1.3.3: Adding and Subtracting Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Be Prepared

Before you get started, take this readiness quiz.

1. Add $\frac{7}{10}+\frac{8}{15}$.
2. Subtract $\frac{3x}{4}-\frac{8}{9}$.
3. Subtract $6(2x+1)-4(x-5)$.

Add and Subtract Rational Expressions with a Common Denominator

What is the first step we take when we add numerical fractions? We check if they have a common denominator. If they do, we add the numerators and place the sum over the common denominator. If they do not have a common denominator, we find one before we add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, we add the numerators and place the sum over the common denominator.

Rational Expression Addition and Subtraction

If p, q, and r are polynomials where $r \neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}.$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after subtracting the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of x should be excluded in the next example?

Example 1.3.3.1

Add: $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$.

Solution

Since the denominator is $x+4$, we must exclude the value $x=-4$.

$$\frac{11x+28}{x+4}+\frac{x^2}{x+4},\ x\neq -4$$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator. $\frac{11x+28+x^2}{x+4}$

Write the degrees in descending order. $\frac{x^2+11x+28}{x+4}$

Factor the numerator. $\frac{(x+4)(x+7)}{x+4}$

Simplify by removing common factors. $\frac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$

Simplify. $x+7$

This simplification is valid for values of x where $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$ is defined.

As before, we will tacitly understand that the simplifications are valid for all values of the variables for which the original expression is defined.

Try It 1.3.3.2

Simplify $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

Answer

$x+2$

Try It 1.3.3.3

Simplify $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Answer

$x+3$

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, we subtract the numerators and place the difference over the common denominator. Be careful of the signs when subtracting a binomial or trinomial.

Example 1.3.3.4

Subtract: $\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$.

Solution

	$\dfrac{5x^2-7x+3}{x^2-3x+18}-\dfrac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the difference over the common denominator.	$\dfrac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\dfrac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\dfrac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\dfrac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\dfrac{(x-2)\cancel{(x-6)}}{(x+3)\cancel{(x-6)}}$
	$(x-2)(x+3)$

? Try It 1.3.3.5

Subtract: $\dfrac{4x^2-11x+8}{x^2-3x+2}-\dfrac{3x^2+x-3}{x^2-3x+2}$.

Answer

$\dfrac{x-11}{x-2}$

? Try It 1.3.3.6

Subtract: $\dfrac{6x^2-x+20}{x^2-81}-\dfrac{5x^2+11x-7}{x^2-81}$.

Answer

$\dfrac{x-3}{x+9}$

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works when simplifying $\dfrac{2}{x-1}-\dfrac{3}{1-x}$.

	$\dfrac{2}{x-1}-\dfrac{3}{1-x}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\dfrac{2}{x-1}-\dfrac{-3}{-(1-x)}$

The denominators are the same.	$\frac{2}{x-1}-\frac{-3}{x-1}$
Simplify.	$\frac{2+3}{x-1}=\frac{5}{x-1}$

Be careful with the signs as we work with the opposites when the fractions are being subtracted.

Example 1.3.3.7

Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Solution

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-(3m+2)}{-(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2)}{m^2-1)}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	$\frac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m+1)(m-1)}$
Simplify by dividing out common factors.	$\frac{(m-2)}{(m+1)}$
Simplify.	$dfracm-2m+1$

Try It 1.3.3.8

Subtract: $\frac{y^2-5y}{y^2-4}-\frac{6y-6}{4-y^2}$.

Answer

$\frac{y+3}{y+2}$

Try It 1.3.3.9

Subtract: $\frac{2n^2+8n-1}{n^2-1}-\frac{n^2-7n-1}{1-n^2}$.

Answer

$\frac{3n-2}{n-1}$

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example: $\frac{7}{12}+\frac{5}{18}$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes: $12=2\cdot2\cdot3$ and $18=2\cdot3\cdot3$. We recognize each of these factorizations contain a 2 and a 3. But we need two 2s to accomodate 12 and two 3s to accomodate 18 and there are no other factors present. So, the LCD is $2^2\cdot3^2=36$.

We could arrange the factors of 12 and 18 in a table as well as follows:

factors of 12	2	2	3	
factors of 18	2		3	3
factors of the LCD	2	2	3	3

Here we have aligned common factors and then 'brought down the common factor so that the last column consists of the factors of the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number (multiplying the fraction by one doesn't change its value!). We are now ready to add.

$$\frac{7}{12}+\frac{5}{18}=\frac{7\cdot3}{12\cdot3}+\frac{5\cdot2}{18\cdot2}=\frac{21+10}{36}=\frac{31}{36}. \tag{1.3.3.1}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

Find the Least Common Denominator (LCD) of Rational Expressions

1. Factor each denominator completely.
2. List the factors of each denominator. Match factors vertically when possible.
3. Bring down the columns by including all factors, but do not include common factors twice.
4. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of x should we exclude in this next example?

Example 1.3.3.10

a. Find the LCD for the expressions $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$ and b. rewrite them as equivalent rational expressions with the lowest common denominator.

Solution

a.

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.	

Factor each denominator completely, lining up common factors. Bring down the columns.	factors of x^2-2x-3: $x-3$, $x+1$ factors of x^2+4x+3: $x+1$, $x+3$ factors of the LCD: $x-3$, $x+1$, $x+3$
Write the LCD as the product of the factors.	$(x-3)(x+1)(x+3)$

b.

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x-3)(x+1)}, \frac{3x}{(x+1)(x+3)}$
Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x-3)(x+1)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x-3)(x+1)(x+3)}, \frac{3x^2-9x)}{(x+1)(x+3)(x-3)}$

Try It 1.3.3.11

a. Find the LCD for the expressions $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x-4)(x+3)(x+4)$

b. $\frac{2x+8}{(x-4)(x+3)(x+4)}$,
$\frac{x+3}{(x-4)(x+3)(x+4)}$

Try It 1.3.3.12

a. Find the LCD for the expressions $\frac{3x}{x^2-3x+10}, \frac{5}{x^2+3x+2}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x+2)(x-5)(x+1)$

b. $\frac{3x^2+3x}{(x+2)(x-5)(x+1)}$,
$\frac{5x-25}{(x+2)(x-5)(x+1)}$

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

Example 1.3.3.13

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution

$$\frac{3}{x-3}+\frac{2}{x-2}=\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}=\frac{3x-6)}{(x-3)(x-2)}+\frac{2x-6)}{(x-2)(x-3)}$$

Now that we have common denominators, we can add the numerators (the number of 'parts' which are now of equal sizes):

$$\frac{3}{x-3}+\frac{2}{x-2}=\frac{3x-6+2x-6}{(x-2)(x-3)}=\frac{5x}{(x-2)(x-3)} \tag{1.3.3.2}$$

Try It 1.3.3.14

Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Answer

$$\frac{7x-4}{(x-2)(x+3)}$$

Try It 1.3.3.15

Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Answer

$$\frac{7m+25}{(m+3)(m+4)}$$

The steps used to add rational expressions are summarized here.

Add or Subtract Rational Expressions

1. Determine if the expressions have a common denominator.
   - **Yes** – go to step 2.
   - **No** – Rewrite each rational expression with the LCD.
     - Find the LCD.
     - Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as $\frac{3x-6}{(x-3)(x-2)}$ to be able to add it to $\frac{2x-6}{(x-2)(x-3)}$. Simplify *only* after combining the numerators.

Example 1.3.3.16

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\dfrac{8}{x^2-2x-3}+\dfrac{3x}{x^2+4x+3}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	Find the LCD. $\begin{aligned} x^2-2x-3 &= (x+1)(x-3) \\ x^2+4x+3 &= (x+1)\quad(x+3) \end{aligned}$ $LCD=(x+1)(x-3)(x+3)$
Rewrite each rational expression as an equivalent rational expression with the LCD.	\(\dfrac{8}{(x+1)(x−3)}+\dfrac{3x}{(x+1) (x+3)}=\dfrac{8(x+3)}{(x+1)(x−3)(x+3)}+\dfrac{3x(x-3)}{(x+1) (x+3)(x-3)}\)
Simplify the numerators.	\(=\dfrac{8x+24}{(x+1)(x−3)(x+3)}+\dfrac{3x^2-9x)}{(x+1) (x+3)(x-3)\)
Add the rational expressions.	$=\dfrac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$=\dfrac{3x^2-x+24}{(x+1)(x-3)(x+3)}$
	The numerator is prime, so there are no common factors.

Try It 1.3.3.17

Add: $\dfrac{1}{m^2-m-2}+\dfrac{5m}{m^2+3m+2}$.

Answer

$$\frac{5m^2-9m+2}{(m+1)(m-2)(m+2)}$$

Try It 1.3.3.18

Add: $\dfrac{2n}{n^2-3n-10}+\dfrac{6}{n^2+5n+6}$.

Answer

$$\frac{2n^2+12n-30}{(n+2)(n-5)(n+3)}$$

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

Example 1.3.3.19

Subtract: $\dfrac{8y}{y^2-16}-\dfrac{4}{y-4}$.

Solution

	$\dfrac{8y}{y^2-16}-\dfrac{4}{y-4}$
Do the expressions have a common denominator?	No.

Rewrite each expression with the LCD.	Find the LCD. $\begin{array}{l} y^2-16=(y-4)(y+4) \\ \underline{y-4=y-4} \\ LCD=(y-4)(y+4) \end{array}$
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{y^2-16}-\frac{4}{y-4}$ $=\frac{8y}{(y+4)(y-4)}-\frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$=\frac{8y}{(y+4)(y-4)}-\frac{4y+16)}{(y-4)(y+4)}$
Subtract the rational expressions.	$=\frac{8y-(4y+16)}{(y+4)(y-4)}$
Simplify the numerator.	$=\frac{8y-4y-16)}{(y+4)(y-4)}=\frac{4y-16}{(y+4)(y-4)}$
Factor the numerator to look for common factors.	$=\frac{4(y-4)}{(y+4)(y-4)}$
Remove common factors	$=\frac{4}{y+4}$, for values of y that make sense in the original expression.
Simplify. This one is already simplified.	$=\frac{4}{y+4}$

? Try It 1.3.3.20

Subtract: $\frac{2x}{x^2-4}-\frac{1}{x+2}$.

Answer

$\frac{1}{x-2}$

? Try It 1.3.3.21

Subtract: $\frac{3}{z+3}-\frac{6z}{z^2-9}$.

Answer

$\frac{-3}{z-3}$

There are lots of negative signs in the next example. Be extra careful.

? Example 1.3.3.22

Subtract: $\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$.

Solution

	$\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$
Factor the denominator.	$=\frac{-3n-9}{(n+3)(n-2)}-\frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$=\frac{-3n-9}{(n+3)(n-2)}-\frac{-(n+3)}{-(2-n)}$

Simplify. Remember, $a-(-b)=a+b$.	$=\frac{-3n-9}{(n+3)(n-2)}-\frac{-n-3)}{-2+n)}$
Do the rational expressions have a common denominator? No.	
Find the LCD.	Find the LCD. $\begin{array}{r}n^2+n-6=(n-2)(n+3)\\ \underline{n-2=(n-2)}\\ LCD=\quad(n-2)(n+3)\end{array}$
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}=\frac{-3n-9}{(n+3)(n-2)}-\frac{(-n-3)(n+3)}{-2+n)(n+3)}$
Simplify the numerators.	$=\frac{-3n-9}{(n+3)(n-2)}-\frac{-n^2-6n-9}{n-2)(n+3)}$
Add the rational expressions.	$=\frac{-3n-9-(-n^2-6n-9)}{(n+3)(n-2)}$
Simplify the numerator.	$=\frac{-3n-9+n^2+6n+9)}{(n+3)(n-2)}=\frac{n^2+3n)}{(n+3)(n-2)}$
Factor the numerator to look for common factors.	$=\frac{n(n+3)}{(n+3)(n-2)}$
Simplify.	$=\frac{n}{n-2}$ for values of n which make sense in the original expression (all values except -3 and 2).
You can verify this simplification by choosing a few values for n to make sure you get the same result upon simplifying. This isn't mathematically necessary, but a good check for errors.	

Try It 1.3.3.23

Subtract: $\frac{3x-1}{x^2-5x-6}-\frac{2}{6-x}$.

Answer

$$\frac{5x+1}{(x-6)(x+1)}$$

Try It 1.3.3.24

Subtract: $\frac{-2y-2}{y^2+2y-8}-\frac{y-1}{2-y}$.

Answer

$$\frac{y+3}{y+4}$$

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

Example 1.3.3.25

Subtract: $\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$.

Solution

	$\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$

Factor the denominators.	$=\frac{4}{(a+5)(a+1)}-\frac{3}{(a+5)(a+2)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $\begin{array}{l} a^2+6a+5=(a+1)(a+5) \\ \underline{a^2+7a+10=(a+5)(a+2)} \\ LCD=(a+1)(a+5)(a+2) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$=\frac{4(a+2)}{(a+5)(a+1)(a+2)}-\frac{3(a+1)}{(a+5)(a+2)(a+1)}$
Simplify the numerators.	$=\frac{4a+8}{(a+5)(a+1)(a+2)}-\frac{3a+3}{(a+5)(a+2)(a+1)}$
Subtract the rational expressions.	$=\frac{4a+8-(3a+3)}{(a+5)(a+1)(a+2)}$
Simplify the numerator.	$=\frac{4a+8-3a-3}{(a+5)(a+1)(a+2)}=\frac{a+5}{(a+5)(a+1)(a+2)}$
Look for common factors and simplify.	$=\frac{1}{(a+1)(a+2)}$ for values that make sense in the original expression (all values except $-5, -1$, and -2).

? Try It 1.3.3.26

Subtract: $\frac{3}{b^2-4b-5}-\frac{2}{b^2-6b+5}$.

Answer

$$\frac{1}{(b+1)(b-1)}$$

? Try It 1.3.3.27

Subtract: $\frac{4}{x^2-4}-\frac{3}{x^2-x-2}$.

Answer

$$\frac{1}{(x+2)(x+1)}$$

? Writing Exercises 1.3.3.28

1. What is the LCD?
2. When is the LCD used? Why?
3. Why can't $\frac{x+5}{x-5}$ be simplified?
4. When I add the two fractions $\frac{8y}{y^2-16}-\frac{4}{y-4}$ to get $\frac{4}{y+4}$, are the two rational expressions the same for all values of y? Explain.
5. What is the first goal when adding fractions?
6. When adding fractions, why might you prefer a least common denominator rather than another common denominator? Give an example.

Exit Question 1.3.3.29

Simplify the expression $\frac{x-1}{x^2+x-6}-\frac{5}{3x-6}$.

Key Concepts

- **Rational Expression Addition and Subtraction**
  If p, q, and r are polynomials where $r \neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

- **How to find the least common denominator of rational expressions.**
  1. Factor each expression completely.
  2. List the factors of each expression. Match factors vertically when possible.
  3. Bring down the columns.
  4. Write the LCD as the product of the factors.
- **How to add or subtract rational expressions.**
  1. Determine if the expressions have a common denominator.
     - Yes – go to step 2.
     - No – Rewrite each rational expression with the LCD.
       - Find the LCD.
       - Rewrite each rational expression as an equivalent rational expression with the LCD.
  2. Add or subtract the rational expressions.
  3. Simplify, if possible.