1.3.1: Integer Exponents: a Review with Variables

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions using the properties for exponents
- Use the definition of a negative exponent

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $(-2)(-2)(-2)$.

2. Simplify $(2x^5)^3$.

3. Simplify $(2x^2)(4x^3)$.

A Review of Positive Integer Exponents

Remember that a positive integer exponent indicates repeated multiplication of the same quantity. For example, in the expression a^m, the positive integer *exponent* m tells us how many times we use the *base* a as a factor.

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{\text{m factors}}$$

For example,

$$(-9)^5 = \underbrace{(-9) \cdot (-9) \cdot (-9) \cdot (-9) \cdot (-9)}_{\text{5 factors}}$$

Let's review the vocabulary for expressions with exponents.

Definition 1.3.1.1

$$a^m = \underbrace{a \cdot a \cdot a \cdots a}_{m\text{ factors}}$$

This is read a to the m^{th} **power**.

In the expression a^m with positive integer m and $a \neq 0$, the **exponent** m tells us how many times we use the **base** a as a factor.

Recall the following example which leads to the the *Product Property for Positive Integer Exponents*.

	x^2x^3
What does this mean?	$= \underbrace{x \cdot x}_{\text{2 factors}} \cdot \underbrace{x \cdot x \cdot x}_{\text{3 factors}}$ $= \underbrace{x \cdot x \cdot x \cdot x \cdot x}_{\text{5 factors}}$
	$= x^5$

The base stayed the same and we added the exponents.

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Product Property for Positive Integer Exponents

If a is a real number and m and n are positive integers, then

$$a^m a^n = a^{m+n}.$$

To multiply with like bases, add the exponents.

Now we will look at an exponent property for division. As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$=\frac{x\cdot x\cdot x\cdot x\cdot x}{x\cdot x}$		$=\frac{x\cdot x}{x\cdot x\cdot x}$
Use the Equivalent Fractions Property.	$=\frac{\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x}{\cancel{x}\cdot\cancel{x}}$		$=\frac{\cancel{x}\cdot\cancel{x}\cdot 1}{\cancel{x}\cdot\cancel{x}\cdot x}$
Simplify.	$=x^3$		$=\frac{1}{x}$
Note.	$x^3=x^{5-2}$		$\frac{1}{x}=\frac{1}{x^1}=\frac{1}{x^{3-2}}$

Here we see how to use the initial exponents to arrive at the simplified expression. When the larger exponent was in the numerator, we were left with factors in the numerator. When the larger exponent was in the denominator, we were left with factors in the denominator--notice the numerator of 1. When all the factors in the numerator have been removed, remember this is really dividing the factors to one, and so we need a 1 in the numerator: $\frac{\cancel{x}}{\cancel{x}}=1$. This leads to the *Quotient Property for Positive Integer Exponents*.

Quotient Property for Positive Integer Exponents

If a is a real number, $a\neq 0$, and m and n are distinct positive integers, then

$$\frac{a^m}{a^n}=\begin{cases} a^{m-n} & \text{if } m>n \\ \frac{1}{a^{n-m}} & \text{if } n>m. \end{cases}$$

Example 1.3.1.2

Simplify each expression:

a. $\frac{x^9}{x^7}$

b. $\frac{3^{10}}{3^2}$

c. $\frac{b^8}{b^{12}}$

d. $\frac{7^3}{7^5}$

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

a.

	$\frac{x^9}{x^7}$
Since $9 > 7$, there are more factors of x in the numerator. Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$, for $m > n$.	$= x^{9-7}$
Simplify.	$= x^2$

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

b.

	$\frac{3^{10}}{3^2}$
Since $10 > 2$, there are more factors of 3 in the numerator. Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$, for $m > n$.	$= 3^{10-2}$
Simplify.	$= 3^8$

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

c.

	$\frac{b^8}{b^{12}}$
Since $12 > 8$, there are more factors of b in the denominator. Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$, for $n > m$.	$= \frac{1}{b^{12-8}}$
Simplify.	$= \frac{1}{b^4}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

d.

	$\frac{7^3}{7^5}$
Since $5 > 3$, there are more factors of 3 in the denominator. Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ for $n > m$.	$= \frac{1}{7^{5-3}}$
Simplify.	$= \frac{1}{7^2}$
Simplify.	$= \frac{1}{49}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

? Try It 1.3.1.3

Simplify each expression:

a. $\frac{x^{15}}{x^{10}}$

b. $\frac{6^{14}}{6^5}$

c. $\frac{x^{18}}{x^{22}}$

d. $\frac{12^{15}}{12^{30}}$

Answer

a. x^5

b. 6^9

c. $\frac{1}{x^4}$

d. $\frac{1}{12^{15}}$

Try It 1.3.1.4

Simplify each expression:

a. $\frac{y^{43}}{y^{37}}$

b. $\frac{10^{15}}{10^7}$

c. $\frac{m^7}{m^{15}}$

d. $\frac{9^8}{9^{19}}$

Answer

a. y^6

b. 10^8

c. $\frac{1}{m^8}$

d. $\frac{1}{9^{11}}$

Extending the Meaning of Exponent to Integers

Note that, while so far an exponent that is not a positive integer has no meaning, we see that blindly applying the above properties for such exponents leads to a couple definitions.

	1
For m a positive integer (so that x^m has meaning, $x \neq 0$).	$= \frac{x^m}{x^m}$
Blindly follow the quotient rule that we know to be true in another case.	$= x^{m-m}$
Reduce the fraction on the left.	$= x^0$
Conclude.	$x^0 = 1$

So **if** the Quotient Property is also to hold for the exponent zero we **must** define, for $x \neq 0$,

$$x^0 = 1.$$

Similarly, consider this expression where m is a positive integer (so that x^m has meaning).

	$x^{-m}x^m$

Blindly applying the product property.	$= x^{-m+m}$
Simplify exponent.	$= x^0$
Using our new definition: $x^0 = 1$.	$= 1$
Draw conclusion.	$x^{-m}x^m = 1$
Note the property of the reciprocal.	x^{-m} is the reciprocal of x^m.
Rewrite in symbols.	$x^{-m} = \frac{1}{x^m}$

So, **if** the product property of exponents holds also for (so far undefined) negative integers, we **must** define, for m a negative integer,

$$x^{-m} = \frac{1}{x^m}.$$

Also, since taking the reciprocal of both sides preserves the equality we also have, equivalently,

$$\frac{1}{x^{-m}} = x^m.$$

We could also write these two statements above simultaneously as

$$x^{-m} = \frac{1}{x^m}, m \text{ any integer.}$$

So, we **must** define

Definition 1.3.1.5

For a any non-zero real number

$$a^0 = 1$$

and for m any positive integer

$$a^{-m} = \frac{1}{a^m} \text{ or, equivalently, } \frac{1}{a^{-m}} = a^m.$$

In the above, the base can be anything (x can be anything) which we know to be different from zero, and in this text, we assume any variable that we raise to the zero power is not zero.

Example 1.3.1.6

Simplify each expression:

a. 9^0

b. n^0

c. $(-4a^2b)^0$

d. -3^0

Solution

The definition says any non-zero number raised to the zero power is 1.

a. Use the definition of the zero exponent. $9^0 = 1$

b. Use the definition of the zero exponent. $n^0 = 1$

To simplify the expression n raised to the zero power we just use the definition of the zero exponent. The result is 1.

c. Anything raised to the power zero is 1. Here the base is $-4a^2b$, so $(-4a^2b)^0 = 1$

d. Anything raised to the power zero is 1. Here the base is 3, so this is the opposite of 3^0, or, the opposite of 1. So, $-3^0 = -1$

Try It 1.3.1.7

Simplify each expression:

a. 11^0

b. q^0

c. $(-12p^3q^2)^0$

d. -7^0

Answer

a. 1

b. 1

c. 1

d. -1

Try It 1.3.1.8

Simplify each expression:

a. 23^0

b. r^0

c. $(2st^5)^0$

d. $-s^0$

Answer

a. 1

b. 1

c. 1

d. -1

Example 1.3.1.9

Simplify each expression. Write your answer using positive exponents.

a. x^{-5}

b. 10^{-3}

c. $\frac{1}{y^{-4}}$

d. $\dfrac{1}{3^{-2}}$

Solution

a.

	x^{-5}
Use the definition of a negative exponent, $a^{-n}=\dfrac{1}{a^n}$.	$=\dfrac{1}{x^5}$

b.

	10^{-3}
Use the definition of a negative exponent, $a^{-n}=\dfrac{1}{a^n}$.	$=\dfrac{1}{10^3}$
Simplify.	$=\dfrac{1}{1000}$

c.

	$\dfrac{1}{y^{-4}}$
Use the definition of a negative exponent, $\dfrac{1}{a^{-n}}=a^n$.	$=y^4$

d.

	$\dfrac{1}{3^{-2}}$
Use the definition of a negative exponent, $\dfrac{1}{a^{-n}}=a^n$.	$=3^2$
Simplify.	$=9$

? Try It 1.3.1.10

Simplify each expression. Write your answer using positive exponents.

a. z^{-3}

b. 10^{-7}

c. $\dfrac{1}{p^{-8}}$

d. $\dfrac{1}{4^{-3}}$

Answer

a. $\dfrac{1}{z^3}$

b. $\dfrac{1}{10,000,000}$

c. p^8

d. 64

Try It 1.3.1.11

Simplify each expression. Write your answer using positive exponents.

a. n^{-2}

b. 10^{-4}

c. $\frac{1}{q^{-7}}$

d. $\frac{1}{2^{-4}}$

Answer

a. $\frac{1}{n^2}$

b. $\frac{1}{10,000}$

c. q^7

d. 16

Properties of Negative Exponents

The negative exponent tells us we can rewrite the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in *simplest form*. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

Suppose now we have a fraction raised to a negative exponent. Let’s use our definition of negative exponents to lead us to a new property.

	$\left(\frac{3}{4}\right)^{-2}$
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$=\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$=\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$=\frac{16}{9}$
But we know that	$\frac{16}{9}=\left(\frac{4}{3}\right)^2$
This tells us that	$\left(\frac{3}{4}\right)^{-2}=\left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Integer Exponent Property*.

Quotient to a Negative Integer Exponent Property

If a and b are real numbers, $a \neq 0$, $b \neq 0$, and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}.$$

Example 1.3.1.12

Simplify each expression. Write your answer using positive exponents.

a. $\left(\frac{5}{7}\right)^{-2}$

b. $\left(-\frac{x}{y}\right)^{-3}$

Solution

a.

	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Integer Exponent Property, $\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$=\left(\frac{7}{5}\right)^{2}$
Simplify.	$=\frac{49}{25}$

b.

	$\left(-\frac{x}{y}\right)^{-3}$
Use the Quotient to a Negative Integer Exponent Property, $\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$=\left(-\frac{y}{x}\right)^{3}$
Simplify.	$=-\frac{y^3}{x^3}$

Try It 1.3.1.13

Simplify each expression. Write your answer using positive exponents.

a. $\left(\frac{2}{3}\right)^{-4}$

b. $\left(-\frac{m}{n}\right)^{-2}$

Answer

a. $\frac{81}{16}$

b. $\frac{n^2}{m^2}$

Try It 1.3.1.14

Simplify each expression. Write your answer using positive exponents.

a. $\left(\frac{3}{5}\right)^{-3}$

b. $\left(-\frac{a}{b}\right)^{-4}$

Answer

a. $\frac{125}{27}$

b. $\frac{b^4}{a^4}$

We would like to verify that the properties of positive integer exponents can be extended to all integer exponents. We will do some examples. Let m and n be non-negative integers, $m > n$. Let's simplify $x^m x^{-n}$.

	$x^m x^{-n}$
Use the definition: $x^{-n} = \frac{1}{x^n}$.	$= x^m \cdot \frac{1}{x^n}$
Multiply fractions.	$= \frac{x^m}{x^n}$
Use the Quotient Property of exponents.	x^{m-n} or $x^{m+(-n)}$
Conclude.	$x^m x^{-n} = x^{m-n} = x^{m+(-n)}$

So, the product property holds in this case.

And if $m < n$,

	$x^m x^{-n}$
Use the definition: $x^{-n} = \frac{1}{x^n}$.	$= x^m \cdot \frac{1}{x^n}$
Multiply fractions.	$= \frac{x^m}{x^n}$
Use the Quotient Property of exponents.	$= \frac{1}{x^{n-m}}$
Use second variation of the definition of negative exponents.	$= x^{-(n-m)}$
Simplify.	x^{m-n} or $x^{m+(-n)}$
Conclude.	$x^m x^{-n} = x^{m-n} = x^{m+(-n)}$

So the Product Property for Positive Integer Exponents holds in this case, too. We can in a similar way check other combinations to see that the Product Property for Positive Integer Exponents holds for all integers. The Product Property for Integer Exponents follows directly

$$\frac{x^m}{x^n} = x^m \cdot \frac{1}{x^n} = x^m \cdot x^{-n} = x^{m-n}$$

and

$$\frac{x^m}{x^n} = \frac{1}{x^{-m}} \cdot \frac{1}{x^n} = \frac{1}{x^{-m} \cdot x^n} = \frac{1}{x^{-m+n}} \quad \text{or} \quad \frac{1}{x^{n-m}}.$$

So, both the Product Property and the Quotient Property hold for all integer exponents.

Product Property for Integer Exponents

If a is a real number and m and n are integers, then

$$a^m a^n = a^{m+n}.$$

To multiply with like bases, add the exponents.

Quotient Property for Integer Exponents

If a is a real number, $a \neq 0$, and m and n are integers, then

$$\frac{a^m}{a^n} = a^{m-n}$$

and

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}.$$

To divide with like bases, subtract the exponents as above.

We will now use the product property with expressions that have negative exponents. We can choose to use the meaning of negative exponents instead of the quotient property above, so we will focus here on the product property.

Example 1.3.1.15

Simplify each expression:

a. $z^{-5}z^{-3}$

b. $(m^4n^{-3})(m^{-5}n^{-2})$

c. $(2x^{-6}y^8)(-5x^5y^{-3})$

Solution

a.

	$z^{-5}z^{-3}$
Add the exponents, since the bases are the same.	$= z^{-5-3}$
Simplify.	$= z^{-8}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{z^8}$

b.

	$(m^4n^{-3})(m^{-5}n^{-2})$
Use the Commutative Property to get like bases together.	$= m^4m^{-5}n^{-2}n^{-3}$
Add the exponents for each base.	$= m^{-1}n^{-5}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{m^1} \cdot \frac{1}{n^5}$

	$(m^4n^{-3})(m^{-5}n^{-2})$
Simplify.	$=\dfrac{1}{mn^5}$

c.

	$(2x^{-6}y^8)(-5x^5y^{-3})$
Rewrite with the like bases together.	$=2(-5)(x^{-6}x^5)(y^8y^{-3})$
Multiply the coefficients and add the exponents of each variable.	$=-10x^{-1}y^5$
Use the definition of a negative exponent, $a^{-n}=\dfrac{1}{a^n}$.	$=-10\dfrac{1}{x}y^5$
Simplify.	$=-\dfrac{10y^5}{x}$

Try It 1.3.1.16

Simplify each expression:

a. $z^{-4}z^{-5}$

b. $(p^6q^{-2})(p^{-9}q^{-1})$

c. $(3u^{-5}v^7)(-4u^4v^{-2})$

Answer

a. $\dfrac{1}{z^9}$

b. $\dfrac{1}{p^3q^3}$

c. $-\dfrac{12v^5}{u}$

Try It 1.3.1.17

Simplify each expression:

a. $c^{-8}c^{-7}$

b. $(r^5s^{-3})(r^{-7}s^{-5})$

c. $(-6c^{-6}d^4)(-5c^{-2}d^{-1})$

Answer

a. $\dfrac{1}{c^{15}}$

b. $\dfrac{1}{r^2s^8}$

c. $\dfrac{30d^3}{c^8}$

Now let’s look at an exponential expression that contains a power raised to a power. Let's see if we can discover a general property.

$(x^2)^3$

	$(x^2)^3$
What does this mean?	$= x^2 x^2 x^2$
How many factors altogether?	$= \underbrace{x \cdot x}_{2\ factors} \cdot \underbrace{x \cdot x}_{2\text{ factors}} \cdot \underbrace{x \cdot x}_{2\text{ factors}}$ $= \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x}_{6\text{ factors}}$
So we have	$= x^6$

Notice the 6 is the *product* of the exponents, 2 and 3. We see that $(x^2)^3$ is $x^{2 \cdot 3}$ or x^6. We can also see that

$$(x^{-2})^3 = \left(\frac{1}{x^2}\right)^3 = \frac{1}{x^2} \cdot \frac{1}{x^2} \cdot \frac{1}{x^2} = \frac{1}{x^6} = x^{-6}$$

so that $(x^{-2})^3 = x^{-6}$. In these examples we multiplied the exponents.

We can check various combinations of signs of the exponents which leads us to the *Power Property for Integer Exponents.*

Power Property for Integer Exponents

If a is a real number and m and n are integers, then

$$(a^m)^n = a^{mn}.$$

To raise a power to a power, multiply the exponents.

Example 1.3.1.18

Simplify each expression:

a. $(y^5)^9$

b. $(4^{-4})^7$

c. $(y^3)^6 (y^5)^4$

Solution

a.

	$(y^5)^9$
Use the power property, $(a^m)^n = a^{mn}$.	$y^{5 \cdot 9}$
Simplify.	y^{45}

b.

	$(4^{-4})^7$
Use the power property.	$= 4^{-4 \cdot 7}$
Simplify.	$= 4^{-28}$

c.

	$(y^3)^6 (y^5)^4$
Use the power property.	$= y^{18} y^{20}$
Add the exponents.	$= y^{38}$

Try It 1.3.1.19

Simplify each expression:

a. $(b^7)^5$

b. $(5^4)^{-3}$

c. $(a^4)^5(a^7)^4$

Answer

a. b^{35}

b. 5^{-12}

c. a^{48}

Try It 1.3.1.20

Simplify each expression:

a. $(z^6)^9$

b. $(3^{-7})^7$

c. $(q^4)^5(q^3)^3$

Answer

a. z^{54}

b. 3^{-49}

c. q^{29}

We will now look at an expression containing a product that is raised to a power. Can we find this pattern?

	$(2x)^3$
What does this mean?	$= 2x \cdot 2x \cdot 2x$
We group the like factors together.	$= 2 \cdot 2 \cdot 2xxx$
How many factors of 2 and of x?	$= 2^3x^3$

Notice that each factor was raised to the power and $(2x)^3$ is 2^3x^3.

The exponent applies to each of the factors! We can say that the exponent distributes over multiplication. If we were to check various examples with exponents which are negative or zero, then we would find the same pattern emerges. This leads to the *Product to a Power Property for Integer Exponents.*

Product to a Power Property for Integer Exponents

If a and b are real numbers and m is an integer, then

$$(ab)^m = a^m b^m.$$

To raise a product to a power, raise each factor to that power.

Example 1.3.1.21

Simplify each expression:

a. $(-3mn)^3$

b. $(6k^3)^{-2}$

c. $(5x^{-3})^2$

Solution

a.

	$(-3mn)^3$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$= (-3)^3 m^3 n^3$
Simplify.	$= -27m^3n^3$

b.

	$(6k^3)^{-2}$
Use the Power of a Product Property, $(ab)^m = a^m b^m$.	$= 6^{-2}(k^3)^{-2}$
Use the Power Property, $(a^m)^n = a^{mn}$.	$= 6^{-2}k^{-6}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{6^2} \cdot \frac{1}{k^6}$
Simplify.	$= \frac{1}{36k^6}$

c.

	$(5x^{-3})^2$
Use the power of a product property, $(ab)^m = a^m b^m$.	$= 5^2(x^{-3})^2$
Simplify.	$= 25x^{-6}$
Rewrite x^{-6} using $a^{-n} = \frac{1}{a^n}$.	$= 25 \frac{1}{x^6}$
Simplify.	$= \frac{25}{x^6}$

Try It 1.3.1.22

Simplify each expression:

a. $(2wx)^5$

b. $(2b^3)^{-4}$

c. $(8a^{-4})^2$

Answer

a. $32w^5x^5$

b. $\frac{1}{16b^{12}}$

c. $\frac{64}{a^8}$

? Try It 1.3.1.23

Simplify each expression:

a. $(-3y)^3$

b. $(-4x^4)^{-2}$

c. $(2c^{-4})^3$

Answer

a. $-27y^3$

b. $\frac{1}{16x^8}$

c. $8c^{12}$

Now we will look at an example that will lead us to the quotient to a power property.

	$\left(\frac{x}{y}\right)^3$
This means	$=\frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y}$
Multiply the fractions.	$=\frac{xxx}{yyy}$
Write with exponents.	$=\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We see that $\left(\frac{x}{y}\right)^3$ is $\frac{x^3}{y^3}$.

This leads to the *Quotient to a Power Property for Integer Exponents*. We can say that the exponent distributes over division as well as multiplication.

Quotient to a Power Property for Integer Exponents

If a and b are real numbers, $b \neq 0$, and m is an integer, then

$$\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}.$$

To raise a fraction to a power, raise the numerator and denominator to that power.

✓ Example 1.3.1.24

Simplify each expression:

a. $\left(\frac{b}{3}\right)^4$

b. $\left(\frac{k}{j}\right)^{-3}$

c. $\left(\frac{2xy^2}{z}\right)^3$

d. $\left(\frac{4p^{-3}}{q^2}\right)^2$

Solution

a.

	$\left(\frac{b}{3}\right)^4$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$= \frac{b^4}{3^4}$
Simplify.	$= \frac{b^4}{81}$

b.

	$\left(\frac{k}{j}\right)^{-3}$
Raise the numerator and denominator to the power.	$= \frac{k^{-3}}{j^{-3}}$
Use the definition of negative exponent, $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.	$= \frac{1}{k^3} \cdot \frac{j^3}{1}$
Multiply.	$= \frac{j^3}{k^3}$

c.

	$\left(\frac{2xy^2}{z}\right)^3$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$= \frac{(2xy^2)^3}{z^3}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$, and the Power Property, $(a^m)^n = a^{mn}$.	$= \frac{8x^3y^6}{z^3}$

d.

	$\left(\frac{4p^{-3}}{q^2}\right)^2$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$= \frac{(4p^{-3})^2}{(q^2)^2}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$= \frac{16p^{-6}}{q^4}$
Use the definition of negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{16}{q^4} \cdot \frac{1}{p^6}$
Simplify.	$= \frac{16}{p^6q^4}$

Try It 1.3.1.25

Simplify each expression:

a. $\left(\frac{p}{10}\right)^4$

b. $\left(\frac{m}{n}\right)^{-7}$

c. $\left(\frac{3ab^3}{c^2}\right)^4$

d. $\left(\frac{3x^{-2}}{y^3}\right)^3$

Answer

a. $\frac{p^4}{10000}$

b. $\frac{n^7}{m^7}$

c. $\frac{81a^4b^{12}}{c^8}$

d. $\frac{27}{x^6y^9}$

Try It 1.3.1.26

Simplify each expression:

a. $\left(\frac{-2}{q}\right)^3$

b. $\left(\frac{w}{x}\right)^{-4}$

c. $\left(\frac{xy^3}{3z^2}\right)^2$

d. $\left(\frac{2m^{-2}}{n^{-2}}\right)^3$

Answer

a. $\frac{-8}{q^3}$

b. $\frac{x^4}{w^4}$

c. $\frac{x^2y^6}{9z^4}$

d. $\frac{8n^6}{m^6}$

We now have several properties for exponents. Let's summarize them and then we'll do some more examples that use more than one of the properties. We note that there are many ways to simplify the expressions, but the final simplification should be equivalent.

Summary of Integer Exponent Definitions and Properties

If a and b are real numbers, and m and n are integers, then

Definition	Description
Definition of Zero Exponent	$a^0 = 1, a \neq 0$
Definition of Negative Exponents	$a^{-n} = \frac{1}{a^n}$, or equivalently, $\frac{1}{a^{-n}} = a^n$

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^n = a^n b^n$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Example 1.3.1.27

Simplify each expression by applying several properties:

a. $(3x^2y)^4(2xy^2)^3$

b. $\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$

c. $\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$

Solution

a.

	$(3x^2y)^4(2xy^2)^3$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$= (3^4x^8y^4)(2^3x^3y^6)$
Simplify.	$= (81x^8y^4)(8x^3y^6)$
Use the Commutative Property.	$= 81 \cdot 8x^8x^3y^4y^6$
Multiply the constants and add the exponents.	$= 648x^{11}y^{10}$

b.

	$\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$= \frac{x^{12}x^{-10}}{x^{30}}$

	$\dfrac{(x^3)^4(x^{-2})^5}{(x^6)^5}$
Add the exponents in the numerator.	$=\dfrac{x^2}{x^{30}}$
Use the Quotient Property, $\dfrac{a^m}{a^n}=\dfrac{1}{a^{n-m}}$	$=\dfrac{1}{x^{28}}$

c.

	$\left(\dfrac{2xy^2}{x^3y^{-2}}\right)^2\left(\dfrac{12xy^3}{x^3y^{-1}}\right)^{-1}$
Simplify inside the parentheses first.	$=\left(\dfrac{2y^4}{x^2}\right)^2\left(\dfrac{12y^4}{x^2}\right)^{-1}$
Use the Quotient to a Power Property, $\left(\dfrac{a}{b}\right)^m=\dfrac{a^m}{b^m}$.	$=\dfrac{(2y^4)^2}{(x^2)^2}\dfrac{(12y^4)^{-1}}{(x^2)^{-1}}$
Use the Product to a Power Property, $(ab)^m=a^mb^m$.	$=\dfrac{4y^8}{x^4}\cdot\dfrac{12^{-1}y^{-4}}{x^{-2}}$
Simplify.	$=\dfrac{4y^4}{12x^2}$
Simplify.	$=\dfrac{y^4}{3x^2}$

? Try It 1.3.1.28

Simplify each expression:

a. $(c^4d^2)^5(3cd^5)^4$

b. $\dfrac{(a^{-2})^3(a^2)^4}{(a^4)^5}$

c. $\left(\dfrac{3xy^2}{x^2y^{-3}}\right)^2$

Answer

a. $81c^{24}d^{30}$

b. $\dfrac{1}{a^{18}}$

c. $\dfrac{9y^{10}}{x^2}$

? Try It 1.3.1.29

Simplify each expression:

a. $(a^3b^2)^6(4ab^3)^4$

b. $\dfrac{(p^{-3})^4(p^5)^3}{(p^7)^6}$

c. $\left(\dfrac{4x^3y^2}{x^2y^{-1}}\right)^2\left(\dfrac{8xy^{-3}}{x^2y}\right)^{-1}$

Answer

a. $256a^{22}b^{24}$

b. $\dfrac{1}{p^{39}}$

c. $2x^3y^{10}$

? Writing Exercises 1.3.1.30

1. Give an example of distributing division over subtraction.
2. Give an example to show that you can not distribute division over multiplication.
3. How is the negative exponent related to reciprocals? Give an example.
4. How are positive and negative exponents used in science to express large or small numbers?
5. What is the purpose in writing numbers this way?

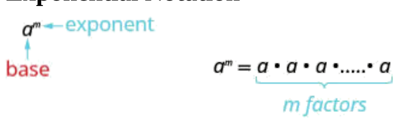
Exit Problem 1.3.1.31

a. Simplify $\left(\dfrac{3a^{-3}}{b^{-5}}\right)^{-3}$. Write your final answer with positive exponents only.

b. Simplify $\dfrac{36x^5y^{10}}{70x^{15}y^5}$. Write your final answer with positive exponents only.

Key Concepts

- **Exponential Notation**

  This is read a to the m^{th} power.
  In the expression a^m, the *exponent* m (when positive) tells us how many times we use the *base* a as a factor.
- **Zero Exponent (Definition)** If a is a non-zero number, then $a^0 = 1$.
- **Negative Exponent (Definition)** If n is an integer and $a \neq 0$, then $a^{-n} = \dfrac{1}{a^n}$ or, equivalently, $\dfrac{1}{a^{-n}} = a^n$.
- **Product Property for Exponents**
  If a is a real number and m and n are integers, then

  $$a^m a^n = a^{m+n}$$

  To multiply with like bases, add the exponents.
- **Quotient Property for Exponents**
  If a is a real number, $a \neq 0$, and m and n are integers, then

  $$\frac{a^m}{a^n} = a^{m-n}, \quad m > n \quad \text{and} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad n > m$$

- **Quotient to a Negative Exponent Property**
  If a and b are real numbers, $a \neq 0$, $b \neq 0$ and n is an integer, then

  $$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

- **Power Property for Exponents**
  If a is a real number and m and n are integers, then

$$(a^m)^n = a^{mn}$$

To raise a power to a power, multiply the exponents.

- **Product to a Power Property for Exponents**
  If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m$$

  To raise a product to a power, raise each factor to that power.

- **Quotient to a Power Property for Exponents**
  If a and b are real numbers, $b \neq 0$, and m is an integer, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

  To raise a fraction to a power, raise the numerator and denominator to that power.

- **Summary of Exponent Properties**
  If a and b are real numbers, and m and n are integers, then

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Property	Description
Definition of Zero Exponent	$a^0 = 1, a \neq 0$
Definition of Negative Exponents	$a^{-n} = \frac{1}{a^n}$, or equivalently, $\frac{1}{a^{-n}} = a^n$

-

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
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Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$
