

8.2: A Single Population Mean using the Normal Distribution

A confidence interval for a population mean with a known standard deviation is based on the fact that the sample means follow an approximately normal distribution. Suppose that our sample has a mean of $\bar{x} = 10$ and we have constructed the 90% confidence interval (5, 15) where $EBM = 5$.

Calculating the Confidence Interval

To construct a confidence interval for a single unknown population mean μ , where the population standard deviation is known, we need \bar{x} as an estimate for μ and we need the margin of error. Here, the margin of error (EBM) is called the error bound for a population mean (abbreviated EBM). The sample mean \bar{x} is the point estimate of the unknown population mean μ .

The confidence interval estimate will have the form:

$$(\text{point estimate} - \text{error bound}, \text{point estimate} + \text{error bound})$$

or, in symbols,

$$(\bar{x} - EBM, \bar{x} + EBM)$$

The **margin of error** (EBM) depends on the confidence level (abbreviated CL). The confidence level is often considered the probability that the calculated confidence interval estimate will contain the true population parameter. However, it is more accurate to state that the confidence level is the percent of confidence intervals that contain the true population parameter when repeated samples are taken. Most often, it is the choice of the person constructing the confidence interval to choose a confidence level of 90% or higher because that person wants to be reasonably certain of his or her conclusions.

There is another probability called alpha (α). α is related to the confidence level, CL . α is the probability that the interval does not contain the unknown population parameter. Mathematically,

$$\alpha + CL = 1.$$

✓ Example 8.2.1

Suppose we have collected data from a sample. We know the sample mean but we do not know the mean for the entire population. The sample mean is seven, and the error bound for the mean is 2.5: $\bar{x} = 7$ and $EBM = 2.5$

The confidence interval is $(7 - 2.5, 7 + 2.5)$ and calculating the values gives (4.5, 9.5). If the confidence level (CL) is 95%, then we say that, "We estimate with 95% confidence that the true value of the population mean is between 4.5 and 9.5."

? Exercise 8.2.1

Suppose we have data from a sample. The sample mean is 15, and the error bound for the mean is 3.2. What is the confidence interval estimate for the population mean?

Answer

(11.8, 18.2)

A confidence interval for a population mean with a known standard deviation is based on the fact that the sample means follow an approximately normal distribution. Suppose that our sample has a mean of $\bar{x} = 10$, and we have constructed the 90% confidence interval (5, 15) where $EBM = 5$. To get a 90% confidence interval, we must include the central 90% of the probability of the normal distribution. If we include the central 90%, we leave out a total of $\alpha = 10$ in both tails, or 5% in each tail, of the normal distribution.

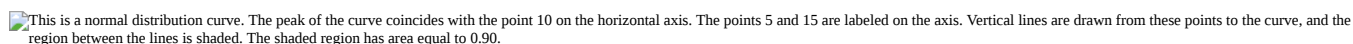
 This is a normal distribution curve. The peak of the curve coincides with the point 10 on the horizontal axis. The points 5 and 15 are labeled on the axis. Vertical lines are drawn from these points to the curve, and the region between the lines is shaded. The shaded region has area equal to 0.90.

Figure 8.2.1

To capture the central 90%, we must go out 1.645 "standard deviations" on either side of the calculated sample mean. The value 1.645 is the z-score from a standard normal probability distribution that puts an area of 0.90 in the center, an area of 0.05 in the far left tail, and an area of 0.05 in the far right tail.

It is important that the "standard deviation" used must be appropriate for the parameter we are estimating, so in this section we need to use the standard deviation that applies to sample means, which is

$$\frac{\sigma}{\sqrt{n}}$$

This fraction is commonly called the "standard error of the mean" to distinguish clearly the standard deviation for a mean from the population standard deviation σ .

In summary, as a result of the central limit theorem:

- \bar{X} is normally distributed, that is, $\bar{X} \sim N(\mu_x, \frac{\sigma}{\sqrt{n}})$.
- When the population standard deviation σ is known, we use a normal distribution to calculate the error bound.

Calculating the Confidence Interval

To construct a confidence interval estimate for an unknown population mean, we need data from a random sample. The steps to construct and interpret the confidence interval are:

- Calculate the sample mean \bar{x} from the sample data. Remember, in this section we already know the population standard deviation σ .
- Find the z-score that corresponds to the confidence level.
- Calculate the error bound EBM .
- Construct the confidence interval.
- Write a sentence that interprets the estimate in the context of the situation in the problem. (Explain what the confidence interval means, in the words of the problem.)

We will first examine each step in more detail, and then illustrate the process with some examples.

Finding the z-score for the Stated Confidence Level

When we know the population standard deviation σ , we use a standard normal distribution to calculate the error bound EBM and construct the confidence interval. We need to find the value of z that puts an area equal to the confidence level (in decimal form) in the middle of the standard normal distribution $Z \sim N(0, 1)$.

The confidence level, CL , is the area in the middle of the standard normal distribution. $CL = 1 - \alpha$, so α is the area that is split equally between the two tails. Each of the tails contains an area equal to $\frac{\alpha}{2}$.

The z-score that has an area to the right of $\frac{\alpha}{2}$ is denoted by $z_{\frac{\alpha}{2}}$.

For example, when $CL = 0.95$, $\alpha = 0.05$ and $\frac{\alpha}{2} = 0.025$; we write $z_{\frac{\alpha}{2}} = z_{0.025}$.

The area to the right of $z_{0.025}$ is 0.025 and the area to the left of $z_{0.025}$ is $1 - 0.025 = 0.975$.

$$z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$$

using a calculator, computer or a standard normal probability table.

`invNorm (0.975, 0, 1) = 1.96`

Remember to use the area to the LEFT of $z_{\frac{\alpha}{2}}$; in this chapter the last two inputs in the invNorm command are 0, 1, because you are using a standard normal distribution $Z \sim N(0, 1)$.

Calculating the Error Bound

The error bound formula for an unknown population mean μ when the population standard deviation σ is known is

$$EBM = z_{\alpha/2} \left(\frac{\sigma}{\sqrt{n}} \right)$$

Constructing the Confidence Interval

The confidence interval estimate has the format $(\bar{x} - EBM, \bar{x} + EBM)$.

The graph gives a picture of the entire situation.

$$CL + \frac{\alpha}{2} + \frac{\alpha}{2} = CL + \alpha = 1.$$

Figure 8.2.2.

Writing the Interpretation

The interpretation should clearly state the confidence level (CL), explain what population parameter is being estimated (here, a **population mean**), and state the confidence interval (both endpoints). "We estimate with ____% confidence that the true population mean (include the context of the problem) is between ____ and ____ (include appropriate units)."

✓ Example 8.2.2

Suppose scores on exams in statistics are normally distributed with an unknown population mean and a population standard deviation of three points. A random sample of 36 scores is taken and gives a sample mean (sample mean score) of 68. Find a confidence interval estimate for the population mean exam score (the mean score on all exams).

Find a 90% confidence interval for the true (population) mean of statistics exam scores.

Answer

- You can use technology to calculate the confidence interval directly.
- The first solution is shown step-by-step (Solution A).
- The second solution uses the TI-83, 83+, and 84+ calculators (Solution B).

Solution A

To find the confidence interval, you need the sample mean, \bar{x} , and the EBM .

$$\bar{x} = 68$$

$$EBM = \left(z_{\frac{\alpha}{2}} \right) \left(\frac{\sigma}{\sqrt{n}} \right)$$

$$\sigma = 3; n = 36$$

The confidence level is 90% ($CL = 0.90$)

$$CL = 0.90$$

so

$$\alpha = 1 - CL = 1 - 0.90 = 0.10$$

$$\frac{\alpha}{2} = 0.05 \quad z_{\frac{\alpha}{2}} = z_{0.05}$$

The area to the right of $z_{0.05}$ is 0.05 and the area to the left of $z_{0.05}$ is $1 - 0.05 = 0.95$.

$$z_{\frac{\alpha}{2}} = z_{0.05} = 1.645$$

using $\text{invNorm}(0.95, 0, 1)$ on the TI-83,83+, and 84+ calculators. This can also be found using appropriate commands on other calculators, using a computer, or using a probability table for the standard normal distribution.

$$EBM = (1.645) \left(\frac{3}{\sqrt{36}} \right) = 0.8225$$

$$\bar{x} - EBM = 68 - 0.8225 = 67.1775$$

$$\bar{x} + EBM = 68 + 0.8225 = 68.8225$$

The 90% confidence interval is **(67.1775, 68.8225)**.

Solution B

Press **STAT** and arrow over to **TESTS**.

Arrow down to **7:ZInterval**.

Press **ENTER**.

Arrow to **Stats** and press **ENTER**.

Arrow down and enter three for σ , 68 for \bar{x} , 36 for n , and .90 for **C-level**.

Arrow down to **Calculate** and press **ENTER**.

The confidence interval is (to three decimal places)(67.178, 68.822).

Interpretation

We estimate with 90% confidence that the true population mean exam score for all statistics students is between 67.18 and 68.82.

Explanation of 90% Confidence Level

Ninety percent of all confidence intervals constructed in this way contain the true mean statistics exam score. For example, if we constructed 100 of these confidence intervals, we would expect 90 of them to contain the true population mean exam score.

? Exercise 8.2.2

Suppose average pizza delivery times are normally distributed with an unknown population mean and a population standard deviation of six minutes. A random sample of 28 pizza delivery restaurants is taken and has a sample mean delivery time of 36 minutes. Find a 90% confidence interval estimate for the population mean delivery time.

Answer

(34.1347, 37.8653)

✓ Example 8.2.3: Specific Absorption Rate

The Specific Absorption Rate (SAR) for a cell phone measures the amount of radio frequency (RF) energy absorbed by the user's body when using the handset. Every cell phone emits RF energy. Different phone models have different SAR measures. To receive certification from the Federal Communications Commission (FCC) for sale in the United States, the SAR level for a cell phone must be no more than 1.6 watts per kilogram. Table shows the highest SAR level for a random selection of cell phone models as measured by the FCC.

Phone Model	SAR	Phone Model	SAR	Phone Model	SAR
Apple iPhone 4S	1.11	LG Ally	1.36	Pantech Laser	0.74
BlackBerry Pearl 8120	1.48	LG AX275	1.34	Samsung Character	0.5
BlackBerry Tour 9630	1.43	LG Cosmos	1.18	Samsung Epic 4G Touch	0.4
Cricket TXTM8	1.3	LG CU515	1.3	Samsung M240	0.867

Phone Model	SAR	Phone Model	SAR	Phone Model	SAR
HP/Palm Centro	1.09	LG Trax CU575	1.26	Samsung Messenger III SCH-R750	0.68
HTC One V	0.455	Motorola Q9h	1.29	Samsung Nexus S	0.51
HTC Touch Pro 2	1.41	Motorola Razr2 V8	0.36	Samsung SGH-A227	1.13
Huawei M835 Ideos	0.82	Motorola Razr2 V9	0.52	SGH-a107 GoPhone	0.3
Kyocera DuraPlus	0.78	Motorola V195s	1.6	Sony W350a	1.48
Kyocera K127 Marbl	1.25	Nokia 1680	1.39	T-Mobile Concord	1.38

Find a 98% confidence interval for the true (population) mean of the Specific Absorption Rates (SARs) for cell phones. Assume that the population standard deviation is $\sigma = 0.337$.

Solution A

To find the confidence interval, start by finding the point estimate: the sample mean.

$$\bar{x} = 1.024$$

Next, find the *EBM*. Because you are creating a 98% confidence interval, $CL = 0.98$.

Figure 8.2.3.

You need to find $z_{0.01}$ having the property that the area under the normal density curve to the right of $z_{0.01}$ is 0.01 and the area to the left is 0.99. Use your calculator, a computer, or a probability table for the standard normal distribution to find $z_{0.01} = 2.326$.

$$EBM = (z_{0.01}) \frac{\sigma}{\sqrt{n}} = (2.326) \frac{0.337}{\sqrt{30}} = 0.1431$$

To find the 98% confidence interval, find $\bar{x} \pm EBM$.

$$\bar{x} - EBM = 1.024 - 0.1431 = 0.8809$$

$$\bar{x} + EBM = 1.024 + 0.1431 = 1.1671$$

We estimate with 98% confidence that the true SAR mean for the population of cell phones in the United States is between 0.8809 and 1.1671 watts per kilogram.

Solution B

- Press STAT and arrow over to TESTS.
- Arrow down to 7:Z Interval.
- Press ENTER.
- Arrow to Stats and press ENTER.
- Arrow down and enter the following values:
 - σ : 0.337
 - \bar{x} : 1024
 - n : 30
 - C-level: 0.98
- Arrow down to Calculate and press ENTER.
- The confidence interval is (to three decimal places) (0.881, 1.167).

? Exercise 8.2.3

Table shows a different random sampling of 20 cell phone models. Use this data to calculate a 93% confidence interval for the true mean SAR for cell phones certified for use in the United States. As previously, assume that the population standard deviation is $\sigma = 0.337$.

Phone Model	SAR	Phone Model	SAR
Blackberry Pearl 8120	1.48	Nokia E71x	1.53
HTC Evo Design 4G	0.8	Nokia N75	0.68
HTC Freestyle	1.15	Nokia N79	1.4
LG Ally	1.36	Sagem Puma	1.24
LG Fathom	0.77	Samsung Fascinate	0.57
LG Optimus Vu	0.462	Samsung Infuse 4G	0.2
Motorola Cliq XT	1.36	Samsung Nexus S	0.51
Motorola Droid Pro	1.39	Samsung Replenish	0.3
Motorola Droid Razr M	1.3	Sony W518a Walkman	0.73
Nokia 7705 Twist	0.7	ZTE C79	0.869

Answer

$$\bar{x} = 0.940$$

$$\frac{\alpha}{2} = \frac{1 - CL}{2} = \frac{1 - 0.93}{2} = 0.035$$

$$z_{0.035} = 1.812$$

$$EBM = (z_{0.035}) \left(\frac{\sigma}{\sqrt{n}} \right) = (1.812) \left(\frac{0.337}{\sqrt{20}} \right) = 0.1365$$

$$\bar{x} - EBM = 0.940 - 0.1365 = 0.8035$$

$$\bar{x} + EBM = 0.940 + 0.1365 = 1.0765$$

We estimate with 93% confidence that the true SAR mean for the population of cell phones in the United States is between 0.8035 and 1.0765 watts per kilogram.

Notice the difference in the confidence intervals calculated in Example and the following Try It exercise. These intervals are different for several reasons: they were calculated from different samples, the samples were different sizes, and the intervals were calculated for different levels of confidence. Even though the intervals are different, they do not yield conflicting information. The effects of these kinds of changes are the subject of the next section in this chapter.

Changing the Confidence Level or Sample Size

✓ Example 8.2.4

Suppose we change the original problem in Example by using a 95% confidence level. Find a 95% confidence interval for the true (population) mean statistics exam score.

Answer

To find the confidence interval, you need the sample mean, \bar{x} , and the EBM .

$$\bar{x} = 68$$

$$EBM = \left(z_{\frac{\alpha}{2}} \right) \left(\frac{\sigma}{\sqrt{n}} \right)$$

$\sigma = 3; n = 36$; The confidence level is 95% ($CL = 0.95$).

$$CL = 0.95 \text{ so } \alpha = 1 - CL = 1 - 0.95 = 0.05$$

$$\frac{\alpha}{2} = 0.025 \quad z_{\frac{\alpha}{2}} = z_{0.025}$$

The area to the right of $z_{0.025}$ is 0.025 and the area to the left of $z_{0.025}$ is $1 - 0.025 = 0.975$

$$z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$$

when using $\text{invnorm}(0.975, 0, 1)$ on the TI-83, 83+, or 84+ calculators. (This can also be found using appropriate commands on other calculators, using a computer, or using a probability table for the standard normal distribution.)

$$EBM = (1.96) \left(\frac{3}{\sqrt{36}} \right) = 0.98$$

$$\bar{x} - EBM = 68 - 0.98 = 67.02$$

$$\bar{x} + EBM = 68 + 0.98 = 68.98$$

Notice that the EBM is larger for a 95% confidence level in the original problem.

Interpretation

We estimate with 95% confidence that the true population mean for all statistics exam scores is between 67.02 and 68.98.

Explanation of 95% Confidence Level

Ninety-five percent of all confidence intervals constructed in this way contain the true value of the population mean statistics exam score.

Comparing the results

The 90% confidence interval is (67.18, 68.82). The 95% confidence interval is (67.02, 68.98). The 95% confidence interval is wider. If you look at the graphs, because the area 0.95 is larger than the area 0.90, it makes sense that the 95% confidence interval is wider. To be more confident that the confidence interval actually does contain the true value of the population mean for all statistics exam scores, the confidence interval necessarily needs to be wider.

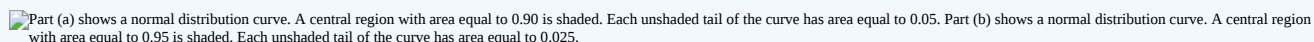
 Part (a) shows a normal distribution curve. A central region with area equal to 0.90 is shaded. Each unshaded tail of the curve has area equal to 0.05. Part (b) shows a normal distribution curve. A central region with area equal to 0.95 is shaded. Each unshaded tail of the curve has area equal to 0.025.

Figure 8.2.4.

Summary: Effect of Changing the Confidence Level

- Increasing the confidence level increases the error bound, making the confidence interval wider.
- Decreasing the confidence level decreases the error bound, making the confidence interval narrower.

? Exercise 8.2.4

Refer back to the pizza-delivery Try It exercise. The population standard deviation is six minutes and the sample mean deliver time is 36 minutes. Use a sample size of 20. Find a 95% confidence interval estimate for the true mean pizza delivery time.

Answer

(33.37, 38.63)

✓ Example 8.2.5

Suppose we change the original problem in Example to see what happens to the error bound if the sample size is changed.

Leave everything the same except the sample size. Use the original 90% confidence level. What happens to the error bound and the confidence interval if we increase the sample size and use $n = 100$ instead of $n = 36$? What happens if we decrease the sample size to $n = 25$ instead of $n = 36$?

- $\bar{x} = 68$
- $EBM = \left(z_{\frac{\alpha}{2}} \right) \left(\frac{\sigma}{\sqrt{n}} \right)$
- $\sigma = 3$; The confidence level is 90% ($CL=0.90$); $z_{\frac{\alpha}{2}} = z_{0.05} = 1.645$.

Answer

Solution A

If we **increase** the sample size n to 100, we **decrease** the error bound.

$$\text{When } n = 100 : EBM = \left(z_{\frac{\alpha}{2}} \right) \left(\frac{\sigma}{\sqrt{n}} \right) = (1.645) \left(\frac{3}{\sqrt{100}} \right) = 0.4935.$$

Solution B

If we **decrease** the sample size n to 25, we **increase** the error bound.

$$\text{When } n = 25 : EBM = \left(z_{\frac{\alpha}{2}} \right) \left(\frac{\sigma}{\sqrt{n}} \right) = (1.645) \left(\frac{3}{\sqrt{25}} \right) = 0.987.$$

Summary: Effect of Changing the Sample Size

- Increasing the sample size causes the error bound to decrease, making the confidence interval narrower.
- Decreasing the sample size causes the error bound to increase, making the confidence interval wider.

? Exercise 8.2.5

Refer back to the pizza-delivery Try It exercise. The mean delivery time is 36 minutes and the population standard deviation is six minutes. Assume the sample size is changed to 50 restaurants with the same sample mean. Find a 90% confidence interval estimate for the population mean delivery time.

Answer

(34.6041, 37.3958)

Working Backwards to Find the Error Bound or Sample Mean

When we calculate a confidence interval, we find the sample mean, calculate the error bound, and use them to calculate the confidence interval. However, sometimes when we read statistical studies, the study may state the confidence interval only. If we know the confidence interval, we can work backwards to find both the error bound and the sample mean.

Finding the Error Bound

- From the upper value for the interval, subtract the sample mean,
- OR, from the upper value for the interval, subtract the lower value. Then divide the difference by two.

Finding the Sample Mean

- Subtract the error bound from the upper value of the confidence interval,
- OR, average the upper and lower endpoints of the confidence interval.

Notice that there are two methods to perform each calculation. You can choose the method that is easier to use with the information you know.

✓ Example 8.2.6

Suppose we know that a confidence interval is **(67.18, 68.82)** and we want to find the error bound. We may know that the sample mean is 68, or perhaps our source only gave the confidence interval and did not tell us the value of the sample mean.

Calculate the Error Bound:

- If we know that the sample mean is 68 : $EBM = 68.82 - 68 = 0.82$.
- If we don't know the sample mean: $EBM = \frac{(68.82 - 67.18)}{2} = 0.82$.

Calculate the Sample Mean:

- If we know the error bound: $\bar{x} = 68.82 - 0.82 = 68$
- If we don't know the error bound: $\bar{x} = \frac{(67.18 + 68.82)}{2} = 68$.

? Exercise 8.2.6

Suppose we know that a confidence interval is (42.12, 47.88). Find the error bound and the sample mean.

Answer

Sample mean is 45, error bound is 2.88

Calculating the Sample Size n

If researchers desire a specific margin of error, then they can use the error bound formula to calculate the required sample size. The error bound formula for a population mean when the population standard deviation is known is

$$EBM = \left(z \frac{\sigma}{\sqrt{n}} \right)$$

The formula for sample size is $n = \frac{z^2 \sigma^2}{EBM^2}$, found by solving the error bound formula for n . In Equation ???, z is $z_{\frac{\alpha}{2}}$, corresponding to the desired confidence level. A researcher planning a study who wants a specified confidence level and error bound can use this formula to calculate the size of the sample needed for the study.

✓ Example 8.2.7

The population standard deviation for the age of Foothill College students is 15 years. If we want to be 95% confident that the sample mean age is within two years of the true population mean age of Foothill College students, how many randomly selected Foothill College students must be surveyed?

Solution

- From the problem, we know that $\sigma = 15$ and $EBM = 2$.
- $z = z_{0.025} = 1.96$, because the confidence level is 95%.
- $n = \frac{z^2 \sigma^2}{EBM^2} = \frac{(1.96)^2 (15)^2}{2^2}$ using the sample size equation.
- Use $n = 217$: Always round the answer UP to the next higher integer to ensure that the sample size is large enough.

Therefore, 217 Foothill College students should be surveyed in order to be 95% confident that we are within two years of the true population mean age of Foothill College students.

? Exercise 8.2.7

The population standard deviation for the height of high school basketball players is three inches. If we want to be 95% confident that the sample mean height is within one inch of the true population mean height, how many randomly selected students must be surveyed?

Answer

35 students

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Glossary

Confidence Level (CL)

the percent expression for the probability that the confidence interval contains the true population parameter; for example, if the $CL = 90$, then in 90 out of 100 samples the interval estimate will enclose the true population parameter.

Error Bound for a Population Mean (EBM)

the margin of error; depends on the confidence level, sample size, and known or estimated population standard deviation.

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