.10 \frac{\alpha}{2} = 0.05, t_{\frac{\alpha}{2}} = t_{0.05} \tag{5.3.3}$$

By definition, the area to the right of $t_{0.05}$ is 0.05 and so the area to the left of $t_{0.05}$ is $1 - 0.05 = 0.95$.

Use a table, calculator, or computer to find that $t_{0.05} = 1.729$.

$$EBM = t_{\frac{\alpha}{2}} \left(\frac{s}{\sqrt{n}}\right) = 1.729 \left(\frac{25.965}{\sqrt{20}}\right) \approx 10.038$$

$$\bar{x} - EBM = 127.45 - 10.038 = 117.412$$

$$\bar{x} + EBM = 127.45 + 10.038 = 137.488$$

We estimate with 90% confidence that the mean number of all targeted industrial chemicals found in cord blood in the United States is between 117.412 and 137.488.

Solution B

Enter the data as a list.

Press `STAT` and arrow over to `TESTS` .
Arrow down to `8:TInterval` and press `ENTER` (or you can just press `8`). Arrow to Data and press `ENTER` .
Arrow down to `List` and enter the list name where you put the data.
Arrow down to `Freq` and enter 1.
Arrow down to `C-level` and enter 0.90
Arrow down to `Calculate` and press `ENTER` .

The 90% confidence interval is (117.41, 137.49).

? Example 5.3.3

A random sample of statistics students were asked to estimate the total number of hours they spend watching television in an average week. The responses are recorded in Table $5.3.2$. Use this sample data to construct a 98% confidence interval for the mean number of hours statistics students will spend watching television in one week.

Table 5.3.2

0	3	1	20	9
5	10	1	10	4
14	2	4	4	5

Solution A

- $\bar{x} = 6.133$,
- $s = 5.514$,
- $n = 15$, and
- $df = 15 - 1 = 14$.

$$CL = 0.98, \text{ so } \alpha = 1 - CL = 1 - 0.98 = 0.02$$

$$\frac{\alpha}{2} = 0.01 t_{\frac{\alpha}{2}} = t_{0.01} 2.624$$

$$EBM = t_{\frac{\alpha}{2}} \left(\frac{s}{\sqrt{n}}\right) = 2.624 \left(\frac{5.514}{\sqrt{15}}\right) - 3.736$$

$$\bar{x} - EBM = 6.133 - 3.736 = 2.397$$

$$\bar{x} + EBM = 6.133 + 3.736 = 9.869$$

We estimate with 98% confidence that the mean number of all hours that statistics students spend watching television in one week is between 2.397 and 9.869.

Solution B

Enter the data as a list.

Press `STAT` and arrow over to `TESTS` .
Arrow down to `8:TInterval` .
Press `ENTER` .
Arrow to `Data` and press `ENTER` .
Arrow down and enter the name of the list where the data is stored.
Enter `Freq` : 1
Enter `C-Level` : 0.98
Arrow down to `Calculate` and press `Enter` .
The 98% confidence interval is (2.3965, 9,8702).

Reference

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Glossary

Degrees of Freedom (df)

the number of objects in a sample that are free to vary

Normal Distribution

a continuous random variable (RV) with pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where μ is the mean of the distribution and σ is the standard deviation, notation: $X \sim N(\mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, the RV is called **the standard normal distribution**.

Standard Deviation

a number that is equal to the square root of the variance and measures how far data values are from their mean; notation: s for sample standard deviation and σ for population standard deviation

Student's t-Distribution

investigated and reported by William S. Gossett in 1908 and published under the pseudonym Student; the major characteristics of the random variable (RV) are:

- It is continuous and assumes any real values.
- The pdf is symmetrical about its mean of zero. However, it is more spread out and flatter at the apex than the normal distribution.
- It approaches the standard normal distribution as *n* get larger.
- There is a "family" of t–distributions: each representative of the family is completely defined by the number of degrees of freedom, which is one less than the number of data.

5.4: A Population Proportion

During an election year, we see articles in the newspaper that state confidence intervals in terms of proportions or percentages. For example, a poll for a particular candidate running for president might show that the candidate has 40% of the vote within three percentage points (if the sample is large enough). Often, election polls are calculated with 95% confidence, so, the pollsters would be 95% confident that the true proportion of voters who favored the candidate would be between 0.37 and 0.43: (0.40 – 0.03,0.40 + 0.03).

Investors in the stock market are interested in the true proportion of stocks that go up and down each week. Businesses that sell personal computers are interested in the proportion of households in the United States that own personal computers. Confidence intervals can be calculated for the true proportion of stocks that go up or down each week and for the true proportion of households in the United States that own personal computers.

The procedure to find the confidence interval, the sample size, the error bound, and the confidence level for a proportion is similar to that for the population mean, but the formulas are different. How do you know you are dealing with a proportion problem? First, the underlying distribution is a binomial distribution. (There is no mention of a mean or average.) If X is a binomial random variable, then

$$X \sim B(n, p)$$

where n is the number of trials and p is the probability of a success.

To form a proportion, take X, the random variable for the number of successes and divide it by n, the number of trials (or the sample size). The random variable P' (read "P prime") is that proportion,

$$P' = \frac{X}{n}$$

(Sometimes the random variable is denoted as $\hat{P}$, read "P hat".)

When n is large and p is not close to zero or one, we can use the normal distribution to approximate the binomial.

$$X \sim N(np, \sqrt{npq})$$

If we divide the random variable, the mean, and the standard deviation by n, we get a normal distribution of proportions with P', called the estimated proportion, as the random variable. (Recall that a proportion as the number of successes divided by n.)

$$\frac{X}{n} = P' \sim N\left(\frac{np}{n}, \frac{\sqrt{npq}}{n}\right)$$

Using algebra to simplify:

$$\frac{\sqrt{npq}}{n} = \sqrt{\frac{pq}{n}}$$

P' follows a normal distribution for proportions:

$$\frac{X}{n} = P' \sim N\left(p, \sqrt{\frac{pq}{n}}\right)$$

The confidence interval has the form

$$(p' – EBP, p' + EBP).$$

where

- EBP is error bound for the proportion.
- $p' = \frac{x}{n}$
- $p' =$ the estimated proportion of successes (p' is a point estimate for p, the true proportion.)
- $x =$ the number of successes
- $n =$ the size of the sample

The error bound (EBP) for a proportion is

$$EBP = \left(z_{\frac{\alpha}{2}}\right)\left(\sqrt{\frac{p'q'}{n}}\right)$$

where $q = 1 - p'$.

This formula is similar to the error bound formula for a mean, except that the "appropriate standard deviation" is different. For a mean, when the population standard deviation is known, the appropriate standard deviation that we use is $\frac{\sigma}{\sqrt{n}}$. For a proportion, the appropriate standard deviation is

$$\sqrt{\frac{pq}{n}}.$$

However, in the error bound formula, we use

$$\sqrt{\frac{p'q'}{n}}$$

as the standard deviation, instead of

$$\sqrt{\frac{pq}{n}}.$$

In the error bound formula, the sample proportions p' and q' are estimates of the unknown population proportions p and q. The estimated proportions p' and q' are used because p and q are not known. The sample proportions p' and q' are calculated from the data: p' is the estimated proportion of successes, and q' is the estimated proportion of failures.

The confidence interval can be used only if the number of successes np' and the number of failures nq' are both greater than five.

Normal Distribution of Proportions

For the normal distribution of proportions, the z-score formula is as follows.

If

$$P' \sim N\left(p, \sqrt{\frac{pq}{n}}\right) \tag{5.4.1}$$

then the z-score formula is

$$z = \frac{p' - p}{\sqrt{\frac{pq}{n}}} \tag{5.4.2}$$

Example 5.4.1

Suppose that a market research firm is hired to estimate the percent of adults living in a large city who have cell phones. Five hundred randomly selected adult residents in this city are surveyed to determine whether they have cell phones. Of the 500 people surveyed, 421 responded yes - they own cell phones. Using a 95% confidence level, compute a confidence interval estimate for the true proportion of adult residents of this city who have cell phones.

Solution A

- The first solution is step-by-step (Solution A).
- The second solution uses a function of the TI-83, 83+ or 84 calculators (Solution B).

Let $X =$ the number of people in the sample who have cell phones. X is binomial.

$$X \sim B(500, \frac{421}{500}).$$

To calculate the confidence interval, you must find p', q', and EBP.

- $n = 500$
- $x =$ the number of successes $= 421$

$$p' = \frac{x}{n} = \frac{421}{500} = 0.842$$

- $p' = 0.842$ is the sample proportion; this is the point estimate of the population proportion.

$$q' = 1 - p' = 1 - 0.842 = 0.158$$

Since $CL = 0.95$, then

$$\alpha = 1 - CL = 1 - 0.95 = 0.05 \left(\frac{\alpha}{2}\right) = 0.025.$$

Then

$$z_{\frac{\alpha}{2}} = z_{0.025=1.96}$$

Use the TI-83, 83+, or 84+ calculator command invNorm(0.975,0,1) to find $z_{0.025}$. Remember that the area to the right of $z_{0.025}$ is 0.025 and the area to the left of $z_{0.025}$ is 0.975. This can also be found using appropriate commands on other calculators, using a computer, or using a Standard Normal probability table.

$$EBP = \left(z_{\frac{\alpha}{2}}\right)\sqrt{\frac{p'q'}{n}} = (1.96)\sqrt{\frac{(0.842)(0.158)}{500}} = 0.032$$

$$p' - EBP = 0.842 - 0.032 = 0.81$$

$$p' + EBP = 0.842 + 0.032 = 0.874$$

The confidence interval for the true binomial population proportion is $(p' - EBP, p' + EBP) = (0.810, 0.874)$.

Interpretation

We estimate with 95% confidence that between 81% and 87.4% of all adult residents of this city have cell phones.

Explanation of 95% Confidence Level

Ninety-five percent of the confidence intervals constructed in this way would contain the true value for the population proportion of all adult residents of this city who have cell phones.

Solution B

Press `STAT` and arrow over to `TESTS` .

Arrow down to `A:1-PropZint` . Press `ENTER` .
Arrow down to xx and enter 421.
Arrow down to nn and enter 500.
Arrow down to `C-Level` and enter .95.
Arrow down to `Calculate` and press `ENTER` .
The confidence interval is (0.81003, 0.87397).

? Exercise 5.4.1

Suppose 250 randomly selected people are surveyed to determine if they own a tablet. Of the 250 surveyed, 98 reported owning a tablet. Using a 95% confidence level, compute a confidence interval estimate for the true proportion of people who own tablets.

Answer

(0.3315, 0.4525)

Example 5.4.2

For a class project, a political science student at a large university wants to estimate the percent of students who are registered voters. He surveys 500 students and finds that 300 are registered voters. Compute a 90% confidence interval for the true percent of students who are registered voters, and interpret the confidence interval.

Answer

- The first solution is step-by-step (Solution A).
- The second solution uses a function of the TI-83, 83+, or 84 calculators (Solution B).

Solution A

- $x = 300$ and
- $n = 500$

$$p' = \frac{x}{n} = \frac{300}{500} = 0.600$$

$$q' = 1 - p' = 1 - 0.600 = 0.400$$

Since $CL = 0.90$, then

$$\alpha = 1 – CL = 1 – 0.90 = 0.10 \left(\frac{\alpha}{2}\right) = 0.05 \tag{5.4.3}$$

$$z_{\frac{\alpha}{2}} = z_{0.05} = 1.645$$

Use the TI-83, 83+, or 84+ calculator command invNorm(0.95,0,1) to find $z_{0.05}$. Remember that the area to the right of $z_{0.05}$ is 0.05 and the area to the left of $z_{0.05}$ is 0.95. This can also be found using appropriate commands on other calculators, using a computer, or using a standard normal probability table.

$$EBP = \left(z_{\frac{\alpha}{2}}\right)\sqrt{\frac{p'q'}{n}} = (1.645)\sqrt{\frac{(0.60)(0.40)}{500}} = 0.036$$

$$p' – EBP = 0.60 - 0.036 = 0.564$$

$$p' + EBP = 0.60 + 0.036 = 0.636$$

The confidence interval for the true binomial population proportion is $(p' – EBP, p' + EBP) = (0.564, 0.636)$.

Interpretation

- We estimate with 90% confidence that the true percent of all students that are registered voters is between 56.4% and 63.6%.
- Alternate Wording: We estimate with 90% confidence that between 56.4% and 63.6% of ALL students are registered voters.

Explanation of 90% Confidence Level

Ninety percent of all confidence intervals constructed in this way contain the true value for the population percent of students that are registered voters.

Solution B

Press `STAT` and arrow over to `TESTS` .

Arrow down to `A:1-PropZint` . Press `ENTER` .
Arrow down to xx and enter 300.
Arrow down to nn and enter 500.
Arrow down to `C-Level` and enter 0.90.
Arrow down to `Calculate` and press `ENTER` .

The confidence interval is (0.564, 0.636).

? Exercise 5.4.2

A student polls his school to see if students in the school district are for or against the new legislation regarding school uniforms. She surveys 600 students and finds that 480 are against the new legislation.

a. Compute a 90% confidence interval for the true percent of students who are against the new legislation, and interpret the confidence interval.
b. In a sample of 300 students, 68% said they own an iPod and a smart phone. Compute a 97% confidence interval for the true percent of students who own an iPod and a smartphone.

Answer a

(0.7731, 0.8269); We estimate with 90% confidence that the true percent of all students in the district who are against the new legislation is between 77.31% and 82.69%.

Answer b

Sixty-eight percent (68%) of students own an iPod and a smart phone.

$$p' = 0.68$$

$$q' = 1 – p' = 1 – 0.68 = 0.32$$

Since $CL = 0.97$, we know

$$\alpha = 1 – 0.97 = 0.03$$

and

$$\frac{\alpha}{2} = 0.015.$$

The area to the left of $z_{0.05}$ is 0.015, and the area to the right of $z_{0.05}$ is 1 – 0.015 = 0.985.

Using the TI 83, 83+, or 84+ calculator function InvNorm(0.985,0,1),

$$z_{0.05} = 2.17$$

$$EPB = \left(z_{\frac{\alpha}{2}}\right)\sqrt{\frac{p'q'}{n}} = 2.17\sqrt{\frac{0.68(0.32)}{300}} \approx 0.0269$$

$$p' – EPB = 0.68 – 0.0269 = 0.6531$$

$$p' + EPB = 0.68 + 0.0269 = 0.7069$$

We are 97% confident that the true proportion of all students who own an iPod and a smart phone is between 0.6531 and 0.7069.

Calculator

Press STAT and arrow over to TESTS.

Arrow down to A:1-PropZint. Press ENTER.
Arrow down to x and enter 300*0.68.
Arrow down to n and enter 300.
Arrow down to C-Level and enter 0.97.
Arrow down to Calculate and press ENTER.

The confidence interval is (0.6531, 0.7069).

"Plus Four" Confidence Interval for p

There is a certain amount of error introduced into the process of calculating a confidence interval for a proportion. Because we do not know the true proportion for the population, we are forced to use point estimates to calculate the appropriate standard deviation of the sampling distribution. Studies have shown that the resulting estimation of the standard deviation can be flawed.

Fortunately, there is a simple adjustment that allows us to produce more accurate confidence intervals. We simply pretend that we have four additional observations. Two of these observations are successes and two are failures. The new sample size, then, is $n+4$, and the new count of successes is $x+2$. Computer studies have demonstrated the effectiveness of this method. It should be used when the confidence level desired is at least 90% and the sample size is at least ten.

✓ Example 5.4.3

A random sample of 25 statistics students was asked: "Have you smoked a cigarette in the past week?" Six students reported smoking within the past week. Use the "plus-four" method to find a 95% confidence interval for the true proportion of statistics students who smoke.

Solution A

Six students out of 25 reported smoking within the past week, so $x=6$ and $n=25$. Because we are using the "plus-four" method, we will use $x=6+2=8$ and $n=25+4=29$.

$$p'=\frac{x}{n}=\frac{8}{29}\approx 0.276$$

$$q'=1-p'=1-0.276=0.724$$

Since $CL=0.95$, we know $\alpha=1-0.95=0.05$ and $\frac{\alpha}{2}=0.025$.

$$z_{0.025}=1.96$$

$$EPB=\left(z_{\frac{\alpha}{2}}\right)\sqrt{\frac{p'q'}{n}}=(1.96)\sqrt{\frac{0.276(0.724)}{29}}\approx 0.163$$

$$p'-EPB=0.276-0.163=0.113$$

$$p'+EPB=0.276+0.163=0.439$$

We are 95% confident that the true proportion of all statistics students who smoke cigarettes is between 0.113 and 0.439.

Solution B

Press STAT and arrow over to TESTS.

Arrow down to A:1-PropZint. Press ENTER.

REMINDER

Remember that the plus-four method assume an additional four trials: two successes and two failures. You do not need to change the process for calculating the confidence interval; simply update the values of x and n to reflect these additional trials.

Arrow down to x and enter eight.

Arrow down to n and enter 29.
Arrow down to C-Level and enter 0.95.
Arrow down to Calculate and press ENTER.

The confidence interval is (0.113, 0.439).

Exercise 5.4.3

Out of a random sample of 65 freshmen at State University, 31 students have declared a major. Use the “plus-four” method to find a 96% confidence interval for the true proportion of freshmen at State University who have declared a major.

Solution A

Using “plus four,” we have $x = 31+2 = 33$ and $n = 65+4 = 69$.

$$p' = 3369 \approx 0.478$$

$$q' = 1 – p' = 1 – 0.478 = 0.522$$

Since $CL = 0.96$, we know $\alpha = 1 – 0.96 = 0.04$ and $\frac{\alpha}{2} = 0.02$.

$$z_{0.02} = 2.054$$

$$EPB = \left(z_{\frac{\alpha}{2}}\right)\sqrt{\frac{p'q'}{n}} = (2.054)\left(\sqrt{\frac{(0.478)(0.522)}{69}}\right) - 0.124$$

$$p' – EPB = 0.478 – 0.124 = 0.354$$

$$p' + EPB = 0.478 + 0.124 = 0.602$$

We are 96% confident that between 35.4% and 60.2% of all freshmen at State U have declared a major.

Solution B

Press STAT and arrow over to TESTS.

Arrow down to A:1-PropZint. Press ENTER.
Arrow down to x and enter 33.
Arrow down to n and enter 69.
Arrow down to C-Level and enter 0.96.
Arrow down to Calculate and press ENTER.

The confidence interval is (0.355, 0.602).

Example 5.4.4

The Berkman Center for Internet & Society at Harvard recently conducted a study analyzing the privacy management habits of teen internet users. In a group of 50 teens, 13 reported having more than 500 friends on Facebook. Use the “plus four” method to find a 90% confidence interval for the true proportion of teens who would report having more than 500 Facebook friends.

Solution A

Using “plus-four,” we have $x = 13+2 = 15$ and $n = 50+4 = 54$.

$$p' = 1554 \approx 0.278$$

$$q' = 1 – p' = 1 - 0.241 = 0.722$$

Since $CL = 0.90$, we know $\alpha = 1 – 0.90 = 0.10$ and $\frac{\alpha}{2} = 0.05$.

$$z_{0.05} = 1.645$$

$$EPB = \left(z_{\frac{\alpha}{2}}\right)\left(\sqrt{\frac{p'q'}{n}}\right) = (1.645)\left(\sqrt{\frac{(0.278)(0.722)}{54}}\right) \approx 0.100$$

$$p' – EPB = 0.278 – 0.100 = 0.178$$

$$p' + EPB = 0.278 + 0.100 = 0.378$$

We are 90% confident that between 17.8% and 37.8% of all teens would report having more than 500 friends on Facebook.

Solution B

Press STAT and arrow over to TESTS.

Arrow down to A:1-PropZint. Press ENTER.
Arrow down to x and enter 15.
Arrow down to n and enter 54.
Arrow down to C-Level and enter 0.90.
Arrow down to Calculate and press ENTER.

The confidence interval is (0.178, 0.378).

Exercise 5.4.4

The Berkman Center Study referenced in Example talked to teens in smaller focus groups, but also interviewed additional teens over the phone. When the study was complete, 588 teens had answered the question about their Facebook friends with 159 saying that they have more than 500 friends. Use the "plus-four" method to find a 90% confidence interval for the true proportion of teens that would report having more than 500 Facebook friends based on this larger sample. Compare the results to those in Example.

Answer

Solution A

Using "plus-four," we have $x = 159 + 2 = 161$ and $n = 588 + 4 = 592$.

$$p' = 161592 \approx 0.272$$

$$q' = 1 – p' = 1 – 0.272 = 0.728$$

Since *CL* = 0.90, we know $\alpha = 1 – 0.90 = 0.10$ and $\frac{\alpha}{2} = 0.05$

$$EPB = \left(z\frac{\alpha}{2}\right)\left(\sqrt{\frac{p'q'}{n}}\right) = (1.645)\left(\sqrt{\frac{(0.272)(0.728)}{592}}\right) \approx 0.030$$

$$p' – EPB = 0.272 – 0.030 = 0.242$$

$$p' + EPB = 0.272 + 0.030 = 0.302$$

We are 90% confident that between 24.2% and 30.2% of all teens would report having more than 500 friends on Facebook.

Solution B

- Press STAT and arrow over to TESTS.
- Arrow down to A:1-PropZint. Press ENTER.
- Arrow down to x and enter 161.
- Arrow down to n and enter 592.
- Arrow down to C-Level and enter 0.90.
- Arrow down to Calculate and press ENTER.
- The confidence interval is (0.242, 0.302).

Conclusion: The confidence interval for the larger sample is narrower than the interval from Example. Larger samples will always yield more precise confidence intervals than smaller samples. The "plus four" method has a greater impact on the smaller sample. It shifts the point estimate from 0.26 (13/50) to 0.278 (15/54). It has a smaller impact on the *EPB*, changing it from 0.102 to 0.100. In the larger sample, the point estimate undergoes a smaller shift: from 0.270 (159/588) to 0.272 (161/592). It is easy to see that the plus-four method has the greatest impact on smaller samples.

Calculating the Sample Size n

If researchers desire a specific margin of error, then they can use the error bound formula to calculate the required sample size. The error bound formula for a population proportion is

$$EBP = \left(z_{\frac{\alpha}{2}}\right)\left(\sqrt{\frac{p'q'}{n}}\right)$$

Solving for n gives you an equation for the sample size.

$$n = \frac{\left(z_{\frac{\alpha}{2}}\right)^2 (p'q')}{EBP^2}$$

Example 5.4.5

Suppose a mobile phone company wants to determine the current percentage of customers aged 50+ who use text messaging on their cell phones. How many customers aged 50+ should the company survey in order to be 90% confident that the estimated (sample) proportion is within three percentage points of the true population proportion of customers aged 50+ who use text messaging on their cell phones.

Answer

From the problem, we know that $\mathbf{EBP = 0.03}$ (3%=0.03) and $z_{\frac{\alpha}{2}}$ $z_{0.05} = 1.645$ because the confidence level is 90%.

However, in order to find n, we need to know the estimated (sample) proportion p'. Remember that $q' = 1 - p'$. But, we do not know p' yet. Since we multiply p' and q' together, we make them both equal to 0.5 because $p'q' = (0.5)(0.5) = 0.25$ results in the largest possible product. (Try other products: $(0.6)(0.4) = 0.24$; $(0.3)(0.7) = 0.21$; $(0.2)(0.8) = 0.16$ and so on). The largest possible product gives us the largest n. This gives us a large enough sample so that we can be 90% confident that we are within three percentage points of the true population proportion. To calculate the sample size n, use the formula and make the substitutions.

$$n = \frac{z^2 p'q'}{EBP^2}$$

gives

$$n = \frac{1.645^2 (0.5)(0.5)}{0.03^2} = 751.7$$

Round the answer to the next higher value. The sample size should be 752 cell phone customers aged 50+ in order to be 90% confident that the estimated (sample) proportion is within three percentage points of the true population proportion of all customers aged 50+ who use text messaging on their cell phones.

Exercise 5.4.5

Suppose an internet marketing company wants to determine the current percentage of customers who click on ads on their smartphones. How many customers should the company survey in order to be 90% confident that the estimated proportion is within five percentage points of the true population proportion of customers who click on ads on their smartphones?

Answer

271 customers should be surveyed. Check the Real Estate section in your local

Glossary

Binomial Distribution

a discrete random variable (RV) which arises from Bernoulli trials; there are a fixed number, n, of independent trials. “Independent” means that the result of any trial (for example, trial 1) does not affect the results of the following trials, and all trials are conducted under the same conditions. Under these circumstances the binomial RV X is defined as the number of successes in n trials. The notation is: $X \sim B(\mathbf{n}, \mathbf{p})$. The mean is $\mu = np$ and the standard deviation is $\sigma = \sqrt{npq}$. The probability of exactly x successes in n trials is $P(X = x = \left(\binom{n}{x}\right))p^x q^{n-x}$.

Error Bound for a Population Proportion ($\boldsymbol{EBP}$)

the margin of error; depends on the confidence level, the sample size, and the estimated (from the sample) proportion of successes.

CHAPTER OVERVIEW

6: Hypothesis Testing with One Sample

One job of a statistician is to make statistical inferences about populations based on samples taken from the population. Confidence intervals are one way to estimate a population parameter. Another way to make a statistical inference is to make a decision about a parameter. For instance, a car dealer advertises that its new small truck gets 35 miles per gallon, on average. A tutoring service claims that its method of tutoring helps 90% of its students get an A or a B. A company says that women managers in their company earn an average of $60,000 per year.

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6.1: Prelude to Hypothesis Testing

CHAPTER OBJECTIVES

By the end of this chapter, the student should be able to:

- Differentiate between Type I and Type II Errors
- Describe hypothesis testing in general and in practice
- Conduct and interpret hypothesis tests for a single population mean, population standard deviation known.
- Conduct and interpret hypothesis tests for a single population mean, population standard deviation unknown.
- Conduct and interpret hypothesis tests for a single population proportion

One job of a statistician is to make statistical inferences about populations based on samples taken from the population. Confidence intervals are one way to estimate a population parameter. Another way to make a statistical inference is to make a decision about a parameter. For instance, a car dealer advertises that its new small truck gets 35 miles per gallon, on average. A tutoring service claims that its method of tutoring helps 90% of its students get an A or a B. A company says that women managers in their company earn an average of $60,000 per year.

Figure 6.1.1: You can use a hypothesis test to decide if a dog breeder’s claim that every Dalmatian has 35 spots is statistically sound. (Credit: Robert Neff)

A statistician will make a decision about these claims. This process is called "hypothesis testing." A hypothesis test involves collecting data from a sample and evaluating the data. Then, the statistician makes a decision as to whether or not there is sufficient evidence, based upon analysis of the data, to reject the null hypothesis. In this chapter, you will conduct hypothesis tests on single means and single proportions. You will also learn about the errors associated with these tests.

Hypothesis testing consists of two contradictory hypotheses or statements, a decision based on the data, and a conclusion. To perform a hypothesis test, a statistician will:

- Set up two contradictory hypotheses.
- Collect sample data (in homework problems, the data or summary statistics will be given to you).
- Determine the correct distribution to perform the hypothesis test.
- Analyze sample data by performing the calculations that ultimately will allow you to reject or decline to reject the null hypothesis.
- Make a decision and write a meaningful conclusion.

> To do the hypothesis test homework problems for this chapter and later chapters, make copies of the appropriate special solution sheets. See Appendix E.

Glossary

Confidence Interval (CI)

an interval estimate for an unknown population parameter. This depends on:

- The desired confidence level.
- Information that is known about the distribution (for example, known standard deviation).
- The sample and its size.

Hypothesis Testing

Based on sample evidence, a procedure for determining whether the hypothesis stated is a reasonable statement and should not be rejected, or is unreasonable and should be rejected.

6.2: Null and Alternative Hypotheses

The actual test begins by considering two **hypotheses**. They are called the **null hypothesis** and the **alternative hypothesis**. These hypotheses contain opposing viewpoints.

H_0: **The null hypothesis:** It is a statement of no difference between the variables—they are not related. This can often be considered the status quo and as a result if you cannot accept the null it requires some action.

H_a: **The alternative hypothesis:** It is a claim about the population that is contradictory to H_0 and what we conclude when we reject H_0. This is usually what the researcher is trying to prove.

Since the null and alternative hypotheses are contradictory, you must examine evidence to decide if you have enough evidence to reject the null hypothesis or not. The evidence is in the form of sample data.

After you have determined which hypothesis the sample supports, you make a **decision.** There are two options for a decision. They are "reject H_0" if the sample information favors the alternative hypothesis or "do not reject H_0" or "decline to reject H_0" if the sample information is insufficient to reject the null hypothesis.

Table 6.2.1: Mathematical Symbols Used in H_0 and H_a:

H_0	H_a
equal (=)	not equal ($\neq$) **or** greater than (>) **or** less than (<)
greater than or equal to ($\geq$)	less than (<)
less than or equal to ($\geq$)	more than (>)

H_0 always has a symbol with an equal in it. H_a never has a symbol with an equal in it. The choice of symbol depends on the wording of the hypothesis test. However, be aware that many researchers (including one of the co-authors in research work) use = in the null hypothesis, even with > or < as the symbol in the alternative hypothesis. This practice is acceptable because we only make the decision to reject or not reject the null hypothesis.

✓ Example 6.2.1

- H_0: No more than 30% of the registered voters in Santa Clara County voted in the primary election. $p \leq 30$
- H_a: More than 30% of the registered voters in Santa Clara County voted in the primary election. $p > 30$

? Exercise 6.2.1

A medical trial is conducted to test whether or not a new medicine reduces cholesterol by 25%. State the null and alternative hypotheses.

Answer

- H_0: The drug reduces cholesterol by 25%. $p = 0.25$
- H_a: The drug does not reduce cholesterol by 25%. $p \neq 0.25$

✓ Example 6.2.2

We want to test whether the mean GPA of students in American colleges is different from 2.0 (out of 4.0). The null and alternative hypotheses are:

- $H_0 : \mu = 2.0$
- $H_a : \mu \neq 2.0$

? Exercise 6.2.2

We want to test whether the mean height of eighth graders is 66 inches. State the null and alternative hypotheses. Fill in the correct symbol $(=, \neq, \geq, <, \leq, >)$ for the null and alternative hypotheses.

- $H_0 : \mu\_66$
- $H_a : \mu\_66$

Answer

- $H_0 : \mu = 66$
- $H_a : \mu \neq 66$

✓ Example 6.2.3

We want to test if college students take less than five years to graduate from college, on the average. The null and alternative hypotheses are:

- $H_0 : \mu \geq 5$
- $H_a : \mu < 5$

? Exercise 6.2.3

We want to test if it takes fewer than 45 minutes to teach a lesson plan. State the null and alternative hypotheses. Fill in the correct symbol (=, ≠, ≥, <, ≤, >) for the null and alternative hypotheses.

a. $H_0 : \mu\_45$
b. $H_a : \mu\_45$

Answer

a. $H_0 : \mu \geq 45$
b. $H_a : \mu < 45$

✓ Example 6.2.4

In an issue of *U. S. News and World Report*, an article on school standards stated that about half of all students in France, Germany, and Israel take advanced placement exams and a third pass. The same article stated that 6.6% of U.S. students take advanced placement exams and 4.4% pass. Test if the percentage of U.S. students who take advanced placement exams is more than 6.6%. State the null and alternative hypotheses.

- $H_0 : p \leq 0.066$
- $H_a : p > 0.066$

? Exercise 6.2.4

On a state driver's test, about 40% pass the test on the first try. We want to test if more than 40% pass on the first try. Fill in the correct symbol $(=, \neq, \geq, <, \leq, >)$ for the null and alternative hypotheses.

a. $H_0 : p\_0.40$
b. $H_a : p\_0.40$

Answer

a. $H_0 : p = 0.40$
b. $H_a : p > 0.40$

COLLABORATIVE EXERCISE

Bring to class a newspaper, some news magazines, and some Internet articles . In groups, find articles from which your group can write null and alternative hypotheses. Discuss your hypotheses with the rest of the class.

Review

In a **hypothesis test**, sample data is evaluated in order to arrive at a decision about some type of claim. If certain conditions about the sample are satisfied, then the claim can be evaluated for a population. In a hypothesis test, we:

1. Evaluate the **null hypothesis**, typically denoted with H_0. The null is not rejected unless the hypothesis test shows otherwise. The null statement must always contain some form of equality $(=, \leq \text{ or } \geq)$
2. Always write the **alternative hypothesis**, typically denoted with H_a or H_1, using less than, greater than, or not equals symbols, i.e., $(\neq, >, \text{or } <)$.
3. If we reject the null hypothesis, then we can assume there is enough evidence to support the alternative hypothesis.
4. Never state that a claim is proven true or false. Keep in mind the underlying fact that hypothesis testing is based on probability laws; therefore, we can talk only in terms of non-absolute certainties.

Formula Review

H_0 and H_a are contradictory.

If H_a has:	equal $(=)$	greater than or equal to $(\geq)$	less than or equal to $(\leq)$
then H_a has:	not equal $(\neq)$ **or** greater than $(>)$ **or** less than $(<)$	less than $(<)$	greater than $(>)$

- If $\alpha \leq p$-value, then do not reject H_0.
- If$\alpha > p$-value, then reject H_0.

α is preconceived. Its value is set before the hypothesis test starts. The p-value is calculated from the data.References

Data from the National Institute of Mental Health. Available online at http://www.nimh.nih.gov/publicat/depression.cfm.

Glossary

Hypothesis

a statement about the value of a population parameter, in case of two hypotheses, the statement assumed to be true is called the null hypothesis (notation H_0) and the contradictory statement is called the alternative hypothesis (notation H_a).

6.3: Outcomes and the Type I and Type II Errors

When you perform a hypothesis test, there are four possible outcomes depending on the actual truth (or falseness) of the null hypothesis H_0 and the decision to reject or not. The outcomes are summarized in the following table:

ACTION	H_0 is Actually True	H_0 is Actually False
Do not reject H_0	Correct Outcome	Type II error
Reject H_0	Type I Error	Correct Outcome

The four possible outcomes in the table are:

1. The decision is **not to reject** H_0 when H_0 **is true (correct decision).**
2. The decision is to **reject** H_0 when H_0 **is true** (incorrect decision known as aType I error).
3. The decision is **not to reject** H_0 when, in fact, H_0 **is false** (incorrect decision known as a Type II error).
4. The decision is to **reject** H_0 when H_0 **is false** (**correct decision** whose probability is called the **Power of the Test**).

Each of the errors occurs with a particular probability. The Greek letters α and β represent the probabilities.

- $\alpha =$ probability of a Type I error $= P(\text{Type I error}) =$ probability of rejecting the null hypothesis when the null hypothesis is true.
- $\beta =$ probability of a Type II error $= P(\text{Type II error}) =$ probability of not rejecting the null hypothesis when the null hypothesis is false.

α and β should be as small as possible because they are probabilities of errors. They are rarely zero.

The *Power of the Test* is $1-\beta$. Ideally, we want a high power that is as close to one as possible. Increasing the sample size can increase the Power of the Test. The following are examples of Type I and Type II errors.

✓ Example 6.3.1: Type I vs. Type II errors

Suppose the null hypothesis, H_0, is: Frank's rock climbing equipment is safe.

- **Type I error**: Frank thinks that his rock climbing equipment may not be safe when, in fact, it really is safe.
- **Type II error**: Frank thinks that his rock climbing equipment may be safe when, in fact, it is not safe.

$\alpha =$ **probability** that Frank thinks his rock climbing equipment may not be safe when, in fact, it really is safe.

$\beta =$ **probability** that Frank thinks his rock climbing equipment may be safe when, in fact, it is not safe.

Notice that, in this case, the error with the greater consequence is the Type II error. (If Frank thinks his rock climbing equipment is safe, he will go ahead and use it.)

? Exercise 6.3.1

Suppose the null hypothesis, H_0, is: the blood cultures contain no traces of pathogen X. State the Type I and Type II errors.

Answer

- **Type I error**: The researcher thinks the blood cultures do contain traces of pathogen X, when in fact, they do not.
- **Type II error**: The researcher thinks the blood cultures do not contain traces of pathogen X, when in fact, they do.

✓ Example 6.3.2

Suppose the null hypothesis, H_0, is: The victim of an automobile accident is alive when he arrives at the emergency room of a hospital.

- **Type I error**: The emergency crew thinks that the victim is dead when, in fact, the victim is alive.
- **Type II error**: The emergency crew does not know if the victim is alive when, in fact, the victim is dead.

$\alpha =$ **probability** that the emergency crew thinks the victim is dead when, in fact, he is really alive $= P(\text{Type I error})$.

$\beta =$ **probability** that the emergency crew does not know if the victim is alive when, in fact, the victim is dead $= P(\text{Type II error})$.

The error with the greater consequence is the Type I error. (If the emergency crew thinks the victim is dead, they will not treat him.)

? Exercise 6.3.2

Suppose the null hypothesis, H_0, is: a patient is not sick. Which type of error has the greater consequence, Type I or Type II?

Answer

The error with the greater consequence is the Type II error: the patient will be thought well when, in fact, he is sick, so he will not get treatment.

✓ Example 6.3.3

It's a Boy Genetic Labs claim to be able to increase the likelihood that a pregnancy will result in a boy being born. Statisticians want to test the claim. Suppose that the null hypothesis, H_0, is: It's a Boy Genetic Labs has no effect on gender outcome.

- **Type I error**: This results when a true null hypothesis is rejected. In the context of this scenario, we would state that we believe that It's a Boy Genetic Labs influences the gender outcome, when in fact it has no effect. The probability of this error occurring is denoted by the Greek letter alpha, α.
- **Type II error**: This results when we fail to reject a false null hypothesis. In context, we would state that It's a Boy Genetic Labs does not influence the gender outcome of a pregnancy when, in fact, it does. The probability of this error occurring is denoted by the Greek letter beta, β.

The error of greater consequence would be the Type I error since couples would use the It's a Boy Genetic Labs product in hopes of increasing the chances of having a boy.

? Exercise 6.3.3

"Red tide" is a bloom of poison-producing algae–a few different species of a class of plankton called dinoflagellates. When the weather and water conditions cause these blooms, shellfish such as clams living in the area develop dangerous levels of a paralysis-inducing toxin. In Massachusetts, the Division of Marine Fisheries (DMF) monitors levels of the toxin in shellfish by regular sampling of shellfish along the coastline. If the mean level of toxin in clams exceeds 800 μg (micrograms) of toxin per kg of clam meat in any area, clam harvesting is banned there until the bloom is over and levels of toxin in clams subside. Describe both a Type I and a Type II error in this context, and state which error has the greater consequence.

Answer

In this scenario, an appropriate null hypothesis would be H_0: the mean level of toxins is at most $800\mu g$, $H_0 : \mu_0 \leq 800\mu g$.

Type I error: The DMF believes that toxin levels are still too high when, in fact, toxin levels are at most $800\mu g$ The DMF continues the harvesting ban.

Type II error: The DMF believes that toxin levels are within acceptable levels (are at least 800 μg) when, in fact, toxin levels are still too high (more than $800\mu g$). The DMF lifts the harvesting ban. This error could be the most serious. If the ban is lifted and clams are still toxic, consumers could possibly eat tainted food.

In summary, the more dangerous error would be to commit a Type II error, because this error involves the availability of tainted clams for consumption.

✓ Example 6.3.4

A certain experimental drug claims a cure rate of at least 75% for males with prostate cancer. Describe both the Type I and Type II errors in context. Which error is the more serious?

- **Type I**: A cancer patient believes the cure rate for the drug is less than 75% when it actually is at least 75%.
- **Type II**: A cancer patient believes the experimental drug has at least a 75% cure rate when it has a cure rate that is less than 75%.

In this scenario, the Type II error contains the more severe consequence. If a patient believes the drug works at least 75% of the time, this most likely will influence the patient's (and doctor's) choice about whether to use the drug as a treatment option.

? Exercise $6.3.4$

Determine both Type I and Type II errors for the following scenario:

Assume a null hypothesis, H_0, that states the percentage of adults with jobs is at least 88%. Identify the Type I and Type II errors from these four statements.

a. Not to reject the null hypothesis that the percentage of adults who have jobs is at least 88% when that percentage is actually less than 88%
b. Not to reject the null hypothesis that the percentage of adults who have jobs is at least 88% when the percentage is actually at least 88%.
c. Reject the null hypothesis that the percentage of adults who have jobs is at least 88% when the percentage is actually at least 88%.
d. Reject the null hypothesis that the percentage of adults who have jobs is at least 88% when that percentage is actually less than 88%.

Answer

Type I error: c

Type II error: b

Summary

In every hypothesis test, the outcomes are dependent on a correct interpretation of the data. Incorrect calculations or misunderstood summary statistics can yield errors that affect the results. A **Type I** error occurs when a true null hypothesis is rejected. A **Type II error** occurs when a false null hypothesis is not rejected. The probabilities of these errors are denoted by the Greek letters α and β, for a Type I and a Type II error respectively. The power of the test, $1-\beta$, quantifies the likelihood that a test will yield the correct result of a true alternative hypothesis being accepted. A high power is desirable.

Formula Review

- $\alpha =$ probability of a Type I error $= P(\text{Type I error}) =$ probability of rejecting the null hypothesis when the null hypothesis is true.
- $\beta =$ probability of a Type II error $= P(\text{Type II error}) =$ probability of not rejecting the null hypothesis when the null hypothesis is false.

Glossary

Type 1 Error

The decision is to reject the null hypothesis when, in fact, the null hypothesis is true.

Type 2 Error

The decision is not to reject the null hypothesis when, in fact, the null hypothesis is false.

6.4: Distribution Needed for Hypothesis Testing

Earlier in the course, we discussed sampling distributions. **Particular distributions are associated with hypothesis testing.** Perform tests of a population mean using a normal distribution or a Student's t-distribution. (Remember, use a Student's t-distribution when the population standard deviation is unknown and the distribution of the sample mean is approximately normal.) We perform tests of a population proportion using a normal distribution (usually n is large or the sample size is large).

If you are testing a single population mean, the distribution for the test is for *means*:

$$\bar{X} \sim N\left(\mu_x, \frac{\sigma_x}{\sqrt{n}}\right) \tag{6.4.1}$$

or

$$t_{df} \tag{6.4.2}$$

The population parameter is μ. The estimated value (point estimate) for μ is $\bar{x}$, the sample mean.

If you are testing a single population proportion, the distribution for the test is for proportions or percentages:

$$P' \sim N\left(p, \sqrt{\frac{p-q}{n}}\right) \tag{6.4.3}$$

The population parameter is p. The estimated value (point estimate) for p is p'. $p' = \frac{x}{n}$ where x is the number of successes and n is the sample size.

Assumptions

When you perform a **hypothesis test of a single population mean** μ using a Student's t-distribution (often called a t-test), there are fundamental assumptions that need to be met in order for the test to work properly. Your data should be a simple random sample that comes from a population that is approximately normally distributed. You use the sample standard deviation to approximate the population standard deviation. (Note that if the sample size is sufficiently large, a t-test will work even if the population is not approximately normally distributed).

When you perform a **hypothesis test of a single population mean** μ using a normal distribution (often called a z-test), you take a simple random sample from the population. The population you are testing is normally distributed or your sample size is sufficiently large. You know the value of the population standard deviation which, in reality, is rarely known.

When you perform a **hypothesis test of a single population proportion** p, you take a simple random sample from the population. You must meet the conditions for a binomial distribution which are: there are a certain number n of independent trials, the outcomes of any trial are success or failure, and each trial has the same probability of a success p. The shape of the binomial distribution needs to be similar to the shape of the normal distribution. To ensure this, the quantities np and nq must both be greater than five $(np > 5$ and $nq > 5)$. Then the binomial distribution of a sample (estimated) proportion can be approximated by the normal distribution with $\mu = p$ and $\sigma = \sqrt{\frac{pq}{n}}$. Remember that $q = 1 - p$.

Summary

In order for a hypothesis test's results to be generalized to a population, certain requirements must be satisfied.

When testing for a single population mean:

1. A Student's t-test should be used if the data come from a simple, random sample and the population is approximately normally distributed, or the sample size is large, with an unknown standard deviation.
2. The normal test will work if the data come from a simple, random sample and the population is approximately normally distributed, or the sample size is large, with a known standard deviation.

When testing a single population proportion use a normal test for a single population proportion if the data comes from a simple, random sample, fill the requirements for a binomial distribution, and the mean number of successes and the mean number of failures satisfy the conditions: $np > 5$ and $nq > 5$ where n is the sample size, p is the probability of a success, and q is the probability of a failure.

Formula Review

If there is no given preconceived α, then use $\alpha = 0.05$.

Types of Hypothesis Tests

- Single population mean, **known** population variance (or standard deviation): **Normal test**.
- Single population mean, **unknown** population variance (or standard deviation): **Student's t-test**.
- Single population proportion: **Normal test**.
- For a **single population mean**, we may use a normal distribution with the following mean and standard deviation. Means: $\mu = \mu_{\bar{x}}$ and $sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}$
- A **single population proportion**, we may use a normal distribution with the following mean and standard deviation. Proportions: $\mu = p$ and $\sigma = \sqrt{\frac{pq}{n}}$.

Glossary

Binomial Distribution

a discrete random variable (RV) that arises from Bernoulli trials. There are a fixed number, n, of independent trials. “Independent” means that the result of any trial (for example, trial 1) does not affect the results of the following trials, and all trials are conducted under the same conditions. Under these circumstances the binomial RV X is defined as the number of successes in n trials. The notation is: $X \sim B(n, p) \mu = np$ and the standard deviation is $\sigma = \sqrt{npq}$. The probability of exactly x successes in n trials is $P(X = x) = \binom{n}{x} p^x q^{n-x}$.

Normal Distribution

a continuous random variable (RV) with pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, where μ is the mean of the distribution, and σ is the standard deviation, notation: $X \sim N(\mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, the RV is called **the standard normal distribution**.

Standard Deviation

a number that is equal to the square root of the variance and measures how far data values are from their mean; notation: s for sample standard deviation and σ for population standard deviation.

Student's *t*-Distribution

investigated and reported by William S. Gossett in 1908 and published under the pseudonym Student. The major characteristics of the random variable (RV) are:

- It is continuous and assumes any real values.
- The pdf is symmetrical about its mean of zero. However, it is more spread out and flatter at the apex than the normal distribution.
- It approaches the standard normal distribution as n gets larger.
- There is a "family" of t-distributions: every representative of the family is completely defined by the number of degrees of freedom which is one less than the number of data items.

6.5: Rare Events, the Sample, Decision and Conclusion

Establishing the type of distribution, sample size, and known or unknown standard deviation can help you figure out how to go about a hypothesis test. However, there are several other factors you should consider when working out a hypothesis test.

Rare Events

Suppose you make an assumption about a property of the population (this assumption is the null hypothesis). Then you gather sample data randomly. If the sample has properties that would be very *unlikely* to occur if the assumption is true, then you would conclude that your assumption about the population is probably incorrect. (Remember that your assumption is just an assumption—it is not a fact and it may or may not be true. But your sample data are real and the data are showing you a fact that seems to contradict your assumption.)

For example, Didi and Ali are at a birthday party of a very wealthy friend. They hurry to be first in line to grab a prize from a tall basket that they cannot see inside because they will be blindfolded. There are 200 plastic bubbles in the basket and Didi and Ali have been told that there is only one with a $100 bill. Didi is the first person to reach into the basket and pull out a bubble. Her bubble contains a $100 bill. The probability of this happening is $\frac{1}{200} = 0.005$. Because this is so unlikely, Ali is hoping that what the two of them were told is wrong and there are more $100 bills in the basket. A "rare event" has occurred (Didi getting the $100 bill) so Ali doubts the assumption about only one $100 bill being in the basket.

Using the Sample to Test the Null Hypothesis

Use the sample data to calculate the actual probability of getting the test result, called the p-value. The p-value is the probability that, if the null hypothesis is true, the results from another randomly selected sample will be as extreme or more extreme as the results obtained from the given sample.

A large p-value calculated from the data indicates that we should not reject the null hypothesis. The smaller the p-value, the more unlikely the outcome, and the stronger the evidence is against the null hypothesis. We would reject the null hypothesis if the evidence is strongly against it.

Draw a graph that shows the p-value. The hypothesis test is easier to perform if you use a graph because you see the problem more clearly.

✓ Example 6.5.1

Suppose a baker claims that his bread height is more than 15 cm, on average. Several of his customers do not believe him. To persuade his customers that he is right, the baker decides to do a hypothesis test. He bakes 10 loaves of bread. The mean height of the sample loaves is 17 cm. The baker knows from baking hundreds of loaves of bread that the standard deviation for the height is 0.5 cm. and the distribution of heights is normal.

- The null hypothesis could be $H_0 : \mu \leq 15$
- The alternate hypothesis is $H_a : \mu > 15$

The words **"is more than"** translates as a ">" so "$\mu > 15$" goes into the alternate hypothesis. The null hypothesis must contradict the alternate hypothesis.

Since σ **is known** ($\sigma = 0.5 cm.$), the distribution for the population is known to be normal with mean $\mu = 15$ and standard deviation

$$\frac{\sigma}{\sqrt{n}} = \frac{0.5}{\sqrt{10}} = 0.16.$$

Suppose the null hypothesis is true (the mean height of the loaves is no more than 15 cm). Then is the mean height (17 cm) calculated from the sample unexpectedly large? The hypothesis test works by asking the question how **unlikely** the sample mean would be if the null hypothesis were true. The graph shows how far out the sample mean is on the normal curve. The p-value is the probability that, if we were to take other samples, any other sample mean would fall at least as far out as 17 cm.

The p-value, then, is the probability that a sample mean is the same or greater than 17 cm. when the population mean is, in fact, 15 cm. We can calculate this probability using the normal distribution for means.

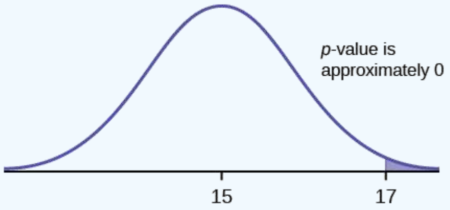

Figure 6.5.1

$p\text{-value} = P(\bar{x} > 17)$ which is approximately zero.

A p-value of approximately zero tells us that it is highly unlikely that a loaf of bread rises no more than 15 cm, on average. That is, almost 0% of all loaves of bread would be at least as high as 17 cm. **purely by CHANCE** had the population mean height really been 15 cm. Because the outcome of 17 cm. is so **unlikely (meaning it is happening NOT by chance alone)**, we conclude that the evidence is strongly against the null hypothesis (the mean height is at most 15 cm.). There is sufficient evidence that the true mean height for the population of the baker's loaves of bread is greater than 15 cm.

? Exercise 6.5.1

A normal distribution has a standard deviation of 1. We want to verify a claim that the mean is greater than 12. A sample of 36 is taken with a sample mean of 12.5.

- $H_0 : \mu \leq 12$
- $H_a : \mu > 12$

The p-value is 0.0013

Draw a graph that shows the p-value.

Answer

$p\text{-value} = 0.0013$

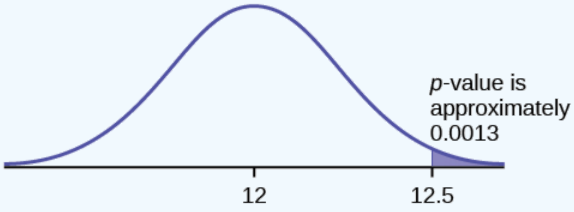

Figure 6.5.2

Decision and Conclusion

A systematic way to make a decision of whether to reject or not reject the null hypothesis is to compare the p-value and a preset or preconceived α (also called a "**significance level**"). A preset α is the probability of a Type I error (rejecting the null hypothesis when the null hypothesis is true). It may or may not be given to you at the beginning of the problem.

When you make a decision to reject or not reject H_0, do as follows:

- If $\alpha > p\text{-value}$, reject H_0. The results of the sample data are significant. There is sufficient evidence to conclude that H_0 is an incorrect belief and that the alternative hypothesis, H_a, may be correct.
- If $\alpha \leq p\text{-value}$, do not reject H_0. The results of the sample data are not significant.There is not sufficient evidence to conclude that the alternative hypothesis,H_a, may be correct.

When you "do not reject H_0", it does not mean that you should believe that H_0 is true. It simply means that the sample data have failed to provide sufficient evidence to cast serious doubt about the truthfulness of H_0.

Conclusion: After you make your decision, write a thoughtful conclusion about the hypotheses in terms of the given problem.

Example 6.5.2

When using the p-value to evaluate a hypothesis test, it is sometimes useful to use the following memory device

- If the p-value is low, the null must go.
- If the p-value is high, the null must fly.

This memory aid relates a p-value less than the established alpha (the p is low) as rejecting the null hypothesis and, likewise, relates a p-value higher than the established alpha (the p is high) as not rejecting the null hypothesis.

Fill in the blanks.

Reject the null hypothesis when ____.

The results of the sample data ____.

Do not reject the null when hypothesis when ____.

The results of the sample data ____.

Answer

Reject the null hypothesis when **the p-value is less than the established alpha value**. The results of the sample data **support the alternative hypothesis**.

Do not reject the null hypothesis when **the p-value is greater than the established alpha value**. The results of the sample data **do not support the alternative hypothesis**.

Exercise 6.5.2

It's a Boy Genetics Labs claim their procedures improve the chances of a boy being born. The results for a test of a single population proportion are as follows:

- $H_0 : p = 0.50, H_a : p > 0.50$
- $\alpha = 0.01$
- $p\text{-value} = 0.025$

Interpret the results and state a conclusion in simple, non-technical terms.

Answer

Since the p-value is greater than the established alpha value (the p-value is high), we do not reject the null hypothesis. There is not enough evidence to support It's a Boy Genetics Labs' stated claim that their procedures improve the chances of a boy being born.

Review

When the probability of an event occurring is low, and it happens, it is called a rare event. Rare events are important to consider in hypothesis testing because they can inform your willingness not to reject or to reject a null hypothesis. To test a null hypothesis, find the p-value for the sample data and graph the results. When deciding whether or not to reject the null the hypothesis, keep these two parameters in mind:

- $\alpha > p - value$, reject the null hypothesis
- $\alpha \leq p - value$, do not reject the null hypothesis

Glossary

Level of Significance of the Test

probability of a Type I error (reject the null hypothesis when it is true). Notation: α. In hypothesis testing, the Level of Significance is called the preconceived α or the preset α.

p-value

the probability that an event will happen purely by chance assuming the null hypothesis is true. The smaller the p-value, the stronger the evidence is against the null hypothesis.

6.6: Additional Information and Full Hypothesis Test Examples

- In a hypothesis test problem, you may see words such as "the level of significance is 1%." The "1%" is the preconceived or preset α.
- The statistician setting up the hypothesis test selects the value of α to use before collecting the sample data.
- If no level of significance is given, a common standard to use is $\alpha = 0.05$.
- When you calculate the p-value and draw the picture, the p-value is the area in the left tail, the right tail, or split evenly between the two tails. For this reason, we call the hypothesis test left, right, or two tailed.
- The alternative hypothesis, H_a, tells you if the test is left, right, or two-tailed. It is the key to conducting the appropriate test.
- H_a never has a symbol that contains an equal sign.
- Thinking about the meaning of the p-value: A data analyst (and anyone else) should have more confidence that he made the correct decision to reject the null hypothesis with a smaller p-value (for example, 0.001 as opposed to 0.04) even if using the 0.05 level for alpha. Similarly, for a large p-value such as 0.4, as opposed to a p-value of 0.056 ($\alpha = 0.05$ is less than either number), a data analyst should have more confidence that she made the correct decision in not rejecting the null hypothesis. This makes the data analyst use judgment rather than mindlessly applying rules.

The following examples illustrate a left-, right-, and two-tailed test.

✓ Example 6.6.1

$H_0 : \mu = 5, H_a : \mu < 5$

Test of a single population mean. H_a tells you the test is left-tailed. The picture of the p-value is as follows:

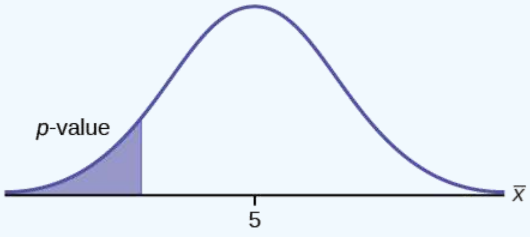

Figure 6.6.1

? Exercise 6.6.1

$H_0 : \mu = 10, H_a : \mu < 10$

Assume the p-value is 0.0935. What type of test is this? Draw the picture of the p-value.

Answer

left-tailed test

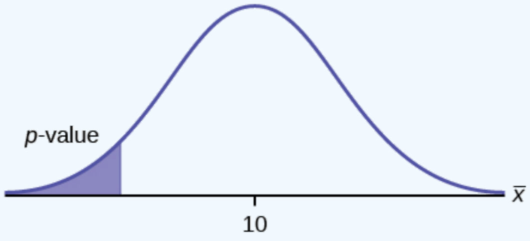

Figure 6.6.2

✓ Example 6.6.2

$H_0 : \mu \leq 0.2, H_a : \mu > 0.2$

This is a test of a single population proportion. H_a tells you the test is **right-tailed**. The picture of the p-value is as follows:

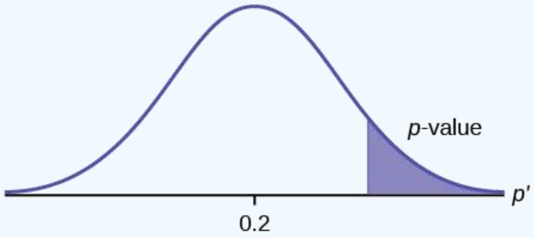

Figure 6.6.3

? Exercise 6.6.2

$H_0 : \mu \leq 1, H_a : \mu > 1$

Assume the p-value is 0.1243. What type of test is this? Draw the picture of the p-value.

Answer

right-tailed test

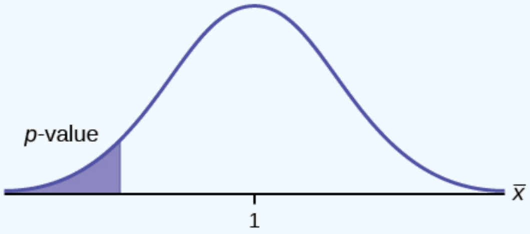

Figure 6.6.4

✓ Example 6.6.3

$H_0 : \mu = 50, H_a : \mu \neq 50$

This is a test of a single population mean. H_a tells you the test is **two-tailed**. The picture of the p-value is as follows.

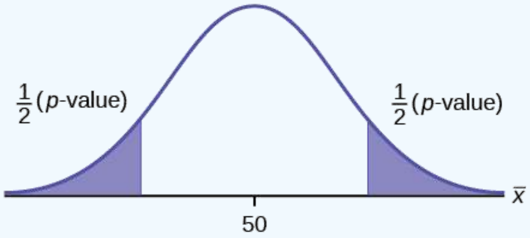

Figure 6.6.5

? Exercise 6.6.3

$H_0 : \mu = 0.5, H_a : \mu \neq 0.5$

Assume the p-value is 0.2564. What type of test is this? Draw the picture of the p-value.

Answer

two-tailed test

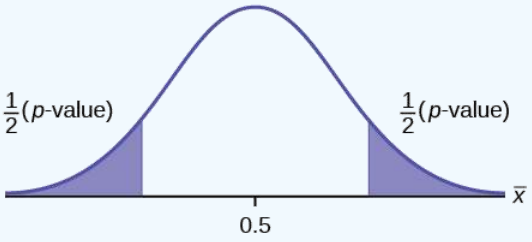

Figure 6.6.6

Full Hypothesis Test Examples

Example 6.6.4

Jeffrey, as an eight-year old, **established a mean time of 16.43 seconds** for swimming the 25-yard freestyle, with a **standard deviation of 0.8 seconds**. His dad, Frank, thought that Jeffrey could swim the 25-yard freestyle faster using goggles. Frank bought Jeffrey a new pair of expensive goggles and timed Jeffrey for **15 25-yard freestyle swims**. For the 15 swims, **Jeffrey's mean time was 16 seconds. Frank thought that the goggles helped Jeffrey to swim faster than the 16.43 seconds.**Conduct a hypothesis test using a preset α = 0.05. Assume that the swim times for the 25-yard freestyle are normal.

Answer

Set up the Hypothesis Test:

Since the problem is about a mean, this is a **test of a single population mean**.

$H_0 : \mu = 16.43, H_a : \mu < 16.43$

For Jeffrey to swim faster, his time will be less than 16.43 seconds. The "<" tells you this is left-tailed.

Determine the distribution needed:

Random variable: $\bar{X} =$ the mean time to swim the 25-yard freestyle.

Distribution for the test: $\bar{X}$ is normal (population standard deviation is known: $\sigma = 0.8$)

$\bar{X} - N\left(\mu, \frac{\sigma_x}{\sqrt{n}}\right)$ Therefore, $\bar{X} - N\left(16.43, \frac{0.8}{\sqrt{15}}\right)$

$\mu = 16.43$ comes from H_0 and not the data. $\sigma = 0.8$, and $n = 15$.

Calculate the $p - \text{value}$ using the normal distribution for a mean:

$p\text{-value} = P(\bar{x} < 16) = 0.0187$ where the sample mean in the problem is given as 16.

$p\text{-value} = 0.0187$ (This is called the **actual level of significance**.) The $p - \text{value}$ is the area to the left of the sample mean is given as 16.

Graph:

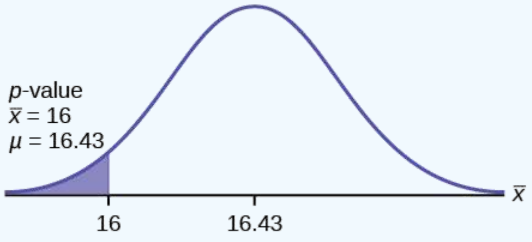

Figure 6.6.7

$\mu = 16.43$ comes from H_0. Our assumption is $\mu = 16.43$.

Interpretation of the $\boldsymbol{p} - \text{value}$: If $\boldsymbol{H_0}$ is true, there is a 0.0187 probability (1.87%) that Jeffrey's mean time to swim the 25-yard freestyle is 16 seconds or less. Because a 1.87% chance is small, the mean time of 16 seconds or less is unlikely to have happened randomly. It is a rare event.

Compare α and the $p-$value:

$\alpha = 0.05 p\text{-value} = 0.0187 \alpha > p\text{-value}$

Make a decision: Since $\alpha > p\text{-value}$, reject H_0.

This means that you reject $\mu = 16.43$. In other words, you do not think Jeffrey swims the 25-yard freestyle in 16.43 seconds but faster with the new goggles.

Conclusion: At the 5% significance level, we conclude that Jeffrey swims faster using the new goggles. The sample data show there is sufficient evidence that Jeffrey's mean time to swim the 25-yard freestyle is less than 16.43 seconds.

The *p*-value can easily be calculated.

Press `STAT` and arrow over to `TESTS` . Press `1:Z-Test` . Arrow over to `Stats` and press `ENTER` . Arrow down and enter 16.43 for μ_0 (null hypothesis), .8 for *σ*, 16 for the sample mean, and 15 for *n*. Arrow down to μ : (alternate hypothesis) and arrow over to $< \mu_0$. Press `ENTER` . Arrow down to `Calculate` and press `ENTER` . The calculator not only calculates the *p*-value ($p = 0.0187$) but it also calculates the test statistic (*z*-score) for the sample mean. $\mu < 16.43$ is the alternative hypothesis. Do this set of instructions again except arrow to `Draw` (instead of `Calculate`). Press `ENTER` . A shaded graph appears with $z = -2.08$ (test statistic) and $p = 0.0187$ $(p-\text{value})$. Make sure when you use `Draw` that no other equations are highlighted in $Y =$ and the plots are turned off.

When the calculator does a Z-Test, the `Z-Test` function finds the *p*-value by doing a normal probability calculation using the central limit theorem:

$P(\bar{X} < 16)$ `2nd DISTR normcdf` $((-10^{99}, 16, 16.43, \frac{0.8}{\sqrt{15}})$

The Type I and Type II errors for this problem are as follows:

The Type I error is to conclude that Jeffrey swims the 25-yard freestyle, on average, in less than 16.43 seconds when, in fact, he actually swims the 25-yard freestyle, on average, in 16.43 seconds. (Reject the null hypothesis when the null hypothesis is true.)

The Type II error is that there is not evidence to conclude that Jeffrey swims the 25-yard free-style, on average, in less than 16.43 seconds when, in fact, he actually does swim the 25-yard free-style, on average, in less than 16.43 seconds. (Do not reject the null hypothesis when the null hypothesis is false.)

? Exercise 6.6.4

The mean throwing distance of a football for a Marco, a high school freshman quarterback, is 40 yards, with a standard deviation of two yards. The team coach tells Marco to adjust his grip to get more distance. The coach records the distances for 20 throws. For the 20 throws, Marco’s mean distance was 45 yards. The coach thought the different grip helped Marco throw farther than 40 yards. Conduct a hypothesis test using a preset $\alpha = 0.05$. Assume the throw distances for footballs are normal.

First, determine what type of test this is, set up the hypothesis test, find the *p*-value, sketch the graph, and state your conclusion.

Press STAT and arrow over to TESTS. Press 1: Z-Test. Arrow over to Stats and press ENTER. Arrow down and enter 40 for μ_0 (null hypothesis), 2 for σ, 45 for the sample mean, and 20 for n. Arrow down to μ: (alternative hypothesis) and set it either as $<$, $\neq$, or $>$. Press ENTER. Arrow down to Calculate and press ENTER. The calculator not only calculates the *p*-value but it also calculates the test statistic (*z*-score) for the sample mean. Select $<$, $\neq$, or $>$ for the alternative hypothesis. Do this set of instructions again except arrow to Draw (instead of Calculate). Press ENTER. A shaded graph appears with test statistic and *p*-value. Make sure when you use Draw that no other equations are highlighted in $Y =$ and the plots are turned off.

Answer

Since the problem is about a mean, this is a test of a single population mean.

- $H_0 : \mu = 40$
- $H_a : \mu > 40$
- $p = 0.0062$

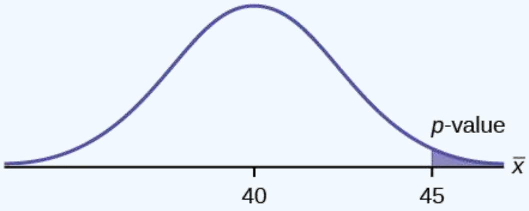

Figure 6.6.8

Because $p < \alpha$, we reject the null hypothesis. There is sufficient evidence to suggest that the change in grip improved Marco's throwing distance.

Historical Note

The traditional way to compare the two probabilities, α and the $p-\text{value}$, is to compare the critical value (z-score from α) to the test statistic (z-score from data). The calculated test statistic for the p-value is –2.08. (From the Central Limit Theorem, the test statistic formula is $z = \frac{\bar{x}-\mu_x}{\left(\frac{\sigma_x}{\sqrt{n}}\right)}$. For this problem, $\bar{x} = 16$, $\mu_x = 16.43$ from the null hypotheses is, $\sigma_x = 0.8$, and $n = 15$.) You can find the critical value for $\alpha = 0.05$ in the normal table (see **15.Tables** in the Table of Contents). The z-score for an area to the left equal to 0.05 is midway between –1.65 and –1.64 (0.05 is midway between 0.0505 and 0.0495). The z-score is –1.645. Since –1.645 > –2.08 (which demonstrates that $\alpha > p-\text{value}$), reject H_0. Traditionally, the decision to reject or not reject was done in this way. Today, comparing the two probabilities α and the p-value is very common. For this problem, the $p-\text{value}$, 0.0187 is considerably smaller than $\alpha = 0.05$. You can be confident about your decision to reject. The graph shows α, the $p-\text{value}$, and the test statistics and the critical value.

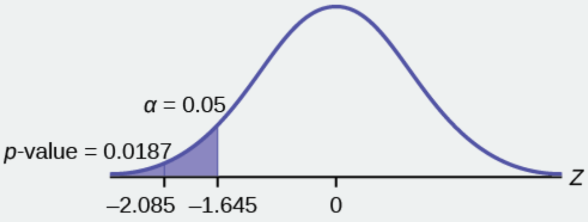

Figure 6.6.9

Example 6.6.5

A college football coach thought that his players could bench press a **mean weight of 275 pounds**. It is known that the **standard deviation is 55 pounds**. Three of his players thought that the mean weight was **more than** that amount. They asked **30** of their teammates for their estimated maximum lift on the bench press exercise. The data ranged from 205 pounds to 385 pounds. The actual different weights were (frequencies are in parentheses) 205(3); 215(3); 225(1); 241(2); 252(2); 265(2); 275(2); 313(2); 316(5); 338(2); 341(1); 345(2); 368(2); 385(1).

Conduct a hypothesis test using a 2.5% level of significance to determine if the bench press mean is more than 275 pounds.

Answer

Set up the Hypothesis Test:

Since the problem is about a mean weight, this is a test of a single population mean.

- $H_0 : \mu = 275$
- $H_a : \mu > 275$

This is a right-tailed test.

Calculating the distribution needed:

Random variable: $\bar{X} =$ the mean weight, in pounds, lifted by the football players.

Distribution for the test: It is normal because σ is known.

- $\bar{X} - N\left(275, \frac{55}{\sqrt{30}}\right)$
- $\bar{x} = 286.2$ pounds (from the data).
- $\sigma = 55$ pounds **(Always use σ if you know it.)** We assume $\mu = 275$ pounds unless our data shows us otherwise.

Calculate the p-value using the normal distribution for a mean and using the sample mean as input (see [link] for using the data as input):

$$p\text{-value} = P(\bar{x} > 286.2) = 0.1323.$$

Interpretation of the p-value: If H_0 is true, then there is a 0.1331 probability (13.23%) that the football players can lift a mean weight of 286.2 pounds or more. Because a 13.23% chance is large enough, a mean weight lift of 286.2 pounds or more is not a rare event.

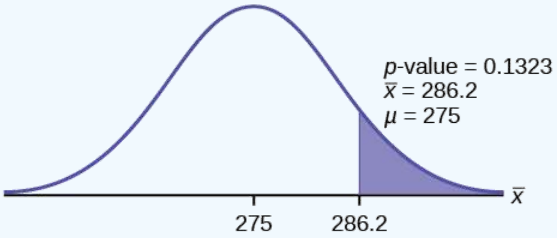

Figure 6.6.10

Compare α and the $p-\text{value}$:

$\alpha = 0.025 p-value = 0.1323$

Make a decision: Since $\alpha < p\text{-value}$, do not reject H_0.

Conclusion: At the 2.5% level of significance, from the sample data, there is not sufficient evidence to conclude that the true mean weight lifted is more than 275 pounds.

The $p-\text{value}$ can easily be calculated.

Put the data and frequencies into lists. Press `STAT` and arrow over to `TESTS`. Press `1:Z-Test`. Arrow over to `Data` and press `ENTER`. Arrow down and enter 275 for μ_0, 55 for σ, the name of the list where you put the data, and the name of the list where you put the frequencies. Arrow down to μ: and arrow over to $> \mu_0$. Press `ENTER`. Arrow down to `Calculate` and press `ENTER`. The calculator not only calculates the $p-\text{value}$ ($p = 0.1331$), a little different from the previous calculation - in it we used the sample mean rounded to one decimal place instead of the data) but it also calculates the test statistic (z-score) for the sample mean, the sample mean, and the sample standard deviation. $\mu > 275$ is the alternative hypothesis. Do this set of instructions again except arrow to `Draw` (instead of `Calculate`). Press `ENTER`. A shaded graph appears with $z = 1.112$ (test statistic) and $p = 0.1331$ ($p-\text{value}$). Make sure when you use `Draw` that no other equations are highlighted in $Y =$ and the plots are turned off.

✓ Example 6.6.6

Statistics students believe that the mean score on the first statistics test is 65. A statistics instructor thinks the mean score is higher than 65. He samples ten statistics students and obtains the scores 65 65 70 67 66 63 63 68 72 71. He performs a hypothesis test using a 5% level of significance. The data are assumed to be from a normal distribution.

Answer

Set up the hypothesis test:

A 5% level of significance means that $\alpha = 0.05$. This is a test of a **single population mean**.

$H_0 : \mu = 65 \qquad H_a : \mu > 65$

Since the instructor thinks the average score is higher, use a ">". The ">" means the test is right-tailed.

Determine the distribution needed:

Random variable: $\bar{X} =$ average score on the first statistics test.

Distribution for the test: If you read the problem carefully, you will notice that there is **no population standard deviation given**. You are only given $n = 10$ sample data values. Notice also that the data come from a normal distribution. This means that the distribution for the test is a student's t.

Use t_{df}. Therefore, the distribution for the test is t_9 where $n = 10$ and $df = 10 - 1 = 9$.

Calculate the p-value using the Student's t-distribution:

p-value $= P(\bar{x} > 67) = 0.0396$ where the sample mean and sample standard deviation are calculated as 67 and 3.1972 from the data.

Interpretation of the *p*-value: If the null hypothesis is true, then there is a 0.0396 probability (3.96%) that the sample mean is 65 or more.

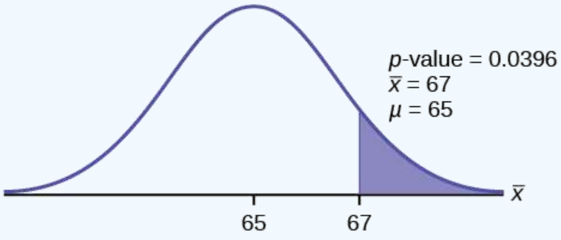

Figure 6.6.11

Compare α and the $p-$value:

Since $\alpha = 0.05$ and p-value $= 0.0396$. $\alpha > p$-value.

Make a decision: Since $\alpha > p$-value, reject H_0.

This means you reject $\mu = 65$. In other words, you believe the average test score is more than 65.

Conclusion: At a 5% level of significance, the sample data show sufficient evidence that the mean (average) test score is more than 65, just as the math instructor thinks.

The p-value can easily be calculated.

Put the data into a list. Press `STAT` and arrow over to `TESTS` . Press `2:T-Test` . Arrow over to `Data` and press `ENTER` . Arrow down and enter 65 for μ_0, the name of the list where you put the data, and 1 for `Freq:` . Arrow down to μ: and arrow over to $> \mu_0$. Press `ENTER` . Arrow down to `Calculate` and press `ENTER` . The calculator not only calculates the p-value (p = 0.0396) but it also calculates the test statistic (*t*-score) for the sample mean, the sample mean, and the sample standard deviation. $\mu > 65$ is the alternative hypothesis. Do this set of instructions again except arrow to `Draw` (instead of `Calculate`). Press `ENTER` . A shaded graph appears with $t = 1.9781$ (test statistic) and $p = 0.0396$ (p-value). Make sure when you use `Draw` that no other equations are highlighted in $Y =$ and the plots are turned off.

? Exercise 6.6.6

It is believed that a stock price for a particular company will grow at a rate of $5 per week with a standard deviation of $1. An investor believes the stock won’t grow as quickly. The changes in stock price is recorded for ten weeks and are as follows: $4, $3, $2, $3, $1, $7, $2, $1, $1, $2. Perform a hypothesis test using a 5% level of significance. State the null and alternative hypotheses, find the p-value, state your conclusion, and identify the Type I and Type II errors.

Answer

- $H_0 : \mu = 5$
- $H_a : \mu < 5$
- $p = 0.0082$

Because $p < \alpha$, we reject the null hypothesis. There is sufficient evidence to suggest that the stock price of the company grows at a rate less than $5 a week.

- Type I Error: To conclude that the stock price is growing slower than $5 a week when, in fact, the stock price is growing at $5 a week (reject the null hypothesis when the null hypothesis is true).

- Type II Error: To conclude that the stock price is growing at a rate of \$5 a week when, in fact, the stock price is growing slower than \$5 a week (do not reject the null hypothesis when the null hypothesis is false).

Example 6.6.7

Joon believes that 50% of first-time brides in the United States are younger than their grooms. She performs a hypothesis test to determine if the percentage is **the same or different from 50%**. Joon samples **100 first-time brides** and **53** reply that they are younger than their grooms. For the hypothesis test, she uses a 1% level of significance.

Answer

Set up the hypothesis test:

The 1% level of significance means that α = 0.01. This is a **test of a single population proportion**.

$H_0 : p = 0.50 \qquad H_a : p \neq 0.50$

The words **"is the same or different from"** tell you this is a two-tailed test.

Calculate the distribution needed:

Random variable: $P\,'=$ the percent of of first-time brides who are younger than their grooms.

Distribution for the test: The problem contains no mention of a mean. The information is given in terms of percentages. Use the distribution for P', the estimated proportion.

$$P' - N\left(p, \sqrt{\frac{p-q}{n}}\right)$$

Therefore,

$$P' - N\left(0.5, \sqrt{\frac{0.5-0.5}{100}}\right)$$

where $p = 0.50, q = 1 - p = 0.50$, and $n = 100$

Calculate the p-value using the normal distribution for proportions:

$$p\text{-value} = P(p\,' < 0.47 \; or \; p\,' > 0.53) = 0.5485$$

where

$$x = 53, p' = \frac{x}{n} = \frac{53}{100} = 0.53$$

.

Interpretation of the p-value: If the null hypothesis is true, there is 0.5485 probability (54.85%) that the sample (estimated) proportion p' is 0.53 or more OR 0.47 or less (see the graph in Figure).

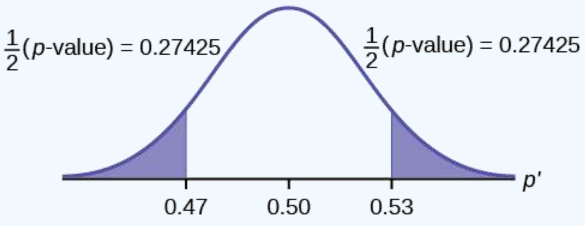

Figure 6.6.12

$\mu = p = 0.50$ comes from H_0, the null hypothesis.

$p\,' = 0.53$. Since the curve is symmetrical and the test is two-tailed, the $p\,'$ for the left tail is equal to $0.50 - 0.03 = 0.47$where $\mu = p = 0.50$. (0.03 is the difference between 0.53 and 0.50.)

Compare α and the p-value:

Since $\alpha = 0.01$ and p-value $= 0.5485$. $\alpha < p$-value.

Make a decision: Since $\alpha < p$-value, you cannot reject H_0.

Conclusion: At the 1% level of significance, the sample data do not show sufficient evidence that the percentage of first-time brides who are younger than their grooms is different from 50%.

The p-value can easily be calculated.

Press `STAT` and arrow over to `TESTS`. Press `5:1-PropZTest`. Enter .5 for p_0, 53 for x and 100 for n. Arrow down to `Prop` and arrow to `not equals` p_0. Press `ENTER`. Arrow down to `Calculate` and press `ENTER`. The calculator calculates the p-value ($p = 0.5485$) and the test statistic (z-score). `Prop not equals` .5 is the alternate hypothesis. Do this set of instructions again except arrow to `Draw` (instead of `Calculate`). Press `ENTER`. A shaded graph appears with $z = 0.6$ (test statistic) and $p = 0.5485$ (p-value). Make sure when you use `Draw` that no other equations are highlighted in $Y =$ and the plots are turned off.

The Type I and Type II errors are as follows:

The Type I error is to conclude that the proportion of first-time brides who are younger than their grooms is different from 50% when, in fact, the proportion is actually 50%. (Reject the null hypothesis when the null hypothesis is true).

The Type II error is there is not enough evidence to conclude that the proportion of first time brides who are younger than their grooms differs from 50% when, in fact, the proportion does differ from 50%. (Do not reject the null hypothesis when the null hypothesis is false.)

? Exercise 6.6.7

A teacher believes that 85% of students in the class will want to go on a field trip to the local zoo. She performs a hypothesis test to determine if the percentage is the same or different from 85%. The teacher samples 50 students and 39 reply that they would want to go to the zoo. For the hypothesis test, use a 1% level of significance.

First, determine what type of test this is, set up the hypothesis test, find the p-value, sketch the graph, and state your conclusion.

Answer

Since the problem is about percentages, this is a test of single population proportions.

- $H_0 : p = 0.85$
- $H_a : p \neq 0.85$
- $p = 0.7554$

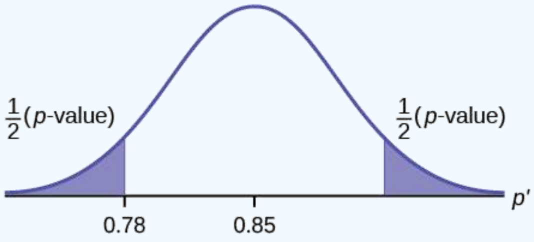

Figure 6.6.13

Because $p > \alpha$, we fail to reject the null hypothesis. There is not sufficient evidence to suggest that the proportion of students that want to go to the zoo is not 85%.

✓ Example 6.6.8

Suppose a consumer group suspects that the proportion of households that have three cell phones is 30%. A cell phone company has reason to believe that the proportion is not 30%. Before they start a big advertising campaign, they conduct a hypothesis test. Their marketing people survey 150 households with the result that 43 of the households have three cell phones.

Answer

Set up the Hypothesis Test:

$H_0 : p = 0.30, H_a : p \neq 0.30$

Determine the distribution needed:

The **random variable** is $P' =$ proportion of households that have three cell phones.

The **distribution** for the hypothesis test is $P' - N\left(0.30, \sqrt{\frac{(0.30 \cdot 0.70)}{150}}\right)$

Exercise 6.6.8.2

a. The value that helps determine the p-value is p'. Calculate p'.

Answer

a. $p' = \frac{x}{n}$ where x is the number of successes and n is the total number in the sample.

$x = 43, n = 150$

$p' = 43150$

Exercise 6.6.8.3

b. What is a **success** for this problem?

Answer

b. A success is having three cell phones in a household.

Exercise 6.6.8.4

c. What is the level of significance?

Answer

c. The level of significance is the preset α. Since α is not given, assume that $\alpha = 0.05$.

Exercise 6.6.8.5

d. Draw the graph for this problem. Draw the horizontal axis. Label and shade appropriately.

Calculate the p-value.

Answer

d. $p\text{-value} = 0.7216$

Exercise 6.6.8.6

e. Make a decision. _____________(Reject/Do not reject) H_0 because____________.

Answer

e. Assuming that $\alpha = 0.05, \alpha < p$-value. The decision is do not reject H_0 because there is not sufficient evidence to conclude that the proportion of households that have three cell phones is not 30%.

Exercise 6.6.8

Marketers believe that 92% of adults in the United States own a cell phone. A cell phone manufacturer believes that number is actually lower. 200 American adults are surveyed, of which, 174 report having cell phones. Use a 5% level of significance. State the null and alternative hypothesis, find the p-value, state your conclusion, and identify the Type I and Type II errors.

Answer

- $H_0 : p = 0.92$
- $H_a : p < 0.92$
- $p\text{-value} = 0.0046$

Because $p < 0.05$, we reject the null hypothesis. There is sufficient evidence to conclude that fewer than 92% of American adults own cell phones.

- Type I Error: To conclude that fewer than 92% of American adults own cell phones when, in fact, 92% of American adults do own cell phones (reject the null hypothesis when the null hypothesis is true).
- Type II Error: To conclude that 92% of American adults own cell phones when, in fact, fewer than 92% of American adults own cell phones (do not reject the null hypothesis when the null hypothesis is false).

The next example is a poem written by a statistics student named Nicole Hart. The solution to the problem follows the poem. Notice that the hypothesis test is for a single population proportion. This means that the null and alternate hypotheses use the parameter p. The distribution for the test is normal. The estimated proportion p' is the proportion of fleas killed to the total fleas found on Fido. This is sample information. The problem gives a preconceived $\alpha = 0.01$, for comparison, and a 95% confidence interval computation. The poem is clever and humorous, so please enjoy it!

✓ Example 6.6.9

My dog has so many fleas,

They do not come off with ease.
As for shampoo, I have tried many types
Even one called Bubble Hype,
Which only killed 25% of the fleas,
Unfortunately I was not pleased.

I've used all kinds of soap,
Until I had given up hope
Until one day I saw
An ad that put me in awe.

A shampoo used for dogs
Called GOOD ENOUGH to Clean a Hog
Guaranteed to kill more fleas.

I gave Fido a bath
And after doing the math
His number of fleas
Started dropping by 3's!
Before his shampoo
I counted 42.

At the end of his bath,
I redid the math
And the new shampoo had killed 17 fleas.
So now I was pleased.

Now it is time for you to have some fun
With the level of significance being .01,
You must help me figure out

Use the new shampoo or go without?

Answer

Set up the hypothesis test:

$H_0 : p \leq 0.25 \qquad H_a : p > 0.25$

Determine the distribution needed:

In words, CLEARLY state what your random variable $\bar{X}$ or P' represents.

$P' =$ The proportion of fleas that are killed by the new shampoo

State the distribution to use for the test.

Normal:

$$N\left(0.25, \sqrt{\frac{(0.25)1-0.25}{42}}\right)$$

Test Statistic: $z = 2.3163$

Calculate the p-value using the normal distribution for proportions:

$$p\text{-value} = 0.0103$$

In one to two complete sentences, explain what the p-value means for this problem.

If the null hypothesis is true (the proportion is 0.25), then there is a 0.0103 probability that the sample (estimated) proportion is 0.4048 $\left(\frac{17}{42}\right)$ or more.

Use the previous information to sketch a picture of this situation. CLEARLY, label and scale the horizontal axis and shade the region(s) corresponding to the p-value.

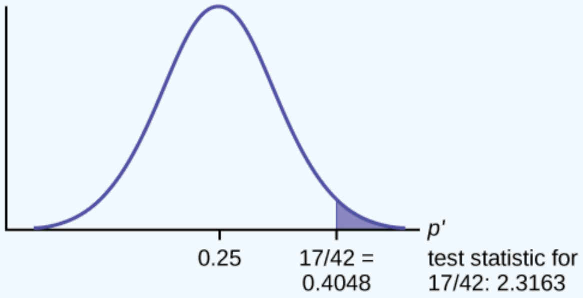

Figure 6.6.14

Compare α and the p-value:

Indicate the correct decision ("reject" or "do not reject" the null hypothesis), the reason for it, and write an appropriate conclusion, using complete sentences.

alpha	decision	reason for decision
0.01	Do not reject H_0	$\alpha < p$-value

Conclusion: At the 1% level of significance, the sample data do not show sufficient evidence that the percentage of fleas that are killed by the new shampoo is more than 25%.

Construct a 95% confidence interval for the true mean or proportion. Include a sketch of the graph of the situation. Label the point estimate and the lower and upper bounds of the confidence interval.

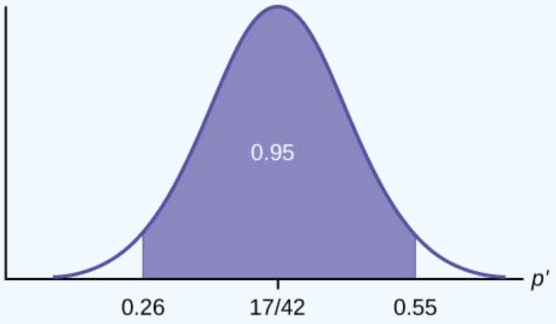

Figure 6.6.15

Confidence Interval: (0.26,0.55) We are 95% confident that the true population proportion *p* of fleas that are killed by the new shampoo is between 26% and 55%.

This test result is not very definitive since the p-value *is very close to alpha. In reality, one would probably do more tests by giving the dog another bath after the fleas have had a chance to return.*

✓ Example 6.6.10

The National Institute of Standards and Technology provides exact data on conductivity properties of materials. Following are conductivity measurements for 11 randomly selected pieces of a particular type of glass.

1.11; 1.07; 1.11; 1.07; 1.12; 1.08; .98; .98 1.02; .95; .95

Is there convincing evidence that the average conductivity of this type of glass is greater than one? Use a significance level of 0.05. Assume the population is normal.

Answer

Let's follow a four-step process to answer this statistical question.

1. **State the Question**: We need to determine if, at a 0.05 significance level, the average conductivity of the selected glass is greater than one. Our hypotheses will be
   a. $H_0 : \mu \leq 1$
   b. $H_a : \mu > 1$
2. **Plan**: We are testing a sample mean without a known population standard deviation. Therefore, we need to use a Student's-t distribution. Assume the underlying population is normal.
3. **Do the calculations**: We will input the sample data into the TI-83 as follows.

alt

Figure 6.6.7.

Figure 6.6.8.

Figure 6.6.9.

alt

Figure 6.6.10.

4. **State the Conclusions**: Since the p-value$(p = 0.036)$ is less than our alpha value, we will reject the null hypothesis. It is reasonable to state that the data supports the claim that the average conductivity level is greater than one.

✓ Example 6.6.11

In a study of 420,019 cell phone users, 172 of the subjects developed brain cancer. Test the claim that cell phone users developed brain cancer at a greater rate than that for non-cell phone users (the rate of brain cancer for non-cell phone users is 0.0340%). Since this is a critical issue, use a 0.005 significance level. Explain why the significance level should be so low in terms of a Type I error.

Answer

We will follow the four-step process.

1. We need to conduct a hypothesis test on the claimed cancer rate. Our hypotheses will be
   a. $H_0 : p \leq 0.00034$
   b. $H_a : p > 0.00034$

   If we commit a Type I error, we are essentially accepting a false claim. Since the claim describes cancer-causing environments, we want to minimize the chances of incorrectly identifying causes of cancer.
2. We will be testing a sample proportion with $x = 172$ and $n = 420,019$. The sample is sufficiently large because we have $np = 420,019(0.00034) = 142.8$ $nq = 420,019(0.99966) = 419,876.2$two independent outcomes, and a fixed probability of success $p = 0.00034$. Thus we will be able to generalize our results to the population.
3. The associated TI results are

   alt

   ***Figure* 6.6.11.**

   ***Figure* 6.6.12.**
4. Since the p-value $= 0.0073$ is greater than our alpha value $= 0.005$, we cannot reject the null. Therefore, we conclude that there is not enough evidence to support the claim of higher brain cancer rates for the cell phone users.

✓ Example 6.6.12

According to the US Census there are approximately 268,608,618 residents aged 12 and older. Statistics from the Rape, Abuse, and Incest National Network indicate that, on average, 207,754 rapes occur each year (male and female) for persons aged 12 and older. This translates into a percentage of sexual assaults of 0.078%. In Daviess County, KY, there were reported 11 rapes for a population of 37,937. Conduct an appropriate hypothesis test to determine if there is a statistically significant difference between the local sexual assault percentage and the national sexual assault percentage. Use a significance level of 0.01.

Answer

We will follow the four-step plan.

1. We need to test whether the proportion of sexual assaults in Daviess County, KY is significantly different from the national average.
2. Since we are presented with proportions, we will use a one-proportion *z*-test. The hypotheses for the test will be
   a. $H_0 : p = 0.00078$
   b. $H_a : p \neq 0.00078$
3. The following screen shots display the summary statistics from the hypothesis test.

   ***Figure* 6.6.13.**

   alt

   ***Figure* 6.6.14.**
4. Since the p-value, $p = 0.00063$, is less than the alpha level of 0.01, the sample data indicates that we should reject the null hypothesis. In conclusion, the sample data support the claim that the proportion of sexual assaults in Daviess County, Kentucky is different from the national average proportion.

Review

The **hypothesis test** itself has an established process. This can be summarized as follows:

1. Determine H_0 and H_a. Remember, they are contradictory.
2. Determine the random variable.
3. Determine the distribution for the test.

4. Draw a graph, calculate the test statistic, and use the test statistic to calculate the $p\text{-value}$. (A z-score and a t-score are examples of test statistics.)
5. Compare the preconceived α with the p-value, make a decision (reject or do not reject H_0), and write a clear conclusion using English sentences.

Notice that in performing the hypothesis test, you use α and not β. β is needed to help determine the sample size of the data that is used in calculating the $p\text{-value}$. Remember that the quantity $1-\beta$ is called the **Power of the Test**. A high power is desirable. If the power is too low, statisticians typically increase the sample size while keeping α the same.If the power is low, the null hypothesis might not be rejected when it should be.

? Exercise 6.6.8

Assume $H_0 : \mu = 9$ and $H_a : \mu < 9$. Is this a left-tailed, right-tailed, or two-tailed test?

Answer

This is a left-tailed test.

? Exercise 6.6.9

Assume $H_0 : \mu \leq 6$ and $H_a : \mu > 6$. Is this a left-tailed, right-tailed, or two-tailed test?

? Exercise 6.6.10

Assume $H_0 : p = 0.25$ and $H_a : p \neq 0.25$. Is this a left-tailed, right-tailed, or two-tailed test?

Answer

This is a two-tailed test.

? Exercise 6.6.11

Draw the general graph of a left-tailed test.

? Exercise 6.6.12

Draw the graph of a two-tailed test.

Answer

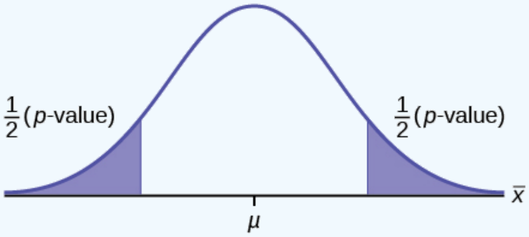

Figure 6.6.16

? Exercise 6.6.13

A bottle of water is labeled as containing 16 fluid ounces of water. You believe it is less than that. What type of test would you use?

? Exercise 6.6.14

Your friend claims that his mean golf score is 63. You want to show that it is higher than that. What type of test would you use?

Answer

a right-tailed test

? Exercise 6.6.15

A bathroom scale claims to be able to identify correctly any weight within a pound. You think that it cannot be that accurate. What type of test would you use?

? Exercise 6.6.16

You flip a coin and record whether it shows heads or tails. You know the probability of getting heads is 50%, but you think it is less for this particular coin. What type of test would you use?

Answer

a left-tailed test

? Exercise 6.6.17

If the alternative hypothesis has a not equals ($\neq$) symbol, you know to use which type of test?

? Exercise 6.6.18

Assume the null hypothesis states that the mean is at least 18. Is this a left-tailed, right-tailed, or two-tailed test?

Answer

This is a left-tailed test.

? Exercise 6.6.19

Assume the null hypothesis states that the mean is at most 12. Is this a left-tailed, right-tailed, or two-tailed test?

? Exercise 6.6.20

Assume the null hypothesis states that the mean is equal to 88. The alternative hypothesis states that the mean is not equal to 88. Is this a left-tailed, right-tailed, or two-tailed test?

Answer

This is a two-tailed test.

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Glossary

Central Limit Theorem

Given a random variable (RV) with known mean μ and known standard deviation σ. We are sampling with size n and we are interested in two new RVs - the sample mean, $\bar{X}$, and the sample sum, $\sum X$. If the size n of the sample is sufficiently large, then $\bar{X} - N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ and $\sum X - N\left(n\mu, \sqrt{n}\sigma\right)$. If the size n of the sample is sufficiently large, then the distribution of the sample means and the distribution of the sample sums will approximate a normal distribution regardless of the shape of the population. The mean of the sample means will equal the population mean and the mean of the sample sums will equal n times the population mean. The standard deviation of the distribution of the sample means, $\frac{\sigma}{\sqrt{n}}$, is called the standard error of the mean.

CHAPTER OVERVIEW

7: Hypothesis Testing with Two Samples

You have learned to conduct hypothesis tests on single means and single proportions. You will expand upon that in this chapter. You will compare two means or two proportions to each other. The general procedure is still the same, just expanded. To compare two means or two proportions, you work with two groups. The groups are classified either as independent or matched pairs. Independent groups consist of two samples that are independent, that is, sample values selected from one population are not related in any way to sample values selected from the other population. Matched pairs consist of two samples that are dependent. The parameter tested using matched pairs is the population mean. The parameters tested using independent groups are either population means or population proportions.

Topic hierarchy

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7.1: Prelude to Hypothesis Testing with Two Samples

Learning Objectives

By the end of this chapter, the student should be able to:

- Classify hypothesis tests by type.
- Conduct and interpret hypothesis tests for two population means, population standard deviations known.
- Conduct and interpret hypothesis tests for two population means, population standard deviations unknown.
- Conduct and interpret hypothesis tests for two population proportions.
- Conduct and interpret hypothesis tests for matched or paired samples.

Studies often compare two groups. For example, researchers are interested in the effect aspirin has in preventing heart attacks. Over the last few years, newspapers and magazines have reported various aspirin studies involving two groups. Typically, one group is given aspirin and the other group is given a placebo. Then, the heart attack rate is studied over several years.

There are other situations that deal with the comparison of two groups. For example, studies compare various diet and exercise programs. Politicians compare the proportion of individuals from different income brackets who might vote for them. Students are interested in whether SAT or GRE preparatory courses really help raise their scores.

Figure 7.1.1. If you want to test a claim that involves two groups (the types of breakfasts eaten east and west of the Mississippi River) you can use a slightly different technique when conducting a hypothesis test. (credit: Chloe Lim)

You have learned to conduct hypothesis tests on single means and single proportions. You will expand upon that in this chapter. You will compare two means or two proportions to each other. The general procedure is still the same, just expanded.

To compare two means or two proportions, you work with two groups. The groups are classified either as **independent** or matched pairs. Independent groups consist of two samples that are independent, that is, sample values selected from one population are not related in any way to sample values selected from the other population. **Matched pairs** consist of two samples that are dependent. The parameter tested using matched pairs is the population mean. The parameters tested using independent groups are either population means or population proportions.

> This chapter relies on either a calculator or a computer to calculate the degrees of freedom, the test statistics, and *p*-values. TI-83+ and TI-84 instructions are included as well as the test statistic formulas. When using a TI-83+ or TI-84 calculator, we do not need to separate two population means, independent groups, or population variances unknown into large and small sample sizes. However, most statistical computer software has the ability to differentiate these tests.

This chapter deals with the following hypothesis tests:

Independent groups (samples are independent)

- Test of two population means.
- Test of two population proportions.

Matched or paired samples (samples are dependent)

- Test of the two population proportions by testing one population mean of differences.

7.2: Two Population Means with Unknown Standard Deviations

1. The two independent samples are simple random samples from two distinct populations.
2. For the two distinct populations:
   - if the sample sizes are small, the distributions are important (should be normal)
   - if the sample sizes are large, the distributions are not important (need not be normal)

> The test comparing two independent population means with unknown and possibly unequal population standard deviations is called the Aspin-Welch t-test. The degrees of freedom formula was developed by Aspin-Welch.

The comparison of two population means is very common. A difference between the two samples depends on both the means and the standard deviations. Very different means can occur by chance if there is great variation among the individual samples. In order to account for the variation, we take the difference of the sample means, $\bar{X}_1 - \bar{X}_2$, and divide by the standard error in order to standardize the difference. The result is a t-score test statistic.

Because we do not know the population standard deviations, we estimate them using the two sample standard deviations from our independent samples. For the hypothesis test, we calculate the estimated standard deviation, or **standard error**, of **the difference in sample means**, $\bar{X}_1 - \bar{X}_2$.

The standard error is:

$$\sqrt{\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}} \tag{7.2.1}$$

The test statistic (*t*-score) is calculated as follows:

$$\frac{(\bar{x} - \bar{x}) - (\mu_1 - \mu_2)}{\sqrt{\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}}} \tag{7.2.2}$$

where:

- s_1 and s_2, the sample standard deviations, are estimates of σ_1 and σ_1, respectively.
- σ_1 and σ_2 are the unknown population standard deviations.
- $\bar{x}_1$ and $\bar{x}_2$ are the sample means. μ_1 and μ_2 are the population means.

The number of *degrees of freedom* (df) requires a somewhat complicated calculation. However, a computer or calculator calculates it easily. The df are not always a whole number. The test statistic calculated previously is approximated by the Student's *t*-distribution with df as follows:

> Degrees of freedom
>
> $$df = \frac{\left(\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}\right)^2}{\left(\frac{1}{n_1 - 1}\right)\left(\frac{(s_1)^2}{n_1}\right)^2 + \left(\frac{1}{n_2 - 1}\right)\left(\frac{(s_2)^2}{n_2}\right)^2} \tag{7.2.3}$$

When both sample sizes n_1 and n_2 are five or larger, the Student's *t* approximation is very good. Notice that the sample variances $(s_1)^2$ and $(s_2)^2$ are not pooled. (If the question comes up, do not pool the variances.)

> It is not necessary to compute the degrees of freedom by hand. A calculator or computer easily computes it.

Example 7.2.1: Independent groups

The average amount of time boys and girls aged seven to 11 spend playing sports each day is believed to be the same. A study is done and data are collected, resulting in the data in Table 7.2.1. Each populations has a normal distribution.

Table 7.2.1

	Sample Size	Average Number of Hours Playing Sports Per Day	Sample Standard Deviation
Girls	9	2	0.8660.866
Boys	16	3.2	1.00

Is there a difference in the mean amount of time boys and girls aged seven to 11 play sports each day? Test at the 5% level of significance.

Answer

The population standard deviations are not known. Let g be the subscript for girls and b be the subscript for boys. Then, μ_g is the population mean for girls and μ_b is the population mean for boys. This is a test of two independent groups, two population means.

Random variable: $\bar{X}_g - \bar{X}_b =$ difference in the sample mean amount of time girls and boys play sports each day.

- $H_0 : \mu_g = \mu_b$
- $H_0 : \mu_g - \mu_b = 0$
- $H_a : \mu_g \neq \mu_b$
- $H_a : \mu_g - \mu_b \neq 0$

The words **"the same"** tell you H_0 has an "=". Since there are no other words to indicate H_a, assume it says **"is different."** This is a two-tailed test.

Distribution for the test: Use t_{df} where df is calculated using the df formula for independent groups, two population means. Using a calculator, df is approximately 18.8462. **Do not pool the variances.**

Calculate the *p*-value using a Student's *t*-distribution: $p\text{-value} = 0.0054$

Graph:

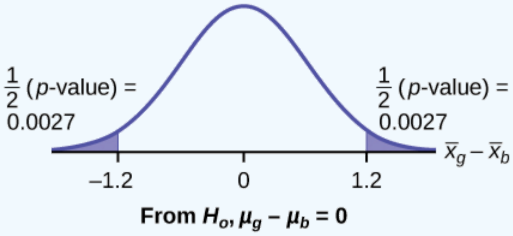


Figure 7.2.1: Normal distribution curve representing the difference in the average amount of time girls and boys play sports all day

$$s_g = 0.866 \tag{7.2.4}$$

$$s_b = 1 \tag{7.2.5}$$

So,

$$\bar{x}_g - \bar{x}_b = 2 - 3.2 = -1.2 \tag{7.2.6}$$

Half the *p*-value is below –1.2 and half is above 1.2.

Make a decision: Since $\alpha > p\text{-value}$, reject H_0. This means you reject $\mu_g = \mu_b$. The means are different.

Press `STAT` . Arrow over to `TESTS` and press `4:2-SampTTest` . Arrow over to Stats and press `ENTER` . Arrow down and enter `2` for the first sample mean, $\sqrt{0.866}$ for Sx1, `9` for n1, `3.2` for the second sample mean, `1` for Sx2, and `16` for n2. Arrow down to µ1: and arrow to `does not equal` µ2. Press `ENTER` . Arrow down to Pooled: and `No` . Press `ENTER` . Arrow down to `Calculate` and press `ENTER` . The p-value is $p = 0.0054$, the dfs are approximately 18.8462, and the test statistic is -3.14. Do the procedure again but instead of Calculate do Draw.

Conclusion: At the 5% level of significance, the sample data show there is sufficient evidence to conclude that the mean number of hours that girls and boys aged seven to 11 play sports per day is different (mean number of hours boys aged seven to 11 play sports per day is greater than the mean number of hours played by girls OR the mean number of hours girls aged seven to 11 play sports per day is greater than the mean number of hours played by boys).

? Exercise 7.2.1

Two samples are shown in Table. Both have normal distributions. The means for the two populations are thought to be the same. Is there a difference in the means? Test at the 5% level of significance.

Table 7.2.2

	Sample Size	Sample Mean	Sample Standard Deviation
Population A	25	5	1
Population B	16	4.7	1.2

Answer

The p-value is 0.4125, which is much higher than 0.05, so we decline to reject the null hypothesis. There is not sufficient evidence to conclude that the means of the two populations are not the same.

When the sum of the sample sizes is larger than $30 (n_1 + n_2 > 30)$ you can use the normal distribution to approximate the Student's t.

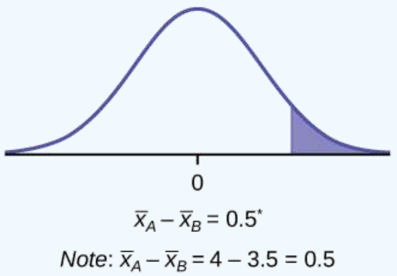
✓ Example 7.2.2

A study is done by a community group in two neighboring colleges to determine which one graduates students with more math classes. College A samples 11 graduates. Their average is four math classes with a standard deviation of 1.5 math classes. College B samples nine graduates. Their average is 3.5 math classes with a standard deviation of one math class. The community group believes that a student who graduates from college A **has taken more math classes,** on the average. Both populations have a normal distribution. Test at a 1% significance level. Answer the following questions.

a. Is this a test of two means or two proportions?
b. Are the populations standard deviations known or unknown?
c. Which distribution do you use to perform the test?
d. What is the random variable?
e. What are the null and alternate hypotheses? Write the null and alternate hypotheses in words and in symbols.
f. Is this test right-, left-, or two-tailed?
g. What is the p-value?
h. Do you reject or not reject the null hypothesis?

Solutions

a. two means
b. unknown
c. Student's t
d. $\bar{X}_A - \bar{X}_B$
e. $H_0 : \mu_A \leq \mu_B$ and $H_a : \mu_A > \mu_B$



f.

right

g. g. 0.1928
h. h. Do not reject.
i. i. At the 1% level of significance, from the sample data, there is not sufficient evidence to conclude that a student who graduates from college A has taken more math classes, on the average, than a student who graduates from college B.

? Exercise 7.2.2

A study is done to determine if Company A retains its workers longer than Company B. Company A samples 15 workers, and their average time with the company is five years with a standard deviation of 1.2. Company B samples 20 workers, and their average time with the company is 4.5 years with a standard deviation of 0.8. The populations are normally distributed.

a. Are the population standard deviations known?
b. Conduct an appropriate hypothesis test. At the 5% significance level, what is your conclusion?

Answer

a. They are unknown.
b. The p-value $= 0.0878$. At the 5% level of significance, there is insufficient evidence to conclude that the workers of Company A stay longer with the company.

✓ Example 7.2.3

A professor at a large community college wanted to determine whether there is a difference in the means of final exam scores between students who took his statistics course online and the students who took his face-to-face statistics class. He believed that the mean of the final exam scores for the online class would be lower than that of the face-to-face class. Was the professor correct? The randomly selected 30 final exam scores from each group are listed in Table $7.2.3$ and Table $7.2.4$.

Table $7.2.3$: Online Class

67.6	41.2	85.3	55.9	82.4	91.2	73.5	94.1	64.7	64.7
70.6	38.2	61.8	88.2	70.6	58.8	91.2	73.5	82.4	35.5
94.1	88.2	64.7	55.9	88.2	97.1	85.3	61.8	79.4	79.4

Table $7.2.4$: Face-to-face Class

77.9	95.3	81.2	74.1	98.8	88.2	85.9	92.9	87.1	88.2
69.4	57.6	69.4	67.1	97.6	85.9	88.2	91.8	78.8	71.8
98.8	61.2	92.9	90.6	97.6	100	95.3	83.5	92.9	89.4

Is the mean of the Final Exam scores of the online class lower than the mean of the Final Exam scores of the face-to-face class? Test at a 5% significance level. Answer the following questions:

a. Is this a test of two means or two proportions?
b. Are the population standard deviations known or unknown?

c. Which distribution do you use to perform the test?
d. What is the random variable?
e. What are the null and alternative hypotheses? Write the null and alternative hypotheses in words and in symbols.
f. Is this test right, left, or two tailed?
g. What is the p-value?
h. Do you reject or not reject the null hypothesis?
i. At the ___ level of significance, from the sample data, there ______ (is/is not) sufficient evidence to conclude that ______.

(See the conclusion in Example, and write yours in a similar fashion)

Be careful not to mix up the information for Group 1 and Group 2!

Answer

a. two means
b. unknown
c. Student's t
d. $\bar{X}_1 - \bar{X}_2$
e. i. $H_0 : \mu_1 = \mu_2$ Null hypothesis: the means of the final exam scores are equal for the online and face-to-face statistics classes.
   ii. $H_a : \mu_1 < \mu_2$ Alternative hypothesis: the mean of the final exam scores of the online class is less than the mean of the final exam scores of the face-to-face class.
f. left-tailed
g. $p\text{-value} = 0.0011$

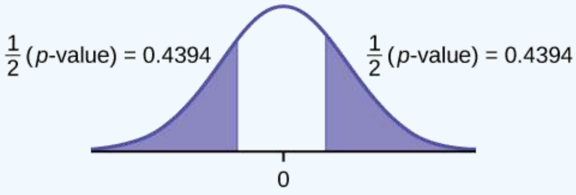


***Figure* $7.2.3$.**

h. Reject the null hypothesis
i. The professor was correct. The evidence shows that the mean of the final exam scores for the online class is lower than that of the face-to-face class.

At the 5% level of significance, from the sample data, there is (is/is not) sufficient evidence to conclude that the mean of the final exam scores for the online class is less than the mean of final exam scores of the face-to-face class.

First put the data for each group into two lists (such as L1 and L2). Press STAT. Arrow over to TESTS and press 4:2SampTTest. Make sure Data is highlighted and press ENTER. Arrow down and enter L1 for the first list and L2 for the second list. Arrow down to μ_1: and arrow to $\neq \mu_1$ (does not equal). Press ENTER. Arrow down to Pooled: No. Press ENTER. Arrow down to Calculate and press ENTER.

Cohen's Standards for Small, Medium, and Large Effect Sizes

Cohen's d is a measure of effect size based on the differences between two means. Cohen’s d, named for United States statistician Jacob Cohen, measures the relative strength of the differences between the means of two populations based on sample data. The calculated value of effect size is then compared to Cohen’s standards of small, medium, and large effect sizes.

Table $7.2.5$: Cohen's Standard Effect Sizes

Size of effect	d
Small	0.2
medium	0.5

Size of effect	*d*
Large	0.8

Cohen's d is the measure of the difference between two means divided by the pooled standard deviation: $d = \frac{\bar{x}_2 - \bar{x}_2}{s_{\text{pooled}}}$ where $s_{pooled} = \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}$

✓ Example 7.2.4

Calculate Cohen’s d for Example. Is the size of the effect small, medium, or large? Explain what the size of the effect means for this problem.

Answer

$$\mu_1 = 4 s_1 = 1.5 n_1 = 11$$

$$\mu_2 = 3.5 s_2 = 1 n_2 = 9$$

$$d = 0.384$$

The effect is small because 0.384 is between Cohen’s value of 0.2 for small effect size and 0.5 for medium effect size. The size of the differences of the means for the two colleges is small indicating that there is not a significant difference between them.

✓ Example 7.2.5

Calculate Cohen’s d for Example. Is the size of the effect small, medium or large? Explain what the size of the effect means for this problem.

Answer

$d = 0.834$; Large, because 0.834 is greater than Cohen’s 0.8 for a large effect size. The size of the differences between the means of the Final Exam scores of online students and students in a face-to-face class is large indicating a significant difference.

✓ Example 10.2.6

Weighted alpha is a measure of risk-adjusted performance of stocks over a period of a year. A high positive weighted alpha signifies a stock whose price has risen while a small positive weighted alpha indicates an unchanged stock price during the time period. Weighted alpha is used to identify companies with strong upward or downward trends. The weighted alpha for the top 30 stocks of banks in the northeast and in the west as identified by Nasdaq on May 24, 2013 are listed in Table and Table, respectively.

Northeast

94.2	75.2	69.6	52.0	48.0	41.9	36.4	33.4	31.5	27.6
77.3	71.9	67.5	50.6	46.2	38.4	35.2	33.0	28.7	26.5
76.3	71.7	56.3	48.7	43.2	37.6	33.7	31.8	28.5	26.0

West

126.0	70.6	65.2	51.4	45.5	37.0	33.0	29.6	23.7	22.6
116.1	70.6	58.2	51.2	43.2	36.0	31.4	28.7	23.5	21.6
78.2	68.2	55.6	50.3	39.0	34.1	31.0	25.3	23.4	21.5

Is there a difference in the weighted alpha of the top 30 stocks of banks in the northeast and in the west? Test at a 5% significance level. Answer the following questions:

a. Is this a test of two means or two proportions?
b. Are the population standard deviations known or unknown?
c. Which distribution do you use to perform the test?
d. What is the random variable?
e. What are the null and alternative hypotheses? Write the null and alternative hypotheses in words and in symbols.
f. Is this test right, left, or two tailed?
g. What is the p-value?
h. Do you reject or not reject the null hypothesis?
i. At the ___ level of significance, from the sample data, there ______ (is/is not) sufficient evidence to conclude that ______.
j. Calculate Cohen's d and interpret it.

Answer

a. two means
b. unknown
c. Student's-t
d. $\bar{X}_1 - \bar{X}_2$
e. i. $H_0 : \mu_1 = \mu_2$ Null hypothesis: the means of the weighted alphas are equal.
   ii. $H_a : \mu_1 \neq \mu_2$ Alternative hypothesis : the means of the weighted alphas are not equal.
f. two-tailed
g. $p\text{-value} = 0.8787$
h. Do not reject the null hypothesis
i. This indicates that the trends in stocks are about the same in the top 30 banks in each region.

This is a normal distribution curve with mean equal to zero. Both the right and left tails of the curve are shaded. Each tail represents 1/2(p-value) = 0.4394.

***Figure* $7.2.4$.**

5% level of significance, from the sample data, there is not sufficient evidence to conclude that the mean weighted alphas for the banks in the northeast and the west are different

j. $d = 0.040$, Very small, because 0.040 is less than Cohen's value of 0.2 for small effect size. The size of the difference of the means of the weighted alphas for the two regions of banks is small indicating that there is not a significant difference between their trends in stocks.

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Review

Two population means from independent samples where the population standard deviations are not known

- Random Variable: $\bar{X}_1 - \bar{X}_2 =$ the difference of the sampling means
- Distribution: Student's t-distribution with degrees of freedom (variances not pooled)

Formula Review

Standard error:

$$SE=\sqrt{\frac{(s_1^2)}{n_1}+\frac{(s_2^2)}{n_2}} \tag{7.2.7}$$

Test statistic (*t*-score):

$$t=\frac{(\bar{x}_1-\bar{x}_2)-(\mu_1-\mu_2)}{\sqrt{\frac{(s_1)^2}{n_1}+\frac{(s_2)^2}{n_2}}} \tag{7.2.8}$$

Degrees of freedom:

$$df=\frac{\left(\frac{(s_1)^2}{n_1}+\frac{(s_2)^2}{n_2}\right)^2}{\left(\frac{1}{n_1-1}\right)\left(\frac{(s_1)^2}{n_1}\right)^2}+\left(\frac{1}{n_2-1}\right)\left(\frac{(s_2)^2}{n_2}\right)^2 \tag{7.2.9}$$

where:

- s_1 and s_2 are the sample standard deviations, and n_1 and n_2 are the sample sizes.
- x_1 and x_2 are the sample means.

Cohen's d is the measure of effect size:

$$d=\frac{\bar{x}_1-\bar{x}_2}{s_{\text{pooled}}} \tag{7.2.10}$$

where

$$s_{\text{pooled}}=\sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}} \tag{7.2.11}$$

Glossary

Degrees of Freedom (df)

the number of objects in a sample that are free to vary.

Standard Deviation

A number that is equal to the square root of the variance and measures how far data values are from their mean; notation: s for sample standard deviation and σ for population standard deviation.

Variable (Random Variable)

a characteristic of interest in a population being studied. Common notation for variables are upper-case Latin letters X, Y, Z,... Common notation for a specific value from the domain (set of all possible values of a variable) are lower-case Latin letters x, y, z,.... For example, if X is the number of children in a family, then x represents a specific integer 0, 1, 2, 3, ... Variables in statistics differ from variables in intermediate algebra in the two following ways.

- The domain of the random variable (RV) is not necessarily a numerical set; the domain may be expressed in words; for example, if $X=$ hair color, then the domain is {black, blond, gray, green, orange}.
- We can tell what specific value x of the random variable X takes only after performing the experiment.

7.3: Matched or Paired Samples

When using a hypothesis test for matched or paired samples, the following characteristics should be present:

1. Simple random sampling is used.
2. Sample sizes are often small.
3. Two measurements (samples) are drawn from the same pair of individuals or objects.
4. Differences are calculated from the matched or paired samples.
5. The differences form the sample that is used for the hypothesis test.
6. Either the matched pairs have differences that come from a population that is normal or the number of differences is sufficiently large so that distribution of the sample mean of differences is approximately normal.

In a hypothesis test for matched or paired samples, subjects are matched in pairs and differences are calculated. The differences are the data. The population mean for the differences, μ_d, is then tested using a Student's t-test for a single population mean with $n-1$ degrees of freedom, where n is the number of differences.

The test statistic (t-score) is:

$$t=\frac{\bar{x}_d-\mu_d}{\left(\frac{s_d}{\sqrt{n}}\right)} \tag{7.3.1}$$

✓ Example 7.3.1

A study was conducted to investigate the effectiveness of hypnotism in reducing pain. Results for randomly selected subjects are shown in Table. A lower score indicates less pain. The "before" value is matched to an "after" value and the differences are calculated. The differences have a normal distribution. Are the sensory measurements, on average, lower after hypnotism? Test at a 5% significance level.

Subject:	A	B	C	D	E	F	G	H
Before	6.6	6.5	9.0	10.3	11.3	8.1	6.3	11.6
After	6.8	2.4	7.4	8.5	8.1	6.1	3.4	2.0

Answer

Corresponding "before" and "after" values form matched pairs. (Calculate "after" – "before.")

After Data	Before Data	Difference
6.8	6.6	0.2
2.4	6.5	-4.1
7.4	9	-1.6
8.5	10.3	-1.8
8.1	11.3	-3.2
6.1	8.1	-2
3.4	6.3	-2.9
2	11.6	-9.6

The data for the test are the differences: $\{0.2,-4.1,-1.6,-1.8,-3.2,-2,-2.9,-9.6\}$

The sample mean and sample standard deviation of the differences are: $\bar{x}_d=-3.13$ and $s_d=2.91$ Verify these values.

Let μ_d be the population mean for the differences. We use the subscript dd to denote "differences."

Random variable:

$\bar{X}_d =$ the mean difference of the sensory measurements

$$H_0 : \mu_d \geq 0 \tag{7.3.2}$$

The null hypothesis is zero or positive, meaning that there is the same or more pain felt after hypnotism. That means the subject shows no improvement. μ_d is the population mean of the differences.

$$H_a : \mu_d < 0 \tag{7.3.3}$$

The alternative hypothesis is negative, meaning there is less pain felt after hypnotism. That means the subject shows improvement. The score should be lower after hypnotism, so the difference ought to be negative to indicate improvement.

Distribution for the test:

The distribution is a Student's *t* with $df = n - 1 = 8 - 1 = 7$. Use t_7. *(Notice that the test is for a single population mean.)*

Calculate the *p*-value using the Student's-t distribution:

$$p\text{-value} = 0.0095 \tag{7.3.4}$$

Graph:

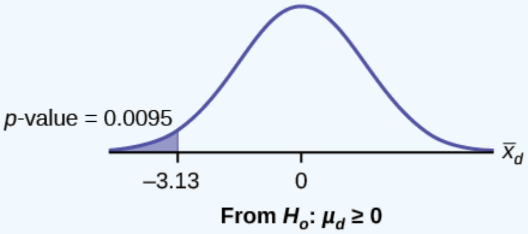

Figure 10.5.1.

$\bar{X}_d$ is the random variable for the differences.

The sample mean and sample standard deviation of the differences are:

$\bar{x}_d = -3.13$

$s_d = 2.91$

Compare α and the p-value

$\alpha = 0.05$ and $p\text{-value} = 0.0095$. $\alpha > p\text{-value}$

Make a decision

Since $\alpha > p\text{-value}$, reject H_0. This means that $\mu_d < 0$ and there is improvement.

Conclusion

At a 5% level of significance, from the sample data, there is sufficient evidence to conclude that the sensory measurements, on average, are lower after hypnotism. Hypnotism appears to be effective in reducing pain.

> For the TI-83+ and TI-84 calculators, you can either calculate the differences ahead of time (**after - before**) and put the differences into a list or you can put the **after** data into a first list and the **before** data into a second list. Then go to a third list and arrow up to the name. Enter 1st list name - 2nd list name. The calculator will do the subtraction, and you will have the differences in the third list.

Use your list of differences as the data. Press `STAT` and arrow over to `TESTS` . Press `2:T-Test` . Arrow over to `Data` and press `ENTER` . Arrow down and enter `0` for μ_0, the name of the list where you put the data, and `1` for Freq:. Arrow down to μ: and arrow over to `<` μ_0. Press `ENTER` . Arrow down to `Calculate` and press `ENTER` . The p-value is 0.0094, and the test statistic is -3.04. Do these instructions again except, arrow to `Draw` (instead of `Calculate`). Press `ENTER` .

Exercise 7.3.1

A study was conducted to investigate how effective a new diet was in lowering cholesterol. Results for the randomly selected subjects are shown in the table. The differences have a normal distribution. Are the subjects' cholesterol levels lower on average after the diet? Test at the 5% level.

Subject	A	B	C	D	E	F	G	H	I
Before	209	210	205	198	216	217	238	240	222
After	199	207	189	209	217	202	211	223	201

Answer

The p-value is 0.0130, so we can reject the null hypothesis. There is enough evidence to suggest that the diet lowers cholesterol.

Example 7.3.2

A college football coach was interested in whether the college's strength development class increased his players' maximum lift (in pounds) on the bench press exercise. He asked four of his players to participate in a study. The amount of weight they could each lift was recorded before they took the strength development class. After completing the class, the amount of weight they could each lift was again measured. The data are as follows:

Weight (in pounds)	Player 1	Player 2	Player 3	Player 4
Amount of weight lifted prior to the class	205	241	338	368
Amount of weight lifted after the class	295	252	330	360

The coach wants to know if the strength development class makes his players stronger, on average.

Record the **differences** data. Calculate the differences by subtracting the amount of weight lifted prior to the class from the weight lifted after completing the class. The data for the differences are: $\{90, 11, -8, -8\}$ Assume the differences have a normal distribution.

Using the differences data, calculate the sample mean and the sample standard deviation.

$$\bar{x}_d = 21.3 \tag{7.3.5}$$

and

$$s_d = 46.7 \tag{7.3.6}$$

> The data given here would indicate that the distribution is actually right-skewed. The difference 90 may be an extreme outlier? It is pulling the sample mean to be 21.3 (positive). The means of the other three data values are actually negative.

Using the difference data, this becomes a test of a single __________ (fill in the blank).

Define the random variable: $\bar{X}$ mean difference in the maximum lift per player.

The distribution for the hypothesis test is t_3.

- $H_0 : \mu_d \leq 0$,
- $H_a : \mu_d > 0$

Graph:

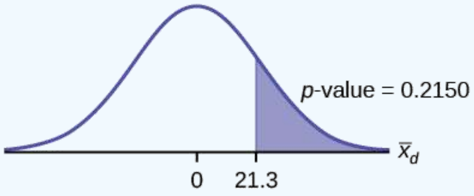

Figure 10.5.2.

Calculate the p**-value:** The p-value is 0.2150

Decision: If the level of significance is 5%, the decision is not to reject the null hypothesis, because $\alpha < p\text{-value}$.

What is the conclusion?

At a 5% level of significance, from the sample data, there is not sufficient evidence to conclude that the strength development class helped to make the players stronger, on average.

? Exercise 7.3.2

A new prep class was designed to improve SAT test scores. Five students were selected at random. Their scores on two practice exams were recorded, one before the class and one after. The data recorded in Table. Are the scores, on average, higher after the class? Test at a 5% level.

SAT Scores	Student 1	Student 2	Student 3	Student 4
Score before class	1840	1960	1920	2150
Score after class	1920	2160	2200	2100

Answer

The p-value is 0.0874, so we decline to reject the null hypothesis. The data do not support that the class improves SAT scores significantly.

✓ Example 7.3.3

Seven eighth graders at Kennedy Middle School measured how far they could push the shot-put with their dominant (writing) hand and their weaker (non-writing) hand. They thought that they could push equal distances with either hand. The data were collected and recorded in Table.

Distance (in feet) using	Student 1	Student 2	Student 3	Student 4	Student 5	Student 6	Student 7
Dominant Hand	30	26	34	17	19	26	20
Weaker Hand	28	14	27	18	17	26	16

Conduct a hypothesis test to determine whether the mean difference in distances between the children's dominant versus weaker hands is significant.

Record the **differences** data. Calculate the differences by subtracting the distances with the weaker hand from the distances with the dominant hand. The data for the differences are: $\{2, 12, 7, -1, 2, 0, 4\}$The differences have a normal distribution.

Using the differences data, calculate the sample mean and the sample standard deviation. $\bar{x} = 3.71$, $s_d = 4.5$.

Random variable: $\bar{X} =$ mean difference in the distances between the hands.

Distribution for the hypothesis test: t_6

$H_0 : \mu_d = 0 \quad H_a : \mu_d \neq 0$

Graph:

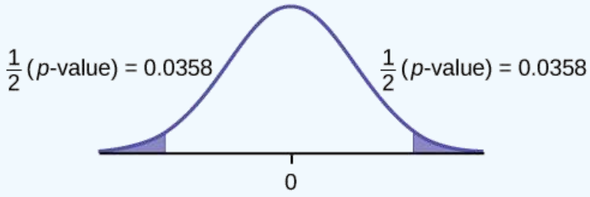

Figure 10.5.3.

Calculate the *p*-value: The p-value is 0.0716 (using the data directly).

(test statistic = 2.18. $p\text{-value} = 0.0719$ using ($\bar{x}_d = 3.71$, $s_d = 4.5$.

Decision: Assume $\alpha = 0.05$. Since $\alpha < p\text{-value}$, Do not reject H_0.

Conclusion: At the 5% level of significance, from the sample data, there is not sufficient evidence to conclude that there is a difference in the children's weaker and dominant hands to push the shot-put.

? Exercise 7.3.3

Five ball players think they can throw the same distance with their dominant hand (throwing) and off-hand (catching hand). The data were collected and recorded in Table. Conduct a hypothesis test to determine whether the mean difference in distances between the dominant and off-hand is significant. Test at the 5% level.

	Player 1	Player 2	Player 3	Player 4	Player 5
Dominant Hand	120	111	135	140	125
Off-hand	105	109	98	111	99

Answer

The p-level is 0.0230, so we can reject the null hypothesis. The data show that the players do not throw the same distance with their off-hands as they do with their dominant hands.

Review

A hypothesis test for matched or paired samples (t-test) has these characteristics:

- Test the differences by subtracting one measurement from the other measurement
- Random Variable: $x_d =$ mean of the differences
- Distribution: Student's t-distribution with $n-1$ degrees of freedom
- If the number of differences is small (less than 30), the differences must follow a normal distribution.
- Two samples are drawn from the same set of objects.
- Samples are dependent.

Formula Review

Test Statistic (*t*-score):

$$t = \frac{\bar{x}_d}{\left(\frac{s_d}{\sqrt{n}}\right)} \tag{7.3.7}$$

where:

x_d is the mean of the sample differences. μ_d is the mean of the population differences. s_d is the sample standard deviation of the differences. n is the sample size.

CHAPTER OVERVIEW

8: The Chi-Square Distribution

A chi-squared test is any statistical hypothesis test in which the sampling distribution of the test statistic is a chi-square distribution when the null hypothesis is true.

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8.1: Prelude to The Chi-Square Distribution

CHAPTER OBJECTIVES

By the end of this chapter, the student should be able to:

- Interpret the chi-square probability distribution as the sample size changes.
- Conduct and interpret chi-square goodness-of-fit hypothesis tests.
- Conduct and interpret chi-square test of independence hypothesis tests.
- Conduct and interpret chi-square homogeneity hypothesis tests.
- Conduct and interpret chi-square single variance hypothesis tests.

Have you ever wondered if lottery numbers were evenly distributed or if some numbers occurred with a greater frequency? How about if the types of movies people preferred were different across different age groups? What about if a coffee machine was dispensing approximately the same amount of coffee each time? You could answer these questions by conducting a hypothesis test.

You will now study a new distribution, one that is used to determine the answers to such questions. This distribution is called the chi-square distribution.

Figure 8.1.1: The chi-square distribution can be used to find relationships between two things, like grocery prices at different stores. (credit: Pete/flickr)

In this chapter, you will learn the three major applications of the chi-square distribution:

a. the goodness-of-fit test, which determines if data fit a particular distribution, such as in the lottery example
b. the test of independence, which determines if events are independent, such as in the movie example
c. the test of a single variance, which tests variability, such as in the coffee example

> Though the chi-square distribution depends on calculators or computers for most of the calculations, there is a table available (see [link]). TI-83+ and TI-84 calculator instructions are included in the text.

COLLABORATIVE CLASSROOM EXERCISE

Look in the sports section of a newspaper or on the Internet for some sports data (baseball averages, basketball scores, golf tournament scores, football odds, swimming times, and the like). Plot a histogram and a boxplot using your data. See if you can determine a probability distribution that your data fits. Have a discussion with the class about your choice.

8.2: Facts About the Chi-Square Distribution

The notation for the chi-square distribution is:

$$\chi \sim \chi^2_{df} \tag{8.2.1}$$

where $df =$ degrees of freedom which depends on how chi-square is being used. (If you want to practice calculating chi-square probabilities then use $df = n - 1$. The degrees of freedom for the three major uses are each calculated differently.)

For the χ^2 distribution, the population mean is $\mu = df$ and the population standard deviation is

$$\sigma = \sqrt{2(df)}. \tag{8.2.2}$$

The random variable is shown as χ^2, but may be any upper case letter. The random variable for a chi-square distribution with k degrees of freedom is the sum of k independent, squared standard normal variables.

$$\chi^2 = (Z_1)^2 + \ldots + (Z_k)^2 \tag{8.2.3}$$

a. The curve is nonsymmetrical and skewed to the right.
b. There is a different chi-square curve for each df.

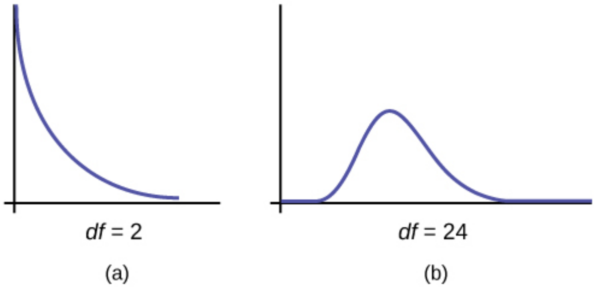

Figure 8.2.1

c. The test statistic for any test is always greater than or equal to zero.
d. When $df > 90$, the chi-square curve approximates the normal distribution. For $\chi \sim \chi^2_{1,000}$ the mean, $\mu = df = 1,000$ and the standard deviation, $\mu = \sqrt{2(1,000)}$. Therefore, $X \sim N(1,000, 44.7)$, approximately.
e. The mean, μ, is located just to the right of the peak.

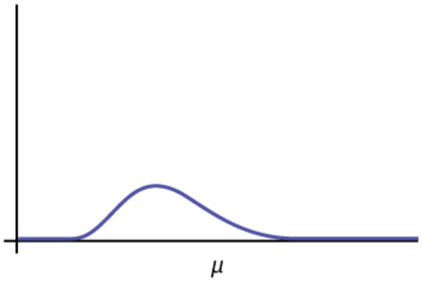

Figure 8.2.2

References

1. Data from *Parade Magazine*.
2. “HIV/AIDS Epidemiology Santa Clara County.”Santa Clara County Public Health Department, May 2011.

Review

The chi-square distribution is a useful tool for assessment in a series of problem categories. These problem categories include primarily (i) whether a data set fits a particular distribution, (ii) whether the distributions of two populations are the same, (iii) whether two events might be independent, and (iv) whether there is a different variability than expected within a population.

An important parameter in a chi-square distribution is the degrees of freedom df in a given problem. The random variable in the chi-square distribution is the sum of squares of *df* standard normal variables, which must be independent. The key characteristics of the chi-square distribution also depend directly on the degrees of freedom.

The chi-square distribution curve is skewed to the right, and its shape depends on the degrees of freedom df. For $df > 90$, the curve approximates the normal distribution. Test statistics based on the chi-square distribution are always greater than or equal to zero. Such application tests are almost always right-tailed tests.

Formula Review

$$\chi^2 = (Z_1)^2 + (Z_2)^2 + \ldots + (Z_{df})^2 \tag{8.2.4}$$

chi-square distribution random variable

$\mu_{\chi^2} = df$ chi-square distribution population mean

$\sigma_{\chi^2} = \sqrt{2(df)}$ Chi-Square distribution population standard deviation

? Exercise 8.2.1

If the number of degrees of freedom for a chi-square distribution is 25, what is the population mean and standard deviation?

Answer

mean $= 25$ and standard deviation $= 7.0711$

? Exercise 8.2.2

If $df > 90$, the distribution is _____________. If $df = 15$, the distribution is ________________.

? Exercise 8.2.3

When does the chi-square curve approximate a normal distribution?

Answer

when the number of degrees of freedom is greater than 90

? Exercise 8.2.4

Where is μ located on a chi-square curve?

? Exercise 8.2.5

Is it more likely the *df* is 90, 20, or two in the graph?

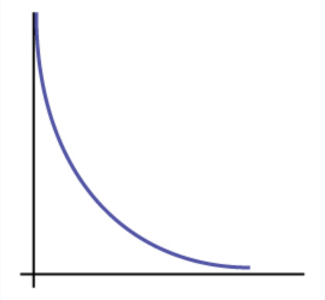

Figure 8.2.3.

Answer

$df = 2$

8.3: Test of Independence

Tests of independence involve using a contingency table of observed (data) values.

The test statistic for a *test of independence* is similar to that of a goodness-of-fit test:

$$\sum_{(i \cdot j)} \frac{(O-E)^2}{E} \tag{8.3.1}$$

where:

- $O =$ observed values
- $E =$ expected values
- $i =$ the number of rows in the table
- $j =$ the number of columns in the table

There are $i \cdot j$ terms of the form $\frac{(O-E)^2}{E}$.

> The expected value for each cell needs to be at least five in order for you to use this test.

A test of independence determines whether two factors are independent or not. You first encountered the term independence in Probability Topics. As a review, consider the following example.

✓ Example 8.3.1

Suppose $A =$ a speeding violation in the last year and $B =$ a cell phone user while driving. If A and B are independent then $P(A \text{ AND } B) = P(A)P(B)$. A AND B is the event that a driver received a speeding violation last year and also used a cell phone while driving. Suppose, in a study of drivers who received speeding violations in the last year, and who used cell phone while driving, that 755 people were surveyed. Out of the 755, 70 had a speeding violation and 685 did not; 305 used cell phones while driving and 450 did not.

Let $y =$ expected number of drivers who used a cell phone while driving and received speeding violations.

If A and B are independent, then $P(A \text{ AND } B) = P(A)P(B)$. By substitution,

$$\frac{y}{755} = \left(\frac{70}{755}\right)\left(\frac{305}{755}\right)$$

Solve for y:

$$y = \frac{(70)(305)}{755} = 28.3$$

About 28 people from the sample are expected to use cell phones while driving and to receive speeding violations.

In a test of independence, we state the null and alternative hypotheses in words. Since the contingency table consists of **two factors**, the null hypothesis states that the factors are **independent** and the alternative hypothesis states that they are **not independent (dependent)**. If we do a test of independence using the example, then the null hypothesis is:

H_0: Being a cell phone user while driving and receiving a speeding violation are independent events.

If the null hypothesis were true, we would expect about 28 people to use cell phones while driving and to receive a speeding violation.

The test of independence is always right-tailed because of the calculation of the test statistic. If the expected and observed values are not close together, then the test statistic is very large and way out in the right tail of the chi-square curve, as it is in a goodness-of-fit.

The number of degrees of freedom for the test of independence is:

$$df = (\text{number of columns} - 1)(\text{number of rows} - 1)$$

The following formula calculates the **expected number** (E):

$$E=\frac{(\text{row total})(\text{column total})}{\text{total number surveyed}}$$

Exercise 8.3.1

A sample of 300 students is taken. Of the students surveyed, 50 were music students, while 250 were not. Ninety-seven were on the honor roll, while 203 were not. If we assume being a music student and being on the honor roll are independent events, what is the expected number of music students who are also on the honor roll?

Answer

About 16 students are expected to be music students and on the honor roll.

Example 8.3.2

In a volunteer group, adults 21 and older volunteer from one to nine hours each week to spend time with a disabled senior citizen. The program recruits among community college students, four-year college students, and nonstudents. In Table 8.3.1 is a **sample** of the adult volunteers and the number of hours they volunteer per week.

Table 8.3.1: Number of Hours Worked Per Week by Volunteer Type (Observed). The table contains **observed (O)** values (data).

Type of Volunteer	1–3 Hours	4–6 Hours	7–9 Hours	Row Total
Community College Students	111	96	48	255
Four-Year College Students	96	133	61	290
Nonstudents	91	150	53	294
Column Total	298	379	162	839

Is the number of hours volunteered **independent** of the type of volunteer?

Answer

The **observed table** and the question at the end of the problem, "Is the number of hours volunteered independent of the type of volunteer?" tell you this is a test of independence. The two factors are **number of hours volunteered** and **type of volunteer**. This test is always right-tailed.

- H_0: The number of hours volunteered is **independent** of the type of volunteer.
- H_a: The number of hours volunteered is **dependent** on the type of volunteer.

The expected results are in Table 8.3.2.

Table 8.3.2: Number of Hours Worked Per Week by Volunteer Type (Expected). The table contains **expected**(E) values (data).

Type of Volunteer	1-3 Hours	4-6 Hours	7-9 Hours
Community College Students	90.57	115.19	49.24
Four-Year College Students	103.00	131.00	56.00
Nonstudents	104.42	132.81	56.77

For example, the calculation for the expected frequency for the top left cell is

$$E=\frac{(\text{row total})(\text{column total})}{\text{total number surveyed}}=\frac{(255)(298)}{839}=90.57$$

Calculate the test statistic: $\chi^2 = 12.99$ (calculator or computer)

Distribution for the test: χ^2_4

$$df = (3 \text{ columns} - 1)(3 \text{ rows} - 1) = (2)(2) = 4$$

Graph:

Nonsymmetrical chi-square curve with values of 0 and 12.99 on the x-axis representing the test statistic of number of hours worked by volunteers of different types. A vertical upward line extends from 12.99 to the curve and the area to the right of this is equal to the p-value.

Figure 8.3.1.

Probability statement: $p\text{-value} = P(\chi^2 > 12.99) = 0.0113$

Compare α and the p-value: Since no α is given, assume $\alpha = 0.05$. $p\text{-value} = 0.0113$. $\alpha > p\text{-value}$.

Make a decision: Since $\alpha > p\text{-value}$, reject H_0. This means that the factors are not independent.

Conclusion: At a 5% level of significance, from the data, there is sufficient evidence to conclude that the number of hours volunteered and the type of volunteer are dependent on one another.

For the example in Table, if there had been another type of volunteer, teenagers, what would the degrees of freedom be?

USING THE TI-83, 83+, 84, 84+ CALCULATOR

Press the `MATRX` key and arrow over to `EDIT`. Press `1:[A]`. Press `3 ENTER 3 ENTER`. Enter the table values by row from Table. Press `ENTER` after each. Press `2nd QUIT`. Press `STAT` and arrow over to `TESTS`. Arrow down to `C:χ2-TEST`. Press `ENTER`. You should see `Observed:[A] and Expected:[B]`. If necessary, use the arrow keys to move the cursor after `Observed:` and press `2nd MATRX`. Press `1:[A]` to select matrix A. It is not necessary to enter expected values. The matrix listed after `Expected:` can be blank. Arrow down to `Calculate`. Press `ENTER`. The test statistic is 12.9909 and the *p*-value = 0.0113. Do the procedure a second time, but arrow down to `Draw` instead of `calculate`.

Exercise 8.3.2

The Bureau of Labor Statistics gathers data about employment in the United States. A sample is taken to calculate the number of U.S. citizens working in one of several industry sectors over time. Table 8.3.3 shows the results:

Table 8.3.3

Industry Sector	2000	2010	2020	Total
Nonagriculture wage and salary	13,243	13,044	15,018	41,305
Goods-producing, excluding agriculture	2,457	1,771	1,950	6,178
Services-providing	10,786	11,273	13,068	35,127
Agriculture, forestry, fishing, and hunting	240	214	201	655
Nonagriculture self-employed and unpaid family worker	931	894	972	2,797
Secondary wage and salary jobs in agriculture and private household industries	14	11	11	36

Industry Sector	2000	2010	2020	Total
Secondary jobs as a self-employed or unpaid family worker	196	144	152	492
Total	27,867	27,351	31,372	86,590

We want to know if the change in the number of jobs is independent of the change in years. State the null and alternative hypotheses and the degrees of freedom.

Answer

- H_0: The number of jobs is independent of the year.
- H_a: The number of jobs is dependent on the year.

$df = 12$

alt

Figure 8.3.2.

Press the `MATRX` key and arrow over to `EDIT` . Press `1:[A]` . Press `3 ENTER 3 ENTER` . Enter the table values by row. Press `ENTER` after each. Press `2nd QUIT` . Press `STAT` and arrow over to `TESTS` . Arrow down to c:χ^2-TEST. Press `ENTER` . You should see `Observed:[A] and Expected:[B]` . Arrow down to `Calculate` . Press `ENTER` . The test statistic is 227.73 and the p-value $= 5.90E - 42 = 0$. Do the procedure a second time but arrow down to `Draw` instead of `calculate` .

Example 8.3.3

De Anza College is interested in the relationship between anxiety level and the need to succeed in school. A random sample of 400 students took a test that measured anxiety level and need to succeed in school. Table shows the results. De Anza College wants to know if anxiety level and need to succeed in school are independent events.

Need to Succeed in School vs. Anxiety Level

Need to Succeed in School	High Anxiety	Med-high Anxiety	Medium Anxiety	Med-low Anxiety	Low Anxiety	Row Total
High Need	35	42	53	15	10	155
Medium Need	18	48	63	33	31	193
Low Need	4	5	11	15	17	52
Column Total	57	95	127	63	58	400

a. How many high anxiety level students are expected to have a high need to succeed in school?
b. If the two variables are independent, how many students do you expect to have a low need to succeed in school and a med-low level of anxiety?
c. $E = \frac{(\text{row total})(\text{column total})}{\text{total surveyed}} =$ ________
d. The expected number of students who have a med-low anxiety level and a low need to succeed in school is about ________.

Solution

a. The column total for a high anxiety level is 57. The row total for high need to succeed in school is 155. The sample size or total surveyed is 400.

$$E = \frac{(\text{row total})(\text{column total})}{\text{total surveyed}} = \frac{155 \cdot 57}{400} = 22.09 \tag{8.3.2}$$

The expected number of students who have a high anxiety level and a high need to succeed in school is about 22.

b. The column total for a med-low anxiety level is 63. The row total for a low need to succeed in school is 52. The sample size or total surveyed is 400.

c. $E = \frac{(\text{row total})(\text{column total})}{\text{total surveyed}} = 8.19$

d. 8

? Exercise 8.3.3

Refer back to the information in Note. How many service providing jobs are there expected to be in 2020? How many nonagriculture wage and salary jobs are there expected to be in 2020?

Answer

12,727, 14,965

References

1. DiCamilo, Mark, Mervin Field, "Most Californians See a Direct Linkage between Obesity and Sugary Sodas. Two in Three Voters Support Taxing Sugar-Sweetened Beverages If Proceeds are Tied to Improving School Nutrition and Physical Activity Programs." The Field Poll, released Feb. 14, 2013. Available online at field.com/fieldpollonline/sub...rs/Rls2436.pdf (accessed May 24, 2013).
2. Harris Interactive, "Favorite Flavor of Ice Cream." Available online at http://www.statisticbrain.com/favori...r-of-ice-cream (accessed May 24, 2013)
3. "Youngest Online Entrepreneurs List." Available online at http://www.statisticbrain.com/younge...repreneur-list (accessed May 24, 2013).

Review

To assess whether two factors are independent or not, you can apply the test of independence that uses the chi-square distribution. The null hypothesis for this test states that the two factors are independent. The test compares observed values to expected values. The test is right-tailed. Each observation or cell category must have an expected value of at least 5.

Formula Review

Test of Independence

- The number of degrees of freedom is equal to $(\text{number of columns - 1})(\text{number of rows - 1})$.
- The test statistic is $\sum_{(i \cdot j)} \frac{(O-E)^2}{E}$ where $O =$ observed values, $E =$ expected values, $i =$ the number of rows in the table, and $j =$ the number of columns in the table.
- If the null hypothesis is true, the expected number $E = \frac{(\text{row total})(\text{column total})}{\text{total surveyed}}$.

Determine the appropriate test to be used in the next three exercises.

? Exercise 8.3.4

A pharmaceutical company is interested in the relationship between age and presentation of symptoms for a common viral infection. A random sample is taken of 500 people with the infection across different age groups.

Answer

a test of independence

? Exercise 8.3.5

The owner of a baseball team is interested in the relationship between player salaries and team winning percentage. He takes a random sample of 100 players from different organizations.

? Exercise 8.3.6

A marathon runner is interested in the relationship between the brand of shoes runners wear and their run times. She takes a random sample of 50 runners and records their run times as well as the brand of shoes they were wearing.

Answer

a test of independence

Use the following information to answer the next seven exercises: Transit Railroads is interested in the relationship between travel distance and the ticket class purchased. A random sample of 200 passengers is taken. Table $8.3.4$ shows the results. The railroad wants to know if a passenger's choice in ticket class is independent of the distance they must travel.

Table 8.3.4

Traveling Distance	Third class	Second class	First class	Total
1–100 miles	21	14	6	41
101–200 miles	18	16	8	42
201–300 miles	16	17	15	48
301–400 miles	12	14	21	47
401–500 miles	6	6	10	22
Total	73	67	60	200

? Exercise 8.3.7

State the hypotheses.

- H_0: _______
- H_a: _______

? Exercise 8.3.8

$df =$ _______

Answer

8

? Exercise 8.3.9

How many passengers are expected to travel between 201 and 300 miles and purchase second-class tickets?

? Exercise 8.3.10

How many passengers are expected to travel between 401 and 500 miles and purchase first-class tickets?

Answer

6.6

? Exercise 8.3.11

What is the test statistic?

? Exercise 8.3.12

What is the p-value?

Answer

0.0435

? Exercise 8.3.13

What can you conclude at the 5% level of significance?

Use the following information to answer the next eight exercises: An article in the New England Journal of Medicine, discussed a study on smokers in California and Hawaii. In one part of the report, the self-reported ethnicity and smoking levels per day were given. Of the people smoking at most ten cigarettes per day, there were 9,886 African Americans, 2,745 Native Hawaiians, 12,831 Latinos, 8,378 Japanese Americans and 7,650 whites. Of the people smoking 11 to 20 cigarettes per day, there were 6,514 African Americans, 3,062 Native Hawaiians, 4,932 Latinos, 10,680 Japanese Americans, and 9,877 whites. Of the people smoking 21 to 30 cigarettes per day, there were 1,671 African Americans, 1,419 Native Hawaiians, 1,406 Latinos, 4,715 Japanese Americans, and 6,062 whites. Of the people smoking at least 31 cigarettes per day, there were 759 African Americans, 788 Native Hawaiians, 800 Latinos, 2,305 Japanese Americans, and 3,970 whites.

? Exercise 8.3.14

Complete the table.

Table 8.3.5: Smoking Levels by Ethnicity (Observed)

Smoking Level Per Day	African American	Native Hawaiian	Latino	Japanese Americans	White	TOTALS
1-10						
11-20						
21-30						
31+						
TOTALS						

Answer

Table 8.3.5B

Smoking Level Per Day	African American	Native Hawaiian	Latino	Japanese Americans	White	Totals
1-10	9,886	2,745	12,831	8,378	7,650	41,490
11-20	6,514	3,062	4,932	10,680	9,877	35,065
21-30	1,671	1,419	1,406	4,715	6,062	15,273
31+	759	788	800	2,305	3,970	8,622
Totals	18,830	8,014	19,969	26,078	27,559	10,0450

? Exercise 8.3.15

State the hypotheses.

- H_0: _______

- H_a: _______

Exercise 8.3.16

Enter expected values in Table. Round to two decimal places.

Calculate the following values:

Answer

Table 8.3.6

Smoking Level Per Day	African American	Native Hawaiian	Latino	Japanese Americans	White
1-10	7777.57	3310.11	8248.02	10771.29	11383.01
11-20	6573.16	2797.52	6970.76	9103.29	9620.27
21-30	2863.02	1218.49	3036.20	3965.05	4190.23
31+	1616.25	687.87	1714.01	2238.37	2365.49

Exercise 8.3.17

$df =$ _______

Exercise 8.3.18

χ^2test statistic $=$ ______

Answer

10,301.8

Exercise 8.3.19

p-value $=$ ______

Exercise 8.3.20

Is this a right-tailed, left-tailed, or two-tailed test? Explain why.

Answer

right

Exercise 8.3.21

Graph the situation. Label and scale the horizontal axis. Mark the mean and test statistic. Shade in the region corresponding to the p-value.

Figure 8.3.3.

State the decision and conclusion (in a complete sentence) for the following preconceived levels of α.

Exercise 8.3.22

$\alpha = 0.05$

a. Decision: ____________________

b. Reason for the decision: ____________________
c. Conclusion (write out in a complete sentence): ____________________

Answer

a. Reject the null hypothesis.
b. p-value $< \alpha$
c. There is sufficient evidence to conclude that smoking level is dependent on ethnic group.

? Exercise $8.3.23$

$\alpha = 0.05$

a. Decision: ____________________
b. Reason for the decision: ____________________
c. Conclusion (write out in a complete sentence): ____________________

Glossary

Contingency Table

a table that displays sample values for two different factors that may be dependent or contingent on one another; it facilitates determining conditional probabilities.

8.4: Test for Homogeneity

The goodness–of–fit test can be used to decide whether a population fits a given distribution, but it will not suffice to decide whether two populations follow the same unknown distribution. A different test, called the test for homogeneity, can be used to draw a conclusion about whether two populations have the same distribution. To calculate the test statistic for a test for homogeneity, follow the same procedure as with the test of independence.

> The expected value for each cell needs to be at least five in order for you to use this test.

Hypotheses

- H_0: The distributions of the two populations are the same.
- H_a: The distributions of the two populations are not the same.

Test Statistic

- Use a χ^2 test statistic. It is computed in the same way as the test for independence.

Degrees of Freedom (df)

- $df = \text{number of columns} - 1$

Requirements

- All values in the table must be greater than or equal to five.

Common Uses

Comparing two populations. For example: men vs. women, before vs. after, east vs. west. The variable is categorical with more than two possible response values.

✓ Example 8.4.1

Do male and female college students have the same distribution of living arrangements? Use a level of significance of 0.05. Suppose that 250 randomly selected male college students and 300 randomly selected female college students were asked about their living arrangements: dormitory, apartment, with parents, other. The results are shown in Table 8.4.1. Do male and female college students have the same distribution of living arrangements?

Table 8.4.1: Distribution of Living Arragements for College Males and College Females

	Dormitory	Apartment	With Parents	Other
Males	72	84	49	45
Females	91	86	88	35

Answer

- H_0: The distribution of living arrangements for male college students is the same as the distribution of living arrangements for female college students.
- H_a: The distribution of living arrangements for male college students is not the same as the distribution of living arrangements for female college students.

Degrees of Freedom (df):

$df = \text{number of columns} - 1 = 4 - 1 = 3$

Distribution for the test: χ^2_3

Calculate the test statistic: $\chi^2 = 10.1287$ (calculator or computer)

Probability statement: $p\text{-value} = P(\chi^2 > 10.1287) = 0.0175$

Press the

```
MATRX
```

key and arrow over to

```
EDIT
```

. Press

```
1:[A]
```

. Press

```
2 ENTER 4 ENTER
```

. Enter the table values by row. Press

```
ENTER
```

after each. Press

```
2nd QUIT
```

. Press

```
STAT
```

and arrow over to

```
TESTS
```

. Arrow down to

```
C:χ2-TEST
```

. Press

```
ENTER
```

. You should see

```
Observed:[A] and Expected:[B]
```

. Arrow down to

```
Calculate
```

. Press

```
ENTER
```

. The test statistic is 10.1287 and the p-value $= 0.0175$. Do the procedure a second time but arrow down to

```
Draw
```

instead of

```
calculate
```

.

Compare α and the p-value: Since no α is given, assume $\alpha = 0.05$. p-value $= 0.0175$. $\alpha > p$-value.

Make a decision: Since $\alpha > p$-value, reject H_0. This means that the distributions are not the same.

Conclusion: At a 5% level of significance, from the data, there is sufficient evidence to conclude that the distributions of living arrangements for male and female college students are not the same.

Notice that the conclusion is only that the distributions are not the same. We cannot use the test for homogeneity to draw any conclusions about how they differ.

? Exercise 8.4.1

Do families and singles have the same distribution of cars? Use a level of significance of 0.05. Suppose that 100 randomly selected families and 200 randomly selected singles were asked what type of car they drove: sport, sedan, hatchback, truck, van/SUV. The results are shown in Table $8.4.2$. Do families and singles have the same distribution of cars? Test at a level of significance of 0.05.

Table 8.4.1

	Sport	Sedan	Hatchback	Truck	Van/SUV
Family	5	15	35	17	28
Single	45	65	37	46	7

Answer

With a p-value of almost zero, we reject the null hypothesis. The data show that the distribution of cars is not the same for families and singles.

✓ Example 11.5.2

Both before and after a recent earthquake, surveys were conducted asking voters which of the three candidates they planned on voting for in the upcoming city council election. Has there been a change since the earthquake? Use a level of significance of 0.05. Table shows the results of the survey. Has there been a change in the distribution of voter preferences since the earthquake?

	Perez	**Chung**	**Stevens**
Before	167	128	135
After	214	197	225

Answer

H_0: The distribution of voter preferences was the same before and after the earthquake.

H_a: The distribution of voter preferences was not the same before and after the earthquake.

Degrees of Freedom (df):

$df = \text{number of columns} - 1 = 3 - 1 = 2$

Distribution for the test: χ^2_2

Calculate the test statistic: $\chi^2 = 3.2603$ (calculator or computer)

Probability statement: $p\text{-value} = P(\chi^2 > 3.2603) = 0.1959$

Press the `MATRX` key and arrow over to `EDIT`. Press `1:[A]`. Press `2 ENTER 3 ENTER`. Enter the table values by row. Press `ENTER` after each. Press `2nd QUIT`. Press `STAT` and arrow over to `TESTS`. Arrow down to `C:χ2-TEST`. Press `ENTER`. You should see `Observed:[A] and Expected:[B]`. Arrow down to `Calculate`. Press `ENTER`. The test statistic is 3.2603 and the *p*-value = 0.1959. Do the procedure a second time but arrow down to `Draw` instead of `calculate`.

Compare α and the p-value: $\alpha = 0.05$ and the $p\text{-value} = 0.1959$. $\alpha < p\text{-value}$.

Make a decision: Since $\alpha < p\text{-value}$, do not reject H_0.

Conclusion: At a 5% level of significance, from the data, there is insufficient evidence to conclude that the distribution of voter preferences was not the same before and after the earthquake.

? Exercise 8.4.2

Ivy League schools receive many applications, but only some can be accepted. At the schools listed in Table, two types of applications are accepted: regular and early decision.

Application Type Accepted	Brown	Columbia	Cornell	Dartmouth	Penn	Yale
Regular	2,115	1,792	5,306	1,734	2,685	1,245
Early Decision	577	627	1,228	444	1,195	761

We want to know if the number of regular applications accepted follows the same distribution as the number of early applications accepted. State the null and alternative hypotheses, the degrees of freedom and the test statistic, sketch the graph of the *p*-value, and draw a conclusion about the test of homogeneity.

Answer

H_0: The distribution of regular applications accepted is the same as the distribution of early applications accepted.

H_a: The distribution of regular applications accepted is not the same as the distribution of early applications accepted.

$df = 5$

$\chi^2 \text{test statistic} = 430.06$

Figure 8.4.1.

Press the `MATRX` key and arrow over to `EDIT`. Press `1:[A]`. Press `3 ENTER 3 ENTER`. Enter the table values by row. Press `ENTER` after each. Press `2nd QUIT`. Press `STAT` and arrow over to `TESTS`. Arrow down to `C:χ2-TEST`. Press `ENTER`. You should see `Observed:[A] and Expected:[B]`. Arrow down to `Calculate`. Press `ENTER`. The test statistic is 430.06 and the *p*-value $= 9.80E - 91$. Do the procedure a second time but arrow down to `Draw` instead of `calculate`.

References

1. Data from the Insurance Institute for Highway Safety, 2013. Available online at www.iihs.org/iihs/ratings (accessed May 24, 2013).
2. “Energy use (kg of oil equivalent per capita).” The World Bank, 2013. Available online at http://data.worldbank.org/indicator/...G.OE/countries (accessed May 24, 2013).
3. “Parent and Family Involvement Survey of 2007 National Household Education Survey Program (NHES),” U.S. Department of Education, National Center for Education Statistics. Available online at http://nces.ed.gov/pubsearch/pubsinf...?pubid=2009030

(accessed May 24, 2013).

4. “Parent and Family Involvement Survey of 2007 National Household Education Survey Program (NHES),” U.S. Department of Education, National Center for Education Statistics. Available online at http://nces.ed.gov/pubs2009/2009030_sup.pdf (accessed May 24, 2013).

Review

To assess whether two data sets are derived from the same distribution—which need not be known, you can apply the test for homogeneity that uses the chi-square distribution. The null hypothesis for this test states that the populations of the two data sets come from the same distribution. The test compares the observed values against the expected values if the two populations followed the same distribution. The test is right-tailed. Each observation or cell category must have an expected value of at least five.

Formula Review

$\sum_{i \cdot j} \frac{(O-E)^2}{E}$ Homogeneity test statistic where: $O =$ observed values

$E =$ expected values

$i =$ number of rows in data contingency table

$j =$ number of columns in data contingency table

$df = (i-1)(j-1)$ Degrees of freedom

Exercise 8.4.3

A math teacher wants to see if two of her classes have the same distribution of test scores. What test should she use?

Answer

test for homogeneity

Exercise 8.4.4

What are the null and alternative hypotheses for Exercise?

Exercise 8.4.5

A market researcher wants to see if two different stores have the same distribution of sales throughout the year. What type of test should he use?

Answer

test for homogeneity

Exercise 8.4.6

A meteorologist wants to know if East and West Australia have the same distribution of storms. What type of test should she use?

Exercise 8.4.7

What condition must be met to use the test for homogeneity?

Answer

All values in the table must be greater than or equal to five.

Use the following information to answer the next five exercises: Do private practice doctors and hospital doctors have the same distribution of working hours? Suppose that a sample of 100 private practice doctors and 150 hospital doctors are selected at random and asked about the number of hours a week they work. The results are shown in Table.

	20–30	30–40	40–50	50–60
Private Practice	16	40	38	6
Hospital	8	44	59	39

? Exercise 8.4.8

State the null and alternative hypotheses.

? Exercise 8.4.9

$df =$ _______

Answer

3

? Exercise 8.4.10

What is the test statistic?

? Exercise 8.4.11

What is the p-value?

Answer

0.00005

? Exercise 8.4.12

What can you conclude at the 5% significance level?

8.5: Comparison of the Chi-Square Tests

You have seen the χ^2 test statistic used in three different circumstances. The following bulleted list is a summary that will help you decide which χ^2 test is the appropriate one to use.

- **Goodness-of-Fit:** Use the goodness-of-fit test to decide whether a population with an unknown distribution "fits" a known distribution. In this case there will be a single qualitative survey question or a single outcome of an experiment from a single population. Goodness-of-Fit is typically used to see if the population is uniform (all outcomes occur with equal frequency), the population is normal, or the population is the same as another population with a known distribution. The null and alternative hypotheses are:
  - H_0: The population fits the given distribution.
  - H_a: The population does not fit the given distribution.
- **Independence:** Use the test for independence to decide whether two variables (factors) are independent or dependent. In this case there will be two qualitative survey questions or experiments and a contingency table will be constructed. The goal is to see if the two variables are unrelated (independent) or related (dependent). The null and alternative hypotheses are:
  - H_0: The two variables (factors) are independent.
  - H_a: The two variables (factors) are dependent.
- **Homogeneity:** Use the test for homogeneity to decide if two populations with unknown distributions have the same distribution as each other. In this case there will be a single qualitative survey question or experiment given to two different populations. The null and alternative hypotheses are:
  - H_0: The two populations follow the same distribution.
  - H_a: The two populations have different distributions.

Review

The goodness-of-fit test is typically used to determine if data fits a particular distribution. The test of independence makes use of a contingency table to determine the independence of two factors. The test for homogeneity determines whether two populations come from the same distribution, even if this distribution is unknown.

? Exercise 8.5.1

Which test do you use to decide whether an observed distribution is the same as an expected distribution?

Answer

a goodness-of-fit test

? Exercise 8.5.2

What is the null hypothesis for the type of test from Exercise?

? Exercise 8.5.3

Which test would you use to decide whether two factors have a relationship?

Answer

a test for independence

? Exercise 8.5.4

Which test would you use to decide if two populations have the same distribution?

? Exercise 8.5.5

How are tests of independence similar to tests for homogeneity?

Answer

Answers will vary. Sample answer: Tests of independence and tests for homogeneity both calculate the test statistic the same way $\sum_{i \cdot j} \frac{(O-E)^2}{E}$. In addition, all values must be greater than or equal to five.

? Exercise 8.5.6

How are tests of independence different from tests for homogeneity?

Bringing It Together

? Exercise 8.5.7

a. Explain why a goodness-of-fit test and a test of independence are generally right-tailed tests.
b. If you did a left-tailed test, what would you be testing?

Answer a

The test statistic is always positive and if the expected and observed values are not close together, the test statistic is large and the null hypothesis will be rejected.

Answer b

Testing to see if the data fits the distribution "too well" or is too perfect.

CHAPTER OVERVIEW

9: Linear Regression and Correlation

Regression analysis is a statistical process for estimating the relationships among variables and includes many techniques for modeling and analyzing several variables. When the focus is on the relationship between a dependent variable and one or more independent variables.

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9.1: Prelude to Linear Regression and Correlation

CHAPTER OBJECTIVES

By the end of this chapter, the student should be able to:

- Discuss basic ideas of linear regression and correlation.
- Create and interpret a line of best fit.
- Calculate and interpret the correlation coefficient.
- Calculate and interpret outliers.

Professionals often want to know how two or more numeric variables are related. For example, is there a relationship between the grade on the second math exam a student takes and the grade on the final exam? If there is a relationship, what is the relationship and how strong is it? In another example, your income may be determined by your education, your profession, your years of experience, and your ability. The amount you pay a repair person for labor is often determined by an initial amount plus an hourly fee.

Figure 9.1.1: Linear regression and correlation can help you determine if an auto mechanic's salary is related to his work experience. (credit: Joshua Rothhaas)

The type of data described in the examples is **bivariate** data — "bi" for two variables. In reality, statisticians use **multivariate** data, meaning many variables. In this chapter, you will be studying the simplest form of regression, "linear regression" with one independent variable (x). This involves data that fits a line in two dimensions. You will also study correlation which measures how strong the relationship is.

9.2: Linear Equations

Linear regression for two variables is based on a linear equation with one independent variable. The equation has the form:

$$y = a + b\text{x}$$

where a and b are constant numbers. The variable x is the *independent variable*, and y is the *dependent variable*. Typically, you choose a value to substitute for the independent variable and then solve for the dependent variable.

✓ Example 9.2.1

The following examples are linear equations.

$$y = 3 + 2\text{x}$$

$$y = -0.01 + 1.2\text{x}$$

? Exercise 9.2.1

Is the following an example of a linear equation?

$$y = -0.125 - 3.5\text{x}$$

Answer

yes

The graph of a linear equation of the form $y = a + b\text{x}$ is a **straight line**. Any line that is not vertical can be described by this equation.

✓ Example 9.2.2

Graph the equation $y = -1 + 2\text{x}$.

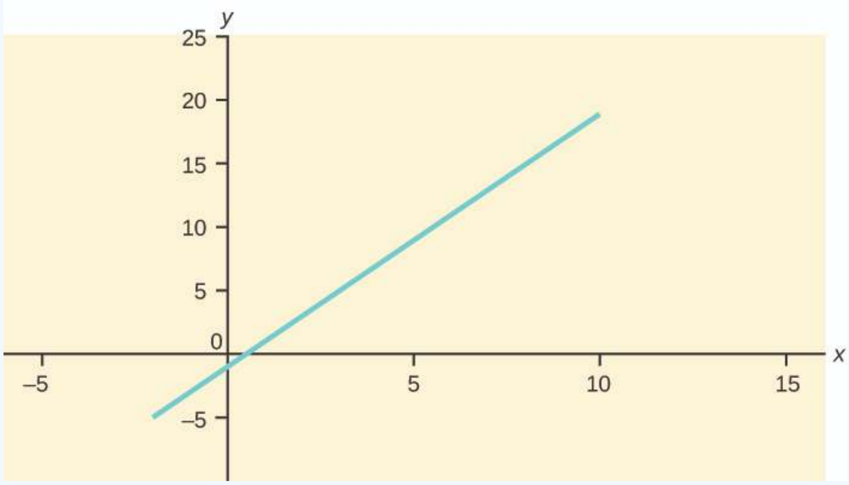


Figure 9.2.1.

? Exercise 9.2.2

Is the following an example of a linear equation? Why or why not?

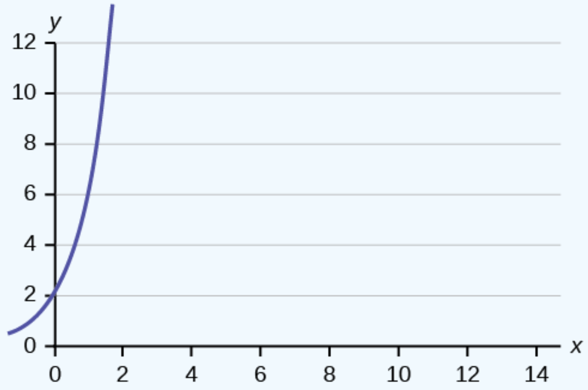


Figure 9.2.2.

Answer

No, the graph is not a straight line; therefore, it is not a linear equation.

✓ Example 9.2.3

Aaron's Word Processing Service (AWPS) does word processing. The rate for services is $32 per hour plus a $31.50 one-time charge. The total cost to a customer depends on the number of hours it takes to complete the job.

Find the equation that expresses the **total cost** in terms of the **number of hours** required to complete the job.

Answer

Let $x =$ the number of hours it takes to get the job done.

Let $y =$ the total cost to the customer.

The $31.50 is a fixed cost. If it takes x hours to complete the job, then $(32)(x)$ is the cost of the word processing only. The total cost is: $y = 31.50 + 32\text{x}$

? Exercise 9.2.3

Emma’s Extreme Sports hires hang-gliding instructors and pays them a fee of $50 per class as well as $20 per student in the class. The total cost Emma pays depends on the number of students in a class. Find the equation that expresses the total cost in terms of the number of students in a class.

Answer

$y = 50 + 20\text{x}$

Slope and Y-Intercept of a Linear Equation

For the linear equation $y = a + b\text{x}$, $b =$ slope and $a = y$-intercept. From algebra recall that the slope is a number that describes the steepness of a line, and the y-intercept is the y coordinate of the point $(0, a)$ where the line crosses the y-axis.

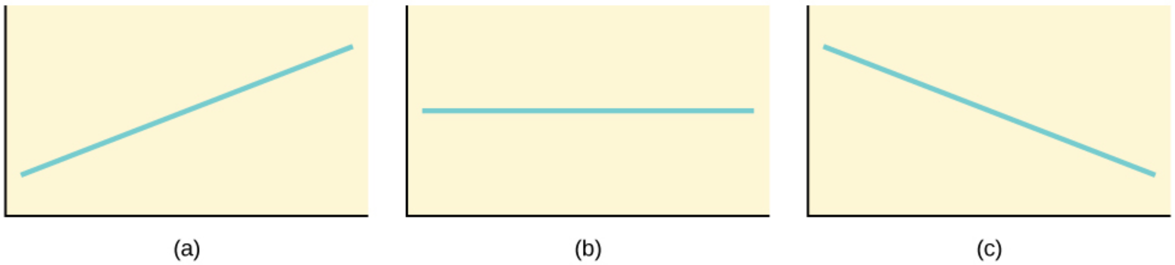


Figure 9.2.3:. Three possible graphs of $y = a + b\text{x}$ (a) If $b > 0$, the line slopes upward to the right. (b) If $b = 0$, the line is horizontal. (c) If $b < 0$, the line slopes downward to the right.

✓ Example 9.2.4

Svetlana tutors to make extra money for college. For each tutoring session, she charges a one-time fee of $25 plus $15 per hour of tutoring. A linear equation that expresses the total amount of money Svetlana earns for each session she tutors is $y = 25 + 15\text{x}$.

What are the independent and dependent variables? What is the y-intercept and what is the slope? Interpret them using complete sentences.

Answer

The independent variable (x) is the number of hours Svetlana tutors each session. The dependent variable (y) is the amount, in dollars, Svetlana earns for each session.

The y-intercept is 25 ($a = 25$). At the start of the tutoring session, Svetlana charges a one-time fee of $25 (this is when $x = 0$). The slope is 15 ($b = 15$). For each session, Svetlana earns $15 for each hour she tutors.

? Exercise 9.2.4

Ethan repairs household appliances like dishwashers and refrigerators. For each visit, he charges $25 plus $20 per hour of work. A linear equation that expresses the total amount of money Ethan earns per visit is $y = 25 + 20\text{x}$.

What are the independent and dependent variables? What is the y-intercept and what is the slope? Interpret them using complete sentences.

Answer

The independent variable (x) is the number of hours Ethan works each visit. The dependent variable (y) is the amount, in dollars, Ethan earns for each visit.

The y-intercept is 25 ($a = 25$). At the start of a visit, Ethan charges a one-time fee of $25 (this is when $x = 0$). The slope is 20 ($b = 20$). For each visit, Ethan earns $20 for each hour he works.

Summary

The most basic type of association is a linear association. This type of relationship can be defined algebraically by the equations used, numerically with actual or predicted data values, or graphically from a plotted curve. (Lines are classified as straight curves.) Algebraically, a linear equation typically takes the form $y = mx + b$, where m and b are constants, x is the independent variable, y is the dependent variable. In a statistical context, a linear equation is written in the form $y = a + bx$, where a and b are the constants. This form is used to help readers distinguish the statistical context from the algebraic context. In the equation $y = a + b\text{x}$, the constant b that multiplies the x variable (b is called a coefficient) is called the **slope**. The constant a is called the y-intercept.

The **slope of a line** is a value that describes the rate of change between the independent and dependent variables. The **slope** tells us how the dependent variable (y) changes for every one unit increase in the independent (x) variable, on average. The **y-intercept** is used to describe the dependent variable when the independent variable equals zero.

Formula Review

$y = a + b\text{x}$ where a is the y-intercept and b is the slope. The variable x is the independent variable and y is the dependent variable.

9.3: Scatter Plots

Before we take up the discussion of linear regression and correlation, we need to examine a way to display the relation between two variables *x* and *y*. The most common and easiest way is a *scatter plot*. The following example illustrates a scatter plot.

✓ Example 9.3.1

In Europe and Asia, m-commerce is popular. M-commerce users have special mobile phones that work like electronic wallets as well as provide phone and Internet services. Users can do everything from paying for parking to buying a TV set or soda from a machine to banking to checking sports scores on the Internet. For the years 2000 through 2004, was there a relationship between the year and the number of m-commerce users? Construct a scatter plot. Let $x =$ the year and let $y =$ the number of m-commerce users, in millions.

Table 9.3.1: Table showing the number of m-commerce users (in millions) by year.

x (year)	y (# of users)
2000	0.5
2002	20.0
2003	33.0
2004	47.0

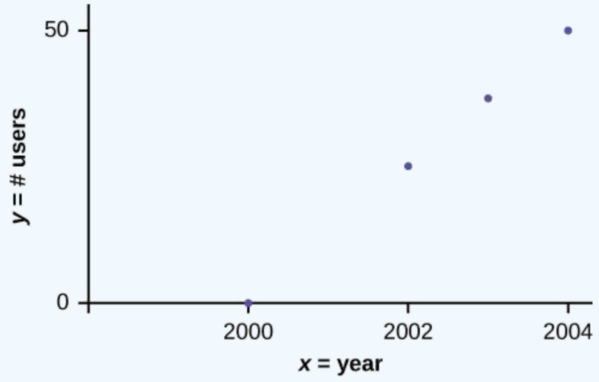


Figure 9.3.1: Scatter plot showing the number of m-commerce users (in millions) by year.

To create a scatter plot

a. Enter your X data into list L1 and your Y data into list L2.
b. Press 2nd STATPLOT ENTER to use Plot 1. On the input screen for PLOT 1, highlight On and press ENTER. (Make sure the other plots are OFF.)
c. For TYPE: highlight the very first icon, which is the scatter plot, and press ENTER.
d. For Xlist:, enter L1 ENTER and for Ylist: L2 ENTER.
e. For Mark: it does not matter which symbol you highlight, but the square is the easiest to see. Press ENTER.
f. Make sure there are no other equations that could be plotted. Press Y = and clear any equations out.
g. Press the ZOOM key and then the number 9 (for menu item "ZoomStat") ; the calculator will fit the window to the data. You can press WINDOW to see the scaling of the axes.

? Exercise 9.3.1

Amelia plays basketball for her high school. She wants to improve to play at the college level. She notices that the number of points she scores in a game goes up in response to the number of hours she practices her jump shot each week. She records the following data:

X (hours practicing jump shot)	Y (points scored in a game)
5	15
7	22
9	28
10	31
11	33
12	36

Construct a scatter plot and state if what Amelia thinks appears to be true.

Answer

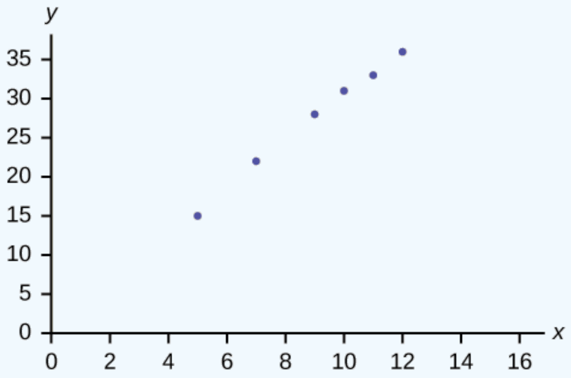


Figure $9.3.2$

Yes, Amelia's assumption appears to be correct. The number of points Amelia scores per game goes up when she practices her jump shot more.

A scatter plot shows the *direction of a relationship* between the variables. A clear direction happens when there is either:

- High values of one variable occurring with high values of the other variable or low values of one variable occurring with low values of the other variable.
- High values of one variable occurring with low values of the other variable.

You can determine the *strength of the relationship* by looking at the scatter plot and seeing how close the points are to a line, a power function, an exponential function, or to some other type of function. For a linear relationship there is an exception. Consider a scatter plot where all the points fall on a horizontal line providing a "perfect fit." The horizontal line would in fact show no relationship.

When you look at a scatter plot, you want to notice the *overall pattern* and any *deviations* from the pattern. The following scatterplot examples illustrate these concepts.

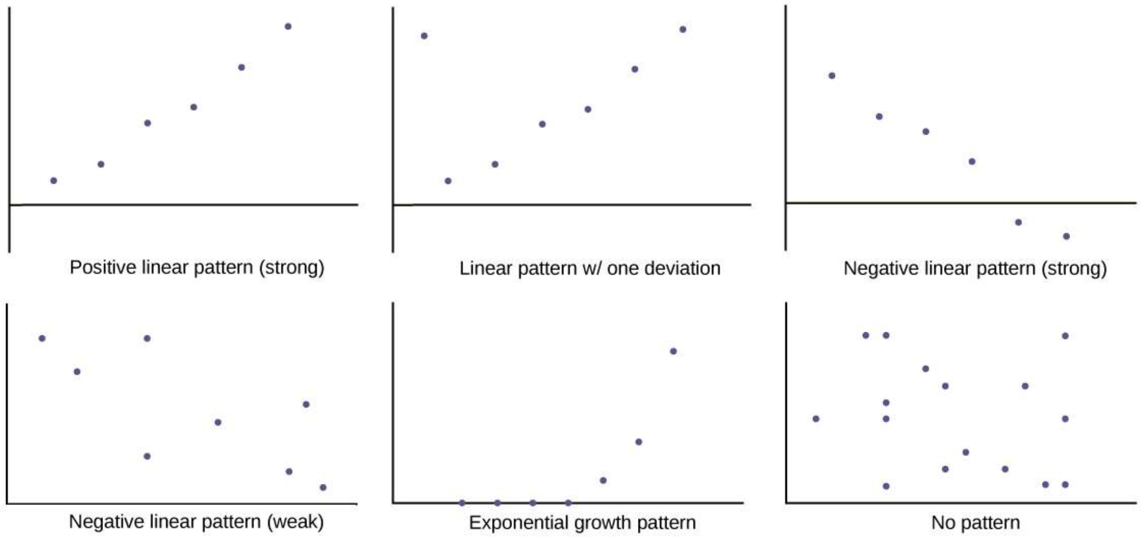


Figure 9.3.3:

In this chapter, we are interested in scatter plots that show a linear pattern. Linear patterns are quite common. The linear relationship is strong if the points are close to a straight line, except in the case of a horizontal line where there is no relationship. If we think that the points show a linear relationship, we would like to draw a line on the scatter plot. This line can be calculated through a process called linear regression. However, we only calculate a regression line if one of the variables helps to explain or predict the other variable. If x is the independent variable and y the dependent variable, then we can use a regression line to predict y for a given value of x

Summary

Scatter plots are particularly helpful graphs when we want to see if there is a linear relationship among data points. They indicate both the direction of the relationship between the x variables and the y variables, and the strength of the relationship. We calculate the strength of the relationship between an independent variable and a dependent variable using linear regression.

9.4: The Regression Equation

Data rarely fit a straight line exactly. Usually, you must be satisfied with rough predictions. Typically, you have a set of data whose scatter plot appears to **"fit"** a straight line. This is called a Line of Best Fit **or** Least-Squares Line.

COLLABORATIVE EXERCISE

If you know a person's pinky (smallest) finger length, do you think you could predict that person's height? Collect data from your class (pinky finger length, in inches). The independent variable, x, is pinky finger length and the dependent variable, y, is height. For each set of data, plot the points on graph paper. Make your graph big enough and **use a ruler**. Then "by eye" draw a line that appears to "fit" the data. For your line, pick two convenient points and use them to find the slope of the line. Find the y-intercept of the line by extending your line so it crosses the y-axis. Using the slopes and the y-intercepts, write your equation of "best fit." Do you think everyone will have the same equation? Why or why not? According to your equation, what is the predicted height for a pinky length of 2.5 inches?

✓ Example 9.4.1

A random sample of 11 statistics students produced the following data, where x is the third exam score out of 80, and y is the final exam score out of 200. Can you predict the final exam score of a random student if you know the third exam score?

1a: Table showing the scores on the final exam based on scores from the third exam.

x (third exam score)	y (final exam score)
65	175
67	133
71	185
71	163
66	126
75	198
67	153
70	163
71	159
69	151
69	159

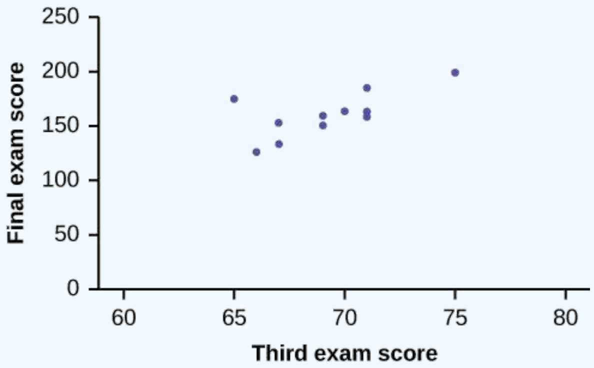


Figure 9.4.1: Scatter plot showing the scores on the final exam based on scores from the third exam.

? Exercise 9.4.1

SCUBA divers have maximum dive times they cannot exceed when going to different depths. The data in Table show different depths with the maximum dive times in minutes. Use your calculator to find the least squares regression line and predict the maximum dive time for 110 feet.

X (depth in feet)	Y (maximum dive time)
50	80
60	55
70	45
80	35
90	25
100	22

Answer

$\hat{y} = 127.24 – 1.11x$

At 110 feet, a diver could dive for only five minutes.

The third exam score, x, is the independent variable and the final exam score, y, is the dependent variable. We will plot a regression line that best "fits" the data. If each of you were to fit a line "by eye," you would draw different lines. We can use what is called a least-squares regression line to obtain the best fit line.

Consider the following diagram. Each point of data is of the the form (x, y) and each point of the line of best fit using least-squares linear regression has the form $(x, \hat{y})$.

The $\hat{y}$ is read **"y hat"** and is the **estimated value of** y. It is the value of y obtained using the regression line. It is not generally equal to y from data.

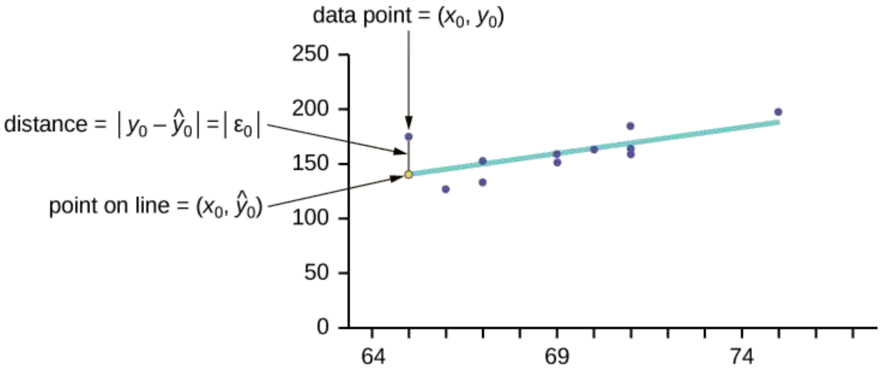


Figure 9.4.2

The term $y_0 – \hat{y}_0 = \varepsilon_0$ is called the **"error" or** residual. It is not an error in the sense of a mistake. The absolute value of a residual measures the vertical distance between the actual value of y and the estimated value of y. In other words, it measures the vertical distance between the actual data point and the predicted point on the line.

If the observed data point lies above the line, the residual is positive, and the line underestimates the actual data value for y. If the observed data point lies below the line, the residual is negative, and the line overestimates that actual data value for y.

In the diagram in Figure, $y_0 – \hat{y}_0 = \varepsilon_0$ is the residual for the point shown. Here the point lies above the line and the residual is positive.

$\varepsilon =$ the Greek letter **epsilon**

For each data point, you can calculate the residuals or errors, $y_i - \hat{y}_i = \varepsilon_i$ for $i = 1, 2, 3, \ldots, 11$.

Each $|\varepsilon|$ is a vertical distance.

For the example about the third exam scores and the final exam scores for the 11 statistics students, there are 11 data points. Therefore, there are 11 ε values. If you square each ε and add, you get

$$(\varepsilon_1)^2 + (\varepsilon_2)^2 + \ldots + (\varepsilon_{11})^2 = \sum_{i=1}^{11} \varepsilon^2 \tag{9.4.1}$$

Equation9.4.1 is called the **Sum of Squared Errors (SSE)**.

Using calculus, you can determine the values of a and b that make the **SSE** a minimum. When you make the **SSE** a minimum, you have determined the points that are on the line of best fit. It turns out that the line of best fit has the equation:

$$\hat{y} = a + bx \tag{9.4.2}$$

where

- $a = \bar{y} - b\bar{x}$ and
- $b = \dfrac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$.

The sample means of the x values and the x values are $\bar{x}$ and $\bar{y}$, respectively. The best fit line always passes through the point $(\bar{x}, \bar{y})$.

The slope b can be written as $b = r\left(\dfrac{s_y}{s_x}\right)$ where $s_y =$ the standard deviation of the y values and $s_x =$ the standard deviation of the x values. r is the correlation coefficient, which is discussed in the next section.

Least Square Criteria for Best Fit

The process of fitting the best-fit line is called **linear regression**. The idea behind finding the best-fit line is based on the assumption that the data are scattered about a straight line. The criteria for the best fit line is that the sum of the squared errors (SSE) is minimized, that is, made as small as possible. Any other line you might choose would have a higher SSE than the best fit line. This best fit line is called the **least-squares regression line** .

> **Note**
>
> Computer spreadsheets, statistical software, and many calculators can quickly calculate the best-fit line and create the graphs. The calculations tend to be tedious if done by hand. Instructions to use the TI-83, TI-83+, and TI-84+ calculators to find the best-fit line and create a scatterplot are shown at the end of this section.

THIRD EXAM vs FINAL EXAM EXAMPLE:

The graph of the line of best fit for the third-exam/final-exam example is as follows:

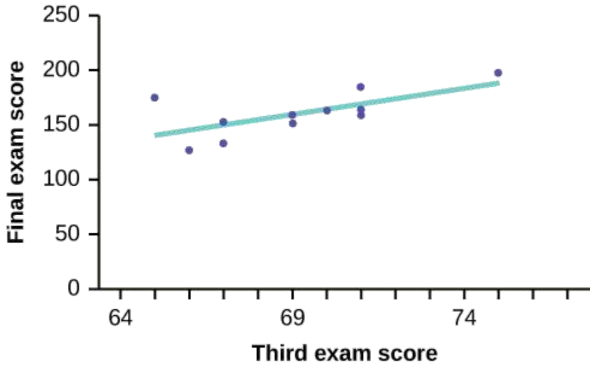


Figure 9.4.3

The least squares regression line (best-fit line) for the third-exam/final-exam example has the equation:

$$\hat{y} = -173.51 + 4.83x \tag{9.4.3}$$

REMINDER

Remember, it is always important to plot a scatter diagram first. If the scatter plot indicates that there is a linear relationship between the variables, then it is reasonable to use a best fit line to make predictions for y given x within the domain of x-values in the sample data, **but not necessarily for x-values outside that domain.** You could use the line to predict the final exam score for a student who earned a grade of 73 on the third exam. You should NOT use the line to predict the final exam score for a student who earned a grade of 50 on the third exam, because 50 is not within the domain of the x-values in the sample data, which are between 65 and 75.

Understanding Slope

The slope of the line, b, describes how changes in the variables are related. It is important to interpret the slope of the line in the context of the situation represented by the data. You should be able to write a sentence interpreting the slope in plain English.

INTERPRETATION OF THE SLOPE: The slope of the best-fit line tells us how the dependent variable (y) changes for every one unit increase in the independent (x) variable, on average.

THIRD EXAM vs FINAL EXAM EXAMPLE

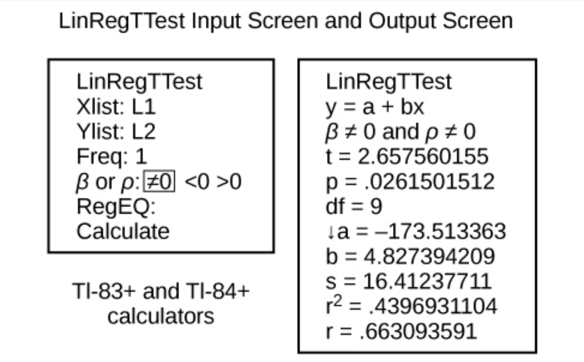
Slope: The slope of the line is $b = 4.83$.

Interpretation: For a one-point increase in the score on the third exam, the final exam score increases by 4.83 points, on average.

USING THE TI-83, 83+, 84, 84+ CALCULATOR

Using the Linear Regression T Test: LinRegTTest

a. In the STAT list editor, enter the X data in list L1 and the Y data in list L2, paired so that the corresponding (x, y) values are next to each other in the lists. (If a particular pair of values is repeated, enter it as many times as it appears in the data.)
b. On the STAT TESTS menu, scroll down with the cursor to select the LinRegTTest. (Be careful to select LinRegTTest, as some calculators may also have a different item called LinRegTInt.)
c. On the LinRegTTest input screen enter: Xlist: L1 ; Ylist: L2 ; Freq: 1
d. On the next line, at the prompt β or ρ, highlight "$\neq 0$" and press ENTER
e. Leave the line for "RegEq:" blank
f. Highlight Calculate and press ENTER.



LinRegTTest Input Screen and Output Screen

```
LinRegTTest
Xlist: L1
Ylist: L2
Freq: 1
β or ρ: ≠0  <0 >0
RegEQ:
Calculate
```

TI-83+ and TI-84+ calculators

```
LinRegTTest
y = a + bx
β ≠ 0 and ρ ≠ 0
t = 2.657560155
p = .0261501512
df = 9
↓a = –173.513363
b = 4.827394209
s = 16.41237711
r² = .4396931104
r = .663093591
```

Figure 9.4.4

The output screen contains a lot of information. For now we will focus on a few items from the output, and will return later to the other items.

The second line says $y = a + bx$. Scroll down to find the values $a = -173.513$, and $b = 4.8273$; the equation of the best fit line is $\hat{y} = -173.51 + 4.83x$

The two items at the bottom are $r_2 = 0.43969$ and $r = 0.663$. For now, just note where to find these values; we will discuss them in the next two sections.

Graphing the Scatterplot and Regression Line

1. We are assuming your X data is already entered in list L1 and your Y data is in list L2
2. Press 2nd STATPLOT ENTER to use Plot 1

3. On the input screen for PLOT 1, highlight **On**, and press ENTER
4. For TYPE: highlight the very first icon which is the scatterplot and press ENTER
5. Indicate Xlist: L1 and Ylist: L2
6. For Mark: it does not matter which symbol you highlight.
7. Press the ZOOM key and then the number 9 (for menu item "ZoomStat") ; the calculator will fit the window to the data
8. To graph the best-fit line, press the "$Y =$" key and type the equation $-173.5 + 4.83X$ into equation Y1. (The X key is immediately left of the STAT key). Press ZOOM 9 again to graph it.
9. Optional: If you want to change the viewing window, press the WINDOW key. Enter your desired window using Xmin, Xmax, Ymin, Ymax

Note

Another way to graph the line after you create a scatter plot is to use LinRegTTest.

a. Make sure you have done the scatter plot. Check it on your screen.
b. Go to LinRegTTest and enter the lists.
c. At RegEq: press VARS and arrow over to Y-VARS. Press 1 for 1:Function. Press 1 for 1:Y1. Then arrow down to Calculate and do the calculation for the line of best fit.
d. Press $Y = (\text{you will see the regression equation})$.
e. Press GRAPH. The line will be drawn."

The Correlation Coefficient r

Besides looking at the scatter plot and seeing that a line seems reasonable, how can you tell if the line is a good predictor? Use the correlation coefficient as another indicator (besides the scatterplot) of the strength of the relationship between x and y. The **correlation coefficient, r,** developed by Karl Pearson in the early 1900s, is numerical and provides a measure of strength and direction of the linear association between the independent variable x and the dependent variable y.

The correlation coefficient is calculated as

$$r = \frac{n\sum(xy) - \left(\sum x\right)\left(\sum y\right)}{\sqrt{\left[n\sum x^2 - \left(\sum x\right)^2\right]\left[n\sum y^2 - \left(\sum y\right)^2\right]}} \tag{9.4.4}$$

where $n =$ the number of data points.

If you suspect a linear relationship between x and y, then r can measure how strong the linear relationship is.

What the VALUE of r tells us:

- The value of r is always between –1 and +1: –1 ≤ r ≤ 1.
- The size of the correlation r indicates the strength of the linear relationship between x and y. Values of r close to –1 or to +1 indicate a stronger linear relationship between x and y.
- If $r = 0$ there is absolutely no linear relationship between x and y **(no linear correlation)**.
- If $r = 1$, there is perfect positive correlation. If $r = -1$, there is perfect negative correlation. In both these cases, all of the original data points lie on a straight line. Of course,in the real world, this will not generally happen.

What the SIGN of r tells us:

- A positive value of r means that when x increases, y tends to increase and when x decreases, y tends to decrease **(positive correlation)**.
- A negative value of r means that when x increases, y tends to decrease and when x decreases, y tends to increase **(negative correlation)**.
- The sign of r is the same as the sign of the slope, b, of the best-fit line.

Note

Strong correlation does not suggest that x causes y or y causes x. We say **"correlation does not imply causation."**

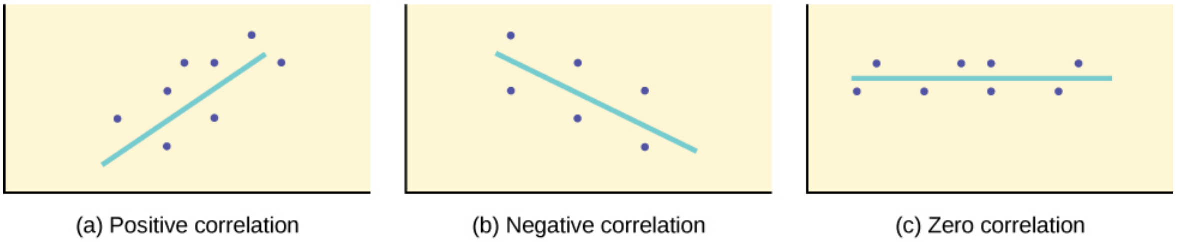


Figure 9.4.5: (a) A scatter plot showing data with a positive correlation. $0 < r < 1$ (b) A scatter plot showing data with a negative correlation. $-1 < r < 0$ (c) A scatter plot showing data with zero correlation. $r = 0$

The formula for r looks formidable. However, computer spreadsheets, statistical software, and many calculators can quickly calculate r. The correlation coefficient r is the bottom item in the output screens for the LinRegTTest on the TI-83, TI-83+, or TI-84+ calculator (see previous section for instructions).

The Coefficient of Determination

The variable r^2 is called *the coefficient of determination* and is the square of the correlation coefficient, but is usually stated as a percent, rather than in decimal form. It has an interpretation in the context of the data:

- r^2, when expressed as a percent, represents the percent of variation in the dependent (predicted) variable y that can be explained by variation in the independent (explanatory) variable x using the regression (best-fit) line.
- $1 - r^2$, when expressed as a percentage, represents the percent of variation in y that is NOT explained by variation in x using the regression line. This can be seen as the scattering of the observed data points about the regression line.

Consider the third exam/final exam example introduced in the previous section

- The line of best fit is: $\hat{y} = -173.51 + 4.83x$
- The correlation coefficient is $r = 0.6631$
- The coefficient of determination is $r^2 = 0.6631^2 = 0.4397$
- **Interpretation of r^2 in the context of this example:**
- Approximately 44% of the variation (0.4397 is approximately 0.44) in the final-exam grades can be explained by the variation in the grades on the third exam, using the best-fit regression line.
- Therefore, approximately 56% of the variation $(1 - 0.44 = 0.56)$ in the final exam grades can NOT be explained by the variation in the grades on the third exam, using the best-fit regression line. (This is seen as the scattering of the points about the line.)

Summary

A regression line, or a line of best fit, can be drawn on a scatter plot and used to predict outcomes for the x and y variables in a given data set or sample data. There are several ways to find a regression line, but usually the least-squares regression line is used because it creates a uniform line. Residuals, also called “errors,” measure the distance from the actual value of y and the estimated value of y. The Sum of Squared Errors, when set to its minimum, calculates the points on the line of best fit. Regression lines can be used to predict values within the given set of data, but should not be used to make predictions for values outside the set of data.

The correlation coefficient r measures the strength of the linear association between x and y. The variable r has to be between –1 and +1. When r is positive, the x and y will tend to increase and decrease together. When r is negative, x will increase and y will decrease, or the opposite, x will decrease and y will increase. The coefficient of determination r^2, is equal to the square of the correlation coefficient. When expressed as a percent, r^2 represents the percent of variation in the dependent variable y that can be explained by variation in the independent variable x using the regression line.

Glossary

Coefficient of Correlation

a measure developed by Karl Pearson (early 1900s) that gives the strength of association between the independent variable and the dependent variable; the formula is:

$$r=\frac{n\sum xy-\left(\sum x\right)\left(\sum y\right)}{\sqrt{\left[n\sum x^{2}-\left(\sum x\right)^{2}\right]\left[n\sum y^{2}-\left(\sum y\right)^{2}\right]}} \tag{9.4.5}$$

where n is the number of data points. The coefficient cannot be more than 1 or less than –1. The closer the coefficient is to ±1, the stronger the evidence of a significant linear relationship between x and y.

9.5: Testing the Significance of the Correlation Coefficient

The correlation coefficient, r, tells us about the strength and direction of the linear relationship between x and y. However, the reliability of the linear model also depends on how many observed data points are in the sample. We need to look at both the value of the correlation coefficient r and the sample size n, together. We perform a hypothesis test of the **"significance of the correlation coefficient"** to decide whether the linear relationship in the sample data is strong enough to use to model the relationship in the population.

The sample data are used to compute r, the correlation coefficient for the sample. If we had data for the entire population, we could find the population correlation coefficient. But because we have only sample data, we cannot calculate the population correlation coefficient. The sample correlation coefficient, r, is our estimate of the unknown population correlation coefficient.

- The symbol for the population correlation coefficient is ρ, the Greek letter "rho."
- $\rho =$ population correlation coefficient (unknown)
- $r =$ sample correlation coefficient (known; calculated from sample data)

The hypothesis test lets us decide whether the value of the population correlation coefficient ρ is "close to zero" or "significantly different from zero". We decide this based on the sample correlation coefficient r and the sample size n.

If the test concludes that the correlation coefficient is significantly different from zero, we say that the correlation coefficient is "significant."

- Conclusion: There is sufficient evidence to conclude that there is a significant linear relationship between x and y because the correlation coefficient is significantly different from zero.
- What the conclusion means: There is a significant linear relationship between x and y. We can use the regression line to model the linear relationship between x and y in the population.

If the test concludes that the correlation coefficient is not significantly different from zero (it is close to zero), we say that correlation coefficient is "not significant".

- Conclusion: "There is insufficient evidence to conclude that there is a significant linear relationship between x and y because the correlation coefficient is not significantly different from zero."
- What the conclusion means: There is not a significant linear relationship between x and y. Therefore, we CANNOT use the regression line to model a linear relationship between x and y in the population.

NOTE

- If r is significant and the scatter plot shows a linear trend, the line can be used to predict the value of y for values of x that are within the domain of observed x values.
- If r is not significant OR if the scatter plot does not show a linear trend, the line should not be used for prediction.
- If r is significant and if the scatter plot shows a linear trend, the line may NOT be appropriate or reliable for prediction OUTSIDE the domain of observed x values in the data.

PERFORMING THE HYPOTHESIS TEST

- **Null Hypothesis:** $H_0 : \rho = 0$
- **Alternate Hypothesis:** $H_a : \rho \neq 0$

WHAT THE HYPOTHESES MEAN IN WORDS:

- **Null Hypothesis H_0:** The population correlation coefficient IS NOT significantly different from zero. There IS NOT a significant linear relationship(correlation) between x and y in the population.
- **Alternate Hypothesis H_a:** The population correlation coefficient IS significantly DIFFERENT FROM zero. There IS A SIGNIFICANT LINEAR RELATIONSHIP (correlation) between x and y in the population.

DRAWING A CONCLUSION:There are two methods of making the decision. The two methods are equivalent and give the same result.

- **Method 1: Using the** p-value
- **Method 2: Using a table of critical values**

In this chapter of this textbook, we will always use a significance level of 5%, $\alpha = 0.05$

NOTE

Using the p-value method, you could choose any appropriate significance level you want; you are not limited to using $\alpha = 0.05$. But the table of critical values provided in this textbook assumes that we are using a significance level of 5%, $\alpha = 0.05$. (If we wanted to use a different significance level than 5% with the critical value method, we would need different tables of critical values that are not provided in this textbook.)

METHOD 1: Using a p-value to make a decision

Using the TI83, 83+, 84, 84+ CALCULATOR

To calculate the p-value using LinRegTTEST:

On the LinRegTTEST input screen, on the line prompt for β or ρ, highlight "$\neq 0$"

The output screen shows the p-value on the line that reads "$p =$".

(Most computer statistical software can calculate the p-value.)

If the p-value is less than the significance level ($\alpha = 0.05$):

- Decision: Reject the null hypothesis.
- Conclusion: "There is sufficient evidence to conclude that there is a significant linear relationship between x and y because the correlation coefficient is significantly different from zero."

If the p-value is NOT less than the significance level ($\alpha = 0.05$)

- Decision: DO NOT REJECT the null hypothesis.
- Conclusion: "There is insufficient evidence to conclude that there is a significant linear relationship between x and y because the correlation coefficient is NOT significantly different from zero."

Calculation Notes:

- You will use technology to calculate the p-value. The following describes the calculations to compute the test statistics and the p-value:
- The p-value is calculated using a t-distribution with $n-2$ degrees of freedom.
- The formula for the test statistic is $t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$. The value of the test statistic, t, is shown in the computer or calculator output along with the p-value. The test statistic t has the same sign as the correlation coefficient r.
- The p-value is the combined area in both tails.

An alternative way to calculate the p-value **(p)** given by LinRegTTest is the command 2*tcdf(abs(t),10^99, n-2) in 2nd DISTR.

THIRD-EXAM vs FINAL-EXAM EXAMPLE: p-value **method**

- Consider the third exam/final exam example.
- The line of best fit is: $\hat{y} = -173.51 + 4.83x$ with $r = 0.6631$ and there are $n = 11$ data points.
- Can the regression line be used for prediction? **Given a third exam score (x value), can we use the line to predict the final exam score (predicted y value)?**

$H_0 : \rho = 0$

$H_a : \rho \neq 0$

$\alpha = 0.05$

- The p-value is 0.026 (from LinRegTTest on your calculator or from computer software).
- The p-value, 0.026, is less than the significance level of $\alpha = 0.05$.
- Decision: Reject the Null Hypothesis H_0
- Conclusion: There is sufficient evidence to conclude that there is a significant linear relationship between the third exam score (x) and the final exam score (y) because the correlation coefficient is significantly different from zero.

Because r is significant and the scatter plot shows a linear trend, the regression line can be used to predict final exam scores.

METHOD 2: Using a table of Critical Values to make a decision

The 95% Critical Values of the Sample Correlation Coefficient Table can be used to give you a good idea of whether the computed value of r **is significant or not**. Compare r to the appropriate critical value in the table. If r is not between the positive and negative critical values, then the correlation coefficient is significant. If r is significant, then you may want to use the line for prediction.

✓ Example 9.5.1

Suppose you computed $r = 0.801$ using $n = 10$ data points. $df = n - 2 = 10 - 2 = 8$. The critical values associated with $df = 8$ are -0.632 and $+0.632$. If $r <$ negative critical value or $r >$ positive critical value, then r is significant. Since $r = 0.801$ and $0.801 > 0.632$, r is significant and the line may be used for prediction. If you view this example on a number line, it will help you.

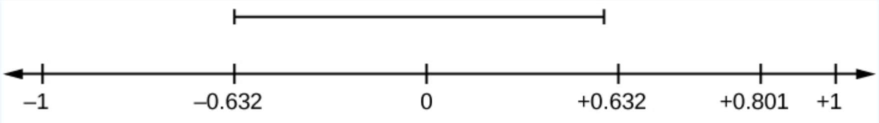


Figure 9.5.1. r is not significant between -0.632 and $+0.632$. $r = 0.801 > +0.632$. Therefore, r is significant.

? Exercise 9.5.1

For a given line of best fit, you computed that $r = 0.6501$ using $n = 12$ data points and the critical value is 0.576. Can the line be used for prediction? Why or why not?

Answer

If the scatter plot looks linear then, yes, the line can be used for prediction, because $r >$ the positive critical value.

✓ Example 9.5.2

Suppose you computed $r = -0.624$ with 14 data points. $df = 14 - 2 = 12$. The critical values are -0.532 and 0.532. Since $-0.624 < -0.532$, r is significant and the line can be used for prediction

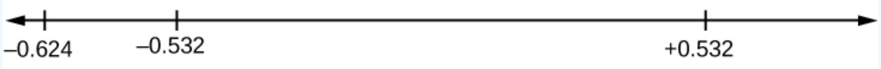


Figure 9.5.2. $r = -0.624 - 0.532$. Therefore, r is significant.

? Exercise 9.5.2

For a given line of best fit, you compute that $r = 0.5204$ using $n = 9$ data points, and the critical value is 0.666. Can the line be used for prediction? Why or why not?

Answer

No, the line cannot be used for prediction, because $r <$ the positive critical value.

✓ Example 9.5.3

Suppose you computed $r = 0.776$ and $n = 6$. $df = 6 - 2 = 4$. The critical values are -0.811 and 0.811. Since $-0.811 < 0.776 < 0.811$, r is not significant, and the line should not be used for prediction.

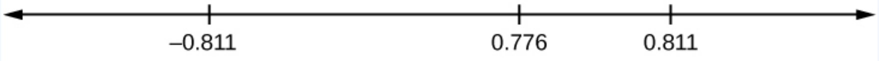


Figure 9.5.3. $-0.811 < r = 0.776 < 0.811$. Therefore, r is not significant.

? Exercise $9.5.3$

For a given line of best fit, you compute that $r = -0.7204$ using $n = 8$ data points, and the critical value is $= 0.707$. Can the line be used for prediction? Why or why not?

Answer

Yes, the line can be used for prediction, because $r <$ the negative critical value.

THIRD-EXAM vs FINAL-EXAM EXAMPLE: critical value method

Consider the third exam/final exam example. The line of best fit is: $\hat{y} = -173.51 + 4.83x$ with $r = 0.6631$ and there are $n = 11$ data points. Can the regression line be used for prediction? **Given a third-exam score (x value), can we use the line to predict the final exam score (predicted y value)?**

- $H_0 : \rho = 0$
- $H_a : \rho \neq 0$
- $\alpha = 0.05$

- Use the "95% Critical Value" table for r with $df = n - 2 = 11 - 2 = 9$.
- The critical values are -0.602 and $+0.602$
- Since $0.6631 > 0.602$, r is significant.
- Decision: Reject the null hypothesis.
- Conclusion:There is sufficient evidence to conclude that there is a significant linear relationship between the third exam score (x) and the final exam score (y) because the correlation coefficient is significantly different from zero.

Because r is significant and the scatter plot shows a linear trend, the regression line can be used to predict final exam scores.

✓ Example $9.5.4$

Suppose you computed the following correlation coefficients. Using the table at the end of the chapter, determine if r is significant and the line of best fit associated with each r can be used to predict a y value. If it helps, draw a number line.

a. $r =– 0.567$ and the sample size, n, is 19. The $df = n - 2 = 17$. The critical value is -0.456. $-0.567 < -0.456$ so r is significant.
b. $r = 0.708$ and the sample size, n, is 9. The $df = n - 2 = 7$. The critical value is 0.666. $0.708 > 0.666$ so r is significant.
c. $r = 0.134$ and the sample size, n, is 14. The $df = 14 - 2 = 12$. The critical value is 0.532. 0.134 is between -0.532 and 0.532 so r is not significant.
d. $r = 0$ and the sample size, n, is five. No matter what the dfs are, $r = 0$ is between the two critical values so r is not significant.

? Exercise $9.5.4$

For a given line of best fit, you compute that $r = 0$ using $n = 100$ data points. Can the line be used for prediction? Why or why not?

Answer

No, the line cannot be used for prediction no matter what the sample size is.

Assumptions in Testing the Significance of the Correlation Coefficient

Testing the significance of the correlation coefficient requires that certain assumptions about the data are satisfied. The premise of this test is that the data are a sample of observed points taken from a larger population. We have not examined the entire population because it is not possible or feasible to do so. We are examining the sample to draw a conclusion about whether the linear relationship that we see between x and y in the sample data provides strong enough evidence so that we can conclude that there is a linear relationship between x and y in the population.

The regression line equation that we calculate from the sample data gives the best-fit line for our particular sample. We want to use this best-fit line for the sample as an estimate of the best-fit line for the population. Examining the scatter plot and testing the significance of the correlation coefficient helps us determine if it is appropriate to do this.

The assumptions underlying the test of significance are:

- There is a linear relationship in the population that models the average value of y for varying values of x. In other words, the expected value of y for each particular value lies on a straight line in the population. (We do not know the equation for the line for the population. Our regression line from the sample is our best estimate of this line in the population.)
- The y values for any particular x value are normally distributed about the line. This implies that there are more y values scattered closer to the line than are scattered farther away. Assumption (1) implies that these normal distributions are centered on the line: the means of these normal distributions of y values lie on the line.
- The standard deviations of the population y values about the line are equal for each value of x. In other words, each of these normal distributions of y values has the same shape and spread about the line.
- The residual errors are mutually independent (no pattern).
- The data are produced from a well-designed, random sample or randomized experiment.

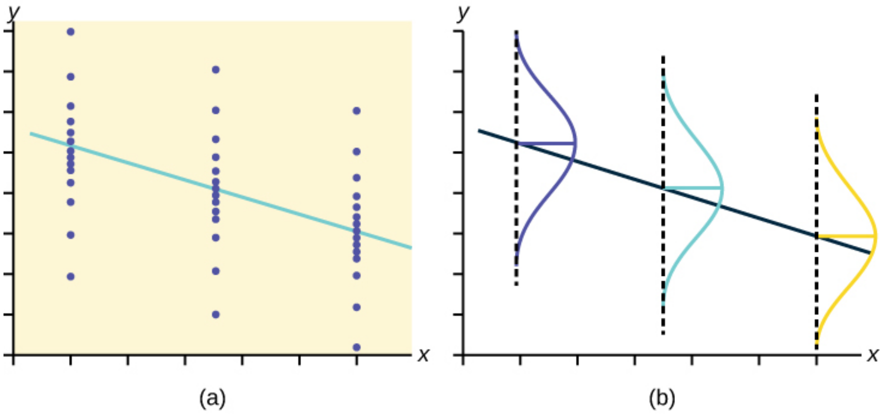


Figure 9.5.4. The y values for each x value are normally distributed about the line with the same standard deviation. For each x value, the mean of the y values lies on the regression line. More y values lie near the line than are scattered further away from the line.

Summary

Linear regression is a procedure for fitting a straight line of the form $\hat{y} = a + bx$ to data. The conditions for regression are:

- **Linear** In the population, there is a linear relationship that models the average value of y for different values of x.
- **Independent** The residuals are assumed to be independent.
- **Normal** The y values are distributed normally for any value of x.
- **Equal variance** The standard deviation of the y values is equal for each x value.
- **Random** The data are produced from a well-designed random sample or randomized experiment.

The slope b and intercept a of the least-squares line estimate the slope β and intercept α of the population (true) regression line. To estimate the population standard deviation of y, σ, use the standard deviation of the residuals, s. $s = \sqrt{\frac{SEE}{n-2}}$. The variable ρ (rho) is the population correlation coefficient. To test the null hypothesis $H_0 : \rho = \textit{hypothesized value}$, use a linear regression t-test. The most common null hypothesis is $H_0 : \rho = 0$ which indicates there is no linear relationship between x and y in the population. The TI-83, 83+, 84, 84+ calculator function LinRegTTest can perform this test (STATS TESTS LinRegTTest).

Formula Review

Least Squares Line or Line of Best Fit:

$$\hat{y} = a + bx \tag{9.5.1}$$

where

$$a = y\text{-intercept} \tag{9.5.2}$$

$$b = \text{slope} \tag{9.5.3}$$

Standard deviation of the residuals:

$$s = \sqrt{\frac{SSE}{n-2}} \tag{9.5.4}$$

where

$$SSE = \text{sum of squared errors} \tag{9.5.5}$$

$$n = \text{the number of data points} \tag{9.5.6}$$

9.6: Prediction

Recall the third exam/final exam example. We examined the scatter plot and showed that the correlation coefficient is significant. We found the equation of the best-fit line for the final exam grade as a function of the grade on the third-exam. We can now use the least-squares regression line for prediction.

Suppose you want to estimate, or predict, the mean final exam score of statistics students who received 73 on the third exam. The exam scores (x-values) range from 65 to 75. Since 73 is between the x-values 65 and 75, substitute $x = 73$ into the equation. Then:

$$\hat{y} = -173.51 + 4.83(73) = 179.08$$

We predict that statistics students who earn a grade of 73 on the third exam will earn a grade of 179.08 on the final exam, on average.

✓ Example 9.6.1

Recall the third exam/final exam example.

a. What would you predict the final exam score to be for a student who scored a 66 on the third exam?
b. What would you predict the final exam score to be for a student who scored a 90 on the third exam?

Answer

a. 145.27

b. The x values in the data are between 65 and 75. Ninety is outside of the domain of the observed x values in the data (independent variable), so you cannot reliably predict the final exam score for this student. (Even though it is possible to enter 90 into the equation for x and calculate a corresponding y value, the y value that you get will not be reliable.)

To understand really how unreliable the prediction can be outside of the observed x-values observed in the data, make the substitution $x = 90$ into the equation.

$$\hat{y} = -173.51 + 4.83(90) = 261.19$$

The final-exam score is predicted to be 261.19. The largest the final-exam score can be is 200.

The process of predicting inside of the observed x values observed in the data is called *interpolation*. The process of predicting outside of the observed x-values observed in the data is called *extrapolation*.

? Exercise 9.6.1

Data are collected on the relationship between the number of hours per week practicing a musical instrument and scores on a math test. The line of best fit is as follows:

$$\hat{y} = 72.5 + 2.8x$$

What would you predict the score on a math test would be for a student who practices a musical instrument for five hours a week?

Answer

86.5

Summary

After determining the presence of a strong correlation coefficient and calculating the line of best fit, you can use the least squares regression line to make predictions about your data.

References

1. Data from the Centers for Disease Control and Prevention.
2. Data from the National Center for HIV, STD, and TB Prevention.

3. Data from the United States Census Bureau. Available online at www.census.gov/compendia/stat...atalities.html
4. Data from the National Center for Health Statistics.

CHAPTER OVERVIEW

10: F Distribution and One-Way ANOVA

For hypothesis tests comparing averages between more than two groups, statisticians have developed a method called "Analysis of Variance" (abbreviated $ANOVA$). In this chapter, you will study the simplest form of $ANOVA$ called single factor or one-way $ANOVA$. You will also study the F distribution, used for one-way $ANOVA$, and the test of two variances. This is just a very brief overview of one-way $ANOVA$. You will study this topic in much greater detail in future statistics courses. One-Way $ANOVA$, as it is presented here, relies heavily on a calculator or computer

Topic hierarchy

Contributors

Barbara Illowsky and Susan Dean (De Anza College) with many other contributing authors. Content produced by OpenStax College is licensed under a Creative Commons Attribution License 4.0 license. Download for free at http://cnx.org/contents/30189442-699...b91b9de@18.114.

10.1: Prelude to F Distribution and One-Way ANOVA

CHAPTER OBJECTIVES

By the end of this chapter, the student should be able to:

- Interpret the F probability distribution as the number of groups and the sample size change.
- Discuss two uses for the F distribution: one-way ANOVA and the test of two variances.
- Conduct and interpret one-way ANOVA.
- Conduct and interpret hypothesis tests of two variances

Many statistical applications in psychology, social science, business administration, and the natural sciences involve several groups. For example, an environmentalist is interested in knowing if the average amount of pollution varies in several bodies of water. A sociologist is interested in knowing if the amount of income a person earns varies according to his or her upbringing. A consumer looking for a new car might compare the average gas mileage of several models.

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E MAGAZINE OF ARCHITECTURE AND DESIGN

Figure 10.1.1: One-way ANOVA is used to measure information from several groups.

For hypothesis tests comparing averages between more than two groups, statisticians have developed a method called "Analysis of Variance" (abbreviated ANOVA). In this chapter, you will study the simplest form of ANOVA called single factor or one-way ANOVA. You will also study the F distribution, used for one-way ANOVA, and the test of two variances. This is just a very brief overview of one-way ANOVA. You will study this topic in much greater detail in future statistics courses. One-Way ANOVA, as it is presented here, relies heavily on a calculator or computer.

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10.2: One-Way ANOVA

The purpose of a one-way ANOVA test is to determine the existence of a statistically significant difference among several group means. The test actually uses variances to help determine if the means are equal or not. To perform a one-way ANOVA test, there are several basic assumptions to be fulfilled:

Five basic assumptions of one-way ANOVA to be fulfilled

1. Each population from which a sample is taken is assumed to be normal.
2. All samples are randomly selected and independent.
3. The populations are assumed to have equal standard deviations (or variances).
4. The factor is a categorical variable.
5. The response is a numerical variable.

The Null and Alternative Hypotheses

The null hypothesis is simply that all the group population means are the same. The alternative hypothesis is that at least one pair of means is different. For example, if there are k groups:

- $H_0: \mu_1 = \mu_2 = \mu_3 = \ldots = \mu_k$
- $H_a:$ At least two of the group means $\mu_2 = \mu_3 = \ldots = \mu_k$ are not equal

The graphs, a set of box plots representing the distribution of values with the group means indicated by a horizontal line through the box, help in the understanding of the hypothesis test. In the first graph (red box plots), $H_0: \mu_1 = \mu_2 = \mu_3$ and the three populations have the same distribution if the null hypothesis is true. The variance of the combined data is approximately the same as the variance of each of the populations.

If the null hypothesis is false, then the variance of the combined data is larger which is caused by the different means as shown in the second graph (green box plots).

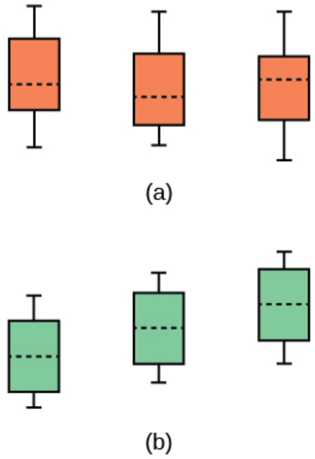

Figure 10.2.1: (a) H_0 is true. All means are the same; the differences are due to random variation. (b) H_0 is not true. All means are not the same; the differences are too large to be due to random variation.

Review

Analysis of variance extends the comparison of two groups to several, each a level of a categorical variable (factor). Samples from each group are independent, and must be randomly selected from normal populations with equal variances. We test the null hypothesis of equal means of the response in every group versus the alternative hypothesis of one or more group means being different from the others. A one-way ANOVA hypothesis test determines if several population means are equal. The distribution for the test is the F distribution with two different degrees of freedom.

Assumptions:

a. Each population from which a sample is taken is assumed to be normal.

b. All samples are randomly selected and independent.
c. The populations are assumed to have equal standard deviations (or variances).

Glossary

Analysis of Variance

also referred to as ANOVA, is a method of testing whether or not the means of three or more populations are equal. The method is applicable if:

- all populations of interest are normally distributed.
- the populations have equal standard deviations.
- samples (not necessarily of the same size) are randomly and independently selected from each population.

The test statistic for analysis of variance is the F-ratio.

One-WayANOVA

a method of testing whether or not the means of three or more populations are equal; the method is applicable if:

- all populations of interest are normally distributed.
- the populations have equal standard deviations.
- samples (not necessarily of the same size) are randomly and independently selected from each population.

The test statistic for analysis of variance is the F-ratio.

Variance

mean of the squared deviations from the mean; the square of the standard deviation. For a set of data, a deviation can be represented as $x-\bar{x}$ where x is a value of the data and $\bar{x}$ is the sample mean. The sample variance is equal to the sum of the squares of the deviations divided by the difference of the sample size and one.

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10.3: The F Distribution and the F-Ratio

The distribution used for the hypothesis test is a new one. It is called the F distribution, named after Sir Ronald Fisher, an English statistician. The F statistic is a ratio (a fraction). There are two sets of degrees of freedom; one for the numerator and one for the denominator.

For example, if F follows an F distribution and the number of degrees of freedom for the numerator is four, and the number of degrees of freedom for the denominator is ten, then $F \sim F_{4,10}$.

> The F distribution is derived from the Student's t-distribution. The values of the F distribution are squares of the corresponding values of the t-distribution. One-Way ANOVA expands the t-test for comparing more than two groups. The scope of that derivation is beyond the level of this course.

To calculate the F ratio, two estimates of the variance are made.

a. **Variance between samples:** An estimate of σ^2 that is the variance of the sample means multiplied by n (when the sample sizes are the same.). If the samples are different sizes, the variance between samples is weighted to account for the different sample sizes. The variance is also called **variation due to treatment or explained variation.**
b. **Variance within samples:** An estimate of σ^2 that is the average of the sample variances (also known as a pooled variance). When the sample sizes are different, the variance within samples is weighted. The variance is also called the **variation due to error or unexplained variation.**

- $SS_{\text{between}} = \text{the sum of squares that represents the variation among the different samples}$
- $SS_{\text{within}} = \text{the sum of squares that represents the variation within samples that is due to chance}$.

To find a "sum of squares" means to add together squared quantities that, in some cases, may be weighted. We used sum of squares to calculate the sample variance and the sample standard deviation in discussed previously.

MS means "mean square." MS_{between} is the variance between groups, and MS_{within} is the variance within groups.

Calculation of Sum of Squares and Mean Square

- $k =$ the number of different groups
- $n_j =$ the size of the j^{th} group}
- $s_j =$ the sum of the values in the j^{th} group
- $n =$ total number of all the values combined (total sample size):

$$n = \sum n_j \tag{10.3.1}$$

- $x =$ one value:

$$\sum x = \sum s_j \tag{10.3.2}$$

- Sum of squares of all values from every group combined:

$$\sum x^2 \tag{10.3.3}$$

- Between group variability:

$$SS_{\text{total}} = \sum x^2 - \frac{\left(\sum x^2\right)}{n} \tag{10.3.4}$$

- Total sum of squares:

$$\sum x^2 - \frac{\left(\sum x\right)^2}{n} \tag{10.3.5}$$

- Explained variation: sum of squares representing variation among the different samples:

$$SS_{\text{between}} = \sum \left[\frac{(s_j)^2}{n_j}\right] - \frac{\left(\sum s_j\right)^2}{n} \tag{10.3.6}$$

- Unexplained variation: sum of squares representing variation within samples due to chance:

$$SS_{\text{within}} = SS_{\text{total}} - SS_{\text{between}} \tag{10.3.7}$$

- df's for different groups (df's for the numerator):

$$df = k - 1 \tag{10.3.8}$$

- Equation for errors within samples (df's for the denominator):

$$df_{\text{within}} = n - k \tag{10.3.9}$$

- Mean square (variance estimate) explained by the different groups:

$$MS_{\text{between}} = \frac{SS_{\text{between}}}{df_{\text{between}}} \tag{10.3.10}$$

- Mean square (variance estimate) that is due to chance (unexplained):

$$MS_{\text{within}} = \frac{SS_{\text{within}}}{df_{\text{within}}} \tag{10.3.11}$$

MS_{between} and MS_{within} can be written as follows:

$$MS_{\text{between}} = \frac{SS_{\text{between}}}{df_{\text{between}}} = \frac{SS_{\text{between}}}{k-1} \tag{10.3.12}$$

$$MS_{\text{within}} = \frac{SS_{\text{within}}}{df_{\text{within}}} = \frac{SS_{\text{within}}}{n-k} \tag{10.3.13}$$

The one-way ANOVA test depends on the fact that MS_{between} can be influenced by population differences among means of the several groups. Since MS_{within} compares values of each group to its own group mean, the fact that group means might be different does not affect MS_{within}.

The null hypothesis says that all groups are samples from populations having the same normal distribution. The alternate hypothesis says that at least two of the sample groups come from populations with different normal distributions. If the null hypothesis is true, MS_{between} and MS_{within} should both estimate the same value.

> The null hypothesis says that all the group population means are equal. The hypothesis of equal means implies that the populations have the same normal distribution, because it is assumed that the populations are normal and that they have equal variances.

F-Ratio or F Statistic

$$F = \frac{MS_{\text{between}}}{MS_{\text{within}}} \tag{10.3.14}$$

If MS_{between} and MS_{within} estimate the same value (following the belief that H_0 is true), then the F-ratio should be approximately equal to one. Mostly, just sampling errors would contribute to variations away from one. As it turns out, MS_{between} consists of the population variance plus a variance produced from the differences between the samples. MS_{within} is an estimate of the population variance. Since variances are always positive, if the null hypothesis is false, MS_{between} will generally be larger than MS_{within}.Then the F-ratio will be larger than one. However, if the population effect is small, it is not unlikely that MS_{within} will be larger in a given sample.

The foregoing calculations were done with groups of different sizes. If the groups are the same size, the calculations simplify somewhat and the F-ratio can be written as:

F-Ratio Formula when the groups are the same size

$$F = \frac{n \cdot s_{\bar{x}}^2}{s^2_{\text{pooled}}} \tag{10.3.15}$$

where ...

- $n = \text{the sample size}$
- $df_{\text{numerator}} = k - 1$
- $df_{\text{denominator}} = n - k$
- $s^2_{\text{pooled}} = \text{the mean of the sample variances (pooled variance)}$
- $s_{\bar{x}^2} = \text{the variance of the sample means}$

Data are typically put into a table for easy viewing. One-Way ANOVA results are often displayed in this manner by computer software.

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (df)	Mean Square (MS)	F

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (df)	Mean Square (MS)	F
Factor (Between)	$SS(\text{Factor})$	$k-1$	$MS(\text{Factor}) = \frac{SS(\text{Factor})}{(k-1)}$	$F = \frac{MS(\text{Factor})}{MS(\text{Error})}$
Error (Within)	$SS(\text{Error})$	$n-k$	$MS(\text{Error}) = \frac{SS(\text{Error})}{(n-k)}$	
Total	$SS(\text{Total})$	$n-1$		

✓ Example 10.3.1

Three different diet plans are to be tested for mean weight loss. The entries in the table are the weight losses for the different plans. The one-way ANOVA results are shown in Table.

Plan 1: $n_1 = 4$	Plan 2: $n_2 = 3$	Plan 3: $n_3 = 3$
5	3.5	8
4.5	7	4
4		3.5
3	4.5	

$$s_1 = 16.5, s_2 = 15, s_3 = 15.7 \tag{10.3.16}$$

Following are the calculations needed to fill in the one-way ANOVA table. The table is used to conduct a hypothesis test.

$$SS(\text{between}) = \sum \left[\frac{(s_j)^2}{n_j}\right] - \frac{(\sum s_j)^2}{n} \tag{10.3.17}$$

$$= \frac{s_1^2}{4} + \frac{s_2^2}{3} + \frac{s_3^2}{3} + \frac{(s_1+s_2+s_3)^2}{10} \tag{10.3.18}$$

where $n_1 = 4, n_2 = 3, n_3 = 3$ and $n = n_1 + n_2 + n_3 = 10$ so

$$SS(\text{between}) = \frac{(16.5)^2}{4} + \frac{(15)^2}{3} + \frac{(5.5)^2}{3} = \frac{(16.5+15+15.5)^2}{10} \tag{10.3.19}$$

$$= 2.2458 \tag{10.3.20}$$

$$S(\text{total}) = \sum x^2 - \frac{(\sum x)^2}{n} \tag{10.3.21}$$

$$= (5^2 + 4.5^2 + 4^2 + 3^2 + 3.5^2 + 7^2 + 4.5^2 + 8^2 + 4^2 + 3.5^2) \tag{10.3.22}$$

$$- \frac{(5 + 4.5 + 4 + 3 + 3.5 + 7 + 4.5 + 8 + 4 + 3.5)^2}{10} \tag{10.3.23}$$

$$= 244 - \frac{47^2}{10} = 244 - 220.9 \tag{10.3.24}$$

$$=23.1 \tag{10.3.25}$$

$$SS(\text{within}) = SS(\text{total}) - SS(\text{between}) \tag{10.3.26}$$

$$= 23.1 - 2.2458 \tag{10.3.27}$$

$$= 20.8542 \tag{10.3.28}$$

One-Way ANOVA Table: The formulas for $SS(\text{Total})$, $SS(\text{Factor}) = SS(\text{Between})$ and $SS(\text{Error}) = SS(\text{Within})$ as shown previously. The same information is provided by the TI calculator hypothesis test function ANOVA in STAT TESTS (syntax is $ANOVA(L1, L2, L3)$ where $L1, L2, L3$ have the data from Plan 1, Plan 2, Plan 3 respectively).

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (df)	Mean Square (MS)	F
Factor (Between)	$SS(\text{Factor}) = SS(\text{Between}) = 2.2458$	$k - 1 = 3$ groups $- 1 = 2$	$MS(\text{Factor}) = \frac{SS(\text{Factor})}{(k-1)} = \frac{2.2458}{2} = 1.1229$	$F = \frac{MS(\text{Factor})}{MS(\text{Error})} = \frac{1.1229}{2.9792} =$
Error (Within)	$SS(\text{Error}) = SS(\text{Within}) = 20.8542$	$n - k = 10$ total data $- 3$ groups $= 7$	$MS(\text{Error})) = \frac{SS(\text{Error})}{(n-k)} = \frac{20.8542}{7} = 2.9792$	

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (df)	Mean Square (MS)	F
Total	$SS(\text{Total}) = 2.2458 + 20.8542 = 23.1$	$n - 1 = 10$ total data $- 1 = 9$		

Exercise 10.3.1

As part of an experiment to see how different types of soil cover would affect slicing tomato production, Marist College students grew tomato plants under different soil cover conditions. Groups of three plants each had one of the following treatments

- bare soil
- a commercial ground cover
- black plastic
- straw
- compost

All plants grew under the same conditions and were the same variety. Students recorded the weight (in grams) of tomatoes produced by each of the $n = 15$ plants:

Bare: $n_1 = 3$	Ground Cover: $n_2 = 3$	Plastic: $n_3 = 3$	Straw: $n_4 = 3$	Compost: $n_5 = 3$
2,625	5,348	6,583	7,285	6,277
2,997	5,682	8,560	6,897	7,818
4,915	5,482	3,830	9,230	8,677

Create the one-way ANOVA table.

Answer

Enter the data into lists L1, L2, L3, L4 and L5. Press STAT and arrow over to TESTS. Arrow down to ANOVA. Press ENTER and enter L1, L2, L3, L4, L5). Press ENTER. The table was filled in with the results from the calculator.

One-Way ANOVA table

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (df)	Mean Square (MS)	F
Factor (Between)	36,648,561	$5 - 1 = 4$	$\frac{36,648,561}{4} = 9,162,140$	$\frac{9,162,140}{2,044,672.6} = 4.4810$
Error (Within)	20,446,726	$15 - 5 = 10$	$\frac{20,446,726}{10} = 2,044,672.6$	
Total	57,095,287	$15 - 1 = 14$		

The one-way ANOVA hypothesis test is always right-tailed because larger F-values are way out in the right tail of the F-distribution curve and tend to make us reject H_0.

Notation

The notation for the F distribution is $F \sim F df(\text{num}), df(\text{denom})$

where $df(\text{num}) = df_{between} and df(\text{denom}) = df_{within}$

The mean for the F distribution is $\mu = \frac{df(\text{num})}{df(\text{denom}) - 1}$

References

1. Tomato Data, Marist College School of Science (unpublished student research)

Review

Analysis of variance compares the means of a response variable for several groups. ANOVA compares the variation within each group to the variation of the mean of each group. The ratio of these two is the F statistic from an F distribution with (number of groups – 1) as the numerator degrees of freedom and (number of observations – number of groups) as the denominator degrees of freedom. These statistics are summarized in the ANOVA table.

Formula Review

$$SS_{between} = \sum \left[\frac{(s_j)^2}{n_j} \right] - \frac{\left(\sum s_j\right)^2}{n}$$

$$SS_{\text{total}} = \sum x^2 - \frac{\left(\sum x\right)^2}{n}$$

$$SS_{\text{within}} = SS_{\text{total}} - SS_{\text{between}}$$

$$df_{\text{between}} = df(\text{num}) = k - 1$$

$$df_{\text{within}} = df(\text{denom}) = n - k$$

$$MS_{\text{between}} = \frac{SS_{\text{between}}}{df_{\text{between}}}$$

$$MS_{\text{within}} = \frac{SS_{\text{within}}}{df_{\text{within}}}$$

$$F = \frac{MS_{\text{between}}}{MS_{\text{within}}}$$

F ratio when the groups are the same size: $F = \dfrac{ns_{\bar{x}}^2}{s_{\text{pooled}}^2}$

Mean of the F distribution: $\mu = \dfrac{df(\text{num})}{df(\text{denom}) - 1}$

where:

- $k =$ the number of groups
- $n_j =$ the size of the j^{th} group
- $s_j =$ the sum of the values in the j^{th} group
- $n =$ the total number of all values (observations) combined
- $x =$ one value (one observation) from the data
- $s_{\bar{x}}^2 =$ the variance of the sample means
- $s_{\text{pooled}}^2 =$ the mean of the sample variances (pooled variance)

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10.4: Facts About the F Distribution

Here are some facts about the F distribution:

a. The curve is not symmetrical but skewed to the right.
b. There is a different curve for each set of dfs.
c. The F statistic is greater than or equal to zero.
d. As the degrees of freedom for the numerator and for the denominator get larger, the curve approximates the normal.
e. Other uses for the F distribution include comparing two variances and two-way Analysis of Variance. Two-Way Analysis is beyond the scope of this chapter.

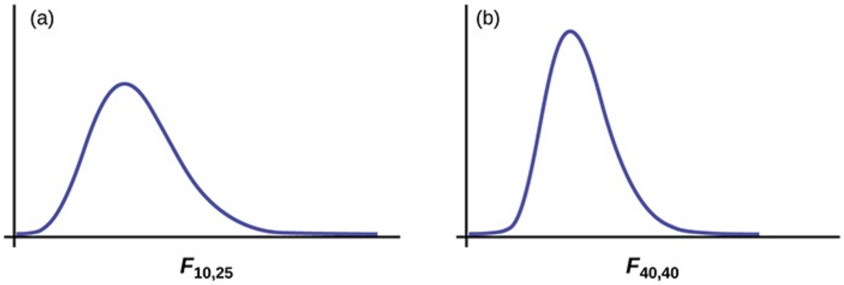

Figure 10.4.1

Example 10.4.1

Let's return to the slicing tomato exercise. The means of the tomato yields under the five mulching conditions are represented by $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$. We will conduct a hypothesis test to determine if all means are the same or at least one is different. Using a significance level of 5%, test the null hypothesis that there is no difference in mean yields among the five groups against the alternative hypothesis that at least one mean is different from the rest.

Answer

The null and alternative hypotheses are:

- $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$
- $H_a : \mu_i \neq \mu_j \text{ some } i \neq j$

The one-way ANOVA results are shown in Table

one-way ANOVA results

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (df)	Mean Square (MS)	F
Factor (Between)	36,648,561	$5-1=4$	$\frac{36,648,561}{4} = 9,162,140$	$\frac{9,162,140}{2,044,672.6} = 4.4810$
Error (Within)	20,446,726	$15-5=10$	$\frac{20,446,726}{10} = 2,044,672.6$	
Total	57,095,287	$15-1=14$		

Distribution for the test: $F_{4,10}$

$$df(\text{num}) = 5 - 1 = 4 \tag{10.4.1}$$

$$df(\text{denom}) = 15 - 5 = 10 \tag{10.4.2}$$

Test statistic: $F = 4.4810$

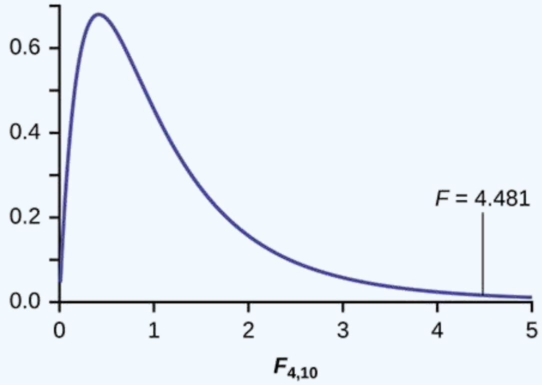

Figure 10.4.2

Probability Statement: $p\text{-value} = P(F > 4.481) = 0.0248.$

Compare α and the p-value: $\alpha = 0.05, p\text{-value} = 0.0248$

Make a decision: Since $\alpha > p\text{-value}$, we reject H_0.

Conclusion: At the 5% significance level, we have reasonably strong evidence that differences in mean yields for slicing tomato plants grown under different mulching conditions are unlikely to be due to chance alone. We may conclude that at least some of mulches led to different mean yields.

To find these results on the calculator:

Press STAT. Press 1:EDIT. Put the data into the lists L_1, L_2, L_3, L_4, L_5.

Press STAT, and arrow over to TESTS, and arrow down to ANOVA. Press ENTER, and then enter L_1, L_2, L_3, L_4, L_5). Press ENTER. You will see that the values in the foregoing ANOVA table are easily produced by the calculator, including the test statistic and the p-value of the test.

The calculator displays:

- $F = 4.4810$
- $p = 0.0248$ (p-value)

Factor

- $df = 4$
- $SS = 36648560.9$
- $MS = 9162140.23$

Error

- $df = 10$
- $SS = 20446726$
- $MS = 2044672.6$

? Exercise 10.4.1

MRSA, or *Staphylococcus aureus*, can cause a serious bacterial infections in hospital patients. Table shows various colony counts from different patients who may or may not have MRSA.

Conc = 0.6	Conc = 0.8	Conc = 1.0	Conc = 1.2	Conc = 1.4
9	16	22	30	27
66	93	147	199	168
98	82	120	148	132

Plot of the data for the different concentrations:

This graph is a scatterplot for the data provided. The horizontal axis is labeled 'Colony counts' and extends from 0 - 200. The vertical axis is labeled 'Tryptone concentrations' and extends from 0.6 - 1.4.

Figure 10.4.3

Test whether the mean number of colonies are the same or are different. Construct the ANOVA table (by hand or by using a TI-83, 83+, or 84+ calculator), find the *p*-value, and state your conclusion. Use a 5% significance level.

Answer

While there are differences in the spreads between the groups (Figure 10.4.1), the differences do not appear to be big enough to cause concern.

We test for the equality of mean number of colonies:

$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$

$H_a : \mu_i \neq \mu_j \text{ some } i \neq j$

The one-way ANOVA table results are shown in Table.

Table 10.4.1

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (df)	Mean Square (MS)	F
Factor (Between)	10,233	$5-1=4$	$\frac{10,233}{4} = 2,558.25$	$\frac{2,558.25}{4,194.9} = 0.6099$
Error (Within)	41,949	$15-5=10$		
Total	52,182	$15-1=14$	$\frac{41,949}{10} = 4,194.9$	

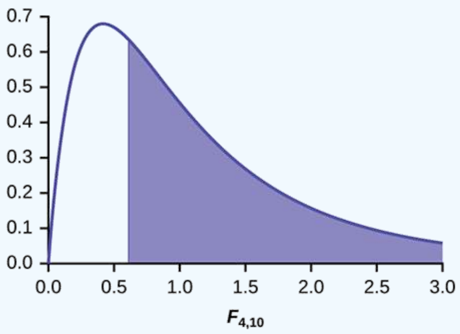

***Figure* 10.4.2**

Distribution for the test: $F_{4,10}$

Probability Statement: $p\text{-value} = P(F > 0.6099) = 0.6649.$

Compare α and the p-value: $\alpha = 0.05, p\text{-value} = 0.669, \alpha > p\text{-value}$

Make a decision: Since $\alpha > p\text{-value}$, we do not reject H_0.

Conclusion: At the 5% significance level, there is insufficient evidence from these data that different levels of tryptone will cause a significant difference in the mean number of bacterial colonies formed.

✓ Example 10.4.2

Four sororities took a random sample of sisters regarding their grade means for the past term. The results are shown in Table.

Figure 10.4.1: MEAN GRADES FOR FOUR SORORITIES

Sorority 1	Sorority 2	Sorority 3	Sorority 4

Sorority 1	Sorority 2	Sorority 3	Sorority 4
2.17	2.63	2.63	3.79
1.85	1.77	3.78	3.45
2.83	3.25	4.00	3.08
1.69	1.86	2.55	2.26
3.33	2.21	2.45	3.18

Using a significance level of 1%, is there a difference in mean grades among the sororities?

Answer

Let $\mu_1, \mu_2, \mu_3, \mu_4$ be the population means of the sororities. Remember that the null hypothesis claims that the sorority groups are from the same normal distribution. The alternate hypothesis says that at least two of the sorority groups come from populations with different normal distributions. Notice that the four sample sizes are each five.

This is an example of a balanced design, because each factor (i.e., sorority) has the same number of observations.

$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$

H_a: Not all of the means $\mu_1, \mu_2, \mu_3, \mu_4$ are equal.

Distribution for the test: $F_{3,16}$

where $k = 4$ groups and $n = 20$ samples in total

$df(\text{num}) = k - 1 = 4 - 1 = 3$

$df(\text{denom}) = n - k = 20 - 4 = 16$

Calculate the test statistic: $F = 2.23$

Graph:

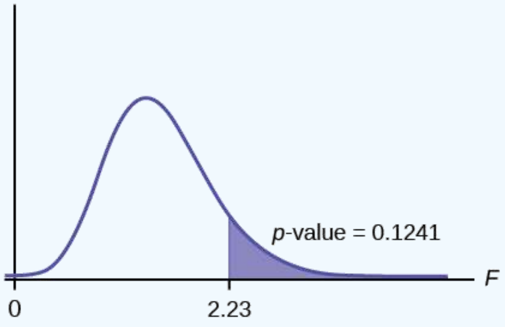

Figure 10.4.5

Probability statement: $p\text{-value} = P(F > 2.23) = 0.1241$

Compare α **and the** p-value: $\alpha = 0.01$

$p\text{-value} = 0.1241$

$\alpha < p\text{-value}$

Make a decision: Since $\alpha < p\text{-value}$, you cannot reject H_0.

Conclusion: There is not sufficient evidence to conclude that there is a difference among the mean grades for the sororities.

Put the data into lists L_1, L_2, L_3, and L_4. Press `STAT` and arrow over to `TESTS`. Arrow down to `F:ANOVA`. Press `ENTER` and Enter (`L1,L2,L3,L4`).

The calculator displays the F statistic, the p-value and the values for the one-way ANOVA table:

$F = 2.2303$

$p = 0.1241$ (p-value)

Factor

$df = 3$

$SS = 2.88732$

$MS = 0.96244$

Error

$df = 1$

$SS = 6.9044$

$MS = 0.431525$

Exercise 10.4.2

Four sports teams took a random sample of players regarding their GPAs for the last year. The results are shown in Table.

GPAs FOR FOUR SPORTS TEAMS

Basketball	Baseball	Hockey	Lacrosse
3.6	2.1	4.0	2.0
2.9	2.6	2.0	3.6
2.5	3.9	2.6	3.9
3.3	3.1	3.2	2.7
3.8	3.4	3.2	2.5

Use a significance level of 5%, and determine if there is a difference in GPA among the teams.

Answer

With a p-value of 0.9271, we decline to reject the null hypothesis. There is not sufficient evidence to conclude that there is a difference among the GPAs for the sports teams.

Example 10.4.3

A fourth grade class is studying the environment. One of the assignments is to grow bean plants in different soils. Tommy chose to grow his bean plants in soil found outside his classroom mixed with dryer lint. Tara chose to grow her bean plants in potting soil bought at the local nursery. Nick chose to grow his bean plants in soil from his mother's garden. No chemicals were used on the plants, only water. They were grown inside the classroom next to a large window. Each child grew five plants. At the end of the growing period, each plant was measured, producing the data (in inches) in Table $10.4.3$.

Table $10.4.3$

Tommy's Plants	Tara's Plants	Nick's Plants
24	25	23
21	31	27
23	23	22
30	20	30
23	28	20

Does it appear that the three media in which the bean plants were grown produce the same mean height? Test at a 3% level of significance.

Answer

This time, we will perform the calculations that lead to the F' statistic. Notice that each group has the same number of plants, so we will use the formula

$$F' = \frac{n \cdot s_{\bar{x}}^2}{s_{\text{pooled}}^2}. \tag{10.4.3}$$

First, calculate the sample mean and sample variance of each group.

	Tommy's Plants	Tara's Plants	Nick's Plants
Sample Mean	24.2	25.4	24.4
Sample Variance	11.7	18.3	16.3

Next, calculate the variance of the three group means (Calculate the variance of 24.2, 25.4, and 24.4). **Variance of the group means** $= 0.413 = s_{\bar{x}}^2$

Then $MS_{\text{between}} = ns_{\bar{x}}^2 = (5)(0.413)$ where $n = 5$ is the sample size (number of plants each child grew).

Calculate the mean of the three sample variances (Calculate the mean of 11.7, 18.3, and 16.3). **Mean of the sample variances** $= 15.433 = s_{\text{pooled}}^2$

Then $MS_{\text{within}} = s_{\text{pooled}}^2 = 15.433$.

The F statistic (or F ratio) is $F = \frac{MS_{\text{between}}}{MS_{\text{within}}} = \frac{ns_{\bar{x}}^2}{s_{\text{pooled}}^2} = \frac{(5)(0.413)}{15.433} = 0.134$

The dfs for the numerator $=$ the number of groups $-1 = 3 - 1 = 2$.

The dfs for the denominator $=$ the total number of samples $-$ the number of groups $= 15 - 3 = 12$

The distribution for the test is $F_{2,12}$ and the F statistic is $F = 0.134$

The p-value is $P(F > 0.134) = 0.8759$.

Decision: Since $\alpha = 0.03$ and the p-value $= 0.8759$, do not reject H_0. (Why?)

Conclusion: With a 3% level of significance, from the sample data, the evidence is not sufficient to conclude that the mean heights of the bean plants are different.

To calculate the p-value:

*Press `2nd DISTR`

*Arrow down to `Fcdf` (and press `ENTER` .

*Enter 0.134, `E99` , 2, 12)

*Press `ENTER`

The p-value is 0.8759

Exercise 10.4.3

Another fourth grader also grew bean plants, but this time in a jelly-like mass. The heights were (in inches) 24, 28, 25, 30, and 32. Do a one-way ANOVA test on the four groups. Are the heights of the bean plants different? Use the same method as shown in Example $10.4.3$

Answer

- $F = 0.9496$

- $p\text{-value} = 0.4402$

From the sample data, the evidence is not sufficient to conclude that the mean heights of the bean plants are different.

Collaborative Exercise

From the class, create four groups of the same size as follows: men under 22, men at least 22, women under 22, women at least 22. Have each member of each group record the number of states in the United States he or she has visited. Run an ANOVA test to determine if the average number of states visited in the four groups are the same. Test at a 1% level of significance. Use one of the solution sheets in [link].

References

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3. Hand, D.J., F. Daly, A.D. Lunn, K.J. McConway, and E. Ostrowski. *A Handbook of Small Datasets.*London: Chapman & Hall, 1994, pg. 50.
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6. Mackowiak, P. A., Wasserman, S. S., and Levine, M. M. (1992), "A Critical Appraisal of 98.6 Degrees F, the Upper Limit of the Normal Body Temperature, and Other Legacies of Carl Reinhold August Wunderlich," *Journal of the American Medical Association*, 268, 1578-1580.

Review

The graph of the F distribution is always positive and skewed right, though the shape can be mounded or exponential depending on the combination of numerator and denominator degrees of freedom. The F statistic is the ratio of a measure of the variation in the group means to a similar measure of the variation within the groups. If the null hypothesis is correct, then the numerator should be small compared to the denominator. A small F statistic will result, and the area under the F curve to the right will be large, representing a large p-value. When the null hypothesis of equal group means is incorrect, then the numerator should be large compared to the denominator, giving a large F statistic and a small area (small p-value) to the right of the statistic under the F curve.

When the data have unequal group sizes (unbalanced data), then techniques discussed earlier need to be used for hand calculations. In the case of balanced data (the groups are the same size) however, simplified calculations based on group means and variances may be used. In practice, of course, software is usually employed in the analysis. As in any analysis, graphs of various sorts should be used in conjunction with numerical techniques. Always look of your data!

10.5: Test of Two Variances

Another of the uses of the F distribution is testing two variances. It is often desirable to compare two variances rather than two averages. For instance, college administrators would like two college professors grading exams to have the same variation in their grading. In order for a lid to fit a container, the variation in the lid and the container should be the same. A supermarket might be interested in the variability of check-out times for two checkers.

> **to perform a F test of two variances, it is important that the following are true:**
>
> - The populations from which the two samples are drawn are *normally* distributed.
> - The two populations are *independent* of each other.

Unlike most other tests in this book, the F test for equality of two variances is very sensitive to deviations from normality. If the two distributions are not normal, the test can give higher p-values than it should, or lower ones, in ways that are unpredictable. Many texts suggest that students not use this test at all, but in the interest of completeness we include it here.

Suppose we sample randomly from two independent normal populations. Let σ_1^2 and σ_2^2 be the population variances and s_1^2 and s_2^2 be the sample variances. Let the sample sizes be n_1 and n_2. Since we are interested in comparing the two sample variances, we use the F ratio:

$$F = \frac{\left[\frac{(s_1)^2}{(\sigma_1)^2}\right]}{\left[\frac{(s_2)^2}{(\sigma_2)^2}\right]} \tag{10.5.1}$$

F has the distribution

$$F \sim F(n_1 - 1, n_2 - 1) \tag{10.5.2}$$

where $n_1 - 1$ are the degrees of freedom for the numerator and $n_2 - 1$ are the degrees of freedom for the denominator.

If the null hypothesis is $\sigma_1^2 = \sigma_2^2$, then the F Ratio becomes

$$F = \frac{\left[\frac{(s_1)^2}{(\sigma_1)^2}\right]}{\left[\frac{(s_2)^2}{(\sigma_2)^2}\right]} = \frac{(s_1)^2}{(s_2)^2}. \tag{10.5.3}$$

> The F ratio could also be $\frac{(s_2)^2}{(s_1)^2}$. It depends on H_a and on which sample variance is larger.

If the two populations have equal variances, then s_1^2 and s_2^2 are close in value and $F = \frac{(s_1)^2}{(s_2)^2}$ is close to one. But if the two population variances are very different, s_1^2 and s_2^2 tend to be very different, too. Choosing s_1^2 as the larger sample variance causes the ratio $\frac{(s_1)^2}{(s_2)^2}$ to be greater than one. If s_1^2 and s_2^2 are far apart, then

$$F = \frac{(s_1)^2}{(s_2)^2} \tag{10.5.4}$$

is a large number.

Therefore, if F is close to one, the evidence favors the null hypothesis (the two population variances are equal). But if F is much larger than one, then the evidence is against the null hypothesis. A test of two variances may be left, right, or two-tailed.

> *A test of two variances may be left, right, or two-tailed.*

Example 10.5.1

Two college instructors are interested in whether or not there is any variation in the way they grade math exams. They each grade the same set of 30 exams. The first instructor's grades have a variance of 52.3. The second instructor's grades have a variance of 89.9. Test the claim that the first instructor's variance is smaller. (In most colleges, it is desirable for the variances of exam grades to be nearly the same among instructors.) The level of significance is 10%.

Answer

Let 1 and 2 be the subscripts that indicate the first and second instructor, respectively.

- $n_1 = n_2 = 30$.
- $H_0 : \sigma_1^2 = \sigma_2^2$ and $H_a : \sigma_1^2 < \sigma_2^2$

Calculate the test statistic: By the null hypothesis $\sigma_1^2 = \sigma_2^2$) , the F statistic is:

$$F = \frac{\left[\frac{(s_1)^2}{(\sigma_1)^2}\right]}{\left[\frac{(s_2)^2}{(s_2)^2}\right]} = \frac{(s_1)^2}{(s_2)^2} = \frac{52.3}{89.9} = 0.5818 \tag{10.5.5}$$

Distribution for the test: $F_{29,29}$ where $n_1 - 1 = 29$ and $n_2 - 1 = 29$.

Graph: This test is left tailed.

Draw the graph labeling and shading appropriately.

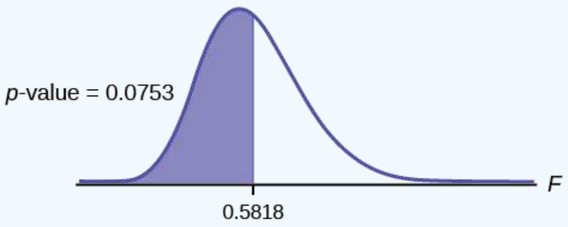

Figure 10.5.1

Probability statement: p-value $= P(F < 0.5818) = 0.0753$

Compare α and the p-value: $\alpha = 0.10 \alpha > p$-value.

Make a decision: Since $\alpha > p$-value, reject H_0.

Conclusion: With a 10% level of significance, from the data, there is sufficient evidence to conclude that the variance in grades for the first instructor is smaller.

Press `STAT` and arrow over to `TESTS` . Arrow down to `D:2-SampFTest` . Press `ENTER` . Arrow to `Stats` and press `ENTER` . For `Sx1` , `n1` , `Sx2` , and `n2` , enter `(52.3)-----√(52.3)` , `30` , `(89.9)-----√(89.9)` , and `30` . Press `ENTER` after each. Arrow to `σ1:` and `<σ2` . Press `ENTER` . Arrow down to `Calculate` and press `ENTER` . F = 0.5818 and p-value = 0.0753. Do the procedure again and try `Draw` instead of `Calculate` .

Exercise 10.5.1

The New York Choral Society divides male singers up into four categories from highest voices to lowest: Tenor1, Tenor2, Bass1, Bass2. In the table are heights of the men in the Tenor1 and Bass2 groups. One suspects that taller men will have lower voices, and that the variance of height may go up with the lower voices as well. Do we have good evidence that the variance of the heights of singers in each of these two groups (Tenor1 and Bass2) are different?

Tenor1	Bass2	Tenor 1	Bass 2	Tenor 1	Bass 2
69	72	67	72	68	67

Tenor1	Bass2	Tenor 1	Bass 2	Tenor 1	Bass 2
72	75	70	74	67	70
71	67	65	70	64	70
66	75	72	66		69
76	74	70	68		72
74	72	68	75		71
71	72	64	68		74
66	74	73	70		75
68	72	66	72		

Answer

The histograms are not as normal as one might like. Plot them to verify. However, we proceed with the test in any case.

Subscripts: $\mathrm{T1} =$ tenor 1 and $\mathrm{B2} =$ bass 2

The standard deviations of the samples are $s_{\mathrm{T1}} = 3.3302$ and $s_{\mathrm{B2}} = 2.7208$.

The hypotheses are

$H_0 : \sigma^2_{\mathrm{T1}} = \sigma^2_{\mathrm{B2}}$ and $H_0 : \sigma^2_{\mathrm{T1}} \neq \sigma^2_{\mathrm{B2}}$ (two tailed test)

The F statistic is 1.4894 with 20 and 25 degrees of freedom.

The $p\text{-value}$ is 0.3430. If we assume alpha is 0.05, then we cannot reject the null hypothesis.

We have no good evidence from the data that the heights of Tenor1 and Bass2 singers have different variances (despite there being a significant difference in mean heights of about 2.5 inches.)

References

1. “MLB Vs. Division Standings – 2012.” Available online at http://espn.go.com/mlb/standings/_/y...ion/order/true.

Review

The F test for the equality of two variances rests heavily on the assumption of normal distributions. The test is unreliable if this assumption is not met. If both distributions are normal, then the ratio of the two sample variances is distributed as an F statistic, with numerator and denominator degrees of freedom that are one less than the samples sizes of the corresponding two groups. A **test of two variances** hypothesis test determines if two variances are the same. The distribution for the hypothesis test is the F distribution with two different degrees of freedom.

Assumptions:

1. The populations from which the two samples are drawn are normally distributed.
2. The two populations are independent of each other.

Formula Review

F has the distribution $F \sim F(n_1 - 1, n_2 - 1)$

$$F = \frac{\frac{s_1^2}{\sigma_1^2}}{\frac{s_2^2}{\sigma_2^2}}$$

If $\sigma_1 = \sigma_2$, then $F = \frac{s_1^2}{s_2^2}$

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