11.2: The Basics of Matrix Algebra

A matrix is a rectangular array of numbers with rows and columns. As noted, operations performed on matrices are performed on all elements of a matrix simultaneously. In this section, we provide the basic understanding of matrix algebra that is necessary to make sense of the expression of multiple regression in matrix form.

11.2.1 Matrix Basics

The individual numbers in a matrix are referred to as “elements”. The elements of a matrix can be identified by their location in a row and column, denoted as $A_{r,c}$. In the following example, m will refer to the matrix row and n will refer to the column.

$$A_{m,n}=\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Therefore, in the following matrix;

$$A=\begin{bmatrix} 10 & 5 & 8 \\ -12 & 1 & 0 \end{bmatrix}$$

element $a_{2,3}=0$ and $a_{1,2}=5$.

11.2.2 Vectors

A vector is a matrix with single column or row. Here are some examples:

$$A=\begin{bmatrix} 6 \\ -1 \\ 8 \\ 11 \end{bmatrix}$$

or

$$A=\begin{bmatrix} 1 & 2 & 8 & 7 \end{bmatrix}$$

11.2.3 Matrix Operations

There are several “operations” that can be performed with and on matrices. Most of the these can be computed with `R`, so we will use `R` examples as we go along. As always, you will understand the operations better if you work the problems in `R` as we go. There is no need to load a data set this time – we will enter all the data we need in the examples.

11.2.4 Transpose

Transposing, or taking the “prime” of a matrix, switches the rows and columns.[21] The matrix

$$A=\begin{bmatrix} 10 & 5 & 8 \\ -12 & 1 & 0 \end{bmatrix}$$

Once transposed is:

$$A'=\begin{bmatrix} 10 & -12 \\ 5 & 1 \\ 8 & 0 \end{bmatrix}$$

Note that the operation “hinges” on the element in the upper right-hand corner of A, $A_{1,1}$, so the first column of A becomes the first row on A'. To transpose a matrix in `R`, create a matrix object then simply use the `t` command.

```
A <- matrix(c(10,-12,5,1,8,0),2,3)
A
```

```
##      [,1] [,2] [,3]
## [1,]   10    5    8
## [2,]  -12    1    0
```

```
t(A)
```

```
##      [,1] [,2]
## [1,]   10  -12
## [2,]    5    1
## [3,]    8    0
```

11.2.5 Adding Matrices

To add matrices together, they must have the same *dimensions*, meaning that the matrices must have the same number of rows and columns. Then, you simply add each element to its counterpart by row and column. For example:

$$A=\begin{bmatrix}4 & -3\\ 2 & 0\end{bmatrix}+B=\begin{bmatrix}8 & 1\\ 4 & -5\end{bmatrix}=A+B=\begin{bmatrix}4+8 & -3+1\\ 2+4 & 0+(-5)\end{bmatrix}=\begin{bmatrix}12 & -2\\ 6 & -5\end{bmatrix}$$

To add matrices together in `R`, simply create two matrix objects and add them together.

```
A <- matrix(c(4,2,-3,0),2,2)
A
```

```
##      [,1] [,2]
## [1,]    4   -3
## [2,]    2    0
```

```
B <- matrix(c(8,4,1,-5),2,2)
B
```

```
##      [,1] [,2]
## [1,]    8    1
## [2,]    4   -5
```

```
A + B
```

```
##      [,1] [,2]
## [1,]   12   -2
## [2,]    6   -5
```

See – how easy is that? No need to be afraid of a little matrix algebra!

11.2.6 Multiplication of Matrices

To multiply matrices they must be **conformable**, which means the number of *columns* in the first matrix must match the number of *rows* in the second matrix.

$$A_{rXq} * B_{qXc} = C_{rXc}$$

Then, multiply column elements by the row elements, as shown here:

$$A=\begin{bmatrix}2 & 5\\ 1 & 0\\ 6 & -2\end{bmatrix} * B=\begin{bmatrix}4 & 2 & 1\\ 5 & 7 & 2\end{bmatrix}=AXB=\begin{bmatrix}(2X4)+(5X5) & (2X2)+(5X7) & (2X1)+(5X2)\\ (1X4)+(0X5) & (1X2)+(0X7) & (1X1)+(0X2)\\ (6X4)+(-2X5) & (6X2)+(-2X7) & (6X1)+(-2X2)\end{bmatrix}=\begin{bmatrix}33 & 39 & 12\\ 4 & 2 & 1\\ 14 & -2 & 2\end{bmatrix}$$

To multiply matrices in `R`, create two matrix objects and multiply them using the `\%*\%` command.

```
A <- matrix(c(2,1,6,5,0,-2),3,2)
A
```

```
##      [,1] [,2]
## [1,]    2    5
## [2,]    1    0
## [3,]    6   -2
```

```
B <- matrix(c(4,5,2,7,1,2),2,3)
B
```

```
##      [,1] [,2] [,3]
## [1,]    4    2    1
## [2,]    5    7    2
```

```
A %*% B
```

```
##      [,1] [,2] [,3]
## [1,]   33   39   12
## [2,]    4    2    1
## [3,]   14   -2    2
```

11.2.7 Identity Matrices

The identity matrix is a square matrix with 1's on the diagonal and 0's elsewhere. For a 4 x 4 matrix, it looks like this:

$$I=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

It acts like a 1 in algebra; a matrix (A) times the identity matrix (I) is A. This can be demonstrated in `R`.

```
A <- matrix(c(5,3,2,4),2,2)
A
```

```
##      [,1] [,2]
## [1,]    5    2
## [2,]    3    4
```

```
I <- matrix(c(1,0,0,1),2,2)
I
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

```
A %*% I
```

```
##      [,1] [,2]
## [1,]    5    2
## [2,]    3    4
```

Note that, if you want to square a column matrix (that is, multiply it by itself), you can simply take the transpose of the column (thereby making it a row matrix) and multiply them. The square of column matrix A is $A'A$.

11.2.8 Matrix Inversion

The matrix inversion operation is a bit like dividing any number by itself in algebra. An inverse of the A matrix is denoted A^{-1}. Any matrix multiplied by its inverse is equal to the identity matrix:

$$AA^{-1} = A^{-1}A = I$$

For example,

$$A = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \text{and } A^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \text{therefore } A * A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

However, matrix inversion is only applicable to a square (i.e., number of rows equals number of columns) matrix; only a square matrix can have an inverse.

Finding the Inverse of a Matrix

To find the inverse of a matrix, the values that will produce the identity matrix, create a second matrix of variables and solve for I.

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} X \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 3a+b & 3c+d \\ 2a+4b & 2c+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Set $3a+b=1$ and $2a+4b=0$ and solve for a and b. In this case $a=\frac{2}{5}$ and $b=-\frac{1}{5}$. Likewise, set $3c+d=0$ and $2c+4d=1$; solving for c and d produces $c=-\frac{1}{10}$ and $d=\frac{3}{10}$. Therefore,

$$A^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{3}{10} \end{bmatrix}$$

Finding the inverse matrix can also be done in `R` using the `solve` command.

```
A <- matrix(c(3,2,1,4),2,2)
A
```

```
##      [,1] [,2]
## [1,]    3    1
## [2,]    2    4
```

```
A.inverse <- solve(A)
A.inverse
```

```
##      [,1] [,2]
## [1,]  0.4 -0.1
## [2,] -0.2  0.3
```

```
A %*% A.inverse
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

OK – now we have all the pieces we need to apply matrix algebra to multiple regression.