11.4: Test of Independence

Tests of independence involve using a contingency table of observed (data) values.

The test statistic for a *test of independence* is similar to that of a goodness-of-fit test:

$$\sum_{(i \cdot j)} \frac{(O-E)^2}{E} \tag{11.4.1}$$

where:

- $O =$ observed values
- $E =$ expected values
- $i =$ the number of rows in the table
- $j =$ the number of columns in the table

There are $i \cdot j$ terms of the form $\frac{(O-E)^2}{E}$.

> The expected value for each cell needs to be at least five in order for you to use this test.

A test of independence determines whether two factors are independent or not. You first encountered the term independence in Probability Topics. As a review, consider the following example.

Example 11.4.1

Suppose $A =$ a speeding violation in the last year and $B =$ a cell phone user while driving. If A and B are independent then $P(A \text{ AND } B) = P(A)P(B)$. A AND B is the event that a driver received a speeding violation last year and also used a cell phone while driving. Suppose, in a study of drivers who received speeding violations in the last year, and who used cell phone while driving, that 755 people were surveyed. Out of the 755, 70 had a speeding violation and 685 did not; 305 used cell phones while driving and 450 did not.

Let $y =$ expected number of drivers who used a cell phone while driving and received speeding violations.

If A and B are independent, then $P(A \text{ AND } B) = P(A)P(B)$. By substitution,

$$\frac{y}{755} = \left(\frac{70}{755}\right)\left(\frac{305}{755}\right)$$

Solve for y:

$$y = \frac{(70)(305)}{755} = 28.3$$

About 28 people from the sample are expected to use cell phones while driving and to receive speeding violations.

In a test of independence, we state the null and alternative hypotheses in words. Since the contingency table consists of **two factors**, the null hypothesis states that the factors are **independent** and the alternative hypothesis states that they are **not independent (dependent)**. If we do a test of independence using the example, then the null hypothesis is:

H_0: Being a cell phone user while driving and receiving a speeding violation are independent events.

If the null hypothesis were true, we would expect about 28 people to use cell phones while driving and to receive a speeding violation.

The test of independence is always right-tailed because of the calculation of the test statistic. If the expected and observed values are not close together, then the test statistic is very large and way out in the right tail of the chi-square curve, as it is in a goodness-of-fit.

The number of degrees of freedom for the test of independence is:

$$df = (\text{number of columns} - 1)(\text{number of rows} - 1)$$

The following formula calculates the **expected number** (E):

$$E=\frac{(\text{row total})(\text{column total})}{\text{total number surveyed}}$$

? Exercise 11.4.1

A sample of 300 students is taken. Of the students surveyed, 50 were music students, while 250 were not. Ninety-seven were on the honor roll, while 203 were not. If we assume being a music student and being on the honor roll are independent events, what is the expected number of music students who are also on the honor roll?

Answer

About 16 students are expected to be music students and on the honor roll.

✔ Example 11.4.2

In a volunteer group, adults 21 and older volunteer from one to nine hours each week to spend time with a disabled senior citizen. The program recruits among community college students, four-year college students, and nonstudents. In Table $11.4.1$ is a **sample** of the adult volunteers and the number of hours they volunteer per week.

Table $11.4.1$: Number of Hours Worked Per Week by Volunteer Type (Observed). The table contains **observed (O)** values (data).

Type of Volunteer	1–3 Hours	4–6 Hours	7–9 Hours	Row Total
Community College Students	111	96	48	255
Four-Year College Students	96	133	61	290
Nonstudents	91	150	53	294
Column Total	298	379	162	839

Is the number of hours volunteered **independent** of the type of volunteer?

Answer

The **observed table** and the question at the end of the problem, "Is the number of hours volunteered independent of the type of volunteer?" tell you this is a test of independence. The two factors are **number of hours volunteered** and **type of volunteer**. This test is always right-tailed.

- H_0: The number of hours volunteered is **independent** of the type of volunteer.
- H_a: The number of hours volunteered is **dependent** on the type of volunteer.

The expected results are in Table $11.4.2$.

Table $11.4.2$: Number of Hours Worked Per Week by Volunteer Type (Expected). The table contains **expected**(E) values (data).

Type of Volunteer	1-3 Hours	4-6 Hours	7-9 Hours
Community College Students	90.57	115.19	49.24
Four-Year College Students	103.00	131.00	56.00
Nonstudents	104.42	132.81	56.77

For example, the calculation for the expected frequency for the top left cell is

$$E=\frac{(\text{row total})(\text{column total})}{\text{total number surveyed}}=\frac{(255)(298)}{839}=90.57$$

Calculate the test statistic: $\chi^2 = 12.99$ (calculator or computer)

Distribution for the test: χ^2_4

$$df = (3 \text{ columns} - 1)(3 \text{ rows} - 1) = (2)(2) = 4$$

Graph:

Nonsymmetrical chi-square curve with values of 0 and 12.99 on the x-axis representing the test statistic of number of hours worked by volunteers of different types. A vertical upward line extends from 12.99 to the curve and the area to the right of this is equal to the p-value.

Figure 11.4.1.

Probability statement: $p\text{-value} = P(\chi^2 > 12.99) = 0.0113$

Compare α and the p-value: Since no α is given, assume $\alpha = 0.05$. $p\text{-value} = 0.0113$. $\alpha > p\text{-value}$.

Make a decision: Since $\alpha > p\text{-value}$, reject H_0. This means that the factors are not independent.

Conclusion: At a 5% level of significance, from the data, there is sufficient evidence to conclude that the number of hours volunteered and the type of volunteer are dependent on one another.

For the example in Table, if there had been another type of volunteer, teenagers, what would the degrees of freedom be?

USING THE TI-83, 83+, 84, 84+ CALCULATOR

Press the `MATRX` key and arrow over to `EDIT`. Press `1:[A]`. Press `3 ENTER 3 ENTER`. Enter the table values by row from Table. Press `ENTER` after each. Press `2nd QUIT`. Press `STAT` and arrow over to `TESTS`. Arrow down to `C:χ2-TEST`. Press `ENTER`. You should see `Observed:[A] and Expected:[B]`. If necessary, use the arrow keys to move the cursor after `Observed:` and press `2nd MATRX`. Press `1:[A]` to select matrix A. It is not necessary to enter expected values. The matrix listed after `Expected:` can be blank. Arrow down to `Calculate`. Press `ENTER`. The test statistic is 12.9909 and the *p*-value = 0.0113. Do the procedure a second time, but arrow down to `Draw` instead of `calculate`.

Exercise 11.4.2

The Bureau of Labor Statistics gathers data about employment in the United States. A sample is taken to calculate the number of U.S. citizens working in one of several industry sectors over time. Table 11.4.3 shows the results:

Table 11.4.3

Industry Sector	2000	2010	2020	Total
Nonagriculture wage and salary	13,243	13,044	15,018	41,305
Goods-producing, excluding agriculture	2,457	1,771	1,950	6,178
Services-providing	10,786	11,273	13,068	35,127
Agriculture, forestry, fishing, and hunting	240	214	201	655
Nonagriculture self-employed and unpaid family worker	931	894	972	2,797
Secondary wage and salary jobs in agriculture and private household industries	14	11	11	36

Industry Sector	2000	2010	2020	Total
Secondary jobs as a self-employed or unpaid family worker	196	144	152	492
Total	27,867	27,351	31,372	86,590

We want to know if the change in the number of jobs is independent of the change in years. State the null and alternative hypotheses and the degrees of freedom.

Answer

- H_0: The number of jobs is independent of the year.
- H_a: The number of jobs is dependent on the year.

$df = 12$

alt

Figure 11.4.2.

Press the `MATRX` key and arrow over to `EDIT`. Press `1:[A]`. Press `3 ENTER 3 ENTER`. Enter the table values by row. Press `ENTER` after each. Press `2nd QUIT`. Press `STAT` and arrow over to `TESTS`. Arrow down to c:χ^2-TEST. Press `ENTER`. You should see `Observed:[A] and Expected:[B]`. Arrow down to `Calculate`. Press `ENTER`. The test statistic is 227.73 and the p-value $= 5.90E - 42 = 0$. Do the procedure a second time but arrow down to `Draw` instead of `calculate`.

Example 11.4.3

De Anza College is interested in the relationship between anxiety level and the need to succeed in school. A random sample of 400 students took a test that measured anxiety level and need to succeed in school. Table shows the results. De Anza College wants to know if anxiety level and need to succeed in school are independent events.

Need to Succeed in School vs. Anxiety Level

Need to Succeed in School	High Anxiety	Med-high Anxiety	Medium Anxiety	Med-low Anxiety	Low Anxiety	Row Total
High Need	35	42	53	15	10	155
Medium Need	18	48	63	33	31	193
Low Need	4	5	11	15	17	52
Column Total	57	95	127	63	58	400

a. How many high anxiety level students are expected to have a high need to succeed in school?
b. If the two variables are independent, how many students do you expect to have a low need to succeed in school and a med-low level of anxiety?
c. $E = \frac{(\text{row total})(\text{column total})}{\text{total surveyed}} =$ ________
d. The expected number of students who have a med-low anxiety level and a low need to succeed in school is about ________.

Solution

a. The column total for a high anxiety level is 57. The row total for high need to succeed in school is 155. The sample size or total surveyed is 400.

$$E = \frac{(\text{row total})(\text{column total})}{\text{total surveyed}} = \frac{155 \cdot 57}{400} = 22.09 \tag{11.4.2}$$

The expected number of students who have a high anxiety level and a high need to succeed in school is about 22.

b. The column total for a med-low anxiety level is 63. The row total for a low need to succeed in school is 52. The sample size or total surveyed is 400.

c. $E = \frac{(\text{row total})(\text{column total})}{\text{total surveyed}} = 8.19$

d. 8

Exercise 11.4.3

Refer back to the information in Note. How many service providing jobs are there expected to be in 2020? How many nonagriculture wage and salary jobs are there expected to be in 2020?

Answer

12,727, 14,965

References

1. DiCamilo, Mark, Mervin Field, “Most Californians See a Direct Linkage between Obesity and Sugary Sodas. Two in Three Voters Support Taxing Sugar-Sweetened Beverages If Proceeds are Tied to Improving School Nutrition and Physical Activity Programs.” The Field Poll, released Feb. 14, 2013. Available online at field.com/fieldpollonline/sub...rs/Rls2436.pdf (accessed May 24, 2013).
2. Harris Interactive, “Favorite Flavor of Ice Cream.” Available online at http://www.statisticbrain.com/favori...r-of-ice-cream (accessed May 24, 2013)
3. “Youngest Online Entrepreneurs List.” Available online at http://www.statisticbrain.com/younge...repreneur-list (accessed May 24, 2013).

Review

To assess whether two factors are independent or not, you can apply the test of independence that uses the chi-square distribution. The null hypothesis for this test states that the two factors are independent. The test compares observed values to expected values. The test is right-tailed. Each observation or cell category must have an expected value of at least 5.

Formula Review

Test of Independence

- The number of degrees of freedom is equal to $(\text{number of columns - 1})(\text{number of rows - 1})$.
- The test statistic is $\sum_{(i \cdot j)} \frac{(O-E)^2}{E}$ where $O =$ observed values, $E =$ expected values, $i =$ the number of rows in the table, and $j =$ the number of columns in the table.
- If the null hypothesis is true, the expected number $E = \frac{(\text{row total})(\text{column total})}{\text{total surveyed}}$.

Determine the appropriate test to be used in the next three exercises.

Exercise 11.4.4

A pharmaceutical company is interested in the relationship between age and presentation of symptoms for a common viral infection. A random sample is taken of 500 people with the infection across different age groups.

Answer

a test of independence

Exercise 11.4.5

The owner of a baseball team is interested in the relationship between player salaries and team winning percentage. He takes a random sample of 100 players from different organizations.

? Exercise 11.4.6

A marathon runner is interested in the relationship between the brand of shoes runners wear and their run times. She takes a random sample of 50 runners and records their run times as well as the brand of shoes they were wearing.

Answer

a test of independence

Use the following information to answer the next seven exercises: Transit Railroads is interested in the relationship between travel distance and the ticket class purchased. A random sample of 200 passengers is taken. Table $11.4.4$ shows the results. The railroad wants to know if a passenger's choice in ticket class is independent of the distance they must travel.

Table 11.4.4

Traveling Distance	Third class	Second class	First class	Total
1–100 miles	21	14	6	41
101–200 miles	18	16	8	42
201–300 miles	16	17	15	48
301–400 miles	12	14	21	47
401–500 miles	6	6	10	22
Total	73	67	60	200

? Exercise 11.4.7

State the hypotheses.

- H_0: _______
- H_a: _______

? Exercise 11.4.8

$df =$ _______

Answer

8

? Exercise 11.4.9

How many passengers are expected to travel between 201 and 300 miles and purchase second-class tickets?

? Exercise 11.4.10

How many passengers are expected to travel between 401 and 500 miles and purchase first-class tickets?

Answer

6.6

? Exercise 11.4.11

What is the test statistic?

Exercise 11.4.12

What is the p-value?

Answer

0.0435

Exercise 11.4.13

What can you conclude at the 5% level of significance?

Use the following information to answer the next eight exercises: An article in the New England Journal of Medicine, discussed a study on smokers in California and Hawaii. In one part of the report, the self-reported ethnicity and smoking levels per day were given. Of the people smoking at most ten cigarettes per day, there were 9,886 African Americans, 2,745 Native Hawaiians, 12,831 Latinos, 8,378 Japanese Americans and 7,650 whites. Of the people smoking 11 to 20 cigarettes per day, there were 6,514 African Americans, 3,062 Native Hawaiians, 4,932 Latinos, 10,680 Japanese Americans, and 9,877 whites. Of the people smoking 21 to 30 cigarettes per day, there were 1,671 African Americans, 1,419 Native Hawaiians, 1,406 Latinos, 4,715 Japanese Americans, and 6,062 whites. Of the people smoking at least 31 cigarettes per day, there were 759 African Americans, 788 Native Hawaiians, 800 Latinos, 2,305 Japanese Americans, and 3,970 whites.

Exercise 11.4.14

Complete the table.

Table 11.4.5: Smoking Levels by Ethnicity (Observed)

Smoking Level Per Day	African American	Native Hawaiian	Latino	Japanese Americans	White	TOTALS
1-10						
11-20						
21-30						
31+						
TOTALS						

Answer

Table 11.4.5B

Smoking Level Per Day	African American	Native Hawaiian	Latino	Japanese Americans	White	Totals
1-10	9,886	2,745	12,831	8,378	7,650	41,490
11-20	6,514	3,062	4,932	10,680	9,877	35,065
21-30	1,671	1,419	1,406	4,715	6,062	15,273
31+	759	788	800	2,305	3,970	8,622
Totals	18,830	8,014	19,969	26,078	27,559	10,0450

Exercise 11.4.15

State the hypotheses.

- H_0: _______

- H_a: _______

Exercise 11.4.16

Enter expected values in Table. Round to two decimal places.

Calculate the following values:

Answer

Table 11.4.6

Smoking Level Per Day	African American	Native Hawaiian	Latino	Japanese Americans	White
1-10	7777.57	3310.11	8248.02	10771.29	11383.01
11-20	6573.16	2797.52	6970.76	9103.29	9620.27
21-30	2863.02	1218.49	3036.20	3965.05	4190.23
31+	1616.25	687.87	1714.01	2238.37	2365.49

Exercise 11.4.17

$df =$ _______

Exercise 11.4.18

χ^2test statistic $=$ ______

Answer

10,301.8

Exercise 11.4.19

p-value $=$ ______

Exercise 11.4.20

Is this a right-tailed, left-tailed, or two-tailed test? Explain why.

Answer

right

Exercise 11.4.21

Graph the situation. Label and scale the horizontal axis. Mark the mean and test statistic. Shade in the region corresponding to the p-value.

Blank graph with vertical and horizontal axes.

Figure 11.4.3.

State the decision and conclusion (in a complete sentence) for the following preconceived levels of α.

? Exercise 11.4.22

$\alpha = 0.05$

a. Decision: ____________________
b. Reason for the decision: ____________________
c. Conclusion (write out in a complete sentence): ____________________

Answer

a. Reject the null hypothesis.
b. p-value $< \alpha$
c. There is sufficient evidence to conclude that smoking level is dependent on ethnic group.

? Exercise 11.4.23

$\alpha = 0.05$

a. Decision: ____________________
b. Reason for the decision: ____________________
c. Conclusion (write out in a complete sentence): ____________________

Glossary

Contingency Table

a table that displays sample values for two different factors that may be dependent or contingent on one another; it facilitates determining conditional probabilities.
