5.3: The Uniform Distribution

The uniform distribution is a continuous probability distribution and is concerned with events that are equally likely to occur. When working out problems that have a uniform distribution, be careful to note if the data is inclusive or exclusive.

Example 5.3.1

The data in Table are 55 smiling times, in seconds, of an eight-week-old baby.

10.4	19.6	18.8	13.9	17.8	16.8	21.6	17.9	12.5	11.1	4.9
12.8	14.8	22.8	20.0	15.9	16.3	13.4	17.1	14.5	19.0	22.8
1.3	0.7	8.9	11.9	10.9	7.3	5.9	3.7	17.9	19.2	9.8
5.8	6.9	2.6	5.8	21.7	11.8	3.4	2.1	4.5	6.3	10.7
8.9	9.4	9.4	7.6	10.0	3.3	6.7	7.8	11.6	13.8	18.6

The sample mean = 11.49 and the sample standard deviation = 6.23.

We will assume that the smiling times, in seconds, follow a uniform distribution between zero and 23 seconds, inclusive. This means that any smiling time from zero to and including 23 seconds is equally likely. The histogram that could be constructed from the sample is an empirical distribution that closely matches the theoretical uniform distribution.

Let $X =$ length, in seconds, of an eight-week-old baby's smile.

The notation for the uniform distribution is

$X \sim U(a, b)$ where $a =$ the lowest value of x and $b =$ the highest value of x.

The probability density function is $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$.

For this example, $X \sim U(0, 23)$ and $f(x) = \frac{1}{23-0}$ for $0 \leq X \leq 23$.

Formulas for the theoretical mean and standard deviation are

$\mu = \frac{a+b}{2}$ and $\sigma = \sqrt{\frac{(b-a)^2}{12}}$

For this problem, the theoretical mean and standard deviation are

$\mu = \frac{0+23}{2} = 11.50$ seconds and $\sigma = \sqrt{\frac{(23-0)^2}{12}} = 6.64$ seconds.

Notice that the theoretical mean and standard deviation are close to the sample mean and standard deviation in this example.

Exercise 5.3.1

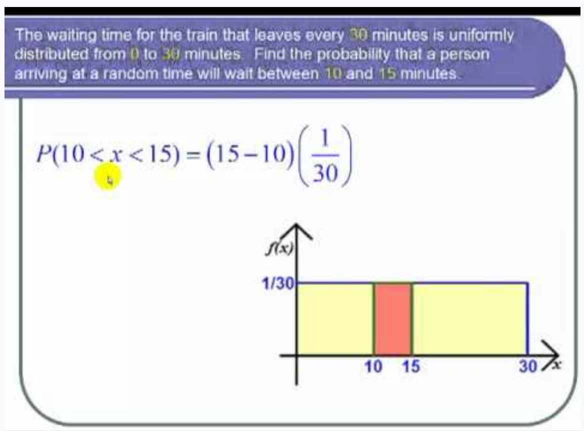
The data that follow are the number of passengers on 35 different charter fishing boats. The sample mean = 7.9 and the sample standard deviation = 4.33. The data follow a uniform distribution where all values between and including zero and 14 are equally likely. State the values of a and b. Write the distribution in proper notation, and calculate the theoretical mean and standard deviation.

1	12	4	10	4	14	11
7	11	4	13	2	4	6
3	10	0	12	6	9	10
5	13	4	10	14	12	11
6	10	11	0	11	13	2

Answer

a is zero; b is 14; $X \sim U(0, 14)$; $\mu = 7$ passengers; $\sigma = 4.04$ passengers

Example 5.3.2



Exercise 5.3.2.1

a. Refer to Example. What is the probability that a randomly chosen eight-week-old baby smiles between two and 18 seconds?

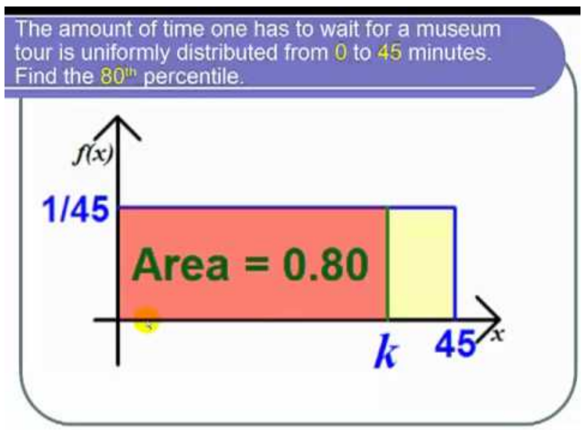
Answer

a. Find $P(2 < x < 18)$.

$P(2 < x < 18) = (\text{base})(\text{height}) = (18 - 2)\left(\frac{1}{23}\right) = \left(\frac{16}{23}\right)$.

This graph shows a uniform distribution. The horizontal axis ranges from 0 to 15. The distribution is modeled by a rectangle extending from x = 0 to x = 15. A region from x = 2 to x = 18 is shaded inside the rectangle.

Figure 5.2.1.



Exercise 5.3.2.2

b. Find the 90th percentile for an eight-week-old baby's smiling time.

Answer

b. Ninety percent of the smiling times fall below the 90th percentile, k, so $P(x < k) = 0.90$

$$P(x<k) = 0.90 \tag{5.3.1}$$

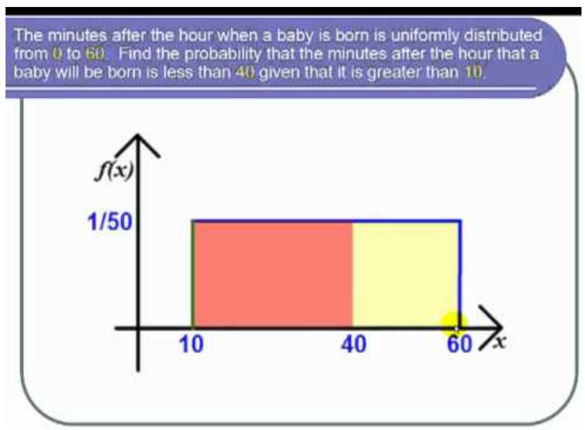
$$(\text{base})(\text{height}) = 0.90 \tag{5.3.2}$$

$$(k-0)\left(\frac{1}{23}\right) = 0.90 \tag{5.3.3}$$

$$k = (23)(0.90) = 20.7 \tag{5.3.4}$$

This shows the graph of the function f(x) = 1/15. A horiztonal line ranges from the point (0, 1/15) to the point (15, 1/15). A vertical line extends from the x-axis to the end of the line at point (15, 1/15) creating a rectangle. A region is shaded inside the rectangle from x = 0 to x = k. The shaded area represents P(x < k) = 0.90.

Figure 5.3.2.



Exercise 5.3.3

c. Find the probability that a random eight-week-old baby smiles more than 12 seconds **KNOWING** that the baby smiles **MORE THAN EIGHT SECONDS**.

Answer

c. This probability question is a **conditional**. You are asked to find the probability that an eight-week-old baby smiles more than 12 seconds when you **already know** the baby has smiled for more than eight seconds.

Find $P(x > 12 | x > 8)$ There are two ways to do the problem. **For the first way**, use the fact that this is a **conditional** and changes the sample space. The graph illustrates the new sample space. You already know the baby smiled more than eight seconds.

Write a new $f(x) : f(x) = \frac{1}{23-8} = \frac{1}{15}$

for $8 < x < 23$

$P(x > 12 | x > 8) = (23 - 12)\left(\frac{1}{15}\right) = \left(\frac{11}{15}\right)$

f(X)=1/15 graph displaying a boxed region consisting of a horizontal line extending to the right from point 1/15 on the y-axis, a vertical upward line from points 8 and 23 on the x-axis, and the x-axis. A shaded region from points 12-23 occurs within this area.

Figure 5.3.3.

For the second way, use the conditional formula from Probability Topics with the original distribution $X \sim U(0, 23)$:

$P(\text{A}|\text{B}) = \frac{P(\text{A AND B})}{P(\text{B})}$

For this problem, A is $(x > 12)$ and B is $(x > 8)$.

So, $P(x > 12|x > 8) = \frac{(x>12 \text{ AND } x>8)}{P(x>8)} = \frac{P(x>12)}{P(x>8)} = \frac{\frac{11}{23}}{\frac{15}{23}} = \frac{11}{15}$

This shows the graph of the function f(x) = 1/23. A horiztonal line ranges from the point (0, 1/23) to the point (23, 1/23). A vertical line extends from the x-axis to the end of the line at point (23, 1/23) creating a rectangle. Vertical lines extend from the horizontal axis to the graph at x = 8 and x = 12.

Figure 5.3.4.

Exercise 5.3.2

A distribution is given as $X \sim U(0, 20)$. What is $P(2 < x < 18)$? Find the 90^th^percentile.

Answer

$P(2 < x < 18) = 0.8$; 90th percentile $= 18$

Example 5.3.3

The amount of time, in minutes, that a person must wait for a bus is uniformly distributed between zero and 15 minutes, inclusive.

Exercise 5.3.3.1

a. What is the probability that a person waits fewer than 12.5 minutes?

Answer

a. Let $X =$ the number of minutes a person must wait for a bus. $a = 0$ and $b = 15$. $X \sim U(0, 15)$. Write the probability density function. $f(x) = \frac{1}{15-0} = \frac{1}{15}$ for $0 \leq x \leq 15$.

Find $P(x < 12.5)$. Draw a graph.

$P(x < k) = (\text{base})(\text{height}) = (12.5 - 0)\left(\frac{1}{15}\right) = 0.8333$

The probability a person waits less than 12.5 minutes is 0.8333.

This shows the graph of the function f(x) = 1/15. A horiztonal line ranges from the point (0, 1/15) to the point (15, 1/15). A vertical line extends from the x-axis to the end of the line at point (15, 1/15) creating a rectangle. A region is shaded inside the rectangle from x = 0 to x = 12.5.

Figure 5.3.5.

Exercise 5.3.3.2

b. On the average, how long must a person wait? Find the mean, μ, and the standard deviation, σ.

Answer

b. $\mu = \frac{a+b}{2} = \frac{15+0}{2} = 7.5$. On the average, a person must wait 7.5 minutes.

$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(12-0)^2}{12}} = 4.3$. The Standard deviation is 4.3 minutes.

Exercise 5.3.3.3

c. Ninety percent of the time, the time a person must wait falls below what value?

Note 5.3.3.3.1

This asks for the 90th percentile.

Answer

c. Find the 90th percentile. Draw a graph. Let $k =$ the 90th percentile.

$P(x < k) = (\text{base})(\text{height}) = (k - 0)\left(\frac{1}{15}\right)$
$0.90 = (k)\left(\frac{1}{15}\right)$

$k = (0.90)(15) = 13.5$

k is sometimes called a critical value.

The 90[th] percentile is 13.5 minutes. Ninety percent of the time, a person must wait at most 13.5 minutes.

f(X)=1/15 graph displaying a boxed region consisting of a horizontal line extending to the right from point 1/15 on the y-axis, a vertical upward line from an arbitrary point on the x-axis, and the x and y-axes. A shaded region from points 0-k occurs within this area. The area of this probability region is equal to 0.90.

Figure 5.3.6.

Exercise 5.3.4

The total duration of baseball games in the major league in the 2011 season is uniformly distributed between 447 hours and 521 hours inclusive.

a. Find a and b and describe what they represent.
b. Write the distribution.
c. Find the mean and the standard deviation.
d. What is the probability that the duration of games for a team for the 2011 season is between 480 and 500 hours?
e. What is the 65[th] percentile for the duration of games for a team for the 2011 season?

Answer

a. a is 447, and b is 521. a is the minimum duration of games for a team for the 2011 season, and b is the maximum duration of games for a team for the 2011 season.
b. $X \sim U(447, 521)$.
c. $\mu = 484$, and $\sigma = 21.36$

Figure 5.3.1.

d. $P(480 < x < 500) = 0.2703$
e. 65[th] percentile is 495.1 hours.

Example 5.3.4

Suppose the time it takes a nine-year old to eat a donut is between 0.5 and 4 minutes, inclusive. Let $X =$ the time, in minutes, it takes a nine-year old child to eat a donut. Then $X \sim U(0.5, 4)$.

a. The probability that a randomly selected nine-year old child eats a donut in at least two minutes is _______.

Solution

a. 0.5714

Exercise 5.3.4.1

b. Find the probability that a different nine-year old child eats a donut in more than two minutes given that the child has already been eating the donut for more than 1.5 minutes.

The second question has a conditional probability. You are asked to find the probability that a nine-year old child eats a donut in more than two minutes given that the child has already been eating the donut for more than 1.5 minutes. Solve the problem two different ways (see Example). You must reduce the sample space. **First way**: Since you know the child has already been eating the donut for more than 1.5 minutes, you are no longer starting at *a* = 0.5 minutes. Your starting point is 1.5 minutes.

Write a new $f(x)$:

$f(x) = \frac{1}{4-1.5} = \frac{2}{5}$ for $1.5 \leq x \leq 4$.

Find $P(x > 2|x > 1.5)$. Draw a graph.

f(X)=2/5 graph displaying a boxed region consisting of a horizontal line extending to the right from point 2/5 on the y-axis, a vertical upward line from points 1.5 and 4 on the x-axis, and the x-axis. A shaded region from points 2-4 occurs within this area.

Figure 5.3.2.

$P(x > 2|x > 1.5) = (\text{base})(\text{new height}) = (4-2)(25)\left(\frac{2}{5}\right) = \ ?$

Answer

b. $\frac{4}{5}$

The probability that a nine-year old child eats a donut in more than two minutes given that the child has already been eating the donut for more than 1.5 minutes is $\frac{4}{5}$.

Second way: Draw the original graph for $X \sim U(0.5, 4)$. Use the conditional formula

$$P(x > 2 | x > 1.5) = \frac{P(x > 2 \text{AND} x > 1.5)}{P(x > 1.5)} = \frac{P(x>2)}{P(x>1.5)} = \frac{\frac{2}{3.5}}{\frac{2.5}{3.5}} = 0.8 = \frac{4}{5}$$

Exercise 5.3.5

Suppose the time it takes a student to finish a quiz is uniformly distributed between six and 15 minutes, inclusive. Let $X =$ the time, in minutes, it takes a student to finish a quiz. Then $X \sim U(6, 15)$.

Find the probability that a randomly selected student needs at least eight minutes to complete the quiz. Then find the probability that a different student needs at least eight minutes to finish the quiz given that she has already taken more than seven minutes.

Answer

$P(x > 8) = 0.7778$

$P(x > 8 | x > 7) = 0.875$

Example 5.3.5

Ace Heating and Air Conditioning Service finds that the amount of time a repairman needs to fix a furnace is uniformly distributed between 1.5 and four hours. Let $x =$ the time needed to fix a furnace. Then $x \sim U(1.5, 4)$.

a. Find the probability that a randomly selected furnace repair requires more than two hours.
b. Find the probability that a randomly selected furnace repair requires less than three hours.
c. Find the 30$^{\text{th}}$ percentile of furnace repair times.
d. The longest 25% of furnace repair times take at least how long? (In other words: find the minimum time for the longest 25% of repair times.) What percentile does this represent?
e. Find the mean and standard deviation

Solution

a. To find $f(x) : f(x) = \frac{1}{4-1.5} = \frac{1}{2.5}$ so $f(x) = 0.4$

$P(x > 2) = (\text{base})(\text{height}) = (4-2)(0.4) = 0.8$

This shows the graph of the function f(x) = 0.4. A horiztonal line ranges from the point (1.5, 0.4) to the point (4, 0.4). Vertical lines extend from the x-axis to the graph at x = 1.5 and x = 4 creating a rectangle. A region is shaded inside the rectangle from x = 2 to x = 4.

Figure 5.3.3. Uniform Distribution between 1.5 and four with shaded area between two and four representing the probability that the repair time x is greater than two

b. $P(x < 3) = (\text{base})(\text{height}) = (3-1.5)(0.4) = 0.6$

The graph of the rectangle showing the entire distribution would remain the same. However the graph should be shaded between $x = 1.5$ and $x = 3$. Note that the shaded area starts at $x = 1.5$ rather than at $x = 0$; since $X \sim U(1.5, 4)$, x can not be less than 1.5.

This shows the graph of the function f(x) = 0.4. A horiztonal line ranges from the point (1.5, 0.4) to the point (4, 0.4). Vertical lines extend from the x-axis to the graph at x = 1.5 and x = 4 creating a rectangle. A region is shaded inside the rectangle from x = 1.5 to x = 3.

Figure 5.3.4. Uniform Distribution between 1.5 and four with shaded area between 1.5 and three representing the probability that the repair time x is less than three

c.

This shows the graph of the function f(x) = 0.4. A horiztonal line ranges from the point (1.5, 0.4) to the point (4, 0.4). Vertical lines extend from the x-axis to the graph at x = 1.5 and x = 4 creating a rectangle. A region is shaded inside the rectangle from x = 1.5 to x = k. The shaded area represents P(x < k) = 0.3.

Figure 5.3.5. Uniform Distribution between 1.5 and 4 with an area of 0.30 shaded to the left, representing the shortest 30% of repair times.

$P(x<k)=0.30$
$P(x<k)=(\text{base})(\text{height})=(k–1.5)(0.4)$
$\mathbf{0.3=(k–1.5)(0.4)}$; Solve to find k:
$0.75=k–1.5$, obtained by dividing both sides by 0.4
$\mathbf{k=2.25}$, obtained by adding 1.5 to both sides

The 30^{th} percentile of repair times is 2.25 hours. 30% of repair times are 2.5 hours or less.

d.

Figure 5.3.6. Uniform Distribution between 1.5 and 4 with an area of 0.25 shaded to the right representing the longest 25% of repair times.

$P(x>k)=0.25$
$P(x>k)=(\text{base})(\text{height})=(4–k)(0.4)$
$\mathbf{0.25=(4–k)(0.4)}$; Solve for k:
$0.625=4-k$,
obtained by dividing both sides by 0.4
$-3.375=-k$,
obtained by subtracting four from both sides: $\mathbf{k=3.375}$
The longest 25% of furnace repairs take at least 3.375 hours (3.375 hours or longer).

Note: Since 25% of repair times are 3.375 hours or longer, that means that 75% of repair times are 3.375 hours or less. 3.375 hours is the **75^{th} percentile** of furnace repair times.

e. $\mu=\frac{a+b}{2}$ and $\sigma=\sqrt{\frac{(b-a)^2}{12}}$

$\mu=\frac{1.5+4}{2}=2.75$ hours and $\sigma=\sqrt{\frac{(4-1.5)^2}{12}}=0.7217$ hours

Exercise 5.3.6

The amount of time a service technician needs to change the oil in a car is uniformly distributed between 11 and 21 minutes. Let $X=$ the time needed to change the oil on a car.

a. Write the random variable X in words. $X=$ ________________.
b. Write the distribution.
c. Graph the distribution.
d. Find $P(x>19)$.
e. Find the 50^{th} percentile.

Answer

a. Let $X=$ the time needed to change the oil in a car.
b. $X\sim U(11,21)$.
c. This graph shows a uniform distribution. The horizontal axis ranges from 405 to 525. The distribution is modeled by a rectangle extending from x = 447 to x = 521.

Figure 5.3.7.

d. $P(x>19)=0.2$
e. the 50^{th} percentile is 16 minutes.

Chapter Review

If X has a uniform distribution where $a<x<b$ or $a\leq x\leq b$, then X takes on values between a and b (may include a and b). All values x are equally likely. We write $X\sim U(a,b)$. The mean of X is $\mu=\frac{a+b}{2}$. The standard deviation of X is $\sigma=\sqrt{\frac{(b-a)^2}{12}}$.

The probability density function of X is $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$. The cumulative distribution function of X is $P(X \leq x) = \frac{x-a}{b-a}$. X is continuous.

The graph shows a rectangle with total area equal to 1. The rectangle extends from x = a to x = b on the x-axis and has a height of 1/(b-a).

Figure 5.3.8.

The probability $P(c < X < d)$ may be found by computing the area under $f(x)$, between c and d. Since the corresponding area is a rectangle, the area may be found simply by multiplying the width and the height.

Formula Review

$X =$ a real number between a and b (in some instances, X can take on the values a and b). $a =$ smallest X; $b =$ largest X

$X \sim U(a, b)$

The mean is $\mu = \frac{a+b}{2}$

The standard deviation is $\sigma = \sqrt{\frac{(b-a)^2}{12}}$

Probability density function: $f(x) = \frac{1}{b-a} \text{for} a \leq X \leq b$

Area to the Left of x: $P(X < x) = (x-a)\left(\frac{1}{b-a}\right)$

Area to the Right of x: P(X > x) = (b – x)$\left(\frac{1}{b-a}\right)$

Area Between c and d: $P(c < x < d) = (\text{base})(\text{height}) = (d-c)\left(\frac{1}{b-a}\right)$

Uniform: $X \sim U(a, b)$ where $a < x < b$

- pdf: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$
- cdf: $P(X \leq x) = \frac{x-a}{b-a}$
- mean $\mu = \frac{a+b}{2}$
- standard deviation $\sigma = \sqrt{\frac{(b-a)^2}{12}}$
- $P(c < X < d) = (d-c)\left(\frac{1}{b-a}\right)$

References

McDougall, John A. The McDougall Program for Maximum Weight Loss. Plume, 1995.

Use the following information to answer the next ten questions. The data that follow are the square footage (in 1,000 feet squared) of 28 homes.

1.5	2.4	3.6	2.6	1.6	2.4	2.0
3.5	2.5	1.8	2.4	2.5	3.5	4.0
2.6	1.6	2.2	1.8	3.8	2.5	1.5
2.8	1.8	4.5	1.9	1.9	3.1	1.6

The sample mean = 2.50 and the sample standard deviation = 0.8302.

The distribution can be written as $X \sim U(1.5, 4.5)$.

Exercise 5.3.7

What type of distribution is this?

Exercise 5.3.8

In this distribution, outcomes are equally likely. What does this mean?

Answer

It means that the value of x is just as likely to be any number between 1.5 and 4.5.

Exercise 5.3.9

What is the height of $f(x)$ for the continuous probability distribution?

Exercise 5.3.10

What are the constraints for the values of x?

Answer

$1.5 \leq x \leq 4.5$

Exercise 5.3.11

Graph $P(2 < x < 3)$.

Exercise 5.3.12

What is $P(2 < x < 3)$?

Answer

0.3333

Exercise 5.3.13

What is $P(x < 3.5 | x < 4)$?

Exercise 5.3.14

What is $P(x = 1.5)$?

Answer

zero

Exercise 5.3.15

What is the 90$^{\text{th}}$ percentile of square footage for homes?

Exercise 5.3.16

Find the probability that a randomly selected home has more than 3,000 square feet given that you already know the house has more than 2,000 square feet.

Answer

0.6

Exercise 5.3.17

What is a? What does it represent?

Exercise 5.3.18

What is b? What does it represent?

Answer

b is 12, and it represents the highest value of x.

Exercise 5.3.19

What is the probability density function?

Exercise 5.3.20

What is the theoretical mean?

Answer

six

Exercise 5.3.21

What is the theoretical standard deviation?

Exercise 5.3.22

Draw the graph of the distribution for $P(x > 9)$.

Answer

This graph shows a uniform distribution. The horizontal axis ranges from 0 to 12. The distribution is modeled by a rectangle extending from x = 0 to x = 12. A region from x = 9 to x = 12 is shaded inside the rectangle.

Figure 5.3.9.

Exercise 5.3.23

Find $P(x > 9)$.

Exercise 5.3.24

Find the 40th percentile.

Answer

4.8

Use the following information to answer the next eleven exercises. The age of cars in the staff parking lot of a suburban college is uniformly distributed from six months (0.5 years) to 9.5 years.

Exercise 5.3.25

What is being measured here?

Exercise 5.3.26

In words, define the random variable X.

Answer

X = The age (in years) of cars in the staff parking lot

Exercise 5.3.27

Are the data discrete or continuous?

Exercise 5.3.28

The interval of values for x is ______.

Answer

0.5 to 9.5

Exercise 5.3.29

The distribution for X is ______.

Exercise 5.3.30

Write the probability density function.

Answer

$f(x) = \frac{1}{9}$ where x is between 0.5 and 9.5, inclusive.

Exercise 5.3.31

Graph the probability distribution.

a. Sketch the graph of the probability distribution.

This is a blank graph template. The vertical and horizontal axes are unlabeled.

Figure 5.3.10.

b. Identify the following values:

i. Lowest value for $\bar{x}$: _______
ii. Highest value for $\bar{x}$: _______
iii. Height of the rectangle: _______
iv. Label for *x*-axis (words): _______
v. Label for *y*-axis (words): _______

Exercise 5.3.32

Find the average age of the cars in the lot.

Answer

μ = 5

Exercise 5.3.33

Find the probability that a randomly chosen car in the lot was less than four years old.

a. Sketch the graph, and shade the area of interest.

Blank graph with vertical and horizontal axes.

Figure 5.3.11.

b. Find the probability. $P(x < 4) =$ _______

Exercise 5.3.34

Considering only the cars less than 7.5 years old, find the probability that a randomly chosen car in the lot was less than four years old.

a. Sketch the graph, shade the area of interest.

This is a blank graph template. The vertical and horizontal axes are unlabeled.

Figure 5.3.12.

b. Find the probability. $P(x < 4|x < 7.5) =$ _______

Answer

a. Check student’s solution.
b. $\frac{3.5}{7}$

Exercise 5.3.35

What has changed in the previous two problems that made the solutions different

Exercise 5.3.36

Find the third quartile of ages of cars in the lot. This means you will have to find the value such that $\frac{3}{4}$, or 75%, of the cars are at most (less than or equal to) that age.

a. Sketch the graph, and shade the area of interest.

Blank graph with vertical and horizontal axes.

Figure 5.3.13.

b. Find the value k such that $P(x < k) = 0.75$.
c. The third quartile is _______

Answer

a. Check student's solution.
b. $k = 7.25$
c. 7.25

Glossary

Conditional Probability
the likelihood that an event will occur given that another event has already occurred

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