1.3.4: Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify: $\dfrac{\frac{3}{5}}{\frac{9}{10}}$.
2. Simplify: $\dfrac{1-\frac{1}{3}}{4^2+4\cdot 5}$.

Simplify a Complex Rational Expression by Writing it as Division

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \quad \text{and} \quad \frac{\frac{x}{2}}{\frac{xy}{6}}.$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

Definition 1.3.4.1

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are some complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}, \quad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \quad \text{and} \quad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}.$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter:

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}.$$

We noted that fraction bars tell us to divide, and rewrote it as the division problem:

$$\left(\frac{6x^2-7x+2}{4x-8}\right) \div \left(\frac{2x^2-8x+3}{x^2-5x+6}\right).$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form where one fraction is over one fraction. We then write it as if we were dividing two fractions.

Example 1.3.4.2

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}.$$

Solution

	$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$
Rewrite the complex fraction as division.	$=\frac{6}{x-4}\div\frac{3}{x^2-16}$
Rewrite as the product of first times the reciprocal of the second.	$=\frac{6}{x-4}\cdot\frac{x^2-16}{3}$
Factor.	$=\frac{3\cdot 2}{x-4}\cdot\frac{(x-4)(x+4)}{3}$
Multiply.	$=\frac{3\cdot 2(x-4)(x+4)}{3(x-4)}$
Remove common factors.	$=\frac{\cancel{3}\cdot 2\,\cancel{(x-4)}\,(x+4)}{\cancel{3}\,\cancel{(x-4)}}$
Simplify.	$=2(x+4)$

Are there any value(s) of x that should not be allowed? The original complex rational expression had denominators of $x-4$ and x^2-16. This expression would be undefined if $x=4$ or $x=-4$.

Try It 1.3.4.3

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}.$$

Answer

$$\frac{2}{3(x-1)}$$

Try It 1.3.4.4

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}.$$

Answer

$$\frac{1}{2(x-3)}$$

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

Example 1.3.4.5

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}.$$

Solution

	$\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.	$=\dfrac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$
Simplify the numerator and denominator.	$=\dfrac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$ $=\dfrac{\frac{3}{6}}{\frac{1}{6}}$
Rewrite the complex rational expression as a division problem.	$=\frac{3}{6}\div\frac{1}{6}$
Multiply the first by the reciprocal of the second.	$=\frac{3}{6}\cdot\frac{6}{1}$
Simplify.	$=3$

Try It 1.3.4.6

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$$

Answer

$$\frac{14}{11}$$

Try It 1.3.4.7

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$$

Answer

$\frac{10}{23}$

We follow the same procedure when the complex rational expression contains variables.

Example 1.3.4.8

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}.$$

Solution

	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$	
Simplify the numerator.	We will simplify the sum in the numerator and the difference in the denominator.	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find the LCD in the numerator and denominator.	$=\frac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
	Simplify.	$=\frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	Add the fractions in the numerator and subtract the fractions in the denominator. We now have just one rational expression in the numerator and one in the denominator.	$=\frac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$=\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$
Divide the expressions.	Multiply the first by the reciprocal of the second.	$=\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$=\frac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$=\frac{\cancel{xy}\,\cancel{(y+x)}}{\cancel{xy}(x-y)\,\cancel{(x+y)}}$
	Simplify.	$=\frac{1}{x-y}$

Try It 1.3.4.9

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}.$$

Answer

$$\frac{y+x}{y-x}$$

Try It 1.3.4.10

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}.$$

Answer

$$\frac{ab}{b-a}$$

We summarize the steps here.

How to Simplify a Complex Rational Expression by Writing It as Division

1. Rewrite the complex rational expression as a division problem.
2. Divide the expressions.

Example 1.3.4.11

Simplify the complex rational expression by writing it as division:

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}.$$

Solution

	$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$
Simplify the numerator and denominator. Find common denominators for the numerator and denominator.	$=\frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	$=\frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$

	$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$$
Subtract the rational expressions in the numerator and add in the denominator.	$$=\frac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$$
Simplify. (We now have one rational expression over one rational expression.)	$$=\frac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$$
Rewrite as fraction division.	$$=\frac{n^2+n}{n+5}\div\frac{2n}{(n+5)(n-5)}$$
Multiply the first times the reciprocal of the second.	$$=\frac{n^2+n}{n+5}\cdot\frac{(n+5)(n-5)}{2n}$$
Factor any expressions if possible.	$$=\frac{n(n+1)(n+5)(n-5)}{(n+5)2n}$$
Remove common factors.	$$=\frac{\cancel{n}(n+1)\cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\cancel{n}}$$
Simplify.	$$=\frac{(n+1)(n-5)}{2}$$

? Try It 1.3.4.12

Simplify the complex rational expression by writing it as division:

$$\frac{b-\frac{3b}{b+5}}{\frac{2}{b+5}+\frac{1}{b-5}}.$$

Answer

$$\frac{b(b+2)(b-5)}{3b-5}$$

? Try It 1.3.4.13

Simplify the complex rational expression by writing it as division:

$$\frac{1-\frac{3}{c+4}}{\frac{1}{c+4}+\frac{c}{3}}.$$

Answer

$$\frac{3}{c+3}$$

Simplify a Complex Rational Expression by Using the LCD

We “cleared” the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let's look at the complex rational expression we simplified one way in Example 7.4.2. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{\text{LCD}}{\text{LCD}}$ we are multiplying by 1, so the value stays the same.

Example 1.3.4.14

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}.$$

Solution

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6. Clear the fractions by multiplying the numerator and denominator by that LCD.	$=\frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$=\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$=\frac{2+1}{3-2}$ $=\frac{3}{1}$ $=3$

Try It 1.3.4.15

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}.$$

Answer

$\frac{7}{3}$

Try It 1.3.4.16

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}.$$

Answer

$\frac{10}{3}$

We will use the same example as in Example 7.4.3. Decide which method works better for you.

Example 1.3.4.17

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}.$$

Solution

		$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Find the LCD of all fractions in the is complex rational expression.	The LCD of all the fractions is xy.	$=\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by xy.	$=\frac{{\color{red}xy}\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{{\color{red}xy}\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$
Simplify the expression.	Distribute.	$=\frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$ $=\frac{y+x}{x^2-y^2}$
	Simplify.	$=\frac{\cancel{(y+x)}}{(x-y)\cancel{(x+y)}}$
	Remove common factors.	$=\frac{1}{x-y}$

Try It 1.3.4.18

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}.$$

Answer

$$\frac{b+a}{a^2+b^2}$$

Try It 1.3.4.19

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}.$$

Answer

$\frac{y-x}{xy}$

How to Simplify a Complex Rational Expression by Using the LCD

1. Multiply the numerator and denominator by the LCD of all rational expressions.
2. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Example 1.3.4.20

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}.$$

Solution

	$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$
Find the LCD of all fractions in the complex rational expression.	The LCD is: $x^2-36=(x+6)(x-6)$.
Multiply the numerator and denominator by the LCD.	$=\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$
Distribute in the denominator.	$=\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$
Simplify.	$=\frac{\cancel{(x+6)}(x-6)\frac{2}{\cancel{x+6}}}{(x+6)\cancel{(x-6)}\left(\frac{4}{\cancel{x-6}}\right)-\cancel{(x+6)(x-6)}\left(\frac{4}{\cancel{(x+6)(x-6)}}\right)}$
Simplify.	$=\frac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$=\frac{2(x-6)}{4x+20}$
Factor the denominator.	$=\frac{2(x-6)}{4(x+5)}$
Remove common factors.	$=\frac{\cancel{2}(x-6)}{\cancel{2}\cdot 2(x+5)}$
Simplify.	$=\frac{x-6}{2(x+5)}$ Notice that there are no more factors common to the numerator and denominator.

Try It 1.3.4.21

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}.$$

Answer

$\frac{3(x-2)}{5x+7}$

Try It 1.3.4.22

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}.$$

Answer

$\frac{x+21}{6x-43}$

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

Example 1.3.4.23

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}.$$

Solution

	$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$
Find the LCD of all fractions in the complex rational expression.	The LCD is $(m-3)(m-4)$.
Multiply the numerator and denominator by the LCD.	$=\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$
Simplify.	$=\frac{\cancel{(m-3)(m-4)}\frac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$
Simplify.	$=\frac{4}{3(m-4)-2(m-3)}$
Distribute.	$=\frac{4}{3m-12-2m+6}$

	$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$$
Combine like terms.	$=\frac{4}{m-6}$

Try It 1.3.4.24

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}.$$

Answer

$$\frac{3}{5x+22}$$

Try It 1.3.4.25

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}.$$

Answer

$$\frac{2\left(2y^2+13y+5\right)}{3y}$$

Example 1.3.4.26

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}.$$

Solution

	$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$$
Find the LCD of all fractions in the complex rational expression.	The LCD is $(y+1)(y-1)$.
Multiply the numerator and denominator by the LCD.	$=\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$

	$\dfrac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$
Distribute in the denominator and simplify.	$=\dfrac{\cancel{(y+1)}(y-1)\frac{y}{\cancel{y+1}}}{(y+1)(y-1)(1)+(y+1)\cancel{(y-1)}\left(\frac{1}{\cancel{y-1}}\right)}$
Simplify.	$=\dfrac{(y-1)y}{(y+1)(y-1)+(y+1)}$
Simplify the denominator and leave the numerator factored.	$=\dfrac{y(y-1)}{y^2-1+y+1}$ $=\dfrac{y(y-1)}{y^2+y}$
Factor the denominator and remove factors common with the numerator.	$=\dfrac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$
Simplify.	$=\dfrac{y-1}{y+1}$

? Try It 1.3.4.27

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}.$$

Answer

$\dfrac{x}{x+4}$

? Try It 1.3.4.28

Simplify the complex rational expression by using the LCD:

$$\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}.$$

Answer

$\dfrac{x(x+1)}{3(x-1)}$

? Writing Exercise 1.3.4.29

1. What is the LCD (least common denominator)?
2. What is the purpose in multiplying numerator and denominator by the LCD in Method II?
3. What happens if you multiply by a common denominator that is not the least common denominator?

Exit Problem 1.3.4.30

Simplify the expression $\dfrac{\dfrac{3}{y^2}+\dfrac{4}{y}+1}{\dfrac{3}{y^2}+\dfrac{1}{y}}$.

Key Concepts

- **Complex rational expression**
- **How to simplify a complex rational expression by writing it as division.**
  1. Simplify the numerator and denominator.
  2. Rewrite the complex rational expression as a division problem.
  3. Divide the expressions.
- **How to simplify a complex rational expression by using the LCD.**
  1. Find the LCD of all fractions in the complex rational expression.
  2. Multiply the numerator and denominator by the LCD.
  3. Simplify the expression.